COMPACTNESS RESULTS FOR NORMAL CURRENTS AND THE PLATEAU PROBLEM IN DUAL BANACH SPACES

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Abstract. We consider the Plateau problem and the corresponding free boundary problem for finite-dimensional surfaces in possibly infinite-dimensional Banach spaces. For a large class of duals and in particular for reflexive spaces we establish the general solvability of these problems in terms of currents. As an auxiliary result we prove a new compactness theorem for currents in dual spaces, which in turn relies on a fine analysis of the w*-topology.

Contents

1. Introduction

Currents are generalized surfaces. More precisely, n-currents in the classical sense of Federer & Fleming [7] are a measure-theoretic generalization of oriented n-dimensional submanifolds inside an Euclidean space $\mathbb{R}^N$ ($N \geq n$); compare the renowned monographs [6, 12]. Even though there is nowadays a much broader interest in currents, they have been designed — from the very beginning — for the treatment of area minimization problems and in particular for the (oriented) Plateau problem, that is the minimization problem for the n-dimensional area subject to a prescribed boundary in $\mathbb{R}^N$. In fact, given a boundary $(n-1)$-current S in $\mathbb{R}^N$ the oriented Plateau problem can always be solved, if it is (re)formulated as the minimization problem for the mass $\mathbf{M}(T)$ among all n-currents T in $\mathbb{R}^N$ with boundary $\partial T = S$.

Ambrosio & Kirchheim [4, 3] extended the theory of currents to complete metric spaces in place of $\mathbb{R}^N$, thus allowing for a much richer geometric structure of the

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ambient space. While this refined theory has also found other remarkable applications (see for instance [14]), in the first instance it has been developed once more for area minimization: indeed, for a large class of dual Banach spaces Y it has been proved in [3, Theorem 10.6] that Plateau's problem is solvable in terms of metric n-currents in Y, provided that the prescribed boundary remains in a *compact* subset of Y. Eventually, still assuming compact support of the boundary Wenger [13, Theorem 1.5] established the same statement for *every* dual Banach space Y. Let us emphasize that these results naturally cover infinite-dimensional spaces Y, and thus on the technical side they overcome the difficulty that Y need not be locally compact and the Riesz-Markov representation theorem is not available.

In this paper, though we follow the basic approach of [3], we propose a novel simplified proof for the solvability of the Plateau problem in Y. It still applies to a large (but slightly different) class of possibly infinite-dimensional dual Banach spaces Y, but it works *without* a compactness assumption on the boundary values. We believe that the latter feature is new and interesting even in the simplest infinite-dimensional spaces, that is in separable Hilbert spaces.

A precise statement of our result can be formulated conveniently in the following terminology (see Section 2 for further notation and in particular Section 2.3 for background definitions concerning metric currents): For a complete metric space Y, $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$, and a metric $(n-1)$-current $S \in \mathbf{M}_{n-1}(Y)$ of finite mass we write

$$\mathrm{Fillmass}(S) := \inf\{\mathbf{M}(C) \,:\, C \in \mathbf{N}_n(Y),\, \partial C = S\}\,, \tag{1.1}$$

$$\mathrm{Fillvol}(S) := \inf\{\mathbf{M}(C) \,:\, C \in \mathbf{I}_n(Y),\, \partial C = S\} \tag{1.2}$$

for the infimum values of the Plateau problem in Y, formulated within the classes $\mathbf{N}_n(Y)$ and $\mathbf{I}_n(Y)$ of normal metric n-currents and integral metric n-currents, respectively. Here, as usual we consider the infima as infinite, if the set of admissible C is empty. For separable dual Banach spaces Y this is actually the only case in which the infima are not attained:

Theorem 1.1 (existence)**.** *Consider a normed space X such that X^* is separable*[1]*, $n \in \mathbb{N}$, and $S \in \mathbf{M}_{n-1}(X^*)$.*

- *If we have* $\mathrm{Fillmass}(S) < \infty$*, then there exists $T \in \mathbf{N}_n(X^*)$ with $\partial T = S$ and* $\mathbf{M}(T) = \mathrm{Fillmass}(S)$*.*
- *Moreover, if we have* $\mathrm{Fillvol}(S) < \infty$*, then there exists $T \in \mathbf{I}_n(X^*)$ with $\partial T = S$ and* $\mathbf{M}(T) = \mathrm{Fillvol}(S)$*.*

Theorem 1.1 is proved in Section 5.1.

Clearly, $\mathrm{Fillmass}(S) < \infty$ implies $S \in \mathbf{N}_{n-1}(X^*)$ with $\partial S \equiv 0$. Moreover, if S is concentrated in a bounded set, then by the cone construction of [3, Section 10] this necessary criterion for the finiteness of the infimum (1.1) is also sufficient. For the finiteness of the infimum in (1.2) one even has a single necessary and sufficient criterion (in which boundedness plays no role): by Wenger's isoperimetric inequalities [13, Corollary 1.3] and the boundary-rectifiability theorem [3, Theorem 8.6] $\mathrm{Fillvol}(S) < \infty$ holds true if and only if one has $S \in \mathbf{I}_{n-1}(X^*)$ and $\partial S \equiv 0$. As a corollary we thus obtain the following formulation of the existence result with more tangible assumptions on S. The main difference to [3, Theorem 10.6] and [13, Theorem 1.5] lies in the fact that the support of S need not be compact.

[1] Notice that separability of X^* implies separability of X, and that X^* is a Banach space. We may assume that also X is a Banach space, since completion of X does not affect X^*.

Corollary 1.2 (existence for bounded or integral boundaries)**.** *Consider a normed space X such that X^* is separable and $n \in \mathbb{N}$.*

- *Then for every $S \in \mathbf{N}_{n-1}(X^*)$ with bounded support and zero boundary there exists $T \in \mathbf{N}_n(X^*)$ with $\partial T = S$ such that*
$$\mathbf{M}(T) \leq \mathbf{M}(C) \qquad \text{for all } C \in \mathbf{N}_n(X^*) \text{ with } \partial C = S\,.$$
- *Moreover, for every $S \in \mathbf{I}_{n-1}(X^*)$ with zero boundary there exists $T \in \mathbf{I}_n(X^*)$ with $\partial T = S$ and*
$$\mathbf{M}(T) \leq \mathbf{M}(C) \qquad \text{for all } C \in \mathbf{I}_n(X^*) \text{ with } \partial C = S\,.$$

Precise references for the deduction of Corollary 1.2 are given at the end of Section 5.1.

By quite similar methods we also obtain the following statement about mass-minimization with only partially prescribed boundary.

Theorem 1.3 (existence with a free boundary)**.** *Consider a normed space X such that X^* is separable, $n \in \mathbb{N}$, and $S \in \mathbf{M}_{n-1}(X^*)$.*

- *Then there exists $T \in \mathbf{N}_n(X^*)$ with*
$$\mathbf{M}(T) + \mathbf{M}(\partial T - S) \leq \mathbf{M}(C) + \mathbf{M}(\partial C - S) \qquad \text{for all } C \in \mathbf{N}_n(X^*)\,.$$
- *Moreover, there exists $T \in \mathbf{I}_n(X^*)$ with*
$$\mathbf{M}(T) + \mathbf{M}(\partial T - S) \leq \mathbf{M}(C) + \mathbf{M}(\partial C - S) \qquad \text{for all } C \in \mathbf{I}_n(X^*)\,.$$

Theorem 1.3 is proved in Section 5.2.

Next, we briefly discuss our assumptions on the ambient space X^*. First of all, all reflexive spaces Y are duals (up to isometric isomorphism) and the existence results apply[2] in them. Moreover, they also apply in some non-reflexive duals, for instance in the space $\ell^1 \cong (\mathrm{c}_0)^*$ of absolutely summable series. Nevertheless, comparing with [3, Theorem 10.6] and [13, Theorem 1.5] our separability assumption on X^* is somewhat restrictive. Indeed this hypothesis rules out some basic dual spaces such as L^∞ and spaces of measures, but — somewhat surprisingly — it is in some sense essential for our approach; see Example B.1 in the appendix.

The decisive ingredient in the proofs of the existence theorems is a refined compactness result for the w*-convergence of currents in the sense of Definition 2.15. We believe that the latter result, which extends [3, Theorem 6.6], is of some interest in itself, and thus we formulate the statement at this stage.

Theorem 1.4 (compactness)**.** *Consider a normed space X such that X^* is separable, $n \in \mathbb{N} \cup \{0\}$, and a sequence $(T_h)_{h\in\mathbb{N}}$ in $\mathbf{N}_n(X^*)$ such that we have*

$$M := \sup_{h\in\mathbb{N}} \mathbf{M}(T_h) < \infty\,, \qquad M_\partial := \sup_{h\in\mathbb{N}} \mathbf{M}(\partial T_h) < \infty\,, \tag{1.3}$$

and the w**-tightness condition*

$$\lim_{R\to\infty} \sup_{h\in\mathbb{N}} \left[\|T_h\|(X^* \setminus B_R(0)) + \|\partial T_h\|(X^* \setminus B_R(0))\right] = 0\,. \tag{1.4}$$

Then there exists a subsequence $(T_{h(k)})_{k\in\mathbb{N}}$ w-converging to some $T \in \mathbf{N}_n(X^*)$ with $\mathbf{M}(T) \leq M$ and $\mathbf{M}(\partial T) \leq M_\partial$.*

[2] For non-separable Y see Appendix B.2.

For the particular case of equi-bounded supports Theorem 1.4 is proved in Section 4.1. The general statement is deduced in Section 4.2.

Notice that in the case $n = 0$ we understand $\mathbf{N}_0(X^*) = \mathbf{M}_0(X^*)$ and consider the assumptions on ∂T_h as void (compare Section 2.3 once more). In this particular case and for separable Hilbert spaces Theorem 1.4 recovers a statement from [2, Lemma 5.1.12], a weak-topology version of Prokhorov's theorem, which has also inspired our strategy of proof.

As a general rule compactness for currents arises from compactness properties of the ambient space. Accordingly, compactness results in the spirit of the Riesz-Markov theorem are available in locally compact spaces (see [9, Theorem 5.4] and compare Section 2.2). Moreover, there is an extension [3, Theorem 5.2] of Prokhorov's theorem to metric currents, while for the Plateau problem in infinite-dimensional dual spaces X^* a more sophisticated strategy has been proposed in [3, Sections 6 and 10]: it relies on Gromov's isometric embeddings [8, Section 6] into a compact space and also exploits local w*-compactness in X^*. In our proof of Theorem 1.4 we will follow a modified strategy, which works more directly with local w*-compactness, but without Gromov's embeddings and without leaving the ambient space X^* at all.

Indeed, let us summarize the core of the compactness proof in Section 4.1. We consider the case that the T_h are supported in a fixed ball in X^*, we regard this ball with the metrizable w*-topology as a compact metric space K, and we first exhibit a limit object T in K. Initially the action of T is then limited to w*-continuous (generalized) forms $\varphi\,\mathrm{d}\pi$. However, our reasoning shows that the action of T extends in a unique way to non-w*-continuous $\varphi\,\mathrm{d}\pi$, and we prove that the extended T has the properties of a normal current. A crucial technical ingredient is Lemma 3.1, which provides a w*-separability property of the space of Lipschitz functions.

With view towards the existence results it is also relevant that our extension procedure carries w*-rectifiable currents (with finite mass and boundary mass) into norm-rectifiable ones. This is indeed true and it readily follows from the rectifiability criterion [3, Remark 8.2] once we have proven that the extension is a normal current. Accordingly, rectifiability enters our arguments only through the known results of [3], while the technically delicate parts of this paper are concerned with normal currents.

We stress that Theorem 1.4 should also be compared to Wenger's compactness result [15, Theorem 1.2] for integral currents with equi-bounded diameter and its local variant [10, Theorem 1.1]. These results yield compactness for isometric embeddings in Gromov's style, in other words they identify a limit current in an abstract metric space. After the submission of the present paper Wenger [16] has pointed out that the latter results can indeed be applied in order to retrieve compactness in the ambient space itself. By such a reasoning he has extended our results for the integral currents case to non-separable dual spaces.

Finally, we believe that with the preceding results at hand one may also ask for additional (regularity) properties of mass-minimizing currents in infinite-dimensional spaces. We plan to address such issues in a continuative project.

