A variational approach to the Navier-Stokes equations

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Abstract

We propose a time discretization approach to the Navier-Stokes equations inspired by the theory of gradient flows. This discretization produces Leray/Hopf solutions in any dimension and suitable solutions in dimension 3. We also show that in dimension 3 and for initial datum in H^1, the scheme converges to strong solutions in some interval $[0, T)$ and, if the datum satisfies the classical smallness condition, it produces the smooth solution in $[0, \infty)$.

1 Introduction

We consider the Navier-Stokes equations on the $d-$dimensional flat torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$:

$$\begin{cases} \partial_t u_t + (u_t \cdot \nabla)u_t + \nabla p_t = \Delta u_t, & \text{in } [0,\infty) \times \mathbb{T}^d, \\ \nabla \cdot u_t = 0, & \text{in } \mathbb{T}^d\ \forall t, \\ u_0 = \overline{u}, & \text{in } \mathbb{T}^d, \end{cases} \tag{1}$$

where the initial datum $\overline{u}$ is a given divergence free vector field, say smooth.

The purpose of this paper is to present a time-discretization argument, inspired by the gradient flows theory, which allows to quickly reproduce several known results about (1). The idea is the following. Fix a parameter $\tau > 0$, which we think as time step and, given $\overline{u}$, define its flow map $\mathbb{R} \times \mathbb{T}^d \ni (t, x) \mapsto X_t^{\overline{u}}(x) \in \mathbb{T}^d$ as the only solution of

$$\begin{cases} \partial_t X_t^{\overline{u}} = \overline{u} \circ X_t^{\overline{u}}, \\ X_0^{\overline{u}} = Id. \end{cases}$$

Now minimize

$$v \quad \mapsto \quad \frac{1}{2}\int_{\mathbb{T}^d} |\nabla v|^2 d\mathcal{L}^d + \frac{\|v \circ X_\tau^{\overline{u}} - \overline{u}\|_{L^2}^2}{2\tau},$$

among all L^2 and divergence free vector fields v. It is not hard to check (see Proposition 3.1) that the unique minimum u^τ satisfies

$$\frac{u^\tau - \overline{u} \circ X_{-\tau}^{\overline{u}}}{\tau} + \nabla p^\tau = \Delta u^\tau, \tag{2}$$

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where p^τ is identified, up to additive constants, by

$$\Delta p^\tau = \nabla \cdot \left(\frac{\overline{u} \circ X^{\overline{u}}_{-\tau}}{\tau} \right).$$

We claim that (2) is a time discretization of (1). Indeed, the the term

$$\frac{u^\tau - \overline{u} \circ X^{\overline{u}}_{-\tau}}{\tau} = \left(\frac{u^\tau \circ X^{\overline{u}}_{\tau} - \overline{u}}{\tau} \right) \circ X^{\overline{u}}_{-\tau},$$

is the time discretization of the convective derivative

$$\partial_t u_t + (u_t \cdot \nabla) u_t = \Big(\partial_t (u_t \circ T_t) \Big) \circ T_t^{-1},$$

where here $[0, \infty) \times \mathbb{T}^d \ni (t, x) \mapsto T_t(x) \in \mathbb{T}^d$ is the flow map (or particle-trajectory map) associated to (u_t), i.e.:

$$\begin{cases} \partial_t T_t &= u_t \circ T_t, \\ T_0 &= Id. \end{cases}$$

and the pressure term satisfies

$$\Delta p^\tau = \nabla \cdot \left(\frac{\overline{u} \circ X^{\overline{u}}_{-\tau}}{\tau} \right) = \nabla \cdot \left(\frac{\overline{u} \circ X^{\overline{u}}_{-\tau} - \overline{u}}{\tau} \right),$$

which is a time discretization of

$$\Delta p_t = \nabla \cdot \big((u_t \cdot \nabla) u_t \big) = \nabla \cdot \big(\partial_t u_t + (u_t \cdot \nabla) u_t \big),$$

the latter being the formula identifying the pressure in (1).

The idea is then to repeat the minimization procedure with u^τ in place of $\overline{u}$, then with the new minimizer in place of u^τ and so on. This argument produces a *discrete solution* (u_t^τ) and our goal is to show that letting $\tau \downarrow 0$ the discrete solutions converge, in a sense to be specified, to certain solutions of (1).

We remark that a time discretization based on (2) is not entirely new in this setting: O. Pirroneau [7] used the same equation (without pointing out its variational structure) in the setting of numerical analysis to investigate the rate of convergence of the discrete solutions under the assumption that a smooth solution of (1) exists on some interval $[0, T]$.

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2 Notation and preliminaries

With $\mathbb{T}^d$ we denote the $d-$dimensional flat torus $\mathbb{R}^d/\mathbb{Z}^d$. A time dependent vector field $[0,\infty) \ni (t,x) \mapsto u_t(x) \in \mathbb{R}^d$ will be typically denoted by (u_t), while we write u_t for the static vector field $x \mapsto u_t(x)$. The subscript t will never stand for time derivative, which will be usually denoted by ∂_t. When not specified, the integral symbol without further specification on the domain will stand for integration over $\mathbb{T}^d$ (resp. $[0,+\infty)$) when performed w.r.t. the measure $d\mathcal{L}^d$ (resp. dt). We will also shorten $L^p(\mathbb{T}^d,\mathbb{R}^d)$ with L^p when the space is clear from the context.

Given a smooth vector vector field $u \in C^\infty(\mathbb{T}^d,\mathbb{R}^d)$, the flow map $\mathbb{R}\times\mathbb{T}^d \ni (t,x) \mapsto X_t^u(x) \in \mathbb{T}^d$ is the unique solution of

$$\begin{cases} \partial_t X_t^u(x) & = u \circ X_t^u(x), \\ X_0^u(x) & = x. \end{cases}$$

The classic Cauchy-Lipschitz theory ensures that X^u is C^∞ in space and time as soon as u is C^∞.

We will frequently use the fact that $\nabla \cdot u = 0$ implies $(X_t^u)_\# \mathcal{L}^d = \mathcal{L}^d$ for any $t \in \mathbb{R}$.

Given a vector field $u \in C^\infty(\mathbb{T}^d,\mathbb{R}^d)$, its Helmholtz decomposition is given by

$$u = \nabla p + w,$$

where $\nabla \cdot w = 0$ and $\int p d\mathcal{L}^d = 0$. It is not hard to see that the Helmholtz decomposition is unique and is an orthogonal decomposition of L^2. For the existence, just solve

$$\Delta p = \nabla \cdot u,$$

by also ensuring $\int p d\mathcal{L}^d = 0$, and define

$$w := u - \nabla p.$$

Classical elliptic regularity theory ensures that p, w are C^∞ as soon as u is.

Given $u \in C^\infty(\mathbb{T}^d,\mathbb{R}^d)$, with $|\nabla u|$ we will always mean the Hilbert-Smith norm of its gradient, given by

$$|\nabla u|^2(x) := \sum_{i,j} |\partial_i u^j(x)|^2.$$

We can, and will, suppose that at any time $t \geq 0$, the velocity field has zero mean value. Indeed, integrating over $\mathbb{T}^d$ equation (1) and integrating by parts gives for any solution of (1) it holds

$$\frac{d}{dt}\int u_t d\mathcal{L}^d = 0 \qquad \forall t \geq 0.$$

Thus, if $\overline{v} := \int \overline{u} d\mathcal{L}^d$, one can look for solutions (w_t) of (1) with initial data $\overline{w} = \overline{u} - \overline{v}$, thus having zero mean velocity for any $t \geq 0$. Letting then $u_t(x) = w_t(x - \overline{v}t) + \overline{v}$, it is easily checked that (u_t) is a solution of the original problem. The additional condition $\int_{\mathbb{T}^d} u d\mathcal{L}^d = 0$ implies the first of the two frequently used estimates

$$\begin{aligned} \|u\|_{L^{2^*}(\mathbb{T}^d)} &\leq C \|\nabla u\|_{L^2(\mathbb{T}^d)}, \\ \|\nabla u\|_{L^{2^*}(\mathbb{T}^d)} &\leq C \|\Delta u\|_{L^2(\mathbb{T}^d)}, \end{aligned} \tag{3}$$

where $2^* = \frac{2d}{d-2}$, while the second one follows from standard elliptic estimates, since ∇u has zero mean on $\mathbb{T}^d$ by periodicity.

To show the convergence of the discretization scheme, we will use the Aubin-Lions lemma:

Lemma 2.1 (Aubin-Lions) *Let $X \subset Y \subset Z$ be three Banach spaces such that: X and Z are reflexive, the embedding of X in Y is compact and the one of Y into Z is continuous. Then for any $p, q \in (1, \infty)$ the space*

$$\Big\{ u \in L^p([0,T], X) \; : \; \partial_t u \in L^q([0,T], Z) \Big\},$$

is compactly embedded in $L^p([0,T], Y)$.

For a proof see for instance [10] Chapter 3, Theorem 2.1.

3 Discrete solutions

Let $u \in C^\infty(\mathbb{T}^d, \mathbb{R}^d)$ be a smooth vector field and $\tau > 0$. The functional $F(v; u, \tau)$ is defined as

$$F(v; u, \tau) := \int_{\mathbb{T}^d} |\nabla v|^2 d\mathcal{L}^d + \frac{1}{\tau} \int_{\mathbb{T}^d} |v \circ X^u_\tau - u|^2 d\mathcal{L}^d. \tag{4}$$

Proposition 3.1 *Let $u \in C^\infty(\mathbb{T}^d, \mathbb{R}^d)$ be a smooth vector field such that $\nabla \cdot u = 0$ and $\tau > 0$. Then there exists a unique minimizer u^τ of $v \mapsto F(v; u, \tau)$ in the class of L^2 vector fields such that $\nabla \cdot v = 0$. The minimum u^τ is C^∞ and satisfies:*

$$\frac{u^\tau - u \circ X^u_{-\tau}}{\tau} + \nabla p^\tau = \Delta u^\tau, \tag{5}$$

for some $p^\tau \in C^\infty$ with $\int p^\tau d\mathcal{L}^d = 0$.

Proof. Existence follows by standard weak compactness-lower semicontinuity arguments. For uniqueness, observe that the map $v \mapsto \int |\nabla v|^2$ is convex and

$$v \mapsto \int |v \circ X^u_\tau - u|^2 d\mathcal{L}^d = \int |v - u \circ X^u_{-\tau}|^2 d\mathcal{L}^d,$$

is strictly convex.

