The two well problem for piecewise affine maps

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Abstract

In the *two well problem* we look for a map u which satisfies Dirichlet boundary conditions and whose gradient Du assumes values in $SO(2)\,A \cup SO(2)\,B = \mathbb{S}_A \cup \mathbb{S}_B$, for two given invertible matrices A, B (an element of $SO(2)\,A$ is of the form RA where R is a rotation). In the original approach by Ball and James [1], [2] A, B are two matrices such that $\det B > \det A > 0$ and $\operatorname{rank}\{A - B\} = 1$. It was proved in the '90 (see [4], [5], [6], [7], [17]) that a map u satisfying given boundary conditions and such that $Du \in \mathbb{S}_A \cup \mathbb{S}_B$ exist in the Sobolev class $W^{1,\infty}(\Omega; \mathbb{R}^2)$ of Lipschitz continuous maps. However, for *orthogonal matrices* it was also proved (see [3], [8], [9], [10], [11], [12], [16]) that solutions exist in the class of piecewise C^1 maps, in particular in the class of piecewise *affine* maps. We prove here that this possibility does not exist for other nonsingular matrices A, B: precisely, the two well problem can be solved by means of piecewise affine maps only for orthogonal matrices.

1 Introduction

The *two well problem* is relevant in nonlinear elasticity and is a model for vector-valued differential inclusions. A two dimensional *Dirichlet problem* for a two well problem can be formulated as follows: given two matrices A, B, find a map u which satisfies some boundary conditions and whose gradient Du assumes values in $SO(2)\,A \cup SO(2)\,B$. Here an element of $SO(2)\,A$ (similarly for $SO(2)\,B$) is of the form RA where $R \in SO(2)$, i.e. R is a *rotation*. From now on we will write

$$\mathbb{S}_A = SO(2)\,A \quad \text{and} \quad \mathbb{S}_B = SO(2)\,B.$$

The problem is also important in geometry: rigid maps, Nash-Kuiper theorem, origami (see [9], [12] for details). However the general problem of potential wells has been introduced by Ball and James [1], [2] in the study of crystallographic models involving fine microstructures. The study of the Dirichlet problem is in general very difficult. Also difficult is the mathematical characterization of approximating solutions, which involves *minimizing sequences* in some problems of the calculus of variations (as in Ball and James [1], [2]) and characterization of the *rank-one convex*, or *the quasiconvex hull* of the vector-valued set of the wells (see Sverak [18]).

Existence of solutions to the Dirichlet problem for the *two well problem* has been obtained in dimension 2 by Dacorogna and Marcellini [4], [5], [6] (see also the book [7]) and $-$ at the same time and with a different method $-$ by Müller and Sverak [17]. The solutions considered in the quoted papers belong to the class of *Lipschitz continuous maps*, for which the gradient exists on a set whose complement has null measure. However in some other similar situations, when A, B are *orthogonal matrices*, it is possible (see [3], [8], [9], [10], [11], [12], [16]) to solve the Dirichlet problem by means of locally piecewise affine maps. The natural question that can be raised is to know if, for more general matrices, the two well problem can be solved by means of locally piecewise C^1 maps, or even locally piecewise affine maps.

In this paper we show that this possibility does not exist. Precisely, the Dirichlet problem for two invertible wells can be solved by means of piecewise affine maps only for *orthogonal matrices.* The particular case of singular matrices is considered in [13].

As we said, *orthogonal matrices* have been studied in this context in [3], [8], [9], [11], [12] pointing out the existence of piecewise smooth solutions, in particular of piecewise *affine* solutions, which assume the boundary datum in a *fractal way.* In this case the matrices A, B have the form

$$A = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = I_- = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Thus, surprisingly, the above case is the only possible one of non-trivial non-degenerate matrices for which the Dirichlet problem for two wells can be solved by piecewise smooth maps.

2 Singular values and notations

We now give the definition of the singular values.

Definition 1 *Let $A \in \mathbb{R}^{n\times n}$. The* singular values *of A, denoted by*

$$0 \leq \lambda_1(A) \leq \cdots \leq \lambda_n(A),$$

are defined to be the square root of the eigenvalues of the symmetric and positive semi definite matrix $A^t A \in \mathbb{R}^{n\times n}$.

The following theorem is the standard decomposition theorem (see Theorem 7.3.5 in [14] or Theorem 3.1.1 in [15], for example).

Theorem 2 (Singular values decomposition theorem) *Let $A \in \mathbb{R}^{n\times n}$ and $0 \leq \lambda_1(A) \leq \cdots \leq \lambda_n(A)$ be its singular values. Then there exist $R, Q \in O(n)$ such that*

$$RAQ = \operatorname{diag}(\lambda_1(A), \cdots, \lambda_n(A)) = \begin{pmatrix} \lambda_1(A) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n(A) \end{pmatrix}.$$

Moreover if the singular values are all different, then R and Q are unique if, for example, $R \in SO(n)$.

We gather below some elementary facts about the singular values, that can be deduced easily from the above theorem.

Proposition 3 *(i) The matrix $A \in \mathbb{R}^{n \times n}$ is invertible if and only if $\lambda_1(A) > 0$. In this case*

$$\lambda_i\left(A^{-1}\right) = \frac{1}{\lambda_{n-i+1}(A)} \quad \textit{for every } i = 1, \cdots, n.$$

(ii) Any matrix $A \in \mathbb{R}^{n \times n}$ satisfies

$$|\det A| = \prod_{i=1}^{n} \lambda_i(A).$$

(iii) Let $A \in \mathbb{R}^{n \times n}$, then

$$A \in O(n) \ \Leftrightarrow\ \lambda_1(A) = \cdots = \lambda_n(A) = 1.$$

If moreover $\det A = -1$, *then $A \in SO(n)\, I_-$ where*

$$I_- = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 \end{pmatrix}.$$

(iv) The eigenvectors of $A^t A$ are orthogonal.

(v) Let $A, B \in \mathbb{R}^{n \times n}$. Then

$$\left[\begin{array}{c} BA^{-1} \in O(n) \\ A \textit{ invertible} \end{array}\right] \Leftrightarrow \left[\begin{array}{c} AB^{-1} \in O(n) \\ B \textit{ invertible} \end{array}\right] \Leftrightarrow \left[\begin{array}{c} A^t A = B^t B \\ A \textit{ and } B \textit{ invertible} \end{array}\right].$$

In particular

$$BA^{-1} \in O(n) \ \Rightarrow\ \lambda_i(B) = \lambda_i(A).$$

Proof. We only prove (v). Assume that A is invertible and $BA^{-1} \in O(n)$, then

$$\left|\det\left(BA^{-1}\right)\right| = 1$$

and therefore B is also invertible. According to (i) and since $BA^{-1} \in O(n)$

$$\lambda_i\left(AB^{-1}\right) = \frac{1}{\lambda_{n-i+1}\left(BA^{-1}\right)} = 1 \quad \text{for every } i = 1, \cdots, n,$$

thus, in view of (iii), $AB^{-1} \in O(n)$.

Assume now that A is invertible and $BA^{-1} = R \in O(n)$. We therefore deduce that $B = RA$ and hence

$$B^t B = (RA)^t (RA) = A^t A$$

and thus $\lambda_i(B) = \lambda_i(A)$. Conversely if A is invertible and

$$A^t A = B^t B,$$

then

$$\begin{aligned} B^t B A^{-1} = A^t \;\Rightarrow\; A^{-t} B^t B A^{-1} &= \left(BA^{-1}\right)^t \left(BA^{-1}\right) = I \\ \Rightarrow\; \lambda_1\left(BA^{-1}\right) &= \cdots = \lambda_n\left(BA^{-1}\right) = 1 \end{aligned}$$

which readily, according to (iii), implies that $BA^{-1} \in O(n)$. ■

3 From C^1 to piecewise affine maps

Definition 4 *If A is an $n \times n$ matrix, we denote with $\mathbb{S}_A$ and $\mathbb{O}_A$ the sets of matrices*

$$\mathbb{S}_A = SO(n) \cdot A = \{RA\colon R \in SO(n)\}$$
$$\mathbb{O}_A = O(n) \cdot A = \{RA\colon R \in O(n)\}.$$

Lemma 5 *Let $\Omega \subset \mathbb{R}^n$ be open and connected. Let A, B be two fixed $n \times n$ matrices such that $\mathbb{S}_A \neq \mathbb{S}_B$. Let $u \in C^1(\Omega; \mathbb{R}^n)$ such that*

$$Du(x) \in \mathbb{S}_A \cup \mathbb{S}_B \quad \textit{for every } x \in \Omega.$$

Then either $Du(x) \in \mathbb{S}_A$ for every $x \in \Omega$ or $Du(x) \in \mathbb{S}_B$ for every $x \in \Omega$.

