Bounded variation with respect to a log-concave measure

Luigi Ambrosio*, Giuseppe Da Prato*, Ben Goldys†, Diego Pallara‡

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Abstract

Let H be a separable Hilbert space and let $A : \mathscr{D}(A) \subset H \to H$ be a self-adjoint operator with $A \le \omega I$, $\omega > 0$ and $\mathrm{Tr}\left(-A^{-1}\right) < \infty$. We endow H with the centered Gaussian measure μ with covariance operator $Q = -\frac{1}{2}A^{-1}$ and consider a funtion $U \in C^3(H)$ with bounded derivatives up to the order 3, the SDE $dX = (AX - DU(X))dt + dW(t)$, $X(0) = x$ and the associated transition semigroup P_t. We define the class $BV(H,\gamma)$ of bounded variation functions with respect to the probability measure $\gamma(dx) = Z^{-1}e^{-2U(x)}\mu(dx)$, where Z is the normalization constant, through an integration by parts formula and prove that $P_t u \in W^{1,1}(H,\gamma)$ for $t > 0$, $u \in BV(H,\gamma)$, and that $u \in BV(H,\gamma)$ if and only if the limit of $\|DP_t u\|_{L^1(H,\gamma)}$ as $t \to 0$ is finite.

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1 Notations and preliminaries

Consider the stochastic differential equation in a separable Hilbert space H

$$\left\{\begin{array}{l} dX = (AX - DU(X))dt + dW(t), \\ X(0) = x, \end{array}\right. \tag{1.1}$$

*Scuola Normale Superiore, Piazza dei Cavalieri,7, 56126 Pisa, Italy, e–mail: l.ambrosio@sns.it, g.daprato@sns.it

†School of Mathematics and Statistics, University of New South Wales, Sydney NSW 2052, Australia, e–mail: B.Goldys@unsw.edu.au

‡Dipartimento di Matematica “Ennio De Giorgi”, Università del Salento, C.P.193, 73100, Lecce, Italy, e–mail: diego.pallara@unisalento.it

where $A : \mathscr{D}(A) \subset H \to H$ is self-adjoint,

$$A \le -\omega I, \quad \omega > 0, \quad \text{and} \quad \mathrm{Tr}\left(-A^{-1}\right) < \infty. \tag{1.2}$$

We denote by $\{e_k\}$ an orthonormal basis on H and by $\{\alpha_k\}$ a sequence of positive numbers such that

$$Ae_k = -\alpha_k e_k, \quad k \in \mathbb{N}.$$

The potential U belongs to $C^3(H)$ is convex, DU, D^2U, D^3U are uniformly continuous and bounded, and W is a cylindrical Wiener process in H. We denote by $C_b(H; H)$ the set of uniformly continuous and bounded mappings from H into H and by $FC_b(H; H)$ the set of *cylindrical* vector fields from H to H, i.e., those of the form $\sum_{k=1}^n f_k e_k$ with f_k dependent only on $\langle x, e_1\rangle, \dots, \langle x, e_n\rangle$. As an example, we can consider $H = L^2(0, 1)$ and the potential U given by

$$U(x) = \int_0^1 \sin(x(\xi))d\xi, \qquad x \in H;$$

in this case, equation (1.1) reduces to the reaction-diffusion equation

$$dX = (AX - \cos X)dt + dW(t).$$

By assumption (1.2) the stochastic integral

$$\int_0^t e^{(t-s)A} dW(s), \quad t \ge 0,$$

is well defined in H, see Theorem 5.2 in [8] and continuous in H, see [12]. Therefore, for every $x \in H$ equation (1.1) has a unique continuous in H solution defined as a solution to the integral equation

$$X(t, x) = e^{tA}x - \int_0^t e^{(t-s)A} DU(X(s, x))ds + \int_0^t e^{(t-s)A} dW(s), \quad t \ge 0.$$

