Partial regularity of strong local minimizers of quasiconvex integrals with \((p, q)\)-growth

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We consider strictly quasiconvex integrals

\[
F[u] := \int_\Omega f(Du) \, dx \quad \text{for } u : \mathbb{R}^n \supset \Omega \to \mathbb{R}^N
\]

in the multi-dimensional calculus of variations. For the \(C^2\)-integrand \(f : \mathbb{R}^{Nn} \to \mathbb{R}\) we impose \((p, q)\)-growth conditions

\[
\gamma |\xi|^p \leq f(\xi) \leq \Gamma (1 + |\xi|^q) \quad \text{for all } \xi \in \mathbb{R}^{Nn}
\]

with \(\gamma, \Gamma > 0\) and \(1 < p \leq q < \min \left\{ p + \frac{1}{n}, \frac{2n-1}{2n-2p} \right\}\). Under these assumptions we prove partial \(C^{1,\alpha}_{loc}\)-regularity for strong local minimizers of \(F\) and the associated relaxed functional \(\mathcal{F}\).

1. Introduction

In this paper we investigate the regularity of strong local minimizers of autonomous variational integrals

\[
F[u] := \int_\Omega f(Du) \, dx
\]

defined on vector-valued maps \(u : \Omega \to \mathbb{R}^N, N \geq 1\). Here \(\Omega\) denotes a bounded open set in \(\mathbb{R}^n, n \geq 2\), and \(f : \mathbb{R}^{Nn} \to \mathbb{R}\) is a \(C^2\)-function satisfying suitable assumptions described below.

The existence and the partial regularity of minimizers of \(F\) are classical issues in the modern calculus of variations, and they have been extensively studied, especially over the last twenty years. Specifically, in the present paper we will focus on gradient regularity under the basic assumption that \(f\) is quasiconvex; that is

\[
\int_B f(\xi + D\varphi) \, dx \geq f(\xi)
\]
holds for all $\xi \in \mathbb{R}^n$ and all $\varphi \in C^\infty_c(B;\mathbb{R}^N)$, where $B$ denotes the unit ball in $\mathbb{R}^n$. Quasiconvexity, introduced by Morrey in his seminal paper [39], generalizes the classical convexity assumption in the calculus of variations and has turned out to be a key concept for both, the existence and the partial regularity of minimizers. In addition, the central role of quasiconvexity in nonlinear elasticity has been pointed out in the fundamental work of Ball [5].

Before presenting our theorems let us briefly describe some previous existence and regularity results. Primarily, imposing the standard growth conditions

$$\gamma |\xi|^p \leq f(\xi) \leq \Gamma(1 + |\xi|^p) \quad (1.2)$$

for some $p > 1$, Morrey [39] proved that quasiconvexity is a necessary and sufficient condition for lower semicontinuity of $F$ with respect to weak $W^{1,p}$-convergence (see also [37, 1, 33, 30, 38]). This semicontinuity property is in turn, via the direct method of the calculus of variations, intimately linked to the existence of minimizers of $F$.

As for regularity, classical examples of minimizers with singularities of the gradient can be constructed [15, 41, 49, 50], even for smooth convex functionals and $n = 3$, showing that in the vectorial case everywhere regularity of minimizers in the interior of $\Omega$ does not hold. Therefore, one is led to consider partial regularity, i.e. regularity outside a negligible closed subset of $\Omega$, called the singular set. For quasiconvex functionals and $p \geq 2$ partial $C^{1,\alpha}_{\text{loc}}$-regularity of minimizers has first been shown by Evans [21]; see [23, 25, 2, 4, 17, 31] for extensions and variants. Partial regularity in the subquadratic case $1 < p < 2$ has been eventually proved by Carozza & Fusco & Mingione [13]; see also [48, 16].

Contrary to convex functionals, quasiconvex functionals may — in general — admit non-trivial critical points, i.e. weak solutions of the Euler equation which are not (absolutely) minimizing. Actually, Müller & Sverák [40] have even constructed examples of critical points which are nowhere $C^1$. This result is in sharp contrast to the partial regularity of minimizers and leads to the investigation of an intermediate notion, namely local minimizers of $F$ in the sense of the following definition.

**Definition 1.1** ($W^{1,q}$ local minimizer, [32]). Let $1 \leq p \leq \infty$, $1 \leq q \leq \infty$. A map $\bar{u} \in W^{1,p}(\Omega, \mathbb{R})$ with $F[\bar{u}] < \infty$ is called a $W^{1,q}$ local minimizer of $F$ if there exists some $\delta > 0$ such that

$$F[\bar{u}] \leq F[\bar{u} + \varphi]$$

holds for all $\varphi \in W^{1,q}_{0}(\Omega, \mathbb{R}^N)$ with $\|D\varphi\|_{L^q(\Omega, \mathbb{R}^N)} \leq \delta$. In particular, for $1 \leq q < \infty$ we call $\bar{u}$ a strong local minimizer and for $q = \infty$ a weak local minimizer.

Having introduced this notion it is natural to ask whether non-trivial local minimizers of $F$ exist and if they are still regular or not. Actually, the investigation of the existence (and non-existence) of local minimizers has followed previous developments [26, 27] for critical points, and has, up to now, focussed on the case of $L^1$ local minimizers with affine boundary data. For instance, if the underlying domain $\Omega$ is an annulus in $\mathbb{R}^2$ there exist non-trivial $L^1$ local minimizers [44, 32], while for a starshaped $\Omega$ every $L^1$ local minimizer is already absolutely minimizing [51].
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fact, generalizing these ideas, Taheri [52, 53] has provided multiplicity bounds for local minimizers in terms of topological invariants of \(\Omega\).

Clearly, the examples of \(L^1\) local minimizers are also \(W^{1,q}\) local minimizers for every \(1 \leq \bar{q} \leq \infty\). However, in the light of [32, Section 2] it would be interesting to discuss whether non-trivial examples of \(W^{1,q}\) local minimizers still exist in the simple case that \(\Omega\) is a ball. Moreover, in view of Remark 2.5 below they should ideally possess the additional feature that they are not \(W^{1,p}\) local minimizers. Indeed, for \(\bar{q} > p\) we are not aware of any theoretical obstruction but no such examples seem to be present in the literature.

Assuming standard growth (1.2), the regularity theory for \(W^{1,q}\) local minimizers has been started in a recent interesting work of Kristensen & Taheri [32]. Let us restate their result:

**Theorem 1.2** ([32]). Let \(2 \leq p < \infty, 1 \leq \bar{q} < \infty\). Assume that \(f \in C^2(\mathbb{R}^{Nn})\) is uniformly strictly quasiconvex with (1.2) and that \(\bar{u} \in W^{1,q}_{\text{loc}}(\Omega, \mathbb{R}^N) \cap W^{1,p}(\Omega, \mathbb{R}^N)\) is a \(W^{1,q}\) local minimizer of \(F\). Then there exists an open set \(\Omega_0 \subset \Omega\) with \(|\Omega \setminus \Omega_0| = 0\) such that \(\bar{u} \in C^{1,\alpha}_{\text{loc}}(\Omega_0, \mathbb{R}^N)\) for every \(0 < \alpha < 1\).

A similar result for weak local minimizers is contained in [32] and an analogous statement in the subquadratic case \(1 < p < 2\) has been established in [12]. We stress that in the light of the counterexamples from [40] these theorems treat a borderline case of regularity. Finally, we mention that it would be desirable to remove the technical assumption \(\bar{u} \in W^{1,q}_{\text{loc}}(\Omega, \mathbb{R}^N)\) in Theorem 1.2. However, at present it seems quite hard to achieve this.

Next we turn to a generalization of (1.2). Actually, starting with a series of papers by Marcellini (see e.g. [35, 36]) an increasing interest in more flexible growth conditions than (1.2) has emerged. In this paper we concentrate on the so-called \((p,q)\)-growth conditions

\[ \gamma |\xi|^p \leq f(\xi) \leq \Gamma(1 + |\xi|^q) \]  \hfill (1.3)

with two growth exponents \(1 < p \leq q < \infty\). In the general vectorial setting the regularity theory for (1.3) was started in [42]. Assuming that \(f\) is strictly convex and \(q < \min\{p + 1, \frac{pn}{n - 1}\}\) the authors showed partial \(C^{1,\alpha}_{\text{loc}}\)-regularity of minimizers of \(F\). Subsequently, considering less restrictive conditions on the growth exponents, higher integrability results for the gradient of minimizers have been given in [18, 19], and partial regularity has been established in [9]. Here, the most general condition on the exponents, appearing in [18, 9], reads \(q < \frac{n + 2}{n - 1} p\). For results concerning non-autonomous functionals we refer to [20, 10, 14, 8].

The quasiconvex case is more recent. In [7, 34, 22, 28, 29] the semicontinuity properties in \(W^{1,p}(\Omega, \mathbb{R}^N)\) of quasiconvex functionals satisfying (1.3) have been investigated. For our approach the following notion from [7] has turned out to be crucial:

**Definition 1.3** (\(W^{1,p}\)-Quasiconvexity, [7]). We say that \(f\) is \(W^{1,p}\)-quasiconvex iff

\[ \int_B f(\xi + D\varphi) \, dx \geq f(\xi) \]

holds for all \(\xi \in \mathbb{R}^{Nn}\) and all \(\varphi \in W^{1,p}_0(B; \mathbb{R}^N)\).
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Relying on [7, 22] it has been shown in [46] that (strict) \( W^{1,p} \)-quasiconvexity together with (1.3) and some restrictions on the exponents \( p \) and \( q \) allows to establish both, existence and partial regularity, of absolute minimizers of \( F \). Precisely, the restriction on the exponents reads \( q < \frac{np}{n-1} \) for the existence and

\[
1 < p \leq q < p + \frac{\min\{2,p\}}{2n}
\]

(1.4)

for the regularity. We refer to the forthcoming paper [45] for similar results in the higher order case.

