Conformal metrics on $\mathbb{R}^{2m}$ with constant $Q$-curvature and large volume

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Abstract
We study conformal metrics $g_u = e^{2u}|dx|^2$ on $\mathbb{R}^{2m}$ with constant $Q$-curvature $Q_{g_u} \equiv (2m-1)!$ (notice that $(2m-1)!$ is the $Q$-curvature of $S^{2m}$) and finite volume. When $m = 3$ we show that there exists $V^*$ such that for any $V \in [V^*, \infty)$ there is a conformal metric $g_u = e^{2u}|dx|^2$ on $\mathbb{R}^6$ with $Q_{g_u} = 5!$ and $\text{vol}(g_u) = V$. This is in sharp contrast with the four-dimensional case, treated by C-S. Lin. We also prove that when $m$ is odd and greater than 1, there is a constant $V_m > \text{vol}(S^{2m})$ such that for every $V \in (0, V_m]$ there is a conformal metric $g_u = e^{2u}|dx|^2$ on $\mathbb{R}^{2m}$ with $Q_{g_u} \equiv (2m-1)!$, $\text{vol}(g) = V$. This extends a result of A. Chang and W-X. Chen. When $m$ is even we prove a similar result for conformal metrics of negative $Q$-curvature.

Keywords: $Q$-curvature, Paneitz operators, GMJS operators, conformal geometry.

1 Introduction and statement of the main theorems
We consider solutions to the equation

$$(-\Delta)^m u = (2m-1)!e^{2mu} \quad \text{in } \mathbb{R}^{2m},$$

satisfying

$$V := \int_{\mathbb{R}^{2m}} e^{2mu(x)} dx < +\infty,$$

with particular emphasis on the role played by $V$.

Geometrically, if $u$ solves (1) and (2), then the conformal metric $g_u := e^{2u}|dx|^2$ has $Q$-curvature $Q_{g_u} \equiv (2m-1)!$ and volume $V$ (by $|dx|^2$ we denote the Euclidean metric). For the definition of $Q$-curvature and related remarks, we refer to Chapter 4 in [Cha] or to [FG] and [FH]. Notice that given a solution $u$ to (1) and $\lambda > 0$, the function $v := u - \frac{1}{2m} \log \lambda$ solves

$$(-\Delta)^m v = \lambda(2m-1)!e^{2mv} \quad \text{in } \mathbb{R}^{2m}, \quad \int_{\mathbb{R}^{2m}} e^{2mv(x)} dx = \frac{V}{\lambda},$$

hence there is no loss of generality in the particular choice of the constant $(2m-1)!$ in (1). On the other hand this constant has the advantage of being the $Q$-curvature of the round sphere.
This implies that the function \( u_1(x) = \log \frac{2}{1 + |x|^2} \), which satisfies \( e^{2u_1}|dx|^2 = (\pi^{-1})^* g_{S^{2m}} \)
(here \( \pi: S^{2m} \to \mathbb{R}^{2m} \) is the stereographic projection) is a solution to (1)-(2) with \( V = \text{vol}(S^{2m}) \).
Translations and dilations (i.e. Möbius transformations) actually give us a large family of solutions to (1)-(2) with \( V = \text{vol}(S^{2m}) \), namely
\[
 u_{x_0, \lambda}(x) := u_1(\lambda(x - x_0)) + \log \lambda = \log \frac{2\lambda}{1 + \lambda^2|x - x_0|^2}, \quad x_0 \in \mathbb{R}^{2m}, \lambda > 0. \quad (3)
\]
We shall call the functions \( u_{x_0, \lambda} \) standard or spherical solutions to (1)-(2).

The question whether the family of spherical solutions in (3) exhausts the set of solutions to (1)-(2) has raised a lot of interest and is by now well understood. W. Chen and C. Li [CL] proved that on \( \mathbb{R}^2 (m = 1) \) every solution to (1)-(2) is spherical, while for every \( m > 1 \), i.e. in dimension 4 and higher, it was proven by A. Chang and W-X. Chen [CC] that Problem (1)-(2) admits solutions which are non-spherical. In fact they proved

**Theorem A (A. Chang-W-X. Chen [CC] 2001).** For every \( m > 1 \) and \( V \in (0, \text{vol}(S^{2m})) \) there exists a solution to (1)-(2).

Several authors have tried to classify spherical solutions or, in other words, to give analytical and geometric conditions under which a solution to (1)-(2) is spherical (see [CY], [WX], [Xu]), and to understand some properties of non-spherical solutions, such as their asymptotic behavior, their volume and their symmetry (see [Lin], [Mar1], [WY]). In particular C-S. Lin proved:

**Theorem B (C-S. Lin [Lin] 1998).** Let \( u \) solve (1)-(2) with \( m = 2 \). Then either \( u \) is spherical (i.e. as in (3)) or \( V < \text{vol}(S^4) \).

Both spherical solutions and the solutions given by Theorem A are radially symmetric (i.e. of the form \( u(|x - x_0|) \) for some \( x_0 \in \mathbb{R}^{2m} \)). On the other hand there also exist plenty of non-radial solutions to (1)-(2) when \( m = 2 \).

**Theorem C (J. Wei and D. Ye [WY] 2006).** For every \( V \in (0, \text{vol}(S^4)) \) there exist (several) non-radial solutions to (1)-(2) for \( m = 2 \).

**Remark D** Probably the proof of Theorem C can be extended to higher dimension \( 2m \geq 2 \), yielding several non-symmetric solutions to (1)-(2) for every \( V \in (0, \text{vol}(S^{2m})) \), but failing to produce non-symmetric solutions for \( V \geq \text{vol}(S^{2m}) \). As in the proof of Theorem A, the condition \( V < \text{vol}(S^{2m}) \) plays a crucial role.