To write the Euler equation, notice that for $\xi \in C^\infty(\mathbb{T}^d, \mathbb{R}^d)$ with $\nabla \cdot \xi = 0$, the standard perturbation argument gives

$$\int \frac{u^\tau - u \circ X^u_{-\tau}}{\tau} \cdot \xi d\mathcal{L}^d = - \int \nabla u^\tau \nabla \xi d\mathcal{L}^d,$$

thus u^τ, p^τ is a weak solution of the Stokes problem (5). Standard regularity theory for the stokes operator guarantees that both u^τ and p^τ are C^∞ and are classical solutions of (5). $\square$

Now we use this minimization problem to build a time-discretized solution of the Navier-Stokes equation:

Definition 3.2 (Discrete solutions) *Let $\overline{u} \in C^\infty(\mathbb{T}^d, \mathbb{R}^d)$ be a smooth vector field with $\nabla \cdot \overline{u} = 0$ and $\tau > 0$. Define the vector field (u_t^τ) recursively by:*

$$
\begin{aligned}
u_0^\tau &:= \overline{u}, \\
u_{(n+1)\tau}^\tau &:= \underset{\nabla\cdot v = 0}{\operatorname{argmin}} F(v; u_{n\tau}^\tau, \tau), \qquad \forall n \in \mathbb{N} \\
u_t^\tau &:= u_{\tau[\frac{t}{\tau}]}^\tau, \qquad \forall t \geq 0.
\end{aligned}
$$

The discrete pressure field (p_t^τ) is defined, for every $t \geq 0$, by

$$
\begin{aligned}
\Delta p_t^\tau &= \nabla \cdot \left(\frac{u_t^\tau \circ X_{-\tau}^{u_t^\tau}}{\tau} \right), \\
\int p_t^\tau \, d\mathcal{L}^d &= 0.
\end{aligned}
$$

Notice that since the smoothness of a vector field implies the smoothness of the corresponding minimizer for any $\tau > 0$, the flow maps are always well defined and so are the discrete solutions. Also u_t^τ, p_t^τ are smooth for any $t \geq 0$.

4 The results

4.1 Weak solutions

Here we prove that discrete solutions produce, when $\tau \downarrow 0$, weak solutions of the Navier-Stokes equations in any dimension.

Definition 4.1 (Hopf Solutions) *We say that (u_t) is a Hopf solution of the Navier-Stokes equation starting from $\overline{u}$ provided it satisfies*

$$
\iint_{(0,\infty)\times\mathbb{T}^d} - \langle u_t, \partial_t \xi_t \rangle - \langle u_t, \nabla \xi_t \cdot u_t \rangle + \langle \nabla u_t, \nabla \xi_t \rangle \, dt \, d\mathcal{L}^d = \int_{\mathbb{T}^d} \langle \overline{u}, \xi_0 \rangle \, d\mathcal{L}^d, \tag{6}
$$

for any $\xi \in C_c^\infty([0,\infty) \times \mathbb{T}^d, \mathbb{R}^d)$ with $\nabla \cdot \xi_t = 0$ for any $t \geq 0$,

$$
\begin{aligned}
\frac{1}{2}\|u_s\|_{L^2}^2 + \int_0^s \|\nabla u_r\|_{L^2}^2 dr &\leq \frac{1}{2}\|\overline{u}\|_{L^2}^2, \qquad \forall s \geq 0, \\
\frac{1}{2}\|u_s\|_{L^2}^2 + \int_t^s \|\nabla u_r\|_{L^2}^2 dr &\leq \frac{1}{2}\|u_t\|_{L^2}^2, \qquad a.e.\ t > 0,\ \forall s \geq t,
\end{aligned}
$$

and $t \mapsto u_t$ is continuous w.r.t. the weak topology of $L^2(\mathbb{T}^d, \mathbb{R}^d)$.

Proposition 4.2 (One step estimates) *With the same notation and assumptions of Proposition 3.1 it holds:*

- Discrete energy inequality.

$$\frac{1}{2}\|u^\tau\|_{L^2}^2 + \tau\|\nabla u^\tau\|_{L^2}^2 \le \frac{1}{2}\|u\|_{L^2}^2. \tag{7}$$

- Discrete distributional solution. *For every $\xi \in C^\infty(\mathbb{T}^d, \mathbb{R}^d)$ it holds*

$$\left\langle \frac{u^\tau - u}{\tau}, \xi \right\rangle_{L^2} - \left\langle u, \frac{\xi \circ X_\tau^u - \xi}{\tau} \right\rangle_{L^2} + \langle p^\tau, \nabla \cdot \xi \rangle_{L^2} = \langle u^\tau, \Delta\xi \rangle_{L^2} = - \langle \nabla u^\tau, \nabla \xi \rangle_{L^2} \tag{8}$$

- Discrete uniform weak continuity. *For any $\xi \in C^\infty(\mathbb{T}^d, \mathbb{R}^d)$ with $\nabla \cdot \xi = 0$ it holds*

$$| \langle u^\tau - u, \xi \rangle_{L^2} | \le \tau C(\xi) \Big(\|u\|_{L^2} + \|u\|_{L^2}^2 \Big), \tag{9}$$

  where $C(\xi) := \max\{\mathrm{Lip}(\xi), \|\Delta\xi\|_{L^2}\}$

- Rough estimate on the discrete time derivative.

$$\left\| \frac{u^\tau - u}{\tau} \right\|_{H_{Df}^{-m}} \le C(\|u\|_{L^2} + \|u\|_{L^2}^2) \tag{10}$$

  for any $m > \frac{n}{2} + 2$, for some constant C independent on u, τ, where H_{Df}^{-m} is the dual space of the space of divergence free vector fields in $H^m(\mathbb{T}^d, \mathbb{R}^d)$ with 0 mean, endowed with the norm $\|u\|_{H^m}^2 := \sum_\alpha \|\partial_\alpha u\|_{L^2}^2$, where α varies over all the multiindexes of length m.

Proof. To get (7) multiply (5) by τu^τ and integrate to get

$$\|u^\tau\|_{L^2}^2 - \left\langle u^\tau, u \circ X_{-\tau}^u \right\rangle_{L^2} = \tau \langle u^\tau, \Delta u^\tau \rangle_{L^2},$$

and conclude noticing that

$$\langle u^\tau, \Delta u^\tau \rangle_{L^2} = -\|\nabla u^\tau\|_{L^2}^2,$$
$$\left\langle u^\tau, u \circ X_{-\tau}^u \right\rangle_{L^2} \le \frac{1}{2}\|u^\tau\|_{L^2}^2 + \frac{1}{2}\|u \circ X_{-\tau}^u\|_{L^2}^2 = \frac{1}{2}\|u^\tau\|_{L^2}^2 + \frac{1}{2}\|u\|_{L^2}^2.$$

To get (8) we sum and subtract u/τ, multiply (5) by ξ, and integrate by parts the terms involving the pressure and Δu^τ. For the discrete convective term we use the fact that $(X_t^u)_\# \mathcal{L}^d = \mathcal{L}^d$ for any t and thus, by the semigroup property of (X_t^u),

$$\left\langle u \circ X_{-\tau}^u, \xi \right\rangle_{L^2} = \langle u, \xi \circ X_\tau^u \rangle_{L^2}.$$

For (9) we observe that

$$
\begin{aligned}
\int \left|\xi \circ X^u_\tau - \xi\right|^2 d\mathcal{L}^d &= \int \left|\int_0^\tau \partial_t(\xi \circ X^u_t)dt\right|^2 d\mathcal{L}^d \\
&= \int \left|\int_0^\tau \nabla\xi \circ X^u_t \cdot u \circ X^u_t dt\right|^2 d\mathcal{L}^d \\
&\le \tau \int\int_0^\tau |\nabla\xi \circ X^u_t \cdot u \circ X^u_t|^2 dt d\mathcal{L}^d \\
&= \tau \int_0^\tau \int |\nabla\xi \circ X^u_t \cdot u \circ X^u_t|^2 d\mathcal{L}^d dt \\
&= \tau \int_0^\tau \int |\nabla\xi \cdot u|^2 d\mathcal{L}^d dt \\
&\le \tau^2 \mathrm{Lip}^2(\xi)\|u\|^2_{L^2},
\end{aligned} \tag{11}
$$

yields

$$
\|\xi \circ X^u_\tau - \xi\|_{L^2} \le \tau \mathrm{Lip}(\xi)\|u\|_{L^2}, \tag{12}
$$

and conclude using (8).

It remains to prove (10). Start recalling that for any $\xi \in H^m(\mathbb{T}^d, \mathbb{R}^d)$, it holds

$$
\|\Delta\xi\|_{L^2} + \mathrm{Lip}(\xi) \le C \|\xi\|_{H^m},
$$

for some constant C. Therefore from (9) we get

$$
|\langle u^\tau - u, \xi\rangle_{L^2}| \le C\tau \|\xi\|_{H^m} \Big(\|u\|_{L^2} + \|u\|^2_{L^2}\Big), \qquad \forall \xi \in C^\infty(\mathbb{T}^d, \mathbb{R}^d) \ s.t.\ \nabla\cdot\xi = 0,
$$

which implies (10). □

Theorem 4.3 (Hopf solutions) *For any sequence $\tau_n \downarrow 0$ there exists a subsequence, not relabeled, such that $u^{\tau_n}_t$ weakly converges (in $L^2(\mathbb{T}^d)$) to some u_t as $n \to \infty$ for any $t \ge 0$, the convergence is strong for a.e. t and $(u^{\tau_n}_t)$ converges strongly in $L^2_{loc}([0, +\infty), L^2(\mathbb{T}^d, \mathbb{R}^d))$ to (u_t).*

Any limit vector field (u_t) found in this way is a Hopf solution of the Navier–Stokes equations.

