# *Regularity of Relaxed Minimizers of Quasiconvex Variational Integrals with $(p,q)$-Growth*

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**Abstract**

We consider autonomous integrals

$$F[u] := \int_\Omega f(Du)\,dx \qquad \text{for } u : \mathbb{R}^n \supset \Omega \to \mathbb{R}^N$$

in the multidimensional calculus of variations, where the integrand $f$ is a strictly quasiconvex $C^2$-function satisfying the $(p,q)$-growth conditions

$$\gamma |A|^p \le f(A) \le \Gamma(1+|A|^q) \qquad \text{for every } A \in \mathbb{R}^{nN}$$

with exponents $1 < p \le q < \infty$.

We examine the Lebesgue-Serrin extension

$$\mathscr{F}_{\mathrm{loc}}[u] := \inf\left\{ \liminf_{k\to\infty} F[u_k] \,:\, W^{1,q}_{\mathrm{loc}} \ni u_k \xrightarrow[k\to\infty]{} u \text{ weakly in } W^{1,p} \right\}$$

of $F$ and establish an existence result for minimizers of $\mathscr{F}_{\mathrm{loc}}$. Furthermore, we prove a corresponding partial $C^{1,\alpha}$-regularity theorem for $q < p + \frac{\min\{2,p\}}{2n}$, which is the first regularity result for this class of integrands.

**Key words.** Calculus of variations – Partial regularity – Quasiconvexity – Non-standard growth – Relaxation

## 1. Introduction

Throughout this paper let $n, N \in \mathbb{N}$ with $n \ge 2$, let $\Omega$ denote a bounded open set in $\mathbb{R}^n$ and $\mathbb{O}_\Omega$ the family of all open subsets of $\Omega$. We study variational integrals

$$F[u;O] := \int_O f(Du)\,dx \in [0,\infty] \qquad \text{for } O \in \mathbb{O}_\Omega \text{ and } u \in W^{1,1}_{\mathrm{loc}}(O;\mathbb{R}^N),$$

where the integrand $f : \mathbb{R}^{nN} \to [0,\infty[$ is a continuous function.

The existence and the regularity of minimizers of $F$ (see [D2, G] for a general introduction) have been successfully investigated if one of the following sets of assumptions is available:

(I) $f$ is quasiconvex and satisfies the standard growth conditions

$$\gamma|A|^p \le f(A) \le \Gamma(1+|A|^p) \tag{1.1}$$

for all $A \in \mathbb{R}^{nN}$ and some $1 < p < \infty$.

(II) $f$ is convex and satisfies the $(p,q)$-growth conditions

$$\gamma|A|^p \le f(A) \le \Gamma(1+|A|^q) \tag{1.2}$$

for all $A \in \mathbb{R}^{nN}$ and certain exponents $1 < p \le q < \infty$, where $p$ and $q$ are not too far apart.

We will now briefly outline some results in these settings. However, we do not intend to give a complete overview and refer the reader to the survey [Mi] for a much broader introduction and further references.

First, we describe the setting (I). We start recalling Morrey's notion of quasiconvexity:

**Definition 1.1 (Quasiconvexity).** *$f$ is called quasiconvex iff*

$$\fint_{B_1} f(A + D\varphi) \ge f(A)$$

*holds for every $A \in \mathbb{R}^{nN}$ and every smooth $\varphi : B_1 \to \mathbb{R}^N$ with compact support in the open unit ball $B_1$ in $\mathbb{R}^n$.*

By Jensen's inequality quasiconvexity is a generalization of convexity. It was originally introduced by Morrey [Mo] as a notion for proving the lower semicontinuity and the existence of minimizers of variational integrals. Actually, assuming (1.1) quasiconvexity has turned out to be a necessary and sufficient condition for the sequential weak lower semicontinuity of $F[-;\Omega]$ on $W^{1,p}(\Omega;\mathbb{R}^N)$. This semicontinuity property is, in turn, the crucial ingredient when proving the existence of minimizers in the setting (I) via the direct method of the calculus of variations. The first theorems in this regard have been proved already in [Mo]. Eventually, these results have been considerably extended in [Me, AF1, Ma1] providing a satisfactory theory even for integrals with $x$- and $u$-dependence. Moreover, relaxation theorems for non-quasiconvex integrands have been given in [D1, AF1] and it has been pointed out by Ball [B1] that polyconvexity implies quasiconvexity identifying a large class of natural examples of quasiconvex integrands.

Quasiconvexity is also a fundamental condition for matters of regularity theory: In the setting (I) it was known since the famous counterexamples of De Giorgi [DG] and Nečas [N] that there is no hope for proving everywhere regularity of minimizers of $F$ in the vectorial case $N > 1$. Nevertheless, extending previous regularity results Evans [E] proved partial $C^{1,\alpha}$-regularity of minimizers $u$ of $F$, i. e. local Hölder continuity of the derivative $Du$ outside a negligible set called the singular set. Precisely, he assumed that $f$ is of class $C^2$ and strictly quasiconvex with (1.1) for $p \geq 2$ and an additional bound for the second derivative $D^2 f$. Later this result was generalized to integrals with $x$- and $u$-dependence in [FH, GM] and the bound for $D^2 f$ was removed in [AF2]. An extension to the subquadratic case $1 < p < 2$ has been obtained in [CFM].

Once partial regularity is proved it is natural to ask what can be said about the singular set except that it is negligible. Indeed, if $f$ is strictly convex it can be shown by a well-known method of Giusti [G, Proposition 2.7] that the Hausdorff dimension of the singular set does not exceed $n-2$ and related results in the presence of $x$- and $u$-dependence have been obtained in [KM1]. In contrast, the problem of establishing any nontrivial bound in the general quasiconvex setting (I) seems to be unsolved. However, recently it has been shown in [KM2] that at least for Lipschitz continuous minimizers of quasiconvex integrals the dimension of the singular set is strictly smaller than $n$.

Now we turn to the setting (II): Here, the existence of minimizers can be proved by some classical arguments from functional analysis based on the convexity of $F[-;\Omega]$. In fact, in the presence of convexity the right inequality in (1.2) is not even involved in the existence theory.

Concerning regularity theory the investigation of the $(p,q)$-growth conditions (1.2) was started by Marcellini [Ma3, Ma4] in the scalar case $N = 1$. In the more general vectorial case partial $C^{1,\alpha}$-regularity has been proved in [AF4] for integrands with a particular structure and later in [PS] for general convex integrands satisfying (1.2) with $2 \leq p \leq q < \left\{\frac{np}{n-1}, p+1\right\}$. Eventually, the results of [PS] have been extended in [BF1] assuming just $1 < p \leq q < \frac{n+2}{n} p$ for the exponents. Related results in the presence of $x$-dependence have been obtained e. g. in [ELM, BF2]. Finally, relying strongly on convexity bounds for the dimension of the singular set in the setting (II) can be given. Basically, this issue has been discussed in [L].

In the present paper we treat a more general situation than (I) and (II): We assume that $f$ is quasiconvex with $(p,q)$-growth. Imposing a stronger quasiconvexity condition, namely the $W^{1,p}$-quasiconvexity of [BM1] (see Definition 7.4), an existence and regularity theory for minimizers of $F$ in this setting has been established in [S2]. In the following we will apply a natural relaxation procedure to show that similar results hold even if this stronger quasiconvexity assumption is not available.

In our situation the well-known semicontinuity theorems of [Mo, Me, AF1, Ma1], which were developed for the setting (I), guarantee weak lower semicontinuity of $F[-;\Omega]$ on $W^{1,q}(\Omega;\mathbb{R}^N)$ but, in general, not on $W^{1,p}(\Omega;\mathbb{R}^N)$. This semicontinuity property is not sufficient to prove the existence of minimizers of $F$. However, some improved semicontinuity results are available. Extending previous results of Marcellini [Ma2] it has been shown by Fonseca & Malý [FM1, Theorem 4.1] and Kristensen [K, Corollary 3.3] that $F[-;\Omega]$ is lower semicontinuous with respect to weak $W^{1,p}$-convergence of $W^{1,q}_{\mathrm{loc}}$-functions under the assumption $q < \frac{np}{n-1}$. By the direct method we deduce that every Dirichlet class $\mathscr{D}$ contains a function $u$ with $F[u;\Omega] \leq F[v;\Omega]$ for all $W^{1,q}_{\mathrm{loc}}$-functions $v \in \mathscr{D}$ (see [S2, Corollary 4.2]). However, this minimizing property is quite weak and no corresponding regularity theory is known.

Here, we follow a different approach. Relaxing with respect to the preceding semicontinuity property as in [FM1], we introduce the following functional:

**Definition 1.2.** *For* $1 < p \leq q < \infty$, $O \in \mathbb{O}_\Omega$ *and* $u \in W^{1,p}(O;\mathbb{R}^N)$ *we define*

$$\mathscr{F}_{\mathrm{loc}}[u;O] := \inf\left\{ \liminf_{k\to\infty} F[u_k;O] \,:\, \begin{array}{l} u_k \in W^{1,q}_{\mathrm{loc}}(O;\mathbb{R}^N)\cap W^{1,p}(O;\mathbb{R}^N), \\ u_k \xrightarrow[k\to\infty]{} u \text{ weakly in } W^{1,p}(O;\mathbb{R}^N) \end{array} \right\}.$$

Let us state some remarks to clarify the meaning of this definition:

- Since smooth functions are dense in $W^{1,p}$, such sequences $u_k$ always exist.
- For $u \in W^{1,q}_{\mathrm{loc}}(O;\mathbb{R}^N)\cap W^{1,p}(O;\mathbb{R}^N)$ and $q < \frac{np}{n-1}$ one has $\mathscr{F}_{\mathrm{loc}}[u;O] = F[u;O]$ (Remark 4.13). Thus, the definition of $\mathscr{F}_{\mathrm{loc}}$ can be interpreted as a way to extend $F$ from $W^{1,q}_{\mathrm{loc}}$ to $W^{1,p}$ by semicontinuity, following the classical idea of the Lebesgue-Serrin-extension. We refer to [Ma2] for a discussion of this point of view.
- $\mathscr{F}_{\mathrm{loc}}$ and $F$ coincide if either $f$ is convex (Theorem 4.15) or $f$ has standard growth (Lemma 4.12).
- However, examples show that, in general, $\mathscr{F}_{\mathrm{loc}}$ and $F$ do not coincide, even though we are assuming quasiconvexity of $f$; see [AD, FM1, FM2].

Our first simple observation is now that - assuming just (1.2) - $\mathscr{F}_{\mathrm{loc}}[-;\Omega]$ is sequentially weakly lower semicontinuous on $W^{1,p}(\Omega;\mathbb{R}^N)$. Consequently, we get an existence theorem for minimizers of $\mathscr{F}_{\mathrm{loc}}$ (Theorem 5.3). Combining methods and results from [BFM, BM1, FM1, S2] we will prove, as our main result, a partial $C^{1,\alpha}$-regularity theorem (Main Theorem 6.4) for these minimizers. This regularity result holds for

$$1 < p \leq q < p + \frac{\min\{2,p\}}{2n}$$

and has been obtained in the author's thesis [S1]. The reader should note at this stage that some of the regularity results in [Ma3, ELM] for minimizers of convex functionals with $(p,q)$-growth are based on a similar relaxation procedure. However, the methods used in these papers rely on convexity and can not be used in the quasiconvex case. Moreover, our regularity theory should be compared to the results of [AM] in the setting of $p(x)$-growth conditions.