Theorems A, B, C and Remark D strongly suggest that also in dimension 6 and higher all non-spherical solutions to (1)-(2) satisfy \( V < \text{vol}(S^{2m}) \), i.e. (1)-(2) has no solution for \( V > \text{vol}(S^{2m}) \) and the only solutions with \( V = \text{vol}(S^{2m}) \) are the spherical ones. Quite surprisingly we found out that this is not at all the case. In fact in dimension 6 we found solutions to (1)-(2) with arbitrarily large \( V \):

\[ S^{2m} \]
**Theorem 1** For $m = 3$ there exists $V^* > 0$ such that for every $V \geq V^*$ there is a solution $u$ to (1)-(2), i.e. there exists a metric on $\mathbb{R}^6$ of the form $g_u = e^{2u}|dx|^2$ satisfying $Q_{g_u} \equiv 5!$ and $\text{vol}(g_u) = V$.

In order to prove Theorem 1 we will consider only rotationally symmetric solutions to (1)-(2), so that (1) reduces to an ODE. Precisely, given $a, b \in \mathbb{R}$ let $u = u_{a,b}(r)$ be the solution of

$$
\begin{align*}
\Delta^3 u &= -e^{6u} \text{ in } \mathbb{R}^6 \\
u(0) &= u'(0) = u'''(0) = u''''(0) = 0 \\
u''(0) &= \frac{\Delta u(0)}{6} = a \\
u''''(0) &= \frac{\Delta^2 u(0)}{16} = b.
\end{align*}
$$

(4)

Here and in the following we will always (by a little abuse of notation) see a rotationally symmetric function $f$ both as a function of one variable $r \in [0, \infty)$ (when writing $f'$, $f''$, etc...) and as a function of $x \in \mathbb{R}^6$ (when writing $\Delta f$, $\Delta^2 f$, etc...). We also used that $\Delta f(0) = 6f''(0)$, $\Delta^2 f(0) = 16f''''(0)$, see e.g. [Mar1, Lemma 17]. Also notice that in (4) we replaced $5!$ by $1$ to make the computations lighter. As we already noticed, this is not a problem.

**Theorem 2** Let $u = u_{a,3}$ solve (4) for a given $a < 0$ and $b = 3$. Then

$$
\int_{\mathbb{R}^6} e^{6u_{a,3}} dx < \infty \text{ for } -a \text{ large;} \quad \lim_{a \to -\infty} \int_{\mathbb{R}^6} e^{6u_{a,3}} dx = \infty.
$$

(5)

In particular the conformal metric $g_{u_{a,3}} = e^{2u_{a,3}}|dx|^2$ of constant $Q$-curvature $Q_{g_{u_{a,3}}} \equiv 1$ satisfies

$$
\lim_{a \to -\infty} \text{vol}(g_{u_{a,3}}) = \infty.
$$

Theorem 1 will follow from Theorem 2 and a continuity argument (Lemma 8 below).

Going through the proof of Theorem A it is clear that it does not extend to the case $V > \text{vol}(S^{2m})$. With a different approach, we are able to prove that, at least when $m \geq 3$ is odd, one can extend Theorem A as follows.

**Theorem 3** For every $m \geq 3$ odd there exists $V_m > \text{vol}(S^{2m})$ such that for every $V \in (0, V_m]$ there is a non-spherical solution $u$ to (1)-(2), i.e. there exists a metric on $\mathbb{R}^{2m}$ of the form $g_u = e^{2u}|dx|^2$ satisfying $Q_{g_u} \equiv (2m - 1)!$ and $\text{vol}(g_u) = V$.

The condition $m \geq 3$ odd is (at least in part) necessary in view of Theorem B and [CL], but the case $m \geq 4$ even is open. Notice also that when $m = 3$, Theorems 1 and 3 guarantee the existence of solutions to (1)-(2) for

$$
V \in (0, V_m] \cup [V^*, \infty),
$$

1The choice $b = 3$ is convenient in the computations, but any other $b > 0$ would work.
but we cannot rule out that $V_m < V^*$ (the explicit value of $V_m$ is given in (38) below) and the existence of solutions to (1)-(2) is unknown for $V \in (V_m, V^*)$. Could there be a gap phenomenon?

We now briefly investigate how large the volume of a metric $g_u = e^{2u}|dx|^2$ on $\mathbb{R}^{2m}$ can be when $Q_{g_u} \equiv \text{const} < 0$. Again with no loss of generality we assume $Q_{g_u} \equiv -(2m - 1)!$. In other words consider the problem

$$(-\Delta)^m u = -(2m - 1)! e^{2mu} \text{ on } \mathbb{R}^{2m}. \quad (6)$$

Although for $m = 1$ it is easy to see that Problem (6)-(2) admits no solutions for any $V > 0$, when $m \geq 2$ Problem (6)-(2) has solutions for some $V > 0$, as shown in [Mar2]. Then with the same proof of Theorem 3 we get:

**Theorem 4** For every $m \geq 2$ even there exists $V_m > \text{vol}(S^{2m})$ such that for $V \in (0, V_m]$ there is a solution $u$ to (6)-(2), i.e. there exists a metric on $\mathbb{R}^{2m}$ of the form $g_u = e^{2u}|dx|^2$ satisfying

$$Q_{g_u} \equiv -(2m - 1)!, \quad \text{vol}(g_u) = V.$$

The cases of solutions to (1)-(2) with $m$ even, or (6)-(2) and $m$ odd seem more difficult to treat since the ODE corresponding to (1) or (6), in analogy with (4) becomes

$$\Delta^m u(r) = (2m-1)! e^{2mu(r)},$$

whose solutions can blow up in finite time (i.e. for finite $r$) if the initial data are not chosen carefully (contrary to Lemma 5 below).