Proof.
Compactness. From (7) we immediately get

$$
\|u^\tau_t\|_{L^2} \le \|\overline{u}\|_{L^2}, \qquad \forall t, \tau > 0. \tag{13}
$$

Thus with a diagonalization argument, for each sequence $\tau_n \downarrow 0$ we can find a subsequence, not relabeled, such that for each rational t, the sequence $n \mapsto u^{\tau_n}_t$ weakly converges to some $u_t \in L^2(\mathbb{T}^d, \mathbb{R}^d)$. From (9) we easily get that for every $\xi \in C^\infty(\mathbb{T}^d, \mathbb{R}^d)$ such that $\nabla\cdot\xi = 0$ it holds,

$$
\left|\langle u^\tau_t - u^\tau_s, \xi\rangle_{L^2}\right| \le (t - s + \tau)C(\xi)\big(\|\overline{u}\|_{L^2} + \|\overline{u}\|^2_{L^2}\big), \qquad \forall \tau > 0,\ \forall t, s \in [0, \infty), \tag{14}
$$

which is enough to conclude that there is weak convergence for every $t \geq 0$ as $\tau_n \downarrow 0$.

To get the strong convergence we use the Aubin-Lions lemma. In order to do so it is better to introduce the piecewise affine interpolation of the $\{u^\tau_{n\tau}\}_{n\in\mathbb{N}}$ in place of the piecewise constant one:

$$w^\tau_t := \left(1 - \frac{t}{\tau} + \left[\frac{t}{\tau}\right]\right) u^\tau_{\tau[\frac{t}{\tau}]} + \left(\frac{t}{\tau} - \left[\frac{t}{\tau}\right]\right) u^\tau_{\tau([\frac{t}{\tau}]+1)}. \tag{15}$$

It is immediate to verify that the compactness of $\{(w^\tau_t)\}_\tau$ implies that of $\{(u^\tau_t)\}_\tau$. To get the compactness of $\{(w^\tau_t)\}_\tau$ in $L^2_{loc}([0,\infty), L^2)$ we apply the Aubin-Lions lemma to the spaces

$$\begin{aligned} X &:= \Big\{u \in H^1(\mathbb{T}^d, \mathbb{R}^d) \ : \ \nabla\cdot u = 0, \ \int_{\mathbb{T}^d} u \, \mathcal{L}^d = 0\Big\}, \\ Y &:= \Big\{u \in L^2(\mathbb{T}^d, \mathbb{R}^d) \ : \ \nabla\cdot u = 0, \ \int_{\mathbb{T}^d} u \, \mathcal{L}^d = 0\Big\}, \\ Z &:= H^{-m}_{Df}, \end{aligned}$$

where X, Y are endowed with the H^1 and the L^2 norms respectively. Then from (10) and the definition of (w^τ_t) we get that

$$\|\partial_t w^\tau_t\|_{H^{-m}_{Df}} = \left\| \frac{u^\tau_{t+\tau} - u^\tau_t}{\tau} \right\|_{H^{-m}_{Df}} \leq C(\|\overline{u}\|_{L^2} + \|\overline{u}\|^2_{L^2}), \qquad a.e.\ t,$$

which is sufficient to conclude.

Now that we have compactness in $L^2_{loc}([0,+\infty), L^2(\mathbb{T}^d, \mathbb{R}^d))$, we know that for any sequence $\tau_n \downarrow 0$ there exists a subsequence τ_{n_k} such that $u^{\tau_{n_k}}_t$ converges strongly to u_t for a.e. $t > 0$.

Any limit is a Hopf solution. Let $\tau_n \downarrow 0$ be a sequence for which we have weak convergence for all times and strong convergence for a.e. times and let (u_t) be the limit vector field. We have the uniform bound

$$\|u_t\|_{L^2} \leq \varliminf_{n\to\infty} \|u^{\tau_n}_t\|_{L^2} \leq \|\overline{u}\|_{L^2}, \qquad \forall t \geq 0,$$

and passing to the limit in (14) we obtain

$$\big|\, \langle u_t - u_s, \xi\rangle_{L^2} \,\big| \leq (t-s) C(\xi)\big(\|\overline{u}\|_{L^2} + \|\overline{u}\|^2_{L^2}\big), \qquad \forall t, s \in [0,\infty),$$

which is enough to get the weak continuity of $t \mapsto u_t$.

Now let $A \subset [0,\infty)$ the set of t's such that $u^{\tau_n}_t$ converges strongly to u_t as $n \to \infty$ and notice that certainly $0 \in A$ and $\mathcal{L}^1([0,\infty)\setminus A) = 0$. Choose $t \in A$ and $s > t$ and observe that from (7) we get

$$\frac{1}{2}\|u^\tau_s\|^2_{L^2} + \int_{\tau[\frac{t}{\tau}]+\tau}^{\tau[\frac{s}{\tau}]+\tau} \|\nabla u^\tau_r\|^2_{L^2} dr \leq \frac{1}{2}\|u^\tau_t\|^2_{L^2}, \qquad \forall \tau > 0. \tag{16}$$

The choice of t ensures that $\|u_t^{\tau_n}\|_{L^2} \to \|u_t\|_{L^2}$, so that from

$$\varliminf_{n\to\infty}\left(\frac{1}{2}\|u_s^{\tau_n}\|_{L^2}^2 + \int\limits_{\tau_n[\frac{t}{\tau_n}]+\tau_n}^{\tau_n[\frac{s}{\tau_n}]+\tau_n} \|\nabla u_r^{\tau_n}\|_{L^2}^2 dr\right) \geq \frac{1}{2}\|u_s\|_{L^2}^2 + \int\limits_t^s \varliminf_{n\to\infty}\|\nabla u_r^{\tau_n}\|_{L^2}^2 dr$$

$$\geq \frac{1}{2}\|u_s\|_{L^2}^2 + \int\limits_t^s \|\nabla u_r\|_{L^2}^2 dr,$$

we get the energy inequality.

To conclude we need to show that (u_t) satisfies (6). Fix $\xi \in C_c^\infty([0,\infty)\times\mathbb{T}^d,\mathbb{R}^d)$ such that $\nabla\cdot\xi_t = 0$ for every $t\geq 0$, and for any $k = 0,1\ldots$ we consider (8) for $u = u_{k\tau}^\tau$ tested with $\xi_{k\tau}$. Adding up one gets

$$-\iint_\tau^{+\infty} u_t^\tau \cdot \frac{\xi_{\tau[\frac{t}{\tau}]} - \xi_{\tau([\frac{t}{\tau}]-1)}}{\tau} + u_t^\tau\cdot\frac{\xi_{\tau[\frac{t}{\tau}]}\circ X_\tau^{u_t^\tau} - \xi_{\tau[\frac{t}{\tau}]}}{\tau} + u_t^\tau\cdot\Delta\xi_{\tau[\frac{t}{\tau}]} dt\, d\mathcal{L}^d = \int \overline{u}\cdot\xi_0 d\mathcal{L}^d.$$

From the smoothness of ξ we know that

$$\frac{\xi_{\tau[\frac{t}{\tau}]} - \xi_{\tau([\frac{t}{\tau}]-1)}}{\tau} \quad\to\quad \partial_t\xi_t,$$

$$\Delta\xi_{\tau[\frac{t}{\tau}]} \quad\to\quad \Delta\xi_t,$$

uniformly in $[0,\infty)\times\mathbb{T}^d$ as $\tau\downarrow 0$. Thus from the strong convergence of $(u_t^{\tau_n})$ to (u_t) in $L^2_{loc}([0,\infty),L^2)$ we get

$$\iint_\tau^{+\infty} u_t^{\tau_n}\cdot\frac{\xi_{\tau_n[\frac{t}{\tau_n}]} - \xi_{\tau_n([\frac{t}{\tau_n}]-1)}}{\tau_n} dt\, d\mathcal{L}^d \quad\to\quad \iint_0^{+\infty} u_t\cdot\partial_t\xi_t dt\, d\mathcal{L}^d,$$

$$\iint_\tau^{+\infty} u_t^{\tau_n}\cdot\Delta\xi_{\tau_n[\frac{t}{\tau_n}]} dt\, d\mathcal{L}^d \quad\to\quad \iint_0^{+\infty} u_t\cdot\Delta\xi_t dt\, d\mathcal{L}^d = -\iint_0^{+\infty}\nabla u_t\cdot\nabla\xi_t dt\, d\mathcal{L}^d,$$

as $n\to\infty$. Thus to conclude it is sufficient to check that

$$\frac{\xi_{\tau_n[\frac{t}{\tau_n}]}\circ X_\tau^{u_t^{\tau_n}} - \xi_{\tau_n[\frac{t}{\tau_n}]}}{\tau_n} \quad\to\quad \nabla\xi_t\cdot u_t,$$

weakly in $L^2([0,T],L^2)$ as $n\to\infty$, where T is such that $\mathrm{supp}(\xi)\subset[0,T]\times\mathbb{T}^d$. From (12) and (13) it is easy to deduce that

$$\frac{1}{\tau_n}\left\|\xi_{\tau_n[\frac{t}{\tau_n}]}\circ X_\tau^{u_t^{\tau_n}} - \xi_{\tau_n[\frac{t}{\tau_n}]}\right\|_{L^2([0,\infty),L^2)} \leq T\|\overline{u}\|_{L^2}\sup_t \mathrm{Lip}(\xi_t).$$

Thus to prove the desired weak convergence it is sufficient to prove that for every $\tilde{\xi}\in C_c^\infty([0,\infty)\times\mathbb{T}^d,\mathbb{R}^d)$ it holds

$$\frac{1}{\tau_n}\iint\left\langle\tilde{\xi}_t, \xi_{\tau_n[\frac{t}{\tau_n}]}\circ X_{\tau_n}^{u_t^{\tau_n}} - \xi_{\tau_n[\frac{t}{\tau_n}]}\right\rangle dt\, d\mathcal{L}^d \quad\to\quad \int\left\langle\tilde{\xi}_t, (u_t\cdot\nabla)\xi_t\right\rangle dt\, d\mathcal{L}^d, \qquad \text{as } n\to\infty.$$