Next, let us outline the proof of our regularity theorem:

Following [S2] we will base our proof upon a new kind of Caccioppoli inequality (cf. Lemma 7.13) adapted to the $(p,q)$-growth of $f$.

Contrary to [S2], we are working with $\mathscr{F}_{\mathrm{loc}}$ instead of $F$ and with quasiconvexity instead of $W^{1,p}$-quasiconvexity. However, in [S2] the full strength of the minimizing property and of the $W^{1,p}$-quasiconvexity assumption have both been used only in the proof of the Caccioppoli inequality. Therefore, apart from this proof, we can follow closely the arguments of [S2], which use only the Euler equation of $F$ and the Legendre-Hadamard condition for $D^2 f$. Indeed, the Legendre-Hadamard condition is immediately available in our setting. In addition, relying heavily on representation results for $\mathscr{F}_{\rm loc}$ from [FM1, BFM], we will show that - surprisingly - the Euler equation of the original functional $F$ is valid also for minimizers of $\mathscr{F}_{\rm loc}$ (see Lemma 7.3).

Thus, the main problem is to establish the Caccioppoli inequality. Here, we shall essentially follow Evans' first regularity proof for quasiconvex $f$ in [E]. Since we are working with minimizers of $\mathscr{F}_{\rm loc}$ we need a quasiconvexity condition for $\mathscr{F}_{\rm loc}$. In fact, $\mathscr{F}_{\rm loc}$ is sequentially weakly lower semicontinuous on $W^{1,p}(\Omega;\mathbb{R}^N)$ and from [BM1] we know that $W^{1,p}$-quasiconvexity is necessary for this semicontinuity property. Unfortunately, this has been proven in [BM1] just for integral functionals, while $\mathscr{F}_{\rm loc}$ cannot be represented by an integral. Invoking measure representation results from [FM1] once more, we will adapt the arguments from [BM1] to $\mathscr{F}_{\rm loc}$. Precisely, we obtain

$$\mathscr{F}_{\rm loc}[u_A+\varphi;B_1] \geq \mathscr{F}_{\rm loc}[u_A;B_1]$$

for all $A \in \mathbb{R}^{nN}$ and all $\varphi \in W^{1,p}(B_1;\mathbb{R}^N)$ with compact support in $B_1$ where we have set $u_A(x) := Ax$. This $W^{1,p}$-quasiconvexity condition can now be employed using the same test functions as in [S2].

However, since $\mathscr{F}_{\rm loc}$ is not a variational integral we are faced with some further technical problems. In a series of lemmas, based upon a smoothing technique from [FM1], we prove

$$\mathscr{F}_{\rm loc}[u;\Omega] = \mathscr{F}_{\rm loc}[u;\Omega\setminus\overline{B_\rho(x_0)}] + \mathscr{F}_{\rm loc}[u;B_\rho(x_0)]$$

for every ball $\overline{B_\rho(x_0)} \subset \Omega$ and all $u \in W^{1,p}(\Omega;\mathbb{R}^N)$ satisfying a boundary regularity condition near $\partial B_\rho(x_0)$. Using this additivity property together with some subtle choices of radii we finally overcome these last difficulties and complete the proof of the Caccioppoli inequality and, therewith, the regularity proof.

Let us mention that another relaxed functional has been introduced in [FM1]:

$$\mathscr{F}[u;O] := \inf\left\{ \liminf_{k\to\infty} F[u_k;O] \,:\, \begin{array}{c} u_k \in W^{1,q}(O;\mathbb{R}^N), \\ u_k \underset{k\to\infty}{\rightharpoonup} u \text{ weakly in } W^{1,p}(O;\mathbb{R}^N) \end{array} \right\}.$$

Moreover, we believe that it is natural to consider also

$$\mathscr{F}_0[u;O] := \inf\left\{ \liminf_{k\to\infty} F[u_k;O] \,:\, \begin{array}{c} u_k \in W^{1,q}_{\rm loc}(O;\mathbb{R}^N) \cap \left[u + W^{1,p}_0(O;\mathbb{R}^N)\right], \\ u_k - u \underset{k\to\infty}{\rightharpoonup} 0 \text{ weakly in } W^{1,p}_0(O;\mathbb{R}^N) \end{array} \right\}.$$

Existence results for these functionals can be proven similarly as for $\mathscr{F}_{\rm loc}$. However, the known measure and integral representation results for $\mathscr{F}$ are much weaker than for $\mathscr{F}_{\rm loc}$, while no representation of $\mathscr{F}_0$ is known at all. Hence, it is not clear if any regularity result holds for minimizers of $\mathscr{F}$ or $\mathscr{F}_0$.

The plan of this paper is now as follows:

In Section 2 we give a precise statement of our assumptions, while Section 3 provides examples of integrands $f$ satisfying these conditions. Some preliminaries are collected in Section 4. Finally, we state an existence result in Section 5 and our regularity theorem in Section 6. The regularity proof is presented in Section 7.

## 2. Assumptions

We recall that $f : \mathbb{R}^{nN} \to [0,\infty[$ is always assumed to be continuous and the functionals $F$ and $\mathscr{F}_{\text{loc}}$ are defined by

$$F[u;O] := \int_O f(Du)\,dx \qquad \text{and}$$

$$\mathscr{F}_{\text{loc}}[u;O] := \inf\left\{ \liminf_{k\to\infty} F[u_k;O] \,:\, \begin{array}{l} u_k \in W^{1,q}_{\text{loc}}(O;\mathbb{R}^N) \cap W^{1,p}(O;\mathbb{R}^N), \\ u_k \xrightarrow[k\to\infty]{} u \text{ weakly in } W^{1,p}(O;\mathbb{R}^N) \end{array} \right\}$$

$$\text{for } O \in \mathbb{O}_\Omega \text{ and } u \in W^{1,p}(O;\mathbb{R}^N).$$

Concerning the existence of minimizers we will work with the following set of assumptions:

**(f1) q-Growth:**
There is a bound $\Gamma > 0$ such that we have

$$f(A) \le \Gamma(1+|A|^q) \qquad \text{for every } A \in \mathbb{R}^{nN}.$$

**(f2) p-Coercivity:**
There is a coercivity constant $\gamma > 0$ such that we have

$$f(A) \ge \gamma |A|^p \qquad \text{for every } A \in \mathbb{R}^{nN}.$$

**(f3) Quasiconvexity:**
$f$ is quasiconvex in the sense of Definition 1.1.

It is well known that the treatment of the regularity question requires a strict version of the (quasi)convexity assumption. In this regard we shall replace **(f3)** with its following variant:

**(f3s) Strict Quasiconvexity:**
$f$ is strictly nondegenerate $p$-quasiconvex, i. e. for every $M > 0$ there is a convexity constant $\lambda_M > 0$ such that we have

$$⨍_{B_1} f(A + D\varphi)\,dx \ge f(A) + \lambda_M ⨍_{B_1} (1+|D\varphi|^2)^{\frac{p-2}{2}} |D\varphi|^2\,dx$$

for all $A \in \mathbb{R}^{nN}$ with $|A| \le M+1$ and all smooth $\varphi : B_1 \to \mathbb{R}^N$ with compact support in $B_1$.

Finally, we mention that our results still hold if $f$ is allowed to take negative values, but remains bounded from below. In this context **(f2)** has to be replaced by $f(A) \ge \gamma|A|^p - c$ with a fixed $c \in \mathbb{R}$. This generalization is obvious since adding a constant to $f$ does not change the minimizers of $\mathscr{F}_{\text{loc}}$.

## 3. Examples of Integrands

An important class of examples is given by integrands of the form

$$\begin{aligned} f^0(A) &:= |A|^p + h(\det A) \\ f^1(A) &:= (1+|A|^2)^{\frac{p}{2}} + h(\det A) \end{aligned} \qquad \text{for } A \in \mathbb{R}^{nN},$$

where $N = n \geq 2$, $1 < p < \infty$ and $h$ is a smooth convex function satisfying

$$0 \leq h(d) \leq \Gamma_\kappa (1 + |d|^\kappa)$$

with $1 \leq \kappa < \infty$. Interest in such integrands arises from problems in nonlinear elasticity; cf. [B1, B2, BM1, Ma2].

Choosing $q := \kappa n$ it is well known that $f^0$ and $f^1$ satisfy **(f1)**, **(f2)**, and **(f3)**. In addition, $f^1$ satisfies **(f3s)** (cf. Lemma 4.3). Thus, the regularity result of this paper will cover $f^1$ (requiring $p \leq \kappa n < p + \frac{\min\{2,p\}}{2n}$). Moreover, we will report on an extension to degenerate cases including the integrand $f^0$ in a forthcoming paper. We stress that our analysis includes the case $p < n$, in which $f^0$ and $f^1$ are not $W^{1,p}$-quasiconvex and minimizers may be discontinuous. For a more detailed discussion of these examples we refer to [BM1, S2].

## 4. Auxiliary Results

### *4.1. The p-energy $e_p$*

**Definition 4.1.** *For $1 \leq p < \infty$ we define the nondegenerate p-energy*

$$e_p : \mathbb{R}^{nN} \to \mathbb{R},\, A \mapsto (1+|A|^2)^{\frac{p}{2}}.$$

**Lemma 4.2.** *For $1 < p < \infty$ and all $A, B \in \mathbb{R}^{nN}$ one has*

$$c^{-1} \leq \frac{\int_0^1 (1+|A+tB)|^2)^{\frac{p-2}{2}}\,dt}{(1+|A|^2+|B|^2)^{\frac{p-2}{2}}} \leq c,$$

*where $c$ depends only on $p$.*

A proof can be found in [GM, Lemma 2.1] and [AF3, Lemma 2.1] for $p \geq 2$ and $p \leq 2$ respectively.

**Lemma 4.3 (Quasiconvexity and Growth of $\mathbf{e_p}$).** *For $1 < p < \infty$, $M > 0$, $A \in \mathbb{R}^{nN}$ with $|A| \leq M+1$, a bounded open subset $O$ of $\mathbb{R}^n$, and $\varphi \in W_0^{1,p}(O;\mathbb{R}^N)$ we have*

$$\begin{aligned} C_1^{-1} &\int_O (1+|D\varphi|^2)^{\frac{p-2}{2}} |D\varphi|^2\,dx \\ &\leq \int_O \Big[ e_p(A+D\varphi) - e_p(A) \Big]\,dx \leq C_1 \int_O (1+|D\varphi|^2)^{\frac{p-2}{2}} |D\varphi|^2\,dx, \end{aligned}$$

*where $C_1$ is a positive constant depending only on $p$ and $M$.*

**Proof.** Calculating $D^2 e_p$ explicitly it is not difficult to derive the estimate

$$c^{-1}(1+|B|^2)^{\frac{p-2}{2}}|C|^2 \leq D^2 e_p(B)(C,C) \leq c(1+|B|^2)^{\frac{p-2}{2}}|C|^2 \quad \text{for } B,C \in \mathbb{R}^{nN}.$$

Applying this estimate and Lemma 4.2 to the right-hand side of the equality

$$\int_O \Big[e_p(A+D\varphi) - e_p(A)\Big]\,dx = \int_O \int_0^1 \int_0^1 D^2 e_p(A+stD\varphi)\,ds\,t\,dt\,(D\varphi, D\varphi)\,dx,$$

we obtain the claim. □

### *4.2. Smoothing with Variable Radius*

This subsection is concerned with a smoothing technique introduced in [FM1].