For \( n = N \) an important class of examples is given by the polyconvex integrands

\[
f(\xi) = (1 + |\xi|^2)^{\frac{p}{2}} + h(\det \xi),
\]

(1.5)

where \( h \) is a convex function of growth rate \( \frac{q}{n} \). These integrands are of some interest in non-linear elasticity as pointed out in [5, 6, 7, 34]. Moreover, we recall from [7] that \( f \) from (1.5) is \( W^{1,p} \)-quasiconvex if and only if \( p \geq n \) holds. Thus, in this case the above existence and regularity results apply. Let us mention, at this stage, that polyconvex integrands with a structure related to the one in (1.5) and \( p > n - 1 \), but with a completely different growth behaviour, have previously been treated in [24] by means of more specific methods taking into account the peculiar nature of the functional.

In the case \( p < n \) the integrands (1.5) are not \( W^{1,p} \)-quasiconvex and the above mentioned results do not apply. However, this case, in which \( F \) can potentially admit discontinuous minimizers, is of particular physical interest. To extend the existence and regularity results, a relaxation method, which is closely related to the classical idea of the Lebesgue-Serrin extension, has been introduced in [34, 22, 11, 47]. Precisely, one considers the relaxed functional

\[
F[u] := \inf \left\{ \liminf_{k \to \infty} F[u_k] : \text{\( W^{1,q}_{\text{loc}}(\Omega, \mathbb{R}^N) \ni u_k \rightharpoonup u \text{ weakly in } W^{1,p}(\Omega, \mathbb{R}^N) \)} \right\}
\]

(1.6)

for \( u \in W^{1,p}(\Omega, \mathbb{R}^N) \). It is not hard to see from this definition that the minimum of \( F \) is attained on every Dirichlet class; see [47]. Furthermore, invoking representation results from [22, 11], the approach of [46] has been carried over to minimizers of \( F \). Precisely, assuming that \( f \) is strictly quasiconvex with (1.3) and (1.4) partial regularity of minimizers of \( F \) has been established in [47].

The aim of the present paper is now to examine the regularity properties of local minimizers of quasiconvex functionals satisfying \((p, q)\)-growth conditions. Clearly, our main interest remains in the model case (1.5), where examples of \( L^1 \) local minimizers have been provided in [53, Section 4]. Precisely, our results are the following: If \( f \) is strictly \( W^{1,q} \)-quasiconvex, we prove partial \( C^{1,\alpha}_{\text{loc}} \)-regularity for \( W^{1,q} \) local minimizers \( \tilde{u} \) of \( F \); see Theorem 2.1. In addition, if \( f \) is only strictly quasiconvex, following the approach of [47] we prove a similar regularity theorem for \( W^{1,q} \) local minimizers \( \tilde{u} \) of the relaxed functional \( F \); see Theorem 2.2. In both cases we are mainly interested in the case \( \tau > p \) where, as in [32], we need to impose the technical integrability assumption \( \tilde{u} \in W^{1,\tau}_{\text{loc}}(\Omega, \mathbb{R}^N) \). Note that, as a byproduct, this assumption allows to work with \( W^{1,q} \)-quasiconvexity instead of \( W^{1,p} \)-quasiconvexity in Theorem 2.1. Our results generalize those obtained in [32, 12, 46, 47].
Finally, let us briefly comment on some technical issues: In the subquadratic case $1 < p < 2$ we improve the condition (1.4) replacing it by

\begin{equation}
1 < p \leq q < \min \left\{ \frac{2n}{2n-2} p, p + \frac{1}{n} \right\}.
\end{equation}

Note that in the model case (1.5) with $q = n = N = 2$ the bound (1.7) allows to replace the condition $p > \frac{8}{5}$ from (1.4) by $p > \frac{3}{2}$. However, the reason for this improvement is mainly a technical one. Moreover, we mention that following [32, 12] we use a blow-up argument based on the excess

\begin{equation}
E(x, r) = \int_{B_r(x)} (1 + |D\bar{u} - (D\bar{u})_{x,r}|^2)^{\frac{p-2}{2}} |D\bar{u} - (D\bar{u})_{x,r}|^2 dx
\end{equation}

to prove the partial regularity. In particular, even in the case of absolute minimizers, we provide an alternative proof of the results in [46, 47], where the $\mathcal{A}$-harmonic approximation method has been used.

2. Statement of the results

In this section we state our main results concerning partial regularity of strong local minimizers. Starting with a growth and a coercivity condition we will now supply precise statements of our assumptions.

(H1) $q$-Growth: There exists a bound $\Gamma > 0$ such that we have

\[ 0 \leq f(\xi) \leq \Gamma (1 + |\xi|^q) \quad \text{for every } \xi \in \mathbb{R}^{Nn}. \]

(H2) $p$-Coercivity: There is a coercivity constant $\gamma > 0$ such that we have

\[ f(\xi) \geq \gamma |\xi|^p \quad \text{for every } \xi \in \mathbb{R}^{Nn}. \]

Next we state two quasiconvexity conditions, which will be imposed in Theorem 2.1 and Theorem 2.2, respectively.

(H3) Strict $W^{1,p}$-Quasiconvexity: For each $L > 0$ there is a convexity constant $\nu_L > 0$ such that we have

\[ \int_{B_r(x_0)} (f(\xi + D\varphi) - f(\xi)) \geq \nu_L \int_{B_r(x_0)} (1 + |D\varphi|^2)^{\frac{p-2}{2}} |D\varphi|^2 dx \]

for all balls $B_r(x_0) \subset \mathbb{R}^n$, for all $\xi \in \mathbb{R}^{Nn}$ with $|\xi| \leq L + 1$ and for all $\varphi \in W_0^{1,p}(B_r(x_0), \mathbb{R}^N)$.

(H4) Strict Quasiconvexity: For each $L > 0$ there is a convexity constant $\nu_L > 0$ such that we have

\[ \int_{B_r(x_0)} (f(\xi + D\varphi) - f(\xi)) \geq \nu_L \int_{B_r(x_0)} (1 + |D\varphi|^2)^{\frac{p-2}{2}} |D\varphi|^2 dx \]

for all balls $B_r(x_0) \subset \mathbb{R}^n$, for all $\xi \in \mathbb{R}^{Nn}$ with $|\xi| \leq L + 1$ and for all $\varphi \in C^{\infty}_c(B_r(x_0), \mathbb{R}^N)$.  


Now we present our first main result, a regularity result for strong local minimizers of the functional $F$ defined in (1.1).

**Theorem 2.1.** Let $\bar{q} \in [1, \infty)$ and

$$1 < p \leq q < \min \left\{ p + \frac{1}{n}, \frac{2n - 1}{2n - 2} \right\}.$$  \hfill (2.1)

Assume that $f \in C^2(\mathbb{R}^{Nn})$ satisfies (H1) and (H3) and that $\bar{u} \in W^{1,\bar{q}}_{\text{loc}}(\Omega, \mathbb{R}^N) \cap W^{1,p}(\Omega, \mathbb{R}^N)$ is a $W^{1,\bar{q}}$ local minimizer of $F$ in the sense of Definition 1.1. Then there exists an open set $\Omega_0 \subset \Omega$ with $|\Omega \setminus \Omega_0| = 0$ such that $\bar{u} \in C^{1,\alpha}_{\text{loc}}(\Omega_0, \mathbb{R}^N)$ for every $0 < \alpha < 1$.

Our second main result concerns strong local minimizers of $F$ from (1.6), where we define local minimizers of $F$ along the lines of Definition 1.1 with $F$ replaced by $F$. Before stating the result we recall some properties of the functional $F$: Assuming (H1) with $1 < p \leq q < \min \left\{ p + \frac{1}{n}, \frac{2n - 1}{2n - 2} \right\}$ one has from [22, 47]

$$F[u] \geq \int_{\Omega} Qf(Du) \, dx \quad \text{for } u \in W^{1,p}(\Omega, \mathbb{R}^N),$$  \hfill (2.2)

$$F[u] = \int_{\Omega} Qf(Du) \, dx \quad \text{for } u \in W^{1,q}(\Omega, \mathbb{R}^N),$$  \hfill (2.3)

where $Qf$ denotes the quasiconvex envelope of $f$. Furthermore, it has been shown in [22, 11] that $F$ depends on the domain $\Omega$ like a Radon measure, whose absolutely continuous part has density $Qf(Du)$. These facts will be crucial in the proof of the following result. In particular, we mention that by an argument of [47] they can be used to prove the validity of Euler’s equation for minimizers of $F$. This is an important observation for the proof of the following theorem.

**Theorem 2.2.** Let $\bar{q} \in [1, \infty)$ and

$$1 < p \leq q < \min \left\{ p + \frac{1}{n}, \frac{2n - 1}{2n - 2} \right\}.$$  \hfill (2.4)

Assume that $f \in C^2(\mathbb{R}^{Nn})$ satisfies (H1), (H2) and (H4) and that $\bar{u} \in W^{1,\bar{q}}_{\text{loc}}(\Omega, \mathbb{R}^N) \cap W^{1,p}(\Omega, \mathbb{R}^N)$ is a $W^{1,\bar{q}}$ local minimizer of $F$ on $\Omega$. Then there exists an open set $\Omega_0 \subset \Omega$ with $|\Omega \setminus \Omega_0| = 0$ such that $\bar{u} \in C^{1,\alpha}_{\text{loc}}(\Omega_0, \mathbb{R}^N)$ for every $0 < \alpha < 1$.

We highlight some features of the previous theorems.

**Remark 2.3.** In some sense $W^{1,\bar{q}}$-quasiconvexity is necessary for the existence of $W^{1,\bar{q}}$ local minimizers. Precisely, adapting the proof of [53, Proposition 4.1] one finds: If $\bar{u} \in W^{1,p}(\Omega, \mathbb{R}^N)$ is a $W^{1,\bar{q}}$ local minimizer of $F$ then for every $\varphi \in W^{1,\bar{q}}(B, \mathbb{R}^N)$ there holds

$$\int_{B} f(D\bar{u}(y) + D\varphi(x)) \, dx \geq f(D\bar{u}(y)) \quad \text{for a.e. } y \in \Omega.$$
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Remark 2.4. If \(\tilde{q} \geq q\) then (strict) \(W^{1,q}\)-quasiconvexity is equivalent to (strict) quasiconvexity. Combining this with \((2.3)\), Theorem 2.1 and Theorem 2.2 turn out to be equivalent in this case.