2. Preliminaries

2.1. **General notation.** Suppose that (E, d_E) and $(\widetilde{E}, \mathrm{d}_{\widetilde{E}})$ are metric spaces. For $x \in E$ and $R > 0$ we write $B_R(x)$ for the open ball $\{y \in E : \mathrm{d}_E(y, x) < R\}$ and

$\overline{B}_R(x)$ for the closed ball $\{y \in E : \mathrm{d}_E(y,x) \le R\}$. Furthermore, for subsets P and Q of E we set $\mathrm{dist}(Q,P) := \inf\{\mathrm{d}_E(q,p) : p \in P, q \in Q\}$.

For $\pi\colon E \to \widetilde{E}$ we define the Lipschitz constant $\mathrm{Lip}(\pi)$ as the smallest (but possibly infinite) constant such that $\mathrm{d}_{\widetilde{E}}(\pi(y),\pi(x)) \le \mathrm{Lip}(\pi)\,\mathrm{d}_E(y,x)$ holds for all $x,y \in E$. We write $\mathrm{Lip}(E;\widetilde{E})$ for the space of all $\pi\colon E \to \widetilde{E}$ with $\mathrm{Lip}(\pi) < \infty$, $\mathrm{Lip}_\mathrm{b}(E;\widetilde{E})$ for the subspace of functions in $\mathrm{Lip}(E;\widetilde{E})$ with bounded image, and $\mathrm{Lip}_1(E;\widetilde{E})$ for the collection of all $\pi\colon E \to \widetilde{E}$ with $\mathrm{Lip}(\pi) \le 1$. Finally, we set $\mathrm{Lip}(E) := \mathrm{Lip}(E;\mathbb{R})$, $\mathrm{Lip}_\mathrm{b}(E) := \mathrm{Lip}_\mathrm{b}(E;\mathbb{R})$, and $\mathrm{Lip}_1(E) := \mathrm{Lip}_1(E;\mathbb{R})$.

If X is a normed space, we write $\|\cdot\|_X$ for its norm and X^* for its topological dual endowed with the operator norm. For $y \in X^*$ and $x \in X$ we use the notation[3] $\langle y,x\rangle$ for the evaluation of y at x. We briefly recall that the w*-topology on X^* is the coarsest topology such that the evaluation maps $X^* \to \mathbb{R}, y \mapsto \langle y,x\rangle$ are continuous on X^* for all $x \in X$.

By a finite Borel measure μ on E we mean a σ-additive function $\mu\colon \mathscr{B}(E) \to [0,\infty)$ on the Borel-σ-algebra $\mathscr{B}(E)$ of E. Analogously, a finite *signed* Borel measure is a σ-additive function $\mu\colon \mathscr{B}(E) \to \mathbb{R}$, and we write $|\mu|$ for the variation measure of μ. The support $\mathrm{spt}\,\mu$ of μ is defined as the closed set of all $x \in E$ satisfying $|\mu|(B_R(x)) > 0$ for all $R > 0$. Moreover, if $|\mu|(E \setminus \Sigma) = 0$ holds for $\Sigma \in \mathscr{B}(E)$, we say that μ is concentrated on Σ, and if $\psi\colon E \to \widetilde{E}$ is a continuous function, we introduce a Borel measure $\psi_\sharp\mu$ on $\widetilde{E}$ by $\psi_\sharp\mu(B) := \mu(\psi^{-1}B)$ for $B \in \mathscr{B}(\widetilde{E})$.

Finally, by $\mathrm{C_b}(E)$ we denote the Banach space of bounded continuous functions $E \to \mathbb{R}$ with the norm of uniform convergence.

2.2. **Weak-$*$-topologies.** We start with the remark that for separable normed spaces X the w*-topology can be metricized on bounded sets in X^*. More precisely, if $\{x_1,x_2,x_3,\ldots\}$ is dense in $X \setminus \{0\}$, we define a new norm $\|\cdot\|_{\mathrm{w}^*}$ on X^* by

$$\|y\|_{\mathrm{w}^*} := \sum_{i=1}^{\infty} 2^{-i}\,|\langle y,x_i\rangle|\,/\|x_i\|_X \qquad \text{for } y \in X^*\,, \tag{2.1}$$

and $\|\cdot\|_{\mathrm{w}^*}$ induces the w*-topology on all bounded sets in X^*, but not globally. In particular, we will use the following consequence of this remark: for bounded sets in the dual of a separable space w*-topological concepts are characterized by sequences, as usual in a metric space.

Our compactness result relies heavily on the following two well-known results (see for instance [11, 3.15] and [1, Theorem 1.54]): the Banach-Alaoglu-Bourbaki compactness theorem and the Riesz-Markov representation theorem in compact metric spaces.

Theorem 2.1. *Suppose that X is a normed space. Then the closed balls $\overline{B}_R(y_0) = \{y \in X^* : \|y-y_0\|_{X^*} \le R\}$ in X^* are compact in the* w**-topology.*

Theorem 2.2. *Suppose that K is a compact metric space. Then every $F \in \mathrm{C_b}(K)^*$ can be represented by a finite signed Borel measure μ on K in the sense of*

$$\langle F,\varphi\rangle = \int_K \varphi\,\mathrm{d}\mu \qquad \textit{for all } \varphi \in \mathrm{C_b}(K)$$

and

$$\|F\|_{\mathrm{C_b}(K)^*} = |\mu|(K)\,.$$

[3] Later on we also use angle brackets with three arguments in a completely different meaning.

We briefly record some well-known consequences of these results: Theorem 2.2 implies that the space of finite signed Borel measures $\mathcal{M}(K)$ on K (with the total variation norm) coincides up to isometric isomorphism with the dual of $\mathrm{C_b}(K)$. It follows by Theorem 2.1 that closed balls in $\mathcal{M}(K)$ are compact in the w*-topology arising from this duality. Moreover, the operator norm on a dual, and thus in particular the total variation of measures, is (almost by definition) lower semicontinuous with respect to w*-convergence.

In addition, we provide two simple lemmas, which will eventually be useful. In the former lemma we crucially use separability of X^* (compare Example B.1 below), while separability of X suffices for all other arguments in this paper.

Lemma 2.3. *Suppose that X is a normed space and write $\mathscr{B}(X^*)$ and $\mathscr{B}_{\mathrm{w}^*}(X^*)$ for the Borel-σ-algebras generated by the norm-topology and the* w*-*topology on X^*, respectively. Then there holds $\mathscr{B}_{\mathrm{w}^*}(X^*) \subset \mathscr{B}(X^*)$, and if X^* is separable, we even have equality $\mathscr{B}_{\mathrm{w}^*}(X^*) = \mathscr{B}(X^*)$.*

Proof. Since every w*-closed set in X^* is also closed, we generally have $\mathscr{B}_{\mathrm{w}^*}(X^*) \subset \mathscr{B}(X^*)$. If X^* is separable, then the closed balls generate $\mathscr{B}(X^*)$, also X is separable, and by Theorem 2.1 the closed balls are also w*-closed. In conclusion, for separable X^* we infer $\mathscr{B}(X^*) \subset \mathscr{B}_{\mathrm{w}^*}(X^*)$. □

Lemma 2.4. *Let $p \le q$ in $\mathbb{R}$, and suppose that P and Q are two compact subsets in the dual X^* of a normed space X with $\mathrm{dist}(Q,P) > 0$. Then there exists a* w*-*continuous Lipschitz function $\eta\colon X^* \to [p,q]$ with $\eta \equiv p$ on P, $\eta \equiv q$ on Q, and $\mathrm{Lip}(\eta) \le 8(q-p)/\mathrm{dist}(Q,P)$.*

Proof. We assume $p = 0$, $q = 1$.

In a first step, instead of compact sets we consider balls P and Q in X^* with centers y_P, y_Q, and radii r_P, r_Q. The condition $\|y_Q - y_P\|_{X^*} - r_P - r_Q = \mathrm{dist}(Q,P) > 0$ implies $\|y_Q - y_P\|_{X^*} > \frac{1}{2}(\|y_Q - y_P\|_{X^*} + r_P + r_Q)$ and thus yields some $x \in X$ with $\|x\|_X = 1$ and $\langle y_Q - y_P, x\rangle \ge \frac{1}{2}(\|y_Q - y_P\|_{X^*} + r_P + r_Q)$. Since $\langle y - y_P, x\rangle - r_P \ge \langle y_Q - y_P, x\rangle - r_P - r_Q \ge \frac{1}{2}(\|y_Q - y_P\|_{X^*} - r_P - r_Q)$ holds for $y \in Q$, one finds that

$$\eta(y) := \min\left\{1, 2\frac{\big(\langle y - y_P, x\rangle - r_P\big)_+}{\|y_Q - y_P\|_{X^*} - r_P - r_Q}\right\} \qquad \text{for } y \in X^*$$

defines a function η with all the required properties (and even $\mathrm{Lip}(\eta) \le 2/\mathrm{dist}(Q,P)$).

In a second step, if P is any compact set and Q is a ball, we cover P by finitely many balls $P_1, P_2, \ldots, P_n$ with $\mathrm{dist}(Q,P_i) \ge \frac{1}{2}\mathrm{dist}(Q,P)$. Then we write η_i for the functions from the first step corresponding to P_i and Q, and we define η as the pointwise minimum of the η_i. It is straightforward to check the required properties of η (and even $\mathrm{Lip}(\eta) \le 4/\mathrm{dist}(Q,P)$).

In an analogous third step we consider compact sets P and Q, we cover Q by finitely many balls, and we choose η as the pointwise maximum of functions obtained from the second step. Again it is straightforward to check that η has the claimed properties. □

2.3. **Metric currents with finite mass.** Here, we partially recall the Ambrosio-Kirchheim-theory [3] of currents with finite mass in metric spaces, fixing by the way notation and terminology. While we refer the reader to the original paper for more comments and motivation, now we briefly restate some basic definitions and results. In this way we keep the present paper essentially self-contained up to the end of Section 4.1, where the proof of Theorem 1.4 is completed at least in the basic case of equi-bounded supports. Anyway, in the subsequent sections we will directly quote some more results from [3].

We suppose for the remainder of this section that $n \in \mathbb{N} \cup \{0\}$ and E is a complete metric space. For a bounded Borel function $\varphi\colon E \to \mathbb{R}$ and $\pi \in \mathrm{Lip}(E)^n$ we permanently use the notation

$$\varphi \,\mathrm{d}\pi := (\varphi, \pi)\,,$$

which emphasizes the far-reaching analogy to Euclidean differential forms. For $n = 0$ we always identify $\mathrm{Lip}(E)^0 := \{0\}$, $\mathrm{Lip_b}(E) \times \mathrm{Lip}(E)^0 := \mathrm{Lip_b}(E)$, and $\varphi \,\mathrm{d}\pi := \varphi$ in the following.

Definition 2.5 (metric functionals, [3, Definition 2.2])**.** *An n-dimensional metric functional on E is a map $T\colon \mathrm{Lip_b}(E) \times \mathrm{Lip}(E)^n \to \mathbb{R}$ such that*

- *T is linear in its first argument from $\mathrm{Lip_b}(E)$ and*
- *$|T|$ is 1-homogeneous and convex[4] in each of the n arguments from $\mathrm{Lip}(E)$.*

Definition 2.6 (boundary, [3, Definition 2.3])**.** *Let $n \geq 1$. For an n-dimensional[5] metric functional T on E the boundary $\partial T\colon \mathrm{Lip_b}(E) \times \mathrm{Lip}(E)^{n-1} \to \mathbb{R}$ is defined by*

$$\partial T(\varphi \,\mathrm{d}\pi) := T(1 \,\mathrm{d}(\varphi, \pi)) \qquad \text{for } \varphi \in \mathrm{Lip_b}(E) \text{ and } \pi \in \mathrm{Lip}(E)^{n-1}\,.$$

If an n-dimensional metric functional T on E is linear in all arguments, then ∂T is an $(n{-}1)$-dimensional metric functional on E.

Definition 2.7 (pushforward, [3, Definition 2.4])**.** *The pushforward $\psi_\sharp T$ of an n-dimensional metric functional T on E along a Lipschitz map $\psi\colon E \to \widetilde{E}$ into another complete metric space $\widetilde{E}$ is the n-dimensional metric functional on $\widetilde{E}$ defined by*

$$\psi_\sharp T(\varphi \,\mathrm{d}\pi) := T((\varphi \circ \psi) \,\mathrm{d}(\pi \circ \psi)) \qquad \text{for } \varphi \in \mathrm{Lip_b}(\widetilde{E}) \text{ and } \pi \in \mathrm{Lip}(\widetilde{E})^n\,.$$

For $n = 0$ Definition 2.7 is compatible with the notation for measures from Section 2.1, but allows less general functions ψ.