To prove this, recall that

$$\frac{\xi_{\tau_n[\frac{t}{\tau_n}]}\circ X^{u_t^{\tau_n}}_{\tau_n}-\xi_{\tau_n[\frac{t}{\tau_n}]}}{\tau_n}=\frac{1}{\tau_n}\int_0^{\tau_n}\Big((u_t^{\tau_n}\cdot\nabla)\xi_{\tau_n[\frac{t}{\tau_n}]}\Big)\circ X_s^{u_t^{\tau_n}}\,ds,$$

and therefore

$$\begin{aligned}
\frac{1}{\tau_n}\iint\Big\langle\tilde\xi_t,\xi_{\tau_n[\frac{t}{\tau_n}]}\circ X^{u_t^{\tau_n}}_{\tau_n}-\xi_{\tau_n[\frac{t}{\tau_n}]}\Big\rangle dt\,d\mathcal{L}^d\\
&=\frac{1}{\tau_n}\iiint_0^{\tau_n}\Big\langle\tilde\xi_t,\Big((u_t^{\tau_n}\cdot\nabla)\xi_{\tau_n[\frac{t}{\tau_n}]}\Big)\circ X_s^{u_t^{\tau_n}}\Big\rangle\,ds\,dt\,d\mathcal{L}^d\\
&=\frac{1}{\tau_n}\int_0^{\tau_n}\iint\Big\langle\tilde\xi_t,\Big((u_t^{\tau_n}\cdot\nabla)\xi_{\tau_n[\frac{t}{\tau_n}]}\Big)\circ X_s^{u_t^{\tau_n}}\Big\rangle\,d\mathcal{L}^d dt\,ds\\
&=\frac{1}{\tau_n}\int_0^{\tau_n}\iint\Big\langle\tilde\xi_t\circ X_{-s}^{u_t^{\tau_n}},(u_t^{\tau_n}\cdot\nabla)\xi_{\tau_n[\frac{t}{\tau_n}]}\Big\rangle\,d\mathcal{L}^d dt\,ds\\
&=\iint\left\langle\frac{1}{\tau_n}\int_0^{\tau_n}\tilde\xi_t\circ X_{-s}^{u_t^{\tau_n}}\,ds,\,(u_t^{\tau_n}\cdot\nabla)\xi_{\tau_n[\frac{t}{\tau_n}]}\right\rangle d\mathcal{L}^d\,dt.
\end{aligned}\tag{17}$$

To conclude notice that the strong convergence of $(u_t^{\tau_n})$ to (u_t) and the smoothness of ξ ensure that $(u_t^{\tau_n}\cdot\nabla)\xi_{\tau_n[\frac{t}{\tau_n}]}$ strongly converges to $(u_t\cdot\nabla)\xi_t$ in $L^2([0,\infty),L^2)$. Also, from (12) with $\tau=s$ one gets

$$\begin{aligned}
\iint\left(\frac{1}{\tau_n}\int_0^{\tau_n}\tilde\xi_t\circ X_{-s}^{u_t^{\tau_n}}\,ds-\tilde\xi_t\right)^2 d\mathcal{L}^d dt&=\frac{1}{\tau_n}\int_0^{\tau_n}\int\Big\|\tilde\xi_t-\tilde\xi_t\circ X_s^{u_t^{\tau_n}}\Big\|^2_{L^2}\,dtds\\
&\le\frac{1}{\tau_n}\int_0^{\tau_n}\int_0^{\widetilde T}s^2\,\mathrm{Lip}(\tilde\xi)^2\,\|u_t^{\tau_n}\|^2_{L^2}\,dtds\\
&\le\frac{\tau_n^2}{3}\,\|\overline{u}\|^2_{L^2}\,\widetilde T\mathrm{Lip}(\tilde\xi)^2,
\end{aligned}\tag{18}$$

where $\widetilde T$ is such that $\mathrm{supp}(\tilde\xi)\subseteq[0,\widetilde T]\times\mathbb{T}^d$. Therefore $\big\|\frac{1}{\tau_n}\int_0^{\tau_n}\tilde\xi_t\circ X_{-s}^{u_t^{\tau_n}}\,ds-\tilde\xi_t\big\|_{L^2([0,\infty),L^2)}=O(\tau_n)$, which completes the proof. □

4.2 Suitable solutions in dimension 3

Here we show that in dimension 3 the discrete solutions converge to suitable solutions of the Navier-Stokes equations.

Definition 4.4 (Suitable solution) *We say that the pair (u_t,p_t) is a suitable solution of the Navier-Stokes equations starting from $\overline{u}$ provided it is a distributional solution of the*

Navier-Stokes equation, (u_t) *is a Hopf solution starting from* $\overline{u}$*, and in addition* $(u_t) \in L^3_{loc}([0,\infty), L^3(\mathbb{T}^3,\mathbb{R}^3))$*,* $(p_t) \in L^{3/2}_{loc}([0,\infty), L^{3/2}(\mathbb{T}^3))$ *and it holds*

$$\partial_t \frac{|u_t|^2}{2} + \nabla \cdot \left(u_t \Big(\frac{|u_t|^2}{2} + p_t \Big) \right) + |\nabla u_t|^2 \leq \Delta \frac{|u_t|^2}{2}, \tag{19}$$

in the sense of distributions, that is

$$\iint -\frac{|u_t|^2}{2}\partial_t \varphi - \nabla\varphi \cdot u_t \Big(\frac{|u_t|^2}{2} + p_t \Big) + \varphi |\nabla u_t|^2 dt\, d\mathcal{L}^3 \leq \iint \Delta\varphi \frac{|u_t|^2}{2} dt\, d\mathcal{L}^3,$$

for any non negative $\varphi \in C^\infty_c((0,\infty)\times\mathbb{T}^3)$*.*

We recall that the importance of suitable solutions is due to the work [2] where important partial regularity results for these solutions have been achieved.

Lemma 4.5 (A time step) *With the same notation and assumptions of Proposition 3.1 it holds*

$$\int \frac{1}{2}(|u^\tau|^2 - |u|^2)\varphi - \frac{1}{2}|u|^2(\varphi \circ X^u_\tau - \varphi) - \nabla\varphi \cdot u^\tau p^\tau + |\nabla u^\tau|^2 \varphi d\mathcal{L}^d \leq \tau \int \frac{|u^\tau|^2}{2} \Delta\varphi d\mathcal{L}^d,$$

for any nonnegative $\varphi \in C^\infty(\mathbb{T}^d)$*.*

Proof. Multiply (5) by u^τ, by $\varphi \in C^\infty(\mathbb{T}^d)$ with $\varphi \geq 0$ and integrate to get

$$\int |u^\tau|^2\varphi - u^\tau \cdot u \circ X^u_{-\tau}\varphi + \varphi u^\tau \cdot \nabla p^\tau d\mathcal{L}^d = \tau \int u^\tau \cdot \Delta u^\tau \varphi d\mathcal{L}^d.$$

It holds

$$\begin{aligned}
\int u^\tau \cdot \Delta u^\tau \varphi d\mathcal{L}^d &= -\int |\nabla u^\tau|^2 \varphi - \sum_{i,j} u^i \partial_j u^i \partial_j \varphi \; d\mathcal{L}^d \\
&= -\int |\nabla u^\tau|^2 \varphi d\mathcal{L}^d + \int \frac{|u^\tau|^2}{2}\Delta\varphi d\mathcal{L}^d, \\
\int \varphi u^\tau \cdot \nabla p^\tau d\mathcal{L}^d &= -\int \nabla\varphi \cdot u^\tau p^\tau d\mathcal{L}^d, \\
\int u^\tau \cdot u \circ X^u_{-\tau}\varphi d\mathcal{L}^d &\leq \frac{1}{2}\int |u^\tau|^2\varphi d\mathcal{L}^d + \frac{1}{2}\int |u\circ X^u_{-\tau}|^2 \varphi d\mathcal{L}^d \\
&= \frac{1}{2}\int |u^\tau|^2\varphi d\mathcal{L}^d + \frac{1}{2}\int |u|^2 \varphi\circ X^u_\tau d\mathcal{L}^d,
\end{aligned}$$

(notice that in the last inequality we used the fact that $\varphi \geq 0$). The conclusion follows. ☐

Lemma 4.6 (Estimate on the pressure) *For any divergence free vector field* $u \in L^3(\mathbb{T}^d, \mathbb{R}^d) \cap C^\infty(\mathbb{T}^d, \mathbb{R}^d)$ *and* $\tau > 0$ *define* p_u^τ *as the only solution of*

$$\begin{cases} \Delta p_u^\tau = \nabla \cdot \left(\dfrac{u \circ X_{-\tau}^u}{\tau} \right), \\ \int p_u^\tau \, d\mathcal{L}^d = 0. \end{cases} \tag{20}$$

Then for some $C > 0$ *independent on* u *it holds*

$$\|p_u^\tau\|_{L^{3/2}} \le C \|u\|_{L^3}^2. \tag{21}$$

Proof. For any $\varphi \in C^\infty(\mathbb{T}^d)$ we have

$$\begin{aligned} \int p_u^\tau \Delta\varphi \, d\mathcal{L}^d &= \int \frac{u \circ X_{-\tau}^u - u}{\tau} \cdot \nabla\varphi \, d\mathcal{L}^d \\ &= \int u \cdot \frac{\nabla\varphi \circ X_\tau^u - \nabla\varphi}{\tau} \, d\mathcal{L}^d \\ &= \int u \cdot \left(\frac{1}{\tau} \int_0^\tau (\nabla^2\varphi \cdot u) \circ X_s^u \, ds \right) d\mathcal{L}^d \\ &\le \|u\|_{L^3} \left\| \frac{1}{\tau} \int_0^\tau (\nabla^2\varphi \cdot u) \circ X_s^u \, ds \right\|_{L^{3/2}} \\ &\le \|u\|_{L^3} \frac{1}{\tau} \int_0^\tau \left\| (\nabla^2\varphi \cdot u) \circ X_s^u \right\|_{L^{3/2}} \, ds \\ &= \|u\|_{L^3} \left\| \nabla^2\varphi \cdot u \right\|_{L^{3/2}} \\ &\le \|u\|_{L^3}^2 \left\| \nabla^2\varphi \right\|_{L^3}. \end{aligned} \tag{22}$$

Standard elliptic regularity results ensure that $\|\nabla^2\varphi\|_{L^3} \le C\|\Delta\varphi\|_{L^3}$ for some $C > 0$. Thus we get

$$\left| \int p_u^\tau \Delta\varphi d\mathcal{L}^d \right| \le C \|u\|_{L^3}^2 \left\| \Delta\varphi \right\|_{L^3}.$$

Again by standard arguments, we know that the equation $\Delta\varphi = \psi$ has a smooth solution for every smooth ψ such that $\int \psi d\mathcal{L}^d = 0$. Thus the above bound yields

$$\left| \int p_u^\tau \psi \, d\mathcal{L}^d \right| \le C \|u\|_{L^3}^2 \|\psi\|_{L^3}, \qquad \forall \psi \in C^\infty(\mathbb{T}^d) \text{ such that } \int \psi \, d\mathcal{L}^d = 0.$$

Since $\int p_u^\tau \, d\mathcal{L}^d = 0$, from the last inequality we conclude that (21) is true. □

Theorem 4.7 (Suitable solutions in dimension 3) *Let* (u_t^τ) *and* (p_t^τ) *be defined by 3.2. Then* $\{(u_t^\tau)\}_\tau$ *is compact in* $L^3_{loc}([0,\infty), L^3(\mathbb{T}^3, \mathbb{R}^3))$ *and* $\{(p_t^\tau)\}_\tau$ *is weakly compact in* $L^{3/2}_{loc}([0,\infty), L^{3/2}(\mathbb{T}^3))$. *For any sequence* $\tau_n \downarrow 0$ *such that* $(u_t^{\tau_n})$ *strongly converges to some* (u_t) *in* $L^3_{loc}([0,\infty), L^3(\mathbb{T}^3, \mathbb{R}^3))$ *and* $(p_t^{\tau_n})$ *weakly converges in* $L^{3/2}_{loc}([0,\infty), L^{3/2}(\mathbb{T}^3))$ *to some* (p_t), *the couple* (u_t), (p_t) *is a suitable solution of the Navier–Stokes equation.*

Proof.