Remark 2.5. In the case \(\tilde{q} \leq p\) it follows from [32, Section 2] that Theorem 2.1 can be reduced to the case of absolute minimizers.

Remark 2.6. Under the additional assumption

\[
\lim sup_{r \to 0^+} \|Du - (Du)_{x,r}\|_{L^\infty(B_r(x),\mathbb{R}^N)} < \delta
\]

Theorem 2.1 and Theorem 2.2 hold also in the case of weak local minimizers, i.e. for \(\tilde{q} = \infty\). This generalization is straightforward along the lines of [32, 12].

Remark 2.7. The proofs of the theorems will show that we can choose \(\Omega_0\) such that

\[
\Omega \setminus \Omega_0 \subset \{ x \in \Omega : \lim inf_{r \to 0^+} E(x,r) > 0 \text{ or } \lim sup_{r \to 0^+} |(Du)_{x,r}| = \infty \}
\]

holds, where \(E(x,r)\) is defined in \((1.8)\).

Remark 2.8. Under the assumptions of Theorem 2.1 or Theorem 2.2, if \(f \in C^{\infty}(\mathbb{R}^N)\) then we have \(u \in C^{\infty}(\Omega_0,\mathbb{R}^N)\). Once \(C^{1,\alpha}_{loc}\)-regularity is proved, this higher regularity result follows from the application of linear theory to the Euler equation.

3. Preliminaries

Throughout this paper we denote by \(a\) a positive constant possibly varying from line to line. The dependences of such constants will only occasionally be highlighted. We write \(B_r(x)\) for the open ball with center \(x\) and radius \(r\) in \(\mathbb{R}^n\) and abbreviate \(B_r := B_r(0)\) and \(B := B_1\). In addition, we will use the common abbreviations

\[
u_{x,r} := \int_{B_r(x)} u \, dx := \frac{1}{|B_r(x)|} \int_{B_r(x)} u \, dx\]

and abbreviate \(r := u_{0,r}\) for mean values, where \(\cdot | \cdot\) denotes the \(n\)-dimensional Lebesgue measure. Moreover, for \(\beta > 0\) we define the functions \(V_\beta : \mathbb{R}^k \to \mathbb{R}^k\) and \(W_\beta : \mathbb{R}^k \to \mathbb{R}^k\) by

\[
V_\beta(\xi) = \left(1 + |\xi|^2\right)^{\frac{\beta - 1}{2}} \xi, \quad W_\beta(\xi) = \left(1 + |\xi|\right)^{\frac{\beta - 1}{2}} \xi
\]

for \(\xi \in \mathbb{R}^k, k \in \mathbb{N}\). Since we are mostly dealing with \(\beta = \frac{p}{2}\), where \(p\) is a fixed exponent, we abbreviate \(V = V_\frac{p}{2}\) and \(W = W_\frac{p}{2}\). Next, we will collect some useful properties of \(V\) and \(W\): Clearly, we have \(|V_\beta(\xi)| = V_\beta(|\xi|)|, |W_\beta(\xi)| = W_\beta(|\xi|)\) and

\[
c^{-1}|W_\beta(\xi)| \leq |V_\beta(\xi)| \leq c|W_\beta(\xi)|, \quad 1 \leq p \leq 2
\]

where \(c\) depends only on \(\beta\). Furthermore, the functions \(V_\beta(t)\) and \(W_\beta(t)\) are both non-decreasing in \(t \geq 0\) and some elementary calculations show that \(|W|^2\) is convex for \(1 \leq p < \infty\) and \(|W|^{\frac{2}{p}}\) is convex for \(1 \leq p \leq 2\) (in contrast to \(|V|^2\) and \(|V|^{\frac{2}{p}}\)). Some additional properties are summarized in the following lemma.
Lemma 3.1. Let $\beta > 0$, $1 < p < \infty$, and $M > 0$. Then, for all $\xi, \eta \in \mathbb{R}^k$ and $t > 0$ we have

(i) $|V(t\xi)| \leq \max\{t, t^p\}|V(\xi)|$;

(ii) $|V_\beta(\xi + \eta)| \leq c(|V_\beta(\xi)| + |V_\beta(\eta)|)$;

(iii) $(1 + |\xi|^2 + |\eta|^2)^{\frac{p}{2}} \leq c(1 + |V(\xi)|^2 + |V(\eta)|^2)$;

(iv) $|V_{p-1}(\xi)||\eta| \leq |V(\xi)|^2 + |V(\eta)|^2$.

Here, $c$ depends only on $\beta$ and $p$, respectively.

Proof. The assertions (i) and (iii) are easy to check. (ii) has been proved for $\frac{1}{2} < \beta < 1$ in [13, Lemma 2.1] and is easily seen to hold for all $\beta > 0$. (iv) follows from the fact that $V_{p-1}$ is non-decreasing. □

Next we restate an integral inequality for $V$; see for instance [46].

Lemma 3.2. Let $1 < p < \infty$ and $u \in W^{1,p}(\Omega, \mathbb{R}^N)$. Then we have

$$\int_{\Omega} |V(Du - (Du)_\Omega)|^2 dx \leq c \int_{\Omega} |V(Du)|^2 dx$$

with a constant $c$ depending only on $p$. □

Furthermore, we recall a Poincaré type inequality and a Sobolev-Poincaré type inequality for $V$.

Lemma 3.3. We consider $1 < p < \infty$, a ball $B_r(x_0)$ in $\mathbb{R}^n$ and a function $u \in W^{1,p}(B_r(x_0), \mathbb{R}^N)$. Then, we have

$$\int_{B_r(x_0)} \left| V\left(\frac{u - u_{x_0,r}}{r}\right)\right|^2 dx \leq c \int_{B_r(x_0)} |V(Du)|^2 dx. \quad (3.3)$$

In addition, setting $p^\# := \frac{2n}{n-p} > 2$ for $1 < p < 2$ we have

$$\left( \int_{B_r(x_0)} \left| V\left(\frac{u - u_{x_0,r}}{r}\right)^{p^\#}\right|^\frac{2}{p^\#} dx \right)^\frac{p}{2} \leq c \left( \int_{B_r(x_0)} |V(Du)|^2 dx \right)^{\frac{p}{2}}. \quad (3.4)$$

The constant $c$ depends only on $n$, $N$ and $p$ in both inequalities. □

Here, (3.4) has been proved in [16, Theorem 2] and (3.3) follows easily from the standard Poincaré inequality for $p \geq 2$ and from (3.4) for $1 < p < 2$. The reader should note that a weaker version of (3.4) has previously been established in [13].

Next we restate some estimates for smoothing operators, which will be crucial for our approach. These estimates, introduced first in [22, Lemma 2.2], have already been used in the regularity theory of integrals with $(p,q)$-growth; see [42, 46, 47]. We state them in the form of [46, Lemma 6.3].
Proof. An elementary proof is given in [22].
Remark 3.6. Assume that $\Xi$ is absolutely continuous and non-decreasing and a set $N \subset \mathbb{R}$ of Lebesgue measure zero is given. Then, we can choose $\tilde{r}$ and $\tilde{s}$ as in Lemma 3.5 even with the additional property $\tilde{r}, \tilde{s} \notin N$; see [47, Lemma 4.6].

Finally, we state another useful lemma concerning the smoothing operator $T_{r,s}$:

Lemma 3.7. Let $1 < p < \infty$, $0 < r < s$ and $B_s(x_0) \subset \Omega$. Then, for $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ we have

\[ |V(DT_{r,s}u)|^2 \leq c_{T_{r,s}} \left[ |V(Du)|^2 \right] \quad \text{a.e. on } \Omega, \tag{3.15} \]

where $c$ depends only on $n$ and $p$.

Proof. Due to (3.2) it suffices to show the claim with $V$ replaced by $W$. Since $|W|^2$ is a non-decreasing and convex function we obtain using (3.5), (3.8) and Jensen’s inequality

\[ |W(DT_{r,s}u)|^2 \leq c |W(T_{r,s}|Du|)|^2 \leq c_{T_{r,s}} |W(Du)|^2 \quad \text{a.e. on } \Omega. \]

This proves the claim. \qed

4. Proof of Theorem 2.1

First we note that the definition (1.8) of the excess reads

\[ E(x,r) = \int_{B_r(x)} |V(D\bar{u} - (D\bar{u})_{x,r})|^2 \, dx \]

in the terminology of Section 3. We will establish a decay estimate for this excess in the following proposition, which we prove by an indirect blow-up argument.

Proposition 4.1. Under the assumptions of Theorem 2.1 for every $L > 0$ there is a constant $C > 0$ with the following property: For each $0 < \tau \leq \frac{1}{2}$ there exists a number $\varepsilon > 0$ such that the conditions

\[ |(D\bar{u})_{x,r}| \leq L, \quad r < \varepsilon \quad \text{and} \quad E(x,r) < \varepsilon \]

for a ball $B_r(x) \subset \subset \Omega$ imply

\[ E(x,\tau r) \leq C\tau^2 E(x,r). \]

Proof. We argue by contradiction. Assuming the proposition to be false there are $L > 0$ and $0 < \tau \leq \frac{1}{2}$, corresponding to a constant $C$ which will be chosen later, such that the following holds: There is a sequence of balls $B_{r_j}(x_j) \subset \subset \Omega$ with $r_j \to 0$ such that

\[ |(D\bar{u})_{x_j,r_j}| \leq L \quad \text{and} \quad 0 < \lambda_j := \sqrt{E(x_j,r_j)} \to 0 \quad \text{as} \quad j \to \infty, \tag{4.1} \]

but

\[ E(x_j,\tau r_j) > C\tau^2 E(x_j,r_j). \tag{4.2} \]
Step 1: Blow-up. We define \( \xi_j := (D\bar{u})_{x_j, r_j} \) and
\[
u_j(y) := \frac{1}{\lambda_j r_j} [\bar{u}(x_j + r_j y) - (\bar{u})_{x_j, r_j} - \xi_j r_j y] \quad \text{for } y \in B.
\]

Then we have
\[
\lambda_j Du_j(y) = D\bar{u}(x_j + r_j y) - \xi_j \quad \text{for } y \in B, \quad (u_j)_{0,1} = 0, \quad (Du_j)_{0,1} = 0
\]
and
\[
\int_B \left| \frac{V(\lambda_j Du_j)}{\lambda_j} \right|^2 dx = 1. \tag{4.3}
\]

Since \( \lambda_j (Du_j)_{0, r} = (D\bar{u})_{x_j, r} - \xi_j \) we get from (4.2)
\[
\lambda_j^{-2} \int_B |V(\lambda_j Du_j - (Du_j)_{0, r})|^2 dx > C r^2. \tag{4.4}
\]

Further we define
\[
f_j(\xi) := f(\xi + D\phi) - f(\xi) - Df(\xi)\lambda_j \xi
\]
for \( \xi \in \mathbb{R}^{Nn} \).