Definition 2.8 (mass and support, [3, Definitions 2.6 and 2.8])**.** *An n-dimensional metric functional T on E has finite mass if there exists a finite Borel measure μ on E such that*

$$|T(\varphi \,\mathrm{d}\pi)| \leq \int_E |\varphi| \,\mathrm{d}\mu \prod_{i=1}^{n} \mathrm{Lip}(\pi_i) \qquad \text{holds for all } \varphi \in \mathrm{Lip_b}(E) \text{ and } \pi \in \mathrm{Lip}(E)^n\,.$$

The least such measure μ is called the mass $\|T\|$ of T, and the total mass is $\mathbf{M}(T) := \|T\|(E)$. If T has finite mass, then the support $\mathrm{spt}\, T$ of T is defined as the support of the measure $\|T\|$.

Occasionally, we will use the assertion that every finite Borel measure and in particular the mass of a metric functional on a complete metric space E is tight, that means concentrated on a σ-compact set. In order to prove the results of the introduction it suffices to utilize this assertion for separable E, in which case it is known as Ulam's theorem and is not too hard to prove; see [5, 7.1.7]. However, in Section 3 and Appendix B we state some results in a more general framework, and

[4] In other words, $|T|$ is a pseudonorm in each argument from $\mathrm{Lip}(E)$.

[5] A boundary ∂T of a 0-dimensional metric functional T, which formally occurs in some of our statements, is identified with the constant $T(1)$. Definitions 2.8 and 2.11 are partially extended by letting $\mathbf{M}(\partial T) := |T(1)| \leq \mathbf{M}(T)$ in this case and $\mathbf{N}_0(E) := \mathbf{M}_0(E)$, while $\|\partial T\|$ and $\mathrm{spt}\,\partial T$ are left undefined.

we will then exploit the same assertion even for non-separable E. This is indeed possible imposing at those points the extra assumption that the cardinality of a dense subset of E is an Ulam number in the sense of Federer [6, 2.1.6]; compare also [6, 2.2.16] and the comments around [3, Lemma 2.9] for a discussion of this assumption.

For every metric functional T on E with finite mass there is a canonical extension — which we still denote by T — to those $\varphi\,\mathrm{d}\pi$ for which $\varphi\colon E \to \mathbb{R}$ is just a bounded Borel function; in fact, it suffices to extend T as a continuous linear functional in φ with respect to the norm of $\mathrm{L}^1(E;\|T\|)$. We use this extension in the next definition.

Definition 2.9 (restriction)**.** *The restriction* $T \llcorner A\colon \mathrm{Lip_b}(E) \times \mathrm{Lip}(E)^n \to \mathbb{R}$ *of an* n*-dimensional metric functional* T *on* E *with finite mass to a Borel subset* A *of* E *is the* n*-dimensional metric functional on* E *defined by*

$$(T \llcorner A)(\varphi\,\mathrm{d}\pi) := T(\chi_A \varphi\,\mathrm{d}\pi) \qquad \text{for } \varphi \in \mathrm{Lip_b}(E) \text{ and } \pi \in \mathrm{Lip}(E)^n\,,$$

where $\chi_A\colon E \to \{0,1\}$ *denotes the characteristic function of* A*.*

Definition 2.10 (currents, [3, Definition 3.1])**.** *An* n*-current in* E *with finite mass is an* n*-dimensional metric functional* T *on* E *with finite mass satisfying moreover the following three axioms:*

- multilinearity: T *is* $(1{+}n)$*-linear;*
- continuity: $T(\varphi\,\mathrm{d}\pi^l)$ *converges to* $T(\varphi\,\mathrm{d}\pi)$ *for* $\varphi \in \mathrm{Lip_b}(E)$ *and* $\pi^l, \pi \in \mathrm{Lip}(E)^n$*, whenever* π^l *converges to* π *pointwise on* E *and* $\mathrm{Lip}(\pi^l)$ *stays bounded;*
- locality: *for* $\varphi \in \mathrm{Lip_b}(E)$ *and* $\pi \in \mathrm{Lip}(E)^n$ *there holds* $T(\varphi\,\mathrm{d}\pi) = 0$ *whenever a component function* π_{i_0} *is constant on an* ε*-neighborhood of* $\operatorname{spt}\varphi$*.*

The space of all n*-currents in* E *with finite mass is denoted by* $\mathbf{M}_n(E)$*.*

We observe that in view of multilinearity it is equivalent to require the continuity and locality axioms only for $\varphi \in \mathrm{Lip}_1(E;[-1,1])$ and $\pi^l, \pi \in \mathrm{Lip}_1(E)^n$. When verifying the axioms later on, we will make use of this fact without further comments. A similar remark applies to the above definition of mass.

Moreover, starting from the fact that the space of finite Borel measures on E is a Banach space with the total variation norm, it can be verified that the mass $\mathbf{M}$ is a norm on $\mathbf{M}_n(E)$, and that $\mathbf{M}_n(E)$ is a Banach space, when endowed with this norm.

Definition 2.11 (normal currents, [3, Definition 3.4])**.** *A current* $T \in \mathbf{M}_n(E)$ *with* $\partial T \in \mathbf{M}_{n-1}(E)$ *is called a normal current, and the space of normal* n*-currents is denoted by* $\mathbf{N}_n(E)$*.*

As pointed out after Definition 3.4 in [3], the boundary of $T \in \mathbf{M}_n(E)$ is always an $(n{-}1)$-dimensional metric functional and satisfies the axioms of an $(n{-}1)$-current. Thus, in order to show that $T \in \mathbf{M}_n(E)$ is normal, it suffices to check that ∂T has finite mass.

The space $\mathbf{I}_n(E)$ of integral n-currents is defined as in [3, Definition 4.2]. We will use the closure theorem [3, Theorem 8.5] for $\mathbf{I}_n(E)$, but we will never work explicitly with the definition, and thus we do not repeat it here.

Lemma 2.12 (equi-continuity, [3, Lemma 5.1])**.** *For* $T \in \mathbf{N}_n(E)$*,* $\varphi \in \mathrm{Lip_b}(E)$ *and* $\pi, \widetilde{\pi} \in \mathrm{Lip}(E)^n$ *there holds*

$$\big|T(\varphi\,\mathrm{d}\widetilde{\pi}) - T(\varphi\,\mathrm{d}\pi)\big| \le \big(\sup_E |\varphi| + \mathrm{Lip}(\varphi)\big) \sum_{i=1}^n \int_{\operatorname{spt}\varphi} |\widetilde{\pi}_i - \pi_i|\,\mathrm{d}\big(\|T\| {+} \|\partial T\|\big)\,.$$

In particular, choosing $\widetilde{\pi}_{i_0} \equiv s$ constant and $\widetilde{\pi}_i = \pi_i$ for $i \neq i_0$ we infer:

Lemma 2.13. *For $T \in \mathbf{N}_n(E)$, $\varphi \in \mathrm{Lip_b}(E)$ and $\pi \in \mathrm{Lip}(E)^n$ there holds*

$$\big|T(\varphi\,\mathrm{d}\pi)\big| \leq \big(\sup_E |\varphi| + \mathrm{Lip}(\varphi)\big) \int_{\mathrm{spt}\,\varphi} |\pi_{i_0} - s|\,\mathrm{d}\big(\|T\|+\|\partial T\|\big)$$

for all $i_0 \in \{1, 2, \ldots, n\}$ and $s \in \mathbb{R}$.

Definition 2.14 (weak convergence, [3, Definition 3.6])**.** *A sequence $(T_h)_{h\in\mathbb{N}}$ in $\mathbf{M}_n(E)$ is said to converge weakly to $T \in \mathbf{M}_n(E)$ if $T_h(\varphi\,\mathrm{d}\pi)$ converges to $T(\varphi\,\mathrm{d}\pi)$ for all $(\varphi, \pi) \in \mathrm{Lip_b}(E) \times \mathrm{Lip}(E)^n$.*

Notice that the uniqueness of weak limits is immediate by definition, while lower semicontinuity of the mass along weakly convergent sequences follows readily from [3, Proposition 2.7]; see the remark after [3, Definition 3.6]. For the following notion of w*-convergence the same assertions are still true, but less immediate; compare Corollary 3.4 below.

Definition 2.15 (w*-convergence, [3, Definition 6.1])**.** *Consider a normed space X. A sequence $(T_h)_{h\in\mathbb{N}}$ in $\mathbf{M}_n(X^*)$ is said to* w**-converge to $T \in \mathbf{M}_n(X^*)$, if $T_h(\varphi\,\mathrm{d}\pi)$ converges to $T(\varphi\,\mathrm{d}\pi)$ for all* w**-continuous $(\varphi, \pi) \in \mathrm{Lip_b}(X^*) \times \mathrm{Lip}(X^*)^n$.*

We remark that the w*-convergence of currents is actually a weak convergence in two regards: on the one hand, it is a distributional convergence of functionals, and on the other hand, even on the base space X^* the w*-topology is used. In contrast, weak convergence is also a distributional convergence, but uses the strong topology of the base space. In particular, these convergences correspond to w* and strong convergence of the basepoints when n equals 0 and the currents are Dirac measures.

Finally, we point out that w*-convergence cannot be defined in a localized way, since a w*-continuous functions φ with bounded support in X^* necessarily vanishes everywhere.

3. A separability lemma and w*-convergence

In this section we establish an auxiliary lemma and we collect a couple of related observations on the notion of w*-convergence, which appears in our compactness results.

As in the proof of [3, Theorem 5.2] we will exploit the fact that the set of Lip_1-functions on a separable domain is itself separable and thus contains a countable dense subset A. The following lemma shows that, when the base space is a dual space, A can be chosen to consist only of w*-continuous functions. This fact will turn out to be crucial for our purposes.

Lemma 3.1 (separability lemma)**.** *Suppose that X is a separable normed space and that there are subsets $\Sigma \subset E \subset X^*$ of its dual, where Σ is separable. Then there exist a countable collection A of* w**-continuous functions in $\mathrm{Lip}_1(E)$ and a pseudodistance*[6] *$\mathrm{d_p}$ on $\mathrm{Lip}_1(E)$ such that A is dense with respect to $\mathrm{d_p}$ and $\mathrm{d_p}$ induces pointwise convergence on Σ.*

[6] A pseudometric space differs from a metric space only in the possibility that two different points may have zero pseudodistance, and the quotient of a pseudometric space by the zero pseudodistance relation is always a metric space. Here, the usage of a pseudodistance is of secondary importance, but we think that it keeps our terminology clean and convenient, since we look at functions on E with a concept of convergence on the possibly smaller set Σ; in fact, if $\pi, \widetilde{\pi} \in \mathrm{Lip}_1(E)$ coincide on Σ, but differ elsewhere, then $\mathrm{d_p}(\widetilde{\pi}, \pi)$ vanishes.

In our applications we use Lemma 3.1 for $\mathbb{R}^{1+n}$-valued maps, in the form that $A_{[-1,1]} \times A^n$ is dense in $\mathrm{Lip}_1(E;[-1,1]) \times \mathrm{Lip}_1(E)^n$ with respect to pointwise convergence on Σ. Here, the set $A_{[-1,1]} := \{\min\{1, \max\{-1,\varphi\}\} : \varphi \in A\}$ of cut-offs still contains only w*-continuous functions, and the latter convergence is induced by the pseudodistance $\mathrm{d}_\mathrm{p}^{1+n}((\widetilde{\varphi},\widetilde{\pi}),(\varphi,\pi)) := \mathrm{d}_\mathrm{p}(\widetilde{\varphi},\varphi) + \sum_{i=1}^n \mathrm{d}_\mathrm{p}(\widetilde{\pi}_i,\pi_i)$.

