Regularity of Minimizers of $W^{1,p}$-Quasiconvex Variational Integrals with (p,q)-Growth

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Abstract We consider autonomous integrals

$$F[u] := \int_{\Omega} f(Du)\,dx \qquad \text{for } u : \mathbb{R}^n \supset \Omega \to \mathbb{R}^N$$

in the multidimensional calculus of variations, where the integrand f is a strictly $W^{1,p}$-quasiconvex C^2-function satisfying the (p,q)-growth conditions

$$\gamma|A|^p \le f(A) \le \Gamma(1+|A|^q) \qquad \text{for every } A \in \mathbb{R}^{nN}$$

with exponents $1 < p \le q < \infty$.

Under these assumptions we establish an existence result for minimizers of F in $W^{1,p}(\Omega;\mathbb{R}^N)$ provided $q < \frac{np}{n-1}$. We prove a corresponding partial $C^{1,\alpha}$-regularity theorem for $q < p + \frac{\min\{2,p\}}{2n}$. This is the first regularity result for autonomous quasiconvex integrals with (p,q)-growth.

Keywords Calculus of Variations · Minimizer · Partial Regularity · Quasiconvexity · Nonstandard Growth

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1 Introduction

Throughout this paper let $n, N \in \mathbb{N}$ with $n \ge 2$, let Ω denote a bounded open set in $\mathbb{R}^n$ and $\mathbb{O}_\Omega$ the system of all open subsets of Ω. We consider variational integrals

$$F[u;O] := \int_O f(Du)\,dx \in [0,\infty] \qquad \text{for } O \in \mathbb{O}_\Omega \text{ and } u \in W^{1,1}_{\text{loc}}(O;\mathbb{R}^N),$$

where the integrand $f : \mathbb{R}^{nN} \to [0,\infty[$ is a continuous function.

We are interested in the minimization of F with respect to some fixed boundary values. More precisely we adopt the following notion of a minimizer:

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Definition 1.1 (Minimizer). *Let $p \geq 1$. $u \in W^{1,p}(\Omega;\mathbb{R}^N)$ is called a $W^{1,p}$-minimizer of F on Ω iff one has $F[u;\Omega] < \infty$ and*

$$F[u;\Omega] \leq F[u+\varphi;\Omega] \qquad \text{for every } \varphi \in W_0^{1,p}(\Omega;\mathbb{R}^N).$$

The direct method in the calculus of variations allows to prove the existence of minimizers of F provided f is convex and satisfies the standard growth assumptions

$$\gamma|A|^p \leq f(A) \leq \Gamma(1+|A|^p) \qquad \text{for every } A \in \mathbb{R}^{nN} \tag{1.1}$$

with $p > 1$. Furthermore if $f \in C^2_{\text{loc}}(\mathbb{R}^{nN})$ satisfies an additional strict convexity assumption it is well known that these minimizers are locally $C^{1,\alpha}$-regular outside a closed set of Hausdorff dimension $< n-2$.

In his famous paper [Mo] C. B. MORREY introduced the following notion:

Definition 1.2 (Quasiconvexity). *f is called quasiconvex iff*

$$\fint_{B_1} f(A+D\varphi) \geq f(A)$$

holds for every $A \in \mathbb{R}^{nN}$ and every smooth $\varphi : B_1 \to \mathbb{R}^N$ with compact support in the open unit ball B_1 in $\mathbb{R}^n$.

Additionally, MORREY proved that quasiconvexity of f still allows to obtain the existence of minimizers by means of the direct method. L. C. EVANS showed in [E] that $C^{1,\alpha}$-regularity of minimizers of F still remains true outside a closed set of measure zero if f is strictly quasiconvex with (1.1) (cf. also [AF2], [CFM]).

Various generalizations of the growth conditions (1.1) have been attracting more and more attention during the last decades. We shall focus on the so-called (p,q)-growth conditions

$$\gamma|A|^p \leq f(A) \leq \Gamma(1+|A|^q) \qquad \text{for every } A \in \mathbb{R}^{nN} \tag{1.2}$$

with two growth exponents $1 < p \leq q < \infty$. These growth conditions are quite flexible, but technically difficult to handle since, estimating f from above and below, one obtains completely different terms.

Existence of $W^{1,p}$-minimizers can still be proven as before, if f is strictly convex with (1.2). However, a regularity theory working under these assumptions is not immediate. Various papers have been treating this subject. Introducing some additional notation let us mention two results:

Definition 1.3 (Regular and Singular Set). *For $u \in L^1_{\text{loc}}(\Omega;\mathbb{R}^N)$ we call*

$$\text{Reg}(u) := \left\{ x \in \Omega \,:\, u|_{B_\rho(x)} \in C^1(B_\rho(x);\mathbb{R}^N) \text{ for some } \rho > 0 \right\}$$

the regular set of u and

$$\text{Sing}(u) := \Omega \setminus \text{Reg}(u)$$

the singular set of u.

Theorem 1.4 (Partial Regularity, [PS]). *Assume that* $f \in C^2_{\mathrm{loc}}(\mathbb{R}^{nN})$ *satisfies* (1.2) *together with*

$$\lambda(1+|A|^2)^{\frac{p-2}{2}}|B|^2 \le D^2 f(A)(B,B) \qquad \text{for all } A,B \in \mathbb{R}^{nN},$$

where $\lambda > 0$ *and*

$$2 \le p \le q < \min\left\{p+1, \frac{np}{n-1}\right\}.$$

Then, for each $W^{1,p}$*-minimizer* u *of* F *on* Ω *one has* $\mathrm{Reg}(u) \in \mathbb{O}_\Omega$ *and* $u \in C^{1,\alpha}_{\mathrm{loc}}(\mathrm{Reg}(u);\mathbb{R}^N)$ *for every* $\alpha \in]0,1[$. *In addition, the Lebesgue measure* $|\mathrm{Sing}(u)|$ *of the singular set vanishes.*

Theorem 1.5 (Partial Regularity, [BF]). *Assume that* $f \in C^2_{\mathrm{loc}}(\mathbb{R}^{nN})$ *satisfies* (1.2) *together with*

$$\lambda(1+|A|^2)^{\frac{p-2}{2}}|B|^2 \le D^2 f(A)(B,B) \le \Lambda(1+|A|^2)^{\frac{q-2}{2}}|B|^2 \qquad \text{for all } A,B \in \mathbb{R}^{nN},$$

where $\lambda,\Lambda > 0$ *and*

$$1 < p \le q < \frac{n+2}{n}p.$$

Then, for each $W^{1,p}$*-minimizer* u *of* F *on* Ω *one has* $\mathrm{Reg}(u) \in \mathbb{O}_\Omega$, $u \in C^{1,\alpha}_{\mathrm{loc}}(\mathrm{Reg}(u);\mathbb{R}^N)$ *for every* $\alpha \in]0,1[$, *and* $|\mathrm{Sing}(u)| = 0$.

Once these results are established standard methods of regularity theory can be applied to show:

- In addition to the hypotheses of Theorem 1.4 or 1.5 let $f \in C^\infty_{\mathrm{loc}}(\mathbb{R}^{nN})$. Then, one has $u \in C^\infty_{\mathrm{loc}}(\mathrm{Reg}(u);\mathbb{R}^N)$.
- In the situation of Theorem 1.5 the Hausdorff dimension of the singular set does not exceed $n-2$.

We shall now consider a more general situation: We assume that f is quasiconvex with (p,q)-growth in the sense of (1.2). In contrast to the convex situation the well-known semicontinuity theorems (cf. [Mo], [AF1], [Ma1]) guarantee weak lower semicontinuity of $F[-;\Omega]$ on $W^{1,q}(\Omega;\mathbb{R}^N)$ but, in general, not on $W^{1,p}(\Omega;\mathbb{R}^N)$. Unfortunately, this semicontinuity property is not adequate to prove the existence of $W^{1,p}$-minimizers of F.

However, some progress has been made concerning this problem: I. FONSECA and J. MALÝ in [FM, Theorem 4.1] and J. KRISTENSEN in [K, Corollary 3.3] showed that $F[-;\Omega]$ is lower semicontinuous with respect to weak $W^{1,p}$-convergence of $W^{1,q}$-functions under the condition $q < \frac{np}{n-1}$. This leads to an existence result, which we will state later, but no corresponding regularity result is known.

The aim of this paper is to show that strengthening the quasiconvexity assumption in the following manner, due to J. M. BALL and F. MURAT, one can obtain an improved existence and regularity theory for minimizers of quasiconvex variational integrals with (p,q)-growth:

Definition 1.6 ($W^{1,p}$-Quasiconvexity, [BM]). *Let* $p \ge 1$. f *is called* $W^{1,p}$*-quasiconvex iff*

$$\fint_{B_1} f(A+D\varphi) \ge f(A)$$

holds for every $A \in \mathbb{R}^{nN}$ *and every* $\varphi \in W^{1,p}_0(B_1;\mathbb{R}^N)$.

Our first observation is that assuming $W^{1,p}$-quasiconvexity and $q < \frac{np}{n-1}$ methods from [FM] show weak lower semicontinuity of $F[-;\Omega]$ on $W^{1,p}(\Omega;\mathbb{R}^N)$. As an immediate consequence we obtain an existence theorem for $W^{1,p}$-minimizers of F.

Furthermore, we will show that strict $W^{1,p}$-quasiconvexity permits the proof of $C^{1,\alpha}$-regularity of these minimizers outside a closed set of measure zero, provided

$$q < p + \frac{\min\{2,p\}}{2n}. \tag{1.3}$$

This partial regularity theorem has been obtained in the author's thesis [S1] and relies on methods from [BM] and [FM]. It is the main result of this paper.