## 2 Proof of Theorem 2

Set $\omega_{2m-1} := \text{vol}(S^{2m-1})$ and let $B_r$ denote the unit ball in $\mathbb{R}^{2m}$ centered at the origin. Given a smooth radial function $f = f(r)$ in $\mathbb{R}^{2m}$ we will often use the divergence theorem in the form

$$\int_{B_r} \Delta f dx = \int_{\partial B_r} \frac{\partial f}{\partial \nu} d\sigma = \omega_{2m-1} r^{2m-1} f'(r). \quad (7)$$

Dividing by $\omega_{2m-1} r^{2m-1}$ into (7) and integrating we also obtain

$$f(t) - f(s) = \int_s^t \frac{1}{\omega_{2m-1} r^{2m-1}} \int_{B_\rho} \Delta f dx d\rho, \quad 0 \leq s \leq t. \quad (8)$$

When no confusion can arise we will simply write $u$ instead of $u_{a,3}$ or $u_{a,b}$ to denote the solution to (4). In what follows, also other quantities (e.g. $R$, $r_0$, $r_1$, $r_2$, $r_3$, $\phi$, $\xi_1$, $\xi_2$) will depend on $a$ and $b$, but this dependence will be omitted from the notation.

**Lemma 5** Given any $a, b \in \mathbb{R}$, the solution $u$ to the ODE (4) exists for all times.
Figure 1: The functions $\phi(r) = \frac{a}{2} r^2 + \frac{1}{8} r^4$ (above) and $u_{a,3}(r) \leq \phi(r)$.

*Proof.* Applying (8) to $f = \Delta^2 u$, and observing that $\Delta(\Delta^2 u) = -e^{6u} \leq 0$ we get

$$\Delta^2 u(t) \leq \Delta^2 u(s) \leq \Delta^2 u(0) = 16b \quad 0 \leq s \leq t,$$

i.e. $\Delta^2 u(r)$ is monotone decreasing. This and (8) applied to $\Delta u$ yield

$$\Delta u(r) \leq \Delta u(0) + \int_0^r \frac{1}{\omega_5 \rho^5} \int_{B_\rho} 16b dx d\rho = 6a + \int_0^r \frac{8}{3} b \rho d\rho = 6a + \frac{4}{3} b r^2.$$

A further application of (8) to $u$ finally gives

$$u(r) \leq \int_0^r \frac{1}{\omega_5 \rho^5} \int_{B_\rho} (6a + \frac{4}{3} b |x|^2) dx d\rho = \int_0^r \int_{B_\rho} (a \rho + \frac{b \rho^2}{6}) d\rho = \frac{a}{2} r^2 + \frac{b}{24} r^4 =: \phi(r).$$

Similar lower bounds can be obtained by observing that $-e^{6u} \geq -1$ for $u \leq 0$. This proves that $u(r)$ cannot blow-up in finite time and, by standard ODE theory, $u(r)$ exists for every $r \geq 0$. □

*Proof of (5) (completed).* Fix $b = 3$ and take $a < 0$. The function $\phi(r) = \frac{a}{2} r^2 + \frac{1}{8} r^4$ vanishes for $r = R = R(a) := 2\sqrt{-a}$. In order to prove (5) we shall investigate the behavior of $u$ in a neighborhood of $R$. The heuristic idea is that $u(r)$ stays close to $\phi$ for some $r > R$ (hence in a region where $\phi$ is positive and $\Delta^3 u$ is not necessarily small) and long enough to make the second integral in (5) blow up as $a \to -\infty$.

**Step 1: Estimates of $u(R)$, $\Delta u(R)$ and $\Delta^2 u(R)$.** From (10) we infer

$$\Delta^3 u = -e^{6u} \geq -e^{6\phi},$$
which, together with (8), gives
\[
\Delta^2 u(r) = \Delta^2 u(0) + \int_0^r \frac{1}{\omega_5 \rho^5} \int_{B_\rho} \Delta^3 u \, dx \, d\rho \geq 48 - \int_0^r \frac{1}{\omega_5 \rho^5} \int_{B_\rho} e^{6\phi(|x|)} \, dx \, d\rho.
\] (11)

We can explicitly compute (see Lemma 6 below and simplify (29) using that \(\phi(R) = 0\) and \(\int_{-\sqrt{3}a}^{\sqrt{3}a} e^2 \, dt = 2 \int_0^{\sqrt{3}a} e^2 \, dt\))
\[
\int_0^R \frac{1}{\omega_5 \rho^5} \int_{B_\rho} e^{6\phi(|x|)} \, dx \, d\rho = \frac{1}{48a} + \frac{(18a^2 + 1)\sqrt{3}}{144a^2} e^{-3a^2} \int_0^{-\sqrt{3}a} e^2 \, dt.
\]

Then by (9) and Lemma 7 below we conclude that
\[
\Delta^2 u(r) \geq \Delta^2 u(R) \geq 48(1 + O(a^{-1})) \quad \text{for } 0 \leq r \leq R = 2\sqrt{-a}.
\] (12)

where here and in the following \(|a^k O(a^{-k})| \leq C = C(k)\) as \(a \to -\infty\) for every \(k \in \mathbb{R}\). Then applying (8) as before we also obtain
\[
\Delta u(r) \geq 6a + 4(1 + O(a^{-1}))r^2 \quad \text{for } 0 \leq r \leq R
\]

and
\[
u(r) \geq \frac{a}{2} r^2 + \frac{1 + O(a^{-1})}{8} r^4 = \phi(r) + O(a^{-1}) r^4 \quad \text{for } 0 \leq r \leq R.
\]

At \(r = R\) this reduces to
\[
u(R) \geq O(a).
\]

**Step 2: Behavior of \(u(r), \Delta u(r), \Delta^2 u(r)\) for \(r \geq R\).** Define \(r_0\) (depending on \(a < 0\)) as
\[
r_0 := \inf \{ r > 0 : u(r) = 0 \} \in [R, \infty].
\]

We first claim that \(r_0 < \infty\). We have by Lemma 6 and Lemma 7
\[
\int_{B_R} e^{6\phi} \, dx = \omega_5 \left( -\frac{4a}{3} + \frac{4(6a^2 - 1)\sqrt{3}}{9} e^{-3a^2} \int_0^{-\sqrt{3}a} e^2 \, dt \right) = O(a).
\] (13)