Noting \( |\xi_j| \leq L \) we get from [2, Lemma II.3] that there is a positive constant \( k(L) \) with
\[
|f_j(\xi)| \leq k\lambda_j^{-2} |V_q(\lambda_j \xi)|^2,
\]
\[
|Df_j(\xi)| \leq k\lambda_j^{-1} |V_{q-1}(\lambda_j \xi)| \tag{4.5}
\]
for all \( \xi \in \mathbb{R}^{Nn} \). Moreover, we rewrite the quasiconvexity hypothesis \((H3)\) in the following form:
\[
\nu_L \int_B \left| \frac{V(\lambda_j D\phi)}{\lambda_j} \right|^2 dx \leq \int_B (f_j(\xi + D\phi) - f_j(\xi)) dx \tag{4.6}
\]
for all \( \xi \in \mathbb{R}^{Nn} \) with \( |\lambda_j \xi| \leq 1 \) and for all \( \varphi \in W_0^{1,q}(B, \mathbb{R}^N) \). In addition, setting
\[
F_j[u] = \int_B f_j(Du) dx
\]
the minimizing property of \( \bar{u} \) can be rephrased as follows: For all \( \varphi \in W_0^{1,q}(B, \mathbb{R}^N) \) with
\[
\|D\varphi\|_{L^q(B, \mathbb{R}^{Nn})} \leq \frac{\delta}{\lambda_j r_j^2} \tag{4.7}
\]
we have
\[
F_j[u_j] \leq F_j[u_j + \varphi]. \tag{4.8}
\]
Next we claim
\[ \int_B |Du_j|^{\min\{2,p\}} \, dx \leq c. \quad (4.9) \]
Actually, (4.9) follows immediately from (4.3) for \( p \geq 2 \). In contrast, for \( p \leq 2 \) we first deduce \( \int_B |V(Du_j)|^2 \, dx \leq c \) from (4.3) by virtue of Lemma 3.1 (i) and then get (4.9) by Lemma 3.1 (iii). Thus, passing to subsequences we may assume that for some \( u \in W^{1,\min\{2,p\}}(B,\mathbb{R}^N) \) and some \( \xi_\infty \in \mathbb{R}^{Nn} \) we have
\[
\begin{align*}
& u_j \to u \quad \text{weakly in } W^{1,\min\{2,p\}}(B,\mathbb{R}^N); \\
& u_j \to u \quad \text{strongly in } L^{\min\{2,p\}}(B,\mathbb{R}^N); \\
& \lambda_j Du_j \to 0 \quad \text{a.e. on } B; \\
& \xi_j \to \xi_\infty \quad \text{in } \mathbb{R}^{Nn}. 
\end{align*}
\quad (4.10)
\]

**Step 2: Linearization.** In this step we will show that \( u \) is a weak solution of a linear system. Precisely, we claim
\[
\int_B D^2 f(\xi_\infty)(Du,D\varphi) \, dx = 0 \quad \text{for all } \varphi \in C^1_c(B,\mathbb{R}^N). \quad (4.11)
\]
Actually, the derivation of the limit equation (4.11) is well-known (see for instance [21, 2, 42, 9, 32]) and we will only sketch it. From the minimality property of \( u_j \) in (4.8) we get the following Euler-Lagrange equation:
\[
\int_B Df_j(Du_j)D\varphi \, dx = 0 \quad \text{for all } \varphi \in C^1_c(B,\mathbb{R}^N).
\]
We will show that the preceding equation converges to (4.11) as \( j \to \infty \). Setting
\[
B_j^+ := \{ x \in B : |\lambda_j Du_j(x)| > 1 \}
\]
and using (4.5) and \( q \leq p + 1 \) we obtain
\[
\left| \int_{B_j^+} Df_j(Du_j)D\varphi \, dx \right| \leq \frac{c}{\lambda_j} \int_{B_j^+} |V_{q-1}(\lambda_j Du_j)| \, dx \sup_B |D\varphi| \\
\leq c \sup_B |D\varphi| \lambda_j \int_B |V(\lambda_j Du_j)|^2 \, dx.
\]
By (4.3) we infer that this term vanishes as \( j \to \infty \) and it remains to treat the integral over \( B_j^- := \{ x \in B : |\lambda_j Du_j(x)| \leq 1 \} \). Here, noting \( |B_j^+| \leq c \lambda_j^2 \to 0 \) as in [32] we have
\[
\int_{B_j^-} Df_j(Du_j)D\varphi \, dx = \frac{1}{\lambda_j} \int_{B_j^-} \left( Df(\xi_j + \lambda_j Du_j) - Df(\xi_j) \right) D\varphi \, dx \\
= \int_{B_j^-} \int_0^1 D^2 f(\xi_j + t\lambda_j Du_j) \, dt \, (Du_j, D\varphi) \, dx \\
\to \int_B D^2 f(\xi_\infty)(Du,D\varphi)
\]
and (4.11) follows. The condition (H3) implies that $D^2 f(\xi_\infty)$ is elliptic in the sense of Legendre-Hadamard with ellipticity constant $2\nu_L$ and upper bound $K_L := \sup_{|\xi| \leq L} D^2 f(\xi)$. Thus, we can apply linear theory to deduce that $u$ is $C^1$ on $B$ and

$$\int_{B_r} |Du - (Du)_0|^2 \, dx \leq cr^2$$

(4.12)

is valid. Here, $c$ depends only on $n$, $N$, $\nu_L$ and $K_L$. The remainder of the proof is now mostly devoted to showing

$$\chi_j^{-2} \int_{B_r} |V(\lambda_j (Du_j - Du))|^2 \, dx \to 0 \quad \text{as} \quad j \to \infty.$$  

(4.13)

Once we have proved (4.13) we will see that (4.12) contradicts (4.4).

**Step 3: Construction of test functions and preliminary estimates.** We fix $\tau < \sigma < 1$, $0 < \alpha < 1$ and consider $B_\sigma(x_0) \subset B_\tau$. We define affine functions $a_j(x) := (u_j)_{x_0,r} + (Du_j)_{x_0,r}(x - x_0)$ and set

$$v_j(x) := u_j(x) - a_j(x).$$

Moreover, we introduce the abbreviation

$$\Xi_j(t) := \chi_j^{-2} \int_{B_t(x_0)} \left( |V\left( \frac{\lambda_j v_j}{(1 - \alpha)^r} \right)|^2 + |V(\lambda_j Dv_j)|^2 \right) \, dx$$

and choose for this function $\alpha r \leq \tilde{\tau}_j \leq \tilde{s}_j \leq r$ as in Lemma 3.5. In particular we have

$$\frac{1}{3} (1 - \alpha) r \leq \tilde{s}_j - \tilde{\tau}_j \leq (1 - \alpha) r.$$  

(4.14)

Now we consider smooth cut-off functions $\eta_j : \mathbb{R}^n \to [0, 1]$ which satisfy $\eta_j \equiv 1$ in a neighborhood of $\overline{B_{\tilde{\tau}_j}(x_0)}$, $\eta_j = 0$ in a neighborhood of $\mathbb{R}^n \setminus B_{\tilde{s}_j}(x_0)$ and $|\nabla \eta_j| \leq \frac{2}{\tilde{s}_j - \tilde{\tau}_j}$ on $B_{\tilde{\tau}_j}(x_0)$. We define

$$\chi_j = [(1 - \eta_j)v_j], \quad \psi_j := T_{\tilde{\tau}_j, \tilde{s}_j} \chi_j \quad \text{and} \quad \varphi_j := v_j - \psi_j,$$

where the smoothing operator $T$ is defined in Lemma 3.4. According to (3.6) and (3.7) of Lemma 3.4 we have $\varphi_j \in W^{1,p}(B_{\tilde{s}_j}(x_0), \mathbb{R}^N)$, $\varphi_j = v_j$, $\varphi_j = 0$ on $B_{\tilde{s}_j}(x_0)$ and

$$Du_j - Da_j = D\varphi_j + D\psi_j \quad \text{on} \quad B.$$  

(4.15)

In addition, the product rule and (4.14) give

$$|D\chi_j| \leq |Dv_j| + \frac{|v_j|}{(1 - \alpha) r}.$$  

(4.16)