Note that another regularity result for quasiconvex integrals under nonstandard growth conditions, namely $p(x)$-growth conditions, has been established in [AM1]. However, this result concerns integrands with an additional x-dependence, while $p(x)$-growth reduces to (1.1) in our autonomous setting. Therefore, a direct comparison to our result is not possible.

Now, let us explain the ideas of the regularity proof:

Following classical methods of regularity theory our proof relies heavily on a so-called Caccioppoli inequality, an integral inequality estimating the first derivatives of a minimizer u in terms of u itself. In the case $p = q$ of standard growth conditions such an inequality has been proven by L. C. EVANS in [E]. His proof uses certain test functions constructed from u and a cut-off function. For $p < q$ these test functions are in $W^{1,p}$ but, in general, not in $W^{1,q}$ and we are not allowed to use them in the definition of the standard quasiconvexity. This is why we need the stronger $W^{1,p}$-quasiconvexity assumption. Moreover, EVANS' proof relies heavily on the fact that certain terms in the estimates from below and above coincide. This poses some serious difficulties. To overcome them we involve certain smoothing operators from [FM] in the construction of the test functions which allow to obtain an L^q-L^p-estimate. Finally, using these modified test functions some terms can be treated in a similar way as before, while some others remain as a perturbation of the right-hand side of the Caccioppoli inequality. However, we will show that these additional terms do not seriously affect the further regularity proof.

We stress that the proof of the Caccioppoli inequality is the only point where we need the condition (1.3). Actually, all other parts of the proof work assuming just $q \le p+1$.

Proceeding with the proof we will use the $\mathscr{A}$-harmonic approximation method developed in [DS]. Nevertheless we believe that just as well a blow-up argument could be used instead, once the Caccioppoli inequality is established. In any case, it is essential to acquire growth estimates for the so-called excess, a certain integral quantity. An excess Φ_q linked to the exponent q has been used in the proof of Theorem 1.5 and seems to yield the best results. However, this relies on higher integrability of the minimizer which is available in the convex but not in the quasiconvex case. Therefore, since we do not know if the minimizer is in $W^{1,q}$, we have to use the excess Φ_p from the proof of Theorem 1.4 linked to the exponent p. Now the rest of the proof follows essentially well-known standard arguments. The perturbation terms from the Caccioppoli inequality can be treated using common smallness assumptions for the excess Φ_p. Using CAMPANATO's integral characterization of Hölder continuity we finally complete the proof.

Let us mention that our proof possesses some analogies with the proof of Theorem 1.4, namely the use of the excess Φ_p and the smoothing operators mentioned above. Therefore it is not surprising that our methods allow to retrieve a somewhat simplified proof of Theorem 1.4 while we cannot reach the condition $q < \frac{n+2}{n}p$ from Theorem 1.5. Anyway, the use of our Caccioppoli type inequality together with a (p,q)-growth condition seems to be new, even in the convex case. The same remark applies concerning the use of the $W^{1,p}$-quasiconvexity condition in regularity theory.

Finally, let us mention that there is another way of treating the existence and regularity of minimizers in the situation above: Retaining the original quasiconvexity condition one can introduce certain relaxed functionals, as already done in [FM], and relaxed minimizers. Some results in this direction concerning semicontinuity and existence are contained in [FM]. We plan to present an extension of these results and a corresponding regularity theory in the forthcoming paper [S2].

The plan of this paper is now as follows:

In section 2 we give a more precise statement of our assumptions while section 3 provides simple examples of integrands f satisfying these conditions. The semicontinuity and existence results are contained in section 4. Our main theorem concerning partial regularity is stated in section 5 while its proof is carried out in sections 6 and 7.

2 Assumptions

We recall that $f:\mathbb{R}^{nN}\to[0,\infty[$ is always assumed to be continuous and the corresponding functional F is given by

$$F[u;O]:=\int_O f(Du)\,dx\in[0,\infty]\qquad\text{for } O\in\mathbb{O}_\Omega \text{ and } u\in W^{1,1}_{\mathrm{loc}}(O;\mathbb{R}^N).$$

Concerning the existence of minimizers we will work with the following set of assumptions:

(f1) q-Growth:
There is a bound $\Gamma>0$ such that we have

$$f(A)\le\Gamma(1+|A|^q)\qquad\text{for every } A\in\mathbb{R}^{nN}.$$

(f2) p-Coercivity:
There is a coercivity constant $\gamma>0$ such that we have

$$f(A)\ge\gamma|A|^p\qquad\text{for every } A\in\mathbb{R}^{nN}.$$

(f3) $\mathbf{W^{1,p}}$-Quasiconvexity:
f is $W^{1,p}$-quasiconvex in the sense of Definition 1.6.

It is well known that the treatment of the regularity question requires a strict version of the (quasi)convexity assumption. Therefore in this regard we shall replace **(f3)** by the following variant:

(f3s) Strict $\mathbf{W^{1,p}}$-Quasiconvexity:
f is strictly nondegenerate $W^{1,p}$-quasiconvex, i. e. for each $M>0$ there is a convexity constant $\lambda_M>0$ such that we have

$$\fint_{B_1} f(A+D\varphi)\,dx\ge f(A)+\lambda_M\fint_{B_1}(1+|D\varphi|^2)^{\frac{p-2}{2}}|D\varphi|^2\,dx$$

for all $A\in\mathbb{R}^{nN}$ with $|A|\le M+1$ and for all $\varphi\in W^{1,p}_0(B_1;\mathbb{R}^N)$.

Finally, we mention that all our results still hold if f is allowed to take negative values but remains bounded below. In this context **(f2)** has to be replaced by $f(A)\ge\gamma|A|^p-c$ with a fixed $c\in\mathbb{R}$. This generalization is obvious since adding a constant to f does not change the minimizers of F.

3 Examples of Integrands

An important class of examples is given by integrands of the form

$$\begin{aligned} f^0(A) &:= |A|^p + h(\det A) \\ f^1(A) &:= (1+|A|^2)^{\frac{p}{2}} + h(\det A) \end{aligned} \qquad \text{for } A \in \mathbb{R}^{nN}, \tag{3.1}$$

where $N = n \geq 2$, $1 < p < \infty$ and h is a convex function satisfying

$$0 \leq h(d) \leq \Gamma_\kappa(1+|d|^\kappa) \qquad \text{for } d \in \mathbb{R}$$

with $1 \leq \kappa < \infty$. Interest in such integrands arises from problems in nonlinear elasticity (cf. [B], [BM] and [Ma2]).

For $p > \kappa n$ these integrands have standard growth and can be treated using standard existence and regularity theorems. Therefore we shall be interested in the case $p < \kappa n$ where we set $q := \kappa n$.

In the latter case **(f1)** and **(f2)** are obviously satisfied. In addition, it is well known that f^0 and f^1 are polyconvex and hence quasiconvex in the sense of Definition 1.2.

Concerning the question if f^0 and f^1 satisfy **(f3)** and **(f3s)** we recall from [BM]:

Theorem 3.1. *Fix $N = n \geq 2$, $1 < p < \infty$ and $1 \leq \kappa < \infty$. Then, for the integrands given by* (3.1) *with h convex, non-constant and nonnegative we have:*

$$f^0 \textit{ satisfies } \mathbf{(f3)} \iff p \geq n,$$
$$f^1 \textit{ satisfies } \mathbf{(f3s)} \iff f^1 \textit{ satisfies } \mathbf{(f3)} \iff p \geq n.$$

Therefore the results of this paper will cover the integrands in the case $n \leq p \leq q := \kappa n$ where we will need the additional bounds $q < \frac{np}{n-1}$ in the existence theory and $q < p + \frac{\min\{2,p\}}{2n}$ in the regularity theory. Let us remark that our regularity theory in the form stated here covers f^1 for smooth h, but not the degenerate integrands f^0. Nevertheless we believe that our methods can be combined with the regularity methods developed to handle degeneracy to cover also integrands like f^0.

In the case $p < n$, f is quasiconvex, but does not satisfy **(f3)**. We will use a refinement of our methods to include this case in the paper [S2].

4 Existence of Minimizers

As aforementioned $F[-;\Omega]$ is lower semicontinuous with respect to weak $W^{1,p}$-convergence of $W^{1,q}$-functions. More precisely we have:

Theorem 4.1 (Semicontinuity, [FM], [K])**.** *Assume that f is quasiconvex and satisfies* **(f1)** *with $1 < p \leq q < \frac{np}{n-1}$. Then, for $u_k \in W^{1,q}(\Omega;\mathbb{R}^N)$ and $u \in W^{1,p}(\Omega;\mathbb{R}^N)$ we have:*

$$u_k \xrightarrow[k\to\infty]{} u \text{ weakly in } W^{1,p}(\Omega;\mathbb{R}^N) \implies F[u;\Omega] \leq \liminf_{k\to\infty} F[u_k;\Omega].$$

Consequently we get:

Corollary 4.2. *Assume that f is quasiconvex and satisfies* **(f1)** *and* **(f2)** *with $1 < p \leq q < \frac{np}{n-1}$. Then, for each $u_0 \in W^{1,p}(\Omega;\mathbb{R}^N)$ we can find an $u \in u_0 + W_0^{1,p}(\Omega;\mathbb{R}^N)$ such that*

$$F[u;\Omega] \leq F[v;\Omega] \qquad \textit{for every } v \in W^{1,q}_{\text{loc}}(\Omega;\mathbb{R}^N) \cap \left[u_0 + W_0^{1,p}(\Omega;\mathbb{R}^N)\right].$$

Proof. Approximating Ω by subdomains we see that the conclusion of Theorem 4.1 still holds if the assumption $u_k \in W^{1,q}(\Omega;\mathbb{R}^N)$ is replaced by $u_k \in W^{1,q}_{\text{loc}}(\Omega;\mathbb{R}^N) \cap W^{1,p}(\Omega;\mathbb{R}^N)$. Now we choose a minimizing sequence for $F[-;\Omega]$ in $W^{1,q}_{\text{loc}}(\Omega;\mathbb{R}^N) \cap W^{1,p}(\Omega;\mathbb{R}^N)$. Applying the direct method of the calculus of variations to this sequence we obtain the result. □

However, the existence theorem just presented is not completely satisfactory. Therefore we will now investigate the consequences of $W^{1,p}$-quasiconvexity for the existence of $W^{1,p}$−minimizers. First we analyze the relation of $W^{1,p}$-quasiconvexity of f and sequential weak lower semicontinuity of $F[-;\Omega]$ on $W^{1,p}(\Omega;\mathbb{R}^N)$. The first result in this direction is:

Theorem 4.3 ([BM]). *Assume that $F[-;\Omega]$ is sequentially weakly lower semicontinuous on $W^{1,p}(\Omega;\mathbb{R}^N)$ where $p \geq 1$. Then f satisfies* **(f3)**.