Proof of Lemma 3.1. Suppose that $\{r_1, r_2, r_3, \ldots\}$ is dense in $\mathbb{R}$, $\{s_1, s_2, s_3, \ldots\}$ is dense in Σ, and $\{x_1, x_2, x_3, \ldots\}$ is dense in $X \setminus \{0\}$. We define $a_{i,j,k} \in \mathrm{Lip}_1(E)$ by

$$a_{i,j,k}(y) := \max\{r_i + |\langle y - s_j, x_h\rangle|/\|x_h\|_X \,:\, h \in \{1,2,\ldots,k\}\}$$

for $y \in E$ and choose A as the set of all pointwise minima of finitely many $a_{i,j,k}$. Then A is a countable collection of w*-continuous functions in $\mathrm{Lip}_1(E)$. We now prove that A is dense in a suitable sense. For $\pi \in \mathrm{Lip}_1(E)$ we define $\widetilde{\pi}_l \in \mathrm{Lip}_1(E)$ and $\pi_{k,l} \in A$ by

$$\widetilde{\pi}_l(y) := \min\{r_i + \|y - s_j\|_{X^*} \,:\, i,j \in \{1,2,\ldots,l\}, r_i \geq \pi(s_j)\}\,,$$
$$\pi_{k,l}(y) := \min\{a_{i,j,k}(y) \,:\, i,j \in \{1,2,\ldots,l\}, r_i \geq \pi(s_j)\}$$

for $y \in E$. It is easy to check that $a_{i,j,k}(y)$ monotonically converges from below to $r_i + \|y - s_j\|_{X^*}$ as $k \to \infty$ and thus $\pi_{k,l}$ converges pointwise to $\widetilde{\pi}_l$ as $k \to \infty$. For every $l \in \mathbb{N}$ we may therefore choose $k(l) \in \mathbb{N}$ such that there holds $|\pi_{k(l),l}(s_j) - \widetilde{\pi}_l(s_j)| < \frac{1}{l}$ for $j \in \{1,2,\ldots,l\}$. Moreover, recalling $\pi \in \mathrm{Lip}_1(E)$ we observe that $\widetilde{\pi}_l$ converges to π from above, pointwise on $\{s_1, s_2, s_3, \ldots\}$. All in all we may conclude that $\pi_l := \pi_{k(l),l}$ belong to A and converge pointwise to π on $\{s_1, s_2, s_3, \ldots\}$. Thus, introducing the pseudodistance

$$\mathrm{d}_\mathrm{p}(\widetilde{\pi},\pi) := \sum_{j=1}^{\infty} 2^{-j} \frac{|\widetilde{\pi}(s_j) - \pi(s_j)|}{1 + |\widetilde{\pi}(s_j) - \pi(s_j)|} \tag{3.1}$$

we have $\lim_l \mathrm{d}_\mathrm{p}(\pi_l,\pi) = 0$ and A is dense with respect to d_p. Since $\{s_1, s_2, s_3, \ldots\}$ is dense in Σ and $\mathrm{Lip}_1(E)$ is a class of equi-continuous functions, d_p induces pointwise convergence on Σ. □

In fact, the stated version of Lemma 3.1 is sufficient for our needs. Nevertheless, let us add a brief comment on variants for general metric spaces E.

Remark 3.2. *In particular, all weakly separable metric spaces E in the sense of* [3, Definition 1.1] *can be isometrically embedded into duals of separable spaces, and Lemma 3.1 applies to them. However, a more plain and well-known version of Lemma 3.1 without* w**-continuity holds for arbitrary metric spaces (E, d_E) not necessarily embedded into any normed space. In fact, replacing $\|y - s_j\|_{X^*}$ with $\mathrm{d}_E(y, s_j)$ and changing the definition of $a_{i,j,k}$ to $a_{i,j,k}(y) := r_i + \mathrm{d}_E(y, s_j)$ this can be proved along the lines of the preceding reasoning, which in fact considerably simplifies in this case, since $a_{i,j,k}$ and $\pi_{k,l} = \widetilde{\pi}_l$ do not depend on k.*

Next we record a continuity property of currents.

Proposition 3.3. *Consider a complete metric space E, $n \in \mathbb{N} \cup \{0\}$, $T \in \mathbf{M}_n(E)$, and denote by Σ a σ-compact set on which $\|T\|$ is concentrated. Then T is continuous on* $\mathrm{Lip}_1(E;[-1,1]) \times \mathrm{Lip}_1(E)^n$ *with respect to pointwise convergence on Σ.*

Proof. For $\varphi^l, \varphi \in \mathrm{Lip}_1(E;[-1,1])$ and $\varphi^l, \varphi \in \mathrm{Lip}_1(E)$ we suppose that (φ^l, π^l) converges pointwise to (φ, π) on Σ. Fixing $\varepsilon > 0$ we find a compact subset C of Σ with $\|T\|(E \setminus C) \leq \varepsilon$, and we observe that by the Arzelà-Ascoli theorem

(φ^l, π^l) converges uniformly to (φ, π) on C. Next we consider a Lipschitz extension operator[7] $\mathcal{E}\colon \mathrm{Lip}_1(C)^n \to \mathrm{Lip}_1(E)^n$ which carries uniform convergence on C into pointwise convergence on E. We notice that $\mathcal{E}(\pi|_C) = \pi$ holds on C and thus by the locality property[8] [3, Theorem 3.5] we have $(T\llcorner C)(\varphi\,\mathrm{d}\pi) = (T\llcorner C)(\varphi\,\mathrm{d}\mathcal{E}(\pi|_C))$. By the preceding choices and the Definition 2.8 of mass we have

$$\begin{aligned}
&|T(\varphi^l\,\mathrm{d}\pi^l) - T(\varphi\,\mathrm{d}\pi)| \\
&\quad\le |(T\llcorner C)((\varphi^l-\varphi)\,\mathrm{d}\pi^l)| + |(T\llcorner C)(\varphi\,\mathrm{d}\pi^l) - (T\llcorner C)(\varphi\,\mathrm{d}\pi)| + 2\|T\|(E\setminus C) \\
&\quad\le \int_C |\varphi^l - \varphi|\,\mathrm{d}\|T\| + |(T\llcorner C)(\varphi\,\mathrm{d}\mathcal{E}(\pi^l|_C)) - (T\llcorner C)(\varphi\,\mathrm{d}\mathcal{E}(\pi|_C))| + 2\varepsilon\,.
\end{aligned}$$

Since φ^l converges uniformly to φ on C the first term on the right-hand side vanishes in the limit $l \to \infty$. For the second term we get the same conclusion from the continuity axiom since $\mathcal{E}(\pi^l|_C)$ converges pointwise to $\mathcal{E}(\pi|_C)$ by the choice of $\mathcal{E}$. Sending ε to 0, we arrive at the claimed continuity property of T. □

We remark that for normal currents Proposition 3.3 can alternatively be inferred from the equi-continuity property of Lemma 2.12 without using a Lipschitz extension operator. A refinement of this approach will be used in the following proof of the compactness result.

Combining Lemma 3.1 and Proposition 3.3 we get some useful conclusions about w*-convergence. Here, the second and the third assertion have already been proved in [3, Proposition 6.3], but we believe that the present approach naturally yields these claims by a slightly different line of argument, which does not employ the extension theorem [3, Theorem 6.2].

Corollary 3.4. *Consider a separable normed space X and $n \in \mathbb{N}\cup\{0\}$. Then*

- *every $T \in \mathbf{M}_n(X^*)$ is already determined by its action on the* w**-continuous $(\varphi,\pi) \in \mathrm{Lip_b}(X^*) \times \mathrm{Lip}(X^*)^n$,*
- w**-limits in the sense of Definition 2.15 are unique,*
- *whenever $T_h \in \mathbf{M}_n(X^*)$* w**-converges to $T \in \mathbf{M}_n(X^*)$ we have lower semi-continuity of the mass*

$$\mathbf{M}(T) \le \liminf_h \mathbf{M}(T_h)\,.$$

Proof. We choose some σ-compact (and hence separable) set $\Sigma \subset X^*$ on which $\|T\|$ is concentrated. Applying Lemma 3.1 with $E := X^*$ we use $A_{[-1,1]} \times A^n$ and $\mathrm{d}_{\mathrm{p}}^{1+n}$ as defined after the statement of the lemma. By Proposition 3.3 the restriction of T to $\mathrm{Lip}_1(X^*;[-1,1]) \times \mathrm{Lip}_1(X^*)^n$ is continuous and uniquely determined by its values on the dense subset $A_{[-1,1]} \times A^n$. Since $A_{[-1,1]} \times A^n$ contains only w*-continuous functions, the first claim follows by multilinearity, and the second one is a direct consequence. To derive the third claim we fix $m \in \mathbb{N}$ and $L \in [0,1]$ for the moment. From Definitions 2.15 and 2.8 we get

$$\sum_{i=1}^m |\varphi^i| \le L \text{ on } X^* \implies \sum_{i=1}^m T(\varphi^i\,\mathrm{d}\pi^i) \le L \liminf_h \mathbf{M}(T_h)\,, \tag{3.2}$$

first for all w*-continuous $(\varphi^i,\pi^i) \in \mathrm{Lip}_1(X^*)^{1+n}$. Next, following an argument of [3] we take a Lipschitz projection p_L on the convex set $\{z \in \mathbb{R}^m : \sum_{i=1}^m |z_i| \le L\}$ in

[7] A well-known version of such an operator is defined by $(\mathcal{E}\pi)_i(y) := \min_{x\in C}\left[\pi_i(x) + \mathrm{d}_E(y,x)\right]$ for $y \in E$ and $i \in \{1,2,\dots,n\}$.

[8] Alternatively one may directly use the axioms in a brief approximation argument.

$\mathbb{R}^m$ and consider the collection $A_L^m := \{p_L \circ (\psi^1, \psi^2, \ldots, \psi^m) \,:\, \psi^i \in A\}$ of joint cut-offs. For all $(\varphi^i, \pi^i) \in \mathrm{Lip}_1(X^*)^{1+n}$ which satisfy the hypothesis of (3.2) we can now approximate π^i from A^n as before and $(\varphi^1, \varphi^2, \ldots, \varphi^m)$ from A_L^m pointwise on Σ. Then (3.2) carries over[9] to such (φ^i, π^i) by the continuity of T from Proposition 3.3, and by multilinearity (3.2) finally extends to all $(\varphi^i, \pi^i) \in \mathrm{Lip}_b(X^*) \times \mathrm{Lip}_1(X^*)^n$. As $m \in \mathbb{N}$ is arbitrary, the claim now follows by the characterization of mass from [3, Proposition 2.7]. □

4. Proof of the compactness result

Relying on Lemma 3.1 we now implement the proof of our compactness result.

4.1. **The case of equi-bounded supports.** We start with the following particular case of Theorem 1.4.

Theorem 4.1. *Theorem 1.4 holds true if we strengthen* (1.4) *to the requirement that*

$$\bigcup_{h\in\mathbb{N}} \operatorname{spt} T_h \textit{ is bounded.}$$

Proof of Theorem 4.1. We assume that a neighborhood of $\bigcup_{h\in\mathbb{N}} \operatorname{spt} T_h$ is contained in some fixed closed ball $\overline{B}_R(0)$ in X^*. Relying on the locality axiom, we may and will regard[10] the T_h as normal currents in $\overline{B}_R(0)$ without changing $\mathbf{M}(T_h)$ or $\mathbf{M}(\partial T_h)$.

Step 1. There is a limit functional T on $A_{[-1,1]} \times A^n$.
We endow the ball $\overline{B}_R(0)$ with a $\|\cdot\|_{\mathrm{w}^*}$-distance inducing the w*-topology; see (2.1). By Theorem 2.1 $\overline{B}_R(0)$ is a compact metric space with this distance, which is denoted by K in the following. In contrast, if we make use of the norm distance, then we will write E for the ball $\overline{B}_R(0)$. By the separability of X^* and Lemma 2.3 we do not need to distinguish between finite (signed) Borel measures on E and K. Consequently, we may regard $\|T_h\|$ and $\|\partial T_h\|$ as measures on K. Taking into account (1.3) and the above remarks after Theorems 2.1 and 2.2, some subsequences $\|T_{h(k)}\|$ and $\|\partial T_{h(k)}\|$ w*-converge to finite Borel measures μ and ν on K, respectively, in the duality with $\mathrm{C_b}(K)$ according to Theorem 2.2. By w*-semicontinuity of the norm we furthermore infer

$$\mu(K) \le \liminf_k \|T_{h(k)}\|(K) \le M\,, \qquad \nu(K) \le \liminf_k \|\partial T_{h(k)}\|(K) \le M_\partial\,. \tag{4.1}$$

Next we view μ and ν as finite Borel measures on E, and by Ulam's theorem [5, 7.1.7] we choose some σ-compact (and in particular separable) set $\Sigma \subset E$ on which $\mu+\nu$ is concentrated. We denote by $A_{[-1,1]} \times A^n$ the corresponding dense subset of $\mathrm{Lip}_1(E;[-1,1]) \times \mathrm{Lip}_1(E)^n$, as explained after Lemma 3.1, and for $\pi \in A^n$ we consider the functionals $F_h^\pi \in \mathrm{C_b}(K)^*$ defined by

$$\langle F_h^\pi, \varphi\rangle := T_h(\varphi \,\mathrm{d}\pi) \qquad \text{for } \varphi \in \mathrm{C_b}(K)\,.$$

[9] Actually, the Lipschitz constants of the approximations of φ^i are uniformly bounded by the Euclidean Lipschitz constant of p_L, but in general not by 1. However, in view of multilinearity, Proposition 3.3 still guarantees continuity along such approximations.

