Graphs of vector valued maps: decomposition of the boundary

Domenico Mucci

Abstract

We consider non-smooth vector valued maps such that the current carried by the graph has finite mass. We give a suitable decomposition of the boundary of the graph-current, provided that it has finite mass, too. Every such a component is a nice current whose support projects on a subset of the domain space that has integer dimension. This structure property is a consequence of a more general one that is proved for "vertical" integer multiplicity rectifiable currents satisfying a null-boundary condition. As a consequence, for wide classes of Sobolev maps such that the graph is a normal current, we shall prove that the singular part of the distributional minors of order k is concentrated on a countably rectifiable set of codimension k, hence no "Cantor-type" part appears. The corresponding class of functions of bounded higher variation is studied, too. Finally, we discuss a possible notion of singular set in our framework, and illustrate several examples.

Contents

1 Introduction

In this paper we discuss a decomposition property for the boundary of the graph of suitable classes of vector valued maps $u : \Omega \to \mathbb{R}^N$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain and $n, N \geq 2$.

Following the theory by Giaquinta-Modica-Souček [19], we shall consider Lebesgue summable maps u that are approximately differentiable a.e. in Ω and such that each minor of the Jacobian matrix ∇u of the approximate gradient is summable. This class of maps is denoted by $\mathcal{A}^1(\Omega, \mathbb{R}^N)$. The membership of u to $\mathcal{A}^1(\Omega, \mathbb{R}^N)$ yields that the *area of the graph* of u is finite, namely

$$A(u, \Omega) := \int_\Omega J_n(\mathrm{Id} \bowtie u)\, dx < \infty\,,$$

where $J_n(\mathrm{Id} \bowtie u)$ is the n-dimensional Jacobian of the graph map $(\mathrm{Id} \bowtie u)(x) := (x, u(x))$. By the area formula, it turns out that $A(u, \Omega)$ agrees with the *mass* $\mathbf{M}(G_u)$ of the integer multiplicity (say i.m.) rectifiable n-current G_u in $\mathcal{R}_n(\Omega \times \mathbb{R}^N)$ carried by the *n-rectifiable graph* $\mathcal{G}_u$ of u.

If u is smooth, the *n-current* G_u is defined by the integration of compactly supported smooth n-forms ω in $\Omega \times \mathbb{R}^N$ over the naturally oriented n-manifold given by the graph $\mathcal{G}_u$ of u:

$$G_u(\omega) := \int_{\mathcal{G}_u} \omega\,, \qquad \omega \in \mathcal{D}^n(\Omega \times \mathbb{R}^N)\,. \tag{1.1}$$

Following [19, Vol. I], see also [20], and writing explicitly the action of ω on $\mathcal{G}_u$ in terms of the pull-back via the graph map $(\mathrm{Id} \bowtie u)$, the above definition extends to the more general class $\mathcal{A}^1(\Omega, \mathbb{R}^N)$, by using the approximate gradient ∇u.[1]

We recall that the *boundary current* ∂G_u is defined by duality as

$$\partial G_u(\eta) := G_u(d\eta)$$

for every compactly supported smooth $(n-1)$-forms η in $\Omega \times \mathbb{R}^N$. If u is smooth, by Stokes' theorem the current G_u has null boundary inside $\Omega \times \mathbb{R}^N$, as for every $\eta \in \mathcal{D}^{n-1}(\Omega \times \mathbb{R}^N)$

$$\partial G_u(\eta) = \int_{\mathcal{G}_u} d\eta = \int_{\partial \mathcal{G}_u} \eta = 0\,.$$

By a density argument, one then obtains that the null-boundary condition

$$\partial G_u(\eta) := G_u(d\eta) = 0 \qquad \forall\, \eta \in \mathcal{D}^{n-1}(\Omega \times \mathbb{R}^N) \tag{1.2}$$

is always satisfied if u is Lipschitz continuous or at least in the Sobolev class $W^{1,p}(\Omega, \mathbb{R}^N)$, where $p = \min\{n, N\}$. However, in general (1.2) is violated, as the following example from [19, Vol. I, Sec. 3.2.2] shows.

Example 1.1 If $\Omega = B^n$, the unit ball centered at the origin, $N = n$, so that $\mathbb{R}^N = \widehat{\mathbb{R}}^n$, and $u(x) := x/|x|$, then $u \in W^{1,p}(B^n, \widehat{\mathbb{R}}^n)$ for each $p < n$ and $\det \nabla u = 0$ a.e., hence $u \in \mathcal{A}^1(B^n, \widehat{\mathbb{R}}^n)$. Moreover, we have

$$\partial G_u = -\delta_0 \times [\![\, \mathbb{S}^{n-1} \,]\!] \qquad \text{on} \quad \mathcal{D}^{n-1}(B^n \times \widehat{\mathbb{R}}^n)\,, \tag{1.3}$$

where δ_0 is the unit Dirac mass at the origin and $[\![\, \mathbb{S}^{n-1} \,]\!]$ in the $(n-1)$-current integration on the (positively oriented) unit $(n-1)$-sphere in the target space $\widehat{\mathbb{R}}^n$. Notice that $u \notin W^{1,n}(B^n, \widehat{\mathbb{R}}^n)$.

In this paper, we deal with non-smooth maps in the class $\mathcal{A}^1(\Omega, \mathbb{R}^N)$ that satisfy the following additional assumptions:

(H_1) (Dirichlet-type condition) *There exists a convex open subset A of Ω, with closure $\overline{A} \subset\subset \Omega$, and a smooth map $\varphi : \Omega \to \mathbb{R}^N$, such that*

$$(G_u - G_\varphi) \llcorner (\Omega \setminus A) \times \mathbb{R}^N = 0\,; \tag{1.4}$$

(H_2) (Finite boundary mass) *The boundary current ∂G_u has finite mass in $\Omega \times \mathbb{R}^N$, i.e.,*

$$\mathbf{M}((\partial G_u) \llcorner \Omega \times \mathbb{R}^N) < \infty\,. \tag{1.5}$$

Therefore, G_u is a *normal* current, and by the boundary rectifiability theorem [29, 30.3], see Theorem 3.4 below, property (1.5) yields that the boundary ∂G_u is an i.m. rectifiable current in $\mathcal{R}_{n-1}(\Omega \times \mathbb{R}^N)$ that is supported in $\overline{A} \times \mathbb{R}^N$, by (1.4). Notice that we shall not assume that the current G_u has compact support in $\overline{\Omega} \times \mathbb{R}^N$, a condition equivalent to a bound on the L^∞-norm of u.

A decomposition property. We shall prove the following decomposition formula for the boundary current ∂G_u of any map u as above. Each component T_k in (1.6) is a boundaryless i.m. rectifiable current in $\mathcal{R}_{n-1}(\Omega \times \mathbb{R}^N)$, with mass controlled by the mass of the boundary of G_u. Furthermore, the set of points of positive multiplicity of T_k is contained into a "cylinder" $S_{n-1-k} \times \mathbb{R}^N$, where S_{n-1-k} is a *countably $\mathcal{H}^{n-1-k}$-rectifiable* subset of Ω. For this reason, we may speak of a *stratification* of the boundary of G_u.

[1]For $u \in \mathcal{A}^1(\Omega, \mathbb{R}^N)$, the n-rectifiable set $\mathcal{G}_u$ is the subset of $\Omega \times \mathbb{R}^N$ given by the points $(x, u(x))$, where x is a Lebesgue point of both u and ∇u and $u(x)$ is the Lebesgue value of u.

Theorem 1.2 (Decomposition Theorem) *Let $n, N \geq 2$ and $\Omega \subset \mathbb{R}^n$ a bounded domain. Let $u : \Omega \to \mathbb{R}^N$ be a vector valued map in $L^1(\Omega, \mathbb{R}^N)$ that is approximately differentiable a.e. in Ω and such that each minor of the Jacobian matrix ∇u of the approximate gradient is summable, i.e., $u \in \mathcal{A}^1(\Omega, \mathbb{R}^N)$. Assume in addition that (H_1) and (H_2) hold. Then we have*

$$(\partial G_u) \llcorner \Omega \times \mathbb{R}^N = \sum_{k=0}^{\underline{n}} T_k \,, \qquad \underline{n} := \min\{n-1, N\} \tag{1.6}$$

where the currents T_k satisfy the following properties:

i) *each T_k is an i.m. rectifiable current in $\mathcal{R}_{n-1}(\Omega \times \mathbb{R}^N)$ with finite mass, $\mathbf{M}(T_k) < \infty$;*

ii) *each T_k has no boundary, $\partial T_k = 0$, and support* $\operatorname{spt}(T_k) \subset \overline{A} \times \mathbb{R}^N$*;*

iii) *the sum of the masses of the T_k's is bounded in terms of the mass of the boundary of G_u, i.e.,*

$$\sum_{k=0}^{\underline{n}} \mathbf{M}(T_k) \leq C \cdot \mathbf{M}((\partial G_u) \llcorner \Omega \times \mathbb{R}^N) < \infty \tag{1.7}$$

for some dimensional constant $C = C(n, N)$;

iv) *if $k > 0$, then $T_k(\eta) = 0$ for every form $\eta \in \mathcal{D}^{n-1}(\Omega \times \mathbb{R}^N)$ that only contains less than k differentials dy^j in the vertical directions $y = (y_1, \dots, y_N)$;*

v) *denoting by* $\operatorname{set}(T_k)$ *the set of points of* positive multiplicity *of T_k, then there exists a* countably $\mathcal{H}^{n-1-k}$-rectifiable *subset S_{n-1-k} of $\overline{A}$ such that*

$$\operatorname{set}(T_k) \subset S_{n-1-k} \times \mathbb{R}^N \,; \tag{1.8}$$

vi) *if $N \geq n$ and $k = \underline{n} = n-1$, then there exists an at most countable set of points $\{a_i\}_i \subset \overline{A}$ and of i.m. rectifiable currents $\Sigma_i \in \mathcal{R}_{n-1}(\mathbb{R}^N)$, with $\partial \Sigma_i = 0$ for every i, such that*

$$T_{n-1} = \sum_{i=1}^{\infty} \delta_{a_i} \times \Sigma_i \,, \qquad \mathbf{M}(T_{n-1}) = \sum_{i=0}^{\infty} \mathbf{M}(\Sigma_i) < \infty \,,$$

where δ_a denotes the unit Dirac mass at the point $a \in \overline{A}$, hence in (1.8) *we correspondingly choose*

$$S_0 := \{a_i \in \overline{A} \mid \Sigma_i \neq 0\} \,; \tag{1.9}$$

vii) *if $N < n$, hence $\underline{n} = N$, we have $T_{\underline{n}} = T_N = 0$, so that $S_0 = \emptyset$ in* (1.9)*;*

viii) *if $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ for some integer exponent $1 \leq p \leq \underline{n}+1$, then $T_k = 0$ for every $k = 0, \dots, p-1$.*

More precisely, we shall denote by $T_{(h)}$ the restriction of a current $T \in \mathcal{D}_{n-1}(\Omega \times \mathbb{R}^N)$ to the forms in $\mathcal{D}^{n-1}(\Omega \times \mathbb{R}^N)$ that contain exactly h differentials dy^j in the vertical directions y.[2] Notice that if T has finite mass, the following decomposition in mass holds:

$$T = \sum_{h=0}^{\underline{n}} T_{(h)} \,, \qquad \mathbf{M}(T) = \sum_{h=0}^{\underline{n}} \mathbf{M}(T_{(h)}) \,. \tag{1.10}$$

In the proof of Theorem 1.2, we shall estimate the mass of each component T_k in (1.6) by

$$\mathbf{M}(T_k) \leq 2 \sum_{h=k}^{\underline{n}} 2^{h-k} \, \mathbf{M}((\partial G_u)_{(h)} \llcorner \Omega \times \mathbb{R}^N) < \infty \,. \tag{1.11}$$

[2]For example, the above property iv) reads as "$(T_k)_{(h)} = 0$ for every $h \leq k-1$".

Therefore, a dimensional constant in the above mass estimate (1.7) is immediately obtained by applying (1.10) to the boundary current $(\partial G_u) \llcorner \Omega \times \mathbb{R}^N$. Notice however that (1.6) is not a decomposition in mass itself, as in general

$$\mathbf{M}((\partial G_u) \llcorner \Omega \times \mathbb{R}^N) \le \sum_{k=0}^{n} \mathbf{M}(T_k) < \infty \,.$$

A STRUCTURE PROPERTY. The main tool that will be used in the proof of Theorem 1.2 is a structure property concerning "vertical" i.m. rectifiable $(n-1)$-currents satisfying the null-boundary condition

$$(\partial T) \llcorner \Omega \times \mathbb{R}^N = 0 \,. \tag{1.12}$$

Notice that property (1.12) is verified by the boundary $T = \partial G_u$ for any u as in Theorem 1.2.

We shall first consider the case of "completely vertical" currents T, i.e., satisfying

$$T_{(h)} = 0 \qquad \text{for } h = 0, \ldots, n-2 \,. \tag{1.13}$$

Theorem 1.3 (Structure property I) *Let $\Omega \subset \mathbb{R}^n$ a bounded domain, and let $N \ge n \ge 2$. Let $T \in \mathcal{R}_{n-1}(\Omega \times \mathbb{R}^N)$ be an i.m. rectifiable current satisfying the null-boundary condition* (1.12) *and the "verticality" property* (1.13)*. Then there exists an at most countable set of points $\{a_i\}_i \subset \Omega$ and of i.m. rectifiable currents $\Sigma_i \in \mathcal{R}_{n-1}(\mathbb{R}^N)$ such that*

$$T = \sum_{i=1}^{\infty} \delta_{a_i} \times \Sigma_i \qquad \text{on} \quad \mathcal{D}^{n-1}(\Omega \times \mathbb{R}^N) \tag{1.14}$$

and

$$\mathbf{M}(T) = \sum_{i=0}^{\infty} \mathbf{M}(\Sigma_i) < \infty \,, \qquad \partial \Sigma_i = 0 \quad \forall\, i \,. \tag{1.15}$$

An inspection to the proof from Sec. 6 below gives that Theorem 1.3 extends to maps defined in bounded domains $\Omega \subset \mathbb{R}^m$, for any $m \ge 2$. As a direct consequence of Theorem 1.3, we also infer:

Corollary 1.4 *Let $n \ge 2$ and $N < n$. Let $T \in \mathcal{R}_{n-1}(\Omega \times \mathbb{R}^N)$ an i.m. rectifiable current satisfying the null-boundary condition* (1.12) *and property* (1.13)*. Then $T = 0$.*

By weakening the null-boundary assumption (1.12) to the boundedness of the boundary mass

$$\mathbf{M}((\partial T) \llcorner \Omega \times \mathbb{R}^N) < \infty \tag{1.16}$$

to our purposes we shall correspondingly obtain:

Proposition 1.5 *Let $T \in \mathcal{R}_{n-1}(\Omega \times \mathbb{R}^N)$ an i.m. rectifiable satisfying* (1.16) *and the "verticality" property* (1.13)*. Then there exists an at most countable set $S_0 \subset \Omega$ such that* $\mathrm{set}(T) \subset S_0 \times \mathbb{R}^N$*. Moreover, $T = 0$ if $N < n-1$.*

A MORE GENERAL STRUCTURE PROPERTY. We now replace the "verticality" assumption (1.13) with the following more general one:

$$T_{(h)} = 0 \qquad \text{for } h = 0, \ldots, k-1 \,, \tag{1.17}$$

where k is any suitable positive integer. We shall then prove:

Theorem 1.6 (Structure property II) *Assume that $\underline{n} := \min\{n-1, N\} > 2$, and let k a positive integer, with $0 < k < \underline{n}$. Let $T \in \mathcal{R}_{n-1}(\Omega \times \mathbb{R}^N)$ an i.m. rectifiable current satisfying the null-boundary condition* (1.12) *and the "verticality" property* (1.17)*. Assume in addition that the support $\mathrm{spt}\, T \subset \overline{A} \times \mathbb{R}^N$ for some open subset A of Ω satisfying $\overline{A} \subset\subset \Omega$. Then there exists a countably $\mathcal{H}^{n-1-k}$-rectifiable subset S_{n-1-k} of $\overline{A}$ such that*

$$\mathrm{set}(T) \subset S_{n-1-k} \times \mathbb{R}^N \,.$$

Discussion on the hypotheses. We first point out that we do not require that the current T in the structure theorems 1.3 and 1.6 have compact support. As a consequence, we do not require e.g. an L^∞-bound for maps u in the decomposition theorem 1.2.

The assumption (H_1) is technical, and it has been introduced in order to apply the structure theorem 1.6 to the boundary of the graph current G_u. In fact, it ensures that the support of $(\partial G_u) \llcorner \Omega \times \mathbb{R}^N$ is contained in $\overline{A} \times \mathbb{R}^N$. On the other hand, the assumption (H_2) that the boundary current ∂G_u has finite mass is a *necessary condition* to the validity of our decomposition theorem 1.2. In fact, S. Müller [26] showed that for $n = N = 2$ the singular part of the *distributional determinant*, first introduced by J.M. Ball [10], may in general concentrate on a set of Hausdorff dimension α, for any prescribed $0 < \alpha < 1$. More precisely, there exist bounded Hölder continuous Sobolev functions u in $W^{1,p}(\Omega, \mathbb{R}^2)$ for every $p < 2$, where $\Omega = (0,1)^2 \subset \mathbb{R}^2$, such that $\det \nabla u = 0$ and $|\nabla u^1| |\nabla u^2| = 0$ a.e. in Ω, but $\mathrm{Det}\, \nabla u = V' \otimes V'$, where V is the Cantor-Vitali function. Therefore, the derivatives of u have no masses, but the distributional determinant has a "Cantor-type" part and the role played by V' in the Cantor set C is here played by $\mathrm{Det}\, \nabla u$ in $C \times C$. The "graph" of u is very similar to the graph of the Cantor-Vitali function V and, actually, has infinitely many holes. As a consequence, the boundary of the current G_u carried by the graph of u cannot satisfy a property as in our decomposition theorem 1.2. In fact, in [19, Vol. I, Sec. 4.2.5] it is shown that in such an example one has $\mathbf{M}((\partial G_u) \llcorner \Omega \times \mathbb{R}^2) = +\infty$, i.e., the mass bound (1.5) is violated.

Finally, notice that the property viii) in Theorem 1.2 yields that the decomposition (1.6) is non-trivial. Actually, it is a crucial point in our new results concerning the singular part of the distributional determinant and minors, see Theorems 1.8 and 1.9 below.

Special functions of bounded variation. If $u = (u^1, \dots, u^N)$ satisfies the hypotheses of Theorem 1.2, it turns out that each component u^j is a *special function of bounded variation*. The class $SBV(\Omega)$ is given by the BV-functions v with null Cantor part of the derivative, $(Dv)^C = 0$,[3] and it was introduced by De Giorgi and Ambrosio in [12] in order to provide a weak formulation to a large class of free discontinuity problems. In fact, the compactness and lower semicontinuity results in SBV, see [3, 4, 1], lead to an existence theory for e.g. the Mumford-Shah functional [28].

The proof of the compactness in [4] starts from a characterization of the subclass of SBV-functions v with finite jump sets, $\mathcal{H}^{n-1}(J_v) < \infty$. In fact, the membership of v to such a class is equivalent to saying that the "horizontal" component $(\partial G_v)_{(0)}$ of the boundary of the current G_v has finite mass in $\Omega \times \mathbb{R}$, and actually, compare [19, Vol. I, Sec. 4.4],

$$2\mathcal{H}^{n-1}(J_v) = \mathbf{M}((\partial G_v)_{(0)} \llcorner \Omega \times \mathbb{R}) < \infty \,. \tag{1.18}$$

Following this approach, Ambrosio-Braides-Garroni studied in [5] the subclass $SBV_0(\Omega)$ of scalar SBV functions v such that every component of the boundary of G_v is representable by integration, i.e.,

$$SBV_0(\Omega) := \{ v \in \mathcal{A}^1(\Omega, \mathbb{R}) \mid \mathbf{M}((\partial G_v) \llcorner \Omega \times \mathbb{R}) < \infty \} \,. \tag{1.19}$$

They proved a compactness property of the class $SBV_0(\Omega)$ and that the traces $v^\pm$ of a function in $SBV_0(\Omega)$ are weakly differentiable on J_v.

Denote now $J_u := \bigcup_{j=1}^N J_{u^j}$. By requiring some mild regularity on the Jump set of u, we have:

Corollary 1.7 *Assume in addition to the hypotheses of Theorem* 1.2 *that the Jump set of u agrees $\mathcal{H}^{n-1}$-essentially with its closure, i.e.,*

$$\mathcal{H}^{n-1}((\overline{J_u} \cap \Omega) \setminus J_u) = 0 \,. \tag{1.20}$$

Then the set S_{n-1} obtained in Theorem 1.2 *agrees $\mathcal{H}^{n-1}$-essentially with the Jump set J_u and hence*

$$\mathcal{H}^{n-1}(S_{n-1}) = \mathcal{H}^{n-1}(J_u) < \infty \,. \tag{1.21}$$

[3] We recall that a Lebesgue function $v \in L^1(\Omega)$ is a *function of bounded variation* if the distributional derivatives $D_i v$ are finite Radon measures. If $v \in BV(\Omega)$, then v is approximately differentiable at almost every point $x \in \Omega$, and the approximate gradient $\nabla v \in L^1(\Omega, \mathbb{R}^n)$, hence $v \in \mathcal{A}^1(\Omega, \mathbb{R})$. Moreover, the vector measure $Dv := (D_1 v, \dots, D_n v)$ decomposes as

$$Dv = \nabla v \, \mathcal{L}^n + (v^+(x) - v^-(x))\, \nu(x)\, \mathcal{H}^{n-1} \llcorner J_v + (Dv)^C \,.$$

The *Jump set* J_v is a countably $\mathcal{H}^{n-1}$-rectifiable set that is σ-finite with respect to the $\mathcal{H}^{n-1}$-measure, $\nu(x) := (\nu_1, \dots, \nu_n)$ is a *unit normal* to J_v at a point x, whereas the $v^\pm(x)$'s denote the one-sided approximate limits of v. Finally, the Cantor-type part satisfies $(Dv)^C(L) = 0$ for every subset $L \subset \Omega$ that is σ-finite w.r.t. the Hausdorff measure $\mathcal{H}^{n-1}$, compare [7].

By Proposition 3.8 below, if $u \in W^{1,1}(\Omega, \mathbb{R}^N)$ we shall deduce that $S_{n-1} = \emptyset$, see property viii). In general, we expect that the claim in Corollary 1.7 holds true even without assuming the regularity property (1.20), but we are not able to prove this fact. Notice that the set J_u being $(n-1)$-rectifiable, property (1.20) holds true if e.g. each component u^j satisfies the classical density lower bound in the sense of De Giorgi-Carriero-Leaci [13], see also [9, 6], a condition verified by minimizers of free discontinuity problems.

The singular set. Our decomposition theorem 1.2 may suggest a definition of $(n-1-k)$-dimensional "singular set" of u by means of the countably $\mathcal{H}^{n-1-k}$-rectifiable set S_{n-1-k} from the above property (1.8), e.g., by (1.9) for S_0. To this purpose, we notice that a part from the case $k = 0$, where trivially $\mathcal{H}^{n-1}(S_{n-1}) < \infty$, for $k \geq 1$ in general we have $\mathcal{H}^{n-1-k}(S_{n-1-k}) \leq \infty$, see Example 9.5 below and also [24, Sec. 7]. However, such an attempt has some drawbacks: namely, we are not able to find a unique "optimal" decomposition in (1.6), see Sec. 10 below.

In fact, roughly speaking, our construction in Sec. 8 relies on the addition of vertical currents to suitable pieces of the boundary current ∂G_u. Such vertical currents are obtained by means of solutions to a mass minimization problem, Proposition 8.2, which fail to be unique, in general. This yields that the singular set S_0 in (1.9) is not uniquely determined, as in general it may depend on the choice of the decomposition in (1.6), see Remark 10.1 below.

Another feature arises even for Sobolev maps $u \in W^{1,n-1}(\Omega, \widehat{\mathbb{R}}^n)$ satisfying the hypotheses of Theorem 1.2. In this case, the sets S_{n-1-k} in (1.8) are all empty but possibly S_0, see (1.9), an at most countable set that in this case is uniquely determined by the function u. In fact, it may happen that S_0 is non-empty, even if the singular part of the distributional determinant $\operatorname{Det} \nabla u$ is equal to zero, see Example 11.5 below.

This fact does not happen if one restricts to Sobolev maps in $W^{1,n-1}(\Omega, \widehat{\mathbb{R}}^n)$ satisfying the "invertibility condition" in the sense of Müller-Spector [27]. In fact, Conti-De Lellis [11] showed that in the framework of such a theory of elasticity, the singular part of the distributional determinant is a purely atomic measure concentrated on an at most countable set of points that agrees with the set S_0 in (1.9), so that

$$(\partial G_u) \llcorner \Omega \times \widehat{\mathbb{R}}^n \subset S_0 \times \widehat{\mathbb{R}}^n \,, \qquad S_0 = \{\text{atoms of } (\operatorname{Det} \nabla u)^s\} \,.$$

Notwithstanding, in Proposition 10.2 we shall prove that *the set* S_0 *in* (1.9) *is concentrated in the set of the atoms of the measure*

$$\mu_u(B) := \mathbf{M}((\partial G_u)_{(n-1)} \llcorner B \times \mathbb{R}^N) \,. \tag{1.22}$$

Plan of the paper. For the sake of clearness, in Sec. 2 we give two basic examples. The first one deals with an SBV-map in the case $n = N = 2$, and the second one with a Sobolev map in the case $n = N = 3$. In Sec. 3, we collect some notation and preliminary results. In Sec. 4, we extend the *isoperimetric inequality* from [25, Prop. 2.1]. In Sec. 5, we consider a *projection argument* that allows to recover the action of a current in terms of the action of the projected currents onto suitable coordinate subspaces. In Sec. 6, we shall prove the structure theorem 1.3 and its consequences, Corollary 1.4 and Proposition 1.5. In Sec. 7, we shall prove the more general structure theorem 1.6. In Sec. 8, making use of both Theorems 1.3 and 1.6, we shall prove Theorem 1.2. In Sec. 9, we shall return to our two basic examples from Sec. 2, and collect some further examples. In Sec. 10, we shall see that the decomposition in (1.6) fails to be unique. Our counterexample is based on a classical non-uniqueness property concerning minimal surfaces. As already remarked, this yields that we are not able to find a good definition of "optimal" singular sets S_{n-1-k} in (1.8). However, in Proposition 10.2 we shall prove that the set S_0 in (1.9) is concentrated in the set of the atoms of the measure μ_u in (1.22).

Distributional minors and the class $\mathrm{B_N V}$. As an application of our decomposition theorem 1.2, in Sec. 11 below we shall discuss some new properties concerning the singular part of the distributional determinant $\operatorname{Det} \nabla u$ and distributional minors in our framework, that we now briefly present.

Let $u \in L^q \cap W^{1,p}(\Omega, \widehat{\mathbb{R}}^n)$ for some exponents q and p satisfying

$$N - 1 \leq p < N \qquad \text{and} \qquad \frac{1}{q} + \frac{N-1}{p} \leq 1 \,, \tag{1.23}$$

where we have chosen $N = n$. De Lellis-Ghiraldin [15] extended a decomposition property first obtained by Müller-Spector [27], showing that if in addition the pointwise determinant $\det \nabla u$ is summable, then

$\operatorname{Det}\nabla u$ is a measure, the density w.r.t. the Lebesgue measure being given by $\det\nabla u$. We shall recover this property, showing in addition that if (1.5) holds, the singular part of $\operatorname{Det}\nabla u$ is concentrated on an at most countable set of points, hence no "Cantor-type" part appears. More precisely, we shall prove:

Theorem 1.8 *Let $n\geq 2$ and $\Omega\subset\mathbb{R}^n$ be a bounded domain. Let $u:\Omega\to\widehat{\mathbb{R}}^n$ be a Sobolev map in $u\in L^q\cap W^{1,p}(\Omega,\widehat{\mathbb{R}}^n)$, where q and p satisfy (1.23), with $N=n$. Assume in addition that $\det\nabla u\in L^1(\Omega)$ and that the boundary of the graph current G_u has finite mass, i.e., (1.5) holds true. Then $(\operatorname{Det}\nabla u)^a=(\det\nabla u)\,\mathcal{L}^n$, and the singular part $(\operatorname{Det}\nabla u)^s$ w.r.t. the Lebesgue measure $\mathcal{L}^n$ is concentrated on the at most countable set S_0 defined by (1.9).*

Alternatively, for maps u in $L^\infty\cap\mathcal{A}^1(\Omega,\widehat{\mathbb{R}}^n)$ and satisfying (1.5), the claim in Theorem 1.8 holds true if (and only if) we require in addition that $(\partial G_u)_{(n-1)}\llcorner\Omega\times\widehat{\mathbb{R}}^n=0$, compare [25]. Under similar assumptions, we shall then discuss the distributional minors of order m, see [24, 25], proving that their singular set is concentrated on the countably $\mathcal{H}^{n-m}$-rectifiable set S_{n-m} obtained from Theorem 1.2, where $k=m-1$.

Finally, we shall discuss the corresponding class $\mathrm{B_NV}$ of *functions of bounded higher variation*, first studied by Jerrard-Soner [22], and consequently used in the asymptotic analysis of functionals of Ginzburg-Landau type. More precisely, for suitable maps $u:\Omega\to\mathbb{R}^N$, where $n\geq N\geq 2$, one can define the current $j_u\in\mathcal{D}_{n-N+1}(\Omega)$ given by

$$\langle j_u,\eta\rangle:=(-1)^N G_u(\omega_N\wedge\eta)\,,\qquad \eta\in\mathcal{D}^{n-N+1}(\Omega)\,,$$

where $\omega_N:=\frac{1}{N}\sum_{j=1}^N(-1)^{j-1}y_j\,\widehat{dy^j}$ is a smooth $(N-1)$-form in $\mathbb{R}^N$, see (11.21) below. For example, j_u is well-defined and has finite mass provided that $u\in L^q\cap W^{1,p}(\Omega,\mathbb{R}^N)$ for some exponents q and p satisfying (1.23). Then u is said to be in $\mathrm{B_NV}(\Omega,\mathbb{R}^N)$ if in addition *the boundary current $\mathfrak{J}_u:=(\partial j_u)\llcorner\Omega$ has finite mass.* Therefore, each map u satisfying the hypotheses of Theorem 1.2 is a function of bounded higher variation. In the case $n>N$, if $u\in\mathrm{B_NV}(\Omega,\mathbb{R}^N)$ we can decompose the current $\mathfrak{J}_u=\sum_{|\alpha|=n}(\mathfrak{J}_u)^\alpha$, where α ranges on the ordered multi-indices of length N in $(1,\dots,n)$. Each component $(\mathfrak{J}_u)^\alpha$ is a Radon measure with finite total variation, the density w.r.t. the Lebesgue measure being given by the $(N\times N)$-minor of the matrix ∇u with columns determined by α, compare [22, 15]. In addition, we shall prove:

Theorem 1.9 *Let u satisfy the hypotheses of Theorem 1.2, where $n\geq N\geq 2$. Assume in addition that $u\in L^q\cap W^{1,p}(\Omega,\mathbb{R}^N)$, where q and p satisfy (1.23). Then the singular part $((\mathfrak{J}_u)^\alpha)^s$ of each component of the current $\mathfrak{J}_u$ is concentrated on the countably $\mathcal{H}^{n-N}$-rectifiable set S_{n-N} given by (1.8), where $k=N-1$.*

Finally, concerning Sobolev maps with values into the unit sphere $\mathbb{S}^{N-1}$ of $\mathbb{R}^N$, for the sake of completeness we shall recover the property first proved in [22] that (up to a real constant factor) *the current $\mathfrak{J}_u$ is i.m. rectifiable in $\mathcal{R}_{n-N}(\Omega)$*, Proposition 11.11.

Acknowledgements. I thank M. Giaquinta for some useful discussions, and L. Biliotti for giving me suggestions concerning a previous version of the projection argument from Sec. 5.

2 Basic examples

In this section we describe two basic examples. The first one deals with an SBV-map in the case $n=N=2$, and the second one with a Sobolev map in the case $n=N=3$. In both the examples, the Dirichlet-type condition (H_1) is clearly verified.

AN EXAMPLE ABOUT SBV-MAPS. Let $n=N=2$, $\Omega=(-2,2)\times(-1,1)$, and $u:\Omega\to\widehat{\mathbb{R}}^2$ given by

$$u(x):=\begin{cases} u_P(x) & \text{if}\quad x_1\leq 0\\ u_Q(x) & \text{if}\quad x_1\geq 0\end{cases}\qquad x=(x_1,x_2)\in\Omega$$

where $P=(-1,0)$, $Q=(1,0)$, and

$$u_P(x):=\left(\operatorname{sgn}(x_2)\sqrt{\frac{|x-P|+(x_1+1)}{2|x-P|}},\sqrt{\frac{|x-P|-(x_1+1)}{2|x-P|}}\right)\,,$$
$$u_Q(x):=\left(\operatorname{sgn}(x_2)\sqrt{\frac{|x-Q|-(x_1-1)}{2|x-Q|}},\sqrt{\frac{|x-Q|+(x_1-1)}{2|x-Q|}}\right)\,.$$

The maps u_P and u_Q glue together on the line $x_1 = 0$ as

$$u_P(0,x_2) = u_Q(0,x_2) = \left(\mathrm{sgn}(x_2)\sqrt{\frac{|(1,x_2)|+1}{2|(1,x_2)|}}, \sqrt{\frac{|(1,x_2)|-1}{2|(1,x_2)|}} \right).$$

Moreover, in polar coordinates

$$u_P(x) = \left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\right) \iff x - P = |x-P|\,(\cos\theta, \sin\theta)\,, \qquad \theta \in]0,2\pi[$$

$$u_Q(x) = \left(\cos\frac{3\pi-\theta}{2}, \sin\frac{3\pi-\theta}{2}\right) \iff x - Q = |x-Q|\,(\cos\theta, \sin\theta)\,, \qquad \theta \in]\pi,3\pi[\,.$$

Therefore, u_P and u_Q map each small circle around the singular points P and Q onto the "upper" half circle of the target space

$$\Sigma := \{y \in \widehat{\mathbb{R}}^2 \,:\, |y| = 1\,,\ y_2 > 0\}\,,$$

that is parameterized with the positive and negative orientation, respectively.