Furthermore, it was conjectured in [BM] that $W^{1,p}$-quasiconvexity of f is also sufficient for weak lower semicontinuity of $F[-;\Omega]$ on $W^{1,p}(\Omega;\mathbb{R}^N)$. We do not know if this is true in general but we will show that the methods of [FM] allow to prove this at least under growth restrictions:

Theorem 4.4 (Semicontinuity). *Assume that f satisfies* **(f1)** *and* **(f3)** *with $1 < p \leq q < \frac{np}{n-1}$. Then, $F[-;\Omega]$ is sequentially weakly lower semicontinuous on $W^{1,p}(\Omega;\mathbb{R}^N)$.*

Sketch of Proof. We follow closely the lines of the proof of Theorem 4.1 given in [FM, Theorem 4.1] except for the following slight difference: In contrast to Theorem 4.1 we consider general $u_k \in W^{1,p}(\Omega;\mathbb{R}^N)$ and do not assume $u_k \in W^{1,q}(\Omega;\mathbb{R}^N)$. Therefore the functions used to test the quasiconvexity of f in [FM] are in general not in $W^{1,q}_0$ but only in $W^{1,p}_0$. However, since we are assuming **(f3)** we are allowed to use these test functions anyway. □

Corollary 4.5 (Existence). *Assume that f satisfies* **(f1)**, **(f2)** *and* **(f3)** *with $1 < p \leq q < \frac{np}{n-1}$. Then, for each $u_0 \in W^{1,p}(\Omega;\mathbb{R}^N)$ with $F[u_0;\Omega] < \infty$ we can find a $W^{1,p}$-minimizer $u \in u_0 + W^{1,p}_0(\Omega;\mathbb{R}^N)$ of F on Ω.*

Proof. Applying Theorem 4.4 together with the direct method of the calculus of variations we immediately obtain the statement. □

5 Main Results: Partial Regularity

We will state our regularity result for local minimizers in the sense of the next definition.

Definition 5.1 (Local Minimizer). *$u \in W^{1,p}_{\text{loc}}(\Omega;\mathbb{R}^N)$ is called a local $W^{1,p}$-minimizer of F on Ω iff every $x \in \Omega$ possesses a neighborhood U in Ω such that u is a $W^{1,p}$-minimizer of F on U.*

Our main result is the following partial regularity theorem which can be applied to the minimizers of Corollary 4.5:

Main Theorem 5.2 (Partial Regularity). *Let $1 < p \leq q < p + \frac{\min\{2,p\}}{2n}$ and assume that $f \in C^2_{\text{loc}}(\mathbb{R}^{nN})$ satisfies* **(f1)** *and* **(f3s)**. *Then, for each local $W^{1,p}$-minimizer $u \in W^{1,p}_{\text{loc}}(\Omega;\mathbb{R}^N)$ of F on Ω we have $\text{Reg}(u) \in \mathbb{O}_\Omega$, $u \in C^{1,\alpha}_{\text{loc}}(\text{Reg}(u);\mathbb{R}^N)$ for every $\alpha \in]0,1[$, and $|\text{Sing}(u)| = 0$.*

We will prove the main theorem in section 7.

Once $C^{1,\alpha}$-regularity is proven the growth of f loses its relevance. This allows, locally on $\mathrm{Reg}(u)$, the application of the Riesz-Schauder theory of linear partial differential equations to the Euler equation of F, obtaining the existence of higher derivatives. In particular the following holds:

Corollary 5.3. *In addition to the assumptions of Main Theorem 5.2 let us assume* $f \in C^\infty_{\mathrm{loc}}(\mathbb{R}^{nN})$. *Then we have* $u \in C^\infty_{\mathrm{loc}}(\mathrm{Reg}(u);\mathbb{R}^N)$.

6 Auxiliary Results

6.1 The auxiliary functions V and W

Definition 6.1. *Let* $\beta > 0$, *we define*

$$V^\beta(A) := (1+|A|^2)^{\frac{\beta-1}{2}}A \qquad \textit{and} \qquad W^\beta(A) := (1+|A|)^{\beta-1}A.$$

Note that V and W are comparable in the sense of

$$c^{-1}|W^\beta(A)| \le |V^\beta(A)| \le c|W^\beta(A)| \tag{6.1}$$

for all $A \in \mathbb{R}^{nN}$ and a constant c depending only on $\beta > 0$.

We shall now collect several properties of these functions. For $\beta > 0$, $1 \le p < \infty$, $A, B \in \mathbb{R}^{nN}$ and $t \ge 0$ we have:

$$|V^\beta(A)| \text{ and } |W^\beta(A)| \text{ depend nondecreasing on } |A|, \tag{6.2}$$

$$\left|W^{\frac{p}{2}}\right|^2 \text{ is convex,} \tag{6.3}$$

$$\left|W^{\frac{p}{2}}\right|^{\frac{2}{p}} \text{ is convex if and only if } p \le 2, \tag{6.4}$$

$$c^{-1} \le \frac{\int_0^1 (1+|A+tB)|^2)^{\frac{p-2}{2}}\,dt}{(1+|A|^2+|B|^2)^{\frac{p-2}{2}}} \le c \quad \text{if } p > 1, \tag{6.5}$$

$$|V^\beta(A+B)| \le c\left(|V^\beta(A)| + |V^\beta(B)|\right), \tag{6.6}$$

$$\min\{t^2,t^p\}|V^{\frac{p}{2}}(A)|^2 \le |V^{\frac{p}{2}}(tA)|^2 \le \max\{t^2,t^p\}|V^{\frac{p}{2}}(A)|^2, \tag{6.7}$$

$$(1+|A|^2+|B|^2)^{\frac{p}{2}} \le 1 + (1+|A|^2+|B|^2)^{\frac{p-2}{2}}(|A|^2+|B|^2) \quad \text{if } p \le 2, \tag{6.8}$$

where c depends only on β and p respectively.

(6.2), (6.3) and (6.4) can be verified using one-dimensional calculus. For proofs of (6.5) and (6.6) and further properties we refer to [AF3, Lemma 2.1], [AM2, Lemma 2.3] and [CFM, Lemma 2.1]. (6.7) and (6.8) are obvious.

In the following we will work primarily with the auxiliary functions V. However, sometimes the properties (6.3) and (6.4) of W will be convenient and we will use (6.1) to pass over to W.

Lemma 6.2. *Let* $1 \le p < \infty$ *and* $u \in W^{1,p}(\Omega;\mathbb{R}^N)$. *Then we have*

$$\int_\Omega |V^{\frac{p}{2}}(Du-(Du)_\Omega)|^2\,dx \le c\int_\Omega |V^{\frac{p}{2}}(Du-A)|^2\,dx \qquad \textit{for every } A \in \mathbb{R}^{nN}$$

with a constant c depending only on p.

Proof. By (6.1) it suffices to verify the claim with V replaced by W. From (6.3) we know that $|W^{\frac{p}{2}}|^2$ is convex. Using (6.6) and the previous observation together with Jensen's inequality we infer:

$$\begin{aligned}&\fint_{\Omega} |W^{\frac{p}{2}}(Du-(Du)_{\Omega})|^2\,dx\\ &\qquad\le c\left[\fint_{\Omega}|W^{\frac{p}{2}}(Du-A)|^2\,dx+|W^{\frac{p}{2}}((Du)_{\Omega}-A)|^2\right]\le c\fint_{\Omega}|W^{\frac{p}{2}}(Du-A)|^2\,dx.\end{aligned}$$

This proves the claim. Let us remark that in the case $p\ge\frac{18}{17}$ the above argument works straightforward with V instead of W since in this case $|V^{\frac{p}{2}}|^2$ is convex. □

6.2 Smoothing with Variable Radius

Lemma 6.3 ([FM])**.** *Let $0<r<s$ and $B_s\subset\Omega$. We define a bounded linear smoothing operator*

$$T_{r,s}:W^{1,1}(\Omega;\mathbb{R}^N)\to W^{1,1}(\Omega;\mathbb{R}^N)$$

for $u\in W^{1,1}(\Omega;\mathbb{R}^N)$ and $x\in\Omega$ by

$$T_{r,s}u(x):=\fint_{B_1}u(x+\vartheta(x)y)\,dy,\quad \textit{where } \vartheta(x):=\frac{1}{2}\max\Big\{\min\{|x|-r,s-|x|\},0\Big\}. \tag{6.9}$$