Since on \(B_{r_0}\) we have \(u \leq 0\), hence \(\Delta^3 u \geq -1\), using (7)-(8) and (13) we get for \(r \in [R, r_0]\)
\[
\Delta^2 u(r) \geq \Delta^2 u(R) - \int_R^r \frac{1}{\omega_5 \rho^5} \left( \int_{B_\rho} e^{6\phi} \, dx + \int_{B_\rho \setminus B_R} 1 \, dx \right) \, d\rho
\]
\[
\geq 48 + O(a) \left[ \frac{1}{R^4} - \frac{1}{r^4} \right] - \int_R^r \frac{\rho^6 - R^6}{6\rho^5} \, d\rho
\] (14)

Assuming \(r \in [R, 2R]\) we can now bound with a Taylor expansion
\[
\frac{1}{R^4} - \frac{1}{r^4} = R^{-4} \tilde{O} \left( \frac{R - r}{R} \right)
\] (15)

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and
\[ \rho^6 - R^6 \leq r^6 - R^6 = R^6 \tilde{O}\left( \frac{r - R}{R} \right), \quad \text{for } \rho \in [R, r], \]
which together with (15) yields
\[ \int_R^r \frac{\rho^6 - R^6}{6\rho^5} d\rho \leq \int_R^r \frac{r^6 - R^6}{6\rho^5} d\rho \leq R^2 \tilde{O}\left( \left( \frac{r - R}{R} \right)^2 \right), \]
where for any \( k \in \mathbb{R} \) we have \( |t^{-k} \tilde{O}(t^k)| \leq C = C(k) \) uniformly for \( 0 \leq t \leq 1 \). Using (15) and (16) we bound in (14)

\[ \Delta^2 u(r) \geq 48 + O(a^{-1}) \tilde{O}\left( \frac{r - R}{R} \right) + R^2 \tilde{O}\left( \left( \frac{r - R}{R} \right)^2 \right), \quad r \in [R, \min\{r_0, 2R\}], \]
whence
\[ \Delta^2 u(r) \geq 48 + O(a^{-1}) + R^2 \tilde{O}\left( \left( \frac{r - R}{R} \right)^2 \right) \chi_{\left( R, \infty \right)}(r), \quad r \in [0, \min\{r_0, 2R\}], \]
where \( \chi_{\left( R, \infty \right)}(r) = 0 \) for \( r \in [0, R] \) and \( \chi_{\left( R, \infty \right)}(r) = 1 \) for \( r > R \). Then with (8) we estimate for \( r \in \left[ 0, \min\{r_0, 2R\} \right] \)

\[ \Delta u(r) \geq 6a + 4(1 + O(a^{-1}))r^2 + \chi_{\left( R, \infty \right)}(r) \int_R^r \frac{1}{\omega_5 \rho^5} \int_{B_\rho \setminus B_R} R^2 \tilde{O}\left( \left( \frac{|x| - R}{R} \right)^2 \right) d\rho dx \]
\[ = 6a + 4(1 + O(a^{-1}))r^2 + R^4 \tilde{O}\left( \left( \frac{r - R}{R} \right)^4 \right) \chi_{\left( R, \infty \right)}(r). \]
\[ u(r) \geq \frac{a}{2} r^2 + \frac{1 + O(a^{-1})}{8} r^4 + \chi_{\left( R, \infty \right)}(r) \int_R^r \frac{1}{\omega_5 \rho^5} \int_{B_\rho \setminus B_R} R^4 \tilde{O}\left( \left( \frac{|x| - R}{R} \right)^4 \right) d\rho dx \]
\[ = \phi(r) + O(a^{-1})r^4 + \tilde{O}\left( \left( \frac{r - R}{R} \right)^6 \right) \chi_{\left( R, \infty \right)}(r), \]
where the integrals in (17) and (18) are easily estimated bounding \( |x| \) with \( r \) and applying (16).

Making a Taylor expansion of \( \phi(r) \) at \( r = R \) and using that \( \phi(R) = 0 \), we can further estimate the right-hand side of (18) for \( r \in [R, \min\{r_0, 2R\}] \) as

\[ u(r) \geq \phi'(R)(r - R) + R^2 \tilde{O}\left( \left( \frac{r - R}{R} \right)^2 \right) + O(a^{-1})r^4 + R^6 \tilde{O}\left( \left( \frac{r - R}{R} \right)^6 \right) \]
\[ = -aR(r - R) + O(a^{-1})R^4 + R^2 \tilde{O}\left( \left( \frac{r - R}{R} \right)^2 \right) + R^6 \tilde{O}\left( \left( \frac{r - R}{R} \right)^6 \right) =: \psi_a(r) \]
Now choosing \( r = R(1 + 1/\sqrt{-a}) \), so that \( (r - R)/R \to 0 \) as \( a \to -\infty \), we get

\[ \lim_{a \to -\infty} \psi_a(R(1 + 1/\sqrt{-a})) \geq \lim_{a \to -\infty} \left( 4(-a)^3 + O(a) - C \right) = \infty. \]
In particular

\[ r_0 \in [R, R(1 + 1/\sqrt{-a})]. \]
We now claim that
\[ \lim_{a \to -\infty} \Delta u(r_0) = \infty. \]  
(19)
Indeed we infer from (17)
\[ \Delta u(r_0) \geq 6a + 4(1 + O(a^{-1}))r_0^2 - C \]
\[ \geq 6a + 4(1 + O(a^{-1}))R^2 - C \]
\[ \geq -10a - C, \]
for \(-a\) large enough, whence (19). Set
\[ r_1 = r_1(a) := \inf\{r > r_0 : u(r) = 0\}. \]

Applying (7) to (17), and recalling that \( \frac{am-R}{R} \leq \frac{1}{\sqrt{a}} \), similar to (18) we obtain
\[ u'(r_0) \geq ar_0 + \frac{1 + O(a^{-1})}{2} r_0^3 - C \]
\[ \geq ar_0 + \frac{1 + O(a^{-1})}{2} r_0R^2 - C \]
\[ \geq -ar_0 - C. \]
In particular for \(-a\) large enough we have \( u'(r_0) > 0 \), which implies \( r_1 > r_0 \). Using (7)-(8) and that \( \Delta^3 u(r) \leq -1 \) for \( r \in [r_0, r_1] \), it is not difficult to see that \( r_1 < \infty \). Moreover there exists at least a point \( r_2 = r_2(a) \in (r_0, r_1] \) such that \( u'(r_2) \leq 0 \), which in turn implies that
\[ \Delta u(r_3) < 0 \quad \text{for some } r_3 = r_3(a) \in (r_0, r_2], \]  
(20)

since otherwise we would have by (7)
\[ u'(r_2) = \frac{1}{\omega_5 r_2^5} \int_{B_{r_2}} \Delta u \, dx + \frac{1}{\omega_5 r_2^5} \int_{B_{r_2} \setminus B_{r_0}} \Delta u \, dx \geq \frac{\omega_5}{r_2^5} u'(r_0) > 0, \]
contradiction.