Furthermore, u is a special function of bounded variation in $SBV(\Omega, \widehat{\mathbb{R}}^2)$, with jump set J_u given by the line segment $I_{P,Q}$ connecting the singular points P, Q. Taking the unit normal $\nu = (0,-1)$, the one sided approximate limits of u on J_u are

$$u_+(x) \equiv B := (-1,0)\,, \qquad u_-(x) \equiv A := (1,0)\,, \qquad x \in I_{P,Q}\,.$$

We thus infer that the distributional derivative of u is given by

$$Du = \nabla u\, d\mathcal{L}^2 + (B-A) \otimes (-1,0)\,\mathcal{H}^1 \llcorner I_{P,Q}\,.$$

Now, the approximate gradient ∇u is summable, whereas by the area formula $\det \nabla u = 0$, as the image of u agrees with the 1-dimensional set Σ. Then the area of the graph of u is finite,

$$A(u,\Omega) := \int_\Omega \sqrt{1 + |\nabla u|^2 + |\det \nabla u|^2}\, dx < \infty\,.$$

Whence, the current G_u is i.m. rectifiable in $\mathcal{R}_2(\Omega \times \widehat{\mathbb{R}}^2)$ and has finite mass,

$$\mathbf{M}(G_u) = A(u,\Omega) < \infty\,.$$

Moreover, according to Example 1.1, and integrating by parts, it is not difficult to show that the boundary current ∂G_u is given by

$$(\partial G_u) \llcorner \Omega \times \widehat{\mathbb{R}}^2 = [\![\, I_{P,Q} \,]\!] \times (\delta_A - \delta_B) - \delta_P \times [\![\, \Sigma \,]\!] + \delta_Q \times [\![\, \Sigma \,]\!]\,. \tag{2.1}$$

In this formula, δ_x denotes the unit Dirac mass at x, whereas $[\![\, I_{P,Q} \,]\!]$ and $[\![\, \Sigma \,]\!]$ denote the 1-current integration on the positively oriented line segment $I_{P,Q}$ and half circle Σ, respectively, so that

$$\partial[\![\, I_{P,Q} \,]\!] = \delta_Q - \delta_P\,, \qquad \partial[\![\, \Sigma \,]\!] = \delta_B - \delta_A\,. \tag{2.2}$$

Therefore, the boundary current $(\partial G_u) \llcorner \Omega \times \widehat{\mathbb{R}}^2$ is i.m. rectifiable in $\mathcal{R}_1(\Omega \times \widehat{\mathbb{R}}^2)$ with finite mass:

$$\mathbf{M}((\partial G_u) \llcorner \Omega \times \widehat{\mathbb{R}}^2) = 2\,(|Q-P| + \mathcal{H}^1(\Sigma)) < \infty\,, \qquad \text{see} \quad (1.5)\,.$$

Let now $[\![\, I_{A,B} \,]\!]$ denote the 1-current integration of 1-forms over the oriented line segment $I_{A,B}$ from A to B in the target space $\widehat{\mathbb{R}}^2$, so that $\partial[\![\, I_{A,B} \,]\!] = \delta_B - \delta_A$. Our Theorem 1.2, see Example 9.1 below, yields to the decomposition

$$(\partial G_u) \llcorner \Omega \times \widehat{\mathbb{R}}^2 = T_0 + T_1\,,$$

where T_0, T_1 are the i.m. rectifiable 1-currents in $\mathcal{R}_1(\Omega \times \widehat{\mathbb{R}}^2)$ given by

$$\begin{array}{rcl} T_0 & := & [\![\, I_{P,Q} \,]\!] \times (\delta_A - \delta_B) + (\delta_Q - \delta_P) \times [\![\, I_{A,B} \,]\!]\,, \\ T_1 & := & (\delta_Q - \delta_P) \times ([\![\, \Sigma \,]\!] - [\![\, I_{A,B} \,]\!])\,. \end{array} \tag{2.3}$$

Notice that both T_0 and T_1 have finite mass,

$$\mathbf{M}(T_0) = 2\left(|Q-P| + |B-A|\right), \qquad \mathbf{M}(T_1) = 2\left(\mathcal{H}^1(\Sigma) + |B-A|\right)$$

and no boundary, as by (2.2)

$$\begin{array}{rcl} \partial T_0 & = & \partial [\![I_{P,Q}]\!] \times (\delta_A - \delta_B) + (\delta_Q - \delta_P) \times \partial [\![I_{A,B}]\!] \\ & = & (\delta_Q - \delta_P) \times (\delta_A - \delta_B) + (\delta_Q - \delta_P) \times (\delta_B - \delta_A) = 0\,, \\ \partial T_1 & = & (\delta_Q - \delta_P) \times \partial ([\![\Sigma]\!] - [\![I_{A,B}]\!]) \\ & = & (\delta_Q - \delta_P) \times ((\delta_B - \delta_A) - (\delta_B - \delta_A)) = 0\,. \end{array}$$

We thus have $S_1 = I_{P,Q}$ and $S_0 = \{P, Q\}$, see (1.8) and (1.9).

An example about Sobolev maps. Let $n = N = 3$, $\Omega = (-2,2) \times (-1,1)^2$, and $P = (-1,0,0)$, $Q = (1,0,0)$. Let $u : \Omega \to \widehat{\mathbb{R}}^3$ given for $x = (x_1, x_2, x_3) \in \Omega$ by

$$u(x) := \begin{cases} u_P(x) & \text{if} \quad x_1 \le -1 \\ u_0(x) & \text{if} \quad -1 \le x_1 \le 1 \\ u_Q(x) & \text{if} \quad x_1 \ge 1 \end{cases}$$

where

$$u_P(x) := \frac{x-P}{|x-P|}\,, \qquad u_0(x) := \Big(0, \frac{(x_2,x_3)}{|(x_2,x_3)|}\Big)\,, \qquad u_Q(x) := \frac{x-Q}{|x-Q|}\,. \tag{2.4}$$

Clearly, u is well-defined and continuous outside the singular set given by the closed line segment $I_{P,Q}$ connecting P to Q, as $x - P = (0, x_2, x_3)$ if $x_1 = -1$, and $x - Q = (0, x_2, x_3)$ if $x_1 = 1$. Moreover, the images by u of the three sub-domains of Ω are

$$\mathrm{im}(u_P) = \Sigma^-\,, \qquad \mathrm{im}(u_0) = S^1\,, \qquad \mathrm{im}(u_Q) = \Sigma^+$$

where, for $y = (y_1, y_2, y_3)$,

$$\Sigma^\pm := \{y \in \widehat{\mathbb{R}}^3 \,:\, |y| = 1\,,\ \pm y_1 \ge 0\}\,, \qquad S^1 := \{y \in \widehat{\mathbb{R}}^3 \,:\, |y| = 1\,,\ y_1 = 0\}\,. \tag{2.5}$$

Furthermore, $u_P, u_Q \in W^{1,p}$ for every $p < 3$, whereas by Fubini's theorem $u_0 \in W^{1,p}$ for every $p < 2$, so that u is a Sobolev map in $W^{1,p}(\Omega, \widehat{\mathbb{R}}^3)$ for every $p < 2$.

Let $\mathrm{adj}(\nabla u)$ denote the matrix of the adjoints of the Jacobian matrix ∇u. Since the unit circle S^1 is 1-dimensional, the area formula and Fubini's theorem yield that

$$\mathrm{adj}(\nabla u) = 0 \quad \text{if} \quad -1 < x_1 < 1\,,$$

whereas by the parallelogram inequality

$$|\,\mathrm{adj}(\nabla u)| \le C\,|\nabla u_P|^2 \quad \text{if} \quad x_1 < -1\,, \qquad |\,\mathrm{adj}(\nabla u)| \le C\,|\nabla u_Q|^2 \quad \text{if} \quad x_1 > 1\,,$$

whence $|\,\mathrm{adj}(\nabla u)| \in L^1(\Omega)$. Also, the half-spheres $\Sigma^\pm$ being 2-dimensional, the area formula yields that $\det \nabla u = 0$. As a consequence, the area of the graph of u is finite,

$$A(u,\Omega) := \int_\Omega \sqrt{1 + |\nabla u|^2 + |\,\mathrm{adj}(\nabla u)|^2 + |\det \nabla u|^2}\, dx < \infty\,,$$

and the current G_u carried by the graph of u is i.m. rectifiable in $\mathcal{R}_3(\Omega \times \widehat{\mathbb{R}}^3)$ with finite mass,

$$\mathbf{M}(G_u) = A(u,\Omega) < \infty\,.$$

According to Example 1.1, and integrating by parts, one also deduces that

$$(\partial G_u) \llcorner \Omega \times \widehat{\mathbb{R}}^3 = [\![I_{P,Q}]\!] \times [\![S^1]\!] - \delta_P \times [\![\Sigma^-]\!] - \delta_Q \times [\![\Sigma^+]\!]\,, \tag{2.6}$$

where $[\![\Sigma^\pm]\!]$ denotes the 2-current integration on the half sphere $\Sigma^\pm$, equipped with the induced standard orientation from the 2-sphere $\Sigma^+ \cup \Sigma^-$, and $[\![S^1]\!]$ is the 1-current integration on the 1-circle S^1, oriented in such a way that $\partial[\![\Sigma^\pm]\!] = \pm[\![S^1]\!]$. Therefore, the boundary current $(\partial G_u)\llcorner\Omega\times\widehat{\mathbb{R}}^3$ is i.m. rectifiable in $\mathcal{R}_2(\Omega\times\widehat{\mathbb{R}}^3)$ with finite mass, see (1.5),

$$\mathbf{M}((\partial G_u)\llcorner\Omega\times\widehat{\mathbb{R}}^3) = |Q-P|\cdot\mathcal{H}^1(S^1) + \mathcal{H}^2(\Sigma^+) + \mathcal{H}^2(\Sigma^-) < \infty\,.$$

Let now $[\![D^2]\!]$ denote the 2-current integration on the 2-disk

$$D^2 := \{y\in\widehat{\mathbb{R}}^3\,:\,|y|\le 1\quad\text{and}\quad y_1=0\}\,, \tag{2.7}$$

positively oriented by the 2-vector $(0,1,0)\wedge(0,0,1)$ of the target space. Notice that we have

$$\partial[\![D^2]\!] = [\![S^1]\!] = \partial[\![\Sigma^+]\!] = -\partial[\![\Sigma^-]\!]\,. \tag{2.8}$$

Theorem 1.2, see Example 9.2 below, this time yields to the decomposition

$$(\partial G_u)\llcorner\Omega\times\widehat{\mathbb{R}}^2 = T_0+T_1+T_2\,,$$

where $T_0=0$, as u is a Sobolev map, and T_1, T_2 are the i.m. rectifiable currents in $\mathcal{R}_2(\Omega\times\widehat{\mathbb{R}}^3)$ given by

$$\begin{array}{lll} T_1 & := & [\![I_{P,Q}]\!]\times[\![S^1]\!] + (\delta_P-\delta_Q)\times[\![D^2]\!]\,,\\ T_2 & := & -\delta_P\times([\![\Sigma^-]\!]+[\![D^2]\!]) - \delta_Q\times([\![\Sigma^+]\!]-[\![D^2]\!])\,. \end{array} \tag{2.9}$$

Again, both T_1 and T_2 have finite mass,

$$\mathbf{M}(T_1) = |Q-P|\cdot\mathcal{H}^1(S^1) + 2\,\mathcal{H}^2(D^2)\,,\qquad \mathbf{M}(T_2) = \mathcal{H}^2(\Sigma^-)+\mathcal{H}^2(\Sigma^+)+2\,\mathcal{H}^2(D^2)$$

and no boundary, as by (2.8)

$$\begin{array}{lll} \partial T_1 & = & \partial\,[\![I_{P,Q}]\!]\times[\![S^1]\!] + (\delta_P-\delta_Q)\times\partial\,[\![D^2]\!]\\ & = & (\delta_Q-\delta_P)\times[\![S^1]\!] + (\delta_P-\delta_Q)\times[\![S^1]\!] = 0\,,\\ \partial T_2 & = & -\delta_P\times\partial\,([\![\Sigma^-]\!]+[\![D^2]\!]) - \delta_Q\times\partial\,([\![\Sigma^+]\!]-[\![D^2]\!]) = 0\,. \end{array}$$

We thus have $S_2=\emptyset$, $S_1=I_{P,Q}$, and $S_0=\{P,Q\}$, see (1.8) and (1.9).

3 Notation and preliminary results

In this section we collect some notation and preliminary results. We refer to [16, 23, 29, 2] for general facts about geometric measure theory, and to [19, Vol. I] or [20] for further details on currents carried by graphs.

Rectifiable sets. Let U an open set in $\mathbb{R}^m$ and $\mathcal{H}^k$ the k-dimensional Hausdorff measure on $\mathbb{R}^m$. For $1\le k\le m$ integer, a set $\mathcal{M}\subset U$ is said to be *countably* $\mathcal{H}^k$*-rectifiable* if it is $\mathcal{H}^k$-measurable and $\mathcal{H}^k$-almost all of $\mathcal{M}$ is contained in the union of the images of countably many Lipschitz functions from $\mathbb{R}^k$ to U, compare [16, 3.2.14]. The set $\mathcal{M}$ is said to be *k-rectifiable* if in addition $\mathcal{H}^k(\mathcal{M})<\infty$.

Remark 3.1 The rectifiability criterium by Besicovitch-Marstrand-Mattila, see [16] or [7, Thm. 2.63], states that if $A\subset\mathbb{R}^m$ is a Borel set satisfying $\mathcal{H}^k(A)<\infty$, then A is k-rectifiable if and only if the *k-dimensional density* $\Theta^k(\mathcal{H}^k,A,x)$ is equal to one for $\mathcal{H}^k$-a.e. $x\in A$. This yields that k-rectifiable sets can be "fractured".

General area-coarea formula. The following theorem by Federer [16, 3.2.2] subsumes both the area and coarea formulas, compare [23, 3.13].

Theorem 3.2 *Let $\mathcal{M}\subset\mathbb{R}^{m_1}$ a k-rectifiable set, $\mathcal{N}$ a μ-rectifiable subset of $\mathbb{R}^{m_2}$, where $m_1\ge m_2\ge 1$ and $k\ge\mu$. Let $f:\mathbb{R}^{m_1}\to\mathbb{R}^{m_2}$ a Lipschitz function such that $f(\mathcal{M})=\mathcal{N}$. Then, for any $\mathcal{H}^k\llcorner\mathcal{M}$-integrable function $\psi:\mathcal{M}\to\mathbb{R}$ we have*

$$\int_{\mathcal{M}} J_f^{\mathcal{M}}(w)\,\psi(w)\,d\mathcal{H}^k(w) = \int_{\mathcal{N}}\Big(\int_{\mathcal{M}\cap f^{-1}(\{z\})}\psi\,d\mathcal{H}^{k-\mu}\Big)\,d\mathcal{H}^\mu(z)\,.$$

In this formula, $J_f^{\mathcal{M}}$ denotes the *k-dimensional tangential Jacobian* of $f \llcorner \mathcal{M} : \mathcal{M} \to \mathbb{R}^{m_2}$, compare e.g. [19, Vol. I, Sec. 2.1.5], given for $k = \mu$ by

$$J_f^{\mathcal{M}}(w) := (\det[(d^{\mathcal{M}} f_w)^*(d^{\mathcal{M}} f_w)])^{1/2}, \qquad w \in \mathcal{M}.$$

RECTIFIABLE CURRENTS. We shall denote by $\mathcal{E}^k(U)$, $\mathcal{E}_b^k(U)$, and $\mathcal{D}^k(U)$ the spaces of smooth, bounded smooth, and compactly supported smooth k-forms in U, respectively. The (strong) dual space to $\mathcal{D}^k(U)$ is the class of k-currents $\mathcal{D}_k(U)$.[4] If $T \in \mathcal{D}_k(U)$ has finite mass[5]

$$\mathbf{M}(T) := \sup\{T(\omega) \mid \omega \in \mathcal{D}^k(U),\ \|\omega\| \le 1\} < \infty, \tag{3.1}$$

by dominated convergence it turns out that the action of T extends to forms $\omega \in \mathcal{E}_b^k(U)$, or even to k-forms with bounded Borel coefficients in U. In particular, the restriction $T \llcorner B$ is well-defined for each Borel set $B \in \mathcal{B}(U)$. Since we shall work with *currents with no compact support*,[6] we shall use the symbol "$_{,c}$" when referring to subclasses of currents with compact support. A current $T \in \mathcal{D}_k(U)$ is said to be of the type $(\mathcal{M}, \theta, \overrightarrow{\xi})$, say $T = \tau(\mathcal{M}, \theta, \overrightarrow{\xi})$, if T acts as

$$T(\omega) = \int_{\mathcal{M}} \langle \omega(z), \overrightarrow{\xi}(z) \rangle \, \theta(z) \, d\mathcal{H}^k(z) \qquad \forall\, \omega \in \mathcal{D}^k(U),$$

where $\mathcal{M} \subset U$ is countably $\mathcal{H}^k$-rectifiable, the multiplicity $\theta : \mathcal{M} \to]0, +\infty]$ is $\mathcal{H}^k$-measurable and locally $(\mathcal{H}^k \llcorner \mathcal{M})$-summable and $\overrightarrow{\xi} : \mathcal{M} \to \Lambda_k \mathbb{R}^m$ is $\mathcal{H}^k$-measurable with $|\overrightarrow{\xi}| = 1$ $(\mathcal{H}^k \llcorner \mathcal{M})$-a.e.. Furthermore, T is said to be an *integer multiplicity* (i.m) *rectifiable current*, $T \in \mathcal{R}_k(U)$, if in addition T has finite mass, see (3.1), the density θ takes integer values, and for $\mathcal{H}^k$-a.e. $z \in \mathcal{M}$ the unit k-vector $\overrightarrow{\xi}(z) \in \Lambda_k \mathbb{R}^m$ provides an orientation to the approximate tangent space to $\mathcal{M}$ at z. Moreover, $\mathrm{set}(T)$ denotes the *set of positive multiplicity* θ *in* $\mathcal{M}$, so that $\mathcal{H}^k(\mathrm{set}(T)) \le \mathbf{M}(T) < \infty$ for every $T \in \mathcal{R}_k(U)$. Finally, in this case the support of T agrees with the closure of $\mathrm{set}(T)$, and in general $\mathcal{H}^k(\mathrm{spt}\, T) \le \infty$.

MAIN PROPERTIES. The fundamental theorem by Federer-Fleming [18] makes i.m. rectifiable currents very natural and important, especially in connection with the calculus of variations:[7]

Theorem 3.3 (Closure-compactness) *Let $\{T_j\} \subset \mathcal{R}_k(U)$ a sequence of i.m. rectifiable currents satisfying*

$$\sup_j [\, \mathbf{M}(T_j \llcorner W) + \mathbf{M}((\partial T_j) \llcorner W)\,] < \infty \quad \forall W \quad \textit{open}, \quad W \subset\subset U. \tag{3.2}$$

If T_j weakly converges to some current $T \in \mathcal{D}_k(U)$, then $T \in \mathcal{R}_k(U)$. Otherwise, there exists a subsequence $\{T_{j'}\}$ of $\{T_j\}$ and an i.m. rectifiable current $T \in \mathcal{R}_k(U)$ such that $T_{j'} \rightharpoonup T$.

Since the Deformation theorem holds true for *normal currents* $T \in \mathcal{D}_k(U)$, i.e., satisfying $\mathbf{M}(T) + \mathbf{M}((\partial T) \llcorner U) < \infty$, compare [2, 1.16], we obtain

Theorem 3.4 (Boundary rectifiability) *Let $T \in \mathcal{R}_k(U)$ a normal current, $\mathbf{M}((\partial T) \llcorner U) < \infty$. Then the boundary of T is i.m. rectifiable, too, i.e., $(\partial T) \llcorner U \in \mathcal{R}_{k-1}(U)$.*

The subclass $\mathcal{P}_k(U)$ of *integral polyhedral chains* is the abelian group (with integer coefficients) generated by oriented k-simplices in U. In a similar way, compare [2, 2.11], arguing as in [16, 4.2.20] one has:

Theorem 3.5 (Strong polyhedral approximation) *Let $T \in \mathcal{R}_k(U)$ such that $\mathbf{M}((\partial T) \llcorner U) < \infty$. Then for each $j \in \mathbb{N}^+$ we can find an integral polyhedral chain $P_j \in \mathcal{P}_k(U)$ and a C^1-diffeomorphism g_j of U onto itself such that $\mathrm{Lip}(g_j) \le 1 + 1/j$, $\mathrm{Lip}(g_j^{-1}) \le 1 + 1/j$, and $\mathbf{M}(g_{j\#}T - P_j) + \mathbf{M}(\partial(g_{j\#}T - P_j) \llcorner U) \le 1/j$.*

---

[4] Therefore, $\mathcal{D}_0(U)$ is the usual space of distributions in U.

[5] In (3.1) we have denoted by $\|\omega\|$ the *comass norm* of ω. Using the standard Euclidean norm of ω, one obtains an equivalent notion of mass that agrees with (3.1) for i.m. rectifiable currents.

[6] The support of T is defined exactly as for distributions.

[7] Recall that the weak convergence $T_j \rightharpoonup T$ in $\mathcal{D}_k(U)$ is defined in the dual sense by requiring that $T_j(\omega) \to T(\omega)$ for every test form $\omega \in \mathcal{D}^k(U)$, and that the mass is sequentially weakly lower semicontinuous. Therefore if a sequence $\{T_j\} \subset \mathcal{D}_k(U)$ satisfies $\sup \mathbf{M}(T_j) < \infty$, there exists a subsequence $\{T_{j'}\}$ of $\{T_j\}$ and a current $T \in \mathcal{D}_k(U)$ with finite mass such that $T_{j'} \rightharpoonup T$. Finally, if $k \ge 1$ the boundary current $\partial T \in \mathcal{D}_{k-1}(U)$ is defined by duality for every $T \in \mathcal{D}_k(U)$ by the formula

$$\partial T(\eta) := T(d\eta) \qquad \forall\, \eta \in \mathcal{D}^{k-1}(U).$$

Notation for multi-indices. Let $n, N \geq 2$ integer. If G is an $(N \times n)$-matrix, β and α will always denote the *ordered* multi-indices of row and column of G, respectively. If e.g. $\alpha = (\alpha_1, \dots, \alpha_p)$, where $1 \leq \alpha_1 < \cdots < \alpha_p \leq n$, is a multi-index of length $|\alpha| = p \leq n$, we set $x_\alpha := (x_{\alpha_1}, \dots, x_{\alpha_p})$ and $dx^\alpha := dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_p}$, where $x = (x_1, \dots, x_n)$. We also say that the positive integer i belongs to α if it is one of the indices $\alpha_1, \dots, \alpha_p$. If $i \in \alpha$ we denote by $\alpha - i$ the multi-index of length $p-1$ obtained by removing i from α. Also, $\overline{\alpha}$ is the complement of α in $(1, \dots, n)$, we set $\overline{0} := (1, \dots, n)$, and $\sigma(\alpha, \overline{\alpha})$ is the sign of the permutation which reorders α and $\overline{\alpha}$, e.g., $\sigma(\alpha, \overline{\alpha}) = (-1)^{i-1}$ if $\alpha = i$. For $\alpha = i$ we finally set $\widehat{x_i} := x_{\overline{\alpha}}$ and $\widehat{dx^i} := dx^{\overline{\alpha}}$. A similar notation holds for β and dy^β, with n replaced by N. Moreover, we shall denote by $(e_1, \dots, e_n)$ and $(\varepsilon_1, \dots, \varepsilon_N)$ the canonical bases in $\mathbb{R}^n$ and $\mathbb{R}^N$, respectively, so that e.g. $e_\alpha := e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_p}$.

If $|\alpha| + |\beta| = n$, and $|\beta| \geq 1$, we define by $G^\beta_{\overline{\alpha}}$ the square submatrix obtained by selecting the rows and columns by β and $\overline{\alpha}$, respectively, and by $M^\beta_{\overline{\alpha}}(G)$ its determinant

$$M^\beta_{\overline{\alpha}}(G) := \det G^\beta_{\overline{\alpha}}, \qquad M^{\overline{0}}_0(G) := 1 . \tag{3.3}$$

Finally, we define the *matrix of adjoints* of $G^\beta_{\overline{\alpha}}$ by the formula

$$(\operatorname{adj} G^\beta_{\overline{\alpha}})^j_i := \sigma(j, \beta - j)\, \sigma(i, \overline{\alpha} - i)\, \det G^{\beta - j}_{\overline{\alpha} - i}, \qquad j \in \beta, \quad i \in \overline{\alpha} . \tag{3.4}$$

Of course, if $n = N$ and $|\beta| = n$, i.e, $\beta = \overline{0}$, we simply write $\operatorname{adj} G$ for $\operatorname{adj} G^\beta_{\overline{\alpha}} = \operatorname{adj} G^{\overline{0}}_{\overline{0}}$, so that

$$(\operatorname{adj} G)^j_i := (-1)^{i+j} \det G^{\overline{j}}_{\overline{i}}, \qquad i, j = 1, \dots, n . \tag{3.5}$$

Splitting of currents. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $U := \Omega \times \mathbb{R}^N$. Every $(n-1)$-form $\eta \in \mathcal{D}^{n-1}(\Omega \times \mathbb{R}^N)$ splits as a sum $\omega = \sum_{k=0}^{\underline{n}} \omega^{(k)}$, where $\underline{n} := \min\{n-1, N\}$ and the $\omega^{(k)}$'s are the $(n-1)$-components that contain exactly k differentials in the vertical y-variables.[8] Every current $T \in \mathcal{D}_{n-1}(\Omega \times \mathbb{R}^N)$ then splits as

$$T = \sum_{k=0}^{\underline{n}} T_{(k)}, \qquad \text{where} \quad T_{(k)}(\omega) := T(\omega^{(k)}) .$$

Notice that the above formula is a *decomposition in mass*. More precisely, since for every $\omega \in \mathcal{D}^{n-1}(\Omega \times \mathbb{R}^N)$

$$\|\omega\| = \sup\{\|\omega^{(k)}\| \,:\, k = 0, \dots, \underline{n}\},$$

by (3.1) we deduce that $\mathbf{M}(T) = \sum_{k=0}^{\underline{n}} \mathbf{M}(T_{(k)})$. For $l = 1, \dots, \underline{n}$, we shall also denote

$$T_{(\geq l)} := \sum_{k=l}^{\underline{n}} T_{(k)}, \quad \text{so that} \quad \mathbf{M}(T_{(\geq l)}) = \sum_{k=l}^{\underline{n}} \mathbf{M}(T_{(k)}) . \tag{3.6}$$

A similar decomposition holds for every $(n-1)$-vector field $\overrightarrow{\xi}$ in $\Omega \times \mathbb{R}^N$, say

$$\overrightarrow{\xi} = \sum_{k=0}^{\underline{n}} \xi_{(k)}, \quad \xi_{(k)} := \sum_{\substack{|\alpha|+|\beta|=n-1 \\ |\beta|=k}} \xi^{\alpha,\beta}\, e_\alpha \wedge \varepsilon_\beta . \tag{3.7}$$

We thus have

$$\langle \omega^{(k)}, \overrightarrow{\xi} \rangle = \langle \omega, \xi_{(k)} \rangle = \langle \omega^{(k)}, \xi_{(k)} \rangle \qquad \forall\, k = 0, \dots, \underline{n} .$$

[8] We thus have for some $\omega_{\alpha,\beta} \in C^\infty_c(\Omega \times \mathbb{R}^N)$

$$\omega^{(k)} := \sum_{\substack{|\alpha|+|\beta|=n-1 \\ |\beta|=k}} \omega_{\alpha,\beta}\, dx^\alpha \wedge dy^\beta \qquad \text{if} \qquad \omega = \sum_{|\alpha|+|\beta|=n-1} \omega_{\alpha,\beta}\, dx^\alpha \wedge dy^\beta .$$

Homotopy formula. Let $V \subset \mathbb{R}^m$ be an open set, where $m \geq n+1$. Let $f, g : \Omega \times \mathbb{R}^N \to V$ be two Lipschitz maps, and let $h : \Omega \times \mathbb{R}^N \times [0,1] \to V$ denote the affine homotopy map

$$h(z,t) := t\, f(z) + (1-t)\, g(z)\,, \quad z \in \Omega \times \mathbb{R}^N\,, \quad t \in [0,1]\,.$$

If a current $T \in \mathcal{D}_{n-1}(\Omega \times \mathbb{R}^N)$ has finite mass, by dominated convergence the action of T is well-defined on smooth forms $\omega \in \mathcal{E}_b^{n-1}(\Omega \times \mathbb{R}^N)$ with bounded coefficients, e.g. for $\omega = f^\# \eta$ for any $\eta \in \mathcal{D}^{n-1}(V)$ and for f as above. Hence the image current $f_\# T \in \mathcal{D}_{n-1}(V)$ is well-defined by $f_\# T(\eta) := T(f^\# \eta)$, for $\eta \in \mathcal{D}^{n-1}(V)$. As a consequence, if T is a normal current, i.e., $\mathbf{M}(T) + \mathbf{M}((\partial T) \llcorner \Omega \times \mathbb{R}^N) < \infty$, the image currents $h_\#(T \times [\![0,1]\!])$ and $h_\#(\partial T \times [\![0,1]\!])$ are both well defined provided that f and g are bounded or the restriction of h to the support of $T \times [\![0,1]\!]$ is proper. In particular, if T has compact support, and $V = \Omega \times \mathbb{R}^N$, the *homotopy formula* [29, 26.22] yields

$$\partial h_\#(T \times [\![0,1]\!]) = h_\#(\partial T \times [\![0,1]\!]) + (-1)^{n-1}(f_\# T - g_\# T) \qquad \text{on} \quad \mathcal{D}^{n-1}(\Omega \times \mathbb{R}^N)\,. \tag{3.8}$$

Dealing with currents that are not compactly supported, in the case $V = \Omega \times \mathbb{R}^N$ and for suitable choices of f and g we overcome this problem by restricting the range of t to intervals of the type $[\varepsilon, 1]$.

Proposition 3.6 *Let $\varepsilon \in]0,1[$ and $h_\varepsilon : \Omega \times \mathbb{R}^N \times [\varepsilon,1] \to \Omega \times \mathbb{R}^N$ denote the affine homotopy map*

$$h_\varepsilon(x,y,t) := t\,(x,y) + (1-t)(x,0)\,, \quad (x,y) \in \Omega \times \mathbb{R}^N\,, \quad t \in [\varepsilon,1]\,. \tag{3.9}$$

If $T \in \mathcal{D}_{n-1}(\Omega \times \mathbb{R}^N)$ has finite mass, the image current $h_{\varepsilon\#}(T \times [\![\varepsilon,1]\!])$ is well-defined in $\mathcal{D}_n(\Omega \times \mathbb{R}^N)$ and it has locally finite mass*, i.e., for every compact set $K \subset \Omega \times \mathbb{R}^N$*

$$\mathbf{M}((h_{\varepsilon\#}(T \times [\![\varepsilon,1]\!])) \llcorner K) < \infty\,.$$

Similarly, if $\mathbf{M}((\partial T) \llcorner \Omega \times \mathbb{R}^N) < \infty$ the image current $h_{\varepsilon\#}(\partial T \times [\![\varepsilon,1]\!])$ is well defined in $\mathcal{D}_{n-1}(\Omega \times \mathbb{R}^N)$ and it has locally finite mass. Finally, setting $f_\varepsilon(x,y) := (x, \varepsilon y)$, if $\mathbf{M}(T) + \mathbf{M}((\partial T) \llcorner \Omega \times \mathbb{R}^N) < \infty$, the following homotopy formula *holds:*

$$\partial h_{\varepsilon\#}(T \times [\![\varepsilon,1]\!]) = h_{\varepsilon\#}(\partial T \times [\![\varepsilon,1]\!]) + (-1)^{n-1}(T - f_{\varepsilon\#} T) \qquad \textit{on} \quad \mathcal{D}^{n-1}(\Omega \times \mathbb{R}^N)\,. \tag{3.10}$$

Proof: Since T has finite mass, we deduce that $h_{\varepsilon\#}(T \times [\![\varepsilon,1]\!])$ is well-defined provided that $\|h_\varepsilon^\# \omega\| < \infty$ for every $\omega \in \mathcal{D}^n(\Omega \times \mathbb{R}^N)$. To prove this property, by a density argument we may and do assume that ω is a linear combinations of forms of the type $\varphi(x)\, \psi(y)\, dx^\alpha \wedge dy^\beta$, where $\varphi \in C_c^\infty(\Omega)$, $\psi \in C_c^\infty(\mathbb{R}^N)$, and $|\alpha| + |\beta| = n$. If $|\beta| > 0$, we have

$$h_\varepsilon^\#(\varphi(x)\, \psi(y) dx^\alpha \wedge dy^\beta) = \varphi(x)\, dx^\alpha \wedge \psi(\widetilde{h}_\varepsilon(y,t))\, \widetilde{h}_\varepsilon^\# dy^\beta\,,$$

where $\widetilde{h}_\varepsilon : \mathbb{R}^N \times [\varepsilon,1] \to \mathbb{R}^N$ is given by $\widetilde{h}_\varepsilon(y,t) = ty$, and compute

$$\widetilde{h}_\varepsilon^\# dy^\beta = dy^\beta - (-1)^{|\beta|}\, \widetilde{\omega}_\beta \wedge dt\,, \quad \text{where} \quad \widetilde{\omega}_\beta := \sum_{j \in \beta} \sigma(j, \beta - j)\, y_j\, dy^{\beta - j} \in \mathcal{E}^{|\beta|-1}(\mathbb{R}^N)\,.$$

Since moreover $\psi \in C_c^\infty(\mathbb{R}^N)$, there exists $R > 0$ such that $\psi(y) = 0$ if $|y| > R$, hence $\psi(\widetilde{h}_\varepsilon(y,t)) = 0$ for every $(y,t) \in \mathbb{R}^N \times [\varepsilon,1]$ provided that $|y| > R/\varepsilon$. Using that $|\widetilde{\omega}_\beta(y)| \leq |y|$, this yields

$$\|h_\varepsilon^\#(\varphi(x)\, \psi(y) dx^\alpha \wedge dy^\beta)\| \leq \|\varphi\|_\infty \|\psi\|_\infty \frac{R}{\varepsilon} < \infty \qquad \text{on} \quad \Omega \times \mathbb{R}^N \times [\varepsilon,1]\,.$$

If $|\beta| = 0$, we have $\|h_\varepsilon^\#(\varphi(x)\, \psi(y)\, dx^\alpha)\| = \|\varphi(x)\, \psi(\widetilde{h}_\varepsilon(y,t))\, dx^\alpha\| \leq \|\varphi\|_\infty \|\psi\|_\infty$. In particular, denoting by $\widetilde{B}_R$ the closed ball in $\mathbb{R}^N$ centered at the origin and with radius R, by (3.1) we deduce that for each $R > 1$

$$\mathbf{M}((h_{\varepsilon\#}(T \times [\![\varepsilon,1]\!])) \llcorner \Omega \times \widetilde{B}_R) \leq \frac{R}{\varepsilon}\, \mathbf{M}(T) < \infty\,,$$

and hence that $h_{\varepsilon\#}(T \times [\![\varepsilon,1]\!])$ has locally finite mass. The second assertion is proved in a similar way. As a consequence, if $\mathbf{M}(T) + \mathbf{M}((\partial T) \llcorner \Omega \times \mathbb{R}^N) < \infty$, property (3.10) follows from the standard homotopy formula (3.8), with 0 replaced by ε, using the dominated convergence theorem. □

Remark 3.7 In general the image currents $h_{\varepsilon\#}(T \times [\![\varepsilon, 1]\!])$ and $h_{\varepsilon\#}(\partial T \times [\![\varepsilon, 1]\!])$ from Proposition 3.6 do not have finite mass, if T does not have compact support.