With this definition, for all $1\le p\le q<\frac{n}{n-1}p$ and all $u\in W^{1,p}(\Omega;\mathbb{R}^N)$ the following assertions are true:

$$T_{r,s}u\in W^{1,p}(\Omega;\mathbb{R}^N),$$

$$u=T_{r,s}u\qquad \textit{almost everywhere on } (\Omega\setminus B_s)\cup B_r, \tag{6.10}$$

$$T_{r,s}u\in u+W_0^{1,p}(B_s\setminus\overline{B_r};\mathbb{R}^N), \tag{6.11}$$

$$|DT_{r,s}u|\le c(n)T_{r,s}|Du|\qquad \textit{almost everywhere on } \Omega, \tag{6.12}$$

$$\|T_{r,s}u\|_{p;B_s\setminus B_r}\le c(n,p)\|u\|_{p;B_s\setminus B_r}, \tag{6.13}$$

$$\|DT_{r,s}u\|_{p;B_s\setminus B_r}\le c(n,p)\|Du\|_{p;B_s\setminus B_r}, \tag{6.14}$$

$$\|T_{r,s}u\|_{q;B_s\setminus B_r}\le c(n,p,q)(s-r)^{\frac{n}{q}-\frac{n-1}{p}}\left[\sup_{t\in]r,s[}\frac{\tilde{\Xi}(t)-\tilde{\Xi}(r)}{t-r}+\sup_{t\in]r,s[}\frac{\tilde{\Xi}(s)-\tilde{\Xi}(t)}{s-t}\right]^{\frac{1}{p}}, \tag{6.15}$$

$$\|DT_{r,s}u\|_{q;B_s\setminus B_r}\le c(n,p,q)(s-r)^{\frac{n}{q}-\frac{n-1}{p}}\left[\sup_{t\in]r,s[}\frac{\Xi(t)-\Xi(r)}{t-r}+\sup_{t\in]r,s[}\frac{\Xi(s)-\Xi(t)}{s-t}\right]^{\frac{1}{p}}. \tag{6.16}$$

Here we have used the abbreviations

$$\tilde{\Xi}(t):=\|u\|^p_{p;B_t}\quad \textit{and}\quad \Xi(t):=\|Du\|^p_{p;B_t}.$$

The lemma is a variant of [FM, Lemma 2.2]. All estimates can be proved using the methods developed there, although not all of them have been stated in the same form before; compare also [PS].

Further estimates of the terms on the right-hand side of (6.15) and (6.16) are obtained using the following lemma:

Lemma 6.4. *Let $-\infty < r < s < \infty$ and a continuous nondecreasing function $\Xi : [r,s] \to \mathbb{R}$ be given. Then there are $\tilde{r} \in [r, \frac{2r+s}{3}]$ and $\tilde{s} \in [\frac{r+2s}{3}, s]$ for which hold:*

$$\begin{aligned} \frac{\Xi(t)-\Xi(\tilde{r})}{t-\tilde{r}} &\le 3\frac{\Xi(s)-\Xi(r)}{s-r} \\ \frac{\Xi(\tilde{s})-\Xi(t)}{\tilde{s}-t} &\le 3\frac{\Xi(s)-\Xi(r)}{s-r} \end{aligned} \qquad \text{for every } t \in]\tilde{r},\tilde{s}[. \tag{6.17}$$

In particular we have

$$\frac{s-r}{3} \le \tilde{s}-\tilde{r} \le s-r. \tag{6.18}$$

An elementary proof is contained in [FM].

Concerning the interaction of the smoothing operator and the functions V the next lemma will be useful:

Lemma 6.5. *Consider $1 \le p \le 2$, $0 < r < s$, $B_s \subset \Omega$ and $u \in W^{1,p}(\Omega;\mathbb{R}^N)$. Then we have*

$$\left|V^{\frac{p}{2}}(DT_{r,s}u)\right|^{\frac{2}{p}} \le cT_{r,s}\left[\left|V^{\frac{p}{2}}(Du)\right|^{\frac{2}{p}}\right] \qquad \text{almost everywhere on } \Omega, \tag{6.19}$$

where c depends only on n and p.

Proof. By (6.1) it suffices to verify the claim with V replaced by W. From (6.2) and (6.4) we infer that $|W^{\frac{p}{2}}|^{\frac{2}{p}}$ is nondecreasing and convex. Therefore, using (6.12), (6.9) and Jensen's inequality, we see:

$$\left|W^{\frac{p}{2}}(DT_{r,s}u)\right|^{\frac{2}{p}} \le c\left|W^{\frac{p}{2}}(T_{r,s}|Du|)\right|^{\frac{2}{p}} \le cT_{r,s}\left[\left|W^{\frac{p}{2}}(Du)\right|^{\frac{2}{p}}\right] \qquad \text{almost everywhere on } \Omega.$$

This proves the lemma. □

6.3 An iteration lemma

We will use a variant of the well-known iteration lemma [Gia, Chapter V, Lemma 3.1]:

Lemma 6.6 (Iteration Lemma)**.** *We consider $p \in [1,\infty[$, $\kappa_1,\kappa_2,\kappa_3 \in \mathbb{R}$, $0 \le R < S < \infty$, $v \in L^p(B_S(x_0);\mathbb{R}^N)$, and a bounded function $g : [R,S] \to [0,\infty[$. Let us assume that for certain $G,H,K \ge 0$, $\vartheta \in [0,1[$ and all $R \le r < s \le S$ we have*

$$g(r) \le \vartheta g(s) + G(s-r)^{\kappa_1} + H\int_{B_S(x_0)}\left|V^{\frac{p}{2}}\left(\frac{v}{s-r}\right)\right|^2 dx + K(s-r)^{\kappa_2}\left(\int_{B_S(x_0)}\left|V^{\frac{p}{2}}\left(\frac{v}{s-r}\right)\right|^2 dx\right)^{\kappa_3}.$$

Then the following estimate holds true:

$$g(R) \le c\left[G(S-R)^{\kappa_1} + H\int_{B_S(x_0)}\left|V^{\frac{p}{2}}\left(\frac{v}{S-R}\right)\right|^2 dx + K(S-R)^{\kappa_2}\left(\int_{B_S(x_0)}\left|V^{\frac{p}{2}}\left(\frac{v}{S-R}\right)\right|^2 dx\right)^{\kappa_3}\right].$$

Here c denotes a constant depending only on p, κ_1, κ_2, κ_3, and ϑ.

A slightly less general variant can be found in [CFM, Lemma 2.7]. We shall omit the proof which is completely analogous.

6.4 $\mathscr{A}$-Harmonic Approximation

Throughout this subsection we consider a bilinear form $\mathscr{A}$ on $\mathbb{R}^{nN}$. We assume that the upper bound

$$|\mathscr{A}| \le \Lambda \tag{6.20}$$

with $\Lambda > 0$ holds and that the Legendre-Hadamard condition

$$\mathscr{A}(\zeta x^T, \zeta x^T) \ge \lambda |x|^2 |\zeta|^2 \qquad \text{for all } x \in \mathbb{R}^n \text{ and } \zeta \in \mathbb{R}^N \tag{6.21}$$

with ellipticity constant $\lambda > 0$ is satisfied. We say that $h \in W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ is $\mathscr{A}$-harmonic on Ω iff

$$\int_\Omega \mathscr{A}(Dh, D\varphi)\, dx = 0$$

holds for all smooth $\varphi : \Omega \to \mathbb{R}^N$ with compact support in Ω.

Lemma 6.7. *Assume* (6.20) *and* (6.21). *Consider an $\mathscr{A}$-harmonic $h \in W^{1,1}(B_\rho(x_0);\mathbb{R}^N)$, then we have $h \in C^\infty_{\mathrm{loc}}(B_\rho(x_0);\mathbb{R}^N)$ together with the standard estimate*

$$\sup_{B_{\rho/2}(x_0)} |Dh| + \rho \sup_{B_{\rho/2}(x_0)} |D^2 h| \le c \fint_{B_\rho(x_0)} |Dh|\, dx,$$

where c depends only on n, N, λ, and Λ.

Similar results for $W^{1,2}$-solutions are contained in the standard literature. However, concerning $W^{1,1}$-solutions we have to refer to recent work: [CFM, Proposition 2.10] and [DGK, Lemma 5].

The method of $\mathscr{A}$-harmonic approximation is based on the so-called $\mathscr{A}$-harmonic approximation lemma. The basic variant of this lemma has been proven in [DS] by means of a simple, but indirect proof. For further results and references concerning $\mathscr{A}$-harmonic approximation and its applications we refer to [DGS]. However, we shall use a variant of the lemma which has not been stated in this form before:

Lemma 6.8 ($\mathscr{A}$-Harmonic Approximation, [DGK])**.** *Fix $1 < p < \infty$, $0 < \lambda \le \Lambda < \infty$ and $\varepsilon > 0$. Then there is a $\delta(n, N, p, \lambda, \Lambda, \varepsilon) > 0$ such that the following assertion holds:*
For all $s \in]0,1]$, for all $\mathscr{A}$ satisfying (6.20) *and* (6.21) *and for each $u \in W^{1,p}(B_\rho(x_0);\mathbb{R}^N)$ with*

$$\fint_{B_\rho(x_0)} |V^{\frac{p}{2}}(Du)|^2\, dx \le s^2$$

and

$$\left| \fint_{B_\rho(x_0)} \mathscr{A}(Du, D\varphi)\, dx \right| \le s\delta \sup_{B_\rho(x_0)} |D\varphi|$$

for all smooth $\varphi : B_\rho(x_0) \to \mathbb{R}^N$ with compact support in $B_\rho(x_0)$

there is an $\mathscr{A}$-harmonic $h \in C^\infty_{\mathrm{loc}}(B_\rho(x_0);\mathbb{R}^N)$ with

$$\sup_{B_{\rho/2}(x_0)} |Dh| + \rho \sup_{B_{\rho/2}(x_0)} |D^2 h| \le c$$

and

$$\fint_{B_{\rho/2}(x_0)} \left| V^{\frac{p}{2}} \left(\frac{u - sh}{\rho} \right) \right|^2 dx \le s^2 \varepsilon.$$

Here c denotes a constant depending only on n, N, p, λ, and Λ.