**Step 3: Conclusion.** We now use the estimates obtained in Step 1 and Step 2 to prove (5).

From (8), (19) and (20) we infer
\[ \lim_{a \to -\infty} \int_{r_0}^{r_3} \frac{1}{\omega_5 r^5} \int_{B_r} \Delta^2 u \, dx \, dr = \lim_{a \to -\infty} (\Delta u(r_3) - \Delta u(r_0)) = -\infty, \]  
(21)
hence by the monotonicity of \( \Delta^2 u(r) \) (see (9))
\[ \lim_{a \to -\infty} \Delta^2 u(r_3)(r_3^2 - r_0^2) = -\infty. \]  
(22)
We now claim that
\[ \lim_{a \to -\infty} \int_{B_{r_3}} e^{6a} \, dx = \infty. \]  
(23)
Indeed consider on the contrary an arbitrary sequence $a_k$ with $\lim_{k \to \infty} a_k = -\infty$ and

\[ \lim_{k \to \infty} \int_{B_{r_k}} e^{6u} \, dx < \infty, \quad (24) \]

where here $r_3$ and $u$ depend on $a_k$ instead of $a$ of course. Since $u \geq 0$ in $B_{r_3} \setminus B_{r_0}$ we have

\[ \int_{B_{r_3}} e^{6u} \, dx \geq \int_{B_{r_3} \setminus B_{r_0}} 1 \, dx = \frac{\omega_5}{6} (r_3^6 - r_0^6). \]

Now observe that $(r_3^6 - r_0^6) \geq (r_3^2 - r_0^2)r_0^4$ to conclude that (24) implies

\[ \lim_{k \to \infty} (r_3^2 - r_0^2) \leq \lim_{k \to \infty} \frac{r_3^6 - r_0^6}{r_0^4} = 0. \quad (25) \]

Then (8), (12) and (22) yield

\[ (r_3^2 - r_0^2) \int_{B_{r_3}} e^{6u} \, dx = (r_3^2 - r_0^2)(\Delta^2 u(R) - \Delta^2 u(r_3)) \]
\[ \geq -\Delta^2 u(r_3)(r_3^2 - r_0^2) \to \infty \quad \text{as } k \to \infty. \]

By (25) we also have

\[ \lim_{k \to \infty} \int_{B_{r_3}} \frac{1}{\omega_5 r_5^5} \int_{B_r} e^{6u} \, dxdr = \infty, \]

which implies at once

\[ \lim_{k \to \infty} \int_{B_{r_3}} e^{6u} \, dx \geq \lim_{k \to \infty} 4R^4 \omega_5 \int_{B_{r_3}} \frac{1}{\omega_5 r_5^5} \int_{B_r} e^{6u} \, dxdr = \infty, \]

contradicting (24). Then (23) is proven.

It remains to show that

\[ \int_{\mathbb{R}^6} e^{6u} \, dx < \infty, \]

at least for $-a$ large enough. It follows from (22) and the monotonicity of $\Delta^2 u$ that for $-a$ large enough we have

\[ \Delta^2 u(r) < B < 0, \quad \text{for } r \geq r_3, \quad (26) \]

and, using (7)-(8) as already done several times, we can find $r_a \geq r_3$ such that

\[ (\Delta u)(r) < \frac{B}{6} r, \quad \Delta u(r) < \frac{B}{12} r^2, \quad u'(r) < \frac{B}{96} r^3, \quad u(r) < \frac{B}{384} r^4, \quad \text{for } r \geq r_a. \quad (27) \]

Then

\[ \int_{\mathbb{R}^6} e^{6u} \, dx \leq \int_{B_{r_a}} e^{6u} \, dx + \int_{\mathbb{R}^6 \setminus B_{r_a}} e^{B|x|^2} \, dx < \infty, \]

as wished. \qed
2.1 Two useful lemmas

We now state and prove two lemmas used in the proof of Theorem 2.

**Lemma 6** For \( \phi(r) = \frac{a}{2}r^2 + \frac{1}{8}r^4 \), \( a \leq 0 \), we have

\[
\int_{B_r} e^{6\phi(|x|)} dx = \omega_5 \left[ \frac{2}{3} a + \frac{1}{3} e^{6\phi(r)}(-2a + r^2) + \frac{(12a^2 - 2)\sqrt{3}}{9} e^{-3a^2} \int_{-\sqrt{3}(a + r^2/2)}^{-\sqrt{3}a} e^{t^2} dt \right] =: \xi_1(r)
\]

and

\[
\int_0^r \frac{1}{\omega_5 r^5} \int_{B_r} e^{6\phi(|x|)} dx dp = \frac{-2a - e^{6\phi(r)}(-2a + r^2)}{12r^4}
\]

\[
+ \frac{(2 - 12a^2 + 3r^4)\sqrt{3}}{36r^4} e^{-3a^2} \int_{-\sqrt{3}(a + r^2/2)}^{-\sqrt{3}a} t^2 dt := \xi_2(r)
\]

**Proof.** Patiently differentiating, using that \( e^{-3a^2} \frac{d}{dr} \int_{-\sqrt{3}(a + r^2/2)}^{-\sqrt{3}a} e^{t^2} dt = \sqrt{3}e^{6\phi(r)} \), one sees that

\[
\xi_1'(r) = \omega_5 r^5 e^{6\phi(r)}, \quad \xi_2'(r) = \frac{\xi_1(r)}{\omega_5 r^5}.
\]

Using that \( \phi(0) = 0 \) it is also easy to see that \( \xi_1(0) = 0 \).