Orthogonal projections. We shall denote by $\pi : \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R}^n$ and $\widehat{\pi} : \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R}^N$ the *orthogonal projections* onto the x and y coordinates, respectively. Let $0 \le j \le \min\{n, N\}$ and $0 \le k \le j$ integers. Let $T \in \mathcal{D}_j(\Omega \times \mathbb{R}^N)$ a current with finite mass, $\mathbf{M}(T) < \infty$. For any $\omega \in \mathcal{D}^k(\mathbb{R}^N)$, we shall denote by $\pi_\#(T \llcorner \widehat{\pi}^\# \omega)$ the current in $\mathcal{D}_{j-k}(\Omega)$ such that

$$\langle \pi_\#(T \llcorner \widehat{\pi}^\# \omega), \varphi \rangle := T(\widehat{\pi}^\# \omega \wedge \pi^\# \varphi) = T(\omega \wedge \varphi)\,,^9 \qquad \varphi \in \mathcal{D}^{j-k}(\Omega)\,.$$

Similarly, for $\varphi \in \mathcal{D}^k(\mathbb{R}^N)$, we shall denote by $\widehat{\pi}_\#(T \llcorner \pi^\# \varphi)$ the current in $\mathcal{D}_{j-k}(\mathbb{R}^N)$ such that

$$\langle \widehat{\pi}_\#(T \llcorner \pi^\# \varphi), \omega \rangle := T(\pi^\# \varphi \wedge \widehat{\pi}^\# \omega) = T(\varphi \wedge \omega)\,, \qquad \omega \in \mathcal{D}^{j-k}(\mathbb{R}^N)\,.$$

Currents carried by graphs. We shall denote by $\mathcal{A}^1(\Omega, \mathbb{R}^N)$ the class of vector-valued L^1-maps $u : \Omega \to \mathbb{R}^N$ that are *a.e. approximately differentiable* and such that *all the minors of the Jacobian matrix* ∇u *are summable*,[10] i.e.,

$$\begin{aligned} \mathcal{A}^1(\Omega, \mathbb{R}^N) := \{ u \in L^1(\Omega, \mathbb{R}^N) \mid & \ u \text{ is a.e. approximately differentiable and} \\ & M^\beta_{\overline{\alpha}}(\nabla u) \in L^1(\Omega) \text{ for all } \alpha,\ \beta \text{ with } |\alpha| + |\beta| = n \}\,. \end{aligned} \tag{3.11}$$

If $u \in \mathcal{A}^1(\Omega, \mathbb{R}^N)$ is smooth, the graph current G_u acts on compactly supported smooth n-form ω in $\Omega \times \mathbb{R}^N$ by integration on the naturally oriented graph $\mathcal{G}_u$ of u, see (1.1), so that by the area formula

$$G_u(\omega) = \int_\Omega (\mathrm{Id} \bowtie u)^\# \omega \qquad \forall\, \omega \in \mathcal{D}^n(\Omega \times \mathbb{R}^N)\,, \tag{3.12}$$

where $\mathrm{Id} \bowtie u(x) := (x, u(x))$. Following [19, Vol. I, Sec. 3.2.1], for general maps u in $\mathcal{A}^1(\Omega, \mathbb{R}^N)$, the graph current $G_u \in \mathcal{R}_n(\Omega \times \mathbb{R}^N)$ is again defined by (3.12), where the pull-back $(\mathrm{Id} \bowtie u)^\# \omega$ is intended in the a.e. approximate sense. Formula (1.1) holds true, but this time the *rectifiable graph* is the n-rectifiable set

$$\mathcal{G}_u := \{ (x, \lambda_u(x)) \,:\, x \in \mathcal{L}_u \cap A_D(u) \cap \Omega \}\,.^{11}$$

As a consequence, the mass of G_u, see (3.1), is equal to the area of the rectifiable graph of u, i.e.,

$$\mathbf{M}(G_u) = \mathcal{H}^n(\mathcal{G}_u) = A(u, \Omega) := \int_\Omega J_n(\mathrm{Id} \bowtie u)\, dx < \infty\,,$$

where $J_n(\mathrm{Id} \bowtie u)$ is the n-dimensional Jacobian of the graph map $\mathrm{Id} \bowtie u$.[12]

Boundaries. As we recalled in the introduction, graphs of smooth maps $u : \Omega \to \mathbb{R}^N$ satisfy the null-boundary condition (1.2). On the other hand, see Example 1.1, if $u \in \mathcal{A}^1(\Omega, \mathbb{R}^N)$, in general the interior boundary of G_u is non zero, i.e.,

$$(\partial G_u) \llcorner \Omega \times \mathbb{R}^N \neq 0\,.$$

The membership of u to Sobolev classes yields the following property on the lower components of the boundary ∂G_u, compare [19, Vol. I, Sec. 3.2.3] and also [20, Prop. 4.22].

Proposition 3.8 *Let* $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ *for some* $p \in \mathbb{N}^+$*. Then for every* $k = 0, \dots, p-1$

$$(\partial G_u)_{(k)} \llcorner \Omega \times \mathbb{R}^N = 0\,. \tag{3.13}$$

---

[9] We shall often omit to write the action of the pull-back by π and $\widehat{\pi}$.

[10] Hence $\nabla u(x)$ is an $(N \times n)$-matrix, for a.e. $x \in \Omega$.

[11] In this formula, $\mathcal{L}_u$ is the set of Lebesgue points, $\lambda_u(x)$ is the Lebesgue value at x, and $A_D(u)$ is the set of approximate differentiability points of u. Notice that if u is a Sobolev map in $W^{1,1}$, then the approximate gradient ∇u agrees with the distributional derivative Du.

[12] Using the notation (3.3), we compute $J_n(\mathrm{Id} \bowtie u)^2 = \sum_{|\alpha|+|\beta|=n} M^\beta_{\overline{\alpha}}(\nabla u)^2$. For example, if $n = N = 3$ we have

$$J_n(\mathrm{Id} \bowtie u) = \sqrt{1 + |\nabla u|^2 + |\operatorname{adj} \nabla u|^2 + (\det \nabla u)^2}\,.$$

PROOF: For every form $\eta \in \mathcal{D}^{n-1}(B^n \times \mathbb{R}^N)$ we have

$$\partial G_u(\eta^{(k)}) = G_u(d_x \eta^{(k)}) + G_u(d_y \eta^{(k)}) \,,$$

where $d = d_x + d_y$ denotes the splitting of the exterior differential d in $\Omega \times \mathbb{R}^N$ into the horizontal and vertical differentials. Therefore, (3.12) yields that $\partial G_u(\eta^{(k)})$ is written as an integral involving minors of ∇u of order lower than $k+1$. Let now $\{u_h\} \subset C^\infty(\Omega, \mathbb{R}^N)$ a sequence of smooth maps converging to u strongly in $W^{1,p}$. By the dominated convergence theorem, one infers that $\partial G_{u_h}(\eta^{(k)}) \to \partial G_u(\eta^{(k)})$ provided that $k \le p-1$. Condition (1.2), that holds true for the u_h's, gives (3.13). $\square$

Remark 3.9 The function u from Example 1.1 belongs to $W^{1,p}(B^n, \widehat{\mathbb{R}}^n)$ for any $p < n$, whereas $\det \nabla u = 0$ a.e., so that $u \in \mathcal{A}^1(B^n, \widehat{\mathbb{R}}^n)$, but $u \notin W^{1,n}(B^n, \widehat{\mathbb{R}}^n)$. In particular, (3.13) holds true for $k = 0, \dots, n-2$. However, property (1.3) implies that the function u does not satisfy (3.13) for $k = n-1$.

Finally, we shall need the following

Lemma 3.10 *Let $T \in \mathcal{R}_{n-1}(\Omega \times \mathbb{R}^N)$ satisfying the null boundary condition* (1.12)*. Then we have $T(d\eta) = 0$ for every smooth form η in $\mathcal{E}^{n-2}(\Omega \times \mathbb{R}^N)$ with Lipschitz coefficients and with support contained in the cylinder $\Omega \times \mathbb{R}^N$.*

PROOF: Condition (1.12) means that $T(d\eta) = 0$ if $\eta \in \mathcal{D}^{n-2}(\Omega \times \mathbb{R}^N)$. For $R > 0$, we choose a cut-off function $\chi_R \in C^\infty_c([0, +\infty))$ such that $\chi_R(t) = 1$ for $0 \le t \le R$, $\chi_R(t) = 0$ for $t \ge R+1$, $0 \le \chi_R \le 1$ and $|\chi'_R| \le 2$. Since $\chi_R(|y|)\, \eta \in \mathcal{D}^{n-2}(\Omega \times \mathbb{R}^N)$, by (1.12) we have $T(d[\chi_R(|y|)\, \eta]) = 0$, whence

$$T(d\eta) = T\big(d[(1 - \chi_R(|y|))\, \eta]\big) \,.$$

Set $U_R := \{(x,y) \mid x \in \Omega,\ |y| \ge R\}$ and $W_j := U_j \setminus U_{j+1}$, for $j \in \mathbb{N}$. Since T has finite mass,

$$\lim_{R \to \infty} \mathbf{M}(T \llcorner U_R) = 0 \,, \qquad \liminf_{j \to \infty} j \cdot \mathbf{M}(T \llcorner W_j) = 0 \,. \tag{3.14}$$

Moreover,

$$d[(1 - \chi_R(|y|))\, \eta] = -\chi'_R(|y|)\, d|y| \wedge \eta + (1 - \chi_R(|y|)\, d\eta \,.$$

Therefore, taking $R = j$ we estimate for each j

$$|T(d\eta)| \le c\, \|\eta\|_{\infty, W_j}\, \mathbf{M}(T \llcorner W_j) + \|d\eta\|_\infty\, \mathbf{M}(T \llcorner U_j) \,.$$

Since η has Lipschitz coefficients and support contained in $\Omega \times \mathbb{R}^N$, we get

$$\|\eta(x,y)\|_{\infty, W_j} \le c_1(1 + \|y\|_{\infty, W_j}) \le c_2(1 + j) \,, \qquad \|d\eta\| \le c_3$$

for some absolute constants $c_i > 0$. Hence for each j

$$|T(d\eta)| \le c_2(1+j)\, \mathbf{M}(T \llcorner W_j) + c_3\, \mathbf{M}(T \llcorner U_j)$$

and the claim follows by taking a subsequence according to (3.14). $\square$

4 An isoperimetric inequality

In this section we extend the isoperimetric inequality from [25, Prop. 2.1]. The main difficulty is due to the fact that we do not require the current T in Proposition 4.1 below to have compact support.

We let $\Omega \subset \mathbb{R}^n$ a bounded domain, where $n \ge 2$, and denote by $\widehat{\mathbb{R}}^n$ the target space. Any form $\omega \in \mathcal{D}^{n-1}(\widehat{\mathbb{R}}^n)$ is identified by a compactly supported smooth vector field $g \in C^\infty_c(\widehat{\mathbb{R}}^n, \widehat{\mathbb{R}}^n)$ via the formula

$$\omega_g(y) := \sum_{j=1}^n (-1)^{j-1} g^j(y)\, \widehat{dy^j} \,, \qquad g = (g^1, \dots, g^n) \,, \tag{4.1}$$

where $\widehat{dy^j} := dy^1 \wedge \cdots \wedge dy^{j-1} \wedge dy^{j+1} \wedge \cdots \wedge dy^n$, so that $d\omega_g = \operatorname{div} g\, dy$, where $dy := dy^1 \wedge \cdots \wedge dy^n$. We let μ_g correspondingly denote the signed measure given on Borel sets $B \in \mathcal{B}(\Omega)$ by

$$\langle \mu_g, B\rangle := (-1)^{n-1} \langle \pi_\#(T \llcorner \widehat{\pi}^\# \omega_g), B\rangle \,,$$

so that for functions $\varphi \in C_c^\infty(\Omega)$

$$(-1)^{n-1} \langle \mu_g, \varphi\rangle = (T \llcorner \widehat{\pi}^\# \omega_g)(\pi^\# \varphi) = T(\varphi \wedge \omega_g)\,.$$

Proposition 4.1 *Let $T \in \mathcal{R}_{n-1}(\Omega \times \widehat{\mathbb{R}}^n)$ an i.m. rectifiable current satisfying the property $T_{(n-2)} = 0$ and the null-boundary condition*

$$(\partial T) \llcorner \Omega \times \widehat{\mathbb{R}}^n = 0\,. \tag{4.2}$$

Then for every $x_0 \in \Omega$ and a.e. $r > 0$ such that $B_r(x_0) \subset\subset \Omega$ we have

$$|\langle \mu_g, \overline{B}_r(x_0)\rangle| \le c_n \, \|\operatorname{div} g\|_\infty \, \mathbf{M}(T \llcorner \overline{B}_r(x_0) \times \widehat{\mathbb{R}}^n)^{n/(n-1)} \tag{4.3}$$

for every $g \in C_c^\infty(\widehat{\mathbb{R}}^n, \widehat{\mathbb{R}}^n)$, where $c_n > 0$ is an absolute constant, not depending on g.

Proof: Fix $\varepsilon \in]0,1[$, define $h_\varepsilon : \Omega \times \widehat{\mathbb{R}}^n \times [\varepsilon, 1] \to \Omega \times \widehat{\mathbb{R}}^n$ as in Proposition 3.6, where $N = n$, and denote

$$H_T^\varepsilon := h_{\varepsilon\#}(T \times [\![\varepsilon, 1]\!]) \in \mathcal{D}_n(\Omega \times \widehat{\mathbb{R}}^n)\,. \tag{4.4}$$

By Proposition 3.6, we may and do introduce for $k \in \{0, \ldots, n\}$ and $\eta \in \mathcal{D}^k(\Omega)$ the $(n-k)$-current

$$H_T^\varepsilon \llcorner \eta := \widehat{\pi}_\#(H_T^\varepsilon \llcorner \pi^\# \eta) \in \mathcal{D}_{n-k}(\widehat{\mathbb{R}}^n)\,.$$

Setting $\widetilde{h}_\varepsilon(y,t) := ty$ for $(y,t) \in \widehat{\mathbb{R}}^n \times [\varepsilon, 1]$, we thus equivalently have:

$$H_T^\varepsilon \llcorner \eta(\omega) := (T \times [\![\varepsilon, 1]\!])(\eta \wedge \widetilde{h}_\varepsilon^\# \omega)\,, \qquad \omega \in \mathcal{D}^{n-k}(\widehat{\mathbb{R}}^n)\,, \tag{4.5}$$

where we have omitted to write the pull-back of the orthogonal projection maps. Even if in general the current H_T^ε from (4.4) does not have finite mass, see Remark 3.7, by Proposition 3.6 we deduce that for every $\eta \in \mathcal{D}^k(\Omega)$, the current $H_T^\varepsilon \llcorner \eta$ in $\mathcal{D}_{n-k}(\widehat{\mathbb{R}}^n)$ has locally finite mass. In the case $k = 1$, we shall make use of the following extension of [25, Lemma 2.3], the proof of which is postponed.

Lemma 4.2 *Let $T \in \mathcal{R}_{n-1}(\Omega \times \widehat{\mathbb{R}}^n)$ be such that $T_{(n-2)} = 0$. Then $H_T^\varepsilon \llcorner \eta = 0$ for every $\eta \in \mathcal{D}^1(\Omega)$.*

Setting now $f_\varepsilon(x,y) := (x, \varepsilon y)$, we let μ_g^ε denote for every $g \in C_c^\infty(\widehat{\mathbb{R}}^n, \widehat{\mathbb{R}}^n)$ the signed measure given on Borel sets $B \in \mathcal{B}(\Omega)$ by

$$\langle \mu_g^\varepsilon, B\rangle := (-1)^{n-1} \langle \pi_\#(f_{\varepsilon\#} T \llcorner \widehat{\pi}^\# \omega_g), B\rangle \,,$$

so that for functions $\varphi \in C_c^\infty(\Omega)$

$$(-1)^{n-1} \langle \mu_g^\varepsilon, \varphi\rangle = f_{\varepsilon\#} T(\varphi \wedge \omega_g)\,.$$

Property (4.2) implies that $h_{\varepsilon\#}(\partial T \times [\![\varepsilon, 1]\!]) = 0$ on forms in $\mathcal{D}^{n-1}(\Omega \times \widehat{\mathbb{R}}^n)$. Therefore, using the above definitions, our homotopy formula (3.10) gives

$$\langle \mu_g - \mu_g^\varepsilon, \varphi\rangle = H_T^\varepsilon \llcorner d\varphi(\omega_g) + H_T^\varepsilon \llcorner \varphi(d\omega_g)$$

for every $\varphi \in C_c^\infty(\Omega)$ and $g \in C_c^\infty(\widehat{\mathbb{R}}^n, \widehat{\mathbb{R}}^n)$, whereas Lemma 4.2 yields that $H_T^\varepsilon \llcorner d\varphi(\omega_g) = 0$, so that

$$\langle \mu_g - \mu_g^\varepsilon, \varphi\rangle = H_T^\varepsilon \llcorner \varphi(d\omega_g) = H_T^\varepsilon \llcorner \varphi(\operatorname{div} g(y)\, dy)\,.$$

As a consequence, taking a sequence $\{\varphi_j\} \subset C_c^\infty(\Omega)$ converging in L^1 to the characteristic function χ of the closed ball $\overline{B}_r(x_0)$, and setting $B_r := B_r(x_0)$ for simplicity, we deduce that

$$\langle \mu_g - \mu_g^\varepsilon, \overline{B}_r\rangle = H_T^\varepsilon \llcorner \chi_{\overline{B}_r}(\operatorname{div} g(y)\, dy)\,.$$

In fact, setting $\widetilde{f}_\varepsilon(y)=\varepsilon y$ and $K=\operatorname{spt}\varphi$, since $\|\omega_g\|=\|g\|_\infty$ by (3.1) we have

$$|f_{\varepsilon\#}T(\varphi\wedge\omega_g)|=|T(\varphi\wedge\widetilde{f}_\varepsilon^{\#}\omega_g)|\leq\|\varphi\|_\infty\,\|g\|_\infty\,\varepsilon^{n-1}\,\mathbf{M}(T\llcorner\overline{K}\times\widehat{\mathbb{R}}^n)\,,$$

so that the measures μ_g and μ_g^ε have finite total variation, as

$$\begin{aligned}|\langle\mu_g,\overline{B}_r\rangle|&\leq\|g\|_\infty\,\mathbf{M}(T\llcorner\overline{B}_r\times\widehat{\mathbb{R}}^n)<\infty\,,\\|\langle\mu_g^\varepsilon,\overline{B}_r\rangle|&\leq\varepsilon\,\|g\|_\infty\,\mathbf{M}(T\llcorner\overline{B}_r\times\widehat{\mathbb{R}}^n)<\infty\,.\end{aligned}\tag{4.6}$$

On the other hand, for each $\omega\in\mathcal{D}^n(\widehat{\mathbb{R}}^n)$, by (4.4) and (4.5) we have

$$H_T^\varepsilon\llcorner\chi_{\overline{B}_r}(\omega)=(T\times[\![\,\varepsilon,1\,]\!])(\chi_{\overline{B}_r}\wedge\widetilde{h}_\varepsilon^{\#}\omega)=((T\llcorner\overline{B}_r\times\widehat{\mathbb{R}}^n)\times[\![\,\varepsilon,1\,]\!])(\widetilde{h}_\varepsilon^{\#}\omega)\,.$$

Therefore, since by Proposition 3.6 the current $H_T^\varepsilon\llcorner\varphi$ in $\mathcal{D}_n(\widehat{\mathbb{R}}^n)$ has locally finite mass, and T is i.m. rectifiable in $\mathcal{R}_{n-1}(\Omega\times\widehat{\mathbb{R}}^n)$, we deduce that the current $H_T^\varepsilon\llcorner\chi_{\overline{B}_r}$ is locally i.m. rectifiable in $\mathcal{R}_{n,\mathrm{loc}}(\widehat{\mathbb{R}}^n)$. We then proceed in a way similar to the second part of the proof of [24, Prop. 3.1].

More precisely, by using the *degree theory* from [19, Vol. I, Sec. 4.3.2], for a.e. $r>0$ small there exists an *integer valued* and locally summable function $\Delta_r^\varepsilon\in L^1_{\mathrm{loc}}(\widehat{\mathbb{R}}^n,\mathbb{Z})$ such that

$$H_T^\varepsilon\llcorner\chi_{\overline{B}_r}(\psi(y)\,dy)=\int_{\widehat{\mathbb{R}}^n}\Delta_r^\varepsilon(y)\,\psi(y)\,dy\qquad\forall\,\psi\in C_c^\infty(\widehat{\mathbb{R}}^n)\,.$$

This yields that for every $g\in C_c^\infty(\widehat{\mathbb{R}}^n,\widehat{\mathbb{R}}^n)$

$$\langle\mu_g-\mu_g^\varepsilon,\overline{B}_r\rangle=\int_{\widehat{\mathbb{R}}^n}\Delta_r^\varepsilon(y)\,\operatorname{div}g(y)\,dy\,.$$

Moreover, by (4.6) the measure $\mu_g-\mu_g^\varepsilon$ has finite total variation,

$$|\langle\mu_g-\mu_g^\varepsilon,\overline{B}_r\rangle|\leq\|g\|_\infty\,(1+\varepsilon)\,\mathbf{M}(T\llcorner\overline{B}_r\times\widehat{\mathbb{R}}^n)<\infty\,.$$

Therefore, Δ_r^ε is a function of bounded variation in $\widehat{\mathbb{R}}^n$, with

$$\begin{aligned}|D\Delta_r^\varepsilon|(\widehat{\mathbb{R}}^n)\quad&:=\quad\sup_{\|g\|_\infty\leq1}\int_{\widehat{\mathbb{R}}^n}\Delta_r^\varepsilon(y)\,\operatorname{div}g(y)\,dy\\&=\quad\sup_{\|g\|_\infty\leq1}|\langle\mu_g-\mu_g^\varepsilon,\overline{B}_r\rangle|\leq(1+\varepsilon)\,\mathbf{M}(T\llcorner\overline{B}_r\times\widehat{\mathbb{R}}^n)<\infty\,.\end{aligned}\tag{4.7}$$

By Sobolev embedding theorem, and by density of smooth maps in $BV_{\mathrm{loc}}(\widehat{\mathbb{R}}^n)$,

$$\|\Delta_r^\varepsilon\|_{L^{n/(n-1)}(\widehat{\mathbb{R}}^n)}\leq c_n\,|D\Delta_r^\varepsilon|(\widehat{\mathbb{R}}^n)$$

whereas, taking into account that $\Delta_r^\varepsilon(y)\in\mathbb{Z}$,

$$\int_{\widehat{\mathbb{R}}^n}|\Delta_r^\varepsilon(y)|\,dy\leq\int_{\widehat{\mathbb{R}}^n}|\Delta_r^\varepsilon(y)|^{n/(n-1)}\,dy=\|\Delta_r^\varepsilon\|^{n/(n-1)}_{L^{n/(n-1)}(\widehat{\mathbb{R}}^n)}\,.$$

We thus obtain

$$\begin{aligned}|\langle\mu_g-\mu_g^\varepsilon,\overline{B}_r\rangle|\quad&\leq\quad\int_{\widehat{\mathbb{R}}^n}|\Delta_r^\varepsilon(y)\,\operatorname{div}g(y)|\,dy\leq\|\operatorname{div}g\|_\infty\int_{\widehat{\mathbb{R}}^n}|\Delta_r^\varepsilon(y)|\,dy\\&\leq\quad\|\operatorname{div}g\|_\infty\,c_n\,(|D\Delta_r^\varepsilon|(\widehat{\mathbb{R}}^n))^{n/(n-1)}\end{aligned}$$

and definitively, by (4.7),

$$|\langle\mu_g-\mu_g^\varepsilon,\overline{B}_r\rangle|\leq\|\operatorname{div}g\|_\infty\,c_n\,(1+\varepsilon)^{n/(n-1)}\mathbf{M}(T\llcorner\overline{B}_r\times\widehat{\mathbb{R}}^n)^{n/(n-1)}\,.\tag{4.8}$$

Finally, since $|\langle\mu_g\overline{B}_r\rangle|\leq|\langle\mu_g-\mu_g^\varepsilon,\overline{B}_r\rangle|+|\langle\mu_g^\varepsilon,\overline{B}_r\rangle|$, using the second line in (4.6), the isoperimetric inequality (4.3) follows by letting $\varepsilon\to0$ in the above formula (4.8). $\square$

Proof of Lemma 4.2: We have (4.5) with $k=1$. Using (4.1), write $\omega\in\mathcal{D}^{n-1}(\widehat{\mathbb{R}}^n)$ as $\omega=\omega_g$ for some $g\in C^\infty_c(\widehat{\mathbb{R}}^n,\widehat{\mathbb{R}}^n)$. By linearity, without loss of generality we may and do assume that $g^j=0$ for $j>1$, and let $g^1(y)=f(y)$, so that $\omega_g=\omega:=f(y)\,\widehat{dy^1}$, where $f\in C^\infty_c(\widehat{\mathbb{R}}^n)$. We compute

$$\widetilde{h}^{\#}_{\varepsilon}\omega=f(ty)[t^{n-1}\widehat{dy^1}+(-1)^n\,\omega^1\wedge t^{n-2}dt]\,,\quad\text{where}\quad \omega^1:=\sum_{h=2}^n(-1)^h\,y_h\,dy^{\overline{(1,h)}}\in\mathcal{E}^{n-2}(\widehat{\mathbb{R}}^n)\,.$$

Since the form $\eta\wedge f(ty)\,t^{n-1}\widehat{dy^1}$ does not contain the differential dt, by definition of Cartesian product of currents and the dominated convergence theorem we get $(T\times[\![\varepsilon,1]\!])(\eta\wedge f(ty)\,t^{n-1}\widehat{dy^1})=0$ and hence

$$\begin{array}{rcl} H^\varepsilon_T\llcorner\eta(\omega) &=& (-1)^n(T\times[\![\varepsilon,1]\!])(\eta(x)\wedge f(ty)\,\omega^1\wedge t^{n-2}dt]) \\ &=& (-1)^nT(\eta(x)\wedge\omega^1(y)\,F_\varepsilon(y))\,, \end{array}$$

where $F_\varepsilon(y):=\displaystyle\int_\varepsilon^1 f(ty)\,t^{n-2}\,dt$. Arguing as in the proof of Proposition 3.6, using that $\omega^1\in\mathcal{E}^{n-2}(\widehat{\mathbb{R}}^n)$ satisfies $|\omega^1(y)|\le|y|$, we deduce that

$$\|\eta(x)\wedge\omega^1(y)\,F_\varepsilon(y)\|\le\|\eta\|\cdot\|f\|_\infty\cdot\frac{R}{\varepsilon}<\infty\qquad\text{on}\quad\Omega\times\widehat{\mathbb{R}}^n\times[\varepsilon,1]\,,$$

where $R>0$ is chosen so that $f(y)=0$ if $|y|>R$. Since $\mathbf{M}(T)<\infty$, property $T_{(n-2)}=0$ and the dominated convergence yield that $T(\eta(x)\wedge\omega^1(y)\,F_\varepsilon(y))=0$, as required. $\square$

5 A projection argument

Let $N>n$ and $u\in W^{1,n-1}(\Omega,\mathbb{R}^N)\cap\mathcal{A}^1$ a Sobolev map such that the boundary current $T:=(\partial G_u)\llcorner\Omega\times\mathbb{R}^N$ has finite mass, see (1.5). For any multi-index β of length $|\beta|=n$, by Lemma 5.2 below the boundary $T^\beta:=\partial(G_{u^\beta})\llcorner\Omega\times\widehat{\mathbb{R}}^n_\beta$ of the graph current of the corresponding Sobolev map $u^\beta\in W^{1,n-1}(\Omega,\widehat{\mathbb{R}}^n_\beta)\cap\mathcal{A}^1$ has finite mass, too, hence it satisfies the hypotheses of Proposition 4.1, see Remark 6.2 below. Using arguments from [24], see the proof of Theorem 1.3 in Sec. 6 below, case $N=n$, this yields that $\text{set}(T^\beta)\subset S^\beta_0\times\widehat{\mathbb{R}}^n_\beta$ for an at most countable set of points $S^\beta_0\subset\Omega$. Therefore, making use of the general area-coarea formula, we aim at recovering the action of T in terms of the action of the currents T^β on suitably related forms, see Proposition 5.5 below. This would allow to conclude that $\text{set}(T)\subset S_0\times\mathbb{R}^N$ for an at most countable set of points $S_0\subset\Omega$, Theorem 1.3. However, as the following example from [19, Vol. I, Sec. 3.2.3] shows, in general the above strategy may fail.

Example 5.1 Let $u:B^2\to\mathbb{R}^3$ the homogeneous extension $u(x):=\varphi\Big(\dfrac{x}{|x|}\Big)$ of the Lipschitz map $\varphi:\mathbb{S}^1\to\mathbb{R}^3$ defined in terms of the angle θ by

$$\varphi(\theta):=\left\{\begin{array}{lll} \big(\cos4\theta,\sin4\theta,0\big) & \text{if} & 0\le\theta<\pi/2 \\ \big(1,0,\theta-\pi/2\big) & \text{if} & \pi/2\le\theta<\pi \\ \big(\cos4\theta,-\sin4\theta,\pi/2\big) & \text{if} & \pi\le\theta<3\pi/2 \\ \big(1,0,2\pi-\theta\big) & \text{if} & 3\pi/2\le\theta<2\pi\,. \end{array}\right.$$

Clearly $u\in W^{1,p}(B^2,\mathbb{R}^3)$ for $p<2$ and $M^{\overline{0}}_{\overline{\alpha}}(\nabla u)=0$ for each $|\alpha|=1$, hence $u\in\mathcal{A}^1(B^2,\mathbb{R}^3)$. Moreover

$$(\partial G_u)\llcorner B^2\times\mathbb{R}^3=-\delta_0\times\varphi_\#[\![\,\mathbb{S}^1\,]\!]\,,\qquad\varphi_\#[\![\,\mathbb{S}^1\,]\!]=[\![\,\Sigma^1_0\,]\!]-[\![\,\Sigma^1_{\pi/2}\,]\!]\,,$$

where for $\lambda\in\mathbb{R}$ the unit circle $\Sigma^1_\lambda:=\{y\in\mathbb{R}^3\,:\,|(y_1,y_2)|=1\,,\ y_3=\lambda\}$ is naturally oriented. Using that

$$\Pi^\beta_\#(\varphi_\#[\![\,\mathbb{S}^1\,]\!])=(\Pi^\beta\circ\varphi)_\#[\![\,\mathbb{S}^1\,]\!]=0\qquad\text{if}\quad|\beta|=2\,,$$

see the notation (5.1) below, we have $(\partial G_{u^\beta})\llcorner\Omega\times\widehat{\mathbb{R}}^2_\beta=0$ for every $|\beta|=2$, even if $(\partial G_u)\llcorner\Omega\times\mathbb{R}^3\neq0$.

Roughly speaking, the cancellation phenomenon in the previous example is due to the fact that the two circles Σ^1_0 and $\Sigma^1_{\pi/2}$ have exactly the same shadow on the coordinate planes $\widehat{\mathbb{R}}^2_\beta$. However, by slightly rotating the target space $\mathbb{R}^3$, this problem can be avoided.

More generally, assume $|\beta| = n$, and define the corresponding projection maps

$$\begin{aligned} &\Pi^\beta : \mathbb{R}^N \to \mathbb{R}^n_\beta \simeq \mathbb{R}^n\,, && \Pi^\beta(y) = y_\beta := (y_{\beta_1}, \ldots, y_{\beta_m})\,, \\ &\Psi_\beta : \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R}^n \times \mathbb{R}^n_\beta\,, && \Psi_\beta(x,y) := (\mathrm{Id}_{\mathbb{R}^n} \bowtie \Pi^\beta)(x,y) = (x, \Pi^\beta(y))\,. \end{aligned} \tag{5.1}$$

For any $T \in \mathcal{R}_{n-1}(\Omega \times \mathbb{R}^N)$, let $T^\beta := \Psi_{\beta\#}T \in \mathcal{D}_{n-1}(\Omega \times \mathbb{R}^n_\beta)$ the corresponding image current, see Lemma 5.2 below. Denoting $\mathcal{M} := \mathrm{set}(T)$, the cancellation phenomenon previously described is clearly avoided provided that the multiplicity function $\mathcal{H}^0(\mathcal{M} \cap \Psi_\beta^{-1}(\{z\})$ is equal to one for $\mathcal{H}^{n-1}$-a.e. point z in the shadow $\Psi_\beta(\mathcal{M})$. To this purpose, we shall first consider the case of polyhedral chains, Proposition 5.3.