Proof. In the case $1 < p \le 2$ a stronger version of the lemma has been proven in [DGK, Lemma 6] and our version follows combining it with Lemma 6.7.
In the case $p \ge 2$, using a simple scaling argument, we assume $x_0 = 0$ and $\rho = 1$. Let δ denote the constant of [DS, Lemma 3.3] and consider s, $\mathscr{A}$ and u as in the statement of Lemma 6.8. Then, for $w := \frac{u}{s}$ we have $\fint_{B_1} |Dw|^2 \le 1$ and

$$\left| \fint_{B_1} \mathscr{A}(Dw, D\varphi)\, dx \right| \le \delta \sup_{B_1} |D\varphi|$$

for all smooth $\varphi : B_1 \to \mathbb{R}^N$ with compact support in B_1.

Therefore we can apply [DS, Lemma 3.3] to get an $\mathscr{A}$-harmonic $h \in W^{1,2}(B_1; \mathbb{R}^N)$ with

$$\fint_{B_1} |Dh|^2\, dx \le 1 \qquad \text{and} \qquad \fint_{B_1} |w - h|^2\, dx \le \varepsilon.$$

Without loss of generality we assume $(w - h)_{1/2} = 0$. Using Lemma 6.7 we see $h \in C^\infty_{\text{loc}}(B_1; \mathbb{R}^N)$ and

$$\sup_{B_{1/2}} |Dh| + \sup_{B_{1/2}} |D^2 h| \le c.$$

Now choose an exponent $p^* > p$ depending only on n and p such that the Sobolev embedding $W^{1,p} \hookrightarrow L^{p^*}$ holds and define $t \in [0,1[$ by $\frac{1}{p} = \frac{1-t}{2} + \frac{t}{p^*}$. Applying L^p-interpolation combined with a Sobolev-Poincaré inequality we conclude:

$$\begin{aligned} \fint_{B_{1/2}} |u - sh|^p\, dx &\le \left(\fint_{B_{1/2}} |u - sh|^2\, dx \right)^{(1-t)\frac{p}{2}} \left(\fint_{B_{1/2}} |u - sh|^{p^*}\, dx \right)^{\frac{tp}{p^*}} \\ &\le c(s^2\varepsilon)^{(1-t)\frac{p}{2}} \left(\fint_{B_{1/2}} |Du|^p\, dx + s^p \fint_{B_{1/2}} |Dh|^p\, dx \right)^t \\ &\le c(s^2\varepsilon)^{(1-t)\frac{p}{2}} (s^2 + s^p)^t \le c s^2 \varepsilon^{(1-t)\frac{p}{2}} \end{aligned}$$

Replacing ε by a smaller quantity conveniently we obtain the claim. ☐

7 Proof of the Main Theorem

Throughout this section we will fix exponents $1 < p \le q < \infty$.

We shall work with an excess coupled to the exponent p, precisely we define:

Definition 7.1 (Excess)**.** *For $u \in W^{1,p}(B_\rho(x_0); \mathbb{R}^N)$ and $A \in \mathbb{R}^{nN}$ we set:*

$$\Phi_p(u, x_0, \rho, A) := \fint_{B_\rho(x_0)} \left| V^{\frac{p}{2}}(Du - A) \right|^2 dx,$$

$$\Phi_p(u, x_0, \rho) := \Phi_p(u, x_0, \rho, (Du)_{x_0,\rho}).$$

Introducing the local bounds

$$\Lambda_M := \sup_{|A|\le M+2} |D^2 f(A)| \tag{7.1}$$

we have:

Lemma 7.2 ([AF2]). *Consider $f \in C^2_{\mathrm{loc}}(\mathbb{R}^{nN})$ satisfying* **(f1)** *and* **(f3)** *and $A,B \in \mathbb{R}^{nN}$ with $|A| \le M+1$. Then the following estimates hold:*

$$|f(A+B) - f(A) - Df(A)B| \le c|V^{\frac{q}{2}}(B)|^2,$$
$$|Df(A+B) - Df(A)| \le c|V^{q-1}(B)|,$$

where c depends only on n, N, q, Γ, M, and Λ_M.

7.1 The Caccioppoli Inequality

Lemma 7.3 (Caccioppoli Inequality). *For $q < p + \frac{\min\{2,p\}}{2n}$ consider $M > 0$, $f \in C^2_{\mathrm{loc}}(\mathbb{R}^{nN})$ fulfilling* **(f1)** *and* **(f3s)***, and a $W^{1,p}$-minimizer $u \in W^{1,p}(B_\rho(x_0);\mathbb{R}^N)$ of F on $B_\rho(x_0)$. Then, for all $\zeta \in \mathbb{R}^N$ and $A \in \mathbb{R}^{nN}$ with $|A| \le M+1$ we have*

$$\Phi_p\left(u, x_0, \frac{\rho}{2}, A\right) \le c\left[h\left(⨍_{B_\rho(x_0)}\left|V^{\frac{p}{2}}\left(\frac{v}{\rho}\right)\right|^2 dx\right) + \left(\Phi_p(u,x_0,\rho,A)\right)^{\frac{q}{p}}\right], \tag{7.2}$$

where we have set $h(t) := t + t^{\frac{q}{p}}$ and $v(x) := u(x) - \zeta - A(x-x_0)$ and where c denotes a positive constant depending only on n, N, p, q, Γ, M, λ_M, and Λ_M.

Proof of Lemma 7.3 for $p \ge 2$. We assume $x_0 = 0$ and choose $\frac{\rho}{2} \le r < s \le \rho$. Setting

$$\Xi(t) := \int_{B_t}\left[|Dv|^p + \left|\frac{v}{s-r}\right|^p\right] dx$$

we choose in addition $r \le \tilde{r} < \tilde{s} \le s$ as in Lemma 6.4. Let η denote a smooth cut-off function with support in $B_{\tilde{s}}$, satisfying $\eta \equiv 1$ in a neighborhood of $\overline{B_{\tilde{r}}}$, and $0 \le \eta \le 1$, $|\nabla\eta| \le \frac{2}{\tilde{s}-\tilde{r}}$ on B_ρ. Using the operator from Lemma 6.3 we set

$$\psi := T_{\tilde{r},\tilde{s}}[(1-\eta)v] \qquad \text{and} \qquad \varphi := v - \psi.$$

Due to (6.10) and (6.11) we have $\varphi \in W^{1,p}_0(B_{\tilde{s}};\mathbb{R}^N)$ and $\varphi = v$ on $B_{\tilde{r}}$. Furthermore, we see

$$Du - A = Dv = D\varphi + D\psi \qquad \text{on } B_\rho.$$

Using **(f3s)** we obtain

$$\lambda_M \int_{B_{\tilde{r}}} |V^{\frac{p}{2}}(Dv)|^2\,dx \le \int_{B_{\tilde{s}}} [f(A+D\varphi) - f(A)]\,dx$$
$$= \int_{B_{\tilde{s}}} [f(Du - D\psi) - f(Du)]\,dx + \int_{B_{\tilde{s}}} [f(Du) - f(Du - D\varphi)]\,dx + \int_{B_{\tilde{s}}} [f(A+D\psi) - f(A)]\,dx.$$

Applying the minimality of u together with Lemma 7.2 we conclude

$$\begin{aligned}
&\lambda_M \int_{B_r} |V^{\frac{p}{2}}(Dv)|^2\,dx \\
&\le \int_{B_{\tilde{s}}} \left[\int_0^1 \Big(Df(A) - Df(Du - \tau D\psi) \Big)\, d\tau\, D\psi + f(A + D\psi) - f(A) - Df(A) D\psi \right] dx \\
&\le c \int_{B_{\tilde{s}}} \left[\int_0^1 |V^{q-1}(Dv - \tau D\psi)|\, d\tau\, |D\psi| + |V^{\frac{q}{2}}(D\psi)|^2 \right] dx.
\end{aligned}$$

Setting $R := B_{\tilde{s}} \setminus \overline{B_{\tilde{r}}}$, recalling $\psi \equiv 0$ on $B_{\tilde{r}}$, and (6.6), (6.7) we infer

$$\int_{B_r} |V^{\frac{p}{2}}(Dv)|^2\,dx \le c \left[\int_R |V^{\frac{q}{2}}(D\psi)|^2\,dx + \int_R |V^{q-1}(Dv)||D\psi|\,dx \right] =: c\,[(I) + (II)]. \tag{7.3}$$

Let us introduce the abbreviation

$$\Delta := \int_{B_s \setminus B_r} \left[\left| V^{\frac{p}{2}}(Dv) \right|^2 + \left| V^{\frac{p}{2}}\left(\frac{v}{s-r} \right) \right|^2 \right] dx.$$