Since \( \xi_2(0) \) is not defined, we will compute the limit of \( \xi_2(r) \) as \( r \to 0 \). We first compute the Taylor expansions

\[
e^{6\phi(r)} = 1 + 3ar^2 + \frac{3}{4}(1 + 6a^2)r^4 + r^4 o(1),
\]

and

\[
\sqrt{3}e^{-3a^2} \int_{-\sqrt{3}(a + r^2/2)}^{-\sqrt{3}a} t^2 dt = \frac{3}{2}r^2 + \frac{9}{4}ar^4 + r^4 o(1),
\]

with errors \( o(1) \to 0 \) as \( r \to 0 \). Then

\[
\frac{-2a - e^{6\phi(r)}(-2a + r^2)}{12r^4} = \frac{(1 - 6a^2)r^2 + \left(\frac{3}{2}a - 9a^2\right)r^4}{12r^4} + o(1)
\]

\[
= -\frac{(2 - 12a^2 + 3r^4)\sqrt{3}}{36r^4} e^{-3a^2} \int_{-\sqrt{3}(a + r^2/2)}^{-\sqrt{3}a} e^{t^2} dt,
\]

with \( o(1) \to 0 \) as \( r \to 0 \). Hence \( \lim_{r \to 0} \xi_2(r) = 0 \).

**Lemma 7** We have

\[
\lim_{r \to \infty} re^{-r^2} \int_0^r e^{t^2} dt = \frac{1}{2}. \quad (30)
\]
Proof. Clearly (30) is equivalent to
\[ \lim_{r \to \infty} r e^{-r^2} \int_2^r e^{t^2} dt = \frac{1}{2}. \] (31)

Integrating by parts we get for \( r \geq 2 \)
\[ r e^{-r^2} \int_2^r e^{t^2} dt = \frac{1}{2} - \frac{r e^{-r^2+4}}{4} + r e^{-r^2} \int_2^r \frac{e^{t^2}}{2t^2} dt. \] (32)

Another integration by parts yields
\[ r e^{-r^2} \int_2^r \frac{e^{t^2}}{2t^2} dt = \frac{1}{4} r e^{-r^2} \int_2^r e^{t^2} dt - \frac{r e^{-r^2+4}}{32} + \frac{r e^{-r^2}}{12t^3} dt \to 0 \quad \text{as} \quad r \to \infty, \]
where we used that the function \( t^{-4} e^{t^2} \) is increasing on \([2, r]\), hence
\[ 0 \leq \int_2^r \frac{e^{t^2}}{12t^4} dt \leq \int_2^r \frac{e^{r^2}}{12r^4} dt = (r-2) \frac{e^{r^2}}{12r^4}. \]

We conclude by taking the limit as \( r \to \infty \) in (32). \( \square \)

3 Proof of Theorem 1

We start with the following lemma.

Lemma 8 Set
\[ V(a) = \frac{1}{5!} \int e^{6u_{a,3}} dx \]
where \( u = u_{a,3} \) is the solution to (4) for given \( a < 0 \) and \( b = 3 \). Then there exists \( a^* < 0 \) such that \( V \) is continuous on \((-\infty, a^\ast]\).

Proof. It follows from (21) and the monotonicity of \( \Delta^2 u \) that we can fix \(-a^\ast\) so large that
\[ \lim_{r \to \infty} \Delta^2 u_{a,3}(r) = 0, \quad \text{for every} \quad a \leq a^\ast. \] (33)

Fix now \( \varepsilon > 0 \). Given \( a \leq a^\ast \) it is not difficult to find \( r_a > 0 \) and \( B = B(a) < 0 \) such that
\[ \Delta^2 u_{a,3}(r) < B < 0, \quad \text{for} \quad r \geq r_a \]
and, possibly choosing \( r_a \) larger, using (7)-(8) as already done in the proof of Theorem 2, we get
\[ (\Delta u_{a,3})'(r) < \frac{B}{6} r, \quad \Delta u_{a,3}(r) < \frac{B}{12} r^2, \quad u_{a,3}'(r) < \frac{B}{96} r^3, \quad u_{a,3}(r) < \frac{B}{384} r^4, \quad \text{for} \quad r \geq r_a. \] (34)

By possibly choosing \( r_a \) even larger we can also assume that
\[ \int_{\mathbb{R}^6 \setminus B_{r_a}} e^{\frac{B}{\pi} |x|} dx < \frac{\varepsilon}{2}. \] (35)
By ODE theory the solution \( u_{a,3} \) to (4) is continuous with respect to \( a \) in \( C^k_{\text{loc}}(\mathbb{R}^6) \) for every \( k \geq 0 \), in the sense that for any \( r' > 0 \), \( u_{a',3} \to u_{a,3} \) in \( C^k(B_{r'}) \) as \( a' \to a \). In particular we can find \( \delta > 0 \) (depending on \( \varepsilon \)) such that if \( |a-a'| < \delta \) then (33)-(34) with \( a \) replaced by \( a' \) are still satisfied for \( r = r_a \) (not \( r_{a'} \)) and (33) holds also for every \( r > r_a \) since \( \Delta^2 u_{a',3}(r) \) is decreasing in \( r \) (see (9)). Then, with (7)-(8) we can also get the bounds in (34) for every \( r \geq r_a \) (and \( u_{a',3} \) instead of \( u_{a,3} \)). For instance

\[
(\Delta u_{a',3})'(r) = \frac{1}{\omega_5 r^5} \int_{B_r} \Delta^3 u_{a',3} dx = \left( \frac{r_a}{r} \right)^5 (\Delta u_{a',3})'(r_a) + \frac{1}{\omega_5 r^5} \int_{B_r \setminus B_{r_a}} \Delta^2 u_{a',3} dx
\]