Compactness. Arguing as in the proof of Theorem 4.3 and using the compact embedding of H^1 into L^4, we deduce that $\{(u_t^\tau)\}_\tau$ is relatively compact in $L^2_{loc}([0,\infty),L^4)$. For any $(u_t) \in L^2_{loc}\big([0,\infty),L^4\big) \cap L^\infty\big([0,\infty),L^2\big)$ it holds

$$\iint |u_t|^3 d\mathcal{L}^3 dt \le \int \|u_t\|_{L^2}\, \|u_t\|^2_{L^4}\, dt = \|u\|_{L^\infty([0,\infty),L^2)} \|u\|^2_{L^2([0,+\infty),L^4)},$$

and therefore the uniform bound (13) ensures the desired strong relative compactness of $\{(u_t^\tau)\}_\tau$ in $L^3([0,+\infty),L^3)$. To get the weak compactness of $\{(p_t^\tau)\}_\tau$, notice that p_t^τ is the unique solution of (20) for $u = u_t^\tau$, and from the uniform bound of (u_t^τ) in $L^3([0,+\infty),L^3)$, together with (21), we get weak compactness of (p_t^τ) in $L^{3/2}([0,+\infty),L^{3/2})$.

Limits are suitable solutions. To simplify the notation, we assume that $(u_t^\tau) \to (u_t)$ strongly in $L^3([0,\infty),L^3)$ $(p_t^\tau) \to (p_t)$ weakly in $L^{3/2}([0,\infty),L^{3/2})$ (i.e., we are not considering a sequence $\tau_n \downarrow 0$). Theorem 4.3 guarantees that u is a Hopf solution of the Navier-Stokes system, and clearly $(u_t) \in L^3([0,\infty),L^3)$, $(p_t) \in L^{3/2}([0,\infty),L^{3/2})$.

To prove that (u_t), (p_t) is a distributional solution, fix $\xi \in C_c^\infty((0,\infty)\times\mathbb{T}^3,\mathbb{R}^3)$, and for any $k = 0,1\ldots$ consider (8) for $u = u_{k\tau}^\tau$, tested with $\xi_{k\tau}$. Adding up one gets

$$\iint_\tau^{+\infty} u_t^\tau \cdot \frac{\xi_{\tau[\frac{t}{\tau}]} - \xi_{\tau([\frac{t}{\tau}]-1)}}{\tau} + u_t^\tau \cdot \frac{\xi_{\tau[\frac{t}{\tau}]} \circ X_\tau^{u_t^\tau} - \xi_{\tau[\frac{t}{\tau}]}}{\tau} dt\, d\mathcal{L}^3 = \iint_\tau^{+\infty} u_t^\tau \cdot \Delta\xi_{\tau[\frac{t}{\tau}]} + p_t^\tau \nabla\cdot\xi_{\tau[\frac{t}{\tau}]} dt\, d\mathcal{L}^3.$$

Now since $\nabla\cdot\xi_{\tau[\frac{t}{\tau}]} \to \nabla\cdot\xi_t$ uniformly on $[0,+\infty)\times\mathbb{T}^3$, the weak convergence of (p_t^τ) to (p_t) yields

$$\iint p_t^\tau \nabla\cdot\xi_{\tau[\frac{t}{\tau}]} dt\, d\mathcal{L}^3 \qquad \to \qquad \iint p_t \nabla\cdot\xi_t dt\, d\mathcal{L}^3,$$

while all the other terms are treated as in the proof of theorem 4.3.

To prove the generalized energy inequality, fix a non negative $\varphi \in C_c^\infty((0,\infty)\times\mathbb{T}^3)$. We want to show that

$$\iint -\frac{|u_t|^2}{2}\partial_t\varphi + u_t\cdot\nabla\varphi\left(\frac{|u_t|^2}{2} + p_t\right) + |\nabla u_t|^2\varphi dt d\mathcal{L}^3 \le \iint \frac{|u_t|^2}{2}\Delta\varphi dt\, d\mathcal{L}^3$$

Suppose τ is sufficiently small, such that $\mathrm{supp}(\varphi) \subset [\tau,+\infty)$. From Lemma 4.5 and the definition of u^τ, p^τ we immediately get, with the usual argument

$$\begin{aligned}\iint -\frac{|u_t^\tau|^2}{2}\frac{\varphi_{\tau[\frac{t}{\tau}]} - \varphi_{\tau([\frac{t}{\tau}]-1)}}{\tau} - \frac{|u_t^\tau|^2}{2}\frac{\varphi_{\tau[\frac{t}{\tau}]}\circ X_\tau^{u_t^\tau} - \varphi_{\tau[\frac{t}{\tau}]}}{\tau} - \nabla\varphi_{\tau[\frac{t}{\tau}]}\cdot u_t^\tau p_t^\tau + |\nabla u_t^\tau|^2\varphi_{\tau[\frac{t}{\tau}]} dt d\mathcal{L}^3 \\ \le \iint \frac{|u_t^\tau|^2}{2}\Delta\varphi_{\tau[\frac{t}{\tau}]} dt d\mathcal{L}^3.\end{aligned}$$

The convergence of u^τ ensures that $|u^\tau|^2$ converges to $|u|^2$ in $L^{3/2}([0,\infty),L^{3/2})$, and from the smoothness of φ and the weak convergence of p^τ it is immediate to verify that as $\tau\downarrow 0$ it holds

$$\begin{aligned}
\iint \frac{|u_t^\tau|^2}{2}\frac{\varphi_{\tau[\frac{t}{\tau}]}-\varphi_{\tau([\frac{t}{\tau}]-1)}}{\tau}dtd\mathcal{L}^3 \qquad &\to \qquad \iint \frac{|u_t|^2}{2}\partial_t\varphi\, dtd\mathcal{L}^3,\\
\iint \nabla\varphi_{\tau[\frac{t}{\tau}]}\cdot u_t^\tau p_t^\tau dtd\mathcal{L}^3 \qquad &\to \qquad \iint \nabla\varphi_t\cdot u_t p_t\, dtd\mathcal{L}^3,\\
\iint \frac{|u_t^\tau|^2}{2}\Delta\varphi_{\tau[\frac{t}{\tau}]}dtd\mathcal{L}^3 \qquad &\to \qquad \iint \frac{|u_t|^2}{2}\Delta\varphi_t dtd\mathcal{L}^3.
\end{aligned}$$

Also, the non negativity of φ easily implies

$$\liminf_{\tau\downarrow 0}\iint |\nabla u_t^\tau|^2\varphi_{\tau[\frac{t}{\tau}]}dtd\mathcal{L}^3 \geq \iint |\nabla u_t|^2\varphi_t dtd\mathcal{L}^3.$$

Thus to conclude it is sufficient to show that

$$\iint \frac{|u_t^\tau|^2}{2}\frac{\varphi_{\tau[\frac{t}{\tau}]}\circ X_\tau^{u_t^\tau}-\varphi_{\tau[\frac{t}{\tau}]}}{\tau}dtd\mathcal{L}^3 \qquad \to \qquad \iint \frac{|u_t^\tau|^2}{2}\nabla\varphi_t\cdot u_t dtd\mathcal{L}^3, \tag{23}$$

as $\tau\downarrow 0$. Since $|u_t^\tau|^2\to|u_t|^2$ in $L^{3/2}([0,\infty),L^{3/2})$, to prove (23) it is sufficient to check that

$$\frac{\varphi_{\tau[\frac{t}{\tau}]}\circ X_\tau^{u_t^\tau}-\varphi_{\tau[\frac{t}{\tau}]}}{\tau} \qquad \to \qquad \nabla\varphi_t\cdot u_t \qquad \text{weakly in } L^3([0,\infty),L^3).$$

From

$$\frac{\varphi_{\tau[\frac{t}{\tau}]}\circ X_\tau^{u_t^\tau}-\varphi_{\tau[\frac{t}{\tau}]}}{\tau} = \frac{1}{\tau}\int_0^\tau \nabla\varphi_{\tau[\frac{t}{\tau}]}\circ X_s^{u_t^\tau}\cdot u_t^\tau\circ X_s^{u_t^\tau}ds,$$

we immediately get that it holds

$$\left\|\frac{\varphi_{\tau[\frac{t}{\tau}]}\circ X_\tau^{u_t^\tau}-\varphi_{\tau[\frac{t}{\tau}]}}{\tau}\right\|_{L^3([0,\infty),L^3)} \leq \sup_t \mathrm{Lip}(\varphi_t)\|u\|_{L^3([0,\infty),L^3)},$$

which gives weak convergence for some subsequence. To conclude fix $\psi\in C_c^\infty((0,\infty)\times\mathbb{T}^3)$ and notice that

$$\begin{aligned}
\iint \psi_t\frac{\varphi_{\tau[\frac{t}{\tau}]}\circ X_\tau^{u_t^\tau}-\varphi_{\tau[\frac{t}{\tau}]}}{\tau}dtd\mathcal{L}^3 &= \frac{1}{\tau}\iiint_0^\tau \psi_t\nabla\varphi_{\tau[\frac{t}{\tau}]}\circ X_s^{u_t^\tau}\cdot u_t^\tau\circ X_s^{u_t^\tau}dsdtd\mathcal{L}^3\\
&= \frac{1}{\tau}\int_0^\tau\iint \psi_t\nabla\varphi_{\tau[\frac{t}{\tau}]}\circ X_s^{u_t^\tau}\cdot u_t^\tau\circ X_s^{u_t^\tau}d\mathcal{L}^3dtds\\
&= \frac{1}{\tau}\int_0^\tau\iint \psi_t\circ X_{-s}^{u_t^\tau}\nabla\varphi_{\tau[\frac{t}{\tau}]}\cdot u_t^\tau d\mathcal{L}^3dtds\\
&= \iint\left(\frac{1}{\tau}\int_0^\tau \psi_t\circ X_{-s}^{u_t^\tau}ds\right)\nabla\varphi_{\tau[\frac{t}{\tau}]}\cdot u_t^\tau d\mathcal{L}^3dt.
\end{aligned}$$