Next, we will derive two preparatory estimates for $\chi_j$, namely (4.17) and (4.18): Setting

$$Y_j := \chi_j^{-2} \int_{B_{\tilde{r}_j}(x_0) \setminus B_{\tilde{s}_j}(x_0)} \left( |V\left( \frac{\lambda_j v_j}{(1 - \alpha)^r} \right)|^2 + |V(\lambda_j Dv_j)|^2 \right) \, dx.$$
we apply in turn (3.8), Lemma 3.7, (3.9) (with $p = 1$), (4.16) and Lemma 3.1 (ii) to get the estimate
\[
\lambda_j^{-2} \int_{B_{r_j}(x_0) \setminus B_{r_j}(x_0)} |V(\lambda_j D\psi_j)|^2 \, dx \leq \frac{c\lambda_j^{-2}}{B_{r_j}(x_0) \setminus B_{r_j}(x_0)} \int_{B_{r_j}(x_0) \setminus B_{r_j}(x_0)} |T_{t_j, \tilde{s}_j}[|V(\lambda_j D\chi_j)|^2]| \, dx \\
\leq \frac{c\lambda_j^{-2}}{B_{r_j}(x_0) \setminus B_{r_j}(x_0)} \int_{B_{r_j}(x_0) \setminus B_{r_j}(x_0)} |V(\lambda_j D\chi_j)|^2 \, dx \leq c Y_j, \tag{4.17}
\]
Arguing in a similar way, but using (3.11) (with $p = 1$, $q = \kappa$) instead of (3.9) we find for $1 \leq \kappa < \frac{2n-1}{n-1}$:
\[
\lambda_j^{-2} \int_{B_{r_j}(x_0) \setminus B_{r_j}(x_0)} |V(\lambda_j D\psi_j)|^{2\kappa} \leq \frac{c\lambda_j^{-2}}{B_{r_j}(x_0) \setminus B_{r_j}(x_0)} \int_{B_{r_j}(x_0) \setminus B_{r_j}(x_0)} |T_{t_j, \tilde{s}_j}[|V(\lambda_j D\chi_j)|^2]|^{\kappa} \, dx \\
\leq \frac{c\lambda_j^{-2}}{B_{r_j}(x_0) \setminus B_{r_j}(x_0)} \int_{B_{r_j}(x_0) \setminus B_{r_j}(x_0)} |V(\lambda_j D\chi_j)|^2 \, dx + \frac{1}{B_{r_j}(x_0) \setminus B_{r_j}(x_0)} \int_{B_{r_j}(x_0) \setminus B_{r_j}(x_0)} |V(\lambda_j D\chi_j)|^2 \, dx \\
\leq \frac{c\lambda_j^{-2}}{B_{r_j}(x_0) \setminus B_{r_j}(x_0)} \int_{B_{r_j}(x_0) \setminus B_{r_j}(x_0)} |V(\lambda_j D\chi_j)|^2 \, dx \leq c Y_j. \tag{4.18}
\]

**Step 4: The main estimate.** In this step we will combine ideas of [32] and [46] to establish a key estimate. This estimate will lead to (4.13) later in the proof. Here, our first aim is to verify (4.7) for $\varphi_j$ with $j$ large, which will enable us to use (4.8). We start with the following computation and use for this purpose (4.14), formula (3.9) from Lemma 3.4 and the Poincaré inequality:
\[
\int_B |D\varphi_j|^q \, dx \leq \int_{B_{r_j}(x_0) \setminus B_{r_j}(x_0)} |D\varphi_j|^q \, dx + \int_{B_{r_j}(x_0) \setminus B_{r_j}(x_0)} |D\psi_j|^q \, dx \\
\leq \int_{B_{r_j}(x_0) \setminus B_{r_j}(x_0)} |D\varphi_j|^q \, dx + c \int_{B_{r_j}(x_0) \setminus B_{r_j}(x_0)} |D\chi_j|^q \, dx \\
\leq \int_{B_{r_j}(x_0) \setminus B_{r_j}(x_0)} |D\varphi_j|^q \, dx + c \left( \int_{B_{r_j}(x_0) \setminus B_{r_j}(x_0)} |D\varphi_j|^q \, dx + \int_{B_{r_j}(x_0) \setminus B_{r_j}(x_0)} \left( \frac{|v_j|}{(1 - \alpha)r} \right)^q \, dx \right) \\
\leq \int_{B_{r_j}(x_0) \setminus B_{r_j}(x_0)} |D\varphi_j|^q \, dx + c \left( \int_{B_{r_j}(x_0) \setminus B_{r_j}(x_0)} |D\varphi_j|^q \, dx + \int_{B_{r_j}(x_0) \setminus B_{r_j}(x_0)} \left( \frac{|v_j|}{(1 - \alpha)r} \right)^q \, dx \right)
\]
\[ \leq c\left(1 + \frac{1}{1 - \alpha}\right)^{\frac{q}{r}} \int_{B_r(x_0)} |Dv_j|^q \, dx. \]

Changing coordinates in view of \( B_{rr_j}(x_0 + r_j x_0) \subset B_{r_j}(x_j) \) we obtain

\[ \|D\varphi_j\|_{L^q(B,\mathbb{R}^n)} \leq c \frac{r_j}{\lambda_j} \left(1 + \frac{1}{1 - \alpha}\right) \left( \int_{B_{r_j}(x_j)} |D\tilde{u}|^q \, dx \right)^{\frac{q}{r}}. \]

Therefore the condition (4.7) is fulfilled if

\[ c\left(1 + \frac{1}{1 - \alpha}\right) \left( \int_{B_{r_j}(x_j)} |D\tilde{u}|^q \, dx \right)^{\frac{q}{r}} \leq \delta \]

holds and this is satisfied for sufficiently large \( j \), say for \( j \geq j_1(\alpha) \). Furthermore, from the definition of \( a_j \) and (4.9) we see \( |Da_j| \leq c \rho^a \). Hence, there exists a \( j_2(\rho) \) such that \( |\lambda_j Da_j| \leq 1 \) holds for all \( j \geq j_2(\rho) \). We define \( j_0(\alpha, \rho) := \max\{j_1(\alpha), j_2(\rho)\} \). Then, for \( j \geq j_0(\alpha, \rho) \) we use (4.6), (4.8) and (4.15) to get

\[
\nu_{L} \int_{B_{r_j}(x_0)} \left| \frac{V(\lambda_j Dv_j)}{\lambda_j} \right|^2 \, dx \leq \nu_{L} \int_{B_{r_j}(x_0)} \left| \frac{V(\lambda_j D\varphi_j)}{\lambda_j} \right|^2 \, dx
\]

\[
\leq \int_{B_{r_j}(x_0)} \left( f_j(Da_j + D\varphi_j) - f_j(Da_j) \right) \, dx
\]

\[
= \int_{B_{r_j}(x_0)} \left( f_j(Du_j - D\psi_j) - f_j(Du_j) \right) \, dx + \int_{B_{r_j}(x_0)} \left( f_j(Du_j) - f_j(Du_j - D\varphi_j) \right) \, dx
\]

\[
+ \int_{B_{r_j}(x_0)} \left( f_j(Da_j + D\psi_j) - f_j(Da_j) \right) \, dx
\]

\[
\leq \int_{B_{r_j}(x_0)} \left( f_j(Du_j - D\psi_j) - f_j(Du_j) \right) \, dx + \int_{B_{r_j}(x_0)} \left( f_j(Da_j + D\psi_j) - f_j(Da_j) \right) \, dx.
\]

Recalling \( \psi_j = 0 \) on \( B_{r_j}(x_0) \) we estimate the right-hand side by inequality (4.5) and Lemma 3.1 (ii):

\[
\nu_{L} \int_{B_{r_j}(x_0)} \left| \frac{V(\lambda_j Dv_j)}{\lambda_j} \right|^2 \, dx \leq \nu_{L} \int_{B_{r_j}(x_0)} \left| \frac{V(\lambda_j Df_j(Da_j + tD\psi_j) - Df_j(Da_j - tD\psi_j))}{\lambda_j} \right| \, dt \, dx
\]

\[
\leq c \int_{B_{r_j}(x_0) \setminus B_{r_{j+1}}(x_0)} \left( \left| \frac{V_{q-1}(\lambda_j Da_j)}{\lambda_j} \right| |D\psi_j| + \left| \frac{V_{q-1}(\lambda_j Dv_j)}{\lambda_j} \right| |D\psi_j| + \left| \frac{V_{q}(\lambda_j D\psi_j)}{\lambda_j} \right|^2 \right) \, dx
\]

\[
=: c (I + II + III)
\]
with the obvious labeling.

**Estimation of III.** We estimate the last integral by Lemma 3.1 (iii), (4.17) and (4.18) with \( \kappa = \frac{2}{p} < \frac{n}{n+1} = 1 + \frac{1}{n-1} : \)

\[
III = \int_{B_{r_j}(x_0) \setminus B_{r_j}(x_0)} \left| \frac{V_2(\lambda_j D\psi_j)}{\lambda_j} \right|^2 dx
\]

\[
\leq \lambda_j^{-2} \int_{B_{r_j}(x_0) \setminus B_{r_j}(x_0)} (1 + |\lambda_j D\psi_j|^2)^{\frac{p}{2} - \frac{2}{p}} |V(\lambda_j D\psi_j)|^2 dx
\]

\[
\leq c \lambda_j^{-2} \int_{B_{r_j}(x_0) \setminus B_{r_j}(x_0)} (1 + |V(\lambda_j D\psi_j)|^2)^{\frac{p}{2} - \frac{2}{p}} |V(\lambda_j D\psi_j)|^2 dx
\]

\[
\leq c \lambda_j^{-2} \int_{B_{r_j}(x_0) \setminus B_{r_j}(x_0)} \left( |V(\lambda_j D\psi_j)|^2 + |V(\lambda_j D\psi_j)|^\frac{p}{2} \right) dx
\]

\[
\leq c \left( Y_j + \left( \frac{\lambda_j^2 Y_j}{(1-\alpha)^{\frac{2}{p} - \frac{2}{p}}} \right)^\frac{p}{2} Y_j \right) \leq c \left( Y_j + \left( \frac{\lambda_j^2 Y_j}{(1-\alpha)^{\frac{2}{p} - \frac{2}{p}}} \right)^\frac{p}{2} Y_j \right).
\]

**Estimation of II.** We estimate the integral II distinguishing the cases \( p > \frac{2n}{n-1} \) and \( p \leq \frac{2n}{n-1} \).