Proof. Notice that $\mathbb{S}_A$ and $\mathbb{S}_B$ are closed (and connected) subsets of the space of $n \times n$ matrices. Moreover $\mathbb{S}_A \cap \mathbb{S}_B = \emptyset$ because otherwise there would exist two matrices $Q, R \in SO(n)$ such that $QA = RB$ i.e. $B = R^{-1}QA$ and hence $\mathbb{S}_B = \mathbb{S}_{R^{-1}QA} = \mathbb{S}_A$ which is excluded by hypothesis. So $\mathbb{S}_A$ and $\mathbb{S}_B$ are two different connected components of $\mathbb{S}_A \cup \mathbb{S}_B$. Notice now that $x \to Du(x)$ is a continuous mapping defined on the connected set Ω. Hence its image is connected and being contained in $\mathbb{S}_A \cup \mathbb{S}_B$, which is not connected, we conclude that the image is fully contained either in $\mathbb{S}_A$ or in $\mathbb{S}_B$. ■

Theorem 6 (Liouville theorem) *Let $\Omega \subset \mathbb{R}^n$ be open and connected. Let A be an $n \times n$ invertible matrix. Let $u \in C^1(\Omega; \mathbb{R}^n)$ such that $Du(x) \in \mathbb{S}_A$ for all $x \in \Omega$. Then $x \to Du(x)$ is constant on Ω (hence u is an affine map).*

Remark 7 *If A is not invertible, then the theorem is false. For example if $n = 2$ and*

$$u(x_1, x_2) = \begin{pmatrix} \sin x_1 \\ -\cos x_1 \end{pmatrix} \quad \textit{and} \quad A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

then Du is not a constant matrix, while

$$Du(x_1, x_2) \in \mathbb{S}_A \quad \textit{for every } (x_1, x_2) \in \mathbb{R}^2.$$

Proof. *Step 1.* Suppose that A is orthogonal with $\det A = 1$. In this case $\mathbb{S}_A = SO(n)$ and the theorem is the classical Liouville theorem. Take any $x \in \Omega$. Since $Du(x) \in SO(n)$ we have that $Du(x)$ is invertible and applying the local invertibility theorem we know that there exists a neighborhood V of x such that $u\colon V \to u(V)$ is invertible. Let $v\colon u(V) \to V$ be its inverse. Then $Dv(y) = (Du(v(y)))^{-1} \in SO(n)$. So both u and v have gradient in $SO(n)$ and this implies that both u and v are $1-$Lipschitz continuous, namely

$$|u(x) - u(x')| \leq |x - x'| \quad \text{and} \quad |v(y) - v(y')| \leq |y - y'|.$$

Letting $y = u(x)$, $y' = u(x')$, we thus obtain

$$|x - x'| = |v(y) - v(y')| \leq |y - y'| = |u(x) - u(x')| \leq |x - x'|$$

which means that $|u(x) - u(x')| = |x - x'|$ i.e. u is an isometry in V. By Cartan-Dieudonné theorem (cf. [9]), we conclude that u is affine, hence $Du(x)$ is constant in V. Since Du is locally constant, it is constant.

Step 2. In the general case where A is any invertible matrix, we let

$$\Omega' = A(\Omega) = \{x \in \mathbb{R}^n : x = Ay \text{ with } y \in \Omega\}$$

and consider the map $v\colon \Omega' \to \mathbb{R}^n$ defined by $v(x) = u(A^{-1}x)$. We then have

$$Dv(x) = Du(A^{-1}x)A^{-1} \in \mathbb{S}_A\, A^{-1} = \mathbb{S}_I = SO(n).$$

Hence we can apply Step 1 to conclude that $Dv(x)$ is constant. As a consequence also $Du(x)$ is constant. ■

Definition 8 (Piecewise-C^1) *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $u\colon \Omega \to \mathbb{R}^n$.*

(i) We define the singular set of u *as the set*

$$\Sigma_u = \{x \in \Omega\colon u \text{ is not differentiable in } x\}.$$

(ii) We say that a map $u\colon \Omega \to \mathbb{R}^n$ is piecewise C^1 *in Ω if*
- u is continuous on Ω,
- Σ_u is relatively closed in Ω,
- $\Omega \setminus \Sigma_u$ has a finite number of connected components
- u is C^1 in $\Omega \setminus \Sigma_u$.

(iii) We say that u is locally piecewise C^1 *if for every open set Ω' such that $\overline{\Omega'}$ is a compact subset of Ω, we have that u is piecewise C^1 in Ω'.*

Definition 9 (Relatively convex and polyhedral sets) *Let U be an open subset of a open set $\Omega \subset \mathbb{R}^n$.*

(i) We say that U is convex relative to Ω *if for every $x, y \in U$ such that the segment $[x, y]$ is contained in Ω, then it is also contained in U.*

(ii) We say that U is a polyhedral set relative to Ω *if* $U = \mathrm{int}\left(\overline{U}\right)$ *and there exist a finite number of hyperplanes* $\Pi_1, \Pi_2, \dots, \Pi_N$ *such that*

$$\partial U \cap \Omega \subset \bigcup_{k=1}^{N} \Pi_k \,.$$

(iii) We say that U is locally polyhedral relative to Ω, *if the previous property holds on every open set Ω' compactly contained in Ω.*

Theorem 10 (Rigidity) *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $u\colon \Omega \to \mathbb{R}^n$ be a locally piecewise C^1 map. Let A, B be two $n \times n$ invertible matrices and suppose that*

$$Du(x) \in \mathbb{S}_A \cup \mathbb{S}_B \quad \textit{for every } x \in \Omega \setminus \Sigma_u \,.$$

Then the connected components Ω_k of $\Omega \setminus \Sigma_u$ are locally polyhedral sets relative to Ω and Du is constant on each connected component. Moreover

$$\Sigma_u = \Omega \cap \bigcup_k \partial\Omega_k \,.$$

Remark 11 *In fact it is possible to prove that the connected components of $\Omega \setminus \Sigma_u$ are locally convex sets relative to Ω*

Proof. *Step 1.* Suppose that u is piecewise C^1 on Ω (not only locally). Then $\Omega \setminus \Sigma_u$ has a finite number of connected components $\Omega_1, \dots, \Omega_N$ and on every component Du is constant by Lemma 5 and Theorem 6. Let $f_k\colon \mathbb{R}^n \to \mathbb{R}^n$ be the affine map such that $u(x) = f_k(x)$ for all $x \in \Omega_k$. Let us now consider a point $x \in \Sigma_u$. Since u is continuous in x, we must have

$$u(x) = f_k(x) = f_j(x)$$

for every $x \in \overline{\Omega}_k \cap \overline{\Omega}_j$. Since x is a singular point of Du, at least two of these maps f_k should be different, otherwise u would be differentiable at x. So there exist j, k such that $f_j(x) = f_k(x)$ and $f_j \neq f_k$. In particular $x \in \Pi_{jk}$ where $\Pi_{jk} = \{y\colon f_j(y) = f_k(y)\}$ is contained in a $(n-1)-$dimensional affine subspace of $\mathbb{R}^n$. By considering all points $x \in \Sigma_u$ we conclude that

$$\Sigma_u \subset \bigcup_{j,k\colon\, f_j \neq f_k} \Pi_{jk} \,. \tag{1}$$

Fix k, notice that $\partial\Omega_k \cap \Omega \subset \Sigma_u$ hence $\partial\Omega_k \cap \Omega$ is contained in the finite union of $(n-1)-$dimensional affine subspaces of $\mathbb{R}^n$. To prove that Ω_k is a polyhedral set relative to Ω, it remains to check the condition $\Omega_k \supset \mathrm{int}\left(\overline{\Omega}_k\right)$ (the other inclusion is always true, since Ω_k is open). Suppose by contradiction that there exists a neighborhood U of x contained in $\overline{\Omega}_k$ but $x \notin \Omega_k$, that is $x \in \overline{\Omega}_k \setminus \Omega_k = \partial\Omega_k$. Since u is continuous, and $u = f_k$ on Ω_k, we know that $u = f_k$ on $\overline{\Omega}_k$ hence $u = f_k$ on the whole set U. Since x is interior to U we conclude that u is differentiable in x which is a contradiction since $x \in \partial\Omega_k \subset \Sigma_u$. To conclude the proof we only need to prove that $\Sigma_u \subset \cup\partial\Omega_k$. If this were not true we could find a point $x \in \Sigma_u$ which is not in any $\overline{\Omega}_k$ hence a whole neighborhood of x is outside every Ω_k and hence Σ_u has non empty interior. But this is in contradiction with (1).