**Lemma 4.4.** *For $0 < r < s$ with $B_s \subset \Omega$ we define a bounded linear smoothing operator*

$$T_{r,s} : W^{1,1}(\Omega;\mathbb{R}^N) \to W^{1,1}(\Omega;\mathbb{R}^N)$$

*by*

$$T_{r,s}u(x) := \fint_{B_1} u(x+\vartheta(x)y)\,dy$$

$$\text{for } u \in W^{1,1}(\Omega;\mathbb{R}^N) \text{ and almost every } x \in \Omega,$$

$$\text{where } \vartheta(x) := \frac{1}{2}\max\Big\{\min\{|x|-r, s-|x|\}, 0\Big\}.$$

*With this definition, for all $1 \leq p \leq q < \frac{np}{n-1}$ and all $u \in W^{1,p}(\Omega;\mathbb{R}^N)$ the following facts hold true:*

$$T_{r,s}u \in W^{1,p}(\Omega;\mathbb{R}^N),$$

$$u = T_{r,s}u \qquad \text{almost everywhere on } (\Omega \setminus B_s) \cup B_r, \tag{4.1}$$

$$T_{r,s}u \in u + W_0^{1,p}(B_s \setminus \overline{B_r};\mathbb{R}^N), \tag{4.2}$$

$$\|DT_{r,s}u\|_{p;B_s\setminus B_r} \leq c\|Du\|_{p;B_s\setminus B_r}, \tag{4.3}$$

$$\|DT_{r,s}u\|_{p;B_\rho\setminus B_r} \leq c\|Du\|_{p;B_{2\rho-r}\setminus B_r} \qquad \text{for } r \leq \rho \leq \frac{r+s}{2}, \tag{4.4}$$

$$\|DT_{r,s}u\|_{p;B_s\setminus B_\rho} \leq c\|Du\|_{p;B_s\setminus B_{2\rho-s}} \qquad \text{for } \frac{r+s}{2} \leq \rho \leq s, \tag{4.5}$$

$$\|T_{r,s}u\|_{q;B_s\setminus B_r} \leq c(s-r)^{\frac{n}{q}-\frac{n-1}{p}}\left[\sup_{t\in]r,s[}\frac{\tilde{\Xi}(t)-\tilde{\Xi}(r)}{t-r} + \sup_{t\in]r,s[}\frac{\tilde{\Xi}(s)-\tilde{\Xi}(t)}{s-t}\right]^{\frac{1}{p}}, \tag{4.6}$$

$$\|DT_{r,s}u\|_{q;B_s\setminus B_r} \leq c(s-r)^{\frac{n}{q}-\frac{n-1}{p}}\left[\sup_{t\in]r,s[}\frac{\Xi(t)-\Xi(r)}{t-r} + \sup_{t\in]r,s[}\frac{\Xi(s)-\Xi(t)}{s-t}\right]^{\frac{1}{p}}. \tag{4.7}$$

*Here $c$ depends only on $n$ and $p$ in* (4.3),(4.4), *and* (4.5), *only on $n$, $p$, and $q$ in* (4.6) *and* (4.7), *and we have used the abbreviations*

$$\tilde{\Xi}(t) := \|u\|_{p;B_t}^p \quad \text{and} \quad \Xi(t) := \|Du\|_{p;B_t}^p.$$

The preceeding lemma is a variant of [FM1, Lemma 2.2]. All estimates can be proved using the methods developed there, although not all of them have been stated in the same form before; compare also [S2, PS].

**Remark 4.5.** *Analyzing the proof of* (4.7) *one finds the more general estimate*

$$\|DT_{r,s}u\|_{q;B_s\setminus B_r} \leq c(s-r)^{\frac{n}{q}-\frac{n-\tau}{p}}\left[\sup_{t\in]r,s[}\frac{\Xi(t)-\Xi(r)}{(t-r)^{\tau}}+\sup_{t\in]r,s[}\frac{\Xi(s)-\Xi(t)}{(s-t)^{\tau}}\right]^{\frac{1}{p}} \quad (4.8)$$

*for all* $\tau \in ]n\frac{q-p}{q},1]$*, where* $c$ *depends only on* $n$*,* $p$*,* $q$ *and* $\tau$*.*

Further estimates of the terms on the right side of (4.6) and (4.7) are obtained using the next lemma. It is a refinement of [FM1, Lemma 2.3].

**Lemma 4.6.** *Let* $-\infty < r < s < \infty$*, an absolutely continuous and nondecreasing function* $\Xi : [r,s] \to \mathbb{R}$*, and a set* $N \subset \mathbb{R}$ *of Lebesgue measure zero be given, then there are* $\tilde{r} \in ]r,\frac{2r+s}{3}[\setminus N$ *and* $\tilde{s} \in ]\frac{r+2s}{3},s[\setminus N$ *such that we have*

$$\frac{\Xi(t)-\Xi(\tilde{r})}{t-\tilde{r}} \leq 4\frac{\Xi(s)-\Xi(r)}{s-r} \qquad \text{for every } t \in ]\tilde{r},s],$$
$$\frac{\Xi(\tilde{s})-\Xi(t)}{\tilde{s}-t} \leq 4\frac{\Xi(s)-\Xi(r)}{s-r} \qquad \text{for every } t \in [r,\tilde{s}[.$$

*In particular it is*

$$\frac{s-r}{3} < \tilde{s}-\tilde{r} < s-r.$$

**Proof.** We assume $\Xi(s) > \Xi(r)$ and consider $\mathscr{G} : [r,s] \to \mathbb{R}^2, t \mapsto (t,\Xi(t))$ and $\tilde{N} := \mathscr{G}(N \cap [r,s]) \subset \mathbb{R}^2$. Using the absolute continuity of $\Xi$ we obtain (for example by means of an elementary covering argument) $\mathscr{H}^1(\tilde{N}) = 0$. Next, we set $m := 4\frac{\Xi(s)-\Xi(r)}{s-r} > 0$ and $v := (-m,1) \in \mathbb{R}^2$. Then, we observe that the orthogonal projection of $\tilde{N}$ on $\mathbb{R}v$ has $\mathscr{H}^1$-measure zero. Hence, we can choose $y \in \big]\Xi(s)-m\frac{2r+s}{3}, \Xi(r)-mr\big[$ such that $\frac{y}{m^2+1}v$ is not contained in this projection, i. e. such that $\tilde{N}$ does not intersect the graph of $l_y : \mathbb{R} \to \mathbb{R}, t \mapsto mt+y$. In particular, we have $l_y(r) < \Xi(r)$ and $l_y(\frac{2r+s}{3}) > \Xi(s)$, therefore

$$\tilde{r} := \min\{t \in [r,s] \,:\, l \geq \Xi \text{ on } [t,s]\}$$

exists and is contained in $]r,\frac{2r+s}{3}[$. A continuity argument provides $l_y(\tilde{r}) = \Xi(\tilde{r})$ and for all $t \in ]\tilde{r},s]$ we see

$$\frac{\Xi(t)-\Xi(\tilde{r})}{t-\tilde{r}} \leq \frac{l_y(t)-l_y(\tilde{r})}{t-\tilde{r}} = m.$$

This proves the claims regarding $\tilde{r}$. Clearly, $\tilde{s}$ can be chosen analogously. □

*4.3. A Poincaré Inequality*

**Lemma 4.7.** *We consider $1 \leq p < \infty$, $0 < R < S < \infty$, and $u \in W^{1,p}(B_S(x_0);\mathbb{R}^N)$ such that $u \equiv 0$ almost everywhere on $B_R(x_0)$. Then, we have the estimate*

$$\int_{B_S(x_0)\setminus B_R(x_0)} \left|\frac{u}{S-R}\right|^p dx \leq \frac{1}{p}\left(\frac{S}{R}\right)^{n-1} \int_{B_S(x_0)\setminus B_R(x_0)} \left|Du(x)\frac{x}{|x|}\right|^p dx.$$

The proof consists of a well-known argument based on radial integration.

*4.4. A Lemma concerning Weak Convergence*

**Definition 4.8.** *Consider a cube $Q$ in $\mathbb{R}^n$. $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^N)$ is called $Q$-periodic iff for all vertices $v$, $w$ of $Q$ we have*

$$u(x+v-w) = u(x) \qquad \text{for almost all } x \in \mathbb{R}^n.$$

**Lemma 4.9.** *Consider $p \in [1,\infty[$, a cube $Q$ in $\mathbb{R}^n$, and $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^N)$. Assume that $Du$ is $Q$-periodic. Then, for $u_k \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^N)$ defined by $u_k(x) := \frac{u(kx)}{k}$ one has*

$$u_k \xrightarrow[k\to\infty]{} u_A \qquad \text{weakly in } W^{1,p}(O;\mathbb{R}^N)$$

*for all bounded regions $O$ in $\mathbb{R}^n$, where we have set $A = \fint_Q Du\,dx$ and $u_A(x) := Ax$.*

For a proof we refer to [BM1, Corollary A.2].

*4.5. Representation Results for $\mathscr{F}_{\mathrm{loc}}$*

We recall that the relaxed functional $\mathscr{F}_{\mathrm{loc}}$ is given by Definition 1.2. As an immediate consequence of this definition we have

$$\mathscr{F}_{\mathrm{loc}}[u;O] \leq F[u;O] \qquad \text{for } O \in \mathbb{O}_\Omega \text{ and } u \in W^{1,q}_{\mathrm{loc}}(O;\mathbb{R}^N) \cap W^{1,p}(O;\mathbb{R}^N). \quad (4.9)$$

In the remainder of this subsection we restate measure and integral representation results, which were obtained in [FM1, BFM]. We start with the measure representation result from [FM1, Theorem 3.1].

**Theorem 4.10 (Measure Representation).** *Assume* **(f1)** *with $1 < p \leq q < \frac{np}{n-1}$. Then, for every $u \in W^{1,p}(\Omega;\mathbb{R}^N)$ with $\mathscr{F}_{\mathrm{loc}}[u;\Omega] < \infty$ there is a uniquely determined finite outer Radon measure $\mu_u$ on $\Omega$ such that*

$$\mathscr{F}_{\mathrm{loc}}[u;-] = \mu_u\big|_{\mathbb{O}_\Omega}.$$

A weaker measure representation result is known for the functional $\mathscr{F}$ defined in the introduction; see [FM1, Theorem 3.2].

Denoting by $Qf$ the quasiconvex envelope of $f$, the relation between $\mathscr{F}$ and $Qf$ has been studied in [FM1, Section 4]. The next lemmas provide analogous statements for $\mathscr{F}_{\mathrm{loc}}$.