**Case** \( p > \frac{2n}{n-1} \): In this case (2.1) reads \( q < p + \frac{1}{p} \) and \( \frac{2}{q} + 1 < p + \frac{1}{p} \). Hence, enlarging \( q \) if necessary we may assume \( q \geq \frac{2}{q} + 1 \) (without destroying \( q < p + \frac{1}{p} \)). Next we give a pointwise estimation of the integrand in II. For \( |\lambda_j D\psi_j| \leq 1 \) we obtain with Young’s inequality

\[
(1 + |\lambda_j D\psi_j|^2)^{\frac{p}{2} - \frac{2}{p}} |D\psi_j| \leq (1 + |\lambda_j D\psi_j|^2)^{\frac{p}{2} - \frac{2}{p}} |\lambda_j D\psi_j| \leq c \left( |V(\lambda_j D\psi_j)|^2 + |\lambda_j D\psi_j|^\frac{p}{2} \right).
\]

while for \(|\lambda_j D\psi_j| > 1\) a similar computation yields

\[
(1 + |\lambda_j D\psi_j|^2)^{\frac{p}{2} - \frac{2}{p}} |D\psi_j| \leq (1 + |\lambda_j D\psi_j|^2)^{\frac{p}{2} - \frac{2}{p}} |\lambda_j D\psi_j| \leq c \left( |V(\lambda_j D\psi_j)|^2 + |\lambda_j D\psi_j|^\frac{p}{2} \right).
\]

Thus, using \( \frac{p}{p+1-q} \geq 2, q < p + \frac{1}{p}, (4.17) \) and (4.18) (with \( \kappa = \frac{1}{p+1-q} < \frac{n}{n-1} \)) we argue essentially as for III:

\[
II \leq c \int_{B_{r_j}(x_0) \setminus B_{r_j}(x_0)} \left| \frac{V(\lambda_j D\psi_j)}{\lambda_j} \right|^2 dx + c \lambda_j^{-2} \int_{B_{r_j}(x_0) \setminus B_{r_j}(x_0)} \left( |\lambda_j D\psi_j|^2 + |\lambda_j D\psi_j|^\frac{p}{2} \right) dx.
\]
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\[ \leq c \int_{B_{r_j}(x_0) \setminus B_r(x_0)} \left( \left| V(\lambda_j Dv_j) \right|^2 + \left| V(\lambda_j D\psi_j) \right|^2 + \lambda_j^{-2} |V(\lambda_j D\psi_j)|^{q+1} \right) dx \]

\[ \leq c \left( Y_j + \left( \frac{\lambda_j^2 Y_j}{((1-\alpha)r)^\alpha} \right)^\frac{2+q-p}{p-1} Y_j \right) \leq c \left( Y_j + \left( \frac{\lambda_j^2 Y_j}{((1-\alpha)r)^\alpha} \right)^\frac{2-1}{p} Y_j \right). \]

\section*{Case \(p \leq 2^{\frac{n-1}{n}}\)}: In this case (2.1) reads \(2^{\frac{n-1}{n}} \leq p\) and we have, in particular, \(q \leq \frac{2n}{p} + 1\). Again we will give an estimate for the integrand in \(II\). In the case \(|Dv_j| \leq |D\psi_j|\) we find since \(V_{q-1}\) is non-decreasing

\[ \left| V_{q-1}(\lambda_j Dv_j) \right| |D\psi_j| \leq \left| V_{\frac{2}{p}}(\lambda_j D\psi_j) \right|^2 \]

and in the case \(|Dv_j| > |D\psi_j|\) we get with Young’s inequality and \(q \leq \frac{2n}{p} + 1\)

\[ \left| V_{q-1}(\lambda_j Dv_j) \right| |D\psi_j| \leq \lambda_j^{-2} (1 + |\lambda_j Dv_j|^2) ^{\frac{2+q-p}{2}} |\lambda_j Dv_j||\lambda_j D\psi_j| \]

\[ \leq \lambda_j^{-2} |V_{\frac{2n}{p}}(\lambda_j D\psi_j)||V(\lambda_j Dv_j)| \]

\[ \leq \lambda_j^{-2} \left( |V_{\frac{2n}{p}}(\lambda_j D\psi_j)|^2 + |V(\lambda_j Dv_j)|^2 \right). \]

Arguing essentially as supplied above and using \(\frac{2}{p} \leq \frac{2n}{p} < \frac{n}{n-1}\) we derive the following estimate for \(II\) in this case:

\[ I \leq c \left( Y_j + \left( \frac{\lambda_j^2 Y_j}{((1-\alpha)r)^\alpha} \right)^\frac{2+q-p}{p-1} Y_j \right) \leq c \left( Y_j + \left( \frac{\lambda_j^2 Y_j}{((1-\alpha)r)^\alpha} \right)^\frac{2-1}{p} Y_j \right). \]

\section*{Estimation of \(I\).} It remains to control \(I\). Here, employing \(|\lambda_j Da_j| \leq 1\) for \(j \geq j_0(\alpha, r)\), Lemma 3.1 (iv) and (4.17) we get

\[ I \leq c \int_{B_{r_j}(x_0) \setminus B_r(x_0)} \left( \left| V(\lambda_j Dv_j) \right|^2 + \left| V(\lambda_j D\psi_j) \right|^2 \right) dx \]

\[ \leq c \left( \int_{B_{r_j}(x_0) \setminus B_r(x_0)} \left| V(\lambda_j Da_j) \right|^2 dx + Y_j \right). \]

Collecting the estimates for \(I, II\) and \(III\) we have proved

\[ \int_{B_{r_j}(x_0)} \left| V(\lambda_j Dv_j) \right|^2 dx \leq c \left( Y_j + \left( \frac{\lambda_j^2 Y_j}{((1-\alpha)r)^\alpha} \right)^\frac{2-1}{p} Y_j \right) + c \int_{B_{r_j}(x_0) \setminus B_{r_j}(x_0)} \left| V(\lambda_j Da_j) \right|^2 dx. \]
By the Poincaré-type inequality (3.3), Lemma 3.1 (i), Lemma 3.2 and (4.3) we have

\[
Y_j \leq \lambda_j^{-2} \int_{B_r(x_0)} \left( |V(\lambda_j Du_j)|^2 + \left| V\left( \frac{\lambda_j v_j}{(1-\alpha)r} \right) \right|^2 \right) dx \\
\leq \left( 1 + \frac{c}{(1-\alpha)^{\max(2,p)}} \right) \int_{B_r(x_0)} \left| V(\lambda_j Du_j) \right|^2 dx \leq \frac{c}{(1-\alpha)^{\max(2,p)}}.
\]

Combining the last two inequalities we find

\[
\int_{B_{\alpha r}(x_0)} \left| V(\lambda_j Du_j) \right|^2 dx \leq c \left( Y_j + \int_{B_r(x_0) \setminus B_{\alpha r}(x_0)} \left| \frac{V(\lambda_j Da_j)}{\lambda_j} \right|^2 dx \right) + c_{\alpha,r} \lambda_j^{2-\tau},
\]

where \( c_{\alpha,r} > 0 \) is a fixed constant depending, in particular, on \( \alpha \) and \( r \). The reader should note that, contrarily, the constants \( c \) in the preceding estimates do not depend on \( \alpha \) or \( r \). Applying Lemma 3.1 (ii) we deduce

\[
\int_{B_{\alpha r}(x_0)} \left| \frac{V(\lambda_j (Du_j - Du))}{\lambda_j} \right|^2 dx \leq c \left( Y_j + \int_{B_r(x_0) \setminus B_{\alpha r}(x_0)} \left| \frac{V(\lambda_j Da_j)}{\lambda_j} \right|^2 dx \right) + c_{\alpha,r} \lambda_j^{2-\tau}.
\]

By Widman’s hole filling trick, that is adding \( c \int_{B_{\alpha r}(x_0)} \left| \frac{V(\lambda_j (Du_j - Du))}{\lambda_j} \right|^2 dx \) on both sides, we finally arrive at the main estimate

\[
\int_{B_{\alpha r}(x_0)} \left| \frac{V(\lambda_j (Du_j - Du))}{\lambda_j} \right|^2 dx \leq \theta \int_{B_r(x_0)} \left| \frac{V(\lambda_j (Du_j - Du))}{\lambda_j} \right|^2 dx + \int_{B_r(x_0)} \left| \frac{V(\lambda_j Da_j)}{\lambda_j} \right|^2 dx \\
+ \int_{B_r(x_0) \setminus B_{\alpha r}(x_0)} \left| \frac{V(\lambda_j Da_j)}{\lambda_j} \right|^2 dx + \lambda_j^{-2} \int_{B_r(x_0)} \left| \frac{\lambda_j v_j}{(1-\alpha)r} \right|^2 dx + c_{\alpha,r} \lambda_j^{2-\tau},
\]

for \( j \geq j_0(\alpha,r) \) with \( \theta = \frac{c}{1+c} < 1 \). We stress that \( \theta \) does not depend on \( \alpha \) or \( r \).

**Step 5: Strong convergence.** Recalling that \( u \) is \( C^1 \) on \( B \) it follows from (4.3) that \( \lambda_j^{-2} \int_{B_r} |V(\lambda_j (Du_j - Du))|^2 dx \) remains bounded as \( j \to \infty \). Thus, there exists
we have
\[ \lambda_j^{-2} |V(\lambda_j(Du_j - Du))|^2 \mathcal{L}^n \rightharpoonup \mu \quad \text{weakly in the sense of measures on } \overline{B}_\sigma. \]

Introducing the affine function \( a(x) := (u)_{x_0,r} + (Du)_{x_0,r}(x - x_0) \) we obviously have \( a_j \rightharpoonup a \) and \( \lambda_j^{-1} V(\lambda_j Da_j) \rightharpoonup Da \). Furthermore, setting \( v := u - a \) we claim
\[
\int_{B_r(x_0)} \lambda_j^{-2} \left| V \left( \frac{\lambda_j(v_j - v)}{(1 - \alpha)r} \right) \right|^2 dx \to 0.
\]

For \( 1 < p < 2 \) we will now prove (4.21) following an argument of [13]: We choose \( t \in (0, 1) \) such that \( 1 \leq t + \frac{1 - t}{p^*} \) holds (recall \( p^* = \frac{2n}{n - p} > 2 \)) and apply the interpolation inequality and the Sobolev type inequality (3.4) to get
\[
\int_{B_r(x_0)} \left| \frac{V(\lambda_j(v_j - v))}{\lambda_j} \right|^2 dx \leq \lambda_j^{2(t-1)} \left( \int_{B_r(x_0)} |v_j - v|^p dx \right)^{2t} \left( \int_{B_r(x_0)} |V(\lambda_j(v_j - v))|^{p^*} dx \right)^{2(1-t)}.
\]

By (4.3) and (4.10) the right-hand side converges to 0 for \( j \to \infty \) and (4.21) is verified for \( p < 2 \). For \( p \geq 2 \), (4.21) follows from (4.10) and (4.3) by a simpler argument and we omit further details.