[10] We record one out of several ways to formalize this change of view: One may replace T_h with $p_\sharp T_h$, where $p\colon X^* \to \overline{B}_R(0)$ is any Lipschitz map with $p(y) = y$ for $y \in \overline{B}_R(0)$. Indeed, from $\operatorname{spt} \partial T_h \subset \operatorname{spt} T_h \subset \overline{B}_R(0)$ and the locality axiom one concludes $\|p_\sharp T_h\| = p_\sharp \|T_h\|$, $\|\partial p_\sharp T_h\| = p_\sharp \|\partial T_h\|$, and $i_\sharp p_\sharp T_h = T_h$ for the inclusion $i\colon \overline{B}_R(0) \to X^*$, which is sufficient in order to justify this replacement.

From Definition 2.8 we see $\|F_h^\pi\|_{\mathrm{C_b}(K)^*} \le \mathbf{M}(T_h)$, and by (1.3) the F_h^π are uniformly bounded in $\mathrm{C_b}(K)^*$. Exploiting by a diagonal argument that A^n is countable we may pass to some further subsequence, still indexed with $h(k)$, such that $F_{h(k)}^\pi$ w*-converge to some F^π in $C(K)^*$ for all $\pi \in A^n$. At this stage we start constructing the limit current T, which is first defined on $A_{[-1,1]} \times A^n$ by

$$T(\varphi\,\mathrm{d}\pi) := \langle F^\pi, \varphi\rangle \qquad \text{for } \varphi \in A_{[-1,1]} \text{ and } \pi \in A^n\,.$$

Here, F^π is well-defined on φ since $A_{[-1,1]} \subset \mathrm{C_b}(K)$ holds by Lemma 3.1.

Step 2. The functional T is continuous and extends to $\mathrm{Lip}_1(E;[-1,1])\times\mathrm{Lip}_1(E)^n$. We will prove that T is uniformly continuous on $A_{[-1,1]} \times A^n$ with respect to the pseudodistance $\mathrm{d}_\mathrm{p}^{1+n}$ introduced after Lemma 3.1, see also (3.1). Actually, for $\varphi, \widetilde{\varphi} \in A_{[-1,1]}$ and $\pi, \widetilde{\pi} \in A^n$ we have by the definition of T, Definition 2.8, Lemma 2.12, and the construction of μ and ν

$$
\begin{aligned}
(4.2)\quad & \big|T(\widetilde{\varphi}\,\mathrm{d}\widetilde{\pi}) - T(\varphi\,\mathrm{d}\pi)\big| \\
&= \lim_k \big|T_{h(k)}(\widetilde{\varphi}\,\mathrm{d}\widetilde{\pi}) - T_{h(k)}(\varphi\,\mathrm{d}\pi)\big| \\
&= \lim_k \big|T_{h(k)}((\widetilde{\varphi}-\varphi)\,\mathrm{d}\widetilde{\pi}) + T_{h(k)}(\varphi\,\mathrm{d}\widetilde{\pi}) - T_{h(k)}(\varphi\,\mathrm{d}\pi)\big| \\
&\le \lim_k \bigg[\int_K |\widetilde{\varphi}-\varphi|\,\mathrm{d}\|T_{h(k)}\| + 2\sum_{i=1}^n \int_K |\widetilde{\pi}_i - \pi_i|\,\mathrm{d}\big(\|T_{h(k)}\|+\|\partial T_{h(k)}\|\big)\bigg] \\
&= \int_\Sigma |\widetilde{\varphi}-\varphi|\,\mathrm{d}\mu + 2\sum_{i=1}^n \int_\Sigma |\widetilde{\pi}_i - \pi_i|\,\mathrm{d}(\mu+\nu)\,,
\end{aligned}
$$

where we made decisive use of the fact that $\varphi, \widetilde{\varphi}, \pi_i, \widetilde{\pi}_i$ are in $\mathrm{C_b}(K)$ by the choice of A. Now we fix $(\varphi,\pi) \in A_{[-1,1]} \times A^n$ and let $(\widetilde{\varphi},\widetilde{\pi})$ tend to (φ,π) with respect to $\mathrm{d}_\mathrm{p}^{1+n}$. Then, by the dominated convergence theorem with bound $\sup_\Sigma |\widetilde{\varphi}-\varphi| \le 2$ the first term on the right-hand side of (4.2) converges to 0. Moreover, by the 1-Lipschitz continuity of $\pi_i, \widetilde{\pi}_i$ on $\Sigma \subset \overline{B}_R(0)$ we have

$$\sup_\Sigma |\widetilde{\pi}_i - \pi_i| \le 2R + |\widetilde{\pi}_i(0) - \pi_i(0)|\,,$$

and thus also the second term on the right-hand side of (4.2) vanishes in the limit by dominated convergence. As $(\varphi,\pi) \in A_{[-1,1]} \times A^n$ is arbitrary, this proves continuity of T on $A_{[-1,1]} \times A^n$ with respect to $\mathrm{d}_\mathrm{p}^{1+n}$. Since both the right-hand side of (4.2) and $\mathrm{d}_\mathrm{p}^{1+n}((\widetilde{\varphi},\widetilde{\pi}),(\varphi,\pi))$ depend on (φ,π) only through the differences $(\widetilde{\varphi}-\varphi, \widetilde{\pi}-\pi)$, we even infer the claimed uniform continuity. Recalling that A is dense in $\mathrm{Lip}_1(E)$ (and $A_{[-1,1]}$ in $\mathrm{Lip}_1(E;[-1,1])$) we may extend T in a unique way to a continuous function, with respect to $\mathrm{d}_\mathrm{p}^{1+n}$, on $\mathrm{Lip}_1(E;[-1,1]) \times \mathrm{Lip}_1(E)^n$.

Step 3. T extends to $\mathrm{Lip}(E)\times\mathrm{Lip}(E)^n$ *and satisfies the axioms of an n-current.* We claim that

$$T((r\varphi+\widetilde{\varphi})\,\mathrm{d}\pi) = rT(\varphi\,\mathrm{d}\pi) + T(\widetilde{\varphi}\,\mathrm{d}\pi)$$

holds for $r \in \mathbb{R}$, whenever $\varphi, \widetilde{\varphi}, r\varphi+\widetilde{\varphi} \in \mathrm{Lip}_1(E;[-1,1])$ and $\pi_i \in \mathrm{Lip}_1(E)$. In fact, this the claim is immediate by the construction of T from the linear functionals F^π: it first follows if the previous functions are all in $A_{[-1,1]}$ and A, respectively, and then it easily extends by continuity. Similarly, one verifies linearity properties of T in the component functions π_i, which carry over from the $T_{h(k)}$ to the F^π. In conclusion, we may extend T to a $(1+n)$-linear map[11] $\mathrm{Lip}(E) \times \mathrm{Lip}(E)^n \to \mathbb{R}$, and

[11]Note that on the ball E we have $\mathrm{Lip_b}(E) = \mathrm{Lip}(E)$.

this extension is unique. We now check T satisfies the axioms of a current from Definition 2.10. The multilinearity axiom is obviously valid by the last extension step, and the continuity axiom follows from the continuity of T with respect to $\mathrm{d}_\mathrm{p}^{1+n}$ on $\mathrm{Lip}_1(E;[-1,1])\times\mathrm{Lip}_1(E)^n$. Finally, with some more effort we verify the locality axiom. We consider $(\varphi,\pi)\in\mathrm{Lip}_1(E;[-1,1])\times\mathrm{Lip}_1(E)^n$ such that some π_{i_0} equals a constant $s\in\mathbb{R}$ on a neighborhood of $\operatorname{spt}\varphi$. Using the σ-compactness of Σ for a given $\varepsilon>0$ we find a compact set C with $(\mu{+}\nu)(\Sigma\setminus C)\le\varepsilon$. Then, by Lemma 2.4 there is some w^*-continuous Lipschitz function $\eta\colon E\to[-1,1]$ with $\eta\equiv-1$ on $C\cap\operatorname{spt}(\pi_{i_0}{-}s)$ and $\eta\equiv1$ on $C\cap\operatorname{spt}\varphi$. For later use we record that the lemma also yields an ε-independent bound[12] for $\mathrm{Lip}(\eta)$. Next, for $(\widetilde\varphi,\widetilde\pi)\in A_{[-1,1]}\times A^n$ we estimate via Definition 2.8 and Lemma 2.13

$$
\begin{aligned}
|T(\widetilde\varphi\,\mathrm{d}\widetilde\pi)| &= \lim_k |T_{h(k)}(\widetilde\varphi\,\mathrm{d}\widetilde\pi)|\\
&= \lim_k |T_{h(k)}((1{-}\eta_+)\widetilde\varphi\,\mathrm{d}\widetilde\pi)+T_{h(k)}(\eta_+\widetilde\varphi\,\mathrm{d}\widetilde\pi)|\\
&\le \limsup_k \bigg[\int_K(1{-}\eta_+)|\widetilde\varphi|\,\mathrm{d}\|T_{h(k)}\|\\
&\qquad + (\sup_E|\eta_+\widetilde\varphi|+\mathrm{Lip}(\eta_+\widetilde\varphi))\int_{\operatorname{spt}\eta_+}|\widetilde\pi_{i_0}{-}s|\,\mathrm{d}\big(\|T_{h(k)}\|{+}\|\partial T_{h(k)}\|\big)\bigg]\\
&\le \int_K(1{-}\eta_+)|\widetilde\varphi|\,\mathrm{d}\mu+(2+\mathrm{Lip}(\eta))\int_{\{\eta\ge0\}}|\widetilde\pi_{i_0}{-}s|\,\mathrm{d}(\mu{+}\nu)\\
&\le \int_{\Sigma\cap\{\eta\ne1\}}|\widetilde\varphi|\,\mathrm{d}\mu+(2+\mathrm{Lip}(\eta))\int_{\Sigma\cap\{\eta\ne-1\}}|\widetilde\pi_{i_0}{-}s|\,\mathrm{d}(\mu{+}\nu)\,.
\end{aligned}
\tag{4.3}
$$

Here, we also used that $\widetilde\varphi$, $\widetilde\pi_{i_0}$, and η_+ are w*-continuous and that the intermediate one of the three sets $\operatorname{spt}\eta_+\subset\{\eta\ge0\}\subset\{\eta\ne-1\}$ is w*-closed in K. Now we send $(\widetilde\varphi,\widetilde\pi)$ to (φ,π) with respect to $\mathrm{d}_\mathrm{p}^{1+n}$ in the resulting estimate. Then the left-hand side of (4.3) converges to $|T(\varphi\,\mathrm{d}\pi)|$ by the continuity of T. By dominated convergence with the bounds

$$
\sup_\Sigma|\widetilde\varphi|\le1 \qquad\text{and}\qquad \sup_\Sigma|\widetilde\pi_{i_0}{-}s|\le R+|\widetilde\pi_{i_0}(0)-s| \tag{4.4}
$$

also the right-hand side of (4.3) converges to its analogue with $\widetilde\varphi$ and $\widetilde\pi$ replaced by φ and π, respectively. Since φ vanishes on $C\cap\{\eta\ne1\}$ and $\pi_{i_0}{-}s$ vanishes on $C\cap\{\eta\ne-1\}$, all in all we come out with

$$
|T(\varphi\,\mathrm{d}\pi)|\le\int_{\Sigma\setminus C}|\varphi|\,\mathrm{d}\mu+(2+\mathrm{Lip}(\eta))\int_{\Sigma\setminus C}|\pi_{i_0}{-}s|\,\mathrm{d}(\mu{+}\nu)\,.
$$

Now we exploit $(\mu{+}\nu)(\Sigma\setminus C)\le\varepsilon$ and use the analogue of (4.4) for φ and π to arrive at

$$
|T(\varphi\,\mathrm{d}\pi)|\le\varepsilon+(2+\mathrm{Lip}(\eta))(R+|\pi_{i_0}(0)-s|)\varepsilon\,.
$$

Recalling that $\mathrm{Lip}(\eta)$ is bounded ε-independently we finally conclude $T(\varphi\,\mathrm{d}\pi)=0$, and the locality axiom is verified.

