Conformal metrics on $\mathbb{R}^{2m}$ with constant Q-curvature and large volume

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Abstract

We study conformal metrics $g_u = e^{2u}|dx|^2$ on $\mathbb{R}^{2m}$ with constant Q-curvature $Q_{g_u} \equiv (2m-1)!$ (notice that $(2m-1)!$ is the Q-curvature of S^{2m}) and finite volume. When $m = 3$ we show that there exists V^* such that for any $V \in [V^*, \infty)$ there is a conformal metric $g_u = e^{2u}|dx|^2$ on $\mathbb{R}^6$ with $Q_{g_u} \equiv 5!$ and $\mathrm{vol}(g_u) = V$. This is in sharp contrast with the four-dimensional case, treated by C-S. Lin. We also prove that when m is odd and greater than 1, there is a constant $V_m > \mathrm{vol}(S^{2m})$ such that for every $V \in (0, V_m]$ there is a conformal metric $g_u = e^{2u}|dx|^2$ on $\mathbb{R}^{2m}$ with $Q_{g_u} \equiv (2m-1)!$, $\mathrm{vol}(g) = V$. This extends a result of A. Chang and W-X. Chen. When m is even we prove a similar result for conformal metrics of *negative* Q-curvature.

KEYWORDS: Q-curvature, Paneitz operators, GMJS operators, conformal geometry.

1 Introduction and statement of the main theorems

We consider solutions to the equation

$$(-\Delta)^m u = (2m-1)! e^{2mu} \quad \text{in } \mathbb{R}^{2m}, \tag{1}$$

satisfying

$$V := \int_{\mathbb{R}^{2m}} e^{2mu(x)} dx < +\infty, \tag{2}$$

with particular emphasis on the role played by V.

Geometrically, if u solves (1) and (2), then the conformal metric $g_u := e^{2u}|dx|^2$ has Q-curvature $Q_{g_u} \equiv (2m-1)!$ and volume V (by $|dx|^2$ we denote the Euclidean metric). For the definition of Q-curvature and related remarks, we refer to Chapter 4 in [Cha] or to [FG] and [FH]. Notice that given a solution u to (1) and $\lambda > 0$, the function $v := u - \frac{1}{2m}\log\lambda$ solves

$$(-\Delta)^m v = \lambda(2m-1)! e^{2mv} \quad \text{in } \mathbb{R}^{2m}, \quad \int_{\mathbb{R}^{2m}} e^{2mv(x)} dx = \frac{V}{\lambda},$$

hence there is no loss of generality in the particular choice of the constant $(2m-1)!$ in (1). On the other hand this constant has the advantage of being the Q-curvature of the round sphere

S^{2m}. This implies that the function $u_1(x) = \log \frac{2}{1+|x|^2}$, which satisfies $e^{2u_1}|dx|^2 = (\pi^{-1})^* g_{S^{2m}}$ (here $\pi : S^{2m} \to \mathbb{R}^{2m}$ is the stereographic projection) is a solution to (1)-(2) with $V = \mathrm{vol}(S^{2m})$. Translations and dilations (i.e. Möbius transformations) actually give us a large family of solutions to (1)-(2) with $V = \mathrm{vol}(S^{2m})$, namely

$$u_{x_0,\lambda}(x) := u_1(\lambda(x - x_0)) + \log \lambda = \log \frac{2\lambda}{1 + \lambda^2 |x - x_0|^2}, \quad x_0 \in \mathbb{R}^{2m}, \lambda > 0. \tag{3}$$

We shall call the functions $u_{x_0,\lambda}$ standard or *spherical* solutions to (1)-(2).

The question whether the family of spherical solutions in (3) exhausts the set of solutions to (1)-(2) has raised a lot of interest and is by now well understood. W. Chen and C. Li [CL] proved that on $\mathbb{R}^2$ ($m = 1$) every solution to (1)-(2) is spherical, while for every $m > 1$, i.e. in dimension 4 and higher, it was proven by A. Chang and W-X. Chen [CC] that Problem (1)-(2) admits solutions which are non spherical. In fact they proved

Theorem A (A. Chang-W-X. Chen [CC] 2001). *For every $m > 1$ and $V \in (0, \mathrm{vol}(S^{2m}))$ there exists a solution to* (1)-(2).

Several authors have tried to classify spherical solutions or, in other words, to give analytical and geometric conditions under which a solution to (1)-(2) is spherical (see [CY], [WX], [Xu]), and to understand some properties of non-spherical solutions, such as their asymptotic behavior, their volume and their symmetry (see [Lin], [Mar1], [WY]). In particular C-S. Lin proved:

Theorem B (C-S. Lin [Lin] 1998). *Let u solve* (1)-(2) *with $m = 2$. Then either u is spherical (i.e. as in* (3)*) or $V < \mathrm{vol}(S^4)$.*

Both spherical solutions and the solutions given by Theorem A are radially symmetric (i.e. of the form $u(|x - x_0|)$ for some $x_0 \in \mathbb{R}^{2m}$). On the other hand there also exist plenty of non-radial solutions to (1)-(2) when $m = 2$.

Theorem C (J. Wei and D. Ye [WY] 2006). *For every $V \in (0, \mathrm{vol}(S^4))$ there exist (several) non-radial solutions to* (1)-(2) *for $m = 2$.*

Remark D Probably the proof of Theorem C can be extended to higher dimension $2m \geq 2$, yielding several non-symmetric solutions to (1)-(2) for every $V \in (0, \mathrm{vol}(S^{2m}))$, but failing to produce non-symmetric solutions for $V \geq \mathrm{vol}(S^{2m})$. As in the proof of Theorem A, the condition $V < \mathrm{vol}(S^{2m})$ plays a crucial role.

Theorems A, B, C and Remark D strongly suggest that also in dimension 6 and higher all non-spherical solutions to (1)-(2) satisfy $V < \mathrm{vol}(S^{2m})$, i.e. (1)-(2) has no solution for $V > \mathrm{vol}(S^{2m})$ and the only solutions with $V = \mathrm{vol}(S^{2m})$ are the spherical ones. Quite surprisingly we found out that this is not at all the case. In fact in dimension 6 we found solutions to (1)-(2) with arbitrarily large V:

Theorem 1 *For $m = 3$ there exists $V^* > 0$ such that for every $V \geq V^*$ there is a solution u to* (1)-(2)*, i.e. there exists a metric on $\mathbb{R}^6$ of the form $g_u = e^{2u}|dx|^2$ satisfying $Q_{g_u} \equiv 5!$ and* $\mathrm{vol}(g_u) = V$.