Since $DU \in C_b^1(H, H)$, the proof of this fact is standard. By Theorem 7.3.6 in [10], for every $t \ge 0$ the function $H \ni x \mapsto X(t, x) \in L^2(\Omega, \mathscr{F}, \mathbb{P}; H)$ is differentiable in all directions and its derivative, denoted $\xi(t, x)$, belongs to $L(H)$ for all $t \ge 0$, $x \in H$. Moreover, for every $h \in H$ the function $\xi^h(t, x) = \xi(t, x)h$ solves the partial differential equation

$$\frac{d}{dt}\,\xi^h(t, x) = (A - D^2U(X(t, x)))\xi^h(t, x), \quad \xi^h(0, x) = h. \tag{1.3}$$

Computing the solution of the above equation we find that

$$\langle \xi(t,x)e_k, e_k\rangle = \langle e^{tA}e_k, e_k\rangle - \int_0^t \langle e^{(t-s)A}D^2U(X(s,x))\xi(s,x)e_k, e_k\rangle ds,$$

whence, summing on k:

$$\mathrm{Tr}\,[\xi(t,x)] = \mathrm{Tr}\,[e^{tA}] - \int_0^t \mathrm{Tr}\,[e^{(t-s)A}D^2U(X(s,x))\xi(s,x)]ds. \tag{1.4}$$

Since

$$\mathrm{Tr}\,[e^{tA}] = \sum_{k=1}^{\infty} e^{-t\alpha_k},$$

the trace of e^{tA} is finite for any $t > 0$ and summable near $t = 0$. Indeed,

$$\int_0^1 \mathrm{Tr}\,[e^{tA}]\,dt = \int_0^1 \sum_{k=1}^{\infty} e^{-t\alpha_k}\,dt \le 2\sum_{k=1}^{\infty} \frac{1}{\alpha_k}. \tag{1.5}$$

As a consequence of (1.4), we have that $\xi(t,x)$ is of trace class and

$$|\mathrm{Tr}\,[\xi(t,x)]| \le \mathrm{Tr}\,[e^{tA}] + \|D^2U\|_{C_b} \int_0^t e^{-\omega(t-s)}\mathrm{Tr}\,[e^{(t-s)A}]]\,ds.$$

Since U is convex we deduce from (1.3) that

$$\|\xi(t,x)\| \le e^{-\omega t}, \quad \forall x \in H,\ t \ge 0. \tag{1.6}$$

We denote by P_t the transition semigroup,

$$P_t\varphi(x) = \mathbb{E}[\varphi(X(t,x))], \quad \varphi \in B_b(H)$$

and by $\pi_t(x,\cdot)$ the law of $X(t,x)$.

Let μ be the zero-mean Gaussian measure on H with the covariance operator $Q = -\frac{1}{2}\,A^{-1}$. We note that from the boundedness hypothesis on DU it follows that U has at most linear growth as $|x| \to \infty$ and by the Fernique's theorem $e^{-2U} \in L^1(H,\mu)$. We may therefore define a log–concave probability measure

$$\gamma(dx) = Z^{-1}e^{-2U(x)}\mu(dx),$$

where Z is the normalization constant. By Theorem 8.6.3 in [9] the measure γ is the unique invariant measure for the semigroup P_t and P_t is symmetric in $L^2(H,\gamma)$. We also set

$$\rho(x) = Z^{-1}e^{-2U(x)}, \quad x \in H,$$

so that

$$D \log \rho(x) = -2U(x).$$

Moreover, P_t is irreducible and Strong Feller, see e.g. [7, Theorems 3.11, 3.13] hence by the Khasminski theorem P_t is regular (see for example Theorem 4.2.1 in [9]). In particular the law $\pi_t(x,\cdot)$ of $X(t,x)$ is equivalent to γ for any $t>0,\ x\in H$.

We denote by N the infinitesimal generator of P_t in $L^2(H,\gamma)$. The generator N is a perturbation of the Ornstein–Uhlenbeck operator

$$L\varphi = \frac{1}{2}\ \mathrm{Tr}\ [D^2\varphi] + \langle x, AD\varphi\rangle, \quad \forall\, \varphi\in \mathscr{E}_A(H),$$

(where $\mathscr{E}_A(H)$ is the space of exponential functions, that is the linear span of the set of all real parts of functions $x\to e^{i\langle x,h\rangle}$, with $h\in\mathscr{D}(A)$), that is

$$N\varphi = L\varphi - \langle DU, D\varphi\rangle, \quad \forall\, \varphi\in \mathscr{E}_A(H).$$

$\mathscr{E}_A(H)$ is a core both for L and for N, see [7].