Furthermore, up to taking \( \delta > 0 \) even smaller, we can assume that

\[
\left| \int_{B_{r_a}} e^{6u_{a',3}} dx - \int_{B_{r_a}} e^{6u_{a,3}} dx \right| < \frac{\varepsilon}{2}.
\]  

(36)

Finally, the last bound in (34) and (35) imply at once

\[
\left| \int_{\mathbb{R}^6 \setminus B_{r_a}} e^{6u_{a',3}} dx - \int_{\mathbb{R}^6 \setminus B_{r_a}} e^{6u_{a,3}} dx \right| < \frac{\varepsilon}{2},
\]

which together with (36) completes the proof. \( \square \)

**Proof of Theorem 1 (completed).** Set \( V^* = V(a^*) \), where \( a^* \) is given by Lemma 8. By Lemma 8, Theorem 2 and the intermediate value theorem, for every \( V \geq V^* \) there exists \( a \leq a^* \) such that

\[
\frac{1}{5!} \int_{\mathbb{R}^6} e^{6u_{a,3}} dx = V,
\]

hence the metric \( g_{u_{a,3}} = e^{2u_{a,3}} |dx|^2 \) has constant \( Q \)-curvature equal to 1 and \( \text{vol}(g_{u_{a,3}}) = 5!V \). Applying the transformation

\[
u = u_{a,3} - \frac{1}{6} \log 5!
\]

it follows at once that the metric \( g_u = e^{2u} |dx|^2 \) satisfies \( \text{vol}(g_u) = V \) and \( Q_{g_u} \equiv 5! \), hence \( u \) solves (1)-(2). \( \square \)

### 4 Proof of Theorems 3 and 4

When \( f : \mathbb{R}^n \to \mathbb{R} \) is radially symmetric we have \( \Delta f(x) = f''(|x|) + \frac{n-1}{|x|} f'(|x|) \). In particular we have

\[
\Delta_{m} r^{2m} = 2^{2m} m (2m - 1)! \text{ in } \mathbb{R}^{2m}.
\]  

(37)

For \( m \geq 2 \) and \( b \leq 0 \) let \( u_b \) solve

\[
\begin{cases}
\Delta^m u_b = -(2m - 1)! e^{2m u_b} & \text{in } \mathbb{R}^{2m} \\
u_b^{(j)}(0) = 0 & \text{for } 0 \leq j \leq 2m - 1, j \neq 2m - 2 \\
u_b^{(2m-2)} = b.
\end{cases}
\]
From (7)-(8) it follows that \( u_0 \leq 0 \), hence \( \Delta^m u_0 \geq -(2m-1)! \). We claim that

\[
u_0(r) \geq \psi(r) := -\frac{r^{2m}}{2^{2m-1} m}.
\]

Indeed according to (37) \( \psi \) solves

\[
\Delta^m \psi = -(2m-1)! \leq \Delta^m u_0 \quad \text{in } \mathbb{R}^{2m}
\]

and

\[
\psi^{(j)}(0) = u_0^{(j)}(0) \quad \text{for } 0 \leq j \leq 2m-1,
\]

which implies

\[
\Delta^j \psi(0) = 0 = \Delta^j u_0(0) \quad \text{for } 0 \leq j \leq m-1,
\]

see [Mar1, Lemma 17]. Then the claim follows from (7)-(8) and a simple induction.

Now integrating we get

\[
\int_{\mathbb{R}^{2m}} e^{2mu_0} \, dx \geq \int_{\mathbb{R}^{2m}} e^{2m\psi} \, dx = \omega_{2m-1} \int_0^\infty r^{2m-1} \exp \left( -\frac{r^{2m}}{2^{2m-1}} \right) \, dr = \frac{2^{2m-2}\omega_{2m-1}}{m} =: V_m.
\]

Using the formulas

\[
\omega_{2m-1} = \text{vol}(S^{2m-1}) = \frac{2\pi^m}{(m-1)!}, \quad \omega_{2m} = \text{vol}(S^{2m}) = \frac{2^{2m}(m-1)!\pi^m}{(2m-1)!} , \quad m \geq 1
\]

we verify

\[
V_m = \frac{(2m)!}{4(m!)^2\omega_{2m}}, \quad \frac{V_{m+1}}{\omega_{2m+2}} \left( \frac{V_m}{\omega_{2m}} \right)^{-1} = \frac{(2m+2)(2m+1)}{(m+1)^2} > 1, \quad (38)
\]

hence by induction

\[
V_m > \text{vol}(S^{2m}) \quad \text{for } m \geq 2. \quad (39)
\]

With the same argument used to prove Lemma 8 we can show that the function

\[
V(b) := \int_{\mathbb{R}^{2m}} e^{6u_b} \, dx, \quad b \in (-\infty, 0]
\]

is finite and continuous. Indeed it is enough to replace (33) with

\[
\Delta^{m-1} u_b(r) \leq B < 0 \quad \text{for } r \geq r_b,
\]

and (34) with

\[
(\Delta^{m-1-j} u_b)'(r) < C_{m,j} B r^{2j-1}, \quad \Delta^{m-1-j} u_b(r) < D_{m,j} B r^{2j}, \quad \text{for } r \geq r_b, \ 1 \leq j \leq m-1
\]

where \( r_b \) is chosen large enough and

\[
C_{m,1} = \frac{1}{2m}, \quad D_{m,j} = \frac{C_{m,j}}{2j}, \quad C_{m,j+1} = \frac{D_{m,j}}{2m+2j},
\]
whence
\[ C_{m,j} = \frac{(m-1)!}{2^{2j-1}(j-1)!(m+j-1)!}, \quad D_{m,j} = \frac{(m-1)!}{2^{2j}j!(m+j-1)!}. \]
Moreover, using that \( \Delta^{m-1}u_b(0) = C_m b \) for some constant \( C_m > 0 \), \( \Delta^m u_b(r) \leq 0 \) for \( r \geq 0 \) and (7)-(8) as before, we easily obtain
\[ u_b(r) \leq E_m br^{2m-2}, \quad (40) \]
where \( E_m := C_mC_{m,m-1} > 0 \), hence
\[ \lim_{b \to -\infty} V(b) \leq \lim_{b \to -\infty} \int_{\mathbb{R}^6} e^{6E_m b|x|^{2m-2}} \, dx = 0. \]
By continuity we conclude that for every \( V \in (0,V_m] \) there exists \( b \leq 0 \) such that \( u = u_b \) satisfies (1)-(2) if \( m \) is odd or (6)-(2) if \( m \) is even. Taking (39) into account it only remains to prove that the solutions \( u_b \) corresponding to \( V = \text{vol}(S^{2m}) \) is not a spherical one. This follows immediately from (40), which is not compatible with (3). \( \square \)