Projection of currents. We first point out the following facts:

Lemma 5.2 *Let $T \in \mathcal{R}_{n-1}(\Omega \times \mathbb{R}^N)$ satisfying the null boundary condition $(\partial T) \llcorner \Omega \times \mathbb{R}^N = 0$. Then the image current $T^\beta := \Psi_{\beta\#}T$ is i.m. rectifiable in $\mathcal{R}_{n-1}(\Omega \times \mathbb{R}^n_\beta)$ and satisfies the null-boundary condition $(\partial T^\beta) \llcorner \Omega \times \mathbb{R}^n_\beta = 0$.*

Proof: Since T is i.m. rectifiable, the first assertion follows if we show that $\mathbf{M}(T^\beta) < \infty$. To prove this, observe that for every $\omega \in \mathcal{D}^{n-1}(\Omega \times \mathbb{R}^n_\beta)$ the pull-back form $\Psi^\#_\beta \omega$ belongs to the class $\mathcal{E}^{n-1}_b(\Omega \times \mathbb{R}^N)$ and satisfies $\|\Psi^\#_\beta \omega\| \le \|\omega\|$. Therefore, by (3.1) and by dominated convergence, we estimate

$$T^\beta(\omega) := T(\Psi^\#_\beta \omega) \le \mathbf{M}(T)\, \|\Psi^\#_\beta \omega\| \le \mathbf{M}(T)\, \|\omega\|$$

that gives $\mathbf{M}(T^\beta) \le \mathbf{M}(T)$. As to the second assertion, for every $\omega \in \mathcal{D}^{n-2}(\Omega \times \mathbb{R}^n_\beta)$ we have

$$\partial T^\beta(\omega) = T^\beta(d\omega) = T(\Psi^\#_\beta d\omega) = T(d\Psi^\#_\beta \omega) = T(d\widetilde{\omega})\,,$$

where the smooth $(n-2)$-form $\widetilde{\omega} := \Psi^\#_\beta \omega$ has support contained in $\Omega \times \mathbb{R}^N$, as $\mathrm{spt}\,\widetilde{\omega} = \Psi_\beta^{-1}(\mathrm{spt}\,\omega)$. Since $\|\widetilde{\omega}\| + \|d\widetilde{\omega}\| < \infty$, Lemma 3.10 gives $T(d\widetilde{\omega}) = 0$, as required. □

Projections of polyhedral chains. Assume $N > k \ge 1$ and denote by $\mathbf{O}^*(N,k)$ the set of orthogonal projections $\mathbf{p}$ of $\mathbb{R}^N$ onto the k-dimensional subspaces of $\mathbb{R}^N$. There is a unique measure on $\mathbf{O}^*(N,k)$ that is invariant under Euclidean motions of $\mathbb{R}^N$ and normalized to have total measure 1.

Proposition 5.3 *Let $N > n$ and $P \in \mathcal{P}_{n-1}(\Omega \times \mathbb{R}^N)$ be an integral polyhedral chain, and let $\mathcal{M} := \mathrm{set}(P)$. Then for a.e. projection $\mathbf{p} \in \mathbf{O}^*(N,n)$ and for $\mathcal{H}^{n-1}$-a.e. $z \in (\mathrm{Id}_{\mathbb{R}^n} \bowtie \mathbf{p})(\mathcal{M})$*

$$N(\mathrm{Id}_{\mathbb{R}^n} \bowtie \mathbf{p} | \mathcal{M}, z) := \mathcal{H}^0(\mathcal{M} \cap (\mathrm{Id}_{\mathbb{R}^n} \bowtie \mathbf{p})^{-1}(\{z\})) = 1\,.$$

Proof: Recall that $(e_1, \ldots, e_n)$ denotes the canonical basis in $\mathbb{R}^n \simeq \mathbb{R}^n \times \{0\} \subset \mathbb{R}^n \times \mathbb{R}^N$. Every projection of the type $\mathrm{Id}_{\mathbb{R}^n} \bowtie \mathbf{q}$, where $\mathbf{q} \in \mathbf{O}^*(N, N-1)$, is clearly determined by a couple $\pm\nu$ of unit normals $\nu \in \mathbb{R}^n \times \mathbb{R}^N$, i.e., $\pm\nu \in \mathbb{S}^{n+N-1}$, and $\nu \cdot e_i = 0$ for each $i = 1, \ldots, n$. Hence, the couple $\pm\nu$ belongs to the "vertical" $(N-1)$-sphere

$$\mathbb{S}^{N-1}_v := \{(x,y) \in \mathbb{S}^{N+n-1} \subset \mathbb{R}^n \times \mathbb{R}^N \mid x = 0\}\,.$$

Using this identification, we write $\mathrm{Id}_{\mathbb{R}^n} \bowtie \mathbf{q} = \pi_{\pm\nu}$.

Since P is an $(n-1)$-dimensional integral polyhedral chain, it is readily checked that the property

$$N(\pi_{\pm\nu} | \mathcal{M}, z) := \mathcal{H}^0(\mathcal{M} \cap {\pi_{\pm\nu}}^{-1}(\{z\})) = 1 \qquad \forall\, z \in \pi_{\pm\nu}(\mathcal{M})$$

holds true for every choice of $\pm\nu \in \mathbb{S}^{N-1}_v$ except for a "bad" set $B \subset \mathbb{S}^{N-1}_v$ of null $\mathcal{H}^n$-measure, $\mathcal{H}^n(B)$=0. This proves the claim in the case $N = n+1$. For $N = n+m$, with $m \ge 2$, it suffices to iterate m times the above argument. □

Remark 5.4 Proposition 5.3 is clearly false for projections $\mathbf{p} \in \mathbf{O}^*(N, n-1)$. If e.g. $N = n+1$, it suffices to take $P = \delta_0 \times [\![Q]\!]$, where Q is an $(n-1)$-dimensional cube in $\mathbb{R}^{n+1}$.

THE AREA-COAREA FORMULA ON CURRENTS. Let now $T \in \mathcal{R}_{n-1}(\Omega \times \mathbb{R}^N)$, where $N > n$, and write $T := \tau(\mathcal{M}, \theta, \overrightarrow{\xi})$. Moreover, for any index β with $|\beta| = m$, where $1 \le m \le N-1$, denote by ξ_β the component of the tangent $(n-1)$-vector field $\overrightarrow{\xi}$ corresponding to the base $(n-1)$-vectors $e_\alpha \wedge \varepsilon_\gamma$, where β contains all the entries of γ, i.e.,

$$\xi_\beta := \sum_{\substack{|\alpha|+|\gamma|=n-1 \\ \gamma \subset \beta}} \xi_\alpha^\gamma \, e_\alpha \wedge \varepsilon_\gamma \qquad \text{if} \qquad \overrightarrow{\xi} = \sum_{|\alpha|+|\gamma|=n-1} \xi_\alpha^\gamma \, e_\alpha \wedge \varepsilon_\gamma \,.$$

Define

$$\mathcal{M}_\beta := \{(x,y) \in \mathcal{M} \mid \xi_\beta(x,y) \neq 0\} \tag{5.2}$$

and observe that the set $\mathcal{M}_\beta$ is $(n-1)$-rectifiable, see Remark 3.1. This yields that $\mathcal{N}_\beta := \Psi_\beta(\mathcal{M}_\beta)$ is an $(n-1)$-rectifiable subset of $\Omega \times \mathbb{R}^n_\beta$. Let $\overrightarrow{\zeta_\beta}$ denote an $\mathcal{H}^{n-1} \llcorner \mathcal{N}_\beta$-measurable function such that $\overrightarrow{\zeta_\beta}(x, y_\beta)$ is a unit $(n-1)$-vector orienting the approximate tangent space to $\mathcal{N}_\beta$ at $\mathcal{H}^{n-1}$-a.e. $(x, y_\beta) \in \mathcal{N}_\beta$. By applying the general area-coarea formula, Theorem 3.2, we obtain:

Proposition 5.5 *Let $|\beta| = m \in \{1, \ldots, N-1\}$. Let $|\alpha| + |\gamma| = n-1$, with $\gamma \subset \beta$. Let $\eta_\alpha^\gamma \in \mathcal{D}^{n-1}(\Omega \times \mathbb{R}^N)$ given by*

$$\eta_\alpha^\gamma := \phi(x)\, f(y_{\overline{\beta}})\, g(y_\beta)\, dx^\alpha \wedge dy^\gamma \,,$$

where $\phi \in C_c^\infty(\Omega)$, $f \in C_c^\infty(\mathbb{R}^{N-m}_{\overline{\beta}})$, $g \in C_c^\infty(\mathbb{R}^n_\beta)$. With the previous notation, we have:

$$T(\eta_\alpha^\gamma) = \int_{\mathcal{N}_\beta} \langle \phi(x)\, \widehat{\Phi}(x, y_\beta)\, g(y_\beta)\, dx^\alpha \wedge dy^\gamma, \overrightarrow{\zeta_\beta}(x, y_\beta) \rangle \; d\mathcal{H}^{n-1}(x, y_\beta) \,, \tag{5.3}$$

where we have set

$$\widehat{\Phi}(x, y_\beta) := \int_{\mathcal{M}_\beta \cap (\psi_\beta^{-1}(\{(x, y_\beta)\})} \sigma(x,y)\, f(y_{\overline{\beta}})\, \theta(x,y)\, d\mathcal{H}^0(x,y) \,, \qquad \sigma(x,y) = \pm 1 \,. \tag{5.4}$$

PROOF: Since $\gamma \subset \beta$, we clearly have

$$T(\eta_\alpha^\gamma) = \int_{\mathcal{M}_\beta} \langle \eta_\alpha^\gamma, \xi_\beta \rangle \, \theta \, d\mathcal{H}^{n-1} \,. \tag{5.5}$$

Moreover, it turns out that the $(n-1)$-dimensional tangential Jacobian of Ψ_β agrees with the norm of the $(n-1)$-vector ξ_β, i.e.,

$$J_{\Psi_\beta}^{\mathcal{M}_\beta}(x,y) = |\xi_\beta(x,y)| \qquad \text{for } \mathcal{H}^{n-1}\text{-a.e. } (x,y) \in \mathcal{M}_\beta \,.$$

Furthermore, for $\mathcal{H}^{n-1}$-a.e. $(x, y_\beta) \in \mathcal{N}_\beta$ and $(x,y) \in \mathcal{M}_\beta \cap \Psi_\beta^{-1}(\{(x, y_\beta)\})$ we have

$$\frac{\xi_\beta(x,y)}{|\xi_\beta(x,y)|} = \sigma(x,y)\, \overrightarrow{\zeta_\beta}(z) \,, \quad \text{where} \quad \sigma(x,y) := \pm 1 \,.$$

We then apply Theorem 3.2, where $\mathcal{M} = \mathcal{M}_\beta$, $\mathcal{N} = \mathcal{N}_\beta$, $k = \mu = n-1$, $m_1 = n+N$, $m_2 = n+m$, $f = \Psi_\beta$, $w = (x,y)$, $z = (x, y_\beta)$, to the $\mathcal{H}^{n-1} \llcorner \mathcal{M}_\beta$-integrable function

$$\Phi(x,y) := \theta(x,y) \langle \phi(x)\, f(y_{\overline{\beta}})\, g(y_\beta)\, dx^\alpha \wedge dy^\gamma, \xi_\beta(x,y) \rangle \, |\xi_\beta(x,y)|^{-1} \,.$$

Since $\langle \eta_\alpha^\gamma, \xi_\beta \rangle \, \theta = J_{\psi_\beta}^{\mathcal{M}_\beta} \cdot \Phi$, by (5.5) we obtain

$$\begin{aligned} T(\eta_\alpha^\gamma) &= \int_{\mathcal{M}_\beta} J_{\psi_\beta}^{\mathcal{M}_\beta}(x,y)\, \Phi(x,y)\, d\mathcal{H}^{n-1}(x,y) \\ &= \int_{\mathcal{N}_\beta} \Big(\int_{\mathcal{M}_\beta \cap \psi_\beta^{-1}(\{(x, y_\beta)\})} \Phi \, d\mathcal{H}^0 \Big) \, d\mathcal{H}^{n-1}(z) \\ &= \int_{\mathcal{N}_\beta} \langle \phi(x)\, \widehat{\Phi}(x, y_\beta)\, g(y_\beta)\, dx^\alpha \wedge dy^\gamma, \overrightarrow{\zeta_\beta}(x, y_\beta) \rangle \; d\mathcal{H}^{n-1}(x, y_\beta) \,, \end{aligned}$$

where $\widehat{\Phi}$ is given by (5.4). $\square$

Good projections. We now restrict to the case $m = n$ of our interest. Assume that $T = P$ is an integral polyhedral chain in $\mathcal{P}_{n-1}(\Omega \times \mathbb{R}^N)$. On account of Proposition 5.3, possibly slightly rotating the target space $\mathbb{R}^N$, and denoting without loss of generality by $(y_1, \ldots, y_n)$ the rotated coordinates, we may and do assume that

$$\mathcal{H}^0(\mathcal{M}_\beta \cap \Psi_\beta^{-1}\{(x, y_\beta)\}) = 1 \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } (x, y_\beta) \in \mathcal{N}_\beta \,.$$

This yields that $\mathcal{N}_\beta = \text{set}(P^\beta)$, where $P^\beta := \Psi_{\beta\#} P \in \mathcal{P}_{n-1}(\Omega \times \mathbb{R}^n_\beta)$, see Lemma 5.2. Writing as before $P := \tau(\mathcal{M}, \theta, \overrightarrow{\xi})$, we also may and do choose the orienting unit $(n-1)$-vector field $\overrightarrow{\zeta_\beta}$ is such a way that $\sigma(x, y) \equiv 1$ in the formula (5.4). We thus have $P^\beta = \tau(\mathcal{N}_\beta, \theta_\beta, \overrightarrow{\zeta_\beta})$, where the multiplicity function $\theta_\beta(x, y_\beta) = \theta(x, y)$ for the unique point $(x, y) \in \mathcal{M}_\beta$ such that $\Psi_\beta(x, y) = (x, y_\beta) \in \mathcal{N}_\beta$. Since (5.4) becomes

$$\widehat{\Phi}(x, y_\beta) = \int_{\mathcal{M}_\beta \cap (\psi_\beta^{-1}(\{(x, y_\beta)\})} f(y_{\overline{\beta}})\, \theta_\beta(x, y_\beta)\, d\mathcal{H}^0(x, y) \,,$$

we conclude that (5.3) can be equivalently written as

$$P(\eta_\alpha^\gamma) = P^\beta(\phi(x)\, \widetilde{\Phi}(x, y_\beta)\, g(y_\beta)\, dx^\alpha \wedge dy^\gamma) \,,$$

where we have set

$$\widetilde{\Phi}(x, y_\beta) := \int_{\mathcal{M}_\beta \cap (\psi_\beta^{-1}(\{(x, y_\beta)\})} f(y_{\overline{\beta}})\, d\mathcal{H}^0(x, y) \,. \tag{5.6}$$

Projections of integral currents. We finally show the way to extend the previous features to i.m. rectifiable currents with finite boundary mass.

Proposition 5.6 *Assume $N > n$. Let $T \in \mathcal{R}_{n-1}(\Omega \times \mathbb{R}^N)$ such that $\mathbf{M}((\partial T) \llcorner \Omega \times \mathbb{R}^N) < \infty$. Following the notation from Proposition 5.5, write for $|\beta| = n$*

$$T = \tau(\mathcal{M}, \theta, \overrightarrow{\xi}) \,, \qquad \Psi_{\beta\#} T = \tau(\mathcal{N}_\beta, \theta_\beta, \overrightarrow{\zeta_\beta}) \,.$$

Then, possibly by slightly rotating the target space, for $|\alpha| + |\gamma| = n - 1$, with $\gamma \subset \beta$, we have

$$T(\eta_\alpha^\gamma) = \Psi_{\beta\#} T(\phi(x)\, \widetilde{\Phi}(x, y_\beta)\, g(y_\beta)\, dx^\alpha \wedge dy^\gamma) \,, \tag{5.7}$$

where $\widetilde{\Phi}(x, y_\beta)$ is defined as in (5.6), with $\mathcal{M}_\beta$ given by (5.2).

Proof: By the strong polyhedral approximation theorem 3.5, for each $j \in \mathbb{N}^+$ we find an integral polyhedral chain $P_j \in \mathcal{P}_{n-1}(\Omega \times \mathbb{R}^N)$ and a C^1-diffeomorphism g_j of $\Omega \times \mathbb{R}^N$ onto itself such that $\text{Lip}(g_j) \le 1 + 1/j$, $\text{Lip}(g_j^{-1}) \le 1 + 1/j$, and $\mathbf{M}(g_{j\#} T - P_j) + \mathbf{M}(\partial(g_{j\#} T - P_j) \llcorner \Omega \times \mathbb{R}^N) \le 1/j$.

Denote $\mathcal{M}_j = \text{set}(P_j)$. By applying Proposition 5.3 to the sequence $\{P_j\}_j$, we deduce that for a.e. projection $\mathbf{p} \in \mathbf{O}^*(N, n)$, for each $j \in \mathbb{N}^+$, and for $\mathcal{H}^{n-1}$-a.e. $z \in (\text{Id}_{\mathbb{R}^n} \bowtie \mathbf{p})(\mathcal{M}_j)$

$$N(\text{Id}_{\mathbb{R}^n} \bowtie \mathbf{p} | \mathcal{M}_j, z) := \mathcal{H}^0(\mathcal{M}_j \cap (\text{Id}_{\mathbb{R}^n} \bowtie \mathbf{p})^{-1}(\{z\})) = 1 \,.$$

As a consequence, possibly by slightly rotating the target space, we deduce that for each multi-index β with $|\beta| = n$ the projections Ψ_β are "good" for each P_j in the above sense, i.e.,

$$N(\Psi_\beta | \mathcal{M}_j, z) := \mathcal{H}^0(\mathcal{M}_j \cap {\Psi_\beta}^{-1}(\{z\})) = 1 \tag{5.8}$$

for each $j \in \mathbb{N}^+$ and for $\mathcal{H}^{n-1}$-a.e. $z \in \Psi_\beta(\mathcal{M}_j)$.

Define now $\widetilde{P}_j := f_{j\#} P_j$, where $f_j = g_j^{-1}$, and write $\widetilde{P}_j = \tau(\widetilde{\mathcal{M}}_j, \theta_j, \xi_j)$, where $\widetilde{\mathcal{M}}_j := \text{set}(\widetilde{P}_j)$. Formula (5.8) yields that for each $j \in \mathbb{N}^+$ and for $\mathcal{H}^{n-1}$-a.e. $z \in \Psi_\beta \circ g_j(\widetilde{\mathcal{M}}_j)$

$$N(\Psi_\beta \circ g_j | \widetilde{\mathcal{M}}_j, z) := \mathcal{H}^0(\widetilde{\mathcal{M}}_j \cap (\Psi_\beta \circ g_j)^{-1}(\{z\})) = 1 \,.$$

By applying the general area-coarea formula, Theorem 3.2, we thus infer that

$$\int_{\widetilde{\mathcal{M}}_j} J^{\widetilde{\mathcal{M}}_j}_{\Psi_\beta\circ g_j}(z)\,\widetilde{\theta}_j(z)\,d\mathcal{H}^{n-1} = \mathbf{M}((\Psi_\beta\circ g_j)_{\#}\widetilde{P}_j)\,.$$

By the strong convergence, and again by the area-coarea formula, we also have

$$\lim_{j\to\infty}\mathbf{M}((\Psi_\beta\circ g_j)_{\#}\widetilde{P}_j) = \mathbf{M}(\psi_{\beta\#}T) \le \int_{\mathcal{M}} J^{\mathcal{M}}_{\Psi_\beta}(z)\,\theta(z)\,d\mathcal{H}^{n-1}$$

where, we recall, $T=\tau(\mathcal{M},\theta,\overrightarrow{\xi})$, and we can assume without loss of generality $\mathcal{M}=\mathrm{set}(T)$. Since moreover $\mathbf{M}(\widetilde{P}_j-T)\to 0$, denoting by $\triangle$ the symmetric difference, we also infer that $\mathcal{H}^{n-1}(\widetilde{\mathcal{M}}_j\triangle\mathcal{M})\to 0$ as $j\to\infty$. Using that $\mathrm{Lip}(g_j)\le 1+1/j$ and $\mathrm{Lip}(g_j^{-1})\le 1+1/j$, we thus deduce that

$$\int_{\mathcal{M}} J^{\mathcal{M}}_{\Psi_\beta}(z)\,\theta(z)\,d\mathcal{H}^{n-1} \le \liminf_{j\to\infty}\int_{\widetilde{\mathcal{M}}_j} J^{\widetilde{\mathcal{M}}_j}_{\Psi_\beta\circ g_j}(z)\,\widetilde{\theta}_j(z)\,d\mathcal{H}^{n-1}$$

and definitively that

$$\mathbf{M}(\psi_{\beta\#}T) = \int_{\mathcal{M}} J^{\mathcal{M}}_{\Psi_\beta}(z)\,\theta(z)\,d\mathcal{H}^{n-1}\,.$$

Using again the general area-coarea formula, this yields that for each β

$$N(\Psi_\beta|\mathcal{M},z) := \mathcal{H}^0(\mathcal{M}\cap{\Psi_\beta}^{-1}(\{z\})) = 1$$

for $\mathcal{H}^{n-1}$-a.e. $z\in\Psi_\beta(\mathcal{M})$. This means exactly that each Ψ_β is a "good" projection in the above sense. The claim follows from Proposition 5.5 and from the above argument concerning "good" projections. $\square$

Remark 5.7 For future use, we notice that the function $\widetilde{\Phi}$ in (5.6) is bounded and $\mathcal{H}^{n-1}\llcorner\mathcal{N}_\beta$-summable, hence it can be extended to a bounded Borel function $\widetilde{\Phi}$ on $\Omega\times\mathbb{R}^n_\beta$. Since moreover T^β has finite mass, the action of T^β is uniquely extended to such class of forms $\omega=\phi(x)\,\widetilde{\Phi}(x,y_\beta)\,g(y_\beta)\,dx^\alpha\wedge dy^\gamma$, namely

$$T^\beta(\omega) = \int_{\mathcal{N}_\beta}\theta_\beta\,\langle\omega,\overrightarrow{\zeta_\beta}\rangle\,d\mathcal{H}^{n-1}\,.$$

Finally, (5.7) can be obtained as the limit of linear combinations of terms of the type

$$T(\phi(x)\,g(y_\beta)\,dx^\alpha\wedge dy^\gamma)\,,\qquad \phi\in C^\infty_c(\Omega)\,,\quad g\in C^\infty_c(\mathbb{R}^n_\beta)\,.$$

6 The structure theorem I

In this section we prove the structure theorem 1.3 and its immediate consequences, Corollary 1.4 and Proposition 1.5.

Proof of Theorem 1.3: We first consider the case $n=N$, where we directly apply Proposition 4.1.

The case $N=n$. We follow the notation from Sec. 4, and make use of the following fact:

Lemma 6.1 *Let λ and μ be respectively a non-negative and a signed Radon measure on Ω, with finite total variation, such that for every $x_0\in\Omega$ and for a.e. $r>0$ for which $B_r(x_0)\subset\subset\Omega$ we have*

$$|\mu(\overline{B}_r(x_0)| \le c\,\lambda(\overline{B}_r(x_0))^\alpha$$

for some fixed constants $c>0$ and $\alpha>1$. Then μ is purely atomic, and it is concentrated on the at most countable set of atoms of λ.

PROOF: See [24, Lemma 4.4], and also [21, Lemma 6.3], where a gap in the proof (the absolute continuity of μ with respect to λ) is filled. $\square$

Now, by Proposition 4.1 we obtain the isoperimetric inequality (4.3). We can thus apply Lemma 6.1 with $\alpha = n/(n-1)$, $\mu = \mu_g$, and $\lambda = \lambda(T)$ given by

$$\langle \lambda(T), B\rangle := \mathbf{M}(T \llcorner B \times \widehat{\mathbb{R}}^n)\,, \qquad B \in \mathcal{B}(\Omega)\,.$$

Denoting by $\{a_i\}_i \subset \Omega$ the at most countable family of atoms of $\lambda(T)$, we deduce that for every i there exists a signed Radon measure λ_i on $\widehat{\mathbb{R}}^n$ such that for every $\phi \in C_c^\infty(\Omega)$ and $g \in C_c^\infty(\widehat{\mathbb{R}}^n, \widehat{\mathbb{R}}^n)$

$$T(\phi \wedge \omega_g) = (-1)^{n-1}\langle \mu_g, \phi\rangle = \sum_{i=1}^\infty \delta_{a_i}(\phi)\cdot \lambda_i(g)\,,$$

where $\omega_g \in \mathcal{D}^{n-1}(\widehat{\mathbb{R}}^n)$ is given by (4.1). Also, forms of the type $\phi\wedge\omega_g$ are dense in the space of forms $\eta = \eta^{(n-1)}$ in $\mathcal{D}^{n-1}(\Omega\times\widehat{\mathbb{R}}^n)$, whereas $T(\eta^{(h)}) = 0$ for $h \le n-2$. Therefore, setting $\Sigma_i \in \mathcal{D}_{n-1}(\widehat{\mathbb{R}}^n)$ by

$$\Sigma_i(\omega_g) := \lambda_i(g)\,, \qquad g \in C_c^\infty(\widehat{\mathbb{R}}^n, \widehat{\mathbb{R}}^n)\,,$$

we obtain (1.14). Furthermore, for every $x \in \Omega$ and for all but an at most countable set of "bad" radii $0 < r < \mathrm{dist}(x, \partial\Omega)$ the boundary $\partial B_r(x)$ does not contain atoms of $\lambda(T)$. Hence, by Lemma 6.1, for any "good" radius we have $\langle \mu_g, \partial B_r(x)\rangle = 0$ for every $g \in C_c^\infty(\widehat{\mathbb{R}}^n, \widehat{\mathbb{R}}^n)$. Taking a smooth sequence $\{\phi_j\} \in C_c^\infty(\Omega)$ strongly converging in L^1 to the characteristic function of $\overline{B}_r(x)$, we find that

$$\lim_{j\to\infty} T(\phi_j \wedge \omega_g) = \sum \{\Sigma_i(\omega_g) \mid i \text{ is such that } a_i \in B_r(x)\} \qquad \forall\, g \in C_c^\infty(\widehat{\mathbb{R}}^n, \widehat{\mathbb{R}}^n)\,.$$

Since $T \in \mathcal{R}_{n-1}(\Omega\times\widehat{\mathbb{R}}^n)$ with $(\partial T)\llcorner \Omega\times\widehat{\mathbb{R}}^n = 0$, this yields (1.15), with $\Sigma_i \in \mathcal{R}_{n-1}(\widehat{\mathbb{R}}^n)$.

Remark 6.2 Let $u : \Omega \to \widehat{\mathbb{R}}^n$ be a Sobolev map in $W^{1,n-1}(\Omega, \widehat{\mathbb{R}}^n)$ such that $\det\nabla u \in L^1(\Omega)$, so that $G_u \in \mathcal{R}_n(\Omega\times\widehat{\mathbb{R}}^n)$. If $\mathbf{M}((\partial G_u)\llcorner \Omega\times\widehat{\mathbb{R}}^n) < \infty$, by the boundary rectifiability theorem 3.4, and by Proposition 3.8, we deduce that the boundary current $T := (\partial G_u)\llcorner\Omega\times\widehat{\mathbb{R}}^n$ satisfies the hypotheses of Theorem 1.3.

THE CASE $N > n$. We make use of the projection argument from Sec. 5. Fix a multi-index β of length $|\beta| = n$, consider the projection map Ψ_β given by (5.1), and define $T^\beta := \Psi_{\beta\#}T$. By the assumption, Lemma 5.2 yields that T^β is i.m. rectifiable in $\mathcal{R}_{n-1}(\Omega\times\mathbb{R}^n_\beta)$ and satisfies $(\partial T^\beta)\llcorner\Omega\times\mathbb{R}^n_\beta = 0$. Moreover, it is readily checked that $T^\beta(\eta) = T^\beta(\eta^{(n-1)})$ for every form $\eta \in \mathcal{D}^{n-1}(\Omega\times\mathbb{R}^n_\beta)$. Then, by using the case $n = N$, we deduce the existence of an at most countable subset S_0^β of Ω such that

$$\mathrm{set}(T^\beta) \subset S_0^\beta \times \mathbb{R}^n_\beta\,. \tag{6.1}$$

We now show that

$$\mathrm{set}(T) \subset S_0\times\mathbb{R}^N\,, \quad \text{where} \quad S_0 := \bigcup_{|\beta|=n} S_0^\beta\,. \tag{6.2}$$

To this purpose, possibly by slightly rotating the target space, we may and do apply Proposition 5.6 with $\gamma = \beta - j$ for some $j \in \beta$, so that $|\alpha| = 0$. The current $T^\beta = \tau(\mathcal{N}_\beta, \theta_\beta, \overrightarrow{\zeta_\beta})$ satisfies (6.1), whereas in (5.7) we have just obtained that

$$T(\phi(x)\, f(y_{\overline{\beta}})\, g(y_\beta)\, dy^{\beta-j}) = T^\beta(\phi(x)\,\widetilde{\Phi}(x, y_\beta)\, g(y_\beta) dy^{\beta-j})\,, \tag{6.3}$$

with $\widetilde{\Phi}$ given by (5.6). Moreover, we observe that linear combinations of forms of the type

$$\phi(x)\, f(y_{\overline{\beta}})\, g(y_\beta)\, dy^{\beta-j}\,, \quad \text{where} \quad \phi\in C_c^\infty(\Omega)\,,\ f \in C_c^\infty(\mathbb{R}^{N-n})\,,\ g \in C_c^\infty(\mathbb{R}^n)\,,$$

yield a dense subclass of forms $\eta = \eta^{(n-1)} \in \mathcal{D}^{n-1}(\Omega\times\mathbb{R}^N)$. Therefore, by Remark 5.7, we deduce that (6.2) follows from (6.1). In conclusion, the structure properties (1.14) and (1.15) are obtained by means of the same argument that is used at the end of the case $n = N$. $\square$

Proof of Corollary 1.4: Assume $N = n-1$, otherwise the claim is trivial, and consider the injection map $\mathfrak{i} : \Omega \times \mathbb{R}^{n-1} \to \Omega \times \widehat{\mathbb{R}}^n$ such that $\mathfrak{i}(x,y) := (x,y,0)$. On account of Lemma 3.10, it is readily checked that the current $\widetilde{T} := \mathfrak{i}_\# T$ satisfies the hypotheses of Theorem 1.3. However, the corresponding currents $\Sigma_i \in \mathcal{R}_{n-1}(\widehat{\mathbb{R}}^n)$ in (1.14) are supported in $\mathbb{R}^{n-1} \times \{0\}$ and satisfy $\partial \Sigma_i = 0$. By the Constancy theorem, see [29, 26.27], any integral $(n-1)$-cycle with finite mass in $\mathbb{R}^{n-1} \times \{0\}$ is equal to zero. Therefore, $\Sigma_i = 0$ for all i, hence $\mathfrak{i}_\# T = 0$ and finally $T = 0$. $\square$

Proof of Proposition 1.5: Consider again the injection map $\mathfrak{i} : \Omega \times \mathbb{R}^N \to \Omega \times \mathbb{R}^{N+1}$ given by $\mathfrak{i}(x,y) := (x,y,0)$. Let $B^N_R \subset \mathbb{R}^N$ denote the open ball of radius $R > 0$ centered at the origin. For a.e. $R > 0$, the restriction $T_R := T \llcorner (\Omega \times B^N_R)$ is a compactly supported i.m. rectifiable current such that $\mathbf{M}((\partial T_R) \llcorner \Omega \times \mathbb{R}^N) < \infty$. Then, the image current $\mathfrak{i}_\# T_R$ belongs to $\mathcal{R}_{n-1,c}(\Omega \times \mathbb{R}^{N+1})$, has compact support contained in $\overline{\Omega} \times \mathbb{R}^N \times \{0\}$, and satisfies $\mathbf{M}((\partial\, \mathfrak{i}_\# T_R) \llcorner \Omega \times \mathbb{R}^{N+1}) < \infty$. Therefore, by the boundary rectifiability theorem 3.4, the current $\overline{T}_R := (\partial\, \mathfrak{i}_\# T_R) \llcorner \Omega \times \mathbb{R}^{N+1}$ is i.m. rectifiable in $\mathcal{R}_{n-2,c}(\Omega \times \mathbb{R}^{N+1})$. Moreover, the assumption (1.13) yields that $\mathfrak{i}_\# T_R = \mathfrak{i}_\# T_{R\,(n-1)}$. Consider the affine homotopy map $\widehat{h} : [0,1] \times (\Omega \times \mathbb{R}^{N+1}) \to \Omega \times \mathbb{R}^{N+1}$ given by

$$\widehat{h}(t,x,y,z) := \widehat{h}_t(x,y,z) := (x, ty, t(z-1)+1)\,, \quad t \in [0,1]\,, \quad (x,y) \in \Omega \times \mathbb{R}^N\,, \quad z \in \mathbb{R}\,,$$

and let $\widehat{T}_R := \widehat{h}_\#([\![0,1]\!] \times \overline{T}_R)$, so that $\widehat{T}_R$ is i.m. rectifiable in $\mathcal{R}_{n-1,c}(\Omega \times \mathbb{R}^{N+1})$. At the end of this section we shall prove the following

Lemma 6.3 *The current $\widehat{T}_R$ is completely vertical, $\widehat{T}_R = \widehat{T}_{R\,(n-1)}$.*

Now, by the definition we have $\widehat{h}_{0\#}\overline{T}_R = \partial\, \widehat{h}_{0\#}(\mathfrak{i}_\# T_R)$ on $\mathcal{D}^{n-2}(\Omega \times \mathbb{R}^{N+1})$, whereas $\mathfrak{i}_\# T_R = \mathfrak{i}_\# T_{R\,(n-1)}$. Since $\widehat{h}_0(x,y,z) = (x,0,1)$ and $n \geq 2$, this yields that $\widehat{h}_{0\#}(\mathfrak{i}_\# T_R) = 0$ and hence $\widehat{h}_{0\#}\overline{T}_R = 0$. Therefore, since $(\partial \overline{T}_R) \llcorner \Omega \times \mathbb{R}^{N+1} = 0$, the homotopy formula (3.8) yields

$$(\partial \widehat{T}_R) \llcorner \Omega \times \mathbb{R}^{N+1} = \widehat{h}_{1\,\#}\overline{T}_R - h_{0\,\#}\overline{T}_R = \overline{T}_R =: (\partial\, \mathfrak{i}_\# T_R) \llcorner \Omega \times \mathbb{R}^{N+1}\,.$$

We thus deduce that the current $\Sigma_R := \mathfrak{i}_\# T_R - \widehat{T}_R \in \mathcal{R}_{n-1,c}(\Omega \times \mathbb{R}^{N+1})$ is completely vertical, $\Sigma_R = \Sigma_{R\,(n-1)}$, and it satisfies the null-boundary condition $(\partial \Sigma_R) \llcorner \Omega \times \mathbb{R}^{N+1} = 0$. By Theorem 1.3, we infer that

$$\text{set}(\Sigma_R) \subset S_R \times \mathbb{R}^{N+1} \tag{6.4}$$

for some countable set of points $S_R \subset \Omega$, and by Corollary 1.4 that $S_R = \emptyset$ in case $N+1 < n$.