Since $\nabla\varphi_{\tau[\frac{t}{\tau}]} \cdot u_t^\tau \to \nabla\varphi_t \cdot u_t$ strongly in $L^3([0,\infty), L^3)$, and, as in (18), we get

$$\frac{1}{\tau}\int_0^\tau \psi_t \circ X_{-s}^{u_t^\tau} ds \qquad \to \qquad \psi_t,$$

in $L^2([0,+\infty), L^2)$, (23) is proved. □

Remark 4.8 We remark that one can actually prove that if $\tau_n \downarrow 0$ is such that $(u_t^{\tau_n})$ strongly converges to some (u_t) in $L^3_{loc}([0,\infty), L^3)$, *then* $(p_t^{\tau_n})$ weakly converges in $L^{3/2}_{loc}([0,\infty), L^{3/2})$ to the distributional solution p of

$$\begin{cases} \Delta p_t = \nabla \cdot \big((u_t \cdot \nabla) u_t\big), \\ \int p_t \, d\mathcal{L}^3 = 0, \end{cases}$$

and thus to the pressure term of the limit equation. Indeed, as in the proof of lemma 4.6, it suffices to check that, for a.e. t, one has

$$\int p_t^{\tau_n} \Delta\varphi \, d\mathcal{L}^3 \qquad \to \qquad \int p_t \Delta\varphi \, d\mathcal{L}^3, \qquad \forall \varphi \in C^\infty(\mathbb{T}^3),$$

as $n \to \infty$. To prove this, start from

$$\int p_t^{\tau_n} \Delta\varphi \, d\mathcal{L}^3 = \int u_t^{\tau_n} \cdot \left(\frac{1}{\tau_n}\int_0^{\tau_n} \big(\nabla^2\varphi \cdot u_t^{\tau_n}\big) \circ X_s^{u_t^{\tau_n}} \, ds\right) d\mathcal{L}^3,$$

(which follows as in (22)), and notice that from the strong convergence of $u_t^{\tau_n}$ to u_t in L^3 it is sufficient to check that it holds

$$\frac{1}{\tau_n}\int_0^{\tau_n} \big(\nabla^2\varphi \cdot u_t^{\tau_n}\big) \circ X_s^{u_t^{\tau_n}} \, ds \qquad \to \qquad \nabla^2\varphi \cdot u_t,$$

weakly in $L^{3/2}$. We already know that the norms are uniformly bounded, thus the conclusion follows along the same lines of the last part of the proof of Theorem 4.3 (see in particular (17) and the conclusion thereafter – in the current situation there is no integral in t), we omit the details. ■

4.3 Strong solutions

Here we show how from the discretization scheme discussed, one can prove two classical results concerning smooth solutions of the Navier-Stokes equations: existence for a time of order $\|\nabla\overline{u}\|_{L^2}^{-4}$, and existence for all times if $\overline{u}$ satisfies the classical smallness condition.

Notice that the calculations that we do here are classical: what we want to show is that the standard approach have a natural 'discrete analogous' in our setting. We recall that the two estimates (3) hold if we seek for solutions u_t of (1) such that $\int u_t d\mathcal{L}^3 \equiv 0$ for any $t \geq 0$, and we can certainly do so, by the discussion preceding the latter formula.

Lemma 4.9 (One step estimates) *Let $d = 3$. With the same notation and assumptions of Proposition 3.1 it holds*

$$\frac{1}{2}\|\nabla u^\tau\|_{L^2}^2 - \frac{1}{2}\|\nabla u\|_{L^2}^2 + \frac{\tau}{2}\|\Delta u^\tau\|_{L^2}^2 \leq C\tau\|\nabla u\|_{L^2}^3 \|\Delta u\|_{L^2} \tag{24}$$

and

$$\frac{1}{2}\|\nabla u^\tau\|_{L^2}^2 - \frac{1}{2}\|\nabla u\|_{L^2}^2 + \frac{\tau}{2}\|\Delta u^\tau\|_{L^2}^2 \leq C\tau\|\nabla u\|_{L^2}^2 \|\Delta u\|_{L^2}^2 . \tag{25}$$

Proof. Multiplying (5) by $-\Delta u^\tau$ and integrating we get

$$-\int u^\tau \cdot \Delta u^\tau + u \cdot \Delta u^\tau + (u \circ X^u_{-\tau} - u) \cdot \Delta u^\tau d\mathcal{L}^3 = -\tau\|\Delta u^\tau\|_{L^2}^2,$$

hence after integration by parts and by Young inequality we have

$$\begin{aligned}\frac{1}{2}\|\nabla u^\tau\|_{L^2}^2 - \frac{1}{2}\|\nabla u\|_{L^2}^2 + \tau\|\Delta u^\tau\|_{L^2}^2 &\leq \|\Delta u^\tau\|_{L^2}\|u \circ X^u_{-\tau} - u\|_{L^2} \\ &\leq \frac{\tau}{2}\|\Delta u^\tau\|_{L^2}^2 + \frac{1}{2\tau}\|u \circ X^u_{-\tau} - u\|_{L^2}^2.\end{aligned} \tag{26}$$

With the same calculations we did for (11) we get

$$\|u \circ X^u_{-\tau} - u\|_{L^2}^2 \leq \tau^2 \int |\nabla u|^2 |u|^2 d\mathcal{L}^3. \tag{27}$$

To get (24) we bound the right hand side as

$$\int |\nabla u|^2|u|^2 d\mathcal{L}^3 \leq \|\nabla u\|_{L^6}\|\nabla u\|_{L^2}\|u\|_{L^6}^2 \leq C\|\Delta u\|_{L^2}\|\nabla u\|_{L^2}^3,$$

and plugging this into (26) we get (24)

To get (25), we bound the right hand side of (27) as

$$\int |\nabla u|^2|u|^2 d\mathcal{L}^3 \leq \|\nabla u\|_{L^6}^2\|u\|_{L^6}\|u\|_{L^2} \leq C\|\Delta u\|_{L^2}^2\|\nabla u\|_{L^2}^2,$$

which gives the statement when inserted into (26). □

Proposition 4.10 (Strong Solutions) *With the same notation and assumptions of Propositon 3.1 and Definition 3.2, there exists a constant $\mathcal{C} > 0$ such that for $T := \frac{\mathcal{C}}{\|\nabla \overline{u}\|^4}$ the discrete solutions converge to a strong solution of the Navier-Stokes equations in $[0, T)$.*

Also, there is another constant $\mathcal{C}'$ such that if $\|\nabla \overline{u}\|_{L^2} \leq \mathcal{C}'$, then the scheme converges to the smooth solution of the Navier-Stokes equations in the whole $[0, \infty)$.

Proof. It is well known (see e.g. [4], Theoreom 5.2 for the case of bounded, smooth open subsets of $\mathbb{R}^3$) that if a weak solution (u_t) of Navier-Stokes belongs to

$$L^\infty([0, T], H^1) \cap L^2([0, T], H^2),$$

then it is smooth on $[0,T]$. Hence to prove the statement it is sufficient to check that for any $\varepsilon > 0$, the discrete solutions are uniformly bounded both in $L^\infty([0,T],H^1)$ and in $L^2([0,T],H^2)$.

Let us fix τ for the moment, and consider (24) and (25) for $u = u^\tau_{i\tau}$, for some nonnegative integer i. Adding up the inequalities (24) from $i = 0 \dots n-1$ we get

$$\begin{aligned}
\|\nabla u^\tau_{n\tau}\| + \tau \sum_{i=1}^{n} \|\Delta u^\tau_{i\tau}\|^2_{L^2} &\le \|\nabla \overline{u}\|^2_{L^2} + 2C\tau \sum_{i=0}^{n-1} \|\nabla u^\tau_{i\tau}\|^3_{L^2} \|\Delta u^\tau_{i\tau}\|_{L^2} \\
&= \|\nabla \overline{u}\|^2_{L^2} + 2C \int_0^{n\tau} \|\nabla u^\tau_t\|^3_{L^2} \|\Delta u^\tau_t\|_{L^2}\, dt \\
&\le \|\nabla \overline{u}\|^2_{L^2} + 2C \sup_{t<n\tau} \|\nabla u^\tau_t\|^3_{L^2} \int_0^{n\tau} \|\Delta u^\tau_t\|_{L^2}\, dt \\
&\le \|\nabla \overline{u}\|^2_{L^2} + 2C \sup_{t<n\tau} \|\nabla u^\tau_t\|^3_{L^2} \sqrt{n\tau} \left(\int_0^{n\tau} \|\Delta u^\tau_t\|^2_{L^2}\, dt \right)^{\frac{1}{2}},
\end{aligned}$$

and therefore

$$\begin{aligned}
&\left(\sup_{t<(n+1)\tau} \|\nabla u^\tau_t\|^2 \right) + \int_0^{(n+1)\tau} \|\Delta u^\tau_t\|^2_{L^2}\, dt \\
&\qquad \le \|\nabla \overline{u}\|^2_{L^2} + \tau \|\Delta \overline{u}\|^2_{L^2} + 2C \sup_{t<n\tau} \|\nabla u^\tau_t\|^3_{L^2} \sqrt{n\tau} \left(\int_0^{n\tau} \|\Delta u^\tau_t\|^2_{L^2}\, dt \right)^{\frac{1}{2}}.
\end{aligned} \tag{28}$$