In order to prove Theorem 1 we will consider only rotationally symmetric solutions to (1)-(2), so that (1) reduces to and ODE. Precisely, given $a, b \in \mathbb{R}$ let $u = u_{a,b}(r)$ be the solution of

$$\begin{cases} \Delta^3 u = -e^{6u} \quad \text{in } \mathbb{R}^6 \\ u(0) = u'(0) = u'''(0) = u'''''(0) = 0 \\ u''(0) = \frac{\Delta u(0)}{6} = a \\ u''''(0) = \frac{\Delta^2 u(0)}{16} = b. \end{cases} \tag{4}$$

Here and in the following we will always (by a little abuse of notation) see a rotationally symmetric function f both as a function of one variable $r \in [0, \infty)$ (when writing f', f'', etc...) and as a function of $x \in \mathbb{R}^6$ (when writing Δf, $\Delta^2 f$, etc...). We also used that

$$\Delta f(0) = 6f''(0), \quad \Delta^2 f(0) = 16 f''''(0),$$

see e.g. [Mar1, Lemma 17]. Also notice that in (4) we replaced 5! by 1 to make the computations lighter. As we already noticed, this is not a problem.

Theorem 2 *Let $u = u_{a,3}$ solve* (4) *for a given $a < 0$ and $b = 3$.*[1] *Then*

$$\int_{\mathbb{R}^6} e^{6u_{a,3}} dx < \infty \text{ for } -a \text{ large}; \quad \lim_{a \to -\infty} \int_{\mathbb{R}^6} e^{6u_{a,3}} dx = \infty. \tag{5}$$

In particular the conformal metric $g_{u_{a,3}} = e^{2u_{a,3}}|dx|^2$ of constant Q-curvature $Q_{g_{u_{a,3}}} \equiv 1$ satisfies

$$\lim_{a \to -\infty} \mathrm{vol}(g_{u_{a,3}}) = \infty.$$

Theorem 1 will follow from Theorem 2 and a continuity argument (Lemma 8 below).

Going through the proof of Theorem A it is clear that it does not extend to the case $V > \mathrm{vol}(S^{2m})$. With a different approach, we are able to prove that, at least when $m \geq 3$ is odd, one can extend Theorem A as follows.

Theorem 3 *For every $m \geq 3$ odd there exists $V_m > \mathrm{vol}(S^{2m})$ such that for every $V \in (0, V_m]$ there is a non-spherical solution u to* (1)-(2)*, i.e. there exists a metric on $\mathbb{R}^{2m}$ of the form $g_u = e^{2u}|dx|^2$ satisfying $Q_{g_u} \equiv (2m-1)!$ and $\mathrm{vol}(g_u) = V$.*

The condition $m \geq 3$ odd is (at least in part) necessary in view of Theorem B and [CL], but the case $m \geq 4$ even is open. Notice also that when $m = 3$, Theorems 1 and 3 guarantee the existence of solutions to (1)-(2) for

$$V \in (0, V_m] \cup [V^*, \infty),$$

[1] The choice $b = 3$ is convenient in the computations, but any other $b > 0$ would work.

but we cannot rule out that $V_m < V^*$ (the explicit value of V_m is given in (38) below) and the existence of solutions to (1)-(2) is unknown for $V \in (V_m, V^*)$. Could there be a gap phenomenon?

We now briefly investigate how large the volume of a metric $g_u = e^{2u}|dx|^2$ on $\mathbb{R}^{2m}$ can be when $Q_{g_u} \equiv const < 0$. Again with no loss of generality we assume $Q_{g_u} \equiv -(2m-1)!$. In other words consider the problem

$$(-\Delta)^m u = -(2m-1)! e^{2mu} \quad \text{on } \mathbb{R}^{2m}. \tag{6}$$

Although for $m = 1$ it is easy to see that Problem (6)-(2) admits no solutions for any $V > 0$, when $m \geq 2$ Problem (6)-(2) has solutions for some $V > 0$, as shown in [Mar2]. Then with the same proof of Theorem 3 we get:

Theorem 4 *For every $m \geq 2$ even there exists $V_m > \mathrm{vol}(S^{2m})$ such that for $V \in (0, V_m]$ there is a solution u to* (6)-(2)*, i.e. there exists a metric on $\mathbb{R}^{2m}$ of the form $g_u = e^{2u}|dx|^2$ satisfying*

$$Q_{g_u} \equiv -(2m-1)!, \quad \mathrm{vol}(g_u) = V.$$

The cases of solutions to (1)-(2) with m even, or (6)-(2) and m odd seem more difficult to treat since the ODE corresponding to (1) or (6), in analogy with (4) becomes

$$\Delta^m u(r) = (2m-1)! e^{2mu(r)},$$

whose solutions can blow up in finite time (i.e. for finite r) if the initial data are not chosen carefully (contrary to Lemma 5 below).

2 Proof of Theorem 2

Set $\omega_{2m-1} := \mathrm{vol}(S^{2m-1})$ and let B_r denote the unit ball in $\mathbb{R}^{2m}$ centered at the origin. Given a smooth radial function $f = f(r)$ in $\mathbb{R}^{2m}$ we will often use the divergence theorem in the form

$$\int_{B_r} \Delta f dx = \int_{\partial B_r} \frac{\partial f}{\partial \nu} d\sigma = \omega_{2m-1} r^{2m-1} f'(r). \tag{7}$$

Dividing by $\omega_{2m-1} r^{2m-1}$ into (7) and integrating we also obtain

$$f(t) - f(s) = \int_s^t \frac{1}{\omega_{2m-1}\rho^{2m-1}} \int_{B_\rho} \Delta f dx d\rho, \quad 0 \leq s \leq t. \tag{8}$$

When no confusion can arise we will simply write u instead of $u_{a,3}$ or $u_{a,b}$ to denote the solution to (4). In what follows, also other quantities (e.g. R, r_0, r_1, r_2, r_3, ϕ, ξ_1, ξ_2) will depend on a and b, but this dependence will be omitted from the notation.