Employing (6.14), (6.16) (using $q < \frac{np}{n-1}$), (6.18), and (6.17) we get:

$$\begin{aligned}
(I) &\le c \left[\int_R |D\psi|^2\,dx + \int_R |D\psi|^q\,dx \right] \\
&\le c \Bigg[\int_R |D[(1-\eta)v]|^2\,dx + (\tilde{s}-\tilde{r})^n \left(\sup_{t \in]\tilde{r},\tilde{s}[} \frac{(\tilde{s}-\tilde{r})^{1-n}}{t-\tilde{r}} \int_{B_t \setminus B_{\tilde{r}}} |D[(1-\eta)v]|^p\,dx \right)^{\frac{q}{p}} \\
&\qquad + (\tilde{s}-\tilde{r})^n \left(\sup_{t \in]\tilde{r},\tilde{s}[} \frac{(\tilde{s}-\tilde{r})^{1-n}}{\tilde{s}-t} \int_{B_{\tilde{s}} \setminus B_t} |D[(1-\eta)v]|^p\,dx \right)^{\frac{q}{p}} \Bigg] \\
&\le c \Bigg[\Delta + (s-r)^n \left((s-r)^{1-n} \sup_{t \in]\tilde{r},\tilde{s}[} \frac{\Xi(t) - \Xi(\tilde{r})}{t-\tilde{r}} \right)^{\frac{q}{p}} \\
&\qquad + (s-r)^n \left((s-r)^{1-n} \sup_{t \in]\tilde{r},\tilde{s}[} \frac{\Xi(\tilde{s}) - \Xi(t)}{\tilde{s}-t} \right)^{\frac{q}{p}} \Bigg] \\
&\le c \left[\Delta + (s-r)^n \left(\frac{\Delta}{(s-r)^n} \right)^{\frac{q}{p}} \right].
\end{aligned} \tag{7.4}$$

Using $q < p + \frac{1}{n} < p+1$ and Hölder's inequality we can treat (II) in a similar fashion:

$$\begin{aligned}
(II) &\le c \int_R \big(|Dv||D\psi| + |Dv|^{q-1}|D\psi| \big)\,dx \\
&\le c \left[\left(\int_R |Dv|^2\,dx \right)^{\frac{1}{2}} \left(\int_R |D\psi|^2\,dx \right)^{\frac{1}{2}} + \left(\int_R |Dv|^p\,dx \right)^{\frac{q-1}{p}} \left(\int_R |D\psi|^{\frac{p}{p+1-q}}\,dx \right)^{\frac{p+1-q}{p}} \right] \\
&\le c \left[\Delta + (s-r)^n \left(\frac{\Delta}{(s-r)^n} \right)^{\frac{q}{p}} \right].
\end{aligned} \tag{7.5}$$

Combining (7.3), (7.4), and (7.5) we arrive at

$$\int_{B_r} |V^{\frac{p}{2}}(Dv)|^2\,dx \le C_1 \left[\Delta + (s-r)^n \left(\frac{\Delta}{(s-r)^n}\right)^{\frac{q}{p}}\right],$$

where $C_1(n,N,p,q,\Gamma,M,\lambda_M,\Lambda_M) > 0$ denotes a fixed constant. Adding $C_1 \int_{B_r} |V^{\frac{p}{2}}(Dv)|^2\,dx$ on both sides and dividing by $1+C_1$ we see:

$$\begin{aligned}\int_{B_r} |V^{\frac{p}{2}}(Dv)|^2\,dx \;\le\; & \frac{C_1}{1+C_1}\int_{B_s} |V^{\frac{p}{2}}(Dv)|^2\,dx + \int_{B_\rho} \left|V^{\frac{p}{2}}\left(\frac{v}{s-r}\right)\right|^2 dx \\ & + (s-r)^n \left(\frac{1}{(s-r)^n}\int_{B_\rho}\left[\left|V^{\frac{p}{2}}(Dv)\right|^2 + \left|V^{\frac{p}{2}}\left(\frac{v}{s-r}\right)\right|^2\right] dx\right)^{\frac{q}{p}}.\end{aligned}$$

Now Lemma 6.6 (with $\vartheta = \frac{C_1}{1+C_1}$) implies

$$\begin{aligned}& \fint_{B_{\rho/2}} |V^{\frac{p}{2}}(Dv)|^2\,dx \\ & \quad \le c\left[\fint_{B_\rho}\left|V^{\frac{p}{2}}\left(\frac{v}{\rho}\right)\right|^2 dx + \left(\fint_{B_\rho}\left[\left|V^{\frac{p}{2}}(Dv)\right|^2 + \left|V^{\frac{p}{2}}\left(\frac{v}{\rho}\right)\right|^2\right] dx\right)^{\frac{q}{p}}\right],\end{aligned}$$

which proves the claim. □

Proof of Lemma 7.3 for $p \le 2$. We use the notations of the previous case except for the following modification:

$$\Xi(t) := \int_{B_t}\left[|V^{\frac{p}{2}}(Dv)|^2 + \left|V^{\frac{p}{2}}\left(\frac{v}{s-r}\right)\right|^2\right] dx.$$

As before we get

$$\int_{B_r} |V^{\frac{p}{2}}(Dv)|^2\,dx \le c\int_R \left[\int_0^1 |V^{q-1}(Dv-\tau D\psi)|\,d\tau\,|D\psi| + |V^{\frac{q}{2}}(D\psi)|^2\right] dx.$$

Combining the previous estimate with (6.5) we have

$$\begin{aligned}& \int_{B_r} |V^{\frac{p}{2}}(Dv)|^2\,dx \\ & \quad \le c\left[\int_R |V^{\frac{q}{2}}(D\psi)|^2\,dx + \int_R (1+|Dv|^2+|D\psi|^2)^{\frac{q-2}{2}}(|Dv|+|D\psi|)|D\psi|\,dx\right] \\ & \quad =: c[(I)+(II)].\end{aligned} \tag{7.6}$$

Now (6.8) leads to

$$\begin{aligned}(I) \le & \int_R \left(1+|V^{\frac{p}{2}}(D\psi)|^2\right)^{\frac{q-p}{p}} |V^{\frac{p}{2}}(D\psi)|^2\,dx \\ & \le c\Big[\int_R |V^{\frac{p}{2}}(D\psi)|^2\,dx + \underbrace{\int_R |V^{\frac{p}{2}}(D\psi)|^{\frac{2q}{p}}\,dx}_{=:(III)}\Big].\end{aligned} \tag{7.7}$$

Using (6.19), (6.15) (based on $q < \frac{np}{n-1}$), (6.18), and (6.17), (III) can be estimated as follows:

$$\begin{aligned}(III) &\le c \int_R \left(T_{\tilde r,\tilde s}\left[|V^{\frac{p}{2}}(D[(1-\eta)v])|^{\frac{2}{p}} \right]\right)^q dx \\ &\le c(\tilde s-\tilde r)^n \Bigg(\sup_{t\in]\tilde r,\tilde s[} \frac{(\tilde s-\tilde r)^{1-n}}{t-\tilde r} \int_{B_t\setminus B_{\tilde r}} |V^{\frac{p}{2}}(D[(1-\eta)v])|^2\,dx \\ &\qquad\qquad + \sup_{t\in]\tilde r,\tilde s[} \frac{(\tilde s-\tilde r)^{1-n}}{\tilde s-t} \int_{B_{\tilde s}\setminus B_t} |V^{\frac{p}{2}}(D[(1-\eta)v])|^2\,dx \Bigg)^{\frac{q}{p}} \\ &\le c(s-r)^n \left[(s-r)^{1-n} \left(\sup_{t\in]\tilde r,\tilde s[} \frac{\Xi(t)-\Xi(\tilde r)}{t-\tilde r} + \sup_{t\in]\tilde r,\tilde s[} \frac{\Xi(\tilde s)-\Xi(t)}{\tilde s-t} \right)\right]^{\frac{q}{p}} \\ &\le c(s-r)^n \left(\frac{\Delta}{(s-r)^n} \right)^{\frac{q}{p}}.\end{aligned}$$

On account of (6.19) and (6.13), the first term on the right-hand side of (7.7) can be estimated by $c\Delta$ similarly. This yields

$$(I) \le c \left[\Delta + (s-r)^n \left(\frac{\Delta}{(s-r)^n} \right)^{\frac{q}{p}} \right]. \tag{7.8}$$

It remains to estimate (II). Recalling (6.8) once again, repeated use of Young's inequality gives:

$$\begin{aligned}(II) &\le c \Bigg[\int_R (1+|Dv|^2+|D\psi|^2)^{\frac{p-2}{2}} (|Dv|+|D\psi|)|D\psi|\,dx \\ &\qquad + \int_R (1+|Dv|^2+|D\psi|^2)^{(p-2)\frac{q}{2p}} (|Dv|^2+|D\psi|^2)^{\frac{q-p}{p}} (|Dv|+|D\psi|)|D\psi|\,dx \Bigg] \\ &\le c \Bigg[\int_R |V^{\frac{p}{2}}(Dv)|^2\,dx + \int_R |V^{\frac{p}{2}}(D\psi)|^2\,dx \\ &\qquad + \underbrace{\int_R |V^{\frac{p}{2}}(D\psi)|^{\frac{2q}{p}}\,dx}_{=(III)} + \underbrace{\int_R |V^{\frac{p}{2}}(Dv)|^{\frac{2q}{p}-1} |V^{\frac{p}{2}}(D\psi)|\,dx}_{=:(IV)} \Bigg].\end{aligned}$$

The first two terms on the right-hand side of the last inequality can be controlled by $c\Delta$ as before. (III) has already been estimated. Therefore it remains to treat (IV). Using $q < \frac{3}{2}p$, Hölder's inequality yields:

$$(IV) \le \left(\int_R |V^{\frac{p}{2}}(Dv)|^2\,dx \right)^{\frac{2q-p}{2p}} \left(\int_R |V^{\frac{p}{2}}(D\psi)|^{\frac{2p}{3p-2q}}\,dx \right)^{\frac{3p-2q}{2p}}$$