We now claim that

$$\text{set}(T_R) \subset S_R \times \mathbb{R}^N\,, \qquad T_R := T \llcorner (\Omega \times B^N_R)\,. \tag{6.5}$$

In fact, using that $\widehat{h}_0(x,y,z) = (x,0,1)$, by our construction

$$\text{set}(\mathfrak{i}_\# T_R) \subset \Omega \times \mathbb{R}^N \times \{0\}\,, \qquad \mathcal{H}^{n-1}(\text{set}(\widehat{T}_R) \cap (\Omega \times \mathbb{R}^N \times \{0\})) = 0\,.$$

Denoting by $\triangle$ the symmetric difference, this yields that

$$\mathcal{H}^{n-1}(\text{set}(\mathfrak{i}_\# T_R) \triangle \text{set}(\widehat{T}_R)) = 0\,.$$

Therefore, there is no cancellation in the sum $\Sigma_R := \mathfrak{i}_\# T_R - \widehat{T}_R$, i.e.,

$$\mathbf{M}(\Sigma_R) = \mathbf{M}(\mathfrak{i}_\# T_R) + \mathbf{M}(\widehat{T}_R)\,.$$

Using (6.4), we can thus conclude that $\text{set}(\mathfrak{i}_\# T_R) \subset S_R \times \mathbb{R}^{N+1}$ and definitely that (6.5) holds true.

Since $\text{set}(T_R)$ is increasing with R, and S_R is at most countable, we obtain

$$\text{set}(T) \subset S_0 \times \mathbb{R}^N\,, \qquad S_0 = \cup_j S_{R_j}\,,$$

by choosing an increasing sequence of "good" radii $R_j \to \infty$. Finally, in the case $N < n-1$ we have $S_0 = \emptyset$ and hence $T = 0$. $\square$

Proof of Lemma 6.3: We first observe that since $\overline{T}_R := (\partial\, \mathrm{i}_\# T_R) \llcorner \Omega \times \mathbb{R}^{N+1}$ is an $(n-2)$-current, property $\mathrm{i}_\# T_R = \mathrm{i}_\# T_{R\,(n-1)}$ yields that $\overline{T}_R = \overline{T}_{R\,(n-2)}$, whereas the current $[\![\, 0,1]\!] \times \overline{T}_R$ has compact support. Moreover, for any $\widetilde{\omega} \in \mathcal{D}^{n-1}(\Omega \times \mathbb{R}^{N+1})$

$$\widehat{T}_R(\widetilde{\omega}) = ([\![\, 0,1]\!] \times \overline{T}_R)(\widehat{h}^\# \widetilde{\omega}) \,.$$

Assume that $\widetilde{\omega} = \widetilde{\omega}^{(k)}$, where $1 \le k \le n-2$, and that $\widetilde{\omega} = \eta \wedge \omega$, where $\eta \in \mathcal{D}^{n-1-k}(\Omega)$ and $\omega \in \mathcal{D}^k(\mathbb{R}^{N+1})$. Setting $\widetilde{y} = (y,z) \in \mathbb{R}^{N+1}$ for simplicity, we can decompose the pull-back of $\widetilde{\omega}$ as

$$\widehat{h}^\# \widetilde{\omega} = \eta(x) \wedge (\Phi(\widetilde{y},t) \wedge dt + \Psi(\widetilde{y},t)) \,,$$

where the forms $\Phi(\cdot,t) \in \mathcal{E}^{k-1}(\mathbb{R}^{N+1})$ and $\Psi(\cdot,t) \in \mathcal{E}^k_b(\mathbb{R}^{N+1})$ for every $t \in (0,1)$. We have

$$([\![\, 0,1]\!] \times \overline{T}_R)(\eta(x) \wedge \Psi(\widetilde{y},t)) = 0 \,,$$

as $\eta \wedge \Psi(\widetilde{y},t)$ does not contain the differential dt, whereas

$$([\![\, 0,1]\!] \times \overline{T}_R)(\eta(x) \wedge \Phi(\widetilde{y},t) \wedge dt) = \overline{T}_R(\eta(x) \wedge \widetilde{\Phi}(\widetilde{y}))$$

for some $(k-1)$-form $\widetilde{\Phi} \in \mathcal{E}^{k-1}(\widehat{\mathbb{R}}^n)$. Since $k \le n-2$, property $\overline{T}_R = (\overline{T}_R)_{(n-2)}$ gives $\overline{T}_R(\eta(x) \wedge \widetilde{\Phi}(\widetilde{y})) = 0$. The case $k=0$ being trivial, the assertion follows by linearity and density. $\square$

7 The structure theorem II

In this section we prove the more general structure theorem 1.6.

Proof of Theorem 1.6: Arguing by induction on $\mathbf{p} \in \mathbb{N}$, at the $\mathbf{p}^{th}$ step we shall prove the claim in Theorem 1.6 in the case $k = n-1-\mathbf{p}$, for any choice of the dimensions n, N. To this purpose, we first observe that for $\mathbf{p} = 0$ the claim has been proved in Theorem 1.3.

We thus fix $\mathbf{p}$ a positive integer, and at the $\mathbf{p}^{th}$ step we assume that we have proved the claim in any dimensions n, N for $k = n-1-\nu$ and for each natural $\nu = 0, 1, \ldots, \mathbf{p}-1$.

Let $T \in \mathcal{R}_{n-1}(\Omega \times \mathbb{R}^N)$ be an i.m. rectifiable current satisfying the null-boundary condition (1.12) and property $T_{(h)} = 0$ for $h = 0, \ldots, k-1$, where $k = n-1-\mathbf{p}$, $\underline{n} := \min\{n-1, N\} \ge 2$, and $0 < k < \underline{n}$. Assume in addition that the support $\operatorname{spt} T \subset \overline{A} \times \mathbb{R}^N$ for some subdomain A, with closure $\overline{A} \subset\subset \Omega$. Since $n-1-k = \mathbf{p}$, we have show the existence of a countably $\mathcal{H}^{\mathbf{p}}$-rectifiable subset $S_{\mathbf{p}}$ of $\overline{A}$ such that

$$\operatorname{set}(T) \subset S_{\mathbf{p}} \times \mathbb{R}^N \,.$$

Every form $\eta \in \mathcal{D}^{n-1}(\Omega \times \mathbb{R}^N)$ decomposes as $\eta = \sum_{m=0}^{\underline{n}} \eta^{(m)}$, where

$$\eta^{(m)} = \sum_{|\alpha| = n-1-m} \eta_\alpha \,, \qquad \eta_\alpha := \sum_{|\beta| = m} \eta^{\alpha,\beta}(x,y)\, dy^\beta \wedge dx^\alpha$$

for some $\eta^{\alpha,\beta} \in C^\infty_c(\Omega \times \mathbb{R}^N)$. By the assumption, we have $T(\eta^{(m)}) = 0$ for $m < k$. For $m = k, \ldots, \underline{n}$, we now analyze the action of T on the component $\eta^{(m)}$. We shall make use of arguments from *slicing theory* of i.m. rectifiable currents, see e.g. [19, Vol. I, Sec. 2.2.5] or [29, Sec. 28].

We first consider the case $m < n-1$. Denote by $\pi_\alpha : \mathbb{R}^n \to \mathbb{R}^{n-1-m}$ the orthogonal projection onto the α-components of x, i.e., $\pi_\alpha(x) = x_\alpha$, and by Ω_{x_α} the $(m+1)$-dimensional section of Ω with the $(m+1)$-plane $\pi_\alpha^{-1}(x_\alpha)$. For $\mathcal{H}^{n-1-m}$-a.e. $x_\alpha \in \mathbb{R}^{n-1-m}$ such that $\Omega_{x_\alpha} \neq \emptyset$, we define the sliced current

$$T_{x_\alpha} := \langle T, \pi_\alpha \bowtie \mathrm{Id}_{\mathbb{R}^N}, x_\alpha \rangle \,.$$

Remark 7.1 As before, recall that linear combinations of forms with coefficients of the type $\eta^{\alpha,\beta}(x,y) = \phi(x)\,\psi(y)$, where $\phi \in C^\infty_c(\Omega)$ and $\psi \in C^\infty_c(\mathbb{R}^N)$, yield a dense sub-class of smooth forms. Since moreover $\operatorname{spt} T \subset \overline{A} \times \mathbb{R}^N$ and $A \subset\subset \Omega$, possibly by enlarging the domain Ω, we deduce that $\varphi(x)$ is the strong limit of linear combination of functions in $C^\infty_c(\Omega)$ that agrees with the product $\varphi(x_{\overline{\alpha}})\,\widetilde{\varphi}(x_\alpha)$ on $\overline{A}$, for some $\varphi \in C^\infty(\mathbb{R}^{m+1})$ and $\widetilde{\varphi} \in C^\infty(\mathbb{R}^{n-m-1})$. In particular, $\varphi \in C^\infty_c(\Omega_{x_\alpha})$ for each x_α as above.

By assumption, and using that the slicing map is an orthogonal projection only involving the "horizontal" coordinates x, we then deduce, for $\mathcal{H}^{n-1-m}$-a.e. $x_\alpha \in \mathbb{R}^{n-1-m}$:

(1) T_{x_α} belongs to $\mathcal{R}_m(\Omega_{x_\alpha} \times \mathbb{R}^N)$;

(2) the boundary of the slice agrees (up to the sign) with the slice of the boundary, hence

$$(\partial T_{x_\alpha}) \llcorner \Omega_{x_\alpha} \times \mathbb{R}^N = (-1)^{n-1-m} \langle (\partial T) \llcorner \Omega \times \mathbb{R}^N, \pi_\alpha \bowtie \mathrm{Id}_{\mathbb{R}^N}, x_\alpha \rangle = 0\,;$$

(3) if $\phi \in C_c^\infty(\Omega)$ agrees on the closure of A with $\varphi(x_{\overline{\alpha}})\,\widetilde{\varphi}(x_\alpha)$, see Remark 7.1, for any $\psi \in C_c^\infty(\mathbb{R}^N)$

$$T(\varphi(x_{\overline{\alpha}})\,\widetilde{\varphi}(x_\alpha)\,\psi(y)\,dy^\beta \wedge dx^\alpha) = \int_{\mathbb{R}^{n-m-1}} \Big(T_{x_\alpha}(\varphi(x_{\overline{\alpha}})\,\psi(y)\,dy^\beta) \Big)\,\widetilde{\varphi}(x_\alpha)\,dx_\alpha\,; \tag{7.1}$$

(4) $T_{x_\alpha}(\eta^{(h)}) = 0$ for every $h < k$ and $\eta \in \mathcal{D}^m(\Omega_{x_\alpha} \times \mathbb{R}^N)$.

This yields that the sliced current T_{x_α} satisfies the hypothesis of Theorem 1.6, with the dimension n replaced by $m+1$, and hence with

$$k = n - 1 - \mathbf{p} = m - \nu\,, \qquad \nu := \mathbf{p} - (m - n - 1)\,.$$

By the assumption $0 < m - n - 1 \le \mathbf{p}$, hence $0 \le \nu < \mathbf{p}$. Therefore, by the inductive hypothesis, we find the existence of a countably $\mathcal{H}^{\mathbf{p}-(m-n-1)}$-rectifiable subset $S_{\mathbf{p}-(m-n-1)}$ of $\overline{A} \cap \Omega_{x_\alpha}$ such that

$$\mathrm{set}(T_{x_\alpha}) \subset S_{\mathbf{p}-(m-n-1)} \times \mathbb{R}^N\,.$$

Using the slicing formula (7.1), we deduce that the claim is proved if $N < n-1$. In this case, in fact, there are no forms $\eta \in \mathcal{D}^{n-1}(\Omega \times \mathbb{R}^N)$ with non-zero vertical components $\eta^{(n-1)}$.

Therefore, it remains to consider the action of T on forms of the type $\eta = \eta^{(n-1)}$. We distinguish among the cases $N = n-1$, $N = n$, and $N > n$.

The case $N = n-1$. We have $\eta^{(n-1)} = \phi(x,y)\,dy$ for some $\phi \in C_c^\infty(\Omega \times \mathbb{R}^{n-1})$. By a density argument, we may and do assume $\phi(x,y) = \varphi(x)\,f(y_1)\,g(\widehat{y_1})$, where $\varphi \in C_c^\infty(\Omega)$, $f \in C_c^\infty(\mathbb{R})$, and $g \in C_c^\infty(\mathbb{R}^{n-2})$. We thus denote by F a primitive of f, and set

$$\xi := \varphi(x)\,F(y_1)\,g(\widehat{y_1})\,\widehat{dy^1} \in \mathcal{E}_b^{n-1}(\Omega \times \mathbb{R}^{n-1})\,.$$

Using the usual convention of summation on the repeated indices, we clearly have

$$d\xi = \varphi_{,x_i}(x)\,F(y_1)\,g(\widehat{y_1})\,dx^i \wedge \widehat{dy^1} + \varphi(x)\,f(y_1)\,g(\widehat{y_1})\,dy\,.$$

Since ξ has support contained in the cylinder $\Omega \times \mathbb{R}^N$ and bounded Lipschitz coefficients, the null-boundary condition (1.12) and Lemma 3.10 yield that $T(d\xi) = 0$, hence

$$T(\varphi(x)\,f(y_1)\,g(\widehat{y_1})\,dy) = -T(\varphi_{,x_i}(x)\,F(y_1)\,g(\widehat{y_1})\,dx^i \wedge \widehat{dy^1})\,.$$

Therefore, the argument that we used for the component $\eta^{(n-2)}$, applied this time to the $(n-1)$-form $\varphi_{,x_i}(x)\,F(y_1)\,g(\widehat{y_1})\,dx^i \wedge \widehat{dy^1}$, yields the assertion, thanks to the dominated convergence theorem.

The case $N = n$. Fix $j \in \{1, \ldots, n\}$. For $t_1 < t_2$, denote

$$\{t_1 < y_j < t_2\} := \{(x,y) \in \Omega \times \widehat{\mathbb{R}}^n \mid t_1 < y_j < t_2\}\,.$$

By slicing theory, for a.e. choice of $t_1 < t_2$ it turns out that the current $T \llcorner \{t_1 < y_j < t_2\}$ is i.m. rectifiable and with boundary of finite mass. Write as usual

$$T \llcorner \{t_1 < y_j < t_2\} = \tau(\mathcal{M}, \theta, \overrightarrow{\xi}\,)\,, \qquad \mathcal{M} = \mathrm{set}(T \llcorner \{t_1 < y_j < t_2\})\,.$$

Proposition 7.2 *For a.e. real numbers* $t_1 < t_2$ *there exists an* $(n-1)$*-rectifiable set* $\widetilde{\mathcal{M}} \subset \Omega \times \widehat{\mathbb{R}}^n$, *with* $\widetilde{\mathcal{M}} \subset \mathrm{set}(T \llcorner \{t_1 < y_j < t_2\})$, *and a countably* $\mathcal{H}^{\mathbf{p}}$*-rectifiable subset* $S_{\mathbf{p}}$ *of* $\overline{A}$ *satisfying*

$$\widetilde{\mathcal{M}} \subset S_{\mathbf{p}} \times \widehat{\mathbb{R}}^n$$

such that for every $(n-1)$*-form* ω *of the type* $\omega := \phi(x, \widehat{y}_j)\, \widehat{dy^j}$, *where* $\phi \in C_c^\infty(\Omega \times \mathbb{R}^{n-1}_{\widehat{y}_j})$, *we have*

$$T \llcorner \{t_1 < y_j < t_2\}(\omega) = \int_{\widetilde{\mathcal{M}}} \langle \omega, \overrightarrow{\xi} \rangle \, \theta \, d\mathcal{H}^{n-1} \,.$$

PROOF: By slicing theory, for a.e. radius $R > 0$ the i.m. rectifiable current

$$T^{j,R} := T \llcorner \{(x,y) \in \Omega \times \widehat{\mathbb{R}}^n \mid |y_h| < R \ \text{ for any } \ h \neq j\}$$

satisfies $\mathbf{M}((\partial T^{j,R}) \llcorner \Omega \times \widehat{\mathbb{R}}^n) < \infty$. Moreover, for any such "good" radius R it turns out that the current

$$T^{j,R}_{s_1,s_2} := T \llcorner \{(x,y) \in \Omega \times \widehat{\mathbb{R}}^n \mid s_1 < y_j < s_2, \ |y_h| < R \ \text{ for any } \ h \neq j\}$$

satisfies $M((\partial T^{j,R}_{s_1,s_2}) \llcorner \Omega \times \widehat{\mathbb{R}}^n) < \infty$ for a.e. $s_1 < s_2$. This yields that for a.e. $t_1 < t_2$ we can find an increasing sequence of good radii $R_h \to \infty$ such that the compactly supported i.m. rectifiable current $T^{j,R_h}_{t_1,t_2} \in \mathcal{R}_{n-1,c}(\Omega \times \widehat{\mathbb{R}}^n)$ satisfies $M((\partial T^{j,R_h}_{t_1,t_2}) \llcorner \Omega \times \widehat{\mathbb{R}}^n) < \infty$ for each h.

Consider the affine homotopy map $h^{j,R_h} : (\Omega \times \widehat{\mathbb{R}}^n) \times [0,1] \to \Omega \times \widehat{\mathbb{R}}^n$

$$h^{j,R_h}(x,y,t) := t\,(x,y) + (1-t)\, f^{j,R_h}(x,y)\,,$$

where $f^{j,R_h}(x,y) := (x, R_h+1, \ldots, R_h+1, y_j, R_h+1, \ldots, R_h+1)$. The current $h^{j,R_h}_{\#}\big(T^{j,R_h}_{t_1,t_2} \times [\![0,1]\!]\big)$ is compactly supported and i.m. rectifiable in $\mathcal{R}_{n,c}(\Omega \times \widehat{\mathbb{R}}^n)$. Similarly, both the currents

$$\begin{array}{rcl} S^{j,R_h}_{t_1,t_2} & := & (-1)^n h^{j,R_h}_{\#}\big(\partial T^{j,R_h}_{t_1,t_2} \times [\![0,1]\!]\big) - f^{j,R_h}_{\#}(T^{j,R_h}_{t_1,t_2})\,, \\ \widetilde{T}^{j,R_h}_{t_1,t_2} & := & T^{j,R_h}_{t_1,t_2} + S^{j,R_h}_{t_1,t_2} \end{array} \tag{7.2}$$

are compactly supported and i.m. rectifiable in $\mathcal{R}_{n-1,c}(\Omega \times \widehat{\mathbb{R}}^n)$. Moreover, by the homotopy formula (3.8) it turns out that $(\partial \widetilde{T}^{j,R_h}_{t_1,t_2}) \llcorner \Omega \times \widehat{\mathbb{R}}^n = 0$.

We claim that

$$\mathcal{H}^{n-1}\big(\mathrm{set}(T^{j,R_h}_{t_1,t_2}) \triangle \,\mathrm{set}(S^{j,R_h}_{t_1,t_2})\big) = 0\,. \tag{7.3}$$

In fact, $\mathrm{set}(f^{j,R_h}_{\#}(T^{j,R_h}_{t_1,t_2}))$ is contained in $\{(x,y) \mid y_h = R_h + 1 \ \text{ for any } \ h \neq j\}$, hence it is disjoint with $\mathrm{set}(T^{j,R_h}_{t_1,t_2})$. Since moreover $M((\partial T^{j,R_h}_{t_1,t_2}) \llcorner \Omega \times \widehat{\mathbb{R}}^n) < \infty$, by our construction we also get

$$\mathcal{H}^{n-1}\big(\mathrm{set}(T^{j,R_h}_{t_1,t_2}) \triangle \,\mathrm{set}\big(h^{j,R_h}_{\#}\big(\partial T^{j,R_h}_{t_1,t_2} \times [\![0,1]\!]\big)\big)\big) = 0\,.$$

By (7.3) we infer that there is no cancellation in the sum in the second line of the definition (7.2), i.e.,

$$\mathbf{M}(\widetilde{T}^{j,R_h}_{t_1,t_2}) = \mathbf{M}\big(T^{j,R_h}_{t_1,t_2}\big) + \mathbf{M}\big(S^{j,R_h}_{t_1,t_2}\big)\,.$$

Therefore, writing as usual

$$T^{j,R_h}_{t_1,t_2} = \tau(\mathcal{M}_h, \theta, \overrightarrow{\xi}\,)\,, \qquad \widetilde{T}^{j,R_h}_{t_1,t_2} = \tau(\mathcal{N}_h, \widetilde{\theta}, \overrightarrow{\zeta}\,)\,,$$

and assuming without loss of generality that $\theta \neq 0$ on $\mathcal{M}_h$ and $\widetilde{\theta} \neq 0$ on $\mathcal{N}_h$, this yields that

$$\mathcal{H}^{n-1}(\mathcal{N}_h) = \mathcal{H}^{n-1}(\mathcal{M}_h) + \mathcal{H}^{n-1}(\mathcal{N}_h \setminus \mathcal{M}_h)\,. \tag{7.4}$$

If e.g. $j \neq 1$, setting $\widetilde{y} := y_{\overline{(1,j)}}$, by a density argument we may and do choose $\phi(x, \widehat{y}_j) = \varphi(x)\, f(y_1)\, g(\widetilde{y})$, where $\varphi \in C_c^\infty(\Omega)$, $f \in C_c^\infty(\mathbb{R})$, and $g \in C_c^\infty(\mathbb{R}^{n-2})$. Denote by F a primitive of f, and let

$$\xi := \varphi(x)\, F(y_1)\, g(\widetilde{y})\, d\widetilde{y}\,, \qquad d\widetilde{y} := dy^{\overline{(1,j)}}\,,$$

so that $\xi \in \mathcal{E}_b^{n-2}(\Omega \times \mathbb{R}^{n-1}_{\widehat{y}_j})$ satisfies $d\xi = \omega + \widetilde{\omega}$, where

$$\omega := \varphi(x)\, f(y_1)\, g(\widetilde{y})\, \widehat{dy^j}\,, \quad \widetilde{\omega} := \varphi_{,x_i}(x)\, F(y_1)\, g(\widetilde{y})\, dx^i \wedge d\widetilde{y}\,.$$

The null-boundary condition $(\partial \widetilde{T}^{j,R_h}_{t_1,t_2}) \llcorner \Omega \times \widehat{\mathbb{R}}^n$ yields $\widetilde{T}^{j,R_h}_{t_1,t_2}(d\xi) = 0$, whence $\widetilde{T}^{j,R_h}_{t_1,t_2}(\omega) = -\widetilde{T}^{j,R_h}_{t_1,t_2}(\widetilde{\omega})$.

Now, denote

$$\begin{aligned} \overrightarrow{\xi}(z) &= \sum_{|\alpha|+|\beta|=n-1} \xi^{\alpha,\beta}(z)\, e_\alpha \wedge \varepsilon_\beta\,, \qquad z \in \mathcal{M}_h \\ \overrightarrow{\zeta}(z) &= \sum_{|\alpha|+|\beta|=n-1} \zeta^{\alpha,\beta}(z)\, e_\alpha \wedge \varepsilon_\beta\,, \qquad z \in \mathcal{N}_h \end{aligned}$$

and correspondingly define

$$\begin{aligned} \widetilde{\mathcal{M}}_h &:= \mathcal{M}_h \setminus \{ z \in \mathcal{M}_h \mid \xi^{\alpha,\beta}(z) = 0 \quad \text{for each } \alpha \text{ and } \beta \text{ s.t. } \beta = \overline{j} \text{ or } \beta = \overline{(1,j)} \ \} \\ \widetilde{\mathcal{N}}_h &:= \mathcal{N}_h \setminus \{ z \in \mathcal{N}_h \mid \zeta^{\alpha,\beta}(z) = 0 \quad \text{for each } \alpha \text{ and } \beta \text{ s.t. } \beta = \overline{j} \text{ or } \beta = \overline{(1,j)} \ \}\,. \end{aligned}$$

On account or Remark 3.1, the set $\widetilde{\mathcal{N}}_h$ is $(n-1)$-rectifiable and moreover

$$\widetilde{T}^{j,R_h}_{t_1,t_2}(\omega) = \int_{\widetilde{\mathcal{N}}_h} \langle \omega, \overrightarrow{\zeta} \rangle\, \widetilde{\theta}\, d\mathcal{H}^{n-1}\,, \quad \widetilde{T}^{j,R_h}_{t_1,t_2}(\widetilde{\omega}) = \int_{\widetilde{\mathcal{N}}_h} \langle \widetilde{\omega}, \overrightarrow{\zeta} \rangle\, \widetilde{\theta}\, d\mathcal{H}^{n-1}\,.$$

Since $\widetilde{\omega}$ "contains" the differentials dx^i, by applying to the term $\widetilde{T}^{j,R_h}_{t_1,t_2}(\widetilde{\omega})$ the slicing argument that we used for the component $\eta^{(n-2)}$, we thus deduce the existence of a countably $\mathcal{H}^{\mathbf{p}}$-rectifiable subset $S^h_{\mathbf{p}}$ of $\overline{A}$ such that $\widetilde{\mathcal{N}}_h \subset S^h_{\mathbf{p}} \times \widehat{\mathbb{R}}^n$. Since moreover the property (7.4) yields

$$\mathcal{H}^{n-1}(\widetilde{\mathcal{N}}_h) = \mathcal{H}^{n-1}(\widetilde{\mathcal{M}}_h) + \mathcal{H}^{n-1}(\widetilde{\mathcal{N}}_h \setminus \widetilde{\mathcal{M}}_h)\,,$$

we also obtain that $\widetilde{\mathcal{M}}_h \subset S^h_{\mathbf{p}} \times \widehat{\mathbb{R}}^n$.

Finally, since $T \llcorner \{t_1 < y_j < t_2\}$ has finite mass, we deduce that $T^{j,R_h}_{t_1,t_2} \rightharpoonup T \llcorner \{t_1 < y_j < t_2\}$ weakly in $\mathcal{D}^{n-1}(\Omega \times \widehat{\mathbb{R}}^n)$ as $h \to \infty$. Therefore, the claim follows by taking $\widetilde{\mathcal{M}} = \cup_h \widetilde{\mathcal{M}}_h$ and $S_{\mathbf{p}} := \cup_h S^h_{\mathbf{p}}$. $\square$

Now, if $n = N$, any completely vertical $(n-1)$-form in $\mathcal{D}^{n-1}(\Omega \times \widehat{\mathbb{R}}^n)$ can be written as

$$\eta^{(n-1)} = \sum_{j=1}^n \psi_j(x,y)\, \widehat{dy^j}\,, \quad \psi_j \in C^\infty_c(\Omega \times \widehat{\mathbb{R}}^n)\,.$$

Fix $j \in \{1, \dots, n\}$. By a density argument, we may and do assume that $\psi_j(x,y) = \phi(x, \widehat{y}_j)\, f(y_j)$ for some $\phi \in C^\infty_c(\Omega \times \mathbb{R}^{n-1}_{\widehat{y}_j})$ and $f \in C^\infty_c(\mathbb{R})$.

For $\nu \in \mathbb{N}$ and $h \in \mathbb{Z}$, denote $t^\nu_h := h\, 2^{-\nu}$. Possibly by slightly moving the points t^ν_h, we may and do assume that for each ν and h we can apply Proposition 7.2 to the restricted current $T \llcorner \{t^\nu_h < y_j < t^\nu_{h+1}\}$. Writing $T = \tau(\mathcal{M}, \theta, \overrightarrow{\xi})$, we then find an $(n-1)$-rectifiable set $\widetilde{\mathcal{M}}^\nu_h \subset \Omega \times \widehat{\mathbb{R}}^n$, with $\widetilde{\mathcal{M}}^\nu_h \subset \mathcal{M}$, and a countably $\mathcal{H}^{\mathbf{p}}$-rectifiable subset $S_{\mathbf{p}}(\nu, h)$ of $\overline{A}$ satisfying

$$\widetilde{\mathcal{M}}^\nu_h \subset S_{\mathbf{p}}(\nu, h) \times \widehat{\mathbb{R}}^n$$

and such that (the sliced current being with finite mass) for every $\phi \in C^\infty_c(\Omega \times \mathbb{R}^{n-1}_{\widehat{y}_j})$

$$T \llcorner \{t^\nu_h < y_j < t^\nu_{h+1}\}(\phi(x, \widehat{y}_j)\, \widehat{dy^j}) = \int_{\widetilde{\mathcal{M}}^\nu_h} \langle \phi(x, \widehat{y}_j)\, \widehat{dy^j}, \overrightarrow{\xi} \rangle\, \theta\, d\mathcal{H}^{n-1}\,.$$

Since moreover $f \in C^\infty_c(\mathbb{R})$, there exists a sequence $\{f_\nu\}_\nu$ of piecewise constant and bounded functions $f_\nu : \mathbb{R} \to \mathbb{R}$ satisfying:

i) f_ν is constant on $I^\nu_h :=]t^\nu_h, t^\nu_{h+1}[$ for each h;

ii) f_ν has compact support contained in the support of f;

iii) $f_\nu \to f$ uniformly as $\nu \to \infty$.

As a consequence, using that $T = \tau(\mathcal{M}, \theta, \overrightarrow{\xi})$ is i.m. rectifiable, we have

$$T(f(y_j)\,\phi(x,\widehat{y}_j)\,\widehat{dy^j}) = \lim_{\nu\to\infty} T(f_\nu(y_j)\,\phi(x,\widehat{y}_j)\,\widehat{dy^j})\,. \tag{7.5}$$

Also, using that $f_\nu(y_j) \equiv a_h^\nu \in \mathbb{R}$ for each $y_j \in I_h^\nu$ and for each h, we have

$$T(f_\nu(y_j)\,\phi(x,\widehat{y}_j)\,\widehat{dy^j}) = \sum_h a_h^\nu \cdot T \llcorner \{t_h^\nu < y_j < t_{h+1}^\nu\}(\phi(x,\widehat{y}_j)\,\widehat{dy^j})\,,$$

where the sum in h is finite for each f_ν. Setting $\widetilde{\mathcal{M}}^\nu := \bigcup_h \widetilde{\mathcal{M}}_h^\nu$ and $S_{\mathbf{p}}(\nu) := \bigcup_h S_{\mathbf{p}}(\nu, h)$, it turns out that $\widetilde{\mathcal{M}}^\nu$ is an $(n-1)$-rectifiable subset of $\mathcal{M}$, and $S_{\mathbf{p}}(\nu)$ a countably $\mathcal{H}^{\mathbf{p}}$-rectifiable subset of $\overline{A}$ satisfying $\widetilde{\mathcal{M}}^\nu \subset S_{\mathbf{p}}(\nu) \times \widehat{\mathbb{R}}^n$ and such that

$$T(f_\nu(y_j)\,\phi(x,\widehat{y}_j)\,\widehat{dy^j}) = \int_{\widetilde{\mathcal{M}}^\nu} \langle f_\nu(y_j)\,\phi(x,\widehat{y}_j)\,\widehat{dy^j}, \overrightarrow{\xi}\rangle\,\theta\,d\mathcal{H}^{n-1}\,.$$

Therefore, setting $\widetilde{\mathcal{M}}^{(j)} := \bigcup_\nu \widetilde{\mathcal{M}}^\nu$ and $S_{\mathbf{p}}^j := \bigcup_\nu S_{\mathbf{p}}(\nu)$, again $\widetilde{\mathcal{M}}^{(j)}$ is an $(n-1)$-rectifiable subset of $\mathcal{M}$, and $S_{\mathbf{p}}^j$ a countably $\mathcal{H}^{\mathbf{p}}$-rectifiable subset of $\overline{A}$ satisfying $\widetilde{\mathcal{M}}^{(j)} \subset S_{\mathbf{p}}^j \times \widehat{\mathbb{R}}^n$ and such that

$$T(f_\nu(y_j)\,\phi(x,\widehat{y}_j)\,\widehat{dy^j}) = \int_{\widetilde{\mathcal{M}}^{(j)}} \langle f_\nu(y_j)\,\phi(x,\widehat{y}_j)\,\widehat{dy^j}, \overrightarrow{\xi}\rangle\,\theta\,d\mathcal{H}^{n-1} \qquad \forall\,\nu \in \mathbb{N}\,.$$

By (7.5), we thus obtain

$$T(f(y_j)\,\phi(x,\widehat{y}_j)\,\widehat{dy^j}) = \int_{\widetilde{\mathcal{M}}^{(j)}} \langle f(y_j)\,\phi(x,\widehat{y}_j)\,\widehat{dy^j}, \overrightarrow{\xi}\rangle\,\theta\,d\mathcal{H}^{n-1}\,.$$

By linearity and density, letting $\widetilde{\mathcal{M}} = \cup_j \widetilde{\mathcal{M}}^{(j)}$ and $S_{\mathbf{p}} := \bigcup_j S_{\mathbf{p}}^j$, we have just shown that

$$T(\eta^{(n-1)}) = \int_{\widetilde{\mathcal{M}}} \langle \eta^{(n-1)}, \overrightarrow{\xi}\rangle\,\theta\,d\mathcal{H}^{n-1}$$

where $\widetilde{\mathcal{M}}$ is an $(n-1)$-rectifiable subset of $\mathcal{M}$, and $S_{\mathbf{p}}$ a countably $\mathcal{H}^{\mathbf{p}}$-rectifiable subset of $\overline{A}$ satisfying $\widetilde{\mathcal{M}} \subset S_{\mathbf{p}} \times \widehat{\mathbb{R}}^n$. This concludes the proof in the case $N = n$.

The case $N > n$. Exactly as for the case $N > n$ in the proof of Theorem 1.3, we make use of the projection argument from Sec. 5. We thus fix a multi-index β of length $|\beta| = n$, consider the projection map Ψ_β given by (5.1), and on account of Lemma 5.2 define

$$T^\beta := \Psi_{\beta\#} T \in \mathcal{R}_{n-1}(\Omega \times \mathbb{R}^n_\beta)\,.$$

By the assumption, we deduce that T^β satisfies the hypotheses of Theorem 1.6, with $k = n - 1 - \mathbf{p}$. Then, by the case $n = N$, we find a countably $\mathcal{H}^{\mathbf{p}}$-rectifiable subset $S_{\mathbf{p}}^\beta$ of $\overline{A}$ such that

$$\mathrm{set}(T^\beta) \subset S_{\mathbf{p}}^\beta \times \mathbb{R}^n_\beta\,, \qquad \mathbb{R}^n_\beta \subset \mathbb{R}^N\,. \tag{7.6}$$

It then remains to show that

$$\mathrm{set}(T) \subset S_{\mathbf{p}} \times \mathbb{R}^N\,, \quad \text{where} \quad S_{\mathbf{p}} := \bigcup_{|\beta|=n} S_{\mathbf{p}}^\beta\,. \tag{7.7}$$

To this purpose, we again apply Proposition 5.6. The current $T^\beta = \tau(\mathcal{N}_\beta, \theta_\beta, \overrightarrow{\zeta_\beta})$ satisfies (7.6), whereas (5.7) holds true, with $\widetilde{\Phi}$ given by (5.6). By Remark 5.7, we conclude that (7.7) follows from (7.6), as required. $\square$

We finally point out that on account of Proposition 3.8, we can apply Theorem 1.6 to the boundary current $T := (\partial G_u) \llcorner \Omega \times \mathbb{R}^N$ for any Sobolev map $u \in W^{1,k}(\Omega, \mathbb{R}^N)$ satisfying the hypotheses of our decomposition theorem 1.2.