Lemma 5 *Given any $a, b \in \mathbb{R}$, the solution u to the ODE* (4) *exists for all times.*

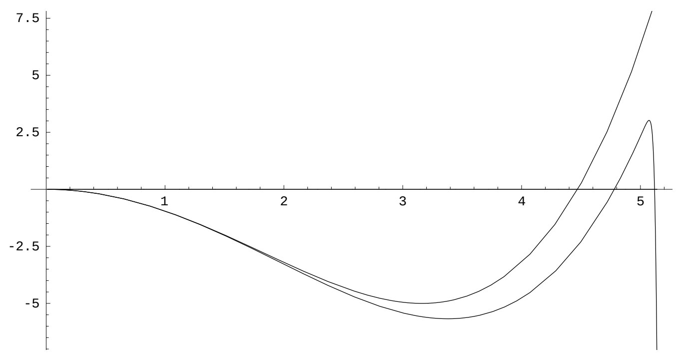

Figure 1: The functions $\phi(r) = \frac{a}{2}r^2 + \frac{1}{8}r^4$ (above) and $u_{a,3}(r) \le \phi(r)$.

Proof. Applying (8) to $f = \Delta^2 u$, and observing that $\Delta(\Delta^2 u) = -e^{6u} \le 0$ we get

$$\Delta^2 u(t) \le \Delta^2 u(s) \le \Delta^2 u(0) = 16b \quad 0 \le s \le t, \tag{9}$$

i.e. $\Delta^2 u(r)$ is monotone decreasing. This and (8) applied to Δu yield

$$\Delta u(r) \le \Delta u(0) + \int_0^r \frac{1}{\omega_5 \rho^5} \int_{B_\rho} 16b dx d\rho = 6a + \int_0^r \frac{8}{3} b\rho d\rho = 6a + \frac{4}{3} br^2.$$

A further application of (8) to u finally gives

$$u(r) \le \int_0^r \frac{1}{\omega_5 \rho^5} \int_{B_\rho} (6a + \frac{4}{3} b|x|^2) dx d\rho = \int_0^r (a\rho + \frac{\rho^3 b}{6}) d\rho = \frac{a}{2} r^2 + \frac{b}{24} r^4 =: \phi(r). \tag{10}$$

Similar lower bounds can be obtained by observing that $-e^{6u} \ge -1$ for $u \le 0$. This proves that $u(r)$ cannot blow-up in finite time and, by standard ODE theory, $u(r)$ exists for every $r \ge 0$. $\square$

Proof of (5) *(completed).* Fix $b = 3$ and take $a < 0$. The function $\phi(r) = \frac{a}{2}r^2 + \frac{1}{8}r^4$ vanishes for $r = R = R(a) := 2\sqrt{-a}$. In order to prove (5) we shall investigate the behavior of u in a neighborhood of R. The heuristic idea is that

$$u^{(j)}(0) = \phi^{(j)}(0), \quad \text{for } 0 \le j \le 5, \qquad \Delta^3 \phi \equiv 0,$$

and for every $\varepsilon > 0$ on $[\varepsilon, R - \varepsilon]$ we have $\phi \le C_\varepsilon a \to -\infty$ and $|\Delta^3 u| \le e^{C_\varepsilon a} \to 0$ as $a \to -\infty$, hence for $r \in [0, R - \varepsilon]$ we expect $u(r)$ to be very close to $\phi(r)$. On the other hand, u cannot stay close to ϕ for r much larger than R because eventually $-\Delta^3 u(r)$ will be large enough to make $\Delta^2 u$, Δu and u negative according to (8) (see Fig. 1). Then it is crucial to show that u stays close to ϕ for some $r > R$ (hence in a region where ϕ is positive and $\Delta^3 u$ is not necessarily small) and long enough to make the second integral in (5) blow up as $a \to -\infty$.

Step 1: Estimates of $u(R)$, $\Delta u(R)$ *and* $\Delta^2 u(R)$. From (10) we infer

$$\Delta^3 u = -e^{6u} \ge -e^{6\phi},$$

which, together with (8), gives

$$\Delta^2 u(r) = \Delta^2 u(0) + \int_0^r \frac{1}{\omega_5 \rho^5} \int_{B_\rho} \Delta^3 u dx d\rho \geq 48 - \int_0^r \frac{1}{\omega_5 \rho^5} \int_{B_\rho} e^{6\phi(|x|)} dx d\rho. \tag{11}$$

We can explicitly compute (see Lemma 6 below and simplify (29) using that $\phi(R) = 0$ and $\int_{\sqrt{3}a}^{-\sqrt{3}a} e^{t^2} dt = 2 \int_0^{-\sqrt{3}a} e^{t^2} dt$)

$$\int_0^R \frac{1}{\omega_5 \rho^5} \int_{B_\rho} e^{6\phi(|x|)} dx d\rho = \frac{1}{48a} + \frac{(18a^2+1)\sqrt{3}}{144a^2} e^{-3a^2} \int_0^{-\sqrt{3}a} e^{t^2} dt.$$

Then by (9) and Lemma 7 below we conclude that

$$\Delta^2 u(r) \geq \Delta^2 u(R) \geq 48(1 + O(a^{-1})) \quad \text{for } 0 \leq r \leq R = 2\sqrt{-a}. \tag{12}$$

where here and in the following $|a^k O(a^{-k})| \leq C = C(k)$ as $a \to -\infty$ for every $k \in \mathbb{R}$. Then applying (8) as before we also obtain

$$\Delta u(r) \geq 6a + 4(1 + O(a^{-1}))r^2 \quad \text{for } 0 \leq r \leq R$$

and

$$u(r) \geq \frac{a}{2} r^2 + \frac{1 + O(a^{-1})}{8} r^4 = \phi(r) + O(a^{-1}) r^4 \quad \text{for } 0 \leq r \leq R.$$