Step 2. In general if u is locally piecewise C^1, we apply Step 1 to every Ω' compactly contained in Ω and obtain the result. ■

4 The singular set

Let Σ be a locally finite union of closed segments in an open set $\Omega \subset \mathbb{R}^2$. We say that a point of Ω is a vertex of Σ if either it is an end point of a segment or a point where at least two segments meet.

Definition 12 (Distorted angle condition − analytical form) *Let A be an invertible 2×2 matrix. We say that a locally polyhedral set $\Sigma \subset \mathbb{R}^2$ satisfies the* distorted angle condition *(with respect to A) if $\Sigma' = A(\Sigma)$ where*

$$\Sigma' = A(\Sigma) = \left\{x \in \mathbb{R}^2 : x = Ay \text{ with } y \in \Sigma\right\}$$

satisfies the angle condition, *namely: at every vertex of Σ' the segments define an even number of angles $\alpha_1, \alpha_2, \ldots, \alpha_{2N}$ which satisfy*

$$\sum_{i=1}^{N} \alpha_{2i-1} = \sum_{i=1}^{N} \alpha_{2i} = \pi. \tag{2}$$

A geometrical interpretation of the *distorted angle condition* with respect to the given 2×2 invertible matrix A is as follows. We recall that $\lambda_1 = \lambda_1(A)$, $\lambda_2 = \lambda_2(A)$ are the square root of the eigenvalues of the symmetric and positive definite matrix $A^t A \in \mathbb{R}^{2\times 2}$; we now also consider the corresponding normalized eigenvectors $v_1 = v_1(A)$, $v_2 = v_2(A)$ and we define the *associated ellipse* E_A whose semi-axes are parallel to the eigenvectors v_1, v_2, of respective length

$$\sqrt{\frac{\lambda_2}{\lambda_1}} \quad \text{and} \quad \sqrt{\frac{\lambda_1}{\lambda_2}}\,.$$

Then E_A can be represented in the form (with the symbol $\langle\cdot;\cdot\rangle$ we denote the scalar product)

$$E_A = \left\{u \in \mathbb{R}^2 : \frac{\lambda_1}{\lambda_2}\langle u; v_1\rangle^2 + \frac{\lambda_2}{\lambda_1}\langle u; v_2\rangle^2 \le 1\right\}. \tag{3}$$

In the geometrical description we will use the following lemma.

Lemma 13 *Let A be an invertible 2×2 matrix and let E_A be the ellipse associated to A as in (3). For every cone Δ in $\mathbb{R}^2$ centered at the origin we have (cf. Figure 1)*

$$|\Delta \cap E_A| = |A(\Delta) \cap C|\,, \tag{4}$$

where C be the unit disk centered at the origin.

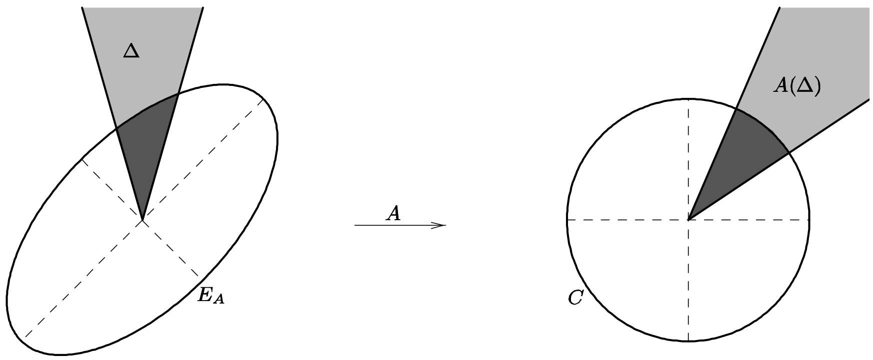

Figure 1: the two sets $\Delta \cap E_A$ and $A(\Delta) \cap C$ have the same area

Proof. *Step 1.* Let us first prove that the image trough A of the ellipse E_A is the disk of radius $\sqrt{|\det A|}$; i.e.,

$$A(E_A) = \sqrt{|\det A|} \cdot C.$$

In fact we have

$$\begin{aligned} A^{-1}(C) &= \left\{ A^{-1}u : u \in \mathbb{R}^2,\ |u| \le 1 \right\} \\ &= \left\{ u \in \mathbb{R}^2 : |Au| \le 1 \right\}. \end{aligned}$$

Then we represent any generic $u \in \mathbb{R}^2$ as linear combination of the normalized orthogonal eigenvectors v_1 , v_2

$$u = \langle u; v_1 \rangle v_1 + \langle u; v_2 \rangle v_2 .$$

Then we have $|Au|^2 = \langle Au; Au \rangle = \langle A^t Au; u \rangle$ and thus

$$\begin{aligned} |Au|^2 &= \left\langle A^t A \left(\langle u; v_1 \rangle v_1 + \langle u; v_2 \rangle v_2 \right); u \right\rangle \\ &= \left\langle \langle u; v_1 \rangle A^t A v_1 + \langle u; v_2 \rangle A^t A v_2; u \right\rangle \\ &= \left\langle \langle u; v_1 \rangle \lambda_1^2 v_1 + \langle u; v_2 \rangle \lambda_2^2 v_2; u \right\rangle \\ &= \langle u; v_1 \rangle^2 \lambda_1^2 + \langle u; v_2 \rangle^2 \lambda_2^2 . \end{aligned}$$

We obtain

$$\begin{aligned} A^{-1}(C) &= \left\{ u \in \mathbb{R}^2 : \langle u; v_1 \rangle^2 \lambda_1^2 + \langle u; v_2 \rangle^2 \lambda_2^2 \le 1 \right\} \\ &= \left\{ u \in \mathbb{R}^2 : \left\langle u\sqrt{\lambda_1 \lambda_2}; v_1 \right\rangle^2 \frac{\lambda_1}{\lambda_2} + \left\langle u\sqrt{\lambda_1 \lambda_2}; v_2 \right\rangle^2 \frac{\lambda_2}{\lambda_1} \le 1 \right\}. \end{aligned}$$

Finally, with the notation $v = u\sqrt{\lambda_1 \lambda_2}$ we get

$$A^{-1}(C) = \left\{ u \in \mathbb{R}^2 : u = \frac{v}{\sqrt{\lambda_1 \lambda_2}}, \quad \langle v; v_1 \rangle^2 \frac{\lambda_1}{\lambda_2} + \langle v; v_2 \rangle^2 \frac{\lambda_2}{\lambda_1} \le 1 \right\}$$

and thus

$$A^{-1}(C) = \left\{ u \in \mathbb{R}^2 : u = \frac{v}{\sqrt{\lambda_1 \lambda_2}}, \;\; v \in E_A \right\}$$
$$= \frac{1}{\sqrt{\lambda_1 \lambda_2}} E_A = \frac{1}{\sqrt{|\det A|}} E_A \, .$$

Step 2. Let Δ be a cone in $\mathbb{R}^2$ centered at the origin and let $A(\Delta)$ be the set

$$A(\Delta) = \{Au : u \in \Delta\} \, .$$

Then by Step 1, since Δ is positively homogeneous, we get

$$A(\Delta) \cap C = A(\Delta) \cap A \left(\frac{E_A}{\sqrt{|\det A|}} \right)$$
$$= A \left(\Delta \cap \frac{E_A}{\sqrt{|\det A|}} \right) = A \left(\frac{\Delta \cap E_A}{\sqrt{|\det A|}} \right) .$$

We now compute the area of the set $A(\Delta) \cap C$ by the change of variables formula and the fact that we are in the $2-$dimensional case. Thus

$$|A(\Delta) \cap C| = \left| A \left(\frac{\Delta \cap E_A}{\sqrt{|\det A|}} \right) \right|$$
$$= |\det A| \cdot \left| \frac{\Delta \cap E_A}{\sqrt{|\det A|}} \right| = |\Delta \cap E_A|$$

which concludes the proof. ■

We are ready to state the geometrical interpretation of the *distorted angle condition* with respect to the given 2×2 invertible matrix A. At every vertex of Σ the segments define an even number of cones $\Delta_1, \Delta_2, \ldots, \Delta_{2N}$ and note that any α_j in (2), for $j = 1, \ldots, 2N$, is the measure of the angle associated to the cone $A(\Delta_j)$; therefore it is equal to $1/2$ the area of the intersection of $A(\Delta_j)$ with the unit disk. That is,

$$\alpha_j = 2 \, |A(\Delta_j) \cap C| \, , \quad \forall \, j = 1, \ldots, 2N.$$

Therefore by Lemma 13, according to (2), we get

$$\sum_{i=1}^{N} |\Delta_{2i-1} \cap E_A| = \sum_{i=1}^{N} |\Delta_{2i} \cap E_A| = \frac{\pi}{2} \, . \tag{5}$$

Thus we can state Definition 12 in the equivalent form:

Proposition 14 (Distorted angle condition – geometrical form) *A locally polyhedral set Σ satisfies the* distorted angle condition *with respect to an invertible 2×2 matrix A if and only if at every vertex of Σ the segments define an even number of cones $\Delta_1, \Delta_2, \ldots, \Delta_{2N}$ satisfying (5).*

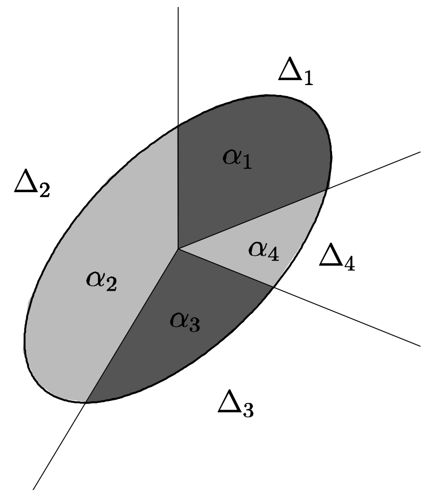

Figure 2: ellipse associated to the matrix

Remark 15 *Recall that* $|\Delta_j \cap E_A|$ *is the area of the intersection of the cone* Δ_j *with the ellipse* E_A *associated to the matrix* A *(see Figure 2).*

Having discussed conditions for vertices, we now consider conditions on edges.

Definition 16 (Lamination condition) *Let* A *and* B *be* 2×2 *matrices with* A *invertible and*

$$\lambda_1 \left(BA^{-1}\right) \leq 1 \leq \lambda_2 \left(BA^{-1}\right),$$

with at least one strict inequality. Let Q *and* R *be the orthogonal matrices which realize the singular value decomposition* $BA^{-1} = Q\Lambda R$, Λ *being a diagonal matrix (notice that* Q *is unique up to the sign of its determinant). We say that* Σ *satisfies the* lamination condition *(with respect to* A *and* B*) if* $\Sigma' = QA(\Sigma)$ *is composed by segments that extend up to the boundary of* Ω*. The two normals* ν_+ *and* ν_- *to the lamination are given by*

$$\nu_\pm = \left(\pm\sqrt{1 - \lambda_1^2 \left(BA^{-1}\right)},\ \sqrt{\lambda_2^2 \left(BA^{-1}\right) - 1}\right).$$

Note that, if either $\lambda_1 \left(BA^{-1}\right) = 1$ or $\lambda_2 \left(BA^{-1}\right) = 1$, then ν_+ and ν_- are parallel to each other, hence in this case all the segments of Σ (and Σ') are parallel to each other. In this case we therefore have a *single* lamination, otherwise we have a *double* lamination.

If in a double lamination two segments of Σ meet at a point $x \in \Omega$, we say that x is a *vertex* of Σ.

Remark 17 *The condition*

$$\lambda_1\left(BA^{-1}\right) \leq 1 \leq \lambda_2\left(BA^{-1}\right)$$

is equivalent to the fact that the two wells $\mathbb{S}_A$ and $\mathbb{S}_B$ are rank one connected (see Lemma 19 below).

5 Notations and preliminary results

We now fix the notations that will be used throughout the remaining part of the article.

Notation 18 *(i) We recall that for $\varphi \in (-\pi, \pi]$, we write a generic matrix in $SO(2)$ as*

$$R_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}.$$

(ii) If $A, B \in \mathbb{R}^{2\times 2}$ are invertible, we let

$$\lambda = \lambda_1\left(BA^{-1}\right)\operatorname{sign}\left[\det\left(BA^{-1}\right)\right] \quad \text{and} \quad \mu = \lambda_2\left(BA^{-1}\right).$$

(iii) We define $\theta \in [0, \pi]$ through

$$\cos\theta = \frac{1+\lambda\mu}{\mu+\lambda} \quad \text{and} \quad \sin\theta = \frac{\sqrt{\mu^2-1}\sqrt{1-\lambda^2}}{\mu+\lambda}. \tag{6}$$

If $|\lambda| \leq 1 \leq \mu$ and at least one strict inequality holds, then θ is well defined, since

$$\left|\frac{1+\lambda\mu}{\mu+\lambda}\right| \leq 1 \Leftrightarrow (1+\lambda\mu)^2 \leq (\mu+\lambda)^2 \Leftrightarrow \left(1-\lambda^2\right)\left(1-\mu^2\right) \leq 0.$$

Thus if $\theta = 0$, then either $\lambda = 1$ or $\mu = 1$; while if $\theta = \pi$, then $\lambda = -1$ (and thus $\mu > 1$).

(iv) We also let

$$\nu_\pm = \left(\pm\sqrt{1-\lambda^2}, \sqrt{\mu^2-1}\right).$$

Before starting our analysis we need two elementary lemmas.

Lemma 19 (Edge) *Let $A = I$ and $B = \Lambda = \operatorname{diag}(\lambda, \mu)$ with $0 < |\lambda| \leq \mu$ and $\Lambda \neq I$. Let $\nu = (\nu_1, \nu_2) \neq 0$ and $\varphi, \psi \in (-\pi, \pi]$. The map $u : \mathbb{R}^2 \to \mathbb{R}^2$ defined by*

$$u(x) = \begin{cases} R_\varphi x & \text{if } \langle x; \nu\rangle = x_1\nu_1 + x_2\nu_2 > 0 \\ R_{\varphi+\psi}\Lambda x & \text{if } \langle x; \nu\rangle = x_1\nu_1 + x_2\nu_2 < 0 \end{cases}$$

can be continuously extended across the line $\langle x; \nu\rangle = 0$ if and only if one of the two following cases happen.

Case 1. The following conditions hold true: $\lambda = -1$, $\mu = 1$ *and* ν *is parallel to* $(1 + \cos\psi, \sin\psi)$ *.*

Case 2. The three following conditions are valid:

$$0 < |\lambda| \leq 1 \leq \mu \quad \textit{and at least one strict inequality holds}$$

$$\psi = \pm\theta \quad \textit{and} \quad \nu \textit{ is parallel to } \nu_\pm \, .$$

Remark 20 *The lemma gives also that*

$$R_\theta \Lambda = I + \alpha_+ \otimes \nu_+ = \begin{pmatrix} 1 - \frac{\mu\left(1-\lambda^2\right)}{\mu+\lambda} & -\frac{\mu\sqrt{\mu^2-1}\sqrt{1-\lambda^2}}{\mu+\lambda} \\ \frac{\lambda\sqrt{\mu^2-1}\sqrt{1-\lambda^2}}{\mu+\lambda} & 1 + \frac{\lambda\left(\mu^2-1\right)}{\mu+\lambda} \end{pmatrix}$$

$$R_{-\theta} \Lambda = I + \alpha_- \otimes \nu_- = \begin{pmatrix} 1 - \frac{\mu\left(1-\lambda^2\right)}{\mu+\lambda} & \frac{\mu\sqrt{\mu^2-1}\sqrt{1-\lambda^2}}{\mu+\lambda} \\ \frac{-\lambda\sqrt{\mu^2-1}\sqrt{1-\lambda^2}}{\mu+\lambda} & 1 + \frac{\lambda\left(\mu^2-1\right)}{\mu+\lambda} \end{pmatrix}$$

where

$$\alpha_\pm = \left(\mp \frac{\mu\sqrt{1-\lambda^2}}{\mu+\lambda}, \frac{\lambda\sqrt{\mu^2-1}}{\mu+\lambda} \right).$$

Proof. The map u is continuous only if

$$\begin{aligned} \det(I - R_\psi \Lambda) &= (1 - \lambda\cos\psi)(1 - \mu\cos\psi) + \lambda\mu\sin^2\psi \\ &= 1 - (\mu + \lambda)\cos\psi + \lambda\mu = 0. \end{aligned}$$

This can happen if and only if

$$-|\mu + \lambda| \leq 1 + \lambda\mu \leq |\mu + \lambda|$$

which is valid if and only if

$$|\lambda| \leq 1 \leq \mu.$$

Let us first examine Case 1. We see that in this case the above condition is true for every ψ. We also infer that u is continuous across the line of discontinuity of the gradient if and only if

$$x_1(1 + \cos\psi) + x_2 \sin\psi = 0$$

which is equivalent to

$$\nu \text{ is parallel to } (1 + \cos\psi, \sin\psi)\, .$$

Let us now examine the second case. In view of the above considerations we see that

$$0 < |\lambda| \leq 1 \leq \mu \quad \text{and at least one strict inequality holds.}$$

This in turn implies that $\mu + \lambda \neq 0$ and therefore

$$\cos\psi = \frac{1+\lambda\mu}{\mu+\lambda}$$

which implies that $\psi = \pm\theta$. It remains to see that ν is parallel to $\nu_{\pm}$. In order to have a continuous map in the case $\psi = \theta$ (the case $\psi = -\theta$ is handled in exactly the same manner) we should therefore have $x = R_\theta \Lambda x$ on the line of discontinuity of the gradient, i.e.

$$\begin{pmatrix} 1-\lambda\cos\theta & \mu\sin\theta \\ -\lambda\sin\theta & 1-\mu\cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0.$$

Solving the first equation of the previous system (we know that the system is degenerate, hence we can drop the second equation) we find

$$x_1(1-\lambda\cos\theta) + x_2\mu\sin\theta = 0$$

which is

$$x_1\left(1-\lambda\frac{1+\lambda\mu}{\mu+\lambda}\right) + x_2\frac{\mu\sqrt{\mu^2-1}\sqrt{1-\lambda^2}}{\mu+\lambda} = 0.$$

We therefore find that

$$x_1\left(\mu-\lambda^2\mu\right) + x_2\mu\sqrt{\mu^2-1}\sqrt{1-\lambda^2} = 0$$

which leads to

$$x_1\sqrt{1-\lambda^2} + x_2\sqrt{\mu^2-1} = 0$$

as wished. The last statements $R_\theta\Lambda = I + \alpha_+ \otimes \nu_+$ and $R_{-\theta}\Lambda = I + \alpha_- \otimes \nu_-$ are immediate. ■

According to the above lemma if $0 < |\lambda| < 1 < \mu$, then any line of discontinuity has to be perpendicular to either ν_+ or ν_- . We therefore deduce that only a vertex of order 4 can exist. This possibility is considered in the next lemma.

Lemma 21 (Vertex) *Let $A = I$ and $B = \Lambda = \operatorname{diag}(\lambda,\mu)$ with*

$$0 < |\lambda| < 1 < \mu.$$

Let $\varphi, \chi, \chi', \psi \in (-\pi,\pi]$. Let

$$\nu_{\pm} = \left(\pm\sqrt{1-\lambda^2}, \sqrt{\mu^2-1}\right).$$

The map $u : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$u(x) = \begin{cases} R_\varphi x & \text{if } \langle x;\nu_+\rangle > 0 \text{ and } \langle x;\nu_-\rangle > 0 \\ R_{\varphi+\chi}\Lambda x & \text{if } \langle x;\nu_+\rangle < 0 \text{ and } \langle x;\nu_-\rangle > 0 \\ R_{\varphi+\psi} x & \text{if } \langle x;\nu_+\rangle < 0 \text{ and } \langle x;\nu_-\rangle < 0 \\ R_{\varphi+\chi'}\Lambda x & \text{if } \langle x;\nu_+\rangle > 0 \text{ and } \langle x;\nu_-\rangle < 0 \end{cases}$$

can be extended by continuity, across the lines of discontinuity of the gradient, if and only if $\lambda\mu = -1$, $\chi = -\chi' = \pi/2$, and $\psi = \pi$.

Remark 22 *Therefore the only possibility of having, under the hypotheses of the lemma, a continuous u is that*

$$R_{-\varphi}\, u\,(x_1,x_2)=\left\{\begin{array}{cl}(x_1,x_2) & \textit{if } x_1+\mu x_2>0 \textit{ and } x_1-\mu x_2<0\\ -(\mu x_2,x_1/\mu) & \textit{if } x_1+\mu x_2<0 \textit{ and } x_1-\mu x_2<0\\ -(x_1,x_2) & \textit{if } x_1+\mu x_2<0 \textit{ and } x_1-\mu x_2>0\\ (\mu x_2,x_1/\mu) & \textit{if } x_1+\mu x_2>0 \textit{ and } x_1-\mu x_2>0.\end{array}\right.$$

Proof. We can apply Lemma 19 to the four lines of discontinuity to find that

$$\chi=-\chi'=\theta,\quad R_\psi=R_{2\chi}=R_{2\chi'}\,.$$

The last condition gives $4\chi=2k\pi$ for some integer k which means that $\chi=k\pi/2$ for $k=-1,0,1,2$. We can exclude $\chi=0$ and $\chi=\pi$ since in that cases we would find $|\lambda|=1$ or $\mu=1$ which are excluded by hypothesis. So, by also matching the sign of χ with the sign of $\nu_\pm$, we find $\chi=\pi/2$, $\chi'=-\pi/2$, $\psi=\pi$. The equation $\cos\chi=0$ hence becomes $1+\lambda\mu=0$ which gives $\lambda=-1/\mu$. ■

6 The structure of the singular set

As in Notation 18, we will let, for $A,B\in\mathbb{R}^{2\times2}$ invertible,

$$\lambda=\lambda_1\left(BA^{-1}\right)\operatorname{sign}\left[\det\left(BA^{-1}\right)\right]\quad\text{and}\quad\mu=\lambda_2\left(BA^{-1}\right)$$

so that $|\lambda|\le\mu$. We have to consider three cases.

Case 1. The first one is when the wells are not rank one connected and then no non-trivial solution exists.

The other two cases are concerned with rank one connected wells. This implies (cf. Remark 17 and Lemma 19) that

$$|\lambda|\le1\le\mu.$$

Case 2. The second case is when at least one of the two above inequalities is strict. We will prove that a double lamination can occur. However an internal vertex can exist only if $\lambda\mu=-1$.

Case 3. Finally we consider the orthogonal case where

$$\lambda=-1\quad\text{and}\quad\mu=1.$$

There the situation is much richer.

Remark 23 *(i) Let $A,B\in\mathbb{R}^{2\times2}$ be invertible. We recall that, using the singular value decomposition, we can find $\varphi,\chi\in(-\pi,\pi]$ so that*

$$BA^{-1}=R_\varphi\Lambda R_\chi\quad\textit{with}\quad\Lambda=\begin{pmatrix}\lambda&0\\0&\mu\end{pmatrix}\tag{7}$$

and

$$\lambda = \lambda_1\left(BA^{-1}\right)\det\left(BA^{-1}\right) \quad \textit{and} \quad \mu = \lambda_2\left(BA^{-1}\right).$$

(ii) In the following proofs, but not in the statements, we will always assume that $A = I$ and $B = \Lambda$ where

$$\Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$$

with $|\lambda| \le \mu$. This is not a loss of generality. Indeed when A and B are general matrices, we have, as said above, that

$$BA^{-1} = R_\varphi \Lambda R_\chi$$

with $\Lambda = \operatorname{diag}(\lambda, \mu)$ and $|\lambda| \le \mu$, $R_\varphi, R_\chi \in SO(2)$. We then consider the map

$$v(x) = u(A^{-1}R_{-\chi}x)$$

which is defined on $R_\chi(A(\Omega))$ and satisfies

$$Dv(x) = Du\left(A^{-1}R_{-\chi}x\right)A^{-1}R_{-\chi}$$

We hence have that

$$Du \in \mathbb{S}_A \cup \mathbb{S}_B \;\Leftrightarrow\; Dv \in \mathbb{S}_I \cup \mathbb{S}_\Lambda$$

since

$$\left(\mathbb{S}_A \cup \mathbb{S}_B\right)A^{-1}R_{-\chi} = \mathbb{S}_{R_{-\chi}} \cup \mathbb{S}_{BA^{-1}R_{-\chi}} = \mathbb{S}_I \cup \mathbb{S}_{R\Lambda} = \mathbb{S}_I \cup \mathbb{S}_\Lambda\,.$$

Therefore any statement on Du and $\mathbb{S}_A \cup \mathbb{S}_B$ becomes a statement on Dv and $\mathbb{S}_I \cup \mathbb{S}_\Lambda$. In particular note that if we let

$$\Sigma' := R_\chi A(\Sigma_u) = \Sigma_v\,,$$

then the lamination condition for Σ_v becomes the lamination condition for Σ_u, according to Definition 16.

We also adopt the notations of the preceding section. Our first theorem concerns the case where the wells are not rank one connected

Theorem 24 (The empty singular set case) *Let Ω be an open set of $\mathbb{R}^2$. Let $A, B \in \mathbb{R}^{2\times 2}$ be invertible and such that one of the following three conditions hold*

$$\lambda_1\left(BA^{-1}\right) \le \lambda_2\left(BA^{-1}\right) < 1$$
$$\lambda_2\left(BA^{-1}\right) \ge \lambda_1\left(BA^{-1}\right) > 1$$
$$\lambda_1\left(BA^{-1}\right) = \lambda_2\left(BA^{-1}\right) = 1 \quad \textit{and} \quad \det\left(BA^{-1}\right) = 1.$$

If u is a piecewise C^1 solution of

$$Du \in \mathbb{S}_A \cup \mathbb{S}_B \quad \textit{a.e. in } \Omega,$$

then u is affine and thus $\Sigma_u = \emptyset$.

Proof. *Step 1.* As already said we assume that $A = I$ and $B = \Lambda$. We start our discussion with the cases where

$$|\lambda| \leq \mu < 1 \quad \text{or} \quad \mu \geq |\lambda| > 1.$$

According to Lemma 19 the map u has to be such that either $Du \in \mathbb{S}_I$ or $Du \in \mathbb{S}_\Lambda$. The claim then follows from Liouville theorem.

Step 2. We next discuss the case

$$\lambda_1 \left(BA^{-1}\right) = \lambda_2 \left(BA^{-1}\right) = 1 \quad \text{and} \quad \det \left(BA^{-1}\right) = 1.$$

We thus have that $BA^{-1} \in SO\,(2)$ and hence

$$\mathbb{S}_A \cup \mathbb{S}_B = \mathbb{S}_A = \mathbb{S}_B$$

and we get the result by Liouville theorem. ■

Our second result is

Theorem 25 (The double lamination case) *Let Ω be an open set of $\mathbb{R}^2$. Let $A, B \in \mathbb{R}^{2\times 2}$ be invertible and such that $\lambda_1 \left(BA^{-1}\right) \leq 1 \leq \lambda_2 \left(BA^{-1}\right)$ and at least one strict inequality holds.*

Necessary conditions. *If u is a piecewise C^1 solution of*

$$Du \in \mathbb{S}_A \cup \mathbb{S}_B \quad \text{a.e. in } \Omega, \tag{8}$$

then Σ_u satisfies the lamination condition (with respect to A and B). Furthermore an internal vertex can exist only if

$$\det \left(BA^{-1}\right) = -1.$$

Sufficient condition. *Conversely if Ω is simply connected and Σ satisfies the lamination condition (with respect to A and B) without internal vertices, then there exists a piecewise affine map u with $\Sigma = \Sigma_u$ satisfying (8).*

Remark 26 *In the case where $\lambda_1 = 1$ or $\lambda_2 = 1$, only one lamination can exist (cf. below).*

Proof. Recall that, without loss of generality, we have

$$A = I \quad \text{and} \quad B = \Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$$

with

$$0 < |\lambda| \leq 1 \leq \mu \quad \text{and at least one strict inequality holds.}$$

In particular we cannot have $|\lambda| = \mu = 1$ and thus

$$\mu + \lambda > 0.$$

Recall also that $\theta \in [0, \pi]$ is defined through

$$\cos\theta = \frac{1 + \lambda\mu}{\mu + \lambda} \quad \text{and} \quad \sin\theta = \frac{\sqrt{\mu^2 - 1}\sqrt{1 - \lambda^2}}{\mu + \lambda}\,.$$

Necessary conditions. We divide the proof into three steps.

Step 1. According to Liouville theorem since Λ is invertible (i.e. $\lambda\mu \neq 0$), then any piecewise C^1 map u such that $Du \in \mathbb{S}_I \cup \mathbb{S}_\Lambda$ is piecewise affine. More precisely, Du is constant on the connected components of $\Omega \backslash \Sigma_u$. In two adjacent regions Du has two different values which must be rank-one connected matrices. Since A and B are invertible, then no rank-one connection can exist in the same well. So, crossing an edge of Σ_u we pass from a gradient in $\mathbb{S}_I$ to a gradient in $\mathbb{S}_\Lambda$ or vice versa.

We start with the case where $\theta \neq 0, \pi/2, \pi$ (these cases will be dealt with in Steps 2 and 3). This implies that

$$0 < |\lambda| < 1 < \mu \quad \text{and} \quad \lambda\mu \neq -1.$$

We assume that Du takes values in both wells, otherwise nothing is to be proved. We can also assume that, up to a rotation, Du is I on a set of positive measure. We then apply Lemma 19 to get that only two laminations are possible, namely those lines that are orthogonal to $\nu_\pm$. This is exactly saying that Σ_u satisfies the lamination condition (with respect to A and B). Lemma 21 implies that there cannot be an internal vertex, since $\lambda\mu \neq -1$. In fact we have, according to Lemma 19, the more precise statements

$$R_{-2\theta} - R_{-\theta}\Lambda = R_{-2\theta}\left(I - R_\theta\Lambda\right) = R_{-2\theta}\left(-\alpha_+ \otimes \nu_+\right) = \left(R_{-2\theta}\left(-\alpha_+\right)\right) \otimes \nu_+$$

and

$$R_{2\theta} - R_\theta\Lambda = R_{2\theta}\left(I - R_{-\theta}\Lambda\right) = R_{2\theta}\left(-\alpha_- \otimes \nu_-\right) = \left(R_{2\theta}\left(-\alpha_-\right)\right) \otimes \nu_-\,.$$

Therefore the only adjacent gradient to $R_{-\theta}\Lambda$ are I for the lamination orthogonal to ν_- and $R_{-2\theta}$ for the lamination orthogonal to ν_+ . This can be summarized in the following diagram

$$I \overset{\nu_-}{\to} R_{-\theta}\Lambda \overset{\nu_-}{\to} I \quad \text{and} \quad R_{-\theta}\Lambda \overset{\nu_+}{\to} R_{-2\theta} \overset{\nu_+}{\to} R_{-\theta}\Lambda.$$

Similarly the only adjacent gradient to $R_\theta\Lambda$ are $R_{2\theta}$ for the lamination orthogonal to ν_- and I for the lamination orthogonal to ν_+ . As before we have the following diagram

$$I \overset{\nu_+}{\to} R_\theta\Lambda \overset{\nu_+}{\to} I \quad \text{and} \quad R_\theta\Lambda \overset{\nu_-}{\to} R_{2\theta} \overset{\nu_-}{\to} R_\theta\Lambda.$$

This concludes the proof of the necessary part when $\theta \neq 0, \pi/2, \pi$.

Step 2. We now consider the case where $\theta = 0$ (the case $\theta = \pi$ is handled similarly). Then either $\lambda = 1$ or $\mu = 1$ (recall that both cannot be equal to 1). Assume that $\mu = 1$ (the case $\lambda = 1$ is dealt with in the same manner). Therefore

ν_+ and ν_- are parallel to $e_1 = (1, 0)$ and hence it follows from Lemma 19 that only one lamination can occur and therefore no internal vertex can exist.

Step 3. We finally deal with the case $\theta = \pi/2$. This implies that

$$0 < |\lambda| < 1 < \mu \quad \text{and} \quad \lambda\mu = -1.$$

So again Lemmas 19 and 21 give the claim. In particular an internal vertex can exist.

Sufficient condition. We suppose that Σ is a given set satisfying the lamination condition with respect to I and Λ and hence Σ is composed by segments perpendicular to $\nu_\pm$. Notice that the vertices of Σ, where several segments meet, can only be composed by four segments (or two if $|\lambda| = 1$ or $\mu = 1$), since we only have two possible normal vectors to the segments.

Since Ω is simply connected and every vertex v of Σ has exactly four neighboring regions (or exactly two if $|\lambda| = 1$ or $\mu = 1$), it is possible (cf, the Recovery Theorem 4.9 in [9]) to make a *two-coloration* of $\Omega \setminus \Sigma$. This means that it is possible to assign a label I or Λ to every region, so that if two regions meet in a segment, then they have a different label. The very same theorem gives the existence of a piecewise affine map u with the required property, in particular the gradient Du belongs to $\mathbb{S}_I$ on every region with label I and to $\mathbb{S}_\Lambda$ on every region with label Λ. ■

We now turn to our third theorem.

Theorem 27 (The orthogonal case) *Let Ω be an open set of $\mathbb{R}^2$. Let $A, B \in \mathbb{R}^{2\times 2}$ with A invertible be such that*

$$\lambda_1 \left(BA^{-1}\right) = \lambda_2 \left(BA^{-1}\right) = 1 \quad \textit{and} \quad \det \left(BA^{-1}\right) = -1$$

or equivalently $\mathbb{S}_A \cup \mathbb{S}_B = \mathbb{O}_A = \mathbb{O}_B$. Then the two following statements hold.

Necessary conditions. *If u is a piecewise C^1 solution of*

$$Du \in \mathbb{S}_A \cup \mathbb{S}_B \quad \textit{a.e. in } \Omega, \tag{9}$$

then u is piecewise affine and Σ_u is locally polyhedral relative to Ω and satisfies the distorted angle condition with respect to A.

Sufficient conditions. *Conversely if Ω is simply connected and Σ is locally polyhedral relative to Ω and satisfies the distorted angle condition with respect to A, then there exists a piecewise affine map u with $\Sigma = \Sigma_u$ satisfying (9).*

Proof. As usual we assume, without loss of generality, that $A = I$ and $B = \Lambda$ where

$$\Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore $\Lambda \in O(2)$ and

$$\mathbb{S}_I \cup \mathbb{S}_\Lambda = \mathbb{O}_I = \mathbb{O}_\Lambda .$$

The distorted angle condition with respect to I is hence the same as the distorted angle condition with respect to Λ, since the associated ellipses are the same. Since $A = I$ and $B = \Lambda$, the distorted angle condition is just the angle condition. We have thus reduced the theorem to Theorems 4.8 and 4.9 in [9]. ■

7 The Dirichlet problem

The main result of this section is the following.

Theorem 28 *Let Ω be a bounded open set of $\mathbb{R}^2$ with piecewise C^1 boundary made of at most seven smooth pieces. Let $A, B \in \mathbb{R}^{2\times 2}$ be invertible matrices such that*

$$\lambda_1\left(BA^{-1}\right) \leq 1 \leq \lambda_2\left(BA^{-1}\right)$$

with at least one strict inequality. Let $C \in \mathbb{R}^{2\times 2}$ with $C \notin \mathbb{S}_A \cup \mathbb{S}_B$. Then there is no locally piecewise C^1 solution of

$$\begin{cases} Du\left(x\right) \in \mathbb{S}_A \cup \mathbb{S}_B & a.e.\ x \in \Omega \\ u\left(x\right) = Cx & x \in \partial\Omega. \end{cases} \tag{10}$$

Proof. Let us assume by contradiction that there exists a locally piecewise C^1 map u whose gradient almost everywhere in Ω satisfies

$$Du\left(x\right) \in \mathbb{S}_A \cup \mathbb{S}_B\,.$$

The singular set Σ_u (where the map u is not differentiable) is not empty, since otherwise the map u would be affine in Ω, but this in impossible since $C \notin \mathbb{S}_A \cup \mathbb{S}_B$. Then, by Theorem 25, Σ_u satisfies the *lamination condition* (with respect to A and B), see Definition 16. Thus Σ_u is composed by segments that extend up to the boundary of Ω, since at any interior vertex a cross forms and the segment cannot end there. These segments are normal to at most two fixed vectors, giving rise to either a *double lamination* (if the two vectors define distinct directions) or to a *single lamination* (if the two vectors define the same direction).

Our first claim is that Σ_u cannot touch the boundary $\partial\Omega$ in a point where the boundary is smooth. In fact let Γ be a relatively open subset of the smooth part of $\partial\Omega$ and consider the set $X = \overline{\Sigma_u} \cap \Gamma$ which we want to prove to be empty. Notice now that X cannot contain isolated points, because the tangential derivative of u at the point x_0 would be different from the left and from the right while it should coincide with the boundary datum C. So every point of X is an accumulation point. But since to every point $x \in X$ there corresponds a segment of Σ_u we would find that some segments of Σ_u accumulate in an internal point of Ω which is excluded by the definition of piecewise-C^1 maps.

Our next claim is that the smooth parts of the boundary are actually straight segments. This is due to the fact that in such parts the boundary datum C and the map u coincide. Since both of these maps are affine, if the boundary has at least two different directions, these two affine maps would coincide which is excluded since C is not an admissible value for Du.

Therefore one of these segments of Σ_u (whose endpoint is x_0) meets the boundary $\partial\Omega$ at a vertex; i.e., were $\partial\Omega$ is not of class C^1. If we consider two consecutive vertices in $\partial\Omega$ (i.e., endpoints of a connected part of the boundary $\partial\Omega$ where it is of class C^1), then in between these two vertices no lamination arrives and u is affine there. Moreover, u being equal to Cx outside Ω, the interface must be affine.

This proves that Ω is a polygon and that the endpoints of the lamination must arrive *only* at the vertices of Ω. Moreover every vertex of $\partial\Omega$ should be an endpoint of a segment in Σ_u . In fact, otherwise, we would have a boundary portion not affine where two affine gradient values meet.

It thus remains to consider this final case, where Ω is a polygon with lamination which arrives at all the vertex of Ω. This conclusion for the polygon comes from the next lemma. ■

Lemma 29 *Let Ω be a polygon with at most seven sides. Let $A, B \in \mathbb{R}^{2\times 2}$ be invertible matrices such that*

$$\lambda_1\left(BA^{-1}\right) \leq 1 \leq \lambda_2\left(BA^{-1}\right)$$

with at least one strict inequality. Let $C \in \mathbb{R}^{2\times 2}$ with $C \notin \mathbb{S}_A \cup \mathbb{S}_B$. Then there is no locally piecewise C^1 solution of the Dirichlet problem (10).

Remark 30 *In the proof below we will use the following elementary fact. Let $a, b, c, d \in \mathbb{R}^2$ then*

$$\det\left(a \otimes b - c \otimes d\right) = 0 \ \Leftrightarrow\ a \| c \text{ or } b \| d.$$

Remark 31 *In the proof of the lemma, we will use the following notion of* extreme point of a set with respect to a given direction. *Let ν be a fixed direction, K be a bounded set of $\mathbb{R}^2$ and $x \in K$. We say that x is an extreme point of K with respect to ν, if a line having $\nu-$direction and containing x leaves K on one side (i.e. K is on one of the half spaces defined by the line).*

If K is a closed polygon in $\mathbb{R}^2$, then it has at least 2 (and at most 4 if it is convex) distinct extreme vertices with respect to any given direction.

Proof. Of course if Ω is a triangle then the singular set is empty, the map u is affine in Ω, $Du = C$ and it does not satisfy the differential inclusion $Du \in \mathbb{S}_A \cup \mathbb{S}_B$ since $C \notin \mathbb{S}_A \cup \mathbb{S}_B$. We therefore consider a polygon Ω with at least 4 sides.

We already proved above that lamination arrives at all the vertices of Ω. We will use this fact to get a contradiction, to show that in fact a solution to the Dirichlet problem (10) cannot have this property. To this aim we will take under special consideration the vertices where a single lamination (not a double one) arrives.

Step 1. Let us first prove that a single lamination arrives in at least 4 vertices of the polygon Ω. As before let us denote by ν_+ and ν_- the two normals to the lamination (we assume here that $\nu_+ \neq \nu_-$, otherwise nothing is to be proved). Then, with respect for instance to ν_+ , let us consider the corresponding lamination of direction $\nu_+^\perp$ (the vector orthogonal to ν_+). The polygon $\overline{\Omega}$ has at least two vertices which are extreme with respect to the direction $\nu_+^\perp$. At these two vertices the lamination cannot arrive from the interior of Ω; thus only the other lamination of direction $\nu_-^\perp$ can arrive there. A similar consideration can

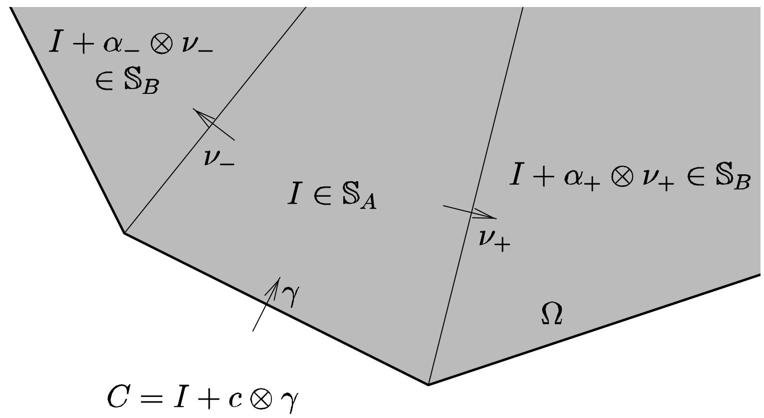

Figure 3: the Dirichlet problem at two consecutive corners

be done for ν_- and we therefore have at least 4 distinct vertices where a single lamination arrives. Note that the extreme points related to $\nu_+^\perp$ and $\nu_-^\perp$ should be distinct one from the other, otherwise at a common extreme point we would have a vertex with no lamination arriving from the inside and this have already been excluded.

Step 2. Since the polygon Ω has at most 7 vertices, then a double lamination arrives at most at 3 vertices and thus at least at two *consecutive* vertices a single lamination arrives. Thus we consider two consecutive vertices of Ω as in Figure 3.

The map u is solution to the Dirichlet problem (10), with $C \notin \mathbb{S}_A \cup \mathbb{S}_B$. Without loss of generality we can consider that

$$A = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$$

with $0 < |\lambda| \leq 1 \leq \mu$ and at least one strict inequality. We know that the normal vectors to the lines of discontinuities are given by

$$\nu_\pm = \left(\pm\sqrt{1-\lambda^2}, \sqrt{\mu^2-1}\right).$$

We also recall that the angle θ is given by

$$\cos\theta = \frac{1+\lambda\mu}{\lambda+\mu} \quad \text{and} \quad \sin\theta = \frac{\sqrt{\mu^2-1}\sqrt{1-\lambda^2}}{\lambda+\mu}.$$

Up to a rotation or to multiplication by B^{-1} we can consider that in between the two lines of discontinuities the value of the gradient of the function is $I \in \mathbb{S}_A$ as in Figure 3. So in the other sides of the discontinuity line we have that the gradient is either $I + \alpha_- \otimes \nu_- \in \mathbb{S}_B$ if the line of discontinuity is orthogonal to

ν_- or $I+\alpha_+\otimes\nu_+\in\mathbb{S}_B$ if the line of discontinuity is orthogonal to ν_+ . Lemma 19 asserts that if

$$\alpha_\pm=(\alpha_1,\alpha_2)=\left(\mp\frac{\mu\sqrt{1-\lambda^2}}{\mu+\lambda},\frac{\lambda\sqrt{\mu^2-1}}{\mu+\lambda}\right)$$

then

$$I+\alpha_-\otimes\nu_-=\begin{pmatrix} 1-\frac{\mu\left(1-\lambda^2\right)}{\mu+\lambda} & \frac{\mu\sqrt{\mu^2-1}\sqrt{1-\lambda^2}}{\mu+\lambda} \\ \frac{-\lambda\sqrt{\mu^2-1}\sqrt{1-\lambda^2}}{\mu+\lambda} & 1+\frac{\lambda\left(\mu^2-1\right)}{\mu+\lambda}\end{pmatrix}=R_{-\theta}\Lambda\in\mathbb{S}_B$$

$$I+\alpha_+\otimes\nu_+=\begin{pmatrix} 1-\frac{\mu\left(1-\lambda^2\right)}{\mu+\lambda} & -\frac{\mu\sqrt{\mu^2-1}\sqrt{1-\lambda^2}}{\mu+\lambda} \\ \frac{\lambda\sqrt{\mu^2-1}\sqrt{1-\lambda^2}}{\mu+\lambda} & 1+\frac{\lambda\left(\mu^2-1\right)}{\mu+\lambda}\end{pmatrix}=R_{\theta}\Lambda\in\mathbb{S}_B\,.$$

Two cases can happen.

Case 1. The lines of discontinuity which arrive at the two consecutive vertices are one orthogonal to ν_- and the other orthogonal to ν_+ (see Figure 3). This necessarily implies that

$$|\lambda|<1<\mu.$$

So we consider two consecutive vertices of $\partial\Omega$ from where one lamination emanates with normal ν_+ and another with normal ν_- (see Figure 3). The boundary datum has the form $u\left(x\right)=Cx$, where $C=I+c\otimes\gamma$. Now let us see what happens for c and γ. We should have (since the map is continuous across the boundary of Ω)

$$\det\left(c\otimes\gamma-\alpha_-\otimes\nu_-\right)=\det\left(c\otimes\gamma-\alpha_+\otimes\nu_+\right)=0.$$

Then, according to Remark 30, we have from the first equation

$$c\|\alpha_-\quad\text{or}\quad\gamma\|\nu_-$$

and from the second equation

$$c\|\alpha_+\quad\text{or}\quad\gamma\|\nu_+$$

Since neither α_+ is parallel to α_- nor ν_+ is parallel to ν_- , we deduce that

$$\gamma\|\nu_+\quad\text{or}\quad\gamma\|\nu_-$$

and thus either the lamination orthogonal to ν_+ or the one orthogonal to ν_- would coincide with the side of the polygon orthogonal to γ and it would not be internal to Ω.

Case 2. At the two consecutive vertices the lines of discontinuity are both orthogonal to ν_- (the case orthogonal to ν_+ is completely analogous) and the figure is then similar to Figure 3. The gradient is then equal on both sides, namely

$$I+\alpha_{-}\otimes\nu_{-}=\begin{pmatrix} 1-\frac{\mu\left(1-\lambda^{2}\right)}{\mu+\lambda} & \frac{\mu\sqrt{\mu^{2}-1}\sqrt{1-\lambda^{2}}}{\mu+\lambda} \\ \frac{-\lambda\sqrt{\mu^{2}-1}\sqrt{1-\lambda^{2}}}{\mu+\lambda} & 1+\frac{\lambda\left(\mu^{2}-1\right)}{\mu+\lambda} \end{pmatrix}\in\mathbb{S}_{B}\,.$$

This is incompatible with the fact that it should match continuously with the boundary datum $u\left(x\right)=Cx$, where $C=I+c\otimes\gamma$ and with the fact that the sides of the polygon cannot be parallel there. The proof is therefore complete. ■

8 Existence of Lipschitz solutions not piecewise affine

The approach for existence that we discuss in this section is a *functional analytic method* based on the *Baire category theorem* and on weak lower semicontinuity of convex and quasiconvex integrals. We mainly refer to the book [7], but we also mention that some comparable results for the two well problem can be found in [17].

The following result about the two well problem can be found in [7], Theorem 7.31.

Theorem 32 *Let Ω be an open set of $\mathbb{R}^2$. Let A, B be two invertible matrices such that* $\det B>\det A>0$ *and* $\operatorname{rank}\{A-B\}=1$. *Let $\varphi\left(x\right)=\xi x$ where ξ is such that*

$$\xi=\alpha\,R_{a}\,A+\beta\,R_{b}\,B$$

where $R_a\,,\ R_b\in SO(2)$ and $\alpha,\beta>0$ are such that

$$\alpha<\frac{\det B-\det\xi}{\det B-\det A}\quad\text{and}\quad\beta<\frac{\det\xi-\det A}{\det B-\det A}\,.$$

Then there exists $u\in W^{1,\infty}(\Omega;\mathbb{R}^2)$ such that

$$\begin{cases} Du\in\mathbb{S}_{A}\cup\mathbb{S}_{B} & a.e.\ in\ \Omega \\ u=\varphi & on\ \partial\Omega. \end{cases}\tag{11}$$

Note that, contrary to the previous sections, here the map u, solution to the Dirichlet problem (11), is a general Lipschitz-continuous solution, and not necessarily a locally piecewise C^1 map; and in fact cannot be!

One of the aims of this paper was to show that solving a Dirichlet problem by means of piecewise C^1 map or even piecewise affine map on one hand gives more information on the solution (since the class of functions under consideration is smaller) but on the other hand it makes the solving of the problem more difficult. Theorem 32 is a relevant example, if compared with Theorem 28, where we proved that it was impossible to solve the same differential problem by means of only piecewise C^1 map.

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