Step 4. There hold $T\in\mathbf{N}_n(E)$, $\mathbf{M}(T)\le M$, and $\mathbf{M}(\partial T)\le M_\partial$.
We observe

$$
|T(\varphi\,\mathrm{d}\pi)|=\lim_k|T_{h(k)}(\varphi\,\mathrm{d}\pi)|\le\lim_k\int_E|\varphi|\,\mathrm{d}\|T_{h(k)}\|=\int_E|\varphi|\,\mathrm{d}\mu
$$

first for $(\varphi,\pi)\in A_{[-1,1]}\times A^n$. By continuity with respect to $\mathrm{d}_\mathrm{p}^{1+n}$ — which we proved for the left-hand side and which is much simpler to check for the right-hand

[12]In fact, the lemma gives $\mathrm{Lip}(\eta)\le16/\mathrm{dist}(\operatorname{spt}\varphi,\operatorname{spt}(\pi_{i_0}{-}s))$.

side — the resulting inequality extends to all $(\varphi,\pi)\in \mathrm{Lip}_1(E;[-1,1])\times \mathrm{Lip}_1(E)^n$, and this proves that T has finite mass in the sense of Definition 2.8 and $\|T\|\le\mu$. Having checked the the axioms of a current in the previous step we infer $T\in\mathbf{M}_n(E)$, and in view of (4.1) we have $\mathbf{M}(T)\le M$. As mentioned after Definition 2.11, to check that T is normal it only remains to verify that the ∂T has finite mass. To this end we first compute

$$\begin{aligned}|T(\mathrm{d}(\varphi,\pi))| &= \lim_k |T_{h(k)}(\mathrm{d}(\varphi,\pi))|\\ &= \lim_k |\partial T_{h(k)}(\varphi\,\mathrm{d}\pi)| \le \lim_k \int_E |\varphi|\,\mathrm{d}\|\partial T_{h(k)}\| = \int_E |\varphi|\,\mathrm{d}\nu\,,\end{aligned}$$

for $(\varphi,\pi)\in A\times A^{n-1}$. The resulting inequality carries over to all $(\varphi,\pi)\in \mathrm{Lip}_1(E)\times\mathrm{Lip}_1(E)^{n-1}$, and from the definitions of boundary and mass we see that ∂T has finite mass and $\|\partial T\|\le\nu$. In view of (4.1) we also get $\mathbf{M}(\partial T)\le M_\partial$.

Step 5. $T_{h(k)}$ w-converges to T in X^* in the sense of Definition 2.15.* We fix $\varphi\in \mathrm{C_b}(K)\cap\mathrm{Lip}_1(E;[-1,1])$, $\pi_i\in\mathrm{C_b}(K)\cap\mathrm{Lip}_1(E)$ and we notice that — repeating in parts a previous reasoning — one may obtain the estimate

$$\lim_{A_{[-1,1]}\times A^n\ni(\widetilde\varphi,\widetilde\pi)\to(\varphi,\pi)}\Big[\lim_k \big|T_{h(k)}(\widetilde\varphi\,\mathrm{d}\widetilde\pi)-T_{h(k)}(\varphi\,\mathrm{d}\pi)\big|\Big]=0\,. \tag{4.5}$$

In fact, the relevant arguments in order to establish (4.5) are precisely those starting from the second line of (4.2). Keeping in mind that φ and π_i are in $\mathrm{C_b}(K)$, these arguments apply unchanged, and we do not repeat them. Proceeding with the proof we notice

$$\begin{aligned}\lim_{A_{[-1,1]}\times A^n\ni(\widetilde\varphi,\widetilde\pi)\to(\varphi,\pi)}\Big[\lim_k T_{h(k)}(\widetilde\varphi\,\mathrm{d}\widetilde\pi)\Big] &= \lim_{A_{[-1,1]}\times A^n\ni(\widetilde\varphi,\widetilde\pi)\to(\varphi,\pi)} T(\widetilde\varphi\,\mathrm{d}\widetilde\pi)\\ &= T(\varphi\,\mathrm{d}\pi)\,,\end{aligned} \tag{4.6}$$

where we used the construction and the continuity of T with respect to $\mathrm{d}_\mathrm{p}^{1+n}$ once more. Combining (4.5) and (4.6) we arrive at

$$\lim_k T_{h(k)}(\varphi\,\mathrm{d}\pi)=T(\varphi\,\mathrm{d}\pi)\,, \tag{4.7}$$

which by multilinearity stays true for all $\varphi,\pi_i\in\mathrm{C_b}(K)\cap\mathrm{Lip}(E)$. At this point we return to our original point of view and we reconsider[13] the $T_{h(k)}$ and T as n-currents in X^* (with supports in $\overline{B}_R(0)$). Then the convergence in (4.7) remains true for all w*-continuous $(\varphi,\pi)\in\mathrm{Lip_b}(X^*)\times\mathrm{Lip}(X^*)^n$, and thus we have proved that $T_{h(k)}$ w*-converges to $T\in\mathbf{N}_n(X^*)$ with $\mathbf{M}(T)\le M$ and $\mathbf{M}(\partial T)\le M_\partial$. ☐

4.2. **The general case.** Following the proof of [3, Theorem 6.6] we now use slicing by large balls in order to establish the full statement of Theorem 1.4. For our purposes it will be sufficient to define slices of normal currents $T\in\mathbf{N}_n(E)$ for a complete metric space E and $n\in\mathbb{N}$ by

$$\langle T,y_0,R\rangle := \partial(T\llcorner B_R(y_0))-(\partial T)\llcorner B_R(y_0) \tag{4.8}$$

for $y_0\in E$ and $0<R<\infty$. The slicing theorem [3, Theorem 5.6] then implies that $\mathrm{spt}\langle T,y_0,R\rangle$ is contained in the sphere $\{y\in E : \mathrm{d}_E(y,y_0)=R\}$, that

$$\langle T,y_0,R\rangle\in\mathbf{N}_{n-1}(E)$$

[13] Formally, one may use the push-forward along the inclusion $i\colon \overline{B}_R(0)\to X^*$ here; compare the footnote at the beginning of the proof.

is a normal current for $\mathcal{L}^1$-almost every $R > 0$, and that we have

$$\int_0^\infty \mathbf{M}(\langle T, y_0, R\rangle)\,\mathrm{d}R \le \mathbf{M}(T) \tag{4.9}$$

(with $\mathcal{L}^1$-measurable integrand on the left-hand side). The next simple lemma will be useful in order to choose good radii.

Lemma 4.2. *Consider a complete metric space E, $y_0 \in E$, $n \in \mathbb{N}$, $T \in \mathbf{N}_n(E)$, $\Gamma > 0$, and a Borel function $f\colon (0,\infty) \to [0,\infty)$. Then for every Borel subset G of $(0,\infty)$ with $\int_G f\,\mathrm{d}R \ge \Gamma$ there is some $R \in G$ such that $\langle T, y_0, R\rangle$ is normal with*

$$\mathbf{M}(\langle T, y_0, R\rangle) \le \Gamma^{-1}\mathbf{M}(T) f(R)\,.$$

Proof. We fix G with $\int_G f\,\mathrm{d}R \ge \Gamma$ and assume the claim to be wrong. Then the integral on the left-hand side of (4.9) would be strictly larger than $\Gamma^{-1}\mathbf{M}(T)\int_G f(R)\,\mathrm{d}R$ and would thus exceed $\mathbf{M}(T)$. Hence, we arrive at a contradiction to (4.9). □

At this point we are in the position to conclude the compactness proof.

Proof of Theorem 1.4. We only deal with the case $n \ge 1$, since the case $n = 0$ can be treated by a much simpler variant of the following reasoning. By Lemma 4.2 with $f \equiv 1$ and $\Gamma = m \in \mathbb{N}$ we can choose some $R_h^m \in [m, 2m)$ with $\mathbf{M}(\langle T_h, 0, R_h^m\rangle) \le m^{-1}\mathbf{M}(T_h)$. Setting

$$T_h^m := T_h \llcorner B_{R_h^m}(0)$$

and keeping (4.8) in mind we thus have $\operatorname{spt} T_h^m \subset B_{2m}(0)$, $\mathbf{M}(T_h^m) \le \mathbf{M}(T_h) \le M$, and $\mathbf{M}(\partial T_h^m) \le \mathbf{M}(\partial T_h) + m^{-1}\mathbf{M}(T_h) \le \mathbf{M}_\partial + m^{-1}M$. All in all, we conclude that $(T_h^m)_{h\in\mathbb{N}}$ satisfies the assumptions of Theorem 4.1, and thus for a subsequence $T_{h(k)}^m$ w*-converges to some $T^m \in \mathbf{N}_n(X^*)$ with

$$\mathbf{M}(T^m) \le M \qquad \text{and} \qquad \mathbf{M}(\partial T^m) \le M_\partial + m^{-1}M\,. \tag{4.10}$$

Here, by a diagonal argument we may assume that this convergence is valid on the same subsequence $h(k)$ independent of $m \in \mathbb{N}$. Using the semicontinuity of Corollary 3.4 we find

$$\begin{aligned}\mathbf{M}(T^m - T^{\widetilde{m}}) &\le \liminf_k \mathbf{M}(T_{h(k)}^m - T_{h(k)}^{\widetilde{m}})\\ &\le \sup_{h\in\mathbb{N}} \left[\mathbf{M}(T_h - T_h^m) + \mathbf{M}(T_h - T_h^{\widetilde{m}})\right]\\ &\le \sup_{h\in\mathbb{N}} \left[\|T_h\|(X^* \setminus B_m(0)) + \|T_h\|(X^* \setminus B_{\widetilde{m}}(0))\right].\end{aligned} \tag{4.11}$$

In view of (1.4) this implies

$$\limsup_{\min\{m,\widetilde{m}\}\to\infty} \mathbf{M}(T^m - T^{\widetilde{m}}) = 0\,,$$

and $(T^m)_{m\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $\mathbf{M}_n(X^*)$ with the mass norm. Consequently, T^m converges to some $T \in \mathbf{M}_n(X^*)$ in mass and in particular weakly. We observe

$$\begin{aligned}&\limsup_k |T_{h(k)}(\varphi\,\mathrm{d}\pi) - T(\varphi\,d\pi)|\\ &\qquad\le \limsup_k |T_{h(k)}(\varphi\,\mathrm{d}\pi) - T_{h(k)}^m(\varphi\,d\pi)| + |T^m(\varphi\,\mathrm{d}\pi) - T(\varphi\,d\pi)|\\ &\qquad\le \sup_{h\in\mathbb{N}} \|T_h\|(X^* \setminus B_m(0)) \sup_{X^*} |\varphi| \prod_{i=1}^n \mathrm{Lip}(\pi_i) + |T^m(\varphi\,\mathrm{d}\pi) - T(\varphi\,d\pi)|\end{aligned}$$

for all w*-continuous $(\varphi,\pi)\in \mathrm{Lip}_{\mathrm{b}}(X^*)\times \mathrm{Lip}(X^*)^n$. Sending $m\to\infty$, using (1.4) once more, and exploiting the construction of T we conclude that $T_{h(k)}$ w*-converges to T. In order to control ∂T we repeat the estimate (4.11) for the boundaries, and involving (1.4) once more we conclude that $(\partial T^m)_{m\in\mathbb{N}}$ is a Cauchy sequence, which converges to some $S\in \mathbf{M}_{n-1}(X^*)$ in mass and weakly. From Definition 2.6 and the weak convergences of T^m and ∂T^m we get $\partial T = S$, and in particular T is normal. Finally we have $\mathbf{M}(T) = \lim_m \mathbf{M}(T_m) \le M$ and $\mathbf{M}(\partial T) = \mathbf{M}(S) = \lim_m \mathbf{M}(\partial T_m) \le \lim_m (M_\partial + m^{-1}M) = M_\partial$ by convergence in mass and (4.10). □

5. Proofs of the existence results

In this section we implement the proofs of Theorem 1.1, Corollary 1.2, and Theorem 1.3.