At $r = R$ this reduces to

$$u(R) \geq O(a).$$

Step 2: Behavior of $u(r)$, $\Delta u(r)$, $\Delta^2 u(r)$ *for* $r \geq R$. Define r_0 (depending on $a < 0$) as

$$r_0 := \inf\{r > 0 : u(r) = 0\} \in [R, \infty].$$

We first claim that $r_0 < \infty$. We have by Lemma 6 and Lemma 7

$$\int_{B_R} e^{6\phi} dx = \omega_5 \Big(-\frac{4a}{3} + \frac{4(6a^2-1)\sqrt{3}}{9} e^{-3a^2} \int_0^{-\sqrt{3}a} e^{t^2} dt \Big) = O(a). \tag{13}$$

Since on B_{r_0} we have $u \leq 0$, hence $\Delta^3 u \geq -1$, using (7)-(8) and (13) we get for $r \in [R, r_0]$

$$\begin{aligned} \Delta^2 u(r) &\geq \Delta^2 u(R) - \int_R^r \frac{1}{\omega_5 \rho^5} \Big(\int_{B_R} e^{6\phi} dx + \int_{B_\rho \setminus B_R} 1 dx \Big) d\rho \\ &\geq 48 + O(a) \Big[\frac{1}{R^4} - \frac{1}{r^4} \Big] - \int_R^r \frac{\rho^6 - R^6}{6\rho^5} d\rho \end{aligned} \tag{14}$$

Assuming $r \in [R, 2R]$ we can now bound with a Taylor expansion

$$\frac{1}{R^4} - \frac{1}{r^4} = R^{-4} \tilde{O}\Big(\frac{r-R}{R} \Big) \tag{15}$$

and

$$\rho^6 - R^6 \leq r^6 - R^6 = R^6 \tilde{O}\Big(\frac{r-R}{R}\Big), \quad \text{for } \rho \in [R, r],$$

which together with (15) yields

$$\int_R^r \frac{\rho^6 - R^6}{6\rho^5} d\rho \leq \int_R^r \frac{r^6 - R^6}{6\rho^5} d\rho \leq R^2 \tilde{O}\Big(\Big(\frac{r-R}{R}\Big)^2\Big), \tag{16}$$

where for any $k \in \mathbb{R}$ we have $|t^{-k}\tilde{O}(t^k)| \leq C = C(k)$ uniformly for $0 \leq t \leq 1$. Using (15) and (16) we bound in (14)

$$\Delta^2 u(r) \geq 48 + O(a^{-1})\tilde{O}\Big(\frac{r-R}{R}\Big) + R^2 \tilde{O}\Big(\Big(\frac{r-R}{R}\Big)^2\Big), \quad r \in [R, \min\{r_0, 2R\}],$$

whence

$$\Delta^2 u(r) \geq 48 + O(a^{-1}) + R^2 \tilde{O}\Big(\Big(\frac{r-R}{R}\Big)^2\Big)\chi_{(R,\infty)}(r), \quad r \in [0, \min\{r_0, 2R\}],$$

where $\chi_{(R,\infty)}(r) = 0$ for $r \in [0, R]$ and $\chi_{(R,\infty)}(r) = 1$ for $r > R$. Then with (8) we estimate for $r \in [0, \min\{r_0, 2R\}]$

$$\begin{aligned} \Delta u(r) &\geq 6a + 4(1 + O(a^{-1}))r^2 + \chi_{(R,\infty)}(r) \int_R^r \frac{1}{\omega_5 \rho^5} \int_{B_\rho \setminus B_R} R^2 \tilde{O}\Big(\Big(\frac{|x| - R}{R}\Big)^2\Big) dx d\rho \\ &= 6a + 4(1 + O(a^{-1}))r^2 + R^4 \tilde{O}\Big(\Big(\frac{r-R}{R}\Big)^4\Big)\chi_{(R,\infty)}(r). \end{aligned} \tag{17}$$

and

$$\begin{aligned} u(r) &\geq \frac{a}{2} r^2 + \frac{1 + O(a^{-1})}{8} r^4 + \chi_{(R,\infty)}(r) \int_R^r \frac{1}{\omega_5 \rho^5} \int_{B_\rho \setminus B_R} R^4 \tilde{O}\Big(\Big(\frac{|x| - R}{R}\Big)^4\Big) dx d\rho \\ &= \phi(r) + O(a^{-1})r^4 + R^6 \tilde{O}\Big(\Big(\frac{r-R}{R}\Big)^6\Big)\chi_{(R,\infty)}(r), \end{aligned} \tag{18}$$

where the integrals in (17) and (18) are easily estimated bounding $|x|$ with r and applying (16).

Making a Taylor expansion of $\phi(r)$ at $r = R$ and using that $\phi(R) = 0$, we can further estimate the right-hand side of (18) for $r \in [R, \min\{r_0, 2R\}]$ as

$$\begin{aligned} u(r) &\geq \phi'(R)(r - R) + R^2 \tilde{O}\Big(\Big(\frac{r-R}{R}\Big)^2\Big) + O(a^{-1})r^4 + R^6 \tilde{O}\Big(\Big(\frac{r-R}{R}\Big)^6\Big) \\ &= -aR(r - R) + O(a^{-1})R^4 + R^2 \tilde{O}\Big(\Big(\frac{r-R}{R}\Big)^2\Big) + R^6 \tilde{O}\Big(\Big(\frac{r-R}{R}\Big)^6\Big) =: \psi_a(r) \end{aligned}$$

Now choosing $r = R(1 + 1/\sqrt{-a})$, so that $(r - R)/R \to 0$ as $a \to -\infty$, we get

$$\lim_{a \to -\infty} \psi_a(R(1 + 1/\sqrt{-a})) \geq \lim_{a \to -\infty} \Big(4(-a)^{\frac{3}{2}} + O(a) - C\Big) = \infty.$$