5 Applications and open questions

Possible gap phenomenon  Theorems 1 and 3 guarantee that for \( m = 3 \) there exists a solution to (1)-(2) for every \( V \in (0,V_3] \cup [V^*, \infty) \), with possibly \( V_3 < V^* \). Could it be that for some \( V \in (V_3,V^*) \) Problem (1)-(2) admits no solution?

If we restrict to rotationally symmetric solutions, some heuristic arguments show that the volume of a solution to (4), i.e. the function
\[ V(a,b) := \int_{\mathbb{R}^6} e^{6u_{a,b}(|x|)} \, dx \]
need not be continuous for all \( (a,b) \in \mathbb{R}^2 \), hence the image of the function \( V \) might not be connected.

Higher dimensions and negative curvature  It is natural to ask whether Theorems 1 and 2 generalize to the case \( m > 3 \) or whether an analogous statement holds when \( m \geq 2 \) and (6) is considered instead of (1). Since the sign on the right-hand side of the ODE (4) plays a crucial role, we would expect that part of the proof of Theorem 2 can be recycled for (1) when \( m \geq 5 \) is odd, or for (6) when \( m \) is even.

For instance let \( u_a = u_a(r) \) be the solution in \( \mathbb{R}^4 \) of
\[
\begin{align*}
\Delta^2 u_a &= -6e^{4u_a} \\
u_a(0) &= u'_a(0) = u''_a(0) = 0 \\
u''_a(0) &= a.
\end{align*}
\]
It should not be difficult to see that \( u_a(r) \) exists for all \( r \geq 0 \) and that \( \int_{\mathbb{R}^4} e^{4u_a(|x|)} \, dx < \infty \). Do we also have
\[ \lim_{a \to +\infty} \int_{\mathbb{R}^4} e^{4u_a(|x|)} \, dx = \infty? \]
Non-radial solutions The proof of Theorem C cannot be extended to provide non-radial solutions to (1)-(2) for \( m \geq 3 \) and \( V \geq \text{vol}(S^{2m}) \), but it is natural to conjecture that they do exist.

Concentration phenomena The classification results of the solutions to (1)-(2), [CL], [Lin], [Xu] and [Mar1], have been used to understand the asymptotic behavior of unbounded sequences of solutions to the prescribed Gaussian curvature problem on 2-dimensional domains (see e.g. [BM] and [LS]), on \( S^2 \) (see [Str4]) and to the prescribed \( Q \)-curvature equation in dimension \( 2m \) (see e.g. [DR], [Mal], [MS], [Ndi], [Rob1], [Rob2], [Mar3], [Mar4]).

For instance consider the following model problem. Let \( \Omega \subset \mathbb{R}^{2m} \) be a connected open set and consider a sequence \((u_k)\) of solutions to the equation

\[
(-\Delta)^m u_k = Q_k e^{2mu_k} \quad \text{in } \Omega,
\]

where

\[
Q_k \to Q_0 \quad \text{in } C^1_{\text{loc}}(\Omega), \quad \limsup_{k \to \infty} \int_{\Omega} e^{2mu_k} \, dx < \infty,
\]

with the following interpretation: \( g_k := e^{2u_k} |dx|^2 \) is a sequence of conformal metrics on \( \Omega \) with \( Q \)-curvatures \( Q_{g_k} = Q_k \) and equibounded volumes.

As shown in [ARS] unbounded sequences of solutions to (41)-(42) can exhibit pathological behaviors in dimension 4 (and higher), contrary to the elegant results of [BM] and [LS] in dimension 2. This is partly due to Theorem A. In fact for \( m \geq 2 \) and \( \alpha \in (0, (2m-1)! \text{vol}(S^{2m})) \) one can found a sequence \((u_k)\) of solutions to (41)-(42) with \( Q_0 > 0 \) and

\[
\lim_{R \to 0} \lim_{k \to \infty} \int_{B_R(x_0)} |Q_k| e^{2mu_k} \, dx = \alpha \quad \text{for some } x_0 \in \Omega.
\]

For \( m = 2 \) this was made very precise by F. Robert [Rob1] in the radially symmetric case. In higher dimension or when \( Q_0 \) is not necessarily positive, thanks to Theorems 1-4 we see that \( \alpha \) can take values larger than \((2m - 1)! \text{vol}(S^{2m})\). Indeed if \( u \) is a solution to (1)-(2) or (6)-(2), then \( u_k := u(kx) + \log k \) satisfies (41)-(42) with \( \Omega = \mathbb{R}^{2m}, Q_k \equiv \pm (2m - 1)! \) and

\[
|Q_k| e^{2mu_k} \, dx \to (2m - 1)! V_{\delta_0}, \quad \text{weakly as measures.}
\]

When \( m = 2, Q_0 > 0 \) (say \( Q_0 \equiv 6 \)) it is unclear whether one could have concentration points carrying more \( Q \)-curvature than \( 6 \text{vol}(S^4) \), i.e. whether one can take \( \alpha > 6 \text{vol}(S^4) \) in (43). Theorem B suggests that if the answer is affirmative, this should be due to the convergence to the same blow-up point of two or more blow-ups. Such a phenomenon is unknown in dimension 4 and higher, but was shown in dimension 2 by Wang [Wan] with a technique which, based on the abundance of conformal transformations of \( \mathbb{C} \) into itself, does not extend to higher dimensions.

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