8 Proof of the Decomposition Theorem

In this section we shall prove the decomposition theorem 1.2 and Corollary 1.7. In order to discuss some properties of the singular part of the distributional determinant and minors, see Sec. 11 below, by slightly modifying the proof of Theorem 1.2 we shall also prove the following

Corollary 8.1 *Under the hypotheses of Theorem* 1.2*, assume in addition that* $(\partial G_u)_{(p)} \llcorner \Omega \times \mathbb{R}^N = 0$ *for some integer* p*. Then we can choose the components* T_k *in such a way that* $T_p = 0$ *and* $(T_k)_{(h)} = 0$ *for every* $k < p$ *and* $h \geq p$*.*

To this purpose, recall that the condition $(\partial G_u)_{(p)} \llcorner \Omega \times \mathbb{R}^N = 0$ is automatically satisfied by Sobolev maps u in $W^{1,p+1}(\Omega, \mathbb{R}^N)$, compare Proposition 3.8.

In order to apply the structure theorems 1.3 and 1.6, the components T_k will be defined by "filling the holes" of suitably defined vertical components of the boundary current ∂G_u. This will be done by choosing solutions to a related minimum problem that we now illustrate.

A MINIMUM PROBLEM. If u satisfies the hypotheses of our decomposition theorem 1.2, by the boundary rectifiability theorem 3.4, the boundary $(\partial G_u) \llcorner \Omega \times \mathbb{R}^N$ is i.m. rectifiable in $\mathcal{R}_{n-1}(\Omega \times \mathbb{R}^N)$. Write $(\partial G_u) \llcorner \Omega \times \mathbb{R}^N = \tau(\partial \mathcal{G}_u, \theta, \overrightarrow{\xi})$, where we assume that $\theta \neq 0$ on $\partial \mathcal{G}_u$, and recall that $\xi_{(h)}$ denotes the component of the $(n-1)$-vector $\overrightarrow{\xi}$ corresponding to $(n-1)$-vectors $e_\alpha \wedge \varepsilon_\beta$, for any α and β with $|\beta| = h$ and $|\alpha| = n-1-h$, see (3.7). For $k = 1, \ldots, \underline{n}$, where $\underline{n} = \min\{n-1, N\}$, define

$$(\partial G_u)^v_k := \tau((\partial \mathcal{G}_u)^v_k, \theta, \overrightarrow{\xi}), \quad \text{where} \quad (\partial \mathcal{G}_u)^v_k := \{x \in \partial \mathcal{G}_u \mid \xi_{(h)} = 0 \quad \forall\, h < k\}\,. \tag{8.1}$$

Since the set $(\partial \mathcal{G}_u)^v_k$ is $(n-1)$-rectifiable, see Remark 3.1, we deduce that the current $(\partial G_u)^v_k$ is i.m. rectifiable in $\mathcal{R}_{n-1}(\Omega \times \mathbb{R}^N)$, with finite mass

$$\mathbf{M}((\partial G_u)^v_k) \leq \mathbf{M}((\partial G_u)_{(\geq k)} \llcorner \Omega \times \mathbb{R}^N) < \infty\,, \tag{8.2}$$

compare (3.6) for the notation. We then define the *integral flat chain*

$$B_k := (\partial(\partial G_u)^v_k) \llcorner \Omega \times \mathbb{R}^N \in \mathcal{D}_{n-2}(\Omega \times \mathbb{R}^N)\,. \tag{8.3}$$

In general, B_k may not have finite mass, even in $u \in L^\infty$, see Example 9.5 below. However, we always find a mass-minimizing current in the (non-empty) class

$$\mathcal{F}_k := \{T \in \mathcal{R}_{n-1}(\Omega \times \mathbb{R}^N) \mid T_{(h)} = 0 \ \forall\, h < k\,,\ (\partial T) \llcorner \Omega \times \mathbb{R}^N = B_k\}\,.$$

Proposition 8.2 *The minimum of the variational problem* $\inf\{\mathbf{M}(T) \mid T \in \mathcal{F}_k\}$ *is attained.*

PROOF: The class $\mathcal{F}_k$ being non-empty, as $(\partial G_u)^v_k \in \mathcal{F}_k$, we consider a minimizing sequence $\{T_j\} \subset \mathcal{F}_k$ for the given problem. Setting $\widetilde{T}_j := T_j - (\partial G_u)^v_k$, the sequence $\{\widetilde{T}_j\}$ belongs to the class $\mathcal{R}_{n-1}(\Omega \times \mathbb{R}^N)$ and satisfies the null-boundary condition $(\partial \widetilde{T}_j) \llcorner \Omega \times \mathbb{R}^N = 0$ for every j. Therefore, possibly passing to a (not relabelled) subsequence, by closure-compactness, Theorem 3.3, we deduce that $\widetilde{T}_j$ weakly converges in $\mathcal{D}_{n-1}$ to some i.m. rectifiable current $\widetilde{T} \in \mathcal{R}_{n-1}(\Omega \times \mathbb{R}^N)$ satisfying $(\partial \widetilde{T}) \llcorner \Omega \times \mathbb{R}^N = 0$. Let now $\eta \in \mathcal{D}^{n-1}(\Omega \times \mathbb{R}^N)$. By the definition $\widetilde{T}_j(\eta^{(h)}) = 0$ for every $h < k$ and every j. Passing to the limit, we get $\widetilde{T}(\eta^{(h)}) = 0$ for $h < k$. Finally, setting $\overline{T} := \widetilde{T} + (\partial G_u)^v_k$, we have $\overline{T} \in \mathcal{F}_k$ and $T_j \rightharpoonup \overline{T}$ weakly in $\mathcal{D}_{n-1}$. The lower semicontinuity of the mass yields the assertion, as $\mathbf{M}(\overline{T}) = \inf\{\mathbf{M}(T) \mid T \in \mathcal{F}_k\}$. □

Definition 8.3 *We shall denote by* $\overline{T}_k \in \mathcal{F}_k$ *a minimum point to the variational problem from Proposition* 8.2*.*

Remark 8.4 As we shall see in Sec. 10 below, it may happen that the above minimum $\overline{T}_k$ is non-unique.

Remark 8.5 For example, if $N \geq n$ and $k = \underline{n} = n-1$, we have

$$(\partial G_u)^v_{n-1} := \tau((\partial \mathcal{G}_u)^v_{n-1}, \theta, \overrightarrow{\xi}), \quad \text{where} \quad (\partial \mathcal{G}_u)^v_{n-1} = \{z \in \partial \mathcal{G}_u \mid \overrightarrow{\xi} = \xi_{(n-1)}\}\,. \tag{8.4}$$

The current $(\partial G_u)^v_{n-1} \in \mathcal{R}_{n-1}(\Omega \times \mathbb{R}^N)$ is "completely vertical" and with finite mass

$$\mathbf{M}((\partial G_u)^v_{n-1}) \leq \mathbf{M}((\partial G_u)_{(n-1)} \llcorner \Omega \times \mathbb{R}^N) < \infty \,.$$

Notice that we cannot replace $(\partial G_u)^v_{n-1}$ by $(\partial G_u)_{(n-1)}$, as in general the current $(\partial G_u)_{(n-1)}$ is not i.m. rectifiable in $\mathcal{R}_{n-1}(\Omega \times \mathbb{R}^N)$. In fact, for any $\eta \in \mathcal{D}^{n-1}(\Omega \times \mathbb{R}^N)$ we clearly have

$$(\partial G_u)_{(n-1)}(\eta) = \partial G_u(\eta^{(n-1)}) = \int_{\widehat{\mathcal{M}}_u} \langle \eta, \xi_{(n-1)} \rangle \, \theta \, d\mathcal{H}^{n-1} \,,$$

where

$$\widehat{\mathcal{M}}_u := \operatorname{set}((\partial G_u)_{(n-1)} \llcorner \Omega \times \mathbb{R}^N) = \{ z \in \partial \mathcal{G}_u \mid \xi_{(n-1)}(z) \neq 0 \} \,.$$

However, writing $(\partial G_u)_{(n-1)} \llcorner \Omega \times \mathbb{R}^N = \tau(\widehat{\mathcal{M}}_u, \theta\,\mu, \xi_{(n-1)}/\mu)$, where $\mu := |\xi_{(n-1)}|$, even if the set $\widehat{\mathcal{M}}_u$ is $(n-1)$-rectifiable, in general the unit $(n-1)$-vector $\xi_{(n-1)}/\mu$ does not provide an orientation to the approximate tangent space at $\widetilde{\mathcal{M}}_u$, see Example 9.3 below.

Proof of Theorem 1.2: We divide the proof in $\underline{n}+2$ steps, where $\underline{n} := \min\{n-1, N\}$. At the first step, we define the "completely vertical" component $T_{\underline{n}}$ and apply Theorem 1.3. At the intermediate steps, we define by iteration the component T_k, for $k = \underline{n}+1-j$ and $j = 2, \ldots, \underline{n}$, and apply Theorem 1.6. At the $(\underline{n}+1)^{th}$ step, we define the remaining component T_0. At the final step, we conclude the proof.

Step 1: the component $T_{\underline{n}}$. In case of codimension $N < n$, we observe that any integral $(n-1)$-cycle in $\mathbb{R}^N$ with finite mass is equal to zero. For this reason, we set $T_{\underline{n}} := 0$, see property viii).

If $N \geq n$, and hence $\underline{n} = n-1$, using the notation from (8.4), we choose $\overline{T}_{n-1} \in \mathcal{R}_{n-1}(\Omega \times \mathbb{R}^N)$ a minimum of the variational problem from Proposition 8.2, where $k = n-1$, and define

$$T_{n-1} := (\partial G_u)^v_{n-1} - \overline{T}_{n-1} \in \mathcal{R}_{n-1}(\Omega \times \mathbb{R}^N) \,. \tag{8.5}$$

By our construction we immediately deduce the mass estimate

$$\mathbf{M}(T_{n-1}) \leq 2\,\mathbf{M}((\partial G_u)^v_{n-1}) \leq 2\,\mathbf{M}((\partial G_u)_{(n-1)} \llcorner \Omega \times \mathbb{R}^N) < \infty \,,$$

see (1.11), the null-boundary condition $(\partial T_{n-1}) \llcorner \Omega \times \mathbb{R}^N = 0$, and that T_{n-1} is "completely vertical", i.e.,

$$T_{n-1}(\eta) = T_{n-1}(\eta^{(n-1)}) \qquad \forall\, \eta \in \mathcal{D}^{n-1}(\Omega \times \mathbb{R}^N) \,.$$

Therefore, T_{n-1} satisfies the hypotheses of Theorem 1.3, that gives the structure property vi). In fact, the assumption (H_1) ensures that the support of $(\partial G_u) \llcorner \Omega \times \mathbb{R}^N$ is contained in $\overline{A} \times \mathbb{R}^N$. The set A being convex, by applying a projection argument to the minimum problem in Proposition 8.2 we deduce that the current T_{n-1} is supported in $\overline{A} \times \mathbb{R}^N$, too.

Remark 8.6 If u is a Sobolev map in $W^{1,n-1}(\Omega, \mathbb{R}^N)$, then $T_{n-1} = (\partial G_u) \llcorner \Omega \times \mathbb{R}^N$. In fact, by Proposition 3.8 we infer that $(\partial G_u)_{(h)} \llcorner \Omega \times \mathbb{R}^N = 0$ for every $h < n-1$. This yields that $(\partial G_u)^v_{n-1} = (\partial G_u) \llcorner \Omega \times \mathbb{R}^N$, hence $B_{n-1} = 0$ and finally $\overline{T}_{n-1} = 0$ in Definition 8.3. Also, if $u \in W^{1,n}(\Omega, \mathbb{R}^N)$, Proposition 3.8 yields $(\partial G_u) \llcorner \Omega \times \mathbb{R}^N = 0$, whence $T_{n-1} = 0$, compare property viii) with $k = \underline{n}$.

We now proceed by iteration. For $j = 2, \ldots, \underline{n}$, we have:

Step j: the component T_k for $k = \underline{n}+1-j$. At the previous steps, we have defined the components T_i, for $i = k+1, \ldots, \underline{n}$, and proved the related structure properties, namely:

(a) $T_i \in \mathcal{R}_{n-1}(\Omega \times \mathbb{R}^N)$;

(b) $\partial T_i = 0$ and $\operatorname{spt}(T_i) \subset \overline{A} \times \mathbb{R}^N = 0$;

(c) $T_i(\eta^{(h)}) = 0$ for every $h < i$ and every form $\eta \in \mathcal{D}^{n-1}(\Omega \times \mathbb{R}^N)$;

(d) $\operatorname{set}(T_i) \subset S_{n-1-i} \times \mathbb{R}^N$ for some countably $\mathcal{H}^{n-1-i}$-rectifiable subset S_{n-1-i} of $\overline{A}$;

(e) $\mathbf{M}(T_i) \le 2 \sum_{l=i}^{n} 2^{l-i} \mathbf{M}((\partial G_u)_{(l)} \llcorner \Omega \times \mathbb{R}^N) < \infty$.

Let $(\partial G_u)^v_k \in \mathcal{R}_{n-1}(\Omega \times \mathbb{R}^N)$ given by (8.1), choose $\overline{T}_k \in \mathcal{F}_k$ a minimum point to the variational problem from Proposition 8.2, see Definition 8.3, and set

$$\widetilde{T}_k := (\partial G_u)^v_k - \overline{T}_k \,. \tag{8.6}$$

The current $\widetilde{T}_k$ belongs to $\mathcal{R}_{n-1}(\Omega \times \mathbb{R}^N)$, satisfies the null-boundary condition $(\partial \widetilde{T}_k) \llcorner \Omega \times \mathbb{R}^N = 0$, and also $\widetilde{T}_k(\eta^{(h)}) = 0$ for every $h < k$ and $\eta \in \mathcal{D}^{n-1}(\Omega \times \mathbb{R}^N)$. Moreover, by (8.2) and (3.6) we deduce the mass estimate

$$\mathbf{M}(\widetilde{T}_k) \le 2\,\mathbf{M}((\partial G_u)^v_k) \le 2 \sum_{i=k}^{n} \mathbf{M}((\partial G_u)_{(i)} \llcorner \Omega \times \mathbb{R}^N) < \infty \,. \tag{8.7}$$

We then define the k^{th} component of the boundary current $(\partial G_u) \llcorner \Omega \times \mathbb{R}^N$ by:

$$T_k := \widetilde{T}_k - \sum_{i=k+1}^{n} T_i \,. \tag{8.8}$$

In fact, by (e) and (8.7) we deduce the mass estimate

$$\mathbf{M}(T_k) \le 2 \sum_{l=k}^{n} 2^{l-k} \mathbf{M}((\partial G_u)_{(l)} \llcorner \Omega \times \mathbb{R}^N) < \infty \,,$$

see (1.11). Moreover, as in Step 1, by the assumption (H_1) we may and do assume that the current $\widetilde{T}_k$ is supported in $\overline{A} \times \mathbb{R}^N$. Therefore, by (a)–(c), and by the above properties of $\widetilde{T}_k$, we infer that the current T_k satisfies the hypotheses of Theorem 1.6. Hence, there exists a countably $\mathcal{H}^{n-1-k}$-rectifiable subset S_{n-1-k} of $\overline{A}$ such that $\mathrm{set}(T_k) \subset S_{n-1-k} \times \mathbb{R}^N$. The properties i)–vii) are verified.

Remark 8.7 Notice that by (8.8), for $k < \overline{n}$ we have

$$\sum_{i=k+1}^{n} T_i = \widetilde{T}_{k+1} \,, \quad \text{hence} \quad T_k = \widetilde{T}_k - \widetilde{T}_{k+1} \,. \tag{8.9}$$

Due to the possible lack of uniqueness of the minimum point $\overline{T}_k$ from Definition 8.3, if $(\partial G_u)^v_k = (\partial G_u)^v_{k+1}$, see (8.1), we choose $\overline{T}_k = \overline{T}_{k+1}$, so that $\widetilde{T}_k = \widetilde{T}_{k+1}$, by (8.6), and hence $T_k = 0$, by (8.9).

Step $\underline{n}+1$: the component T_0. Define

$$T_0 := (\partial G_u) \llcorner \Omega \times \mathbb{R}^N - \sum_{k=1}^{n} T_k \,, \tag{8.10}$$

so that the properties i)–vii) and the mass estimate (1.11) are readily checked, for $k = 0$.

Final step: conclusion. Assume now that u is a Sobolev map in $W^{1,p}(\Omega, \mathbb{R}^N)$ for some positive integer p, and let $0 \le k \le p$ integer. Proposition 3.8 yields that $(\partial G_u)_{(h)} \llcorner \Omega \times \mathbb{R}^N = 0$ for each $h < k$. Therefore $(\partial G_u)^v_k = (\partial G_u) \llcorner \Omega \times \mathbb{R}^N$, see (8.1), hence $B_k = 0$, see (8.3), and definitively $\overline{T}_k = 0$ in Definition 8.3, so that $\widetilde{T}_k = (\partial G_u) \llcorner \Omega \times \mathbb{R}^N$. By (8.9) and (8.10), we get

$$T_p = (\partial G_u) \llcorner \Omega \times \mathbb{R}^N - \sum_{i=p+1}^{n} T_i \,, \qquad T_k = 0 \quad \forall\, k \in \{0, \dots, p-1\} \,,$$

see property viii) and Remark 8.6. $\square$

Proof of Corollary 1.7: Let $\widetilde{\Omega}$ denote the open set $\Omega \setminus \overline{J_u}$. Since the restriction of u to $\widetilde{\Omega}$ belongs to the Sobolev class $W^{1,1}(\widetilde{\Omega}, \mathbb{R}^N)$, by Proposition 3.8 we deduce that $(\partial G_u)_{(0)} \llcorner \widetilde{\Omega} \times \mathbb{R}^N = 0$. Therefore, by (8.10) we deduce that $T_0 \llcorner \widetilde{\Omega} \times \mathbb{R}^N = 0$, and hence that $S_{n-1} \subset \overline{J_u}$. The regularity assumption (1.20) yields the claim, on account of (1.18). $\square$

Proof of Corollary 8.1: Assume that $(\partial G_u)_{(p)} \llcorner \Omega \times \mathbb{R}^N = 0$, condition automatically verified if $u \in W^{1,p+1}(\Omega, \mathbb{R}^N)$, see Proposition 3.8. Using the notation (3.6), denote

$$S_p := (\partial G_u)_{(\geq p)} \llcorner \Omega \times \mathbb{R}^N \in \mathcal{D}_{n-1}(\Omega \times \mathbb{R}^N)\,.$$

Since $(S_p)_{(p)} = 0$, it is readily checked that $(\partial S_p) \llcorner \Omega \times \mathbb{R}^N = 0$. Therefore, S_p being concentrated on a countably $\mathcal{H}^{n-1}$-rectifiable set, see Remark 3.1, by the *rectifiable slices theorem* [8, 14], it turns out that the normal current S_p is i.m. rectifiable in $\mathcal{R}_{n-1}(\Omega \times \mathbb{R}^N)$. We thus modify the proof of Theorem 1.2 by setting $\widetilde{T}_p := S_p$, so that $(\widetilde{T}_p)_{(h)} = 0$ for $h \leq p$. Furthermore, for $1 \leq k < p$, we may and do replace $(\partial G_u)^v_k$ in (8.1) with $(\widetilde{\partial G_u})^v_k := (\partial G_u)^v_k - S_p$, and consider the class

$$\widetilde{\mathcal{F}}_k := \{T \in \mathcal{R}_{n-1}(\Omega \times \mathbb{R}^N) \mid T_{(h)} = 0 \ \forall\, h < k \text{ or } h \geq p\,,\ (\partial T) \llcorner \Omega \times \mathbb{R}^N = \widetilde{B}_k\}$$

where, according to (8.3), the integral flat chain $\widetilde{B}_k$ is defined by

$$\widetilde{B}_k := (\partial(\widetilde{\partial G_u})^v_k) \llcorner \Omega \times \mathbb{R}^N \in \mathcal{D}_{n-2}(\Omega \times \mathbb{R}^N)\,.$$

Since $(\widetilde{\partial G_u})^v_k \in \widetilde{\mathcal{F}}_k$, similarly to Proposition 8.2, it is readily checked that *the minimum of the variational problem* $\inf\{\mathbf{M}(T) \mid T \in \widetilde{\mathcal{F}}_k\}$ *is attained.* Denoting this time by $\widehat{T}_k$ a corresponding minimum point, we correspondingly define

$$\widetilde{T}_k := (\widetilde{\partial G_u})^v_k - \widehat{T}_k\,, \qquad T_k := \widetilde{T}_k - \widetilde{T}_{k+1}\,.$$

This yields that $(T_k)_{(h)} = 0$ for $h \geq p$. Using (8.10), the same property holds true for T_0, as required.

Finally, if the condition $(\partial G_u)_{(p)} \llcorner \Omega \times \mathbb{R}^N = 0$ is verified for more than one integer p, say for $p \in \{p_1, \dots, p_m\}$, where $\underline{n} \geq p_1 > p_2 > \cdots p_m \geq 0$, we iterate the above argument, setting $\widetilde{T}_{p_j} := S_{p_j}$ for $j = 1, \dots, m$, whereas for $p_{j+1} < k < p_j$, where $j = 1, \dots, m$ and $p_{m+1} := 0$, we let $(\widetilde{\partial G_u})^v_k := (\partial G_u)^v_k - S_{p_j}$. We omit any further detail. $\square$

9 Further examples

In this section we consider some examples concerning our decomposition theorem 1.2. In all these examples, the Dirichlet-type condition (H_1) is readily checked.

Example 9.1 In our first example from Sec. 2, by (2.1) we have

$$(\partial G_u)^v_1 = -\delta_P \times [\![\, \Sigma \,]\!] + \delta_Q \times [\![\, \Sigma \,]\!]\,,$$

see (8.4), where $n = N = 2$, hence by (8.3) and Definition 8.3

$$B_1 = (\delta_Q - \delta_P) \times (\delta_B - \delta_A)\,, \quad \overline{T}_1 = (\delta_Q - \delta_P) \times [\![\, I_{A,B} \,]\!]\,.$$

Using (2.1), (8.5), and (8.10), we readily obtain the formulas from (2.3).

Example 9.2 In our second example from Sec. 2, by (2.6) we have

$$(\partial G_u)^v_2 = -\delta_P \times [\![\, \Sigma^- \,]\!] - \delta_Q \times [\![\, \Sigma^+ \,]\!]\,, \tag{9.1}$$

see (8.4), where $n = N = 3$, hence by (8.3) and Definition 8.3, using (2.8) we get

$$B_2 = (\delta_P - \delta_Q) \times [\![\, S^1 \,]\!]\,, \quad \overline{T}_2 = (\delta_P - \delta_Q) \times [\![\, D^2 \,]\!]\,. \tag{9.2}$$

Since $u \in W^{1,1}(\Omega, \widehat{\mathbb{R}}^3)$, in (8.1) we also have

$$(\partial G_u)^v_1 = (\partial G_u) \llcorner \Omega \times \widehat{\mathbb{R}}^3\,, \tag{9.3}$$

whence $B_1=0$, $\overline{T}_1=0$, and $T_1=(\partial G_u)\llcorner\Omega\times\widehat{\mathbb{R}}^3-T_2$. This gives the decomposition

$$(\partial G_u)\llcorner\Omega\times\widehat{\mathbb{R}}^3=T_0+T_1+T_2\,, \tag{9.4}$$

where $T_0=0$ and T_1, T_2 are the i.m. rectifiable currents in $\mathcal{R}_2(\Omega\times\widehat{\mathbb{R}}^3)$ given by (2.9).

Example 9.3 We slightly modify the previous example, by taking

$$u(x):=\begin{cases} u_P(x) & \text{if} \quad x_1\le -1 \\ \widetilde{u}_0(x) & \text{if} \quad -1\le x_1\le 1 \\ u_Q(x) & \text{if} \quad x_1\ge 1 \end{cases}$$

where again $P:=(-1,0,0)$, $Q:=(1,0,0)$, but instead of (2.4)

$$u_P(x):=\frac{x-P}{|x-P|}\,,\qquad \widetilde{u}_0(x):=\Bigl(|x_1|-1,\frac{(x_2,x_3)}{|(x_2,x_3)|}\Bigr)\,,\qquad u_Q(x):=\frac{x-Q}{|x-Q|}\,.$$

Similarly to (2.6), this time we obtain

$$(\partial G_u)\llcorner\Omega\times\widehat{\mathbb{R}}^3=\Gamma_{\#}[\![(-1,1)\times(0,2\pi)]\!]-\delta_P\times[\![\Sigma^-]\!]-\delta_Q\times[\![\Sigma^+]\!]\,, \tag{9.5}$$

where $\Gamma:(-1,1)\times(0,2\pi)\to\Omega\times\widehat{\mathbb{R}}^3$ is given by $\Gamma(t,\theta):=(t,0,0,|t|-1,\cos\theta,\sin\theta)$. Notice that

$$\partial\Gamma_{\#}[\![(-1,1)\times(0,2\pi)]\!]=(\delta_Q-\delta_P)\times[\![S^1]\!]\,.$$

In this case the current $(\partial G_u)_{(2)}$ is not i.m. rectifiable in $\mathcal{R}_2(\Omega\times\widehat{\mathbb{R}}^3)$, see Remark 8.5. More precisely, using the notation $(\partial G_u)\llcorner\Omega\times\widehat{\mathbb{R}}^3=\tau(\partial\mathcal{G}_u,\theta,\overrightarrow{\xi})$, according to (3.7) we have

$$\widehat{\mathcal{M}}_u:=\operatorname{set}((\partial G_u)_{(2)}\llcorner\Omega\times\widehat{\mathbb{R}}^3)=\{z\in\partial\mathcal{G}_u\mid\xi_{(2)}(z)\neq 0\}=\partial\mathcal{G}_u$$

so that $(\partial G_u)_{(2)}\llcorner\Omega\times\widehat{\mathbb{R}}^3=\tau(\widetilde{\mathcal{M}}_u,\theta\,\mu,\xi_{(2)}/\mu)$, where $\mu:=|\xi_{(2)}|$. Therefore, the unit 2-vector $\xi_{(2)}/\mu$ does not provide an orientation to the approximate tangent space at the points in the support of the component $\Gamma_{\#}[\![(-1,1)\times(0,2\pi)]\!]$ of the boundary current.

However, the formulas (9.1), (9.2), and (9.3) continue to hold, so that we obtain the decomposition (9.4), where $T_0=0$ and T_1, T_2 are the boundaryless i.m. rectifiable currents in $\mathcal{R}_2(\Omega\times\widehat{\mathbb{R}}^3)$

$$\begin{array}{rcl} T_1 & := & \Gamma_{\#}[\![(-1,1)\times(0,2\pi)]\!]+(\delta_P-\delta_Q)\times[\![D^2]\!]\,, \\ T_2 & := & -\delta_P\times([\![\Sigma^-]\!]+[\![D^2]\!])-\delta_Q\times([\![\Sigma^+]\!]-[\![D^2]\!])\,. \end{array}$$

We thus have $S_2=\emptyset$, $S_1=I_{P,Q}$ and $S_0=\{P,Q\}$.

Example 9.4 Again we slightly modify the second example from Sec. 2, see Example 9.2. For $\lambda\in\mathbb{R}$, set $y_\lambda:=(\lambda,0,0)\in\widehat{\mathbb{R}}^3$ and

$$u_\lambda(x):=y_\lambda+\widetilde{u}(x)\,,\qquad \widetilde{u}(x):=\begin{cases} \widetilde{u}_P(x) & \text{if} \quad x_1\le -1 \\ u_0(x) & \text{if} \quad -1\le x_1\le 1 \\ \widetilde{u}_Q(x) & \text{if} \quad x_1\ge 1 \end{cases}$$

where this time, instead of (2.4), we set

$$\widetilde{u}_P(x):=\varphi\Bigl(\frac{x-P}{|x-P|}\Bigr)\,,\qquad u_0(x):=\Bigl(0,\frac{(x_2,x_3)}{|(x_2,x_3)|}\Bigr)\,,\qquad \widetilde{u}_Q(x):=\varphi\Bigl(\frac{x-Q}{|x-Q|}\Bigr)\,,$$

the function $\varphi:\widehat{\mathbb{R}}^3\to\widehat{\mathbb{R}}^3$ being $\varphi(y_1,y_2,y_2):=(0,y_2,y_3)$. We have $\varphi_{\#}[\![\Sigma^\pm]\!]=\pm[\![D^2]\!]$, whence

$$(\partial G_{u_\lambda})\llcorner\Omega\times\widehat{\mathbb{R}}^3=[\![I_{P,Q}]\!]\times[\![y_\lambda+S^1]\!]+(\delta_P-\delta_Q)\times[\![y_\lambda+D^2]\!]\,. \tag{9.6}$$

Since $(\partial G_{u_\lambda})_2^v=(\delta_P-\delta_Q)\times[\![y_\lambda+D^2]\!]$, and $[\![y_\lambda+D^2]\!]$ is a mass-minimizer, this time we readily obtain the decomposition (9.4), where $T_0=0$, $T_2=0$, and $T_1:=(\partial G_{u_\lambda})\llcorner\Omega\times\widehat{\mathbb{R}}^3$. We thus have $S_2=\emptyset$, $S_1=\overline{I_{P,Q}}$ and $S_0=\emptyset$.

Example 9.5 Let again $n = N = 3$, $\Omega = (-2,2) \times (-1,1)^2$, and $u : \Omega \to \widehat{\mathbb{R}}^3$ the function of the second example from Sec. 2, see Example 9.2. Let $j \in \mathbb{N}^+$ and

$$u_j(x) := \frac{1}{j}\, u(j^2\,(x - a_j))\,, \quad x \in \Omega_j := a_j + \frac{1}{j^2}\,\Omega\,,$$

where $a_1 := 0$ and $a_j := \sum_{i=1}^{j-1} \frac{4}{i^2}$, for $j \geq 2$.