In particular

$$r_0 \in [R, R(1 + 1/\sqrt{-a})].$$

We now claim that

$$\lim_{a\to-\infty}\Delta u(r_0)=\infty. \tag{19}$$

Indeed we infer from (17)

$$\begin{aligned}\Delta u(r_0)&\geq 6a+4(1+O(a^{-1}))r_0^2-C\\&\geq 6a+4(1+O(a^{-1}))R^2-C\\&\geq -10a-C,\end{aligned}$$

for $-a$ large enough, whence (19). Set

$$r_1=r_1(a):=\inf\{r>r_0:u(r)=0\}.$$

Applying (7) to (17), and recalling that $\frac{r_0-R}{R}\leq\frac{1}{\sqrt{a}}$, similar to (18) we obtain

$$\begin{aligned}u'(r_0)&\geq ar_0+\frac{1+O(a^{-1})}{2}r_0^3-C\\&\geq ar_0+\frac{1+O(a^{-1})}{2}r_0R^2-C\\&\geq -ar_0-C.\end{aligned}$$

In particular for $-a$ large enough we have $u'(r_0)>0$, which implies $r_1>r_0$. Using (7)-(8) and that $\Delta^3u(r)\leq -1$ for $r\in[r_0,r_1]$, it is not difficult to see that $r_1<\infty$. Moreover there exists at least a point $r_2=r_2(a)\in(r_0,r_1]$ such that $u'(r_2)\leq 0$, which in turn implies that

$$\Delta u(r_3)<0\quad\text{for some } r_3=r_3(a)\in(r_0,r_2], \tag{20}$$

since otherwise we would have by (7)

$$u'(r_2)=\frac{1}{\omega_5r_2^5}\int_{B_{r_0}}\Delta u dx+\frac{1}{\omega_5r_2^5}\int_{B_{r_2}\setminus B_{r_0}}\Delta u dx\geq\frac{r_0^5}{r_2^5}u'(r_0)>0,$$

contradiction.

Step 3: Conclusion. We now use the estimates obtained in Step 1 and Step 2 to prove (5).

From (8), (19) and (20) we infer

$$\lim_{a\to-\infty}\int_{r_0}^{r_3}\frac{1}{\omega_5r^5}\int_{B_r}\Delta^2u dx dr=\lim_{a\to-\infty}(\Delta u(r_3)-\Delta u(r_0))=-\infty, \tag{21}$$

hence by the monotonicity of $\Delta^2u(r)$ (see (9))

$$\lim_{a\to-\infty}\Delta^2u(r_3)(r_3^2-r_0^2)=-\infty. \tag{22}$$

We now claim that

$$\lim_{a\to-\infty}\int_{B_{r_3}}e^{6u}dx=\infty. \tag{23}$$

Indeed consider on the contrary an arbitrary sequence a_k with $\lim_{k\to\infty} a_k = -\infty$ and

$$\lim_{k\to\to\infty} \int_{B_{r_3}} e^{6u} dx < \infty, \tag{24}$$

where here r_3 and u depend on a_k instead of a of course. Since $u \geq 0$ in $B_{r_3} \setminus B_{r_0}$ we have

$$\int_{B_{r_3}} e^{6u} dx \geq \int_{B_{r_3}\setminus B_{r_0}} 1dx = \frac{\omega_5}{6}(r_3^6 - r_0^6).$$

Now observe that $(r_3^6 - r_0^6) \geq (r_3^2 - r_0^2) r_0^4$ to conclude that (24) implies

$$\lim_{k\to\infty} (r_3^2 - r_0^2) \leq \lim_{k\to\infty} \frac{r_3^6 - r_0^6}{r_0^4} = 0. \tag{25}$$

Then (8), (12) and (22) yield

$$\begin{aligned}(r_3^2 - r_0^2) \int_R^{r_3} \frac{1}{\omega_5 r^5} \int_{B_r} e^{6u} dx dr &= (r_3^2 - r_0^2)(\Delta^2 u(R) - \Delta^2 u(r_3)) \\ &\geq -\Delta^2 u(r_3)(r_3^2 - r_0^2) \to \infty \quad \text{as } k \to \infty.\end{aligned}$$

By (25) we also have

$$\lim_{k\to\infty} \int_R^{r_3} \frac{1}{\omega_5 r^5} \int_{B_r} e^{6u} dx dr = \infty,$$

which implies at once

$$\lim_{k\to\infty} \int_{B_{r_3}} e^{6u} dx \geq \lim_{k\to\infty} 4R^4 \omega_5 \int_R^{r_3} \frac{1}{\omega_5 r^5} \int_{B_{r_3}} e^{6u} dx dr = \infty,$$

contradicting (24). Then (23) is proven.

It remains to show that

$$\int_{\mathbb{R}^6} e^{6u} dx < \infty,$$

at least for $-a$ large enough. It follows from (22) and the monotonicity of $\Delta^2 u$ that for $-a$ large enough we have

$$\Delta^2 u(r) < B < 0, \quad \text{for } r \geq r_3, \tag{26}$$

and, using (7)-(8) as already done several times, we can find $r_a \geq r_3$ such that

$$(\Delta u)'(r) < \frac{B}{6} r, \quad \Delta u(r) < \frac{B}{12} r^2, \quad u'(r) < \frac{B}{96} r^3, \quad u(r) < \frac{B}{384} r^4, \quad \text{for } r \geq r_a. \tag{27}$$

Then

$$\int_{\mathbb{R}^6} e^{6u} dx \leq \int_{B_{r_a}} e^{6u} dx + \int_{\mathbb{R}^6\setminus B_{r_a}} e^{\frac{B}{64}|x|^2} dx < \infty,$$

as wished. $\square$

2.1 Two useful lemmas

We now state and prove two lemmas used in the proof of Theorem 2.