**Lemma 4.11.** *We assume* **(f1)** *with* $1 < p \leq q < \frac{np}{n-1}$. *Then, for* $O \in \mathbb{O}_\Omega$ *and* $u \in W^{1,p}(O;\mathbb{R}^N)$ *one has*

$$\mathscr{F}_{\mathrm{loc}}[u;O] \geq \int_O Qf(Du)\,dx.$$

**Proof.** Approximating $\Omega$ by subdomains, $W^{1,q}$ can be replaced by $W^{1,q}_{\mathrm{loc}}$ in the semicontinuity theorem [FM1, Theorem 4.1]. Then, the claim follows immediately from this theorem. □

**Lemma 4.12.** *We assume* **(f1)** *with* $1 < p \leq q < \frac{np}{n-1}$. *Then, for* $O \in \mathbb{O}_\Omega$ *and* $u \in W^{1,q}(O;\mathbb{R}^N)$ *one has*

$$\mathscr{F}_{\mathrm{loc}}[u;O] = \int_O Qf(Du)\,dx.$$

**Proof.** We apply the relaxation theorem [AF1, Statement III.7] getting

$$\int_O Qf(Du)\,dx = \inf\Big\{ \liminf_{k\to\infty} F[u_k;O] \,:\, u_k \underset{k\to\infty}{\rightharpoonup} u \text{ weakly in } W^{1,q}(O;\mathbb{R}^N) \Big\}. \quad (4.10)$$

Note that the previous formula follows also from the main result of [D1] and a simple regularization procedure. Anyway, (4.10) implies $\mathscr{F}_{\mathrm{loc}}[u;O] \leq \int_O Qf(Du)\,dx$ and in view of Lemma 4.11 we have proved the claim. □

**Remark 4.13.** *The hypothesis* $u \in W^{1,q}(O;\mathbb{R}^N)$ *in Lemma 4.12 can be weakened. Actually, the equality of Lemma 4.12 is valid for* $u \in W^{1,q}_{\mathrm{loc}}(O;\mathbb{R}^N) \cap W^{1,p}(O;\mathbb{R}^N)$ *if either* $f$ *is quasiconvex or* $\mathscr{F}_{\mathrm{loc}}[u;O] < \infty$ *holds.*

*If* $f$ *is quasiconvex this claim follows from* (4.9) *and Lemma 4.11. If* $\mathscr{F}_{\mathrm{loc}}[u;O] < \infty$ *holds the assertion can be proved approximating* $O$ *by subdomains and taking into account Theorem 4.10 and Lemma 4.12.*

The next result is taken from [BFM] and will turn out to be crucial for our purposes.

**Theorem 4.14.** *Assume* **(f1)** *with* $1 < p \leq q < \frac{np}{n-1}$ *and consider* $u \in W^{1,p}(\Omega;\mathbb{R}^N)$ *with* $\mathscr{F}_{\mathrm{loc}}[u;\Omega] < \infty$. *Then, the absolutely continuous part of the measure* $\mu_u$ *from Theorem 4.10 has density* $Qf(Du)$ *with respect to the Lebesgue measure* $\mathscr{L}^n$ *on* $\Omega$, *i. e.*

$$\frac{d\mu_u}{d\mathscr{L}^n} = Qf(Du) \qquad \text{holds almost everywhere on } \Omega.$$

Finally, for the sake of completeness we record that for a convex integrand $f$ the functionals $\mathscr{F}_{\mathrm{loc}}$ and $F$ coincide. We state a slightly more general assertion:

**Theorem 4.15.** *Assume that* $f$ *satisfies* **(f1)** *with* $1 < p \leq q < \infty$ *and that* $Qf$ *is convex. Furthermore, consider* $O \in \mathbb{O}_\Omega$ *and* $u \in W^{1,p}(O;\mathbb{R}^N)$. *If* $u$ *has a* $W^{1,p}$-*extension on* $\mathbb{R}^n$ *then one has*

$$\mathscr{F}_{\mathrm{loc}}[u;O] = \int_O Qf(Du).$$

The proof of Theorem 4.15 is similar to [FM1, Remark 4.6], where the analogous statement for $\mathscr{F}$ instead of $\mathscr{F}_{\mathrm{loc}}$ has been established. Since we will not use the theorem in the following we omit the proof.

Taking into account Theorem 4.15 and the semicontinuity theorem for $W^{1,p}$-quasiconvex integrands [S2, Theorem 4.4], it seems natural to conjecture that $\mathscr{F}_{\mathrm{loc}}$ and $F$ coincide for every $W^{1,p}$-quasiconvex integrand $f$. At least, there should be a relation between their minimizers. Such a theorem would unify the existence and regularity theory of [S2] and the present paper. Unfortunately, I was not able to prove such a relation.

## 5. Existence of Minimizers

Throughout the remaining sections we fix exponents $1 < p \leq q < \infty$.

In this section we will investigate the existence of minimizers of $\mathscr{F}_{\mathrm{loc}}$. The next lemma will be crucial in the existence proof.

**Lemma 5.1 (Semicontinuity).** *Assume that* **(f2)** *is satisfied and that $\Omega$ has a $C^0$-boundary. Then, $\mathscr{F}_{\mathrm{loc}}[-;\Omega]$ is sequentially weakly lower semicontinuous on $W^{1,p}(\Omega;\mathbb{R}^N)$.*

**Proof.** Let us assume that $\Omega$ is connected. We consider an arbitrary sequence $u^l \underset{l\to\infty}{\rightharpoonup} u$ weakly in $W^{1,p}(\Omega;\mathbb{R}^N)$ such that

$$\sup_{l\in\mathbb{N}} \mathscr{F}_{\mathrm{loc}}[u^l;\Omega] < \infty. \tag{5.1}$$

From Rellich's theorem we have

$$u^l \underset{l\to\infty}{\longrightarrow} u \qquad \text{strongly in } L^p_{\mathrm{loc}}(\Omega;\mathbb{R}^N). \tag{5.2}$$

Using the definition of $\mathscr{F}_{\mathrm{loc}}$, for every $l \in \mathbb{N}$ we find another sequence $u^l_k \in W^{1,q}_{\mathrm{loc}}(\Omega;\mathbb{R}^N) \cap W^{1,p}(\Omega;\mathbb{R}^N)$ with $u^l_k \underset{k\to\infty}{\rightharpoonup} u^l$ weakly in $W^{1,p}(\Omega;\mathbb{R}^N)$ such that

$$\lim_{k\to\infty} F[u^l_k;\Omega] \leq \mathscr{F}_{\mathrm{loc}}[u^l;\Omega] + \frac{1}{l}$$

holds. We assume $(u^l_k)_\Omega = (u^l)_\Omega = u_\Omega$. Using Rellich's theorem again we deduce for every $l \in \mathbb{N}$ the existence of a number $k(l) \in \mathbb{N}$ such that we have

$$F[u^l_{k(l)};\Omega] \leq \mathscr{F}_{\mathrm{loc}}[u^l;\Omega] + \frac{2}{l} \qquad \text{and} \qquad \|u^l_{k(l)} - u^l\|_{p;\Omega_l} \leq \frac{1}{l}, \tag{5.3}$$

where we have set $\Omega_l := \{x \in \Omega : \operatorname{dist}(x,\partial\Omega) > \frac{1}{l}\}$. From (5.1), the first part of (5.3), and **(f2)** we conclude that the diagonal sequence $\left(Du^l_{k(l)}\right)_{l\in\mathbb{N}}$ is bounded in $L^p(\Omega;\mathbb{R}^{nN})$. Taking into account Poincaré's inequality on $\Omega$ (this is the point where the smoothness assumption on the boundary is needed), it follows that

$\left(u^l_{k(l)}\right)_{l\in\mathbb{N}}$ is weakly convergent in $W^{1,p}(\Omega;\mathbb{R}^N)$. Clearly, (5.2) and the second part of (5.3) imply that the weak limit is $u$. Now, from the definition of $\mathscr{F}_{\text{loc}}$ and the first part of (5.3) we get

$$\mathscr{F}_{\text{loc}}[u;\Omega] \le \liminf_{l\to\infty} \mathscr{F}_{\text{loc}}[u^l:\Omega],$$

and the lemma is proved. □

**Definition 5.2 (Minimizer).** *We call $u \in W^{1,p}(\Omega;\mathbb{R}^N)$ a minimizer of $\mathscr{F}_{\text{loc}}$ on $\Omega$ iff we have $\mathscr{F}_{\text{loc}}[u;\Omega] < \infty$ and*

$$\mathscr{F}_{\text{loc}}[u;\Omega] \le \mathscr{F}_{\text{loc}}[u+\varphi;\Omega] \qquad \text{for all } \varphi \in W^{1,p}_0(\Omega;\mathbb{R}^N).$$

**Theorem 5.3 (Existence).** *Assume that* **(f2)** *is satisfied and that $\Omega$ has a $C^0$-boundary. Then, for every $u_0 \in W^{1,p}(\Omega;\mathbb{R}^N)$ such that $\mathscr{F}_{\text{loc}}[u_0;\Omega] < \infty$ there is a minimizer $u \in u_0 + W^{1,p}_0(\Omega;\mathbb{R}^N)$ of $\mathscr{F}_{\text{loc}}$ on $\Omega$.*

**Proof.** By means of the direct method of the calculus of variations, the claim follows from Lemma 5.1. □

Writing $\mathscr{D}$ for the Dirichlet class $u_0 + W^{1,p}_0(\Omega;\mathbb{R}^N)$ and $\mathscr{S}$ for the subclass $\mathscr{D}\cap W^{1,q}_{\text{loc}}(\Omega;\mathbb{R}^N)$, let us briefly comment on the minimum value. By Theorem 5.3 and (4.9) we clearly have

$$\mathscr{F}_{\text{loc}}[u;\Omega] = \min_{\mathscr{D}} \mathscr{F}_{\text{loc}}[-;\Omega] \le \inf_{\mathscr{S}} \mathscr{F}_{\text{loc}}[-;\Omega] \le \inf_{\mathscr{S}} F[-;\Omega]. \tag{5.4}$$

We observe that Lemma 7.7 below implies that the three values in (5.4) coincide at least in some particular cases (namely if $\Omega$ is a ball and $u$ satisfies some boundary regularity condition). Specifically, this means that we have $\min_{\mathscr{D}} \mathscr{F}_{\text{loc}}[-;\Omega] = \inf_{\mathscr{S}} \mathscr{F}_{\text{loc}}[-;\Omega]$, even though $\inf_{\mathscr{D}} F[-;\Omega] < \inf_{\mathscr{S}} F[-;\Omega]$ might hold. In this light the passage from $F$ to $\mathscr{F}_{\text{loc}}$ can be interpreted as a way to rule out the possible occurrence of the so called Lavrentiev gap (see [BM2, Section 5] for a general discussion of these issues). Moreover, our approach should be compared to the point of view of [ELM], where the role of the Lavrentiev gap as an obstruction to regularity is discussed.

## 6. Main Results: Partial Regularity

Before stating our results we introduce some additional terminology. We start with a slightly weakened notion of a minimizer.

**Definition 6.1 (Weak Minimizer).** *We call $u \in W^{1,p}(\Omega;\mathbb{R}^N)$ a weak minimizer of $\mathscr{F}_{\text{loc}}$ on $\Omega$ iff we have $\mathscr{F}_{\text{loc}}[u;\Omega] < \infty$ and*

$$\mathscr{F}_{\text{loc}}[u;\Omega] \le \mathscr{F}_{\text{loc}}[u+\varphi;\Omega]$$
$$\text{for all } \varphi \in W^{1,p}(\Omega;\mathbb{R}^N) \text{ with compact support in } \Omega.$$

Next, denote by $u$ a minimizer of $\mathscr{F}_{\mathrm{loc}}$ on $\Omega$. Then, it is not clear, in general, whether $u$ is also a minimizer of $\mathscr{F}_{\mathrm{loc}}$ on every $U \in \mathbb{O}_\Omega$, but Theorem 4.10 implies that $u$ is a weak minimizer of $\mathscr{F}_{\mathrm{loc}}$ on every $U \in \mathbb{O}_\Omega$. This motivates us to give the following definition of a local minimizer ensuring - at least - that every minimizer is also a local minimizer.