Returning to the general case, for every measurable subset \( A \) of \( \overline{B}_\sigma \) we have
\[
\mu(\text{int}A) \leq \liminf_{j \to \infty} \int_A \lambda_j^{-2} |V(\lambda_j(Du_j - Du))|^2 dx
\]
\[
\leq \limsup_{j \to \infty} \int_A \lambda_j^{-2} |V(\lambda_j(Du_j - Du))|^2 dx \leq \mu(A).
\]

Keeping this in mind and passing to the limit in (4.20) we obtain
\[
\mu(B_{\sigma}(x_0)) \leq \theta \mu(B_r(x_0)) + \int_{B_r(x_0)} |Dv|^2 dx + (1 - \alpha^n) r^n |Da|^2 + c \int_{B_r(x_0)} \frac{v}{(1 - \alpha)r}^2 dx.
\]

Here, for the treatment of the fourth term on the right-hand side we have used Lemma 3.1 (ii) and (4.21). Since \( 0 < \alpha < 1 \) is arbitrary we can, by virtue of a continuity argument, replace \( \mu(B_{\sigma}(x_0)) \) by \( \mu(B_{\sigma}(x_0)) \) on the left-hand side of the previous inequality. Hence, dividing by \( r^n \) we have established
\[
\alpha^n \frac{\mu(B_{\sigma}(x_0))}{\alpha^{\theta r^n}} \leq \theta \frac{\mu(B_r(x_0))}{r^n} + \varepsilon_1(r) + |Da|^2 (1 - \alpha^n) + \frac{\varepsilon_2(r)}{(1 - \alpha)^2},
\]
where we have abbreviated
\[ \varepsilon_1(r) = \frac{1}{r^n} \int_{B_r(x_0)} |Dv|^2 \quad \text{and} \quad \varepsilon_2(r) = \frac{c}{r^{n+2}} \int_{B_r(x_0)} |v|^2 \, dx. \]

Since \( u \) is \( C^1 \), we have \( \varepsilon_1(r) + \varepsilon_2(r) \to 0 \) and \( Da \to Du(x_0) \) as \( r \to 0^+ \). Next we claim
\[ \lim_{r \to 0^+} \mu(B_r(x_0)) r^n = 0. \] (4.23)

To prove (4.23), following [32], we first suppose \( \limsup_{r \to 0^+} \mu(B_r(x_0)) > 0 \). Then, by an argument of [32, p. 78-79] we can pass \( r \to 0^+ \) in (4.22) arriving at
\[ \alpha^n \leq \theta + |Du(x_0)|^2 (1 - \alpha^n) \limsup_{r \to 0^+} \frac{r^n}{\mu(B_r(x_0))} \]
for all \( 0 < \alpha < 1 \). Thus, passing \( \alpha \to 1^- \) (recall that \( \theta < 1 \) is independent of \( \alpha \)) we get (4.23) in any case and for all \( x_0 \in B_\sigma \). Hence, following [32] again, by Vitali’s covering theorem we deduce
\[ \mu(B_r) = 0, \]
which, in turn, implies the strong convergence stated in (4.13).

**Step 6: Conclusion.** Noting \((Du_j)_{0,\tau} \to (Du)_{0,\tau}\) we deduce from Lemma 3.1 (ii), (4.12) and (4.13)
\[ \lim_{j \to \infty} \frac{1}{\lambda_j^2} \int_{B_r} |V(\lambda_j(Du_j - (Du_j)_{0,\tau}))|^2 \, dx \]
\[ \leq \lim_{j \to \infty} \frac{c}{\lambda_j^2} \int_{B_r} \left[ |V(\lambda_j(Du_j - Du))|^2 + |V(\lambda_j(Du - (Du)_0,\tau))|^2 \right. \]
\[ + \left. |V(\lambda_j((Du)_{0,\tau} - (Du)_{0,\tau}))|^2 \right] \, dx \]
\[ = c \int_{B_r} |Du - (Du)_{0,\tau}|^2 \, dx \leq C^* r^2 \]
for some constant \( C^* > 0 \). Finally, the last inequality contradicts (4.4) if we choose \( C = C^* + 1 \) and the proof is finished. The reader should note that \( C \) and \( C^* \) depend only on \( n, N, p, \nu_L \) and \( K_L \).

Once Proposition 4.1 is established, Theorem 2.1 follows by a well-known iteration argument and Campanato’s integral characterization of Hölder continuity. For further details see for instance [21, 13].

5. **Proof of Theorem 2.2**

In this section we present the proof of Theorem 2.2, modifying the proof of Theorem 2.1 along the lines of [47].
First we recall some simple estimates for the non-degenerate $p$-energy
\[ e_p(\xi) := (1 + |\xi|^2)^{\frac{p}{2}}, \tag{5.1} \]
see e. g. [47] for a proof.

**Lemma 5.1.** For $1 < p < \infty$, $L > 0$, $\xi \in \mathbb{R}^n$ with $|\xi| \leq L + 1$, a ball $B_r(x_0)$ in \( \mathbb{R}^n \) and $\varphi \in W^{1,p}_0(B_r(x_0), \mathbb{R}^N)$ we have
\[ C_1^{-1} \int_{B_r(x_0)} |V(D\varphi)|^2 \, dx \leq \int_{B_r(x_0)} [e_p(\xi + D\varphi) - e_p(\xi)] \, dx \leq C_1 \int_{B_r(x_0)} |V(D\varphi)|^2 \, dx \]
for some constant $C_1 > 0$ depending only on $p$ and $L$.

In the following lemmas we collect several properties of the relaxed functional $\mathcal{F}$. These lemmas have been proposed in [47] and rely heavily on (2.2), (2.3) and the measure and integral representation results obtained in [22, 11]. We mention that later in this section we will also apply (2.3) and the measure representation result [22, Theorem 3.1] explicitly. Next we give a reformulation of [47, Lemma 7.1]. Note that the growth condition imposed on $Df$ in [47] follows from (H1) and the quasiconvexity of $f$.

**Lemma 5.2.** We suppose that $f \in C^1$ is quasiconvex with (H1) and $1 < p \leq q < \min\{p + 1, \frac{np}{n-1}\}$ Then, for $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ with $F[u] < \infty$ and $\psi \in W^{1,\frac{p}{p-1}}(\Omega, \mathbb{R}^N)$ we have
\[ \mathcal{F}[u + \psi] - \mathcal{F}[u] = F[u + \psi] - F[u]. \]

As in [47, Lemma 7.3] we see that Lemma 5.2 implies the validity of Euler’s equation for $W^{1,q}$ local minimizers of $\mathcal{F}$.

**Lemma 5.3** (Euler’s equation). Let $1 \leq q \leq \infty$. We suppose that $f \in C^1$ is quasiconvex with (H1) and $1 < p \leq q < \min\{p + 1, \frac{np}{n-1}\}$ Then, every $W^{1,q}$ local minimizer $\bar{u} \in W^{1,q}(\Omega, \mathbb{R}^N)$ of $\mathcal{F}$ is a weak solution of the Euler equation of $F$, i.e.
\[ \int_{\Omega} Df(D\bar{u})D\varphi \, dx = 0 \quad \text{for all } \varphi \in C_c^\infty(\Omega, \mathbb{R}^N). \]

Now we introduce the additional notation
\[ \mathcal{F}[u; O] := \inf \left\{ \liminf_{k \to \infty} \int_O f(Du_k) \, dx : W^{1,q}_{\text{loc}}(O, \mathbb{R}^N) \ni u_k \rightharpoonup u \text{ weakly in } W^{1,p}(O, \mathbb{R}^N) \right\} \]
for open subsets $O$ of $\Omega$. We will need the next two lemmas, which can also be found in [47].

**Lemma 5.4** ($W^{1,p}$-quasiconvexity). Assume (H1) and (H2) with $1 < p \leq q < \frac{np}{n-1}$. Then, the following $W^{1,p}$-quasiconvexity condition holds for $\mathcal{F}$: For every ball $B_r(x_0)$ in $\mathbb{R}^n$, every $\xi \in \mathbb{R}^n$ and every $\varphi \in W^{1,p}(B_r(x_0), \mathbb{R}^N)$ with compact support in $B_r(x_0)$ we have
\[ \mathcal{F}[l_\xi + \varphi; B_r(x_0)] \geq \mathcal{F}[l_\xi; B_r(x_0)], \tag{5.2} \]
where we have set $l_\xi(x) := \xi x$. 
Lemma 5.5 (Additivity property). Assume (H1) and (H2) with $1 < p \leq q < \frac{np}{n-1}$.
We consider a ball $B_s(x_0) \subset \subset \Omega$ and $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ such that the boundary regularity condition
\[
\limsup_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{B_{s+\varepsilon}(x_0) \setminus B_{s-\varepsilon}(x_0)} |Du|^p dx < \infty
\] (5.3)
holds. Then we have
\[
\mathcal{F}[u; \Omega] = \mathcal{F}[u; B_s(x_0)] + \mathcal{F}[u; \Omega \setminus B_s(x_0)].
\]

After these preparations we turn to the proof of Theorem 2.2: As for Theorem 2.1 it suffices to establish the following proposition, whose statement is completely analogous to Proposition 4.1.