The Sobolev map $u_j : \Omega_j \to \widehat{\mathbb{R}}^3$ is smooth outside the line segment connecting the points $P_j := a_j + j^{-2}\,(-1,0,0)$ and $Q_j := a_j + j^{-2}\,(1,0,0)$, and it satisfies the hypotheses of our decomposition theorem 1.2. In fact, using that $\det \nabla u_j = 0$, by a change of variable we find that $A(u_j, \Omega_j) \leq j^{-4}\, A(u, \Omega)$, whereas by (2.6) we check that $(\partial G_{u_j}) \llcorner \Omega_j \times \widehat{\mathbb{R}}^3 = \widetilde{T}_1^j + \widetilde{T}_2^j$, where

$$\widetilde{T}_1^j := [\![\, I_{P_j,Q_j} \,]\!] \times [\![\, j^{-1}\, S^1 \,]\!]\,, \qquad \widetilde{T}_2^j := -\delta_{P_j} \times [\![\, j^{-1}\Sigma^- \,]\!] - \delta_{Q_j} \times [\![\, j^{-1}\Sigma^+ \,]\!]\,,$$

so that

$$\mathbf{M}((\partial G_{u_j}) \llcorner \Omega_j \times \widehat{\mathbb{R}}^3) \leq \frac{1}{j^2}\Big(|Q_1 - P_1| \cdot \mathcal{H}^1(S^1) + \mathcal{H}^2(\Sigma^-) + \mathcal{H}^2(\Sigma^+)\Big) \leq \frac{C}{j^2}\,.$$

The family $\{\Omega_j\}_{j \in \mathbb{N}^+}$ is pairwise disjoint, and its union is contained in the bounded domain $\widetilde{\Omega} := (-2,8) \times (-1,1)$. Therefore, one can easily define a bounded map $\widetilde{u} : \widetilde{\Omega} \to \widehat{\mathbb{R}}^3$ that is smooth outside each open set Ω_j, agrees with u_j in Ω_j for each j, and such that its graph has finite area:

$$A(\widetilde{u}, \widetilde{\Omega}) \leq C + \sum_{j=1}^{\infty} A(u_j, \Omega_j) \leq C + A(u, \Omega) \sum_{j=1}^{\infty} \frac{1}{j^4} < \infty\,.$$

This yields that $G_{\widetilde{u}}$ is i.m. rectifiable in $\mathcal{R}_{3,c}(\widetilde{\Omega} \times \widehat{\mathbb{R}}^3)$, with finite mass, and its boundary is given by

$$(\partial G_{\widetilde{u}}) \llcorner \widetilde{\Omega} \times \widehat{\mathbb{R}}^3 = \sum_{j=1}^{\infty} (\widetilde{T}_1^j + \widetilde{T}_2^j)\,.$$

Notice that

$$\mathbf{M}((\partial G_{\widetilde{u}}) \llcorner \widetilde{\Omega} \times \widehat{\mathbb{R}}^3) = \sum_{j=1}^{\infty} \mathbf{M}(\widetilde{T}_1^j + \widetilde{T}_2^j) \leq C \sum_{j=1}^{\infty} \frac{1}{j^2} < \infty\,,$$

so that property (1.5) holds true, the boundary current is well defined in terms of the mass convergence, and actually it is i.m. rectifiable in $\mathcal{R}_{2,c}(\widetilde{\Omega} \times \widehat{\mathbb{R}}^3)$. Following (8.3), where $k = 2$ and $u = \widetilde{u}$, we have

$$B_2 := (\partial(\partial G_{\widetilde{u}})_2^v) \llcorner \widetilde{\Omega} \times \widehat{\mathbb{R}}^3 = \sum_{j=1}^{\infty} B_2^j\,,$$

where

$$B_2^j := (\partial(\partial G_{u_j})_2^v) \llcorner \Omega_j \times \widehat{\mathbb{R}}^3 = (\delta_{P_j} - \delta_{Q_j}) \times [\![\, j^{-1} S^1 \,]\!]\,.$$

Therefore, *the integral flat chain* B_2 *does not have finite mass*, as

$$\mathbf{M}(B_2) = \sum_{j=1}^{\infty} \mathbf{M}(B_2^j) = 2\,\mathcal{H}^1(S^1) \sum_{j=1}^{\infty} \frac{1}{j} = \infty\,.$$

However, Theorem 1.2 leads to the decomposition

$$(\partial G_{\widetilde{u}}) \llcorner \widetilde{\Omega} \times \widehat{\mathbb{R}}^3 = T_0 + T_1 + T_2\,,$$

where $T_0 = 0$ and T_1, T_2 (according to (2.9)) are the i.m. rectifiable currents in $\mathcal{R}_{2,c}(\widetilde{\Omega} \times \widehat{\mathbb{R}}^3)$

$$T_1 := \sum_{j=1}^{\infty} \big([\![\, I_{P_j,Q_j} \,]\!] \times [\![\, j^{-1}\, S^1 \,]\!] + (\delta_{P_j} - \delta_{Q_j}) \times [\![\, j^{-1} D^2 \,]\!]\big)\,,$$
$$T_2 := \sum_{j=1}^{\infty} \big(-\delta_{P_j} \times ([\![\, j^{-1}\Sigma^- \,]\!] + [\![\, j^{-1} D^2 \,]\!]) - \delta_{Q_j} \times ([\![\, j^{-1}\Sigma^+ \,]\!] - [\![\, j^{-1} D^2 \,]\!])\big)\,.$$

Notice that both T_1 and T_2 have zero boundary and finite mass, as

$$\mathbf{M}(T_1) = \big(|Q_1 - P_1| \cdot \mathcal{H}^1(S^1) + 2\,\mathcal{H}^2(D^2)\big) \cdot \sum_{j=1}^{\infty} \frac{1}{j^2} < \infty\,,$$
$$\mathbf{M}(T_2) = \big(\mathcal{H}^2(\Sigma^-) + \mathcal{H}^2(\Sigma^+) + 2\,\mathcal{H}^2(D^2)\big) \cdot \sum_{j=1}^{\infty} \frac{1}{j^2} < \infty\,.$$

Finally, this time we have $S_2 = \emptyset$, $S_1 = \bigcup_{j=1}^{\infty} I_{P_j,Q_j}$, and $S_0 = \bigcup_{j=1}^{\infty}\{P_j, Q_j\}$, so that

$$\mathcal{H}^1(S_1) = |Q_1 - P_1| \cdot \sum_{j=1}^{\infty} \frac{1}{j} = \infty\,, \quad \mathcal{H}^0(S_0) = \infty\,.$$

10 Non-uniqueness of the singular set

In this section we discuss an example in dimension $n = N = 3$ showing that in general the decomposition in Theorem 1.2 fails to be unique. Our construction is based on a classical example of non-uniqueness concerning minimal surfaces. This yields that the optimal singular set S_0 in (1.9) is not well-defined. Moreover, see Example 11.5 below, it may happen that the singular part of the distributional determinant may be equal to zero, even if the singular set S_0 in (1.9) is non-trivial. Notwithstanding, in Proposition 10.2 we will show that the set S_0 in (1.9) is concentrated in the set of the atoms of the measure

$$\mu_u(B) := \mathbf{M}((\partial G_u)_{(n-1)} \llcorner B \times \mathbb{R}^N)\,. \tag{10.1}$$

NON-UNIQUENESS. For $\lambda > 1$, denote by Σ_λ the *catenoid* surface of revolution in $\widehat{\mathbb{R}}^3$ with equation

$$\sqrt{y_2^2 + y_3^2} = \frac{1}{\lambda}\cosh(\lambda\, y_1)\,, \qquad -a_\lambda < y_1 < a_\lambda\,,$$

where $a_\lambda > 0$ is chosen in such a way that $\cosh(\lambda\, a_\lambda) = \lambda$, i.e., $a_\lambda := \lambda^{-1}\log(\lambda + \sqrt{\lambda^2 - 1})$. Therefore, Σ_λ is a minimal surface with boundary given by the union of the two unit circles $(\pm a_\lambda, 0, 0) + S^1$, where S^1 is defined by (2.5). Moreover, one has

$$f(\lambda) := \mathrm{area}(\Sigma_\lambda) = \frac{4\pi}{\lambda}\int_0^{a_\lambda} \cosh^2(\lambda t)\, dt = \frac{2\pi}{\lambda^2}\big(\log(\lambda + \sqrt{\lambda^2 - 1}) + \lambda\sqrt{\lambda^2 - 1}\big) < \infty\,.$$

Since for $\lambda > 1$

$$f'(\lambda) = \frac{4\pi}{\lambda^3}\Big(\frac{\lambda}{\sqrt{\lambda^2 - 1}} - \log(\lambda + \sqrt{\lambda^2 - 1})\Big)\,,$$

we find $\overline{\lambda} > 1$ such that $f'(\lambda) > 0$ if and only if $\lambda \in]1, \overline{\lambda}[$. Using that $f(\lambda) \to 2\pi$ as $\lambda \to \infty$, we get $f(\overline{\lambda}) > 2\pi$. Therefore, since $f(\lambda) \to 0$ as $\lambda \to 1$, there is a threshold $\lambda_c \in]1, \overline{\lambda}[$ such that the area of the catenoid Σ_λ is greater than 2π (that is the area of two unit disks) if and only if $\lambda > \lambda_c$.

Correspondingly, let $B^3 \subset \mathbb{R}^3$ the unit ball centered at the origin, and $u_\lambda : B^3 \to \widehat{\mathbb{R}}^3$ given by

$$u_\lambda(x) := \begin{cases} (-a_\lambda, 0, 0) + \varphi\Big(\dfrac{x}{|x|}\Big) & \text{if} \quad x_1 < 0 \\ (a_\lambda, 0, 0) + \varphi\Big(\dfrac{x}{|x|}\Big) & \text{if} \quad x_1 > 0 \end{cases} \qquad x = (x_1, x_2, x_3) \in B^3$$

where $\varphi : \widehat{\mathbb{R}}^3 \to \widehat{\mathbb{R}}^3$ is the map $\varphi(y_1, y_2, y_3) := (0, y_2, y_3)$. We now check that u_λ satisfies the hypotheses of Theorem 1.2. In fact, $u_\lambda \in SBV(B^3, \widehat{\mathbb{R}}^3) \cap L^\infty$, with jump set $J_{u_\lambda} = \{x \in B^3 \mid x_1 = 0\}$, and choosing the unit normal $\nu = e_1$, the approximate limits at the Jump points are

$$u_\lambda^\pm(x) = \Big(\pm a_\lambda, \frac{x_2}{|x|}, \frac{x_3}{|x|}\Big)\,, \qquad x \in J_{u_\lambda} \setminus \{0\}\,.$$

Moreover, $\nabla u_\lambda \in L^p$ for every $p < 3$, and $\det \nabla u_\lambda = 0$ a.e., so that $\mathbf{M}(G_{u_\lambda}) = A(u_\lambda, B^3) < \infty$.

Furthermore, it turns out that the boundary of the graph current G_{u_λ} satisfies

$$(\partial G_{u_\lambda}) \llcorner B^3 \times \widehat{\mathbb{R}}^3 = \delta_0 \times \Delta_\lambda + S_\lambda^+ - S_\lambda^- \,.$$

In this formula, we have set

$$\Delta_\lambda := \big(\tau^-_{\lambda\#}[\![\, D^2 \,]\!] - \tau^+_{\lambda\#}[\![\, D^2 \,]\!]\big) \in \mathcal{R}_{2,c}(\widehat{\mathbb{R}}^3)\,,$$

where D^2 is the (positively oriented) 2-disk given by (2.7), and $\tau_\lambda^\pm(y) := (\pm a_\lambda, 0, 0) + y$. Moreover, $S_\lambda^\pm \in \mathcal{R}_{2,c}(B^3 \times \widehat{\mathbb{R}}^3)$ is the i.m. rectifiable current

$$S_\lambda^\pm := \gamma^\pm_{\lambda\#}[\![\, (0,1) \times (0,2\pi) \,]\!]\,,$$

where $\gamma_\lambda^\pm : (0,1) \times (0,2\pi) \to B^3 \times \widehat{\mathbb{R}}^3$ is defined by

$$\gamma_\lambda^\pm(\rho,\theta) := (0, \rho\cos\theta, \rho\sin\theta, \pm a_\lambda, \cos\theta, \sin\theta)\,.$$

Therefore, the assumption (1.5) is verified, whereas the Dirichlet-type condition (H_1) can be obtained by smoothly extending u_λ to a larger ball, without affecting the discussion.

Using (8.4) and (8.3) we get $(\partial G_{u_\lambda})_2^v = \delta_0 \times \Delta_\lambda$ and hence $B_2 = \delta_0 \times \Gamma_\lambda$, where

$$\Gamma_\lambda := \tau^-_{\lambda\#}[\![\, S^1 \,]\!] - \tau^+_{\lambda\#}[\![\, S^1 \,]\!] \in \mathcal{R}_{1,c}(\widehat{\mathbb{R}}^3)\,,$$

the (naturally oriented) unit circle S^1 being given by (2.5). Notice that the currents $S_\lambda^\pm$ are concentrated on $J_{u_\lambda} \times \widehat{\mathbb{R}}^3$, and that

$$\partial(S_\lambda^+ - S_\lambda^-) \llcorner B^3 \times \widehat{\mathbb{R}}^3 = -\delta_0 \times \Gamma_\lambda = -\partial\big(\delta_0 \times \Delta_\lambda\big)\,.$$

Therefore, the current $\overline{T}_2$ from Definition 8.3 is given by $\overline{T}_2 = \delta_0 \times R_\lambda$, where $R_\lambda \in \mathcal{R}_2(\widehat{\mathbb{R}}^3)$ is a *mass-minimizing current* in the class

$$\mathcal{G}_\lambda := \{R \in \mathcal{R}_2(\widehat{\mathbb{R}}^3) \mid \partial R = \Gamma_\lambda\}\,.$$

Since moreover $(\partial G_{u_\lambda})_1^v = (\partial G_{u_\lambda})_2^v$, by Remark 8.7 we correspondingly get $T_1 = 0$. By (8.5), we deduce that Theorem 1.2 yields to the decomposition $(\partial G_{u_\lambda}) \llcorner B^3 \times \widehat{\mathbb{R}}^3 = T_0^\lambda + T_1^\lambda + T_2^\lambda$, where

$$T_2^\lambda := \delta_0 \times (\Delta_\lambda - R_\lambda)\,, \quad T_1^\lambda = 0\,, \quad T_0^\lambda := S_\lambda^+ - S_\lambda^- + \delta_0 \times R_\lambda\,. \tag{10.2}$$

Choosing a suitable orientation on the catenoid surface Σ_λ, the corresponding i.m. rectifiable current satisfies the boundary condition $\partial[\![\, \Sigma_\lambda \,]\!] = \Gamma_\lambda$, hence $[\![\, \Sigma_\lambda \,]\!]$ belongs to the class $\mathcal{G}_\lambda$, and its mass $\mathbf{M}([\![\, \Sigma_\lambda \,]\!])$ agrees with the area of Σ_λ. By the previous construction, this yields that for $\lambda > \lambda_c$ the *unique* mass-minimizing current in $\mathcal{G}_\lambda$ is Δ_λ, and hence

$$T_2^\lambda = 0\,, \qquad T_0^\lambda = S_\lambda^+ - S_\lambda^- + \delta_0 \times \Delta_\lambda\,.$$

Similarly, for $1 < \lambda < \lambda_c$ the *unique* mass-minimizing current in $\mathcal{G}_\lambda$ is given by $[\![\, \Sigma_\lambda \,]\!]$, whence

$$T_2^\lambda = \delta_0 \times \big(\Delta_\lambda - [\![\, \Sigma_\lambda \,]\!]\big)\,, \qquad T_0^\lambda = S_\lambda^+ - S_\lambda^- + \delta_0 \times [\![\, \Sigma_\lambda \,]\!]\,.$$

For $\lambda = \lambda_c$ instead, it turns out that *both* Δ_{λ_c} *and* $[\![\, \Sigma_{\lambda_c} \,]\!]$ *are mass-minimizing* in the class $\mathcal{G}_{\lambda_c}$. As a consequence, *the decomposition of the boundary current in* (10.2) *fails to be unique*, for $\lambda = \lambda_c$.

Remark 10.1 According to (1.8), for any choice of $\lambda > 1$ we have $S_1 = \emptyset$ and $S_2 = J_{u_\lambda}$. Moreover, by (1.9) we have $S_0 = \emptyset$ for $\lambda > \lambda_c$, whereas $S_0 = \{0\}$ for $1 < \lambda < \lambda_c$. However, due to the lack of uniqueness of the decomposition, *for* $\lambda = \lambda_c$ *we do not have a unique choice for the set* S_0.

THE SINGULAR SET S_0. In general it may happen that the singular part of the distributional determinant is zero, even if the singular set S_0 in (1.8) is non-trivial, see Example 11.5 below. Moreover, we have yet seen that the set S_0 is not well-defined, in general. Notwithstanding, we have:

Proposition 10.2 *Let* $u : \Omega \to \mathbb{R}^N$ *satisfy the hypotheses of Theorem* 1.2. *Then the singular set* S_0 *in* (1.9) *is concentrated in the set of the atoms of the measure* μ_u *defined by* (10.1).

PROOF: The claim is trivial if $N < n$, as $S_0 = \emptyset$, see property vii). In the case $N \geq n$, by the structure property vi) we have

$$T_{n-1} = \sum_{i=1}^{\infty} \delta_{a_i} \times \Sigma_i \,, \qquad S_0 := \{a_i \in \overline{A} \mid \Sigma_i \neq 0\} \,.$$

We recall from Sec. 8 that at the first step of the proof of Theorem 1.2 we have chosen $T_{n-1} := G - \overline{T}$, where $G \in \mathcal{R}_{n-1}(\Omega \times \mathbb{R}^N)$ denotes for simplicity the current $(\partial G_u)^v_{n-1}$ in (8.4), and $\overline{T} := \overline{T}_{n-1}$ is a mass minimizing current in the class

$$\mathcal{F}_{n-1} := \{T \in \mathcal{R}_{n-1}(\Omega \times \mathbb{R}^N) \mid T_{(h)} = 0 \ \forall\, h < n-1\,,\ (\partial(T - G)) \llcorner \Omega \times \mathbb{R}^N = 0\} \,,$$

see Definition 8.3. Assume by contradiction that e.g. the point $a_1 \in S_0$ does not belong to the set of atoms of μ_u. By (8.4) and (10.1), we thus have

$$\lim_{r \searrow 0} \mathbf{M}(G \llcorner \overline{B}_r(a_1) \times \mathbb{R}^N) \leq \lim_{r \searrow 0} \mu_u(\overline{B}_r(a_1)) = 0 \,. \tag{10.3}$$

Moreover, we can choose a decreasing sequence $\{r_j\}$ of positive radii such that $r_1 < \operatorname{dist}(a_1, \partial\Omega)$, $r_j \searrow 0$, and for each j the boundary $\partial B_{r_j}(a_1)$ does not contain points of the set S_0. Denoting $B_j = B_{r_j}(a_1)$ for simplicity, this yields

$$(\partial(G - \overline{T})) \llcorner \overline{B}_j \times \mathbb{R}^N = (\partial T_{n-1}) \llcorner \overline{B}_j \times \mathbb{R}^N = 0 \qquad \forall\, j \,. \tag{10.4}$$

Consider the currents

$$\overline{S}_j := \overline{T} + (G - \overline{T}) \llcorner \overline{B}_j \times \mathbb{R}^N \in \mathcal{R}_{n-1}(\Omega \times \mathbb{R}^N) \,.$$

We clearly have $(S_j)_{(h)} = 0$ for all $h < n-1$ whereas, using (10.4) and that $\overline{T} \in \mathcal{F}_{n-1}$,

$$(\partial(\overline{S}_j - G)) \llcorner \Omega \times \mathbb{R}^N = (\partial(\overline{T} - G)) \llcorner \Omega \times \mathbb{R}^N = 0 \qquad \forall\, j \,.$$

Therefore, the sequence $\{\overline{S}_j\}$ belongs to the class $\mathcal{F}_{n-1}$. Writing

$$\overline{S}_j = \overline{T} \llcorner (\Omega \setminus \overline{B}_j) \times \mathbb{R}^N + G \llcorner \overline{B}_j \times \mathbb{R}^N \,,$$

formula (10.3) yields that $\overline{S}_j$ weakly converges in $\mathcal{D}_{n-1}$ to a mass minimizing current $\overline{S}$ in the class $\mathcal{F}_{n-1}$. The corresponding sequence $S_j := G - \overline{S}_j$ satisfies $S_j = (G - \overline{T}) \llcorner (\Omega \setminus \overline{B}_j) \times \mathbb{R}^N$, where $G - \overline{T} = T_{n-1}$, and hence S_j weakly converges to the current $T_{n-1} - \delta_{a_1} \times \Sigma_1$. Since $\overline{S}_j = G - S_j$, we get

$$\overline{S} = G - T_{n-1} + \delta_{a_1} \times \Sigma_1 = \overline{T} + \delta_{a_1} \times \Sigma_1 \,, \qquad \mathbf{M}(\Sigma_1) > 0 \,.$$

Since both $\overline{S}$ and $\overline{T}$ are mass minimizing in the class $\mathcal{F}_{n-1}$, this yields a contradiction, as required. $\square$

11 The distributional minors and the class $\mathrm{B_N V}$

In this final section we discuss some new results concerning the *distributional determinant* $\operatorname{Det} \nabla u$, first introduced by J.M. Ball [10], and the *distributional minors*, see [24, 25]. We also deal with the class $\mathrm{B_N V}$ of *functions of higher bounded variation*, first studied by Jerrard-Soner [22].

THE DISTRIBUTIONAL DETERMINANT. Let $\Omega \subset \mathbb{R}^n$ a bounded domain and $n = N$. Under suitable assumptions, the distributional determinant of a (non-smooth) map $u : \Omega \to \widehat{\mathbb{R}}^n$ is defined by

$$\operatorname{Det} \nabla u := \frac{1}{n} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(u^j \, (\operatorname{adj} \nabla u)^j_i \right) , \tag{11.1}$$

where $\operatorname{adj} \nabla u$ is the matrix of the adjoints of ∇u, see (3.5). More precisely, if u belongs to $L^\infty(\Omega, \widehat{\mathbb{R}}^n)$ the above formula is well-defined (in the distributional sense) provided that $|\operatorname{adj} \nabla u| \in L^1(\Omega)$, e.g. if

$u \in W^{1,n-1}(\Omega, \widehat{\mathbb{R}}^n)$ or $u \in \mathcal{A}^1(\Omega, \widehat{\mathbb{R}}^n)$, see (3.11). If u is not essentially bounded, (11.1) is well defined provided that $u \in L^q \cap W^{1,p}(\Omega, \widehat{\mathbb{R}}^n)$ for some exponents q and p satisfying

$$n-1 < p < n \qquad \text{and} \qquad \frac{1}{q} + \frac{n-1}{p} \leq 1\,. \tag{11.2}$$

Moreover, $\operatorname{Det} \nabla u = \det \nabla u\, \mathcal{L}^n$ if u is Lipschitz and hence if $u \in W^{1,n}(\Omega, \widehat{\mathbb{R}}^n)$, by a standard density argument. In all these cases, moreover, $\operatorname{Det} \nabla u$ is a signed Radon measure with finite total variation.

Denote by ω_n the smooth form in $\mathcal{E}^{n-1}(\widehat{\mathbb{R}}^n)$

$$\omega_n := \frac{1}{n} \sum_{j=1}^{n} (-1)^{j-1} y_j\, \widehat{dy^j}\,, \qquad \widehat{dy^j} := dy^1 \wedge \cdots \wedge dy^{j-1} \wedge dy^{j+1} \wedge \cdots \wedge dy^n\,, \tag{11.3}$$

so that

$$d(\omega_n \wedge \varphi) = d\omega_n \wedge \varphi + (-1)^{n-1} \omega_n \wedge d\varphi\,, \quad d\omega_n = dy^1 \wedge \cdots \wedge dy^n\,. \tag{11.4}$$

By computing the pull-back $u^{\#}\omega_n$, and using that

$$(-1)^{j-1} u^{\#} \widehat{dy^j} \wedge d\varphi = (-1)^{n-1} \sum_{i=1}^{n} (\operatorname{adj} \nabla u)_i^j \frac{\partial \varphi}{\partial x_i}\, dx^1 \wedge \cdots \wedge dx^n\,,$$

see definition (3.5), we equivalently have

$$\langle \operatorname{Det} \nabla u, \varphi \rangle := (-1)^n \int_\Omega u^{\#} \omega_n \wedge d\varphi\,, \qquad \varphi \in C_c^\infty(\Omega)\,.$$

Therefore, by (3.12) it turns out that

$$\langle \operatorname{Det} \nabla u, \varphi \rangle = (-1)^n\, G_u(\omega_n \wedge d\varphi)\,, \tag{11.5}$$

where we have omitted to write action of the pull-back of the vertical and horizontal projections $\widehat{\pi}$ and π on the forms ω_n and $d\varphi$, respectively.

If $u \in L^\infty \cap \mathcal{A}^1(\Omega, \widehat{\mathbb{R}}^n)$, so that the graph current G_u is compactly supported and i.m. rectifiable in $\mathcal{R}_{n,c}(\Omega \times \widehat{\mathbb{R}}^n)$, by (11.4) and (11.5) we readily obtain that

$$\langle \operatorname{Det} \nabla u, \varphi \rangle = G_u(d\omega_n \wedge \varphi) - \partial G_u(\omega_n \wedge \varphi)\,, \tag{11.6}$$

where by (3.12)

$$G_u(d\omega_n \wedge \varphi) = G_u(\varphi\, dy^1 \wedge \cdots \wedge dy^n) = \int_\Omega \varphi(x)\, \det \nabla u(x)\, dx\,.$$

If moreover the boundary of the graph current G_u has finite mass in $\Omega \times \widehat{\mathbb{R}}^n$, i.e., (1.5) holds true, with $n = N$, by the boundary rectifiability theorem 3.4 it turns out that the second addendum in the right-hand side of (11.6) agrees with the *singular part* $(\operatorname{Det} \nabla u)^s$ with respect to the Lebesgue measure $\mathcal{L}^n$, compare the first part of [24, Prop. 4.2]. We thus deduce the decomposition

$$\langle (\operatorname{Det} \nabla u)^a, \varphi \rangle = \langle (\det \nabla u)\, \mathcal{L}^n, \varphi \rangle\,, \qquad \langle (\operatorname{Det} \nabla u)^s, \varphi \rangle = -\partial G_u(\omega_n \wedge \varphi) \tag{11.7}$$

into the absolute continuous and singular parts, for every bounded Borel function φ.

Müller-Spector [27] studied the distributional determinant in the setting of a theory for nonlinear elasticity, showing that the singular part is concentrated on an at most countable set of point. In the same spirit, we now extend [25, Prop. 3.1] in our framework, by *removing the L^∞-condition.*

Theorem 11.1 *Let $u : \Omega \to \widehat{\mathbb{R}}^n$ satisfy the hypotheses of Theorem 1.2. Assume in addition that $u \in L^q \cap W^{1,p}(\Omega, \widehat{\mathbb{R}}^n)$, where q and p satisfy (11.2), or $u \in L^\infty \cap W^{1,n-1}(\Omega, \widehat{\mathbb{R}}^n)$. Alternatively, assume that $u \in L^\infty$ and $(\partial G_u)_{(n-2)} \llcorner \Omega \times \widehat{\mathbb{R}}^n = 0$. Then $(\operatorname{Det} \nabla u)^a = (\det \nabla u)\, \mathcal{L}^n$, and the singular part $(\operatorname{Det} \nabla u)^s$ w.r.t. the Lebesgue measure $\mathcal{L}^n$ is concentrated on the at most countable set S_0 defined by (1.9).*

PROOF: We argue exactly as in the proof of Theorem 11.6 below, where we choose $\mathbb{R}^N = \widehat{\mathbb{R}}^n$, $m = n$, $\omega_\beta = \omega_n$, $y_\beta = y$, $u^\beta = u$, $\mathrm{Div}_{\bar\alpha}^{\beta} u = \mathrm{Det}\,\nabla u$, $\omega_\varphi^\alpha = \varphi$, $M_{\bar\alpha}^{\beta}(\nabla u) = \det \nabla u$, and we use (11.7) and (11.5) instead of (11.14) and (11.12), respectively. For this reason, we omit any further detail. $\square$

Remark 11.2 Theorem 1.8 readily follows by observing that for Sobolev maps u in $W^{1,n-1}$ we do not make use of the assumption (H_1) in the proof of Theorem 1.2. In this case, moreover, the condition $(\partial G_u)_{(n-2)} \llcorner \Omega \times \widehat{\mathbb{R}}^n = 0$ is automatically satisfied, see Proposition 3.8. As we already observed in the introduction, the example by S. Müller [26] shows that the bound (1.5) on the mass of the boundary current is a *necessary condition* to the validity of Theorem 1.2.[13] As a consequence, we deduce that for maps $u \in \mathcal{A}^1(\Omega, \widehat{\mathbb{R}}^n) \cap L^\infty$, the bound (1.5) is as stronger property than requiring that the distributional determinant $\mathrm{Det}\,\nabla u$ is a measure with finite total variation, even for Sobolev maps in $W^{1,p}(\Omega, \widehat{\mathbb{R}}^n)$ for every $p < n$. As we shall see in Proposition 11.11 below, the two properties are equivalent for Sobolev $W^{1,n-1}$-maps that take values into the unit sphere.

EXAMPLES. We now recover the above features in some of the examples from Sec. 9.

In Example 9.1, we have $\det \nabla u = 0$, whereas by (2.1) and (11.7)

$$\langle (\mathrm{Det}\,\nabla u)^s, \varphi \rangle = (\delta_P \times [\![\Sigma]\!] - \delta_Q \times [\![\Sigma]\!])(\varphi \wedge \omega_2) = \big(\varphi(P) - \varphi(Q)\big) \cdot [\![\Sigma]\!](\omega_2)\,,$$

and a direct computation gives $[\![\Sigma]\!](\omega_2) = \pi/2$, so that $\mathrm{Det}\,\nabla u = (\pi/2) \cdot (\delta_P - \delta_Q)$.

In Example 9.2, we have $\det \nabla u = 0$, whereas by (2.6) and (11.7)

$$\langle (\mathrm{Det}\,\nabla u)^s, \varphi \rangle = (\delta_P \times [\![\Sigma^-]\!] + \delta_Q \times [\![\Sigma^+]\!])(\varphi \wedge \omega_3) = \varphi(P) \cdot [\![\Sigma^-]\!](\omega_3) + \varphi(Q) \cdot [\![\Sigma^+]\!](\omega_3)\,,$$

and a direct computation gives $[\![\Sigma^\pm]\!](\omega_3) = 2\pi/3$, so that $\mathrm{Det}\,\nabla u = (2\pi/3) \cdot (\delta_P + \delta_Q)$.

In Example 9.3, we again have $\det \nabla u = 0$. However, this time by (9.5) and (11.7) we find that

$$\langle (\mathrm{Det}\,\nabla u)^s, \varphi \rangle = -\Gamma_{\#}[\![(-1,1) \times (0,2\pi)]\!](\varphi \wedge \omega_3) + \frac{2\pi}{3} \langle \delta_P + \delta_Q, \varphi \rangle\,,$$

where $\Gamma : (-1,1) \times (0,2\pi) \to \Omega \times \widehat{\mathbb{R}}^3$ is given by $\Gamma(t,\theta) := (t, 0, 0, |t| - 1, \cos\theta, \sin\theta)$. Since

$$\Gamma^{\#} dy^1 = \mathrm{sgn}(t)\, dt\,, \quad \Gamma^{\#} dy^2 = -\sin\theta\, d\theta\,, \quad \Gamma^{\#} dy^3 = \cos\theta\, d\theta\,,$$

we have

$$\Gamma^{\#}(\varphi \wedge \omega_3) = -\frac{1}{3}\, \mathrm{sgn}(t)\, \varphi(t,0,0)\, dt \wedge d\theta\,,$$

whence

$$-\Gamma_{\#}[\![(-1,1) \times (0,2\pi)]\!](\varphi \wedge \omega_3) = \frac{1}{3} [\![(-1,1) \times (0,2\pi)]\!]\big(\mathrm{sgn}(t)\, \varphi(t,0,0)\, dt \wedge d\theta\big) = \frac{2\pi}{3} \langle \mu, \varphi \rangle\,,$$

where μ is the signed Radon measure

$$\langle \mu, \varphi \rangle := \int_{-1}^{1} \mathrm{sgn}(t)\, \varphi(t,0,0)\, dt\,.$$

In conclusion, this time we obtain

$$\mathrm{Det}\,\nabla u = (\mathrm{Det}\,\nabla u)^s = \frac{2\pi}{3} \big(\mu + \delta_P + \delta_Q\big)\,.$$

Remark 11.3 This example shows that the additional assumption $(\partial G_u)_{(n-2)} \llcorner \Omega \times \widehat{\mathbb{R}}^n = 0$ cannot be dropped in Theorem 11.1. In fact, $u \notin W^{1,2}(\Omega, \widehat{\mathbb{R}}^3)$, $(\partial G_u)_{(1)} \llcorner \Omega \times \widehat{\mathbb{R}}^3 \neq 0$, and the singular part of the distributional determinant of u is concentrated on the closed line segment connecting the points P and Q.

[13] In fact, he showed the existence of bounded Sobolev functions u in $W^{1,p}(\Omega, \mathbb{R}^2)$ for every $p < 2$, where $\Omega = (0,1)^2 \subset \mathbb{R}^2$, such that $\det \nabla u = 0$ and $|\nabla u^1|\,|\nabla u^2| = 0$ a.e. in Ω, but $\mathrm{Det}\,\nabla u = V' \otimes V'$, where V is the Cantor-Vitali function. Hence, the distributional determinant is a non-negative Radon measure concentrated on $C \times C$ where C is the Cantor set. According to our results, this clearly yields that the boundary current ∂G_u does not have finite mass, i.e., property (1.5) is violated.

As to Example 9.4, since $\det\nabla u_\lambda=0$ and

$$[\![\, y_\lambda + D^2\,]\!](\omega_3) = \frac{\lambda}{3}\,[\![\, D^2\,]\!](dy^2\wedge dy^3) = \frac{\lambda}{3}\,\pi\,,$$

by (9.6) and (11.7) we obtain that $\mathrm{Det}\,\nabla u_\lambda = \lambda(\pi/3)\cdot(\delta_Q-\delta_P)$.

Remark 11.4 Therefore, in this framework our definition (11.1) of distributional determinant differs from the classical one $\widetilde{\mathrm{Det}}\nabla u := \sum_{i=1}^n \dfrac{\partial}{\partial x_i}\big(u^1\,(\mathrm{adj}\,\nabla u)^1_i\big)$. In fact, for maps in L^∞, similarly to (11.6) one obtains

$$\langle\widetilde{\mathrm{Det}}\nabla u,\varphi\rangle = \langle\det\nabla u\,\mathcal{L}^n,\varphi\rangle - (\partial G_u)(y^1\widehat{dy^1}\wedge\varphi)\,.$$

Now, setting $\widetilde u_\lambda := \begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{pmatrix} u_\lambda$, by (9.6) we get $\partial G_{\widetilde u_\lambda}(y^1\widehat{dy^1}\wedge\varphi) = -\partial G_{u_\lambda}(y^2\widehat{dy^2}\wedge\varphi)=0$, hence $\widetilde{\mathrm{Det}}\nabla\widetilde u_\lambda = 0$ for every $\lambda\in\mathbb{R}$, whereas $\mathrm{Det}\,\nabla\widetilde u_\lambda = \mathrm{Det}\,\nabla u_\lambda = \lambda(\pi/3)\cdot(\delta_Q-\delta_P)$.

Finally, in Example 9.5 we similarly deduce

$$(\mathrm{Det}\,\nabla\widetilde u)^s = \frac{2\pi}{3}\cdot\sum_{j=1}^\infty\frac{1}{j^2}(\delta_{P_j}+\delta_{Q_j})\,,\qquad |(\mathrm{Det}\,\nabla\widetilde u)^s|(\widetilde\Omega) = \frac{4\pi}{3}\cdot\sum_{j=1}^\infty\frac{1}{j^2}<\infty\,.$$

Example 11.5 In general it may happen that the singular part of the distributional determinant is zero, even if the boundary current ∂G_u is non-trivial. Take e.g. $n=2$ and $u:B^2\to\widehat{\mathbb{R}}^2$ the homogeneous extension $u(x):=\varphi\Big(\dfrac{x}{|x|}\Big)$ of the Lipschitz map $\varphi:\mathbb{S}^1\to\widehat{\mathbb{R}}^2$ defined in terms of the angle θ by

$$\varphi(\theta) := \begin{cases} \big(-1+\cos 2\theta,\sin 2\theta\big) & \text{if}\quad 0\le\theta<\pi\\ \big(1-\cos 2\theta,\sin 2\theta\big) & \text{if}\quad \pi\le\theta<2\pi\,.\end{cases}$$

Clearly $u\in W^{1,p}(B^2,\widehat{\mathbb{R}}^2)$ for any $p<2$ and $\det\nabla u=0$, hence $u\in\mathcal{A}^1(B^2,\widehat{\mathbb{R}}^2)$. Moreover

$$(\partial G_u)\llcorner B^2\times\mathbb{R}^2 = -\delta_0\times\varphi_\#[\![\,\mathbb{S}^1\,]\!]\,,\qquad \varphi_\#[\![\,\mathbb{S}^1\,]\!] = [\![\,\Sigma^1_-\,]\!]-[\![\,\Sigma^1_+\,]\!]\,,$$

where $\Sigma^1_\pm := \{y\in\widehat{\mathbb{R}}^2 \,:\, |y-(\pm1,0)|=1\}$. Since $[\![\,\Sigma^1_\pm\,]\!](\omega_2)=\pi$, by (11.7) we deduce that $(\mathrm{Det}\,\nabla u)^s=0$, even if $(\partial G_u)\llcorner B^2\times\widehat{\mathbb{R}}^2\neq 0$, and the singular set $S_0=\{0\}$, see (1.8).