Finally, we denote by P_t' the transpose of P_t defined in the dual $C_b(H;H)'$ by

$$\langle \varphi, P_t'\nu\rangle = \langle P_t\varphi,\nu\rangle, \quad \nu\in C_b(H)',$$

and use the same notation when acting (componentwise) on $C_b(H;H)'$.

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2 Functions of bounded variation

For every $k\in\mathbb{N}$, set $\lambda_k = \frac{1}{2\alpha_k}$ and recall the basic integration by parts formula

$$\int_H u\,\langle D\varphi, z\rangle\,d\gamma = \int_H \langle Du, z\rangle\,\varphi\,d\gamma - \int_H u\,\varphi\,\langle D\log\rho, z\rangle\,d\gamma + \int_H \langle Q^{-\frac{1}{2}}z, Q^{-\frac{1}{2}}x\rangle\,u\,\varphi\,d\gamma, \tag{2.1}$$

which is valid for any $u,\varphi\in C_b^1(H)$ and any $z\in Q^{1/2}(H)$. Notice that the series in

$$\langle Q^{-1/2}z, Q^{-1/2}x\rangle = \sum_{k=1}^{\infty} \lambda_k^{-1}\langle z, e_k\rangle\langle x, e_k\rangle,$$

is convergent in $L^2(H,\mu)$ because

$$\int_H |\langle Q^{-1/2}z, Q^{-1/2}x\rangle|^2\mu(dx) = |Q^{-1/2}z|^2.$$

By (2.1) we have in particular that

$$\int_H u\, D_k\varphi\, d\gamma = -\int_H D_k u\,\varphi\, d\gamma - \int_H u\,\varphi\, D_k \log\rho\, d\gamma + \frac{1}{\lambda_k}\int_H x_k\, u\,\varphi\, d\gamma. \tag{2.2}$$

Setting

$$D_k^*\varphi = -D_k\varphi - \varphi\, D_k\log\rho + \frac{1}{\lambda_k} x_k\,\varphi, \tag{2.3}$$

we can write (2.2) as

$$\int_H u\, D_k^*\varphi\, d\gamma = \int_H D_k u\,\varphi\, d\gamma.$$

We shall also introduce the divergence operator div_γ, defined on $FC^1(H,H)$ by

$$\mathrm{div}_\gamma F(x) := \sum_{k\in\mathbb{N}} D_k^*\langle F, e_k\rangle(x), \qquad x\in H.$$

Lemma 2.1. *The gradient operator*

$$D : C_b^1(H) \to L^p(H,\gamma;H), \quad u\mapsto Du,$$

is closable in $L^p(H,\gamma)$ for every $p\in[1,\infty)$.

Proof. Assume that $(u_n)\in C_b^1(H)$ and $F\in L^p(H,\gamma;H)$ are such that

$$\lim_{n\to\infty} u_n = 0, \quad \text{in } L^p(H,\gamma)$$

and

$$\lim_{n\to\infty} Du_n = F, \quad \text{in } L^p(H,\gamma;H).$$

Then by (2.1) it follows that

$$\begin{aligned}\int_H u_n\,\langle D\varphi, z\rangle\, d\gamma = &-\int_H \langle Du_n, z\rangle\,\varphi\, d\gamma - \int_H u_n\,\varphi\,\langle D\log\rho, z\rangle\, d\gamma \\ &+ \int_H \langle Q^{-\frac{1}{2}}z, Q^{-\frac{1}{2}}x\rangle\, u_n\,\varphi\, d\gamma.\end{aligned}$$

Now choose $z \in Q(H)$ and φ such that $\frac{\varphi}{1+|\cdot|} \in C_b^1(H)$. Then we deduce

$$\int_H \langle F, z\rangle \, \varphi \, d\gamma = 0,$$

which implies $F = 0$ because $Q(H)$ is dense in H and the space of all functions $\varphi \in C_b^1(H)$ such that $\frac{\varphi}{1+|\cdot|} \in C_b^1(H)$ is dense in $L^p(H,\gamma)$ by a standard monotone class argument. □

We denote by $H^{1,p}(H,\gamma)$ the domain of the closure of D (which is still denoted by D) in $L^p(H,\gamma)$, $1 \le p < \infty$.