**Definition 6.2 (Local Minimizer).** *We call $u \in W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ a local minimizer of $\mathscr{F}_{\mathrm{loc}}$ on $\Omega$ iff every $x \in \Omega$ has a neighborhood $U$ in $\Omega$ such that $u$ is a weak minimizer of $\mathscr{F}_{\mathrm{loc}}$ on $U$.*

Finally, we define the regular and the singular set:

**Definition 6.3 (Regular and Singular Set).** *For $u \in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ we call*

$$\mathrm{Reg}(u) := \left\{ x \in \Omega \, : \, u|_{B_\rho(x)} \in C^1(B_\rho(x);\mathbb{R}^N) \text{ for some } \rho > 0 \right\}$$

*the regular set of $u$ and*

$$\mathrm{Sing}(u) := \Omega \setminus \mathrm{Reg}(u)$$

*the singular set of $u$.*

Our main result is the following partial regularity theorem which can be applied to the minimizers of Theorem 5.3:

**Main Theorem 6.4 (Partial Regularity).** *Let $1 < p \le q < p + \frac{\min\{2,p\}}{2n}$ and assume that $f \in C^2_{\mathrm{loc}}(\mathbb{R}^{nN})$ satisfies* **(f1)**, **(f2)** *and* **(f3s)**. *Furthermore, consider a local minimizer $u \in W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ of $\mathscr{F}_{\mathrm{loc}}$ on $\Omega$. Then, we have $\mathrm{Reg}(u) \in \mathbb{O}_\Omega$, $u \in C^{1,\alpha}_{\mathrm{loc}}(\mathrm{Reg}(u);\mathbb{R}^N)$ for every $\alpha \in ]0,1[$, and $\mathscr{L}^n(\mathrm{Sing}(u)) = 0$.*

We will prove the main theorem in Section 7.

At this stage, taking into account Lemma 4.12, we can use standard methods to deduce higher regularity on $\mathrm{Reg}(u)$:

**Corollary 6.5.** *In addition to the assumptions of Main Theorem 6.4 let us assume $f \in C^\infty_{\mathrm{loc}}(\mathbb{R}^{nN})$. Then we have $u \in C^\infty_{\mathrm{loc}}(\mathrm{Reg}(u);\mathbb{R}^N)$.* □

**Remark 6.6.** *In view of Lemma 4.11 we can apply Main Theorem 6.4 and Corollary 6.5 to a $W^{1,p}$-minimizer $u$ of $F$ itself on $\Omega$, provided $u$ satisfies*

$$\mathscr{F}_{\mathrm{loc}}[u;\Omega] = F[u;\Omega].$$

## 7. Proof of the Main Theorem

### *7.1. Euler's Equation*

**Lemma 7.1.** *We assume that $f \in C^1_{\mathrm{loc}}(\mathbb{R}^{nN})$ satisfies the growth condition*

$$|Df(A)| \le C_2(1 + |A|^{q-1}) \qquad \text{for all } A \in \mathbb{R}^{nN} \tag{7.1}$$

*with* $C_2 > 0$ *and* $1 < p \le q < \min\left\{\frac{np}{n-1}, p+1\right\}$. *Then, for* $u \in W^{1,p}(\Omega;\mathbb{R}^N)$ *with* $\mathscr{F}_{\mathrm{loc}}[u;\Omega] < \infty$ *and* $\varphi \in W^{1,\frac{p}{p+1-q}}(\Omega;\mathbb{R}^N)$ *we have*

$$\mathscr{F}_{\mathrm{loc}}[u+\varphi;O] - \mathscr{F}_{\mathrm{loc}}[u;O] = \int_O [Qf(Du+D\varphi) - Qf(Du)]\,dx \quad \text{for all } O \in \mathbb{O}_\Omega.$$

**Proof.** As a consequence of (7.1), **(f1)** is satisfied. We consider the Radon measures $\mu_{u+\varphi}$ and $\mu_u$ from Theorem 4.10 and fix $\varepsilon > 0$. Furthermore, we choose $u_k \in W^{1,q}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ with $u_k \xrightarrow[k\to\infty]{} u$ weakly in $W^{1,p}(\Omega;\mathbb{R}^N)$ and $\lim_{k\to\infty} F[u_k;\Omega] \le \mathscr{F}_{\mathrm{loc}}[u;\Omega] + \varepsilon$. For $O \in \mathbb{O}_\Omega$ with $\mu_u(\partial O \cap \Omega) = 0$ we get $\limsup_{k\to\infty} F[u_k;O] \le \mu_u(O) + \varepsilon$ and consequently

$$\begin{aligned}\mu_{u+\varphi}(O) - \mu_u(O) &\le \liminf_{k\to\infty}\Big(F[u_k+\varphi;O] - F[u_k;O]\Big) + \varepsilon\\ &= \liminf_{k\to\infty}\int_O\int_0^1 Df(Du_k + tD\varphi)\,dt\,D\varphi\,dx + \varepsilon \le \int_O h_1 D\varphi\,dx + \varepsilon.\end{aligned}$$

Here we have used the fact that, due to (7.1), $\int_0^1 Df(Du_k + tD\varphi)\,dt$ remains bounded in $L^{\frac{p}{q-1}}(\Omega;\mathbb{R}^{nN})$ as $k \to \infty$, and we have denoted by $h_1$ the weak limit of a subsequence. Furthermore, we have used $D\varphi \in L^{\frac{p}{p+1-q}}(\Omega;\mathbb{R}^N) = \Big(L^{\frac{p}{q-1}}(\Omega;\mathbb{R}^N)\Big)^*$. Note that taking $O = \Omega$ we infer $\mathscr{F}_{\mathrm{loc}}[u+\varphi;\Omega] < \infty$ which justifies the preceding application of Theorem 4.10 to $u+\varphi$. Recalling the regularity properties of $\mu_{u+\varphi}$ and $\mu_u$, by a simple approximation argument we get

$$\mu_{u+\varphi}(O) - \mu_u(O) \le \int_O h_1 D\varphi\,dx + \varepsilon \qquad \text{for all measurable subsets } O \text{ of } \Omega.$$

Similarly we see

$$\mu_u(O) - \mu_{u+\varphi}(O) \le \int_O h_2 D\varphi\,dx + \varepsilon \qquad \text{for all measurable subsets } O \text{ of } \Omega,$$

where $h_2 \in L^{\frac{p}{q-1}}(\Omega;\mathbb{R}^N)$. Thus, we have shown that the singular part of the signed Radon measure $\mu_{u+\varphi} - \mu_u$ with respect to the Lebesgue measure on $\Omega$ has a total variation not exceeding $2\varepsilon$. Letting $\varepsilon \searrow 0$ we conclude that $\mu_{u+\varphi} - \mu_u$ is absolutely continuous with respect to the Lebesgue measure on $\Omega$. Using Theorem 4.14 together with the Lebesgue decomposition of Radon measures we obtain

$$\mu_{u+\varphi}(O) - \mu_u(O) = \int_O [Qf(Du+D\varphi) - Qf(Du)]\,dx$$

for all measurable subsets $O$ of $\Omega$. □

**Lemma 7.2.** *Consider* $f \in C^1_{\mathrm{loc}}(\mathbb{R}^{nN})$ *with* **(f1)** *and* **(f3)**. *Then, one has*

$$|Df(A)| \le c\Gamma(1 + |A|^{q-1}) \qquad \text{for all } A \in \mathbb{R}^{nN},$$

*where c depends only on n, N and q.*

A proof can be found in [Ma1] and [G, Lemma 5.2].

After these preparations we shall now prove the validity of the Euler equation for minimizers of $\mathscr{F}_{\mathrm{loc}}$:

**Lemma 7.3 (Euler Equation).** *Assume that* $f \in C^1_{\mathrm{loc}}(\mathbb{R}^{nN})$ *fulfills* **(f1)** *and* **(f3)** *where* $1 < p \leq q < \min\left\{\frac{np}{n-1}, p+1\right\}$ *and* $u \in W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ *is a local minimizer of* $\mathscr{F}_{\mathrm{loc}}$ *on* $\Omega$*. Then, u satisfies the Euler equation of F on* $\Omega$ *in a weak formulation:*

$$\int_\Omega Df(Du)D\varphi\,dx = 0$$

*for all smooth* $\varphi : \Omega \to \mathbb{R}^N$ *with compact support in* $\Omega$*.*

**Proof.** Consider an arbitrary $O \in \mathbb{O}_\Omega$ such that $u$ is weakly $\mathscr{F}_{\mathrm{loc}}$-minimizing on $O$ and fix a smooth $\varphi : O \to \mathbb{R}^N$ with compact support in $O$. Then we have $\mathscr{F}_{\mathrm{loc}}[u;O] < \infty$ and

$$\mathscr{F}_{\mathrm{loc}}[u;O] \leq \mathscr{F}_{\mathrm{loc}}[u+t\varphi;O] \qquad \text{for all } t \in \mathbb{R}.$$

From Lemma 7.2 we infer that (7.1) holds; hence we can apply Lemma 7.1 to deduce that

$$\mathbb{R} \to \mathbb{R},\, t \mapsto \int_O f(Du + tD\varphi)\,dx$$

has a global minimum at 0. Differentiating we derive

$$\int_O Df(Du)D\varphi\,dx = 0.$$

Now, recalling Definition 6.2, the claim follows by $C^\infty$-partition of unity. □

### *7.2. $W^{1,p}$-Quasiconvexity*

We recall the notion of $W^{1,p}$-quasiconvexity introduced in [BM1]:

**Definition 7.4.** $f$ *is called* $W^{1,p}$*-quasiconvex iff*

$$⨍_O f(A + D\varphi)\,dx \geq f(A)$$

*holds for all bounded regions O in* $\mathbb{R}^n$*, all* $A \in \mathbb{R}^{nN}$ *and all* $\varphi \in W^{1,p}_0(O;\mathbb{R}^N)$*.*

It has been proven in [BM1, Corollary 3.2] that $W^{1,p}$-quasiconvexity is necessary for weak lower semicontinuity on $W^{1,p}(\Omega;\mathbb{R}^N)$:

**Theorem 7.5.** *If* $F[-;\Omega]$ *is sequentially weakly lower semicontinuous on* $W^{1,p}(\Omega;\mathbb{R}^N)$ *then f is* $W^{1,p}$*-quasiconvex.*

This criterion can be generalized to a larger class of functionals. Namely, we have for $\mathscr{F}_{\mathrm{loc}}$:

**Lemma 7.6.** *Assume* **(f1)** *and* **(f2)** *with* $1 < p \le q < \frac{np}{n-1}$*. Then, the following* $W^{1,p}$*-quasiconvexity condition holds for* $\mathscr{F}_{\rm loc}$*: For all* $O \in \mathbb{O}_\Omega$*,* $A \in \mathbb{R}^{nN}$*, and* $\varphi \in W^{1,p}(O;\mathbb{R}^N)$ *with compact support in* $O$ *we have*

$$\mathscr{F}_{\rm loc}[u_A+\varphi;O] \ge \mathscr{F}_{\rm loc}[u_A;O],$$

*where we have set* $u_A(x) := Ax$*.*

**Proof.** Let $\mathbb{K}_{\mathbb{R}^n}$ denote the set of all bounded open subsets of $\mathbb{R}^n$. Clearly, we can define $\mathscr{F}_{\rm loc}[u;O]$ for all $O \in \mathbb{K}_{\mathbb{R}^n}$ and $u \in W^{1,p}(O;\mathbb{R}^N)$. Recalling Lemma 4.12 we observe

$$\mathscr{F}_{\rm loc}[u_A;O] = Qf(A)|O| \qquad \text{for every } O \in \mathbb{K}_{\mathbb{R}^n} \text{ and } A \in \mathbb{R}^{nN}. \tag{7.2}$$