Lemma 6 *For $\phi(r) = \frac{a}{2}r^2 + \frac{1}{8}r^4$, $a \leq 0$, we have*

$$\int_{B_r} e^{6\phi(|x|)}dx = \omega_5\left[\frac{2}{3}a + \frac{1}{3}e^{6\phi(r)}(-2a+r^2) + \frac{(12a^2-2)\sqrt{3}}{9}e^{-3a^2}\int_{-\sqrt{3}(a+r^2/2)}^{-\sqrt{3}a} e^{t^2}dt\right] =: \xi_1(r) \tag{28}$$

and

$$\begin{aligned}\int_0^r \frac{1}{\omega_5\rho^5}\int_{B_\rho} e^{6\phi(|x|)}dxd\rho =& \frac{-2a - e^{6\phi(r)}(-2a+r^2)}{12r^4} \\ &+ \frac{(2-12a^2+3r^4)\sqrt{3}}{36r^4}e^{-3a^2}\int_{-\sqrt{3}(a+r^2/2)}^{-\sqrt{3}a} e^{t^2}dt := \xi_2(r)\end{aligned} \tag{29}$$

Proof. Patiently differentiating, using that $e^{-3a^2}\frac{d}{dr}\int_{-\sqrt{3}(a+r^2/2))}^{-\sqrt{3}a} e^{t^2}dt = \sqrt{3}re^{6\phi(r)}$, one sees that

$$\xi_1'(r) = \omega_5 r^5 e^{6\phi(r)}, \quad \xi_2'(r) = \frac{\xi_1(r)}{\omega_5 r^5}.$$

Using that $\phi(0) = 0$ it is also easy to see that $\xi_1(0) = 0$.

Since $\xi_2(0)$ is not defined, we will compute the limit of $\xi_2(r)$ as $r \to 0$. We first compute the Taylor expansions

$$e^{6\phi(r)} = 1 + 3ar^2 + \frac{3}{4}(1+6a^2)r^4 + r^4 o(1),$$

and

$$\sqrt{3}e^{-3a^2}\int_{-\sqrt{3}(a+r^2/2)}^{-\sqrt{3}a} e^{t^2}dt = \frac{3}{2}r^2 + \frac{9}{4}ar^4 + r^4 o(1),$$

with errors $o(1) \to 0$ as $r \to 0$. Then

$$\begin{aligned}\frac{-2a - e^{6\phi(r)}(-2a+r^2)}{12r^4} &= \frac{(1-6a^2)r^2 + (\frac{3}{2}a - 9a^3)r^4}{12r^4} + o(1) \\ &= -\frac{(2-12a^2+3r^4)\sqrt{3}}{36r^4}e^{-3a^2}\int_{-\sqrt{3}(a+r^2/2)}^{-\sqrt{3}a} e^{t^2}dt,\end{aligned}$$

with $o(1) \to 0$ as $r \to 0$. Hence $\lim_{r\to 0}\xi_2(r) = 0$. □

Lemma 7 *We have*

$$\lim_{r\to\infty} re^{-r^2}\int_0^r e^{t^2}dt = \frac{1}{2}. \tag{30}$$

Proof. Clearly (30) is equivalent to

$$\lim_{r\to\infty} re^{-r^2}\int_2^r e^{t^2}dt = \frac{1}{2}. \tag{31}$$

Integrating by parts we get for $r \geq 2$

$$re^{-r^2}\int_2^r e^{t^2}dt = \frac{1}{2} - \frac{re^{-r^2+4}}{4} + re^{-r^2}\int_2^r \frac{e^{t^2}}{2t^2}dt. \tag{32}$$

Another integration by parts yields

$$re^{-r^2}\int_2^r \frac{e^{t^2}}{2t^2}dt = \frac{1}{4r^2} - \frac{re^{-r^2+4}}{32} + re^{-r^2}\int_2^r \frac{e^{t^2}}{12t^4}dt \to 0 \quad \text{as } r\to\infty,$$

where we used that the function $t^{-4}e^{t^2}$ is increasing on $[2,r]$, hence

$$0 \leq \int_2^r \frac{e^{t^2}}{12t^4}dt \leq \int_2^r \frac{e^{r^2}}{12r^4}dt = (r-2)\frac{e^{r^2}}{12r^4}.$$

We conclude by taking the limit as $r\to\infty$ in (32). $\square$

3 Proof of Theorem 1

We start with the following lemma.

Lemma 8 *Set*

$$V(a) = \frac{1}{5!}\int_{\mathbb{R}^6} e^{6u_{a,3}}dx$$

where $u = u_{a,3}$ is the solution to (4) for given $a < 0$ and $b = 3$. Then there exists $a^* < 0$ such that V is continuous on $(-\infty, a^*]$.

Proof. It follows from (21) and the monotonicity of $\Delta^2 u$ that we can fix $-a^*$ so large that

$$\lim_{r\to\infty}\Delta^2 u_{a,3}(r) < 0, \quad \text{for every } a \leq a^*.$$

Fix now $\varepsilon > 0$. Given $a \leq a^*$ it is not difficult to find $r_a > 0$ and $B = B(a) < 0$ such that

$$\Delta^2 u_{a,3}(r) < B < 0, \quad \text{for } r \geq r_a \tag{33}$$

and, possibly choosing r_a larger, using (7)-(8) as already done in the proof of Theorem 2, we get

$$(\Delta u_{a,3})'(r) < \frac{B}{6}r, \quad \Delta u_{a,3}(r) < \frac{B}{12}r^2, \quad u'_{a,3}(r) < \frac{B}{96}r^3, \quad u_{a,3}(r) < \frac{B}{384}r^4, \quad \text{for } r \geq r_a. \tag{34}$$

By possibly choosing r_a even larger we can also assume that

$$\int_{\mathbb{R}^6\setminus B_{r_a}} e^{\frac{B}{64}|x|^4}dx < \frac{\varepsilon}{2}. \tag{35}$$

By ODE theory the solution $u_{a,3}$ to (4) is continuous with respect to a in $C^k_{\mathrm{loc}}(\mathbb{R}^6)$ for every $k \geq 0$, in the sense that for any $r' > 0$, $u_{a',3} \to u_{a,3}$ in $C^k(B_{r'})$ as $a' \to a$. In particular we can find $\delta > 0$ (depending on ε) such that if $|a - a'| < \delta$ then (33)-(34) with a replaced by a' are still satisfied for $r = r_a$ (not $r_{a'}$) and (33) holds also for every $r > r_a$ since $\Delta^2 u_{a',3}(r)$ is decreasing in r (see (9)). Then, with (7)-(8) we can also get the bounds in (34) for every $r \geq r_a$ (and $u_{a',3}$ instead of $u_{a,3}$). For instance

$$
\begin{aligned}
(\Delta u_{a',3})'(r) &= \frac{1}{\omega_5 r^5} \int_{B_r} \Delta^3 u_{a',3} dx = \left(\frac{r_a}{r}\right)^5 (\Delta u_{a',3})'(r_a) + \frac{1}{\omega_5 r^5} \int_{B_r \setminus B_{r_a}} \Delta^2 u_{a',3} dx \\
&< \left(\frac{r_a}{r}\right)^5 \frac{B r_a}{6} + \frac{B(r^6 - r_a^6)}{6r^5} = \frac{B}{6} r.
\end{aligned}
$$