We now define *weak gradients* and *weak Sobolev functions*. We say that $u \in L^1(H,\gamma)$ possesses a weak gradient if there exists $G \in L^1(H,\gamma;H)$ such that

$$\int_H u(x)\, \mathrm{div}_\gamma F(x)\, \gamma(dx) = \int_H \langle DF(x), G(x)\rangle \, \gamma(dx), \quad \forall\, F \in FC_b^1(H;H). \tag{2.4}$$

In this case we set $Du = G$. Then we denote by $W^{1,1}(H,\gamma)$ the set of all $u \in L^1(H,\gamma)$ which possess a weak gradient. Obviously, the inclusion $H^{1,1}(H,\gamma) \subset W^{1,1}(H,\gamma)$ holds; we don't if the converse is also true. However, the following holds.

Proposition 2.2. *The space $W^{1,1}(H,\gamma)$, endowed with the natural norm*

$$\|u\|_{1,1} = \int_H |u| \, \gamma(dx) + \int_H |Du| \, \gamma(dx),$$

is a Banach space.

The proof is obtained passing to the limit in both sides of (2.4) and using the completeness of $L^1(H,\gamma)$ and $L^1(H,\gamma;H)$.

Recalling that the dual of $L^1(H,\gamma;H)$ is precisely $L^\infty(H,\gamma;H)$, see e.g. [11, Corollary 1, page 282], we denote by D^*_∞ the adjoint of the weak gradient in the duality between $L^1(H,\gamma;H)$ and $L^\infty(H,\gamma;H)$, i.e., $F \in L^\infty(H,\gamma;H)$ belongs to the domain $\mathscr{D}(D^*_\infty)$ if and only if

$$\left|\int_H \langle D\varphi(x), F(x)\rangle \gamma(dx)\right| \le C \int_H |\varphi(x)|\gamma(dx),$$

for all $\varphi \in W^{1,1}(H,\gamma)$ and some constant $C > 0$. In this case, there is $g \in L^\infty(H,\gamma)$ such that

$$\int_H u\, g \, d\gamma = -\int_H \langle Du, F\rangle \, d\gamma, \quad u \in W^{1,1}(H,\gamma),$$

we denote g by $D^*_\infty F$ and notice that the inclusion $FC_b^1(H;H) \subset \mathscr{D}(D^*_\infty)$ and the equality $D^*_\infty F = \mathrm{div}_\gamma F$ hold.

Example 2.3. Let $F(x) = \psi(x)z,\ x \in H$, where $\psi \in W^{1,1}(H,\gamma)$ and $z \in Q^{1/2}(H)$, and assume that $\psi(x)(1+|x|) \in L^\infty(H,\mu)$. Then $F \in \mathscr{D}(D^*_\infty)$ and we have

$$D^*_\infty(F)(x) = -\langle D\psi(x), z\rangle - \psi(x)\langle D\log\rho(x), z\rangle + \psi(x)\langle Q^{-1/2}x, Q^{-1/2}z\rangle.$$

Let us come to BV functions. We recall that a vector-valued measure M is a mapping defined on the Borel σ-algebra of H such that $M(\emptyset) = 0$ and for every sequence (B_n) of pairwise disjoint Borel sets we have

$$M\left(\bigcup_n B_n\right) = \sum_n M\left(B_n\right),$$

where the series converges in the norm topology.

The total variation measure M_{TV} of M is defined by

$$M_{TV}(B) = \sup \sum_n \left|M\left(B_n\right)\right|, \quad B \text{ Borel},$$

where the supremum is taken over all countable Borel partitions (B_n) of B. It is well known that M_{TV} is a countably additive positive measure. If it is finite we say that M has finite total variation. We denote by $\mathscr{M}(H,H) \subset C_b(H;H)'$ the set of all vector-valued measures defined on the Borel σ-algebra of H which are of finite total variation.