Proposition 5.6. Under the assumptions of Theorem 2.2 for every $L > 0$ there is a constant $C > 0$ with the following property: For each $0 < \tau \leq \frac{1}{2}$ there exists a number $\varepsilon > 0$ such that the conditions
\[
|(D\bar{u})_{x,r}| \leq L, \quad r < \varepsilon \quad \text{and} \quad E(x,r) < \varepsilon
\]
for a ball $B_r(x) \subset \subset \Omega$ imply
\[
E(x,\tau r) < C\tau^2 E(x,r).
\]

Sketch of proof. We argue by contradiction. Assuming the proposition to be wrong we proceed by blow-up as for Proposition 4.1 and we will highlight only the necessary modifications in the proof. First we note that by Lemma 5.3 the Euler equation used in Step 2 is available. Thus, the remaining modifications, which will be outlined now, concern only the handling of the quasiconvexity hypothesis and the minimizing property. We use the nomenclature of the proof of Proposition 4.1 up to the following difference: We choose $\tilde{r}_j, \tilde{s}_j$ as in Remark 3.6 avoiding the set
\[
N_j := \left\{ t \in [\alpha r, r] : t \mapsto \int_{B_{\tilde{r}_j}(x_0)} |Du_j|^p dx \text{ is not differentiable at } t \right\}.
\]
Thus, $u_j$ satisfies the condition (5.3) near $\partial B_{\tilde{s}_j}(x_0)$. As explained in the proof of [47, Lemma 7.13] it is easy to see that the same condition holds also for $a_j + \varphi_j$ and $u_j - \varphi_j$. We will use this fact later when applying Lemma 5.5. Next we will rewrite the quasiconvexity hypothesis (H4) in an adequate form for our purposes. To this aim we introduce the auxiliary integrand
\[
g(\xi) := f(\xi) - \frac{\nu \ell}{C_1} e_p(\xi) \quad \text{for } \xi \in \mathbb{R}^{Nn}.
\]
where $e_p$ is defined in (5.1) and $C_1, \gamma$ and $\nu_L$ denote the constants from Lemma 5.1, (H2) and (H4). Moreover, for $W^{1,p}$-functions $w$ we set
\[
G[w] := \int_{\Omega} g(Dw) dx,
\]
\[
G[w] := \inf \left\{ \liminf_{k \to \infty} G[w_k] : W^{1,q}_{\text{loc}}(\Omega, \mathbb{R}^N) \ni w_k \rightharpoonup w \text{ weakly in } W^{1,p}(\Omega, \mathbb{R}^N) \right\},
\]
\[
F_j[w] := \inf \left\{ \liminf_{k \to \infty} F_j[w_k] : W^{1,q}_{\text{loc}}(B, \mathbb{R}^N) \ni w_k \rightharpoonup w \text{ weakly in } W^{1,p}(B, \mathbb{R}^N) \right\},
\]
and we will also use the obvious modifications of these notations for open subsets \( O \) of \( \Omega \) and \( B \), respectively. Following an argument from the proof of [47, Lemma 7.13] it is not hard to see from the definitions of \( \mathcal{F} \) and \( \mathcal{G} \) that we have

\[
\mathcal{G}[w; O] \leq \mathcal{F}[w; O] - \frac{\nu_L}{C_1} \int_{\partial O} g_p(Dw) \, dx
\]

for all open subsets \( O \) of \( \Omega \) and all \( w \in W^{1,p}(O; \mathbb{R}^N) \). In addition, from (H1) and (H2) we see that \( g \) fulfills the growth conditions

\[
\left( \gamma - \frac{2\nu_L}{C_1} \right) |\xi|^p - \frac{2\nu_L}{C_1} \leq g(\xi) \leq \Gamma(1 + |\xi|^p)
\]

for all \( \xi \in \mathbb{R}^n \). Imposing the condition \( 2\nu_L < C_1 \gamma \), which is clearly not restrictive, we infer that \( g \) satisfies (H1) and (H2) up to an additive constant. Obviously, this is sufficient to allow the application of Lemma 5.4 to \( \mathcal{G} \). Furthermore, we deduce from (H4) and Lemma 5.1 that \( g \) is quasiconvex at all \( \xi \) with \( |\xi| \leq L + 1 \). In particular, recalling (2.3) this gives \( \mathcal{G}[\xi; B_{r_j}(x_j)] = \int_{B_{r_j}(x_j)} \Omega(\xi) = \int_{B_{r_j}(x_j)} g(\xi) \) for these \( \xi \), where we have used the notation \( l_\xi \) from Lemma 5.4. Consequently, applying Lemma 5.4 to \( \mathcal{G} \) we get

\[
0 \leq \mathcal{G}[\xi + \varphi; B_{r_j}(x_j)] - \mathcal{G}[\xi; B_{r_j}(x_j)]
\]

\[
\leq \mathcal{F}[\xi + \varphi; B_{r_j}(x_j)] - |B_{r_j}(x_j)| f(\xi) - \frac{\nu_L}{C_1} \int_{B_{r_j}(x_j)} \left[ e_p(\varphi + D\varphi) - e_p(\xi) \right] dx
\]

for \( |\xi| \leq L + 1 \) and all \( \varphi \in W^{1,p}(B_{r_j}(x_j), \mathbb{R}^N) \) with compact support in \( B_{r_j}(x_j) \). By Lemma 5.1 we conclude

\[
\int_{B_{r_j}(x_j)} |V(D\varphi)|^2 \, dx \leq c \left( \mathcal{F}[\xi + \varphi; B_{r_j}(x_j)] - |B_{r_j}(x_j)| f(\xi) \right)
\]

for all \( \xi \in \mathbb{R}^N \) with \( |\xi| \leq L + 1 \) and all \( \varphi \in W^{1,p}(B_{r_j}(x_j), \mathbb{R}^N) \) with compact support in \( B_{r_j}(x_j) \). Rescaling gives us

\[
\int_B \left| \frac{V(\lambda_j D\varphi_j)}{\lambda_j^2} \right|^2 \, dx \leq c(F_j[l_\xi + \varphi; B] - |B| f_j(\xi)),
\]

for all \( \xi \in \mathbb{R}^N \) with \( |\lambda_j \xi| \leq 1 \) and for all \( \varphi \in W^{1,p}(B, \mathbb{R}^N) \) with compact support in \( B \). Next, we recall \( \varphi_j \in W^{1,p}_0(B_{\lambda_j}(x_0), \mathbb{R}^N) \) and \( F_j[a_j; O] = |O| f_j(Da_j) \) for all open subsets \( O \) of \( \Omega \); see Step 3 in the proof of Proposition 4.1 and (2.3). Using these facts, Lemma 5.5 and the choice of \( \lambda_j \) we deduce from the previous inequality

\[
\int_{B_{\lambda_j}(x_0)} \left| \frac{V(\lambda_j D\varphi_j)}{\lambda_j^2} \right|^2 \, dx \leq c \left( F_j[a_j + \varphi_j; B_{\lambda_j}(x_0)] - |B_{\lambda_j}(x_0)| f_j(Da_j) \right)
\]

(5.4)

\[\text{1}\] Actually, this can be verified by a straightforward computation using the definitions of \( F_j \), \( F_j \), \( f_j \), \( F \), \( F \) and the fact that the integral of the linear term in the definition of \( f_j \) is weakly continuous.
for $j \geq j_2(r)$. In the following we will use the quasiconvexity hypothesis in the form (5.4). Next we turn to a reformulation of the minimizing property: We have assumed that $\bar{u}$ is a $W^{1,q}$ local minimizer of $F$ on $\Omega$. By the measure representation theorem [22, Theorem 3.1] this is easily seen to imply

$$\mathcal{F}[\bar{u}; B_{r_j}(x_j)] \leq \mathcal{F}[\bar{u} - \varphi; B_{r_j}(x_j)]$$

for all functions $\varphi \in W^{1,q}(B_{r_j}(x_j), \mathbb{R}^N)$ with compact support in $B_{r_j}(x_j)$ and $\|D\varphi\|_{L^q(B_{r_j}(x_j), \mathbb{R}^N)} \leq \delta$. Recalling as before we get

$$\mathcal{F}_j[u_j; B] \leq \mathcal{F}_j[u_j - \varphi; B]$$

for all $\varphi \in W^{1,q}(B, \mathbb{R}^N)$ with compact support in $B$ and $\|D\varphi\|_{L^q(B, \mathbb{R}^N)} \leq \frac{\delta}{\lambda_j r_j}$. Recalling $\|D\varphi_j\|_{L^q(B, \mathbb{R}^N)} \leq \frac{\delta}{\lambda_j r_j}$ for $j \geq j_1(\alpha)$ (see Step 3 in the proof of Proposition 4.1) we get from Lemma 5.5 and the choice of $\tilde{s}_j$

$$\mathcal{F}_j[u_j; B_{\tilde{s}_j}(x_0)] \leq \mathcal{F}_j[u_j - \varphi_j; B_{\tilde{s}_j}(x_0)].$$  
(5.5)

Finally, recalling (4.15) and putting together (5.4) and (5.5) we find

$$\int_{B_{\alpha r}(x_0)} \left| \frac{V(\lambda_j D\varphi)}{\lambda_j} \right|^2 \, dx \leq c \left( \mathcal{F}_j[u_j - \psi_j; B_{\tilde{s}_j}(x_0)] - \mathcal{F}_j[u_j; B_{\tilde{s}_j}(x_0)] \right)
+ \mathcal{F}_j[u_j + \psi_j; B_{\tilde{s}_j}(x_0)] - \|B_{\tilde{s}_j}(x_0)|f_j(Da_j)\).$$

Since $q \leq \frac{p}{p+1-q} < \frac{np}{n-1}$ we see $\psi_j \in W^{1,\frac{np}{n-1}}(B_{\tilde{s}_j}(x_0), \mathbb{R}^N) \subset W^{1,q}(B_{\tilde{s}_j}(x_0), \mathbb{R}^N)$ from the estimates of Lemma 3.4. Thus, we can apply Lemma 5.2 and (2.3) to simplify the right-hand side of the preceding formula deriving

$$\int_{B_{\alpha r}(x_0)} \left| \frac{V(\lambda_j D\varphi)}{\lambda_j} \right|^2 \, dx \leq c \left( \int_{B_{\tilde{s}_j}(x_0)} (f_j(Du_j - D\psi_j) - f_j(Du_j)) \, dx
+ \int_{B_{\tilde{s}_j}(x_0)} (f_j(Da_j + D\psi_j) - f_j(Da_j)) \, dx \right)$$

for $j \geq j_0(\alpha, r)$. Since the last inequality coincides with the estimate in (4.19) we can now argue exactly as in the proof of Proposition 4.1.

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