Furthermore, it is not difficult to see

$$\mathscr{F}_{\rm loc}[u;O] = \mathscr{F}_{\rm loc}[\tilde{u};x_0+O], \qquad \text{where } \tilde{u}(x) := u(x-x_0)+\zeta, \tag{7.3}$$

$$\mathscr{F}_{\rm loc}[u;O] = r^n \mathscr{F}_{\rm loc}\left[\tilde{u};\frac{1}{r}O\right], \qquad \text{where } \tilde{u}(x) := \frac{u(rx)}{r} \tag{7.4}$$

for all $x_0 \in \mathbb{R}^n$, $\zeta \in \mathbb{R}^N$, $r > 0$, $O \in \mathbb{K}_{\mathbb{R}^n}$, and $u \in W^{1,p}(O;\mathbb{R}^N)$.
Now let us fix an open subset $O$ of $]0,1[^n$, a matrix $A \in \mathbb{R}^{nN}$, and a function $\varphi \in W^{1,p}(O;\mathbb{R}^N)$ with compact support in $O$. We extend $\varphi$ by 0 on $]0,1[^n$ and $]0,1[^n$-periodic in the sense of Definition 4.8 on the whole $\mathbb{R}^n$ denoting this extension by $\tilde{\varphi} \in W^{1,p}_{\rm loc}(\mathbb{R}^n;\mathbb{R}^N)$. Furthermore, letting $u(x) := Ax + \tilde{\varphi}(x)$ and $u_k(x) := \frac{u(kx)}{k}$, we define the functions $u, u_k \in W^{1,p}_{\rm loc}(\mathbb{R}^n;\mathbb{R}^N)$. From Lemma 4.9 we know $u_k \xrightarrow[k\to\infty]{} u_A$ weakly in $W^{1,p}(]0,1[^n;\mathbb{R}^N)$ and, using Lemma 5.1, we obtain

$$\mathscr{F}_{\rm loc}\Big[u_A;]0,1[^n\Big] \le \liminf_{k\to\infty} \mathscr{F}_{\rm loc}\Big[u_k;]0,1[^n\Big].$$

For $\mathscr{F}_{\rm loc}[u;]0,k[^n] < \infty$ let us denote by $\nu_k$ the finite Radon measure on $]0,k[^n$ representing $\mathscr{F}_{\rm loc}[u;-]$ in the sense of Theorem 4.10, while for $\mathscr{F}_{\rm loc}[u;]0,k[^n] = \infty$ we set $\nu_k := \infty \cdot \mathscr{L}^n$. By $Q_k$ let us denote the union of all cubes $x+]0,1[^n$, where $x$ varies within $\{0,1,2,\dots,k-1\}^n$. Clearly, we have $Q_k \subset ]0,k[^n$. Since the support of $\varphi$ is in $]0,1[^n$, we have $u = u_A$ near $]0,k[^n \setminus Q_k$ and (7.2) allows us to conclude $\nu_k(]0,k[^n \setminus Q_k) = 0$. Using in turn (7.4), the additivity of $\nu_k$, and (7.3) together with the periodicity of $u$ we compute

$$\begin{aligned} k^n \mathscr{F}_{\rm loc}\Big[u_k;]0,1[^n\Big] &= \nu_k\Big(]0,k[^n\Big) = \nu_k(Q_k) \\ &= \sum_{x\in\{0,1,2,\dots,k-1\}^n} \mathscr{F}_{\rm loc}[u;x+]0,1[^n] = k^n \mathscr{F}_{\rm loc}\Big[u,]0,1[^n\Big]. \end{aligned}$$

Hence, we have

$$\mathscr{F}_{\rm loc}\Big[u_A;]0,1[^n\Big] \le \mathscr{F}_{\rm loc}\Big[u,]0,1[^n\Big].$$

Using the measure representation again we deduce

$$\mathscr{F}_{\rm loc}[u_A;O] \le \mathscr{F}_{\rm loc}[u_A+\varphi,O].$$

Finally, (7.3) and (7.4) imply that the last inequality remains valid even for an arbitrary $O \in \mathbb{K}_{\mathbb{R}^n}$. $\square$

### *7.3. An Additivity Property*

First we will show that, under technical assumptions, we can restrict ourselves in the definition of $\mathscr{F}_{\mathrm{loc}}[u;O]$ to sequences attaining the boundary values of $u$ itself.

**Lemma 7.7.** *Assume* **(f1)** *and* **(f2)** *with* $1 < p \leq q < \frac{np}{n-1}$. *We consider a* $u \in W^{1,p}(B_s(x_0);\mathbb{R}^N)$ *such that the boundary regularity condition*

$$\limsup_{\varepsilon\searrow 0} \varepsilon^{-\tau} \int_{B_s(x_0)\setminus B_{s-\varepsilon}(x_0)} |Du|^p \, dx < \infty \tag{7.5}$$

*holds for some* $\tau \in ]n\frac{q-p}{q}, 1]$. *Now, if* $\mathscr{F}_{\mathrm{loc}}[u;B_s(x_0)] < \infty$ *holds, then there are*

$$w_k \in W^{1,q}(B_s(x_0);\mathbb{R}^N) \cap \Big[u + W_0^{1,p}(B_s(x_0);\mathbb{R}^N)\Big]$$

*such that we have* $w_k \xrightarrow[k\to\infty]{} u$ *weakly in* $W^{1,p}(B_s(x_0);\mathbb{R}^N)$ *and*

$$\mathscr{F}_{\mathrm{loc}}[u;B_s(x_0)] = \lim_{k\to\infty} F[w_k;B_s(x_0)].$$

Recalling the definition of $\mathscr{F}_0$ given in the introduction the conclusion of Lemma 7.7 can be rephrased as

$$\mathscr{F}_{\mathrm{loc}}[u;B_s(x_0)] = \mathscr{F}_0[u;B_s(x_0)].$$

Thus, the lemma can be seen as a nonstandard growth version of [F, Lemma 2.1]. Anyway, we will use Lemma 7.7 only with $\tau = 1$, but as explained in Section 5 the more general version stated here might be of some independent interest. After these remarks we will now turn to the proof of the lemma.

**Proof.** We assume $x_0 = 0$ and start proving the following assertion:

*Proposition.* For every given $\delta \in ]0,s[$ there is a $v \in W^{1,q}(B_s;\mathbb{R}^N)$ such that $v - u \in W_0^{1,p}(B_s;\mathbb{R}^N)$, $\|v-u\|_{p;B_{s-\delta}} \leq \delta$ and

$$F[v;B_s] \leq \mathscr{F}_{\mathrm{loc}}[u;B_s] + \delta. \tag{7.6}$$

Using **(f2)** similarly as in the proof of Lemma 5.1, it is not difficult to show the existence of a sequence $(u_k)_{k\in\mathbb{N}}$ for which the infimum in the definition of $\mathscr{F}_{\mathrm{loc}}[u;B_s]$ is attained. Passing to a subsequence and applying Rellich's theorem we infer

$$|Du_k|^p \cdot \mathscr{L}^n \xrightarrow[k\to\infty]{*} \mu \qquad \text{weakly in the sense of measures on } B_s,$$
$$u_k \xrightarrow[k\to\infty]{} u \qquad \text{strongly in } L^p(B_s;\mathbb{R}^N).$$

Here $\mu$ denotes a finite nonnegative Radon measure on $B_s$. Now, we consider radii $\frac{s}{2} \leq r \leq R < S \leq \frac{R+s}{2}$ to be fixed later and set

$$\tilde{u} := T_{R,S}u,$$

where $T$ is the operator defined in Lemma 4.4. From (4.1) and (4.2) we deduce

$$\tilde{u} = u \text{ on } B_R \qquad \text{and} \qquad \tilde{u} \in u + W_0^{1,p}(B_s;\mathbb{R}^N).$$

Next we define

$$\Xi_k(t) := \int_{B_t} |Du_k|^p + \left|\frac{u_k - \tilde{u}}{S-R}\right|^p dx$$

and choose radii $R \leq \tilde{R}_k < \tilde{S}_k \leq S$ with the properties of Lemma 4.6 regarding $\Xi_k$ and the empty set. Let $\eta_k : B_s \to \mathbb{R}$ denote a cut-off function with support in $B_{\tilde{S}_k}$ such that $\eta_k \equiv 1$ on $B_{\tilde{R}_k}$, $0 \leq \eta_k \leq 1$ and $|\nabla \eta_k| \leq \frac{2}{\tilde{S}_k - \tilde{R}_k}$. Relying on the operator from Lemma 4.4 once more we construct

$$v_k := \eta_k \left(T_{\tilde{R}_k,\tilde{S}_k} u_k\right) + (1-\eta_k)\left(T_{\tilde{R}_k,\tilde{S}_k} \tilde{u}\right)$$

and observe

$$\begin{aligned} v_k = u_k \text{ on } B_{\tilde{R}_k}, \qquad v_k = \tilde{u} \text{ on } B_s \setminus B_{\tilde{S}_k}, \\ |Dv_k| \leq |DT_{\tilde{R}_k,\tilde{S}_k} u_k| + |DT_{\tilde{R}_k,\tilde{S}_k} \tilde{u}| + 2\left|\frac{T_{\tilde{R}_k,\tilde{S}_k}(u_k - \tilde{u})}{\tilde{S}_k - \tilde{R}_k}\right| \qquad \text{on } B_s. \end{aligned}$$

Using these observations together with **(f1)** and (4.3) we have

$$\begin{aligned} \int_{B_s} \left[f(Dv_k) - f(Du_k)\right] dx &\leq \Gamma \int_{B_s \setminus B_{\tilde{R}_k}} (1 + |Dv_k|^q)\, dx \\ &\leq \Gamma \left[\mathscr{L}^n(B_s \setminus B_r) + \int_{B_s \setminus B_R} |D\tilde{u}|^q\, dx + \int_{B_{\tilde{S}_k} \setminus B_{\tilde{R}_k}} |Dv_k|^q\, dx\right] \\ &\leq c\left[\mathscr{L}^n(B_s \setminus B_r) + \int_{B_s \setminus B_R} |D\tilde{u}|^q\, dx \right. \\ &\qquad \left. + \int_{B_{\tilde{S}_k} \setminus B_{\tilde{R}_k}} \left(|DT_{\tilde{R}_k,\tilde{S}_k} u_k|^q + \left|\frac{T_{\tilde{R}_k,\tilde{S}_k}(u_k - \tilde{u})}{\tilde{S}_k - \tilde{R}_k}\right|^q\right) dx\right] \\ &=: c\left[\mathscr{L}^n(B_s \setminus B_r) + (I) + (II)\right]. \end{aligned} \tag{7.7}$$

Employing (4.7), (4.6), Lemma 4.6, Lemma 4.7, and (4.4) we estimate the last term in the following manner:

$$\begin{aligned}
(II) &\le c(S-R)^{n-n\frac{q}{p}}\left[\int_{B_S\setminus B_R}\left(|Du_k|^p+\left|\frac{u_k-\tilde{u}}{S-R}\right|^p\right)dx\right]^{\frac{q}{p}}\\
&\le c(S-R)^{n-n\frac{q}{p}}\left[\int_{B_S\setminus B_R}\left(|Du_k|^p+\left|\frac{u_k-u}{S-R}\right|^p+\left|\frac{u-\tilde{u}}{S-R}\right|^p\right)dx\right]^{\frac{q}{p}}\\
&\le c(S-R)^{n-n\frac{q}{p}}\left[\int_{B_S\setminus B_R}\left(|Du_k|^p+\left|\frac{u_k-u}{S-R}\right|^p+|Du|^p+|D\tilde{u}|^p\right)dx\right]^{\frac{q}{p}}\\
&\le c(S-R)^{n-n\frac{q}{p}}\left[\int_{B_S\setminus B_R}\left(|Du_k|^p+\left|\frac{u_k-u}{S-R}\right|^p\right)dx\right.\\
&\qquad\qquad\left.+\int_{B_{2S-R}\setminus B_R}|Du|^p\,dx\right]^{\frac{q}{p}}. \qquad (7.8)
\end{aligned}$$

Applying (4.8) we have for the first term

$$(I)\le c(s-r)^{n-(n-\tau)\frac{q}{p}}\left[\sup_{t\in]R,s[}\frac{\Xi^p(t)-\Xi^p(R)}{(t-R)^\tau}+\sup_{t\in]R,s[}\frac{\Xi^p(s)-\Xi^p(t)}{(s-t)^\tau}\right]^{\frac{q}{p}}, \qquad (7.9)$$

where we have abbreviated

$$\Xi^p(t):=\int_{B_t}|Du|^p\,dx.$$

Let us remark that the constants in (7.7), (7.9) and (7.8) depend only on $n$, $p$, $q$, and $\Gamma$.