We can proceed in a similar manner for (25), obtaining

$$\begin{aligned}
&\left(\sup_{t<(n+1)\tau} \|\nabla u^\tau_t\|^2 \right) + \int_0^{(n+1)\tau} \|\Delta u^\tau_t\|^2_{L^2}\, dt \\
&\qquad \le \|\nabla \overline{u}\|^2_{L^2} + \tau \|\Delta \overline{u}\|^2_{L^2} + 2C \sup_{t<n\tau} \|\nabla u^\tau_t\|^2_{L^2} \int_0^{n\tau} \|\Delta u^\tau_t\|^2_{L^2}\, dt.
\end{aligned} \tag{29}$$

Let us fix $T > 0$ and suppose $0 \le n \le \left[\frac{T}{\tau}\right]$. We define

$$\delta_\tau(\overline{u}) := \|\nabla \overline{u}\|^2_{L^2} + \tau \|\Delta \overline{u}\|^2_{L^2}, \qquad a_n := \sup_{t<n\tau} \|\nabla u^\tau_t\|^2 + \int_0^{n\tau} \|\Delta u^\tau_t\|^2_{L^2}\, dt,$$

and notice that Young inequality applied to the last term on the right of both (28) and (29), yields

$$a_{n+1} \le \delta_\tau(\overline{u}) + C\sqrt{T} a_n^2, \qquad a_{n+1} \le \delta_\tau(\overline{u}) + C a_n^2,$$

respectively, and thus

$$a_{n+1} \le \delta_\tau(\overline{u}) + C \min\{1, \sqrt{T}\} a_n^2.$$

Now, suppose that the equation $\lambda = \delta_\tau(\overline{u}) + C\min\{1,\sqrt{T}\}\lambda^2$ has a positive solution, i.e.,

$$\min\{1, \sqrt{T}\}\delta_\tau(\overline{u}) \le \frac{1}{4C}, \tag{30}$$

and let $\overline{\lambda}$ be the smallest one:

$$\overline{\lambda} = \frac{2\delta_\tau(\overline{u})}{1+\sqrt{1-4C\min\{1,\sqrt{T}\}\delta_\tau(\overline{u})}} \leq 2\delta_\tau(\overline{u}).$$

It is easily proved by induction that $a_n \leq \overline{\lambda}$ for any n, since $a_0 \leq \delta_\tau(\overline{u}) \leq \overline{\lambda}$. Therefore the family $\{u_t^\tau\}$ is bounded both in $L^\infty([0,T-\varepsilon],H^1)$ and in $L^2([0,T-\varepsilon],H^2)$ by $2\delta_\tau(\overline{u})$. For $\tau \leq \|\nabla\overline{u}\|_{L^2}^2 / \|\Delta\overline{u}\|_{L^2}^2$ we get $\delta_\tau(\overline{u}) \leq 2\|\nabla\overline{u}\|_{L^2}^2$, and condition (30) reads

$$\min\{1,\sqrt{T}\}\,\|\nabla\overline{u}\|_{L^2}^2 \leq \frac{1}{8C},$$

which gives the claims. □

5 The case of bounded and smooth open sets

All the conclusions of this paper are still true if we work on a bounded smooth open subset Ω of $\mathbb{R}^3$ (weak solutions are produced in any dimension) provided we add the Dirichlet boundary condition $u_t \equiv 0$ on $\partial\Omega$. In this case the minimization problem of proposition 3.1 has to be solved in the set $\mathcal{H}^1(\Omega) := \{v \in H_0^1(\Omega) : \nabla\cdot v = 0\}$, and all the results follow along the same line. The only difference from the periodic case is in the proof of the $L^{3/2}_{loc}([0,+\infty),L^{3/2}(\Omega))$ bound on the pressure p^τ, needed in passing to the limit when one wants to prove suitability of the solution. To this end one has to proceed in a different way, using the mixed $L^{p,s}$ estimates of the evolutionary Stokes problem. The space $L^{s,r}([0,\infty)\times\Omega)) := L^s([0,\infty);L^r(\Omega))$ is equipped with the norm

$$\|v\|_{L^{s,r}} := \left(\int_0^\infty \|v_t\|_{L^r}^s\,dt\right)^{1/s}$$

Lemma 5.1 *Let u_t^τ be a discrete solution of the Navier–Stokes equation, as in definition 3.2, for some smooth initial data $\overline{u}$ with $\nabla\cdot\overline{u}=0$ and $\overline{u}|_{\partial\Omega}=0$. Then*

$$\left\|u^\tau - u^\tau\circ X_{-\tau}^{u^\tau}\right\|_{L^{s,r}} \leq \tau C(\overline{u}) \tag{31}$$

for $3/2 \geq r \geq 1$ and $s \geq 1$ such that

$$4 = \frac{3}{r}+\frac{2}{s} \tag{32}$$

Proof. Denoting by r' the conjugate exponent of r, we follow the proof of (11) with exponent r instead of 2, obtaining, for any smooth u

$$\begin{aligned}\int_\Omega |u-u\circ X_{-\tau}^u|^r d\mathcal{L}^3 &= \int_\Omega\left|\int_0^\tau \frac{d}{dt}u\circ X_{-t}^u dt\right|^r d\mathcal{L}^3\\ &\leq \tau^{r/r'}\int_0^\tau\int_\Omega |\nabla u\circ X_{-t}^u\cdot u\circ X_{-t}^u|^r d\mathcal{L}^3 dt\\ &\leq \tau^r\int_\Omega |\nabla u\cdot u|^r d\mathcal{L}^3.\end{aligned}$$

By Holder inequality with exponent $2/r$ on ∇u, and by the interpolation inequality

$$\|u\|_{L^q(\Omega)} \le C \|\nabla u\|^{1-\alpha}_{L^2(\Omega)} \|u\|^{\alpha}_{L^2(\Omega)}, \qquad 2 \le q \le 6, \quad \alpha = \frac{3}{q} - \frac{1}{2},$$

for $q = 2r/(2-r)$, one gets

$$\|\nabla u \cdot u\|_{L^r(\Omega)} \le \|u\|_{L^{\frac{2r}{2-r}}(\Omega)} \|\nabla u\|_{L^2(\Omega)} \le C\|u\|^{\frac{3}{r}-2}_{L^2(\Omega)} \|\nabla u\|^{4-\frac{3}{r}}_{L^2(\Omega)}.$$

We apply this estimate for $u = u_t^\tau$, integrate in time over $[0,\infty)$, use (16) and (32) to get

$$\begin{aligned}\left\|u^\tau - u^\tau \circ X^{u^\tau}_{-\tau}\right\|^s_{L^{s,r}} &\le C\tau \int_0^\infty \|u_t^\tau\|^{2(s-1)}_{L^2(\Omega)} \|\nabla u_t^\tau\|^2_{L^2(\Omega)} dt \\ &\le C\tau \|\overline{u}\|^{2(s-1)}_{L^2(\Omega)} \left(\int_\tau^\infty \|\nabla u_t^\tau\|^2_{L^2(\Omega)} dt + \tau \|\nabla \overline{u}\|^2_{L^2(\Omega)} \right) \\ &\le C\tau \big(\|\overline{u}\|^{2s}_{L^2(\Omega)} + \tau \|\overline{u}\|^{2(s-1)}_{L^2(\Omega)} \|\nabla \overline{u}\|^2_{L^2(\Omega)} \big).\end{aligned}$$

Finally, the condition $2 \le 2r/(2-r) \le 6$ forces $1 \le r \le 3/2$. □

To obtain a bound for the pressure p^τ one can then proceed as in [8]: one lets

$$g_n^\tau = P\left(\frac{u_n^\tau - u_n^\tau \circ X^{u_n^\tau}_{-\tau}}{\tau} \right) \Big|_{t=n\tau}$$

where P is the projection on the closure in $L^r(\Omega)$ of $\{u \in C_c^\infty : \nabla \cdot u = 0\}$ (the dependance on r will be implicit) and $u_n^\tau = u^\tau(n\tau)$. Then u_n^τ solves the difference equation in $PL^r(\Omega)$

$$\begin{cases} u_{n+1}^\tau - u_n^\tau = \tau S u_{n+1}^\tau + \tau g_n^\tau, \\ u_0^\tau = \overline{u}, \end{cases} \tag{33}$$

where $S = P\Delta$ is the Stokes operator in $PL^r(\Omega)$. We recall here that, (see [9], [5]), S generates an analytic semigroup with optimal regularity in $PL^r(\Omega)$. Setting $\Sigma_\theta := \{\lambda \in \mathbb{C} \setminus \{0\} : |\mathrm{Arg}(\lambda)| \le \theta\}$, this is equivalent, by e.g. [6, Theorem 1.11] to

$$\{\lambda R(\lambda, S) : \lambda \in \Sigma_\theta\} \quad \mathcal{R}\text{-bounded for some } \theta \ge \pi/2,$$

(and thus bounded). We will henceforth fix such a $\theta \ge \pi/2$ and let $\mathcal{R}(S)$ be the corresponding $\mathcal{R}$-bound.