5.1. **Plateau's problem.** At this stage a simple existence proof for the Plateau problem can be given for the bounded boundaries S of Corollary 1.2 in separable Hilbert spaces $\mathcal{H}$. Actually, projections onto balls in $\mathcal{H}$ are Lip_1-functions, and the pushforward under such projections decreases mass. Starting from this observation the boundedness of $\operatorname{spt} S$ implies that one may always pass to a minimizing sequence with equi-bounded supports. For such a sequence Theorem 4.1 yields compactness, and existence is readily proved by the direct method of the calculus of variations. Thus, in this situation we do not need the full strength of Theorem 1.4, but just the weaker statement of Theorem 4.1 for the equi-bounded case; in particular, we can avoid all slicing arguments.

We have not been able to mimic the same approach to Corollary 1.2 for bounded boundaries in Banach spaces without Hilbertian structure, and actually we do not know whether minimizing sequences or the minimizers of Corollary 1.2 have bounded support in this generality. Consequently, we are forced to work with possibly unbounded supports in the following. We will see that in this setup the full statements of Theorems 1.1 and 1.4 and the w*-tightness condition (1.4) appear quite naturally.

Proof of Theorem 1.1. We first prove the claim for normal currents. Since by assumption $\mathrm{Fillmass}(S)$ is finite, the admissible set

$$\mathcal{A} := \{C\in \mathbf{N}_n(X^*) \,:\, \partial C = S\}$$

is non-empty, and for the remainder of the proof we can fix a $V\in\mathbf{N}_n(X^*)$ with $\partial V = S$. Moreover, we may choose a minimizing sequence $(T_h)_{h\in\mathbb{N}}$ in $\mathcal{A}$ with

$$\lim_h \mathbf{M}(T_h) = \inf_{C\in\mathcal{A}} \mathbf{M}(C)\,.$$

In order to apply Theorem 1.4 we will now prove that this sequence satisfies (1.4). For arbitrary $\varepsilon>0$ we choose a radius R_0 with

$$\|V\|(X^*\setminus B_{R_0}(0)) \le \varepsilon\,,$$

and setting $\Gamma := \varepsilon^{-1}\sup_{h\in\mathbb{N}}\mathbf{M}(T_h{-}V) < \infty$ we fix $m\in\mathbb{N}$ with $\int_{R_0}^{R_0+m} R^{-1}\,\mathrm{d}R \ge \Gamma$. Then Lemma 4.2 with $f(R)=R^{-1}$ provides further radii $R_h\in[R_0,R_0+m)$ such that we have $\langle T_h{-}V,0,R_h\rangle \in \mathbf{N}_{n-1}(X^*)$ and

$$\mathbf{M}(\langle T_h{-}V,0,R_h\rangle) \le \Gamma^{-1}\mathbf{M}(T_h{-}V)R_h^{-1} \le \varepsilon R_h^{-1}\,.$$

In view of $\partial T_h = \partial V = S$, (4.8) gives

$$\partial((T_h{-}V)\llcorner B_{R_h}(0) + V) = \langle T_h{-}V,0,R_h\rangle + S\,, \tag{5.1}$$

and in particular we have $\partial\langle T_h - V, 0, R_h\rangle = -\partial S = -\partial\partial V \equiv 0$. Consequently, relying on the cone construction [3, Definition 10.1 and Proposition 10.2] there are $C_h \in \mathbf{N}_n(X^*)$ with

$$\partial C_h = \langle T_h - V, 0, R_h\rangle \tag{5.2}$$

and

$$\mathbf{M}(C_h) \le R_h\,\mathbf{M}(\langle T_h - V, 0, R_h\rangle)\,.$$

It follows from (5.1) and (5.2) that $\partial((T_h - V)\llcorner B_{R_h}(0) + V - C_h) = S$ holds and the currents $(T_h - V)\llcorner B_{R_h}(0) + V - C_h$ are in the admissible class $\mathcal{A}$. Testing with these currents and putting things together we infer

$$\begin{aligned}\inf_{C\in\mathcal{A}} \mathbf{M}(C) &\le \|T_h\|(B_{R_h}(0)) + \|V\|(X^* \setminus B_{R_h}(0)) + \mathbf{M}(C_h)\\ &\le \|T_h\|(B_{R_0+m}(0)) + \|V\|(X^* \setminus B_{R_0}(0)) + R_h\,\mathbf{M}(\langle T_h - V, 0, R_h\rangle)\\ &\le \mathbf{M}(T_h) - \|T_h\|(X^* \setminus B_{R_0+m}(0)) + 2\varepsilon\,.\end{aligned}$$

We rewrite the resulting inequality as

$$\|T_h\|(X^* \setminus B_{R_0+m}(0)) \le 2\varepsilon + \mathbf{M}(T_h) - \inf_{C\in\mathcal{A}} \mathbf{M}(C)$$

and exploit it in the next estimate. Moreover, we use that $\|T_h\|(X^* \setminus B_R(0))$ tends to 0 for $R \to \infty$, and that this convergence stays true for *finite* suprema. Thus, for arbitrarily fixed $k \in \mathbb{N}$ we get

$$\begin{aligned}&\lim_{R\to\infty} \sup_{h\in\mathbb{N}} \|T_h\|(X^* \setminus B_R(0))\\ &\qquad \le \lim_{R\to\infty} \sup_{h>k} \|T_h\|(X^* \setminus B_R(0)) + \lim_{R\to\infty} \sup_{h\le k} \|T_h\|(X^* \setminus B_R(0))\\ &\qquad \le \sup_{h>k} \|T_h\|(X^* \setminus B_{R_0+m}(0))\\ &\qquad \le 2\varepsilon + \sup_{h>k} \mathbf{M}(T_h) - \inf_{C\in\mathcal{A}} \mathbf{M}(C)\,.\end{aligned}$$

Recalling that $(T_h)_{h\in\mathbb{N}}$ is a minimizing sequence, we send $k \to \infty$ and $\varepsilon \searrow 0$ to conclude

$$\lim_{R\to\infty} \sup_{h\in\mathbb{N}} \|T_h\|(X^* \setminus B_R(0)) = 0\,.$$

Since the boundary $\partial T_h = S$ is fixed, we trivially have

$$\lim_{R\to\infty} \sup_{h\in\mathbb{N}} \|\partial T_h\|(X^* \setminus B_R(0)) = \lim_{R\to\infty} \|S\|(X^* \setminus B_R(0)) = 0\,.$$

Therefore we have established the validity of the tightness assumption (1.4) in Theorem 1.4. Exploiting once more that $(T_h)_{h\in\mathbb{N}}$ is a minimizing sequence with fixed boundary, also the boundedness assumptions in Theorem 1.4 are available, and the theorem yields a subsequence $T_{h(k)}$ w*-converging to some $T \in \mathbf{N}_n(X^*)$. From Corollary 3.4 we get $\partial T = S$ and

$$\mathbf{M}(T) \le \liminf_k \mathbf{M}(T_{h(k)}) = \inf_{C\in\mathcal{A}} \mathbf{M}(C)\,.$$

Hence, the proof for the case of normal currents is complete.

For integral currents we use an analogous reasoning. Indeed, the only change is that we make use of a couple of additional results from [3], namely the closure theorem for integer-rectifiable currents and the facts that slicing and cone construction preserve integer-rectifiability; see Theorems 5.7, 8.5, and 10.4 in [3]. □

Proof of Corollary 1.2. It suffices to show for given S that the assumptions of Corollary 1.2 imply those of Theorem 1.1. Indeed, by [3, Definition 10.1 and Proposition 10.2] every $S \in \mathbf{N}_{n-1}(X^*)$ with $\operatorname{spt} S \subset \overline{B}_R(0)$ and $\partial S \equiv 0$ can be written as the boundary ∂C of a cone $C \in \mathbf{N}_n(X^*)$ (with $\mathbf{M}(C) \le R\mathbf{M}(S)$) and thus satisfies $\operatorname{Fillmass}(S) < \infty$. Furthermore, every $S \in \mathbf{I}_0(X^*)$ with $\partial S \equiv 0$ has automatically bounded support, by [3, Theorem 10.4] the corresponding cone is in $\mathbf{I}_1(X^*)$, and we get $\operatorname{Fillvol}(S) < \infty$ in this case. Finally, in the case $n \ge 2$ [13, Corollary 1.3] implies that for every $S \in \mathbf{I}_{n-1}(X^*)$ with $\partial S \equiv 0$ there exists some $C \in \mathbf{I}_n(X^*)$ with $\partial C = S$ (and $\mathbf{M}(C) \le D_n \mathbf{M}(S)^{\frac{n}{n-1}}$ for a dimensional constant D_n). Thus, also in this case there holds $\operatorname{Fillvol}(S) < \infty$. □

5.2. **The free boundary problem.** By quite similar means we next establish our existence statement including free boundaries.

Proof of Theorem 1.3. We start proving the claim for normal currents. We choose a minimizing sequence $(T_h)_{h\in\mathbb{N}}$ in $\mathbf{N}_n(X^*)$ with

$$\lim_h \big[\mathbf{M}(T_h)+\mathbf{M}(\partial T_h - S)\big] = \inf_{C\in\mathbf{N}_n(X^*)} \big[\mathbf{M}(C)+\mathbf{M}(\partial C - S)\big] \le \mathbf{M}(S) < \infty \,. \tag{5.3}$$

In order to apply Theorem 1.4 we will now prove that this sequence satisfies (1.4). For arbitrary $\varepsilon > 0$ we choose a radius R_0 with

$$\|S\|(X^* \setminus B_{R_0}(0)) \le \varepsilon \,,$$

and we set $\Gamma := \varepsilon^{-1} \sup_{h\in\mathbb{N}} \mathbf{M}(T_h) < \infty$. Then Lemma 4.2 with $f \equiv 1$ provides further radii $R_h \in [R_0, R_0+\Gamma)$ such that we have $\langle T_h, 0, R_h\rangle \in \mathbf{N}_{n-1}(X^*)$ and

$$\mathbf{M}(\langle T_h, 0, R_h\rangle) \le \Gamma^{-1}\mathbf{M}(T_h) \le \varepsilon \,.$$

Moreover, from (4.8) we get

$$\partial(T_h \llcorner B_{R_h}(0)) - S = (\partial T_h - S)\llcorner B_{R_h}(0) - S\llcorner(X^* \setminus B_{R_h}(0)) + \langle T_h, 0, R_h\rangle \,.$$

Since the masses of the last two terms on the right-hand side of the preceding formula are controlled by ε, testing with $T_h \llcorner B_{R_h}(0)$ we infer

$$\begin{aligned} \inf_{C\in\mathbf{N}_n(X^*)} &\big[\mathbf{M}(C) + \mathbf{M}(\partial C - S)\big] \\ &\le \mathbf{M}(T_h \llcorner B_{R_h}(0)) + \mathbf{M}(\partial(T_h \llcorner B_{R_h}(0)) - S) \\ &\le (\|T_h\| + \|\partial T_h - S\|)(B_{R_h}(0)) + 2\varepsilon \\ &\le \mathbf{M}(T_h) + \mathbf{M}(\partial T_h - S) - (\|T_h\| + \|\partial T_h\|)(X^* \setminus B_{R_0+\Gamma}(0)) + 3\varepsilon \,. \end{aligned}$$

We rewrite the resulting inequality as

$$\begin{aligned} (\|T_h\| &+ \|\partial T_h\|)(X^* \setminus B_{R_0+\Gamma}(0)) \\ &\le 3\varepsilon + \mathbf{M}(T_h) + \mathbf{M}(\partial T_h - S) - \inf_{C\in\mathbf{N}_n(X^*)} \big[\mathbf{M}(C) + \mathbf{M}(\partial C - S)\big] \end{aligned}$$

and exploit it in the next estimate. We also use that $(\|T_h\| + \|\partial T_h\|)(X^* \setminus B_R(0))$ tends to 0 for $R \to \infty$, and that this convergence stays true for *finite* suprema.