Furthermore, up to taking $\delta > 0$ even smaller, we can assume that

$$
\left| \int_{B_{r_a}} e^{6u_{a',3}} dx - \int_{B_{r_a}} e^{6u_{a,3}} dx \right| < \frac{\varepsilon}{2}. \tag{36}
$$

Finally, the last bound in (34) and (35) imply at once

$$
\left| \int_{\mathbb{R}^6 \setminus B_{r_a}} e^{6u_{a',3}} dx - \int_{\mathbb{R}^6 \setminus B_{r_a}} e^{6u_{a,3}} dx \right| < \frac{\varepsilon}{2},
$$

which together with (36) completes the proof. $\square$

Proof of Theorem 1 (completed). Set $V^* = V(a^*)$, where a^* is given by Lemma 8. By Lemma 8, Theorem 2 and the intermediate value theorem, for every $V \geq V^*$ there exists $a \leq a^*$ such that

$$
\frac{1}{5!} \int_{\mathbb{R}^6} e^{6u_{a,3}} dx = V,
$$

hence the metric $g_{u_{a,3}} = e^{2u_{a,3}} |dx|^2$ has constant Q-curvature equal to 1 and $\mathrm{vol}(g_{u_{a,3}}) = 5!V$. Applying the transformation

$$
u = u_{a,3} - \frac{1}{6} \log 5!
$$

it follows at once that the metric $g_u = e^{2u} |dx|^2$ satisfies $\mathrm{vol}(g_u) = V$ and $Q_{g_u} \equiv 5!$, hence u solves (1)-(2). $\square$

4 Proof of Theorems 3 and 4

When $f : \mathbb{R}^n \to \mathbb{R}$ is radially symmetric we have $\Delta f(x) = f''(|x|) + \frac{n-1}{|x|} f'(|x|)$. In particular we have

$$
\Delta^m r^{2m} = 2^{2m} m (2m-1)! \quad \text{in } \mathbb{R}^{2m}. \tag{37}
$$

For $m \geq 2$ and $b \leq 0$ let u_b solve

$$
\begin{cases}
\Delta^m u_b = -(2m-1)! e^{2m u_b} & \text{in } \mathbb{R}^{2m} \\
u_b^{(j)}(0) = 0 & \text{for } 0 \leq j \leq 2m-1,\ j \neq 2m-2 \\
u_b^{(2m-2)} = b. &
\end{cases}
$$

From (7)-(8) it follows that $u_0 \leq 0$, hence $\Delta^m u_0 \geq -(2m-1)!$. We claim that

$$u_0(r) \geq \psi(r) := -\frac{r^{2m}}{2^{2m} m}.$$

Indeed according to (37) ψ solves

$$\Delta^m \psi = -(2m-1)! \leq \Delta^m u_0 \quad \text{in } \mathbb{R}^{2m}$$

and

$$\psi^{(j)}(0) = 0 = u_0^{(j)}(0) \quad \text{for } 0 \leq j \leq 2m-1,$$

which implies

$$\Delta^j \psi(0) = 0 = \Delta^j u_0(0) \quad \text{for } 0 \leq j \leq m-1,$$

see [Mar1, Lemma 17]. Then the claim follows from (7)-(8) and a simple induction.

Now integrating we get

$$\int_{\mathbb{R}^{2m}} e^{2mu_0} dx \geq \int_{\mathbb{R}^{2m}} e^{2m\psi} dx = \omega_{2m-1} \int_0^\infty r^{2m-1} \exp\Big(-\frac{r^{2m}}{2^{2m-1}}\Big) dr = \frac{2^{2m-2}\omega_{2m-1}}{m} =: V_m.$$

Using the formulas

$$\omega_{2m-1} = \mathrm{vol}(S^{2m-1}) = \frac{2\pi^m}{(m-1)!}, \quad \omega_{2m} = \mathrm{vol}(S^{2m}) = \frac{2^{2m}(m-1)!\pi^m}{(2m-1)!}, \quad m \geq 1$$

we verify

$$V_m = \frac{(2m)!}{4(m!)^2}\omega_{2m}, \quad \frac{V_2}{\omega_4} = \frac{3}{2} > 1, \quad \frac{V_{m+1}}{\omega_{2m+2}}\Big(\frac{V_m}{\omega_{2m}}\Big)^{-1} = \frac{(2m+2)(2m+1)}{(m+1)^2} > 1, \tag{38}$$

hence by induction

$$V_m > \mathrm{vol}(S^{2m}) \quad \text{for } m \geq 2. \tag{39}$$

With the same argument used to prove Lemma 8 we can show that the function

$$V(b) := \int_{\mathbb{R}^{2m}} e^{6u_b} dx, \quad b \in (-\infty, 0]$$

is finite and continuous. Indeed it is enough to replace (33) with

$$\Delta^{m-1} u_b(r) \leq B < 0 \quad \text{for } r \geq r_b,$$

and (34) with

$$(\Delta^{m-1-j} u_b)'(r) < C_{m,j} B r^{2j-1}, \quad \Delta^{m-1-j} u_b(r) < D_{m,j} B r^{2j}, \quad \text{for } r \geq r_b,\ 1 \leq j \leq m-1$$

where r_b is chosen large enough and

$$C_{m,1} = \frac{1}{2m}, \quad D_{m,j} = \frac{C_{m,j}}{2j}, \quad C_{m,j+1} = \frac{D_{m,j}}{2m+2j},$$

whence

$$C_{m,j} = \frac{(m-1)!}{2^{2j-1}(j-1)!(m+j-1)!}, \quad D_{m,j} = \frac{(m-1)!}{2^{2j}j!(m+j-1)!}.$$