Let $F \in B_b(H;H)$ and set $F_k(x) = \langle F(x), e_k\rangle,\ k \in \mathbb{N}$. We define the integral of F with respect to $M \in \mathscr{M}(H,H)$ by setting

$$\int_H \langle F(x), M(dx)\rangle = \sum_{k=1}^\infty \int_H F_k(x) M_k(dx),$$

where $M_k(B) = \langle M(B), e_k\rangle$. Notice that the inequality

$$\Big|\int_H \langle F(x), M(dx)\rangle\Big| \le \int_H |F(x)| M_{TV}(dx) \tag{2.5}$$

holds.

Definition 2.4. *A function $u \in L^1(H,\gamma)$ is said to be of bounded variation if there exists a vector measure $Du \in \mathscr{M}(H;H)$ such that*

$$\int_H u(x)\,\mathrm{div}_\gamma F(x)\,\gamma(dx) = \int_H \langle F(x), Du(dx)\rangle, \qquad \forall F \in FC^1_b(H;H). \tag{2.6}$$

We denote by $BV(H,\gamma)$ the set of all bounded variation functions on H.

If $u \in BV(H,\gamma)$ we can easily show that

$$Du_{TV}(H) = \sup\left\{\int_H \langle u(x), \mathrm{div}_\gamma F(x)\rangle\,\gamma(dx):\ F \in FC^1_b(H;H),\ |F(x)| \le 1\right\}. \quad (2.7)$$

Remark 2.5. A function $u \in L^1(H,\gamma)$ is of bounded variation if and only if there exists a vector measure $Du \in \mathscr{M}(H;H)$,

$$Du(B) = \sum_{h=1}^{\infty} D_k u(B)\, e_k, \quad B \in \mathscr{B}(H),$$

such that

$$\int_H u(x)\, D_k^* \varphi(x)\,\gamma(dx) = \int_H \varphi(x) D_k u(dx), \quad \forall \varphi \in C^1_b(H), \quad k \in \mathbb{N}. \quad (2.8)$$

Of course, (2.8) follows from (2.6) simply taking $F(x) = \varphi(x)e_k$. The converse implication is also clear by linearity.

In order to investigate the space $BV(H,\gamma)$, we need to show that the integration by parts formula (2.6) holds with a larger class of test functions. Therefore, we introduce the class $\mathscr{D}$ as follows

Definition 2.6. *We say that $F : H \to H$ belongs to $\mathscr{D}$ if*

(a) $F \in C^1_b(H,H)$ and $DF \in C_b(H;\mathscr{L}_1(H))$, where $\mathscr{L}_1(H)$ is the space of trace class operators. In this case we define the operator $\mathrm{div}\, F(x) = \mathrm{Tr}\,[DF(x)]$.

(b) $Q^{-1}F \in C_b(H,H)$.

In the class $\mathscr{F}$ the following holds.

Lemma 2.7. *If $F \in \mathscr{D}$, then*

$$D^*_\infty F(x) = -\mathrm{div}\, F(x) + \langle Q^{-1}x, F(x)\rangle - \langle D\log\rho(x), F(x)\rangle \quad (2.9)$$

In addition, if $u \in BV(H,\gamma)$ and $F \in \mathscr{D}$ then the integration by parts formula

$$\int_H u(x)\, D^*_\infty F(x)\,\gamma(dx) = \int_H \langle F(x), Du(dx)\rangle, \quad (2.10)$$

holds.

Proof. Taking into account that equality (2.9) holds for in $FC^1_b(H);H)$, we approximate F with a sequence F_n in $FC^1_b(H);H)$ such that $F_n \to F$ and $\operatorname{div} F_n \to \operatorname{div} F$ pointwise with bounded norms. This allows us to extend (2.9) to $\mathscr{D}$. The convergence of F_n to F ensures that (2.10) passes to the limit and holds in $\mathscr{D}$.