Thus, for arbitrarily fixed $k \in \mathbb{N}$ we get

$$\begin{aligned}
&\lim_{R\to\infty} \sup_{h\in\mathbb{N}} (\|T_h\|+\|\partial T_h\|)(X^* \setminus B_R(0)) \\
&\le \lim_{R\to\infty} \sup_{h>k} (\|T_h\|+\|\partial T_h\|)(X^* \setminus B_R(0)) + \lim_{R\to\infty} \sup_{h\le k} (\|T_h\|+\|\partial T_h\|)(X^* \setminus B_R(0)) \\
&\le \sup_{h>k} (\|T_h\|+\|\partial T_h\|)(X^* \setminus B_{R_0+\Gamma}(0)) \\
&\le 3\varepsilon + \sup_{h>k} \big[\mathbf{M}(T_h) + \mathbf{M}(\partial T_h - S)\big] - \inf_{C\in \mathbf{N}_n(X^*)} \big[\mathbf{M}(C) + \mathbf{M}(\partial C - S)\big]\,.
\end{aligned}$$

Recalling (5.3) we send $k \to \infty$ and $\varepsilon \searrow 0$ to conclude

$$\lim_{R\to\infty} \sup_{h\in\mathbb{N}} (\|T_h\|+\|\partial T_h\|)(X^* \setminus B_R(0)) = 0\,,$$

and hence we have indeed established the validity of the tightness assumption (1.4) in Theorem 1.4. Exploiting (5.3) once more also the boundedness assumptions in Theorem 1.4 are available, and the theorem yields a subsequence $T_{h(k)}$ w*-converging to some $T \in \mathbf{N}_n(X^*)$. As a consequence also $\partial T_{h(k)} - S$ w*-converges to $\partial T - S$ and from Corollary 3.4 we get

$$\begin{aligned}
\mathbf{M}(T) + \mathbf{M}(\partial T - S) &\le \liminf_k \big[\mathbf{M}(T_{h(k)}) + \mathbf{M}(\partial T_{h(k)} - S)\big] \\
&= \inf_{C\in \mathbf{N}_n(X^*)} \big[\mathbf{M}(C) + \mathbf{M}(\partial C - S)\big]\,.
\end{aligned}$$

Hence, the proof for the case of normal currents is complete.

In the case of integral currents we choose the minimizing sequence $(T_h)_{h\in\mathbb{N}}$ in $\mathbf{I}_n(X^*)$ and argue in a completely analogous way. As the only change we additionally apply the closure theorem for integer-rectifiable currents [3, Theorem 8.5] at the very end of our reasoning to conclude $T \in \mathbf{I}_n(X^*)$. □

Appendix A. An abstract extension

In this appendix we consider a separable Banach space Y with an additional topology $\mathcal{W}$ which is weaker than the norm topology. If $\mathcal{W}$ satisfies suitable (compactness) assumptions, then it can indeed take the role of the w*-topology in our arguments, and we have the following abstract extension of our main results.

Theorem A.1. *For a separable Banach space Y suppose that $(Y, \mathcal{W})$ is a topological vector space with a topology $\mathcal{W}$, weaker than the norm topology. Moreover, assume that*

(A.1) *$\mathcal{W}$ is boundedly metrizable and boundedly compact,*

that is closed norm balls in Y are compact with respect to $\mathcal{W}$, and the trace of $\mathcal{W}$ on such balls is induced by a distance. Then our existence results hold verbatim with Y in place of X^*. Likewise, Theorem 1.4 extends when w**-convergence is replaced by the appropriate concept of $\mathcal{W}$-convergence.*

We stress that Theorem A.1 potentially applies to non-dual spaces Y. To mention one concrete example we briefly comment on the choice $Y := \mathrm{L}\log\mathrm{L}([0,1])$ of the Zygmund space (with the Luxemburg norm). We embed[14] $\mathrm{L}\log\mathrm{L}([0,1])$ in the space $\mathcal{M}([0,1]) \cong \mathrm{C_b}([0,1])^*$ of finite signed Borel measures measures on $[0,1]$, and we take $\mathcal{W}$ as the trace of the w*-topology. Then the assumptions of Theorem A.1 are valid and we come out with existence results in $\mathrm{L}\log\mathrm{L}([0,1])$.

[14] We here refer to the continuous linear embedding which assigns to $f \in \mathrm{L}\log\mathrm{L}([0,1])$ the weighted Lebesgue measure on $[0,1]$ with weight function f.

After all, the preceding example still relies on a w*-topology on a w*-closed subspace of a dual, and it could in fact be treated in a less abstract way. Nevertheless — even though we are not aware of a definite application to a true non-w*-topology — we believe that Theorem A.1 may still be of interest in order to single out those properties of duals and the w*-topology which are essential for the existence program.

From the assumptions of Theorem A.1 one can draw a couple of conclusions about the topology $\mathcal{W}$. We mention the following two: On the one hand we observe that $\mathcal{W}$ coincides with the norm topology on all norm-(locally-)compact subsets and in particular on all finite-dimensional subspaces of Y. On the other hand $\mathcal{W}$ sees the norm of Y in the sense of

$$\|y\|_Y = \sup\{\varphi(y) \,:\, \varphi \in \mathrm{Lip}_1(Y) \text{ is } \mathcal{W}\text{-continuous with } \varphi(0) = 0\} \tag{A.2}$$

for all $y \in Y$. In fact, to establish (A.2) it suffices to extend the function φ on $\{0, y\}$ with $\varphi(0) := 0$ and $\varphi(y) := \|y\|_Y$ to a $\mathcal{W}$-continuous Lip_1-function on Y. Such an extension can in turn be found by a straightforward adaption of the arguments in the proof of [3, Theorem 6.2].

On the proof of Theorem A.1. We check that under the present assumptions our preliminary lemmas hold for $\mathcal{W}$ in place of the w*-topology.

First of all, in case of Lemma 2.3 this is obvious from its proof.

Regarding Lemma 2.4 only the first step of the proof needs a slight modification: indeed, we replace X^* with Y and the mapping $y \mapsto \langle y - y_P, x\rangle$ with φ, where $\varphi \in \mathrm{Lip}_1(Y)$ is $\mathcal{W}$-continuous with $\varphi(y_P) = 0$ and $\varphi(y_Q) \geq \frac{1}{2}(\|y_Q - y_P\|_Y + r_P + r_Q)$. Here, the choice of a suitable φ is possible due to (A.2).

In case of Lemma 3.1 we perform a similar modification: by (A.2) we choose $\mathcal{W}$-continuous functions $\varphi_{j,h} \in \mathrm{Lip}_1(E)$ with $\varphi_{j,h}(s_j) = 0$ and $\varphi_{j,h}(s_h) \geq \|s_h - s_j\|_Y$, and we change the definition of $a_{i,j,k} \in \mathrm{Lip}_1(E)$ to

$$a_{i,j,k}(y) := \max\{r_i + \varphi_{j,h}(y) \,:\, h \in \{1, 2, \ldots, k\}\}$$

for $y \in E$. For $y \in \{s_1, s_2, s_3, \ldots\}$ it follows that $a_{i,j,k}(y)$ converges monotonically from below to $r_i + \|y - s_j\|_Y$ as $k \to \infty$, and this convergence suffices to conclude the proof of the lemma in precisely the same way as before.

With all relevant preliminary lemmas at hand, the conclusions follow along the lines of the previous sections: one can first deduce the analogue of Corollary 3.4 for the $\mathcal{W}$-convergence of currents (defined with $\mathcal{W}$-continuous test functions $(\varphi, \pi) \in \mathrm{Lip_b}(Y) \times \mathrm{Lip}(Y)^n$), and then one proves the claimed compactness and existence results for normal currents along the lines of Sections 4 and 5. Here, we exploit (A.1) as a replacement for (2.1) and Theorem 2.1, but apart from that the required changes are mostly notational ones. Regarding the existence results for integral currents we finally rely on $\mathcal{W}$-versions of the results in [3, Section 8]; once more such versions can be established by the same arguments as in the case of the w*-topology. □

Appendix B. Non-separable spaces

In this appendix we show that — in remarkable contrast — our results generally fail in non-separable duals, while they carry over to non-separable reflexive spaces.

B.1. **A counterexample in non-separable duals.** The following example is inspired by an idea of Bogachev [5, 7.14.149] and illustrates the necessity of the separability assumption on X^*.

Example B.1. *We show that the statements of Theorems 1.4 and 4.1 fail in non-separable duals X^*, even for $n = 0$ and even if X is separable. Specifically, we provide counterexamples in the cases $X^* \cong \mathrm{L}^\infty([0,1])$ and $X^* \cong \mathcal{M}([0,1])$. As a byproduct it follows that the inclusion $\mathscr{B}_{\mathrm{w}^*}(X^*) \subset \mathscr{B}(X^*)$ of Lemma 2.3 is strict in these cases.*

Construction of Example B.1. Denote by X one of the separable spaces $\mathrm{L}^1([0,1])$ and $\mathrm{C_b}([0,1])$. Then it is well known that we have $X^* \cong \mathrm{L}^\infty([0,1])$ and $X^* \cong \mathcal{M}([0,1])$, respectively (compare Theorem 2.2 in the latter case). We start with a sequence $(\mu_h)_{h\in\mathbb{N}}$ of Borel probability measures on $[0,1]$ such that each μ_h is concentrated on a finite set and such that μ_h converges weakly to the Lebesgue measure $\mathcal{L}^1$ on $[0,1]$ (for instance one can take $\mu_h = \sum_{i=1}^h \frac{1}{h}\delta_{i/h}$). Similar to the above proof of Theorem 4.1 we write K for the closed ball $\overline{B}_2(0)$ in X^* endowed with the w*-topology. Moreover, we consider the map $i\colon [0,1] \to K$, where $i(x)$ is defined as the characteristic function $\chi_{[0,x)}$ and the Dirac measure δ_x, respectively. In both cases i is continuous and it follows that the $i_\sharp\mu_h$ weakly converge to the finite Borel measure $i_\sharp\mathcal{L}^1$ on K. Now the $i_\sharp\mu_h$ can be regarded as Borel measures or 0-currents on X^* (since they are concentrated on finite sets), while we claim that $i_\sharp\mathcal{L}^1$ cannot be extended as a σ-additive set function from $\mathscr{B}(K) = \mathscr{B}_{\mathrm{w}^*}(\overline{B}_2(0))$ to $\mathscr{B}(\overline{B}_2(0))$. In fact, if this claim were wrong and such an extension existed, then $i_\sharp\mathcal{L}^1$ would be concentrated on a σ-compact and thus separable subset Σ of $\overline{B}_2(0)$. Since we have $\|i(x) - i(y)\|_{X^*} \geq 1$ for $x \neq y$, the preimage $i^{-1}\Sigma$ could contain at most countably many points. Thus, we would arrive at the contradiction $1 = \mathcal{L}^1([0,1]) = \mathcal{L}^1(i^{-1}\Sigma) = 0$. Consequently, the above claim on the non-existence of an extension holds. In conclusion, we have $i_\sharp\mu_h \in \mathbf{M}_0(X^*)$ with $\mathbf{M}(i_\sharp\mu_h) = 1$, but the $i_\sharp\mu_h$ cannot w*-converge to some T in $\mathbf{M}_0(X^*)$, since T would coincide with $i_\sharp\mathcal{L}^1$ as a functional on $\mathrm{C_b}(K)$ and would yield the non-existing extension. □

Notice that Example B.1 also prevents us from covering certain *separable* spaces — like $\mathrm{L}^1([0,1])$ — which are naturally embedded into non-separable duals, but are not w*-closed there; compare [3, Example 10.7]. It remains open whether an analysis of w*-Borel measures (like $i_\sharp\mathcal{L}^1$ in the previous construction) could yield any reasonable existence theorem for these cases.

B.2. **Non-separable, reflexive spaces.** As reflexive spaces are isometrically isomorphic to duals, our results evidently apply in separable, reflexive spaces. However, we will now explain that all results of the introduction remain true for *every* reflexive space Y in place of X^*.

Indeed, for each sequence $(T_h)_{h\in\mathbb{N}}$ in $\mathbf{M}_n(Y)$ there are σ-compact and hence separable sets Σ_h such that T_h is concentrated on Σ_h, and also the closure Z of the span of $\bigcup_{h=1}^\infty \Sigma_h$ is separable. Moreover, Z inherits reflexivity from Y, and the corresponding w*-topology on Z is the trace of the w*-topology on Y. Consequently, Theorem 1.4 easily extends to non-separable, reflexive spaces Y by applying it in such separable subspaces Z. Once Theorem 1.4 is extended, our proofs of the existence results carry over verbatim to non-separable, reflexive spaces Y.

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