The first factor on the right-hand side of the previous inequality can be estimated by $\Delta^{\frac{2q-p}{2p}}$ while, using $q < \frac{3}{2}p$ and $q < p + \frac{p}{2n}$, the second one can be treated exactly as (III). Collecting the estimates we have

$$(IV) \le c(s-r)^n \left(\frac{\Delta}{(s-r)^n} \right)^{\frac{q}{p}}$$

and

$$(II) \leq c\left[\Delta + (s-r)^n\left(\frac{\Delta}{(s-r)^n}\right)^{\frac{q}{p}}\right]. \tag{7.9}$$

Combining (7.6), (7.8), and (7.9) we get

$$\int_{B_r} |V^{\frac{p}{2}}(Dv)|^2\,dx \leq c\left[\Delta + (s-r)^n\left(\frac{\Delta}{(s-r)^n}\right)^{\frac{q}{p}}\right].$$

Now we proceed exactly as in the case $p \geq 2$. □

Remark 7.4. *Let us mention that in the case $q = p$ (7.2) holds without the second term on its right-hand side. This can be inferred directly from the proofs. However, in the case $q > p$ we will see that this second term is arbitrarily small. This is the reason why we call (7.2) a 'Caccioppoli inequality'.*

Remark 7.5. *If in addition to* **(f3s)** *f is convex, in particular if*

$$D^2 f(A)(B,B) \geq \lambda(1+|A|^2)^{\frac{p-2}{2}}|B|^2 \qquad \text{for all } A,B \in \mathbb{R}^{nN},$$

then (7.2) holds under the weaker assumption $q < \frac{np}{n-1}$ for the exponents. This variant of the Caccioppoli inequality provides an alternative proof of Theorem 1.4.

Proof of Remark 7.5. We use the notations from the proofs of Lemma 7.3, introducing the additional abbreviations

$$\tilde{\varphi} := T_{\tilde{r},\tilde{s}}[\eta v] \in W_0^{1,p}(B_{\tilde{s}};\mathbb{R}^N),$$
$$w := T_{\tilde{r},\tilde{s}} v.$$

Then

$$Dw = D\tilde{\varphi} + D\psi \qquad \text{on } B_\rho,$$

and arguing essentially as before we get:

$$\lambda_M \int_{B_{\tilde{r}}} |V^{\frac{p}{2}}(Dv)|^2\,dx \leq \int_{B_{\tilde{s}}} [f(A+D\tilde{\varphi}) - f(A)]\,dx$$
$$= \int_R [f(A+D\tilde{\varphi}) - f(Du)]\,dx + \int_{B_{\tilde{s}}} [f(Du) - f(Du - D\varphi)]\,dx + \int_R [f(A+D\psi) - f(A)]\,dx.$$

Exploiting the convexity of f we have

$$f(Du) \geq f(A) + Df(A)Dv \qquad \text{on } B_\rho.$$

We combine the last two inequalities with the minimizing property of u and use $v - w \in W_0^{1,p}(R;\mathbb{R}^N)$ to obtain:

$$\lambda_M \int_{B_{\tilde{r}}} |V^{\frac{p}{2}}(Dv)|^2\,dx \leq \int_R [f(A+D\tilde{\varphi}) - f(A) - Df(A)Dw]\,dx + \int_R [f(A+D\psi) - f(A)]\,dx$$
$$= \int_R [f(A+D\tilde{\varphi}) - f(A) - Df(A)D\tilde{\varphi}]\,dx + \int_R [f(A+D\psi) - f(A) - Df(A)D\psi]\,dx.$$

Recalling $|A| \leq M+1$, Lemma 7.2 yields

$$\int_{B_r} |V^{\frac{p}{2}}(Dv)|^2\,dx \leq c\left[\int_R |V^{\frac{q}{2}}(D\psi)|^2\,dx + \int_R |V^{\frac{q}{2}}(D\tilde{\varphi})|\,dx\right] =: c[(I) + (II)].$$

The term (I) is the same as in the proofs of Lemma 7.3 and can be treated as before. However, (II) can now be estimated completely analogous to (I) using only the condition $q < \frac{np}{n-1}$. □

7.2 Approximate $\mathscr{A}$-Harmonicity

From the uniform continuity of $D^2 f$ on bounded subsets of $\mathbb{R}^{nN}$ we deduce:

Remark 7.6. *Assume $f \in C^2_{\mathrm{loc}}(\mathbb{R}^{nN})$. Then for each $M > 0$ there is a modulus of continuity $\nu_M : [0,\infty[\to [0,\infty[$ satisfying $\lim_{\omega \searrow 0} \nu_M(\omega) = 0$ such that for all $A, B \in \mathbb{R}^{nN}$ we have:*

$$|A| \leq M, |B| \leq M+1 \implies |D^2 f(A) - D^2 f(B)| \leq \nu_M(|A-B|^2). \tag{7.10}$$

Furthermore, ν_M can be chosen such that the following properties hold:

(I) ν_M is nondecreasing,
(II) ν_M^2 is concave,
(III) $\nu_M^2(\omega) \geq \omega$ for all $\omega \geq 0$.

Lemma 7.7. *Assume $f \in C^2_{\mathrm{loc}}(\mathbb{R}^{nN})$ satisfies* **(f1)** *and* **(f3)**, *and $u \in W^{1,p}(B_\rho(x_0); \mathbb{R}^N)$ is a $W^{1,p}$-minimizer of F on $B_\rho(x_0)$, where $q \leq p+1$. Then, for all $M > 0$, all $A \in \mathbb{R}^{nN}$ with $|A| \leq M$, and all smooth $\varphi : B_\rho(x_0) \to \mathbb{R}^N$ with compact support in $B_\rho(x_0)$ we have*

$$\left| \fint_{B_\rho(x_0)} D^2 f(A)(Du - A, D\varphi)\, dx \right| \leq c \sqrt{\Phi_p}\, \nu_M(\Phi_p) \sup_{B_\rho(x_0)} |D\varphi|.$$

Here we have abbreviated $\Phi_p(u, x_0, \rho, A)$ by Φ_p and c depends only on n, N, p, q, Γ, M, and Λ_M.

Proof. We may assume $x_0 = 0$ and $\sup_{B_\rho} |D\varphi| = 1$. Setting $v(x) := u(x) - Ax$ the Euler equation of F gives

$$\left| \fint_{B_\rho} D^2 f(A)(Dv, D\varphi)\, dx \right| \leq \fint_{B_\rho} \left| D^2 f(A)(Dv, D\varphi) + Df(A)D\varphi - Df(Du)D\varphi \right| dx. \tag{7.11}$$

Now we estimate the integrand on the right-hand side: On $\{x \in B_\rho : |Dv| \leq 1\}$, using Remark 7.6, we have:

$$\begin{aligned} &\left| D^2 f(A)(Dv, D\varphi) + Df(A)D\varphi - Df(Du)D\varphi \right| \\ &\qquad \leq \int_0^1 \left| D^2 f(A) - D^2 f(A + tDv) \right| dt\, |Dv| \\ &\qquad \leq \nu_M(|Dv|^2)|Dv| \leq c\nu_M\left(\left| V^{\frac{p}{2}}(Dv) \right|^2 \right) \left| V^{\frac{p}{2}}(Dv) \right|. \end{aligned} \tag{7.12}$$

On $\{x \in B_\rho : |Dv| \geq 1\}$, Lemma 7.2 implies

$$\begin{aligned} &\left| D^2 f(A)(Dv, D\varphi) + Df(A)D\varphi - Df(Du)D\varphi \right| \\ &\qquad \leq \Lambda_M |Dv| + c|V^{q-1}(Dv)| \leq c|Dv|^{\max\{q-1,1\}} \leq c \left| V^{\frac{p}{2}}(Dv) \right|^2. \end{aligned} \tag{7.13}$$

Combining (7.11), (7.12), and (7.13) with the property (III) from Remark 7.6 we get

$$\left| \fint_{B_\rho} D^2 f(A)(Dv, D\varphi)\, dx \right| \leq c \fint_{B_\rho} \nu_M\left(\left| V^{\frac{p}{2}}(Dv) \right|^2 \right) \left| V^{\frac{p}{2}}(Dv) \right| dx.$$

Now we apply the inequalities of Cauchy-Schwarz and Jensen, using the concavity of ν_M^2, to obtain

$$\left| \fint_{B_\rho} D^2 f(A)(Dv, D\varphi)\, dx \right| \leq c \sqrt{\Phi_p}\, \nu_M(\Phi_p).$$

This completes the proof. ☐

7.3 Excess Estimates

It is well known that quasiconvexity of f implies the Legendre-Hadamard condition for $D^2 f$. More precisely we have:

Lemma 7.8. *Assume that $f \in C^2_{\mathrm{loc}}(\mathbb{R}^{nN})$ satisfies* **(f3s)**. *Let us consider $M > 0$ and $A \in \mathbb{R}^{nN}$ with $|A| \le M$, then $D^2 f(A)$ fulfills the Legendre-Hadamard condition* (6.21) *with ellipticity constant $2\lambda_M$.*

The proof of the lemma is completely analogous to that of [Giu, Proposition 5.2].

Proposition 7.9. *We assume $q < p + \frac{\min\{2,p\}}{2n}$ and we consider $M > 0$, $\alpha \in]0,1[$, and $f \in C^2_{\mathrm{loc}}(\mathbb{R}^{nN})$ with* **(f1)** *and* **(f3s)**. *Then there are constants $\varepsilon_0 > 0$ and $\theta \in]0,1[$ such that the conditions*

$$\Phi_p(u,x_0,\rho) \le \varepsilon_0 \qquad \text{and} \qquad |(Du)_{x_0,\rho}| \le M \tag{7.14}$$

for a $W^{1,p}$-minimizer $u \in W^{1,p}(B_\rho(x_0);\mathbb{R}^N)$ of F on $B_\rho(x_0)$ imply

$$\Phi_p(u,x_0,\theta\rho) \le \theta^{2\alpha} \Phi_p(u,x_0,\rho).$$

Here θ depends only on n, N, p, q, Γ, M, λ_M, Λ_M, and α, and ε_0 depends additionally on ν_M.