Let f be in $\mathscr{D}$. Denoting by P_n the projection onto the linear span of $\{e_1, \dots, e_n\}$ and defining $F_n(x) = P_n\left(F(P_n x)\right)$, let us show that $F_n \to F$ and $\operatorname{div} F_n \to \operatorname{div} F$ pointwise with bounded norms. Since P_n converges to the identity, the stated convergence of F_n to F is trivial. Coming to the divergence, set $f_k = \langle F, e_k\rangle$ and

$$G_n(x) = \left[\sum_{h=1}^{n} D_k f_k\right](x)$$

and notice that $G_n \to \operatorname{div} F$ in $\mathscr{L}_1(H)$ and $\operatorname{div} F_n(x) = G(P_n x)$. From condition (a) in Definition 2.6 we deduce that the functions G_n are equicontinuous. Indeed, given $x_0 \in H$ and $\varepsilon > 0$, there is $\delta > 0$ such that

$$|x - x_0| \le \delta \quad \Rightarrow \quad \|DF(x) - DF(x_0)\|_{\mathscr{L}_1(H)} = \sum_{k=1}^{\infty} |D_k f_k(x) - D_k f_k(x_0)| \le \varepsilon$$

and the equicontinuity of the G_n follows. Then, for x_0, ε, δ as above, it suffices to take n large enough to have $|P_n x_0 - x_0| < \delta$ and $\|\operatorname{div} F - G_n\|_{\mathscr{L}_1(H)} < \varepsilon$ to get

$$|\operatorname{div} F(x_0) - \operatorname{div}_\gamma F_n(x_0)| \le |\operatorname{div} F(x_0) - G_n(x_0)| + |G_n(x_0) - G_n(P_n x_0)| < 2\varepsilon.$$

□

Remark 2.8. A function $u \in L^1(H,\gamma)$ belongs to $W^{1,1}(H,\gamma)$ if and only if $u \in BV(H,\gamma)$ and $Du \ll \gamma$. In this case, we denote by Du the density of the gradient measure with respect to γ, which is the weak gradient of u, and obviously (2.10) holds in the form

$$\int_H u(x)\, D^*F(x)\,\gamma(dx) = \int_H \langle Du(x), F(x)\rangle\,\gamma(dx), \quad \forall\ F \in \mathscr{D}.$$

Besides the space $BV(H,\gamma)$, we can consider the space $BV(H,\mu)$ studied in [1] and defined in an obvious way setting $U = 0$. We compare the two spaces in the next remark. To make clearer the presentation, we denote the two notions of gradient and their adjoint operators by different symbols, namely $D_\gamma u,\ D_\mu u$ and $D^*_\gamma F,\ D^*_\mu F$. Let us start from a result concerning the differentiation of a product. The construction and the relevant properties of the approximating functions u_n in its proof are justified by Theorem 3.4 and Remark 4.3 below. Notice, however, that the argument in Remark 2.10 relies on (2.11) with $\gamma = \mu$, hence the results in [1] could be invoked.

Lemma 2.9. *Assume that $u \in BV(H,\gamma)$ and $f \in C_b^1(H)$. Then $fu \in BV(H,\gamma)$ and*

$$D_\gamma(fu) = fD_\gamma u + uDfd\gamma. \tag{2.11}$$

Proof. Let $(u_n) \subset W^{1,1}(H,\gamma)$ be given by $u_n = P_{1/n}u$, so that $u_n \to u$ in $L^1(H,\gamma)$ and $Du_n \to Du$ in $C_b(H;H)'$ by (4.3). Then

$$\begin{aligned}\int_H fuD_\infty^* F d\gamma &= \lim_{n\to\infty}\int_H fu_n D_\infty^* F d\gamma \\ &= \lim_{n\to\infty}\int_H u_n\langle Df,F\rangle d\gamma + \lim_{n\to\infty}\int_H f\left\langle Du_n,F\right\rangle d\gamma \\ &= \int_H u\left\langle Df,F\right\rangle d\gamma + \int_H f\left\langle F,dDu\right\rangle .\end{aligned}$$

□

Remark 2.10. Recall that the function U can be unbounded below, but with at most linear growth. If U is unbounded, there is no relation between the spaces $BV(H,\mu)$ and $BV(H,\gamma)$. If U is bounded below, a case which is still interesting and non trivial, then the inclusion $BV(H,\mu) \subset BV(H,\gamma)$ holds, and for $u \in BV(H,\mu)$, the equality $D_\gamma u = \rho D_\mu u$ holds. Indeed, the inclusion $L^1(H,\mu) \subset L^1(H,\gamma)$ is trivial because ρ is bounded. Moreover, observing that by (2.1)

$$D_\gamma^* F = D_\mu^* F - \langle F, D\log\rho\rangle = D_\mu^* F(x) + 2\langle DU(