Next we shall derive estimates for the right-hand side of (7.9). From (7.5) we get

$$\sup_{t\in]r,s[}\frac{\Xi^p(s)-\Xi^p(t)}{(s-t)^\tau}\le\sup_{t\in]0,s[}(s-t)^{-\tau}\int_{B_s\setminus B_t}|Du|^p\,dx\le c, \qquad (7.10)$$

where $c$ depends only on $p$, $\tau$, $s$, and $Du$. Now we choose the remaining radii $r$, $R$, and $S$. We recall that we have assumed $\frac{s}{2}\le r\le R<S\le\frac{R+s}{2}$. First choose $r\in]\frac{s}{2},s[$ close enough to $s$ such that the following conditions hold:

$$s-r\le\delta, \qquad (7.11)$$

$$(s-r)^{n-(n-\tau)\frac{q}{p}}\le\delta, \qquad (7.12)$$

$$\mathscr{L}^n(B_s\setminus B_r)\le\delta. \qquad (7.13)$$

Introducing the notations

$$\Xi^\mu(t):=\mu(B_t),$$

$$N:=\{t\in]r,s[:\ \Xi^\mu \text{ or } \Xi^p \text{ is not differentiable at } t.\}$$

we have $\mathscr{L}^1(N)=0$. Next Lemma 4.6 enables us to choose an $R\in]\frac{r+s}{2},s[\setminus N$ such that we have

$$\sup_{t\in]R,s[}\frac{\Xi^p(t)-\Xi^p(R)}{t-R}\leq 4\frac{\Xi^p(s)-\Xi^p(\frac{r+s}{2})}{s-\frac{r+s}{2}}.$$

Combining this with (7.10) we see

$$\sup_{t\in]R,s[}\frac{\Xi^p(t)-\Xi^p(R)}{(t-R)^\tau}\leq c, \tag{7.14}$$

where $c$ depends only on $p$, $\tau$, $s$, and $Du$, but not on $r$ or $R$. Proceeding we fix $S\in]R,\frac{R+s}{2}[$ close enough to $R$ such that the following inequalities are valid:

$$(S-R)^{n-(n-1)\frac{q}{p}}\left[\frac{1}{S-R}\,\mu(\overline{B_S}\setminus B_R)\right]^{\frac{q}{p}}\leq\delta, \tag{7.15}$$

$$(S-R)^{n-(n-1)\frac{q}{p}}\left[\frac{1}{S-R}\int_{B_{2S-R}\setminus B_R}|Du|^p\,dx\right]^{\frac{q}{p}}\leq\delta. \tag{7.16}$$

Note that we used $R\notin N$ and $q<\frac{np}{n-1}$ to achieve (7.15) and (7.16).
Now all the radii are fixed and we can finally find a $k\in\mathbb{N}$ such that we have

$$\|u_k-u\|_{p;B_r}\leq\delta, \tag{7.17}$$

$$F[u_k;B_s]\leq\mathscr{F}_{\mathrm{loc}}[u;B_s]+\delta, \tag{7.18}$$

$$(S-R)^{n-n\frac{q}{p}}\left[\int_{B_S\setminus B_R}\left|\frac{u_k-u}{S-R}\right|^p dx\right]^{\frac{q}{p}}\leq\delta, \tag{7.19}$$

$$(S-R)^{n-n\frac{q}{p}}\left\{\left[\int_{B_S\setminus B_R}|Du_k|^p\,dx\right]^{\frac{q}{p}}-\left[\mu(\overline{B_S}\setminus B_R)\right]^{\frac{q}{p}}\right\}\leq\delta. \tag{7.20}$$

Finally, we combine the above estimates. The right-hand side of (7.8) can be treated using (7.15), (7.16), (7.19), and (7.20), while the terms from (7.9) are estimated in (7.10), (7.12), and (7.14). Together with (7.7), (7.13), and (7.18) these inequalities yield

$$F[v_k;B_s]\,dx\leq\mathscr{F}_{\mathrm{loc}}[u;B_s]+C_3\delta,$$

where $C_3(n,p,q,\tau,\Gamma,s,Du)>0$ is a fixed constant. If necessary we replace $\delta$ by $\frac{\delta}{C_3}$ obtaining (7.6) for $v_k$. From (7.11) and (7.17) we see $\|v_k-u\|_{p;B_{s-\delta}}\leq\delta$ and the remaining claims of the proposition are easy to check.
Once the proposition is proved the claim of the lemma follows easily using **(f2)** again. □

**Lemma 7.8.** *Assume* **(f1)** *and* **(f2)** *with* $1<p\leq q<\frac{np}{n-1}$. *We consider a ball* $\overline{B_s(x_0)}\subset\Omega$ *and a* $u\in W^{1,p}(\Omega\setminus\overline{B_s(x_0)};\mathbb{R}^N)$ *such that the boundary regularity condition*

$$\limsup_{\varepsilon\searrow 0}\varepsilon^{-\tau}\int_{B_{s+\varepsilon}(x_0)\setminus B_s(x_0)}|Du|^p\,dx<\infty$$

*holds for some* $\tau \in ]n\frac{q-p}{q}, 1]$. *Now, if* $\mathscr{F}_{\mathrm{loc}}[u; \Omega \setminus \overline{B_s(x_0)}] < \infty$, *then there are* $w_k \in W^{1,q}_{\mathrm{loc}}(\Omega \setminus \overline{B_s(x_0)}; \mathbb{R}^N)$ *taking on* $\partial B_s(x_0)$ *the boundary values of* $u$ *such that we have* $w_k \xrightarrow[k\to\infty]{} u$ *weakly in* $W^{1,p}(\Omega \setminus \overline{B_s(x_0)}; \mathbb{R}^N)$ *and*

$$\mathscr{F}_{\mathrm{loc}}[u; \Omega \setminus \overline{B_s(x_0)}] = \lim_{k\to\infty} F[w_k; \Omega \setminus \overline{B_s(x_0)}].$$

The proof is essentially the same as for Lemma 7.7. Actually, on closer inspection of the proof we have:

**Remark 7.9.** *In the situation of Lemma 7.8 we can achieve, in addition,* $w_k \in W^{1,q}(\tilde{\Omega} \setminus \overline{B_s(x_0)}; \mathbb{R}^N)$ *for all* $\tilde{\Omega} \in \mathbb{O}_\Omega$ *with* $\tilde{\Omega} \subset\subset \Omega$.

From the measure representation of Theorem 4.10 one can easily deduce that the following additivity property (7.22) holds for almost every ball $\overline{B_s(x_0)} \subset \Omega$. However, relying on the preceding lemmas, we shall now provide a sufficient criterion for a *given* ball, namely the condition (7.21), from which (7.22) can be derived:

**Lemma 7.10 (Additivity Property).** *Assume* **(f1)** *and* **(f2)** *with* $1 < p \le q < \frac{np}{n-1}$. *We consider a ball* $\overline{B_s(x_0)} \subset \Omega$ *and* $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ *such that the boundary regularity condition*

$$\limsup_{\varepsilon \searrow 0} \varepsilon^{-\tau} \int_{B_{s+\varepsilon}(x_0) \setminus B_{s-\varepsilon}(x_0)} |Du|^p \, dx < \infty. \tag{7.21}$$

*holds for some* $\tau \in ]n\frac{q-p}{q}, 1]$. *Then, we have*

$$\mathscr{F}_{\mathrm{loc}}[u; \Omega] = \mathscr{F}_{\mathrm{loc}}[u; B_s(x_0)] + \mathscr{F}_{\mathrm{loc}}[u; \Omega \setminus \overline{B_s(x_0)}]. \tag{7.22}$$

**Proof.** "$\ge$" is a simple consequence of the definition of $\mathscr{F}_{\mathrm{loc}}$. To prove "$\le$" let us assume that the right-hand side of (7.22) is finite. Now, the claim can be obtained composing the sequences from Lemma 7.7 and Remark 7.9. □

**Remark 7.11.** *A modification of the preceding arguments shows that Lemma 7.10 remains valid even without assuming* **(f2)**.

### *7.4. The Caccioppoli Inequality*

**Definition 7.12 (Excess).** *For* $u \in W^{1,p}(B_\rho(x_0); \mathbb{R}^N)$ *and* $A \in \mathbb{R}^{nN}$ *we define*

$$\Phi_p(u, x_0, \rho, A) := \fint_{B_\rho(x_0)} (1 + |Du - A|^2)^{\frac{p-2}{2}} |Du - A|^2 \, dx.$$

Furthermore, we introduce the local bounds

$$\Lambda_M := \sup_{|A| \le M+2} |D^2 f(A)|.$$

Finally, we combine most of the previous results in the proof of the following Caccioppoli type estimate. It has been demonstrated in [S2] that this estimate, in turn, is the crucial step in the regularity proof.

**Lemma 7.13 (Caccioppoli Inequality).** *Let* $1 < p \le q < p + \frac{\min\{2,p\}}{2n}$. *Consider* $M > 0$, $f \in C^2_{\mathrm{loc}}(\mathbb{R}^{nN})$ *satisfying* **(f1)**, **(f2)**, *and* **(f3s)**, *and a weak minimizer* $u \in W^{1,p}(B_\rho(x_0);\mathbb{R}^N)$ *of* $\mathscr{F}_{\mathrm{loc}}$ *on* $B_\rho(x_0)$. *Then, for all* $\zeta \in \mathbb{R}^N$ *and* $A \in \mathbb{R}^{nN}$ *with* $|A| \le M+1$ *we have*

$$\Phi_p\left(u,x_0,\frac{\rho}{2},A\right) \le c\left[h\left(\fint_{B_\rho(x_0)}\left(1+\left|\frac{v}{\rho}\right|^2\right)^{\frac{p-2}{2}}\left|\frac{v}{\rho}\right|^2 dx\right) + \left(\Phi_p(u,x_0,\rho,A)\right)^{\frac{q}{p}}\right],$$

*where we have set* $h(t) := t + t^{\frac{q}{p}}$ *and* $v(x) := u(x) - \zeta - A(x-x_0)$, *and where* $c$ *denotes a positive constant depending only on* $n$, $N$, $p$, $q$, $\gamma$, $\Gamma$, $M$, $\lambda_M$ *and* $\Lambda_M$.