We rewrite the difference equation in the standard form

$$\begin{cases} u_{n+1}^\tau = T_\tau u_n^\tau + f_n^\tau, \\ u_0^\tau = \overline{u}, \end{cases} \tag{34}$$

where

$$T_\tau = (1-\tau S)^{-1}, \qquad f_n^\tau = \tau(1-\tau S)^{-1} g_n^\tau.$$

Now, T_τ is a powerbounded operator in $PL^r(\Omega)$, since

$$\sigma(T_\tau) = \left\{ \frac{1}{1+\tau\lambda_n} : \lambda_n \in \sigma(S) \right\} \subset (-1,0),$$

and it is also analytic, i.e.

$$\{n(T_\tau - 1)T_\tau^n\} \quad \text{is bounded.} \tag{35}$$

To prove analyticity is suffice to observe that

$$T_\tau - 1 = \tau S(1-\tau S)^{-1} := \tau S_\tau \tag{36}$$

is (a multiple of) the Yoshida approximation S_τ of S in $PL^r(\Omega)$, and thus it generates an analytic semigroup; this, together with $\sigma(T_\tau) \subset \{z \in \mathbb{C} : |z| \le 1\}$ is equivalent to (35) by [1, Theorem 2.3]. We can reduce system (34) to zero initial data by subtracting

$$v_n^\tau = T_\tau^n \overline{u},$$

noticing that

$$\left\| v_{n+1}^\tau - v_n^\tau \right\|_{L^r(\Omega)} \le \frac{1}{n} \left\| n(T_\tau - 1)T_\tau^n \right\| \left\| \overline{u} \right\|_{L^r(\Omega)}. \tag{37}$$

We therefore seek for (uniform in τ) optimal $L^s\big(\mathbb{N}, PL^r(\Omega)\big)$-regularity of the discrete time parabolic equation for $w_n := u_n - v_n$

$$\begin{cases} w_{n+1}^\tau = T_\tau w_n^\tau + f_n^\tau, \\ w_0^\tau = 0, \end{cases}$$

in the sense that

$$\sum_{n=1}^{+\infty} \left\| w_{n+1}^\tau - w_n^\tau \right\|_{L^r(\Omega)}^s \le C_{s,r} \sum_{n=1}^{\infty} \left\| f_n^\tau \right\|_{L^r(\Omega)}^s. \tag{38}$$

By [1, Theorem 1.1], the discrete optimal $L^s\big(\mathbb{N}, PL^r(\Omega)\big)$-regularity is equivalent to the optimal $L^s\big([0,+\infty), L^r(\Omega)\big)$ regularity of the analytic semigroup generated by the operator S_τ given in (36). Moreover, the constant in the optimal regularity estimate depends only on the $\mathcal{R}$-bound of

$$\{itR(it, S_\tau) : t \in \mathbb{R} \setminus \{0\}\}, \tag{39}$$

and thus it suffice to prove an $\mathcal{R}$-bound for this set of operators which is indipendent of τ. To this end, a short computation shows that

$$itR(it, S_\tau) = \frac{1}{it}\lambda_\tau^2(t) R(\lambda_\tau(t), S) + \tau\lambda_\tau(t)\mathrm{Id}$$

where

$$\lambda_\tau(t) = \frac{it}{1+\tau it}.$$

Notice that $\lambda(\mathbb{R}\setminus\{0\})$ is the circle through the origin, with center on the real axis and radius $1/\tau$, minus the origin; therefore it is contained in $\Sigma_{\pi/2}$. By the subadditivity and submultiplicativity of the $\mathcal{R}$-bounds, we get that the set in (39) is $\mathcal{R}$-bounded by $\mathcal{R}(S)+1$, since $\{\lambda_\tau(t)R(\lambda_\tau(t),S) : t \in \mathbb{R}\setminus\{0\}\}$ is $\mathcal{R}$-bounded by $\mathcal{R}(S)$ and

$$|\lambda_\tau(t)/t| \leq 1, \quad \tau|\lambda_\tau(t)| \leq 1, \quad \forall t \in \mathbb{R}.$$

This shows that the constant in (38) (and thus, *a fortiori*, the one in (35)) is bounded indipendently of τ. We are now ready to prove an estimate of the pressure, which allows to prove the suitability of the limit solutions, in the case of Ω bounded and smooth.

Theorem 5.2 *Let $\Omega \subset \mathbb{R}^3$ be a smooth and bounded open set. Let u_t^τ, p_t^τ be a discrete solution of the Navier–Stokes equation, as in definition 3.2, for some smooth initial data $\overline{u}$ with $\nabla\cdot\overline{u} = 0$ and $\overline{u}|_{\partial\Omega} = 0$. For any $\varepsilon > 0$, there exists a constant $C = C(\varepsilon,\overline{u})$ such that for any sufficiently small $\tau > 0$*

$$\|p^\tau\|_{L^{5/3}(\Omega\times[\varepsilon,\infty))} \leq C(\varepsilon,\overline{u}).$$

Proof. Let r and s satisfy (32), with $s > 1$, and let $m = [\varepsilon/\tau]$. With the same notation of the preceding discussion we first of all prove that for some constant C independent of τ, it holds

$$\sum_{n=m}^{+\infty} \left\| \frac{u_{n+1}^\tau - u_n^\tau}{\tau} \right\|_{L^r(\Omega)}^s \leq C \sum_{n=0}^{+\infty} \|g_n^\tau\|_{L^r(\Omega)}^s + \frac{C}{\varepsilon^{s-1}\tau} \|\overline{u}\|_{L^r(\Omega)}^s . \tag{40}$$

By (37), for any $n \geq m$ it holds

$$\left\| \frac{v_{n+1}^\tau - v_n^\tau}{\tau} \right\|_{L^r(\Omega)} \leq C\frac{1}{n\tau} \|\overline{u}\|_{L^r(\Omega)} ,$$

and thus

$$\sum_{n=m}^{+\infty} \left\| \frac{v_{n+1}^\tau - v_n^\tau}{\tau} \right\|_{L^r(\Omega)}^s \leq C \sum_{n=m}^{+\infty} \frac{1}{n^s\tau^s} \|\overline{u}\|_{L^r(\Omega)}^s \leq \frac{C}{\tau}\frac{1}{(\tau m)^{s-1}} \|\overline{u}\|_{L^r(\Omega)}^s \leq \frac{C}{\varepsilon^{s-1}\tau} \|\overline{u}\|_{L^r(\Omega)}^s \tag{41}$$

Moreover, by the $L^r(\Omega)$ resolvent estimate for the Stokes operator, it holds $\left\|(1-\tau S)^{-1}\right\| \leq C$, and thus

$$\|f_n^\tau\|_{L^r(\Omega)} \leq C\tau \|g_n^\tau\|_{L^r(\Omega)} .$$

We split $u_n^\tau = w_n^\tau + v_n^\tau$ and use this estimate and (38) for w_n and (41) for v_n, obtaining (40).

Notice that the difference equation (33) readily gives an analogous estimate for $\left\|Su_{n+1}^\tau\right\|_{L^r(\Omega)}$; finally we can take the $L^r(\Omega)$ norm in the original difference equation

$$\nabla p_n^\tau = \Delta u_n^\tau - \frac{u_{n+1}^\tau - u_n^\tau}{\tau} + \frac{u_n^\tau - u_n^\tau \circ X_{-\tau}^{u_n^\tau}}{\tau},$$

and by the coercive estimate $\|u\|_{W^{2,r}(\Omega)} \leq C\,\|Su\|_{L^r(\Omega)}$, obtain the full regularity estimate

$$\sum_{n=m}^{+\infty} \left\| \frac{u_{n+1}^\tau - u_n^\tau}{\tau} \right\|_{L^r(\Omega)}^s + \left\|\Delta u_{n+1}^\tau\right\|_{L^r(\Omega)}^s + \left\|\nabla p_{n+1}^\tau\right\|_{L^r(\Omega)}^s \leq C \sum_{n=0}^{+\infty} \left\| \frac{u_n^\tau - u_n^\tau \circ X_{-\tau}^{u_n^\tau}}{\tau} \right\|_{L^r(\Omega)}^s + \frac{C_\varepsilon}{\tau} \|\overline{u}\|_{L^r(\Omega)}^s ,$$

where $C_\varepsilon = C/\varepsilon^{s-1}$. We now choose $s = 5/3$ and $r = 15/14$, and use lemma 5.1 in this estimate to get

$$\|\nabla p^\tau\|_{L^{5/3}([\varepsilon,\infty);L^{15/14}(\Omega))}^{5/3} \leq \tau \sum_{n=m}^{+\infty} \left\|\nabla p_{n+1}^\tau\right\|_{L^{15/14}(\Omega)}^{5/3} \leq C(\varepsilon, \overline{u}).$$

Since $\int_\Omega p^\tau d\mathcal{L}^3 = 0$, the Sobolev-Poincaré inequality $\|p\|_{L^{5/3}(\Omega)} \leq C\,\|\nabla p\|_{L^{15/14}(\Omega)}$ applied to the latter estimate gives the claim. □

Remark 5.3 (On the smoothness of the initial datum) We point out that the restriction to smooth initial data has been made to simplify the discussion about the existence of flow maps. Actually, this is unnecessary: as showed by DiPerna-Lions in [3], as soon as a vector field u has Sobolev regularity and is divergence free, one has existence and uniqueness of a one parameter group of maps X_t^u such that $(X_t^u)_\# \mathcal{L}^d = \mathcal{L}^d$, $X_0 = Id$ and $\partial_t X_t^u = u \circ X_t^u$ in the sense of distributions. Thus one can use these maps as flow maps and directly get a weak/suitable solution for any initial datum in H^1. Notice, however, that in order to get the discrete estimates needed for the existence of strong solutions, it is required that the initial datum is in H^2, although the H^2 norm actually does not appear in the final statement. ■

References

[1] S. Blunck *Maximal regularity of discrete and continuous time evolution equations.* Studia Math. 146 (2001), 157–176.

[2] L. Caffarelli, R. Kohn, and L. Nirenberg, *Partial regularity of suitable weak solutions of the Navier–Stokes equations.* Comm. Pure & Appl. Math. 35 (1982), 771–831.

[3] R. J. DiPerna, and P.-L. Lions, *Ordinary differential equations, transport theory and Sobolev spaces.* Invent. Math. 98 (1989), 511–547.

[4] G. P. Galdi, *An Introduction to the Navier–Stokes Initial Boundary Value Problem.*

[5] Y. Giga, *The Stokes operator in L_r spaces.* Proc. Japan Acad. 57 (1981), 85–89.

[6] P.C. Kunstmann, and L. Weis, *Maximal L_p regularity fo Parabolic equations, Fourier Multiplier Theorems and H^∞-functional calculus.* in “Functional analytic methods for evolution equations”, M. Iannelli, R. Nagel, S. Piazzera eds, Lecture Notes in Mathematics 1855, Springer-Verlag, Berlin 2004.

[7] O. Pironneau, *On the transport-diffusion algorithm and its applications to the Navier–Stokes equations.* Numer. Math. 38 (1981/82), no. 3, 309–332

[8] H. Sohr, and W. von Wahl, *On the regularity of the pressure of weak solutions of Navier–Stokes equations.* Arch. Math. 46 (1986), 428–439.

[9] V. A. Solonnikov, *Estimates for solutions of nonstationary Navier–Stokes equations.* J. Soviet Math. 8 (1977), 467–529.

[10] R. Temam, *Navier–Stokes equations. Theory and numerical analysis.* North Holland publishing company, 1977