Distributional minors. Let us fix the order $2\le m\le\min(n,N)$. Also, let α and β be any multi-indices with length $|\alpha|=n-m$ and $|\beta|=m$. In a similar way, if $u:\Omega\to\mathbb{R}^N$ is sufficiently smooth, the *distributional minor* of indices $\overline\alpha$ and β of ∇u is well-defined by

$$\mathrm{Div}^\beta_{\overline\alpha}u := \frac{1}{|\beta|}\sum_{j\in\beta}\sum_{i\in\overline\alpha}\frac{\partial}{\partial x_i}\big(u^j\,(\mathrm{adj}(\nabla u)^\beta_{\overline\alpha})^j_i\big)\,,$$

where $\mathrm{adj}(\nabla u)^\beta_{\overline\alpha}$ is the $(m\times m)$-matrix of the adjoints of $(\nabla u)^\beta_{\overline\alpha}$, see (3.4). More precisely, if $u\in L^\infty(\Omega,\mathbb{R}^N)$ the above formula is well defined in the distributional sense provided that $|\,\mathrm{adj}(\nabla u)^\beta_{\overline\alpha}|\in L^1(\Omega)$, e.g. if $u\in\mathcal{A}^1(\Omega,\mathbb{R}^N)$ or $u\in W^{1,m-1}(\Omega,\mathbb{R}^N)$. If u is not essentially bounded, it is well-defined provided that $u\in L^q\cap W^{1,p}(\Omega,\widehat{\mathbb{R}}^n)$ for some exponents q and p satisfying

$$m-1<p<m\qquad\text{and}\qquad \frac{1}{q}+\frac{m-1}{p}\le 1\,. \tag{11.8}$$

In all these cases, $\mathrm{Div}^\beta_{\overline\alpha}u$ is a signed Radon measure with finite total variation. Moreover, it turns out that $\mathrm{Div}^\beta_{\overline\alpha}u = M^\beta_{\overline\alpha}(\nabla u)\,\mathcal{L}^n$ if u is Lipschitz or even $u\in W^{1,m}(\Omega,\mathbb{R}^N)$, where we recall by (3.3) that $M^\beta_{\overline\alpha}(\nabla u) := \det((\nabla u)^\beta_{\overline\alpha})$.

Denote by $\omega^\alpha_\varphi\in\mathcal{D}^{n-m}(\Omega)$ the $(n-m)$-form associated to α and $\varphi\in C^\infty_c(\Omega)$ by

$$\omega^\alpha_\varphi(x) := (-1)^{|\alpha|}\sigma(\alpha,\overline\alpha)\,\varphi(x)\,dx^\alpha\,, \tag{11.9}$$

and set $\omega^\alpha_\varphi := \varphi$ if $m=n$. Moreover, denote by

$$\omega_\beta := \frac{1}{|\beta|}\sum_{j\in\beta}\sigma(j,\beta-j)\,y_j\,dy^{\beta-j} \tag{11.10}$$

the form in $\mathcal{E}^{m-1}(\mathbb{R}^N)$ associated to β, so that

$$d(\omega_\beta\wedge\omega^\alpha_\varphi) = d\omega_\beta\wedge\omega^\alpha_\varphi + (-1)^{m-1}\omega_\beta\wedge d\omega^\alpha_\varphi\,,\quad d\omega_\beta = dy^\beta\,. \tag{11.11}$$

Using the notation (3.4), since

$$u^\# dy^{\beta-j}\wedge d\omega^\alpha_\varphi = (-1)^{m-1}\sum_{i\in\alpha}(\mathrm{adj}(\nabla u)^\beta_{\overline{\alpha}})^j_i\,\frac{\partial\varphi}{\partial x_i}\,dx^1\wedge\cdots\wedge dx^n\,,$$

we deduce that equivalently

$$\langle \mathrm{Div}^\beta_{\overline{\alpha}}\,u,\varphi\rangle := (-1)^m\int_\Omega u^\#\omega_\beta\wedge d\omega^\alpha_\varphi\,,\qquad \varphi\in C^\infty_c(\Omega)\,.$$

Therefore, by (3.12) we similarly obtain

$$\langle \mathrm{Div}^\beta_{\overline{\alpha}}\,u,\varphi\rangle = (-1)^m\,G_u(\omega_\beta\wedge d\omega^\alpha_\varphi)\,. \tag{11.12}$$

If $u\in L^\infty\cap\mathcal{A}^1(\Omega,\mathbb{R}^N)$, by (11.11) and (11.12) this time we get

$$\langle \mathrm{Div}^\beta_{\overline{\alpha}}\,u,\varphi\rangle = G_u(dy^\beta\wedge\omega^\alpha_\varphi) - \partial G_u(\omega_\beta\wedge\omega^\alpha_\varphi)\,. \tag{11.13}$$

Moreover, by (3.12) we have

$$G_u(dy^\beta\wedge\omega^\alpha_\varphi) = \int_\Omega \varphi(x)\,M^\beta_{\overline{\alpha}}(\nabla u(x))\,dx\,.$$

Therefore, compare the first part of [24, Prop. 4.9], if the boundary of the graph current G_u has finite mass in $\Omega\times\mathbb{R}^N$, i.e., (1.5) holds true, by the boundary rectifiability theorem 3.4 we deduce the decomposition

$$\langle(\mathrm{Div}^\beta_{\overline{\alpha}}\,u)^a,\varphi\rangle = \langle M^\beta_{\overline{\alpha}}(\nabla u)\,\mathcal{L}^n,\varphi\rangle\,,\qquad \langle(\mathrm{Div}^\beta_{\overline{\alpha}}\,u)^s,\varphi\rangle = -\partial G_u(\omega_\beta\wedge\omega^\alpha_\varphi) \tag{11.14}$$

into the absolute continuous and singular parts, for every bounded Borel function φ.

We now extend [25, Prop. 5.3] by *removing the L^∞-condition*, showing in particular that $\mathrm{Div}^\beta_{\overline{\alpha}}\,u$ has no "Cantor-type" part, its singular part being concentrated on a countably $\mathcal{H}^{n-m}$-rectifiable subset of Ω.

Theorem 11.6 *Let $u:\Omega\to\mathbb{R}^N$ satisfy the hypotheses of Theorem 1.2. Let $2\le m\le\min(n,N)$. Assume in addition that $u\in L^q\cap W^{1,p}(\Omega,\mathbb{R}^N)$, where q and p satisfy (11.8), or $u\in L^\infty\cap W^{1,m-1}(\Omega,\mathbb{R}^N)$. Alternatively, assume that $u\in L^\infty$ and $(\partial G_u)_{(m-2)}\llcorner\Omega\times\mathbb{R}^N=0$. Then for each α and β such that $|\alpha|=n-m$ and $|\beta|=m$, we have $(\mathrm{Div}^\beta_{\overline{\alpha}}\,u)^a=(M^\beta_{\overline{\alpha}}(\nabla u))\,\mathcal{L}^n$. Moreover, the singular part $(\mathrm{Div}^\beta_{\overline{\alpha}}\,u)^s$ w.r.t. the Lebesgue measure $\mathcal{L}^n$ is concentrated on the countably $\mathcal{H}^{n-m}$-rectifiable set S_{n-m} given by (1.8), where $k=m-1$.*

PROOF: Assume first that $u\in L^\infty$ and $(\partial G_u)_{(m-2)}\llcorner\Omega\times\widehat{\mathbb{R}}^n=0$. By Corollary 8.1 we may and do choose the components T_k in such a way that $T_{m-2}=0$ and $(T_k)_{(h)}=0$ for every $0\le k\le m-2$ and $h\ge m-2$. Using property iv) in Theorem 1.2, this yields that $T_k(\omega_\beta\wedge\omega^\alpha_\varphi)=0$ for $k\ne m-1$. By (11.14) and (1.6), we thus obtain

$$\langle(\mathrm{Div}^\beta_{\overline{\alpha}}\,u)^s,\varphi\rangle = -T_{m-1}(\omega_\beta\wedge\omega^\alpha_\varphi)$$

and hence the claim follows from the property v) of Theorem 1.2, where $k=m-1$.

Assume now that $u\in L^q\cap W^{1,p}(\Omega,\mathbb{R}^N)$, where q and p satisfy (11.8), or $u\in L^\infty\cap W^{1,m-1}(\Omega,\mathbb{R}^N)$. Property iv) in Theorem 1.2 yields that if $k>m-1$, then $T_k(\eta^{(m-1)})=0$ for every $\eta\in\mathcal{D}^{n-1}(\Omega\times\mathbb{R}^N)$. By the properties v) and viii), where $p=m-1$, we thus obtain that

$$(\partial G_u)(\eta^{(m-1)}) = T_{m-1}(\eta^{(m-1)})\,,\qquad \mathrm{set}(T_{m-1})\subset S_{n-m}\times\mathbb{R}^N \tag{11.15}$$

where T_{m-1} is a boundaryless i.m. rectifiable current in $\mathcal{R}_{n-1}(\Omega\times\mathbb{R}^N)$ with finite mass, $\mathbf{M}(T_{m-1})<\infty$, and S_{m-1} is a countably $\mathcal{H}^{n-m}$-rectifiable subset of Ω.

For $R>0$, choose a cut-off function $\chi_R\in C_c^\infty([0,+\infty))$ as in the proof of Lemma 3.10. By (11.12), for every $\varphi\in C_c^\infty(\Omega)$ we have

$$\langle \mathrm{Div}^\beta_{\overline{\alpha}} u,\varphi\rangle = (-1)^m\, G_u(\chi_R(|y_\beta|)\,\omega_\beta\wedge d\omega_\varphi^\alpha) + (-1)^m\, G_u((1-\chi_R(|y_\beta|))\,\omega_\beta\wedge d\omega^\alpha_\varphi)\,. \tag{11.16}$$

The form $\chi_R(|y_\beta|)\,\omega_\beta\wedge\omega^\alpha_\varphi$ has bounded Lipschitz coefficients in $\Omega\times\mathbb{R}^N$, is compactly supported inside the cylinder $\Omega\times\mathbb{R}^N$, and

$$d(\chi_R(|y_\beta|)\,\omega_\beta\wedge\omega^\alpha_\varphi) = d(\chi_R(|y_\beta|)\,\omega_\beta)\wedge\omega^\alpha_\varphi + (-1)^{m-1}\chi_R(|y_\beta|)\,\omega_\beta\wedge d\omega^\alpha_\varphi\,.$$

Using that $\mathbf{M}(G_u)+\mathbf{M}((\partial G_u)\llcorner\Omega\times\mathbb{R}^N)<\infty$, we can write

$$(-1)^m\, G_u(\chi_R(|y_\beta|)\,\omega_\beta\wedge d\omega^\alpha_\varphi) = G_u(d(\chi_R(|y_\beta|)\,\omega_\beta)\wedge\omega^\alpha_\varphi) - \partial G_u(\chi_R(|y_\beta|)\,\omega_\beta\wedge\omega^\alpha_\varphi)\,, \tag{11.17}$$

compare (11.13). Now, we have

$$d(\chi_R(|y_\beta|)\,\omega_\beta) = \chi_R(|y_\beta|)\,d\omega_\beta + \chi_R'(|y_\beta|)\,d|y_\beta|\wedge\omega_\beta$$

whereas by (11.10)

$$d|y_\beta|\wedge\omega_\beta = \Big(\sum_{j\in\beta}\frac{y_j}{|y_\beta|}\,dy^j\Big)\wedge\omega_\beta = \frac{1}{m}\,|y_\beta|\,d\omega_\beta\,.$$

By (3.12), and using that

$$u^\# d\omega_\beta\wedge\omega^\alpha_\varphi = M^\beta_{\overline{\alpha}}(\nabla u)\,\varphi(x)\,dx\,,$$

where $M^\beta_{\overline{\alpha}}(\nabla u)\in L^1(\Omega)$, we find

$$G_u(d(\chi_R(|y_\beta|)\,\omega_\beta)\wedge\omega^\alpha_\varphi) = \int_\Omega \varphi(x)\,\Big(\chi_R(|u^\beta|)+\frac{1}{m}\chi_R'(|u^\beta|)\,|u^\beta|\Big)\,M^\beta_{\overline{\alpha}}(\nabla u(x))\,dx\,.$$

We claim that there exists an increasing sequence $\{R_j\}$ of integer radii $R_j\to\infty$ such that

$$\lim_{j\to\infty} G_u(d(\chi_{R_j}(|y_\beta|)\,\omega_\beta)\wedge\omega^\alpha_\varphi) = \int_\Omega \varphi(x)\,M^\beta_{\overline{\alpha}}(\nabla u(x))\,dx\,. \tag{11.18}$$

In fact, since $u^\beta\in L^1$, we have that $\chi_R(|u^\beta|)\to 1$ a.e. in Ω, whereas by the hypothesis $M^\beta_{\overline{\alpha}}(\nabla u)\in L^1(\Omega)$. Whence, by the dominated convergence for every bounded Borel function φ

$$\lim_{R\to\infty}\int_\Omega \varphi(x)\,\chi_R(|u^\beta|)\,M^\beta_{\overline{\alpha}}(\nabla u(x))\,dx = \int_\Omega \varphi(x)\,M^\beta_{\overline{\alpha}}(\nabla u(x))\,dx\,.$$

Moreover, $\chi_R'(|u^\beta|)$ is uniformly bounded and supported in $A_R:=\{x\in\Omega\mid R\le|u^\beta(x)|<R+1\}$. Setting

$$a_j := \int_{A_j}|M^\beta_{\overline{\alpha}}(\nabla u(x))|\,dx\,,\qquad R=j\in\mathbb{N}\,,$$

condition $M^\beta_{\overline{\alpha}}(\nabla u)\in L^1(\Omega)$ yields that $\sum_j a_j<\infty$, whence $\liminf\limits_{j\to\infty}(j+1)\,a_j=0$. Therefore, the claim (11.18) follows by observing that

$$\left|\int_\Omega \varphi(x)\,\frac{1}{m}\chi_R'(|u^\beta|)\,|u^\beta|\,M^\beta_{\overline{\alpha}}(\nabla u(x))\,dx\right| \le c\,\|\varphi\|_\infty\,(j+1)\,a_j\,.$$

On the other hand, by (11.15) we have

$$(\partial G_u)(\chi_R(|y_\beta|)\,\omega_\beta\wedge\omega^\alpha_\varphi) = T_{m-1}(\chi_R(|y_\beta|)\,\omega_\beta\wedge\omega^\alpha_\varphi)\,. \tag{11.19}$$

Similarly, we get

$$G_u((1-\chi_R(|y_\beta|))\,\omega_\beta\wedge d\omega^\alpha_\varphi) = \int_\Omega((1-\chi_R(|u^\beta|))\,u^\#\omega_\beta\wedge d\omega^\alpha_\varphi\,.$$

Since $u \in L^q \cap W^{1,p}(\Omega, \mathbb{R}^N)$, where q and p satisfy (11.8), or $u \in L^\infty \cap W^{1,m-1}(\Omega, \mathbb{R}^N)$, by dominated convergence, and using that $(1 - \chi_R(|u^\beta|)) \to 0$ a.e. in Ω as $R \to \infty$, we deduce

$$\lim_{R\to\infty} G_u((1 - \chi_R(|y_\beta|))\, \omega_\beta \wedge d\omega_\varphi^\alpha) = 0\,. \tag{11.20}$$

By (11.16), (11.17), (11.18), (11.19), and (11.20), we obtain that

$$\langle \mathrm{Div}_{\overline{\alpha}}^{\beta} u, \varphi\rangle = \int_\Omega M_{\overline{\alpha}}^{\beta}(\nabla u(x))\, \varphi(x)\, dx + \lim_{j\to\infty} \langle \mu_{R_j}, \varphi\rangle\,,$$

where the increasing sequence $R_j \nearrow \infty$ is chosen as in (11.18) and

$$\langle \mu_{R_j}, \varphi\rangle := -T_{m-1}(\chi_{R_j}(|y_\beta|)\, \omega_\beta \wedge \omega_\varphi^\alpha)\,.$$

Since by (11.15) all the measures μ_{R_j} are concentrated on the set S_{n-m}, the claim is proved. $\square$

Remark 11.7 In the case $m = 1$, if $\beta = j$ and $\overline{\alpha} = i$, we have $(\mathrm{adj}\, \nabla u)_{\overline{\alpha}}^{\beta} = 1$ and $\mathrm{Div}_{\overline{\alpha}}^{\beta} u = D_i u^j$. Therefore, Theorem 11.6 is the higher order counterpart of well-known features concerning the class SBV_0 in (1.19), compare Thm. 3.1 and Thm. 3.4 from [5], and also (1.18).

Remark 11.8 If $u : B^2 \to \mathbb{R}^3$ is the map from Example 5.1, we have seen that $(\partial G_{u^\beta}) \llcorner \Omega \times \widehat{\mathbb{R}}_\beta^2 = 0$ for every $|\beta| = 2$. Moreover, by the area formula we infer that $M_{\overline{\alpha}}^{\beta}(\nabla u) = 0$ for $|\alpha| = 1$ and $|\beta| = 2$. This yields that all the corresponding distributional minors $\mathrm{Div}_{\overline{\alpha}}^{\beta} u = 0$, even if $(\partial G_u)_{(1)} \llcorner \Omega \times \mathbb{R}^3 \neq 0$.

Functions of bounded higher variation. Assume now $n \geq N \geq 2$. Jerrard-Soner [22] introduced the class $\mathrm{B_N V}(\Omega, \mathbb{R}^N)$ of functions of *bounded higher variation*. On account of Remark 11.4, in our setting the corresponding definition is the following one.

According to (11.3), consider the $(N-1)$-form in $\mathbb{R}^N$

$$\omega_N := \frac{1}{N} \sum_{j=1}^{N} (-1)^{j-1} y_j\, \widehat{dy^j}\,, \tag{11.21}$$

so that $d\omega_N = dy^1 \wedge \cdots \wedge dy^N$. Notice that it agrees with ω_β in (11.10) when $\beta = (1, \ldots, N)$.

If $u \in \mathcal{A}^1(\Omega, \mathbb{R}^N) \cap L^\infty$, the current $j_u := (-1)^N \pi_\#(G_u \llcorner \widehat{\pi}^\# \omega_N) \in \mathcal{D}_{n-N+1}(\Omega)$, given by

$$\langle j_u, \eta\rangle := (-1)^N G_u(\omega_N \wedge \eta)\,, \qquad \eta \in \mathcal{D}^{n-N+1}(\Omega)\,,$$

is well defined and has finite mass. The same property holds true if $u \in W^{1,N-1}(\Omega, \mathbb{R}^N) \cap L^\infty$ or $u \in W^{1,p}(\Omega, \mathbb{R}^N) \cap L^q$, where the exponents p, q satisfy (11.8), with $m = N$. Consider the boundary current

$$\mathfrak{J}_u := (\partial j_u) \llcorner \Omega \in \mathcal{D}_{n-N}(\Omega)\,.$$

Definition 11.9 *We say that $u \in \mathrm{B_N V}(\Omega, \mathbb{R}^N)$ if the current $\mathfrak{J}_u \in \mathcal{D}_{n-N}(\Omega)$ has finite mass.*

We now see that the current $\mathfrak{J}_u$ corresponds to our notion of distributional minors of higher order. In fact, for every form $\xi \in \mathcal{D}^{n-N}(\Omega)$ we have

$$\langle \mathfrak{J}_u, \xi\rangle := (-1)^N G_u(\omega_N \wedge d\xi)\,.$$

Therefore, by (11.5) we get

$$\langle \mathfrak{J}_u, \xi\rangle = \langle \mathrm{Det}\, \nabla u, \xi\rangle\,, \qquad \text{if } n = N\,.$$

In a similar way, if $n > N$, we can always write

$$\xi = \sum_{|\alpha| = n-N} \omega_{\varphi^\alpha}^{\alpha}\,, \qquad \varphi^\alpha \in C_c^\infty(\Omega)\,, \tag{11.22}$$

where $\omega_\varphi^\alpha \in \mathcal{D}^{n-N}(\Omega)$ is given by (11.9), so that

$$\xi = (-1)^{n-N} \sum_{|\alpha|=n-N} \sigma(\alpha, \overline{\alpha})\, \varphi^\alpha(x)\, dx^\alpha\,.$$

By (11.13), where we choose $\beta = \overline{0} = (1, \ldots, N)$ and $m = N$, this time we obtain the decomposition

$$\langle \mathfrak{J}_u, \xi \rangle = \sum_{|\alpha|=n-N} \langle (\mathfrak{J}_u^\alpha), \xi \rangle\,, \qquad \text{where} \quad \langle (\mathfrak{J}_u^\alpha), \xi \rangle := \langle \mathrm{Div}_{\overline{\alpha}}^{\overline{0}} u, \varphi^\alpha \rangle \tag{11.23}$$

if ξ is written as in (11.22). As a consequence, compare Theorem 1.9, we readily obtain:

Corollary 11.10 *Let $n \geq N \geq 2$ and $u : \Omega \to \mathbb{R}^N$ satisfy the hypotheses of Theorem* 11.6*, where we choose $m = N$. Then u is a function of bounded higher variation. Moreover, each component $(\mathfrak{J}_u)^\alpha$ of the current $\mathfrak{J}_u$ in* (11.23) *can be written as*

$$(\mathfrak{J}_u)^\alpha = M_{\overline{\alpha}}^{\overline{0}}(\nabla u)\, \mathcal{L}^n + ((\mathfrak{J}_u)^\alpha)^s\,,$$

where the singular part $((\mathfrak{J}_u)^\alpha)^s$ w.r.t. the Lebesgue measure $\mathcal{L}^n$ is concentrated on a countably $\mathcal{H}^{n-N}$-rectifiable set. Finally, $((\mathfrak{J}_u)^\alpha)^s = 0$ if u is Lipschitz-continuous or at least in $W^{1,N}(\Omega, \mathbb{R}^N)$.

Maps into the sphere. As we have seen in Remark 11.2, the membership of u to the class $\mathrm{B_N V}(\Omega, \mathbb{R}^N)$ does not imply the bound (1.5) on the mass of the boundary of G_u, even for maps $u \in L^\infty \cap \mathcal{A}^1(\Omega, \mathbb{R}^N)$ that belong to the Sobolev class $W^{1,p}$ for every $p < n$. Denote now

$$W^{1,p}(\Omega, \mathbb{S}^{N-1}) := \{u \in W^{1,p}(\Omega, \mathbb{R}^N) \,:\, |u(x)| = 1 \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in \Omega\}$$

the class of bounded Sobolev maps that take values into the unit sphere $\mathbb{S}^{N-1}$ of the target space. Also, let B^N the unit ball in the target space, and equip $\mathbb{S}^{N-1}$ with the natural orientation, so that

$$[\![\, \mathbb{S}^{N-1}]\!](\omega_N) = \partial [\![\, B^N]\!](\omega_N) = [\![\, B^N]\!](d\omega_N) = [\![\, B^N]\!](dy^1 \wedge \cdots \wedge dy^N) = \mathcal{L}^N(B^N)\,. \tag{11.24}$$

Following the argument from [22, Sec. 6] that is due to M. Giaquinta and G. Modica, we finally recall that *the converse of Corollary* 11.10 *holds true*, provided that $u \in W^{1,N-1}(\Omega, \mathbb{S}^{N-1})$, see Proposition 11.11. To this purpose, notice that in this case by the area formula $M_{\overline{\alpha}}^{\overline{0}}(\nabla u) = 0$ for each multi-index α of length $|\alpha| = n - N$, i.e., for each minor of maximum order N of ∇u. As a consequence,

$$W^{1,N-1}(\Omega, \mathbb{S}^{N-1}) \subset \mathcal{A}^1(\Omega, \mathbb{R}^N) \cap L^\infty\,. \tag{11.25}$$

Moreover, using that for $\xi \in \mathcal{D}^{n-N}(\Omega)$

$$d(\omega_N \wedge \xi) = d\omega_N \wedge \xi + (-1)^{N-1} \omega_N \wedge d\xi\,,$$

and that

$$d\omega_N \wedge \xi = (-1)^{N(N-n)} \xi \wedge d\omega_N\,, \qquad \omega_N \wedge \xi = (-1)^{(N-1)(N-n)} \xi \wedge \omega_N\,,$$

for each $u \in \mathcal{A}^1(\Omega, \mathbb{R}^N) \cap L^\infty$ we have

$$\begin{aligned} \langle \mathfrak{J}_u, \xi \rangle &:= (-1)^N G_u(\omega_N \wedge d\xi) = G_u(d\omega_N \wedge \xi) - \partial G_u(\omega_N \wedge \xi) \\ &= (-1)^{N(N-n)} G_u(\xi \wedge d\omega_N) - (-1)^{(N-1)(N-n)} \partial G_u(\xi \wedge \omega_N)\,. \end{aligned} \tag{11.26}$$

Proposition 11.11 ([22, Sec. 6]) *Let $n \geq N \geq 2$ and $u \in W^{1,N-1}(\Omega, \mathbb{S}^{N-1})$. Then*

$$(\partial G_u) \llcorner \Omega \times \mathbb{R}^N = -\sigma(n, N) \cdot \frac{1}{\alpha_N} \cdot \mathfrak{J}_u \times [\![\, \mathbb{S}^{N-1}]\!]\,, \tag{11.27}$$

where $\sigma(n, N) := (-1)^{(N-1)(n-N)}$ and $\alpha_N := \mathcal{L}^N(B^N)$. Therefore, u is a function of bounded higher variation in $\mathrm{B_N V}$*, i.e., $\mathbf{M}(\mathfrak{J}_u) < \infty$, see Definition* 11.9*, if and only if $\mathbf{M}((\partial G_u) \llcorner \Omega \times \mathbb{R}^N) < \infty$, see* (1.5)*. In this case, moreover, the current $\alpha_N^{-1} \cdot \mathfrak{J}_u$ is i.m. rectifiable in $\mathcal{R}_{n-N}(\Omega)$.*

Proof: The inclusion (11.25) yields that $G_u \in \mathcal{R}_{n,c}(\Omega \times \mathbb{R}^N)$ for every $u \in W^{1,N-1}(\Omega, \mathbb{S}^{N-1})$. Moreover, Federer's flatness theorem [17] yields that actually G_u is an i.m. rectifiable current in $\mathcal{R}_n(\Omega \times \mathbb{S}^{N-1})$. Arguing as in Proposition 3.8, one has $\partial G_u(\eta) = 0$ for every form $\eta \in \mathcal{D}^{n-1}(\Omega \times \mathbb{S}^{N-1})$ such that $\eta^{(N-1)} = 0$. Therefore, for every $(N-2)$-form $\gamma \in \mathcal{D}^{N-2}(\Omega \times \mathbb{S}^{N-1})$, condition $(d_x\gamma)^{(N-1)} = 0$ yields $\partial G_u(d_x\gamma) = 0$. Denote by d_y the tangential differential in $\mathbb{S}^{N-1}$. Since $d_x \circ d_x = d_y \circ d_y = 0$ and $d_x \circ d_y = -d_y \circ d_x$, we thus obtain

$$\partial G_u(d_y\gamma) = G_u(d_x \circ d_y\gamma) = -G_u(d_y \circ d_x\gamma) = -\partial G_u(d_x\gamma) = 0\,. \tag{11.28}$$

By Hodge decomposition theorem, see e.g. [19, Vol. I, Sec. 5.2.5], for every $(N-1)$-form $\alpha \in \mathcal{D}^{N-1}(\mathbb{S}^{N-1})$ there exists a real number $\lambda \in \mathbb{R}$ and an $(N-2)$-form $\beta \in \mathcal{D}^{N-2}(\mathbb{S}^{N-1})$ such that $\alpha = \lambda\,\omega_N + d\beta$. Using (11.28), for every $(n-N)$-form $\xi \in \mathcal{D}^{n-N}(\Omega)$ we thus deduce

$$\partial G_u(\xi \wedge \alpha) = \partial G_u(\lambda\,\xi \wedge \omega_N) + \partial G_u(\xi \wedge d\beta) = \lambda \cdot \partial G_u(\xi \wedge \omega_N) + 0\,.$$

Therefore, by (11.26), and observing that by the area formula

$$G_u(\xi \wedge d\omega_N) = G_u(\xi \wedge dy^1 \wedge \cdots \wedge dy^N) = \int_\Omega \xi \wedge du^1 \wedge \cdots \wedge du^N = 0\,,$$

as $M^{\bar{0}}_{\overline{\alpha}}(\nabla u) = 0$ for each multi-index α of length $|\alpha| = n - N$, we obtain

$$\partial G_u(\xi \wedge \alpha) = -\sigma(n,N) \cdot \lambda \cdot \langle \mathfrak{J}_u, \xi \rangle\,.$$

On the other hand, since $[\![\,\mathbb{S}^{N-1}\,]\!](d\beta) = \partial[\![\,\mathbb{S}^{N-1}\,]\!](\beta) = 0$, using (11.24) we have

$$\mathfrak{J}_u \times [\![\,\mathbb{S}^{N-1}\,]\!](\xi \wedge \alpha) = \langle \mathfrak{J}_u, \xi \rangle \cdot \big(\lambda \cdot [\![\,\mathbb{S}^{N-1}\,]\!](\omega_N) + [\![\,\mathbb{S}^{N-1}\,]\!](d\beta)\big) = \lambda\,\alpha_N \cdot \langle \mathfrak{J}_u, \xi \rangle\,.$$

By density of forms $\xi \wedge \alpha$ as above among the forms $\eta = \eta^{(N-1)} \in \mathcal{D}^{N-1}(\Omega \times \mathbb{S}^{N-1})$, this yields (11.27). Since $G_u \in \mathcal{R}_n(\Omega \times \mathbb{S}^{N-1})$, the last assertion follows from the boundary rectifiability theorem 3.4. $\square$

References

[1] ALBERTI G., MANTEGAZZA C., A note on the theory of *SBV* functions, *Boll. UMI (7-B)* **11** (1997), 375–382.

[2] ALMGREN F.J., *Deformations and multiple-valued functions*, In: "Geometric measure theory and the calculus of variations", Proc. Symposia Pure Math. Amer. Math. Soc., R.I., **44** (1986), 29–130.

[3] AMBROSIO L., A compactness theorem for a special class of functions of bounded variation. *Boll. UMI (3-B)* **7** (1989), 857-881.

[4] AMBROSIO L., A new proof of the *SBV* compactness theorem. *Calc. Var. Partial Differential Equations* **3** (1995), 127-137.

[5] AMBROSIO L., BRAIDES A., GARRONI A., Special functions with bounded variation and with weakly differentiable traces on the jump set, *NoDEA Nonlinear Differential Equations Appl.* **5** (1998), 219–243.

[6] AMBROSIO L., FUSCO N., PALLARA D., *Partial regularity of free discontinuity sets II*, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (4)* **24** (1997), 39–62.

[7] AMBROSIO L., FUSCO N., PALLARA D., *Functions of bounded variation and free discontinuity problems,* Oxford Math. Monographs, Oxford 2000.

[8] AMBROSIO L., KIRCHHEIM B., Currents in metric spaces, *Acta Math.* **185** (2000), 1–80.

[9] AMBROSIO L., PALLARA D., *Partial regularity of free discontinuity sets I*, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (4)* **24** (1997), 1–38.

[10] BALL J.M., Convexity conditions and existence theorems in nonlinear elasticity, *Arch. Rational Mech. Anal.* **63** (1977), 337–403.

[11] CONTI S., DE LELLIS C., Some remarks on the theory of elasticity for compressible Neohookean materials, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **2** (2003), 521–549.

[12] DE GIORGI E., AMBROSIO L., Un nuovo tipo di funzionale del calcolo delle variazioni. *Rend. Accad. Naz. Lincei, Cl. Sci. Fis. Mat. Natur.* **82** (1988), 199-210.

[13] DE GIORGI E., CARRIERO M. LEACI A., Existence theorem for a minimum problem with free discontinuity set. *Arch. Ration. Mech. Anal.* **108** (1989), 195-218.

[14] DE LELLIS C., Some fine properties of currents and applications to distributional Jacobians, *Proc. Royal Soc. Edinburgh Sect. A* **132** (2002), 815–842.

[15] DE LELLIS C., GHIRALDIN F., An extension of the identity Det = det, *C. R. Acad. Sci. Paris, Ser. 1* **348** (2010), 973–976.

[16] FEDERER H., *Geometric measure theory,* Grundlehren math. wissen. 153, Springer, New York (1969).

[17] FEDERER H., Real flat chains, cochains and variational problems, *Indiana Univ. Math. J.* **24** (1974), 351–407.

[18] FEDERER H., FLEMING W., Normal and integral currents, *Ann. of Math.* **72** (1960), 458–520.

[19] GIAQUINTA M., MODICA G., SOUČEK J., *Cartesian currents in the calculus of variations,* I, II. Ergebnisse Math. Grenzgebiete (III Ser), 37, 38, Springer, Berlin (1998).

[20] GIAQUINTA M., MUCCI D., *Maps into manifolds and currents: area and* $W^{1,2}$*-,* $W^{1/2}$*-,* BV*-energies.* Edizioni della Normale, C.R.M. Series, Sc. Norm. Sup. Pisa (2006).

[21] HENAO D., MORA-CORRAL C., Lusin's condition and the distributional determinant for deformations with finite energy. *Adv. Calc. Var.* DOI 10.1515/ACV-2011.016 (2011).

[22] JERRARD R.L., SONER H.M., Functions of higher bounded variation, *Indiana Univ. Math. J.* **51** (2002), 645–677.

[23] MORGAN F., *Geometric measure theory. A beginner's guide,* Academic Press, Inc. Boston, MA (1988).

[24] MUCCI D., Fractures and vector valued maps, *Calc. Var. Partial Differential Equations* **22** (2005), 391–420. Erratum to: Fractures and vector valued maps, *Ibid.* **37** (2010), 547–548.

[25] MUCCI D., A variational problem involving the distributional determinant, *Riv. Mat. Univ. Parma* New Series **1** no. 2 (2010), 321–345.

[26] MÜLLER S., On the singular support of the distributional determinant. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **10** (1993), 657-696.

[27] MÜLLER S., SPECTOR S.J., An existence theory for nonlinear elasticity that allows for cavitation, *Arch. Rational Mech. Anal.* **131** (1995), 1–66.

[28] MUMFORD D., SHAH J., Optimal approximations by piecewise smooth functions and associated variational problems. *Comm. Pure Appl. Math.* **42** (1989), 577-685.

[29] SIMON L., *Lectures on geometric measure theory,* Proc. of the Centre for Math. Analysis, Vol 3, Australian National University, Canberra (1983).

D. Mucci: Dipartimento di Matematica dell'Università di Parma, Parco Area delle Scienze 53/A, I-43124 Parma, Italy. E-mail: domenico.mucci@unipr.it