**Proof.** We may assume $x_0 = 0$, $\zeta = 0$, and $2^{\frac{p}{2}}\lambda_M < C_1\gamma$, where $C_1$ denotes the constant from Lemma 4.3. Setting

$$g(B) := f(B) - \frac{\lambda_M}{C_1}e_p(B) \qquad \text{for } B \in \mathbb{R}^{nN}$$

we see from **(f3s)**, Lemma 4.3, **(f1)**, and **(f2)** that $g$ is quasiconvex at $A$, i. e. $Qg(A) = g(A)$, and

$$\left(\gamma - \frac{2^{\frac{p}{2}}\lambda_M}{C_1}\right)|B|^p - \frac{2^{\frac{p}{2}}\lambda_M}{C_1} \le g(B) \le \Gamma(1+|B|^q) \qquad \text{for } B \in \mathbb{R}^{nN}.$$

For the sake of simplicity let us now assume $g \ge 0$ on $\mathbb{R}^{nN}$. Following the definitions of $F$ and $\mathscr{F}_{\mathrm{loc}}$, but with $g$ instead of $f$, we introduce functionals $G$ and $\mathscr{G}_{\mathrm{loc}}$. Now, we consider a sequence $w_k \in W^{1,q}_{\mathrm{loc}}(O;\mathbb{R}^N) \cap W^{1,p}(O;\mathbb{R}^N)$ such that $w_k \xrightarrow[k\to\infty]{} w$ weakly in $W^{1,p}(O;\mathbb{R}^N)$. Using the definition of $\mathscr{G}_{\mathrm{loc}}$ and the semicontinuity properties of convex functionals we obtain

$$\mathscr{G}_{\mathrm{loc}}[w;O] \le \liminf_{k\to\infty} F[w_k;O] - \frac{\lambda_M}{C_1}\int_O e_p(Dw)\,dx.$$

Remembering the definition of $\mathscr{F}_{\mathrm{loc}}$ we infer

$$\mathscr{G}_{\mathrm{loc}}[w;O] \le \mathscr{F}_{\mathrm{loc}}[w;O] - \frac{\lambda_M}{C_1}\int_O e_p(Dw)\,dx. \tag{7.23}$$

Following essentially the lines of [S2] we consider arbitrary radii $\frac{\rho}{2} \le r < s \le \rho$. Setting

$$\Xi(t) := \begin{cases} \int_{B_t}\left[|Dv|^p + \left|\frac{v}{s-r}\right|^p\right] dx & \text{for } p \ge 2 \\ \int_{B_t}\left[(1+|Dv|^2)^{\frac{p-2}{2}}|Dv|^2 + \left(1+\left|\frac{v}{s-r}\right|^2\right)^{\frac{p-2}{2}}\left|\frac{v}{s-r}\right|^2\right] dx & \text{for } p \le 2 \end{cases},$$

$$N := \{t \in ]r,s[ : t \mapsto \int_{B_t}|Du|^p\,dx \text{ is not differentiable at } t.\},$$

we choose in addition $\tilde r, \tilde s \notin N$ with $r < \tilde r < \tilde s < s$ as in Lemma 4.6. Furthermore, let $\eta$ denote a smooth cut-off function with support in $B_{\tilde s}$ satisfying $\eta \equiv 1$ in a neighborhood of $B_{\tilde r}$ and $0 \le \eta \le 1$, $|\nabla \eta| \le \frac{2}{\tilde s - \tilde r}$ on $B_\rho$. Using the operator from Lemma 4.4 we set

$$\psi := T_{\tilde r, \tilde s}[(1-\eta)v] \qquad \text{and} \qquad \varphi := v - \psi.$$

Due to (4.1) and (4.2) we have

$$\varphi \in W^{1,p}_0(B_{\tilde s}; \mathbb{R}^N) \qquad \text{and} \qquad \varphi = v, \quad \psi = 0 \quad \text{on } B_{\tilde r}.$$

Combining the estimates from Lemma 4.4 and Lemma 4.6 and the condition $q < p + \frac{1}{n}$ with the choices of $\tilde r$ and $\tilde s$ as usual we observe

$$\psi \in W^{1, \frac{p}{p+1-q}}(B_{\tilde s}; \mathbb{R}^N) \subset W^{1,q}(B_{\tilde s}; \mathbb{R}^N).$$

From $\tilde s \notin N$ we infer

$$\limsup_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \int_{B_{\tilde s + \varepsilon} \setminus B_{\tilde s - \varepsilon}} |Du|^p \, dx < \infty.$$

As a consequence of (4.5) we have that this condition is satisfied not only for $u$, but also for $v$ and $T_{\tilde r, \tilde s} v$ and, recalling $\varphi = v - T_{\tilde r, \tilde s} v$ near $\partial B_{\tilde s}$, also for $u_A + \varphi$ and $u - \varphi$. Hence, (7.21) with respect to $B_{\tilde s}$ is satisfied for all these functions.
We shall now combine the $W^{1,p}$-quasiconvexity condition from Lemma 7.6, the additivity property from Lemma 7.10, Lemma 4.12, and (7.23) in the following estimate. Let us stress that we apply these lemmas to $G$ and $\mathscr{G}_{\mathrm{loc}}$ instead of $F$ and $\mathscr{F}_{\mathrm{loc}}$.

$$\begin{aligned}
0 &\le \mathscr{G}_{\mathrm{loc}}[u_A + \varphi; B_\rho] - \mathscr{G}_{\mathrm{loc}}[u_A; B_\rho] \\
&= \mathscr{G}_{\mathrm{loc}}[u_A + \varphi; B_{\tilde s}] + \mathscr{G}_{\mathrm{loc}}[u_A; B_\rho \setminus \overline{B_{\tilde s}}] - \mathscr{G}_{\mathrm{loc}}[u_A; B_\rho] \\
&= \mathscr{G}_{\mathrm{loc}}[u_A + \varphi; B_{\tilde s}] + G[u_A; B_\rho \setminus \overline{B_{\tilde s}}] - G[u_A; B_\rho] \\
&\le \mathscr{F}_{\mathrm{loc}}[u_A + \varphi; B_{\tilde s}] - \int_{B_{\tilde s}} f(A)\, dx - \frac{\lambda_M}{C_1} \int_{B_{\tilde s}} [e_p(A + D\varphi) - e_p(A)]\, dx.
\end{aligned} \tag{7.24}$$

From the weak minimizing property we have $\mathscr{F}_{\mathrm{loc}}[u; B_\rho] < \infty$ and

$$\mathscr{F}_{\mathrm{loc}}[u; B_\rho] \le \mathscr{F}_{\mathrm{loc}}[u - \varphi; B_\rho].$$

Applying Lemma 7.10 we conclude

$$\mathscr{F}_{\mathrm{loc}}[u; B_{\tilde s}] \le \mathscr{F}_{\mathrm{loc}}[u - \varphi; B_{\tilde s}] = \int_{B_{\tilde s}} f(A + D\psi)\, dx < \infty.$$

Here, the integral representation of the right-hand side is a consequence of Lemma 4.12. Moreover, $\psi \in W^{1, \frac{p}{p+1-q}}(B_{\tilde s}; \mathbb{R}^N)$ and Lemma 7.1 (where we are using implicitly also Lemma 7.2) ensure

$$\mathscr{F}_{\mathrm{loc}}[u - \psi; B_{\tilde s}] - \mathscr{F}_{\mathrm{loc}}[u; B_{\tilde s}] = \int_R [f(Du - D\psi) - f(Du)]\, dx,$$

where we introduced the abbreviation $R := B_{\tilde{s}} \setminus \overline{B_{\tilde{r}}}$. From (7.24), Lemma 4.3, and the last two (in)equalities we get

$$\begin{aligned}
&\int_{B_r} (1+|Dv|^2)^{\frac{p-2}{2}} |Dv|^2 \, dx \\
&\qquad \le c \left( \mathscr{F}_{\mathrm{loc}}[u_A + \varphi; B_{\tilde{s}}] - \int_{B_{\tilde{s}}} f(A)\, dx \right) \\
&\qquad = c \Bigg( \mathscr{F}_{\mathrm{loc}}[u-\psi; B_{\tilde{s}}] - \mathscr{F}_{\mathrm{loc}}[u; B_{\tilde{s}}] + \mathscr{F}_{\mathrm{loc}}[u; B_{\tilde{s}}] - \mathscr{F}_{\mathrm{loc}}[u-\varphi; B_{\tilde{s}}] \\
&\qquad\qquad + \int_{B_{\tilde{s}}} [f(A+D\psi) - f(A)]\, dx \Bigg) \\
&\qquad \le c \left( \int_R [f(Du - D\psi) - f(Du)]\, dx + \int_R [f(A+D\psi) - f(A)]\, dx \right).
\end{aligned}$$

Once this estimate is established we can argue exactly as in [S2, Lemma 7.3]. □

### *7.5. Concluding Remarks*

The next lemma is well known (cf. e. g. [D2, 4.1.1.1] or [G, Proposition 5.2]):

**Lemma 7.14.** *Assume that $f \in C^2_{\mathrm{loc}}(\mathbb{R}^{nN})$ satisfies* **(f3s)**. *Let us consider $M > 0$ and $A \in \mathbb{R}^{nN}$ with $|A| \le M$. Then, $D^2 f(A)$ fulfills the Legendre-Hadamard condition*

$$D^2 f(A)(\zeta x^T, \zeta x^T) \ge 2\lambda_M |x|^2 |\zeta|^2 \qquad \text{for all } x \in \mathbb{R}^n \text{ and } \zeta \in \mathbb{R}^N.$$

Investigating well-known regularity methods like blowup or $\mathscr{A}$-harmonic approximation we notice that there are only a few points where the minimizing property and the quasiconvexity condition are used, namely in the proof of the Euler equation, the Caccioppoli inequality, and the Legendre-Hadamard condition. Hence, once Lemma 7.3, Lemma 7.13, and Lemma 7.14 are established, standard methods lead to partial regularity. Actually, to prove Main Theorem 6.4 we can argue exactly as demonstrated in [S2] by means of the $\mathscr{A}$-harmonic approximation method of [DS]. We abandon further details.

Finally, as in [S2, Remark 7.11] we have:

**Remark 7.15.** *Under the assumptions of Main Theorem 6.4 the characterization* $\mathrm{Sing}(u) = \Sigma_1 \cup \Sigma_2$ *of the singular set is valid, where we have set*

$$\Sigma_1 := \left\{ x \in \Omega \,:\, \liminf_{\rho \searrow 0} \Phi_p(u, x, \rho, (Du)_{x,\rho}) > 0 \right\},$$

$$\Sigma_2 := \left\{ x \in \Omega \,:\, \limsup_{\rho \searrow 0} |(Du)_{x,\rho}| = \infty \right\}.$$

*Moreover, on a neighborhood of every $x \in \mathrm{Reg}(u)$ the Hölder seminorm of $Du$ can be bounded in terms of $n$, $N$, $p$, $q$, $\gamma$, $\Gamma$, $M := 1 + 2\limsup_{\rho \searrow 0} |(Du)_{x,\rho}|$, $\lambda_M$, $\Lambda_M$, $\nu_M$, and $\alpha$. Here, $\nu_M$ denotes a local modulus of continuity for $D^2 f$ as in [S2, Remark 7.6].*

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