Proof. We assume $x_0 = 0$ and we abbreviate $A := (Du)_\rho$, $\Phi_p(-) := \Phi_p(u,0,-)$. Then $|A| \le M$. Since the claim is obvious in the case $\Phi_p(\rho) = 0$ we can assume $\Phi_p(\rho) \neq 0$. Setting

$$w(x) := u(x) - Ax \qquad \text{and} \qquad s := \sqrt{\Phi_p(\rho)}$$

we have

$$\fint_{B_\rho} |V^{\frac{p}{2}}(Dw)|^2\,dx = s^2.$$

Next we will approximate w by $\mathscr{A}$-harmonic functions, where

$$\mathscr{A} := D^2 f(A).$$

From (7.1) and Lemma 7.8 we deduce that $\mathscr{A}$ satisfies (6.20) with bound Λ_M and (6.21) with ellipticity constant $2\lambda_M$. Lemma 7.7 yields the estimate

$$\left|\fint_{B_\rho} \mathscr{A}(Dw, D\varphi)\,dx\right| \le s C_2 \nu_M(\Phi_p(\rho)) \sup_{B_\rho} |D\varphi|$$
$$\text{for all smooth } \varphi : B_\rho \to \mathbb{R}^N \text{ with compact support in } B_\rho,$$

where $C_2(n,N,p,q,\Gamma,M,\Lambda_M) > 0$ is a constant. For $\varepsilon > 0$ to be specified later we fix the corresponding constant $\delta > 0$ from Lemma 6.8. Imposing the smallness assumptions

$$C_2 \nu_M(\Phi_p(\rho)) \le \delta, \tag{7.15}$$

$$s = \sqrt{\Phi_p(\rho)} \le 1, \tag{7.16}$$

we apply Lemma 6.8. The lemma ensures the existence of an $\mathscr{A}$-harmonic function $h \in C^\infty_{\mathrm{loc}}(B_\rho;\mathbb{R}^N)$ such that

$$\begin{aligned} \sup_{B_{\rho/2}} |Dh| + \rho \sup_{B_{\rho/2}} |D^2 h| &\le c, \\ \fint_{B_{\rho/2}} \left| V^{\frac{p}{2}}\left(\frac{w - sh}{\rho}\right)\right|^2 dx &\le s^2 \varepsilon. \end{aligned} \tag{7.17}$$

Now fix $\theta \in]0, \frac{1}{4}]$. Taylor expansion implies the estimate

$$\sup_{x\in B_{2\theta\rho}} |h(x) - h(0) - Dh(0)x| \le \frac{1}{2}(2\theta\rho)^2 \sup_{B_{\rho/2}} |D^2 h| \le c\,\theta^2 \rho.$$

Using (6.6) and (6.7) together with (7.17) and the last inequality we get:

$$\begin{aligned} &\fint_{B_{2\theta\rho}} \left| V^{\frac{p}{2}}\left(\frac{w(x) - sh(0) - sDh(0)x}{2\theta\rho}\right)\right|^2 dx \\ &\le c\left[\theta^{-n-\max\{2,p\}} \fint_{B_{\rho/2}} \left| V^{\frac{p}{2}}\left(\frac{w - sh}{\rho}\right)\right|^2 dx + \fint_{B_{2\theta\rho}} \left| V^{\frac{p}{2}}\left(s\frac{h(x) - h(0) - Dh(0)x}{2\theta\rho}\right)\right|^2 dx\right] \\ &\le c\left[\theta^{-n-\max\{2,p\}} s^2 \varepsilon + |V^{\frac{p}{2}}(\theta s)|^2\right] \\ &\le c\left[\theta^{-n-\max\{2,p\}} s^2 \varepsilon + \theta^2 s^2\right]. \end{aligned}$$

Setting $\varepsilon := \theta^{n+2+\max\{2,p\}}$ and recalling the definitions of w and s we have

$$\fint_{B_{2\theta\rho}} \left| V^{\frac{p}{2}}\left(\frac{u(x) - Ax - s(h(0) + Dh(0)x)}{2\theta\rho}\right)\right|^2 dx \le c\,\theta^2 \Phi_p(\rho). \tag{7.18}$$

From (7.17) we conclude

$$|sDh(0)|^2 \le C_3 \Phi_p(\rho) \tag{7.19}$$

with a constant $C_3(n, N, p, M, \lambda_M, \Lambda_M) > 0$. Using (6.6), (7.19), and (7.16) we get:

$$\begin{aligned} &\Phi_p(2\theta\rho, A + sDh(0)) \\ &\qquad \le c\left[(2\theta)^{-n} \fint_{B_\rho} |V^{\frac{p}{2}}(Du - A)|^2 dx + |V^{\frac{p}{2}}(sDh(0))|^2\right] \le c\theta^{-n} \Phi_p(\rho). \end{aligned} \tag{7.20}$$

Next we combine (7.18), (7.20), and the Caccioppoli inequality (7.2) (with $\zeta = sh(0)$ and $A + sDh(0)$ instead of A) to derive

$$\Phi_p(\theta\rho, A + sDh(0)) \le c\left[\theta^2 \Phi_p(\rho) + \theta^{\frac{2q}{p}} \Phi_p(\rho)^{\frac{q}{p}} + \theta^{-n\frac{q}{p}} \Phi_p(\rho)^{\frac{q}{p}}\right]. \tag{7.21}$$

Thereby the condition $|A + sDh(0)| \le M + 1$ of Lemma 7.3 can be deduced from (7.19) together with the additional smallness assumption

$$C_3 \Phi_p(\rho) \le 1. \tag{7.22}$$

For $q > p$ the smallness assumptions (7.16) and

$$\theta^{-n\frac{q}{p}} \Phi_p(\rho)^{\frac{q-p}{p}} \le \theta^2, \tag{7.23}$$

and $\theta \le 1$ imply

$$\Phi_p(\theta\rho, A + sDh(0)) \le c\,\theta^2 \Phi_p(\rho).$$

For $q = p$, however, the last inequality holds without further assumptions since the last term on the right-hand side of (7.21) does not occur (cf. Remark 7.4).
Using Lemma 6.2 we deduce from the previous inequality:

$$\Phi_p(\theta\rho) \le C_4 \theta^2 \Phi_p(\rho), \tag{7.24}$$

where $C_4(n, N, p, q, \Gamma, M, \lambda_M, \Lambda_M) > 0$. Finally, we choose $\theta \in]0, \frac{1}{4}]$ small enough such that

$$C_4 \theta^2 \le \theta^{2\alpha} \tag{7.25}$$

holds, and ε_0 small enough such that (7.15), (7.16), (7.22), and (7.23) follow from the first part of (7.14). Taking into account (7.24) and (7.25) the proof of the proposition is complete. □

Iterating Proposition 7.9 we find:

Lemma 7.10. *We assume $q < p + \frac{\min\{2,p\}}{2n}$ and consider $M > 0$, $\alpha \in]0,1[$, and $f \in C^2_{\mathrm{loc}}(\mathbb{R}^{nN})$ with* **(f1)** *and* **(f3s)***. Then there is a constant $\tilde{\varepsilon}_0 > 0$ such that the assumptions*

$$\Phi_p(u, x_0, \rho) \le \tilde{\varepsilon}_0 \qquad \textit{and} \qquad |(Du)_{x_0,\rho}| \le \frac{M}{2}$$

for a $W^{1,p}$-minimizer $u \in W^{1,p}(B_\rho(x_0); \mathbb{R}^N)$ of F on $B_\rho(x_0)$ imply the growth condition

$$\Phi_p(u, x_0, r) \le c \left(\frac{r}{\rho}\right)^{2\alpha} \Phi_p(u, x_0, \rho) \qquad \textit{for all } r \in]0, \rho].$$

Here c depends only on n, N, p, q, Γ, M, λ_M, Λ_M, and α, and $\tilde{\varepsilon}_0$ depends additionally on ν_M.

This lemma can be deduced from Proposition 7.9 by standard arguments (see [E] and [CFM]); details can be found in [S1].

7.4 Concluding Remarks

Once Lemma 7.10 is established CAMPANATO's integral characterization of Hölder continuity leads to the claims of Main Theorem 5.2. We omit further details.

Analyzing the preceding proof we deduce the following detailed assertions:

Remark 7.11. *Assume the hypotheses of Main Theorem 5.2. Then,*

- *we have* $\mathrm{Sing}(u) = \Sigma_1 \cup \Sigma_2$ *where we have set*

$$\Sigma_1 := \left\{ x \in \Omega \,:\, \liminf_{\rho \searrow 0} \Phi_p(u, x, \rho) > 0 \right\},$$

$$\Sigma_2 := \left\{ x \in \Omega \,:\, \limsup_{\rho \searrow 0} |(Du)_{x,\rho}| = \infty \right\};$$

- *for every $\alpha \in]0,1[$ and $x_0 \in \mathrm{Reg}(u)$ there is a $\sigma > 0$ such that Du is Hölder continuous on $B_\sigma(x_0)$ with exponent α where the Hölder constant depends only on n, N, p, q, Γ, $M := 1 + 2\limsup_{\rho \searrow 0} |(Du)_{x_0,\rho}|$, λ_M, Λ_M, ν_M, and α.*

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