Moreover, using that $\Delta^{m-1}u_b(0) = C_m b$ for some constant $C_m > 0$, $\Delta^m u_b(r) \leq 0$ for $r \geq 0$ and (7)-(8) as before, we easily obtain

$$u_b(r) \leq E_m b r^{2m-2}, \tag{40}$$

where $E_m := C_m C_{m,m-1} > 0$, hence

$$\lim_{b\to-\infty} V(b) \leq \lim_{b\to-\infty} \int_{\mathbb{R}^6} e^{6E_m b|x|^{2m-2}} dx = 0.$$

By continuity we conclude that for every $V \in (0, V_m]$ there exists $b \leq 0$ such that $u = u_b$ solves (1)-(2) if m is odd or (6)-(2) if m is even. Taking (39) into account it only remains to prove that the solutions u_b corresponding to $V = \mathrm{vol}(S^{2m})$ is not a spherical one. This follows immediately from (40), which is not compatible with (3). $\square$

5 Applications and open questions

Possible gap phenomenon Theorems 1 and 3 guarantee that for $m = 3$ there exists a solution to (1)-(2) for every $V \in (0, V_3] \cup [V^*, \infty)$, with possibly $V_3 < V^*$. Could it be that for some $V \in (V_3, V^*)$ Problem (1)-(2) admits no solution?

If we restrict to rotationally symmetric solutions, some heuristic arguments show that the volume of a solution to (4), i.e. the function

$$V(a,b) := \int_{\mathbb{R}^6} e^{6u_{a,b}(|x|)} dx$$

need not be continuous for all $(a,b) \in \mathbb{R}^2$, hence the image of the function V might not be connected.

Higher dimensions and negative curvature It is natural to ask whether Theorems 1 and 2 generalize to the case $m > 3$ or whether an analogous statement holds when $m \geq 2$ and (6) is considered instead of (1). Since the sign on the right-hand side of the ODE (4) plays a crucial role, we would expect that part of the proof of Theorem 2 can be recycled for (1) when $m \geq 5$ is odd, or for (6) when m is even.

For instance let $u_a = u_a(r)$ be the solution in $\mathbb{R}^4$ of

$$\begin{cases} \Delta^2 u_a = -6e^{4u_a} \\ u_a(0) = u_a'(0) = u_a'''(0) = 0 \\ u_a''(0) = a. \end{cases}$$

It should not be difficult to see that $u_a(r)$ exists for all $r \geq 0$ and that $\int_{\mathbb{R}^4} e^{4u_a(|x|)}dx < \infty$. Do we also have

$$\lim_{a\to+\infty} \int_{\mathbb{R}^4} e^{4u_a(|x|)} dx = \infty?$$

Non-radial solutions The proof of Theorem C cannot be extended to provide non-radial solutions to (1)-(2) for $m \geq 3$ and $V \geq \mathrm{vol}(S^{2m})$, but it is natural to conjecture that they do exist.

Concentration phenomena The classification results of the solutions to (1)-(2), [CL], [Lin], [Xu] and [Mar1], have been used to understand the asymptotic behavior of unbounded sequences of solutions to the prescribed Gaussian curvature problem on 2-dimensional domains (see e.g. [BM] and [LS]), on S^2 (see [Str4]) and to the prescribed Q-curvature equation in dimension $2m$ (see e.g. [DR], [Mal], [MS], [Ndi], [Rob1], [Rob2], [Mar3], [Mar4]).

For instance consider the following model problem. Let $\Omega \subset \mathbb{R}^{2m}$ be a connected open set and consider a sequence (u_k) of solutions to the equation

$$(-\Delta)^m u_k = Q_k e^{2mu_k} \quad \text{in } \Omega, \tag{41}$$

where

$$Q_k \to Q_0 \quad \text{in } C^1_{\mathrm{loc}}(\Omega), \quad \limsup_{k\to\infty} \int_\Omega e^{2mu_k} dx < \infty, \tag{42}$$

with the following interpretation: $g_k := e^{2u_k}|dx|^2$ is a sequence of conformal metrics on Ω with Q-curvatures $Q_{g_k} = Q_k$ and equibounded volumes.

As shown in [ARS] unbounded sequences of solutions to (41)-(42) can exhibit pathological behaviors in dimension 4 (and higher), contrary to the elegant results of [BM] and [LS] in dimension 2. This is partly due to Theorem A. In fact for $m \geq 2$ and $\alpha \in (0, (2m-1)!\,\mathrm{vol}(S^{2m})]$ one can found a sequence (u_k) of solutions to (41)-(42) with $Q_0 > 0$ and

$$\lim_{R\to 0} \lim_{k\to\infty} \int_{B_R(x_0)} |Q_k| e^{2mu_k} dx = \alpha \quad \text{for some } x_0 \in \Omega. \tag{43}$$

For $m = 2$ this was made very precise by F. Robert [Rob1] in the radially symmetric case. In higher dimension or when Q_0 is not necessarily positive, thanks to Theorems 1-4 we see that α can take values larger than $(2m-1)!\,\mathrm{vol}(S^{2m})$. Indeed if u is a solution to (1)-(2) or (6)-(2), then $u_k := u(kx) + \log k$ satisfies (41)-(42) with $\Omega = \mathbb{R}^{2m}$, $Q_k \equiv \pm(2m-1)!$ and

$$|Q_k| e^{2mu_k} dx \rightharpoonup (2m-1)! V \delta_0, \quad \text{weakly as measures.}$$

When $m = 2$, $Q_0 > 0$ (say $Q_0 \equiv 6$) it is unclear whether one could have concentration points carrying more Q-curvature than $6\,\mathrm{vol}(S^4)$, i.e. whether one can take $\alpha > 6\,\mathrm{vol}(S^4)$ in (43). Theorem B suggests that if the answer is affirmative, this should be due to the convergence to the same blow-up point of two or more blow-ups. Such a phenomenon is unknown in dimension 4 and higher, but was shown in dimension 2 by Wang [Wan] with a technique which, based on the abundance of conformal transformations of $\mathbb{C}$ into itself, does not extend to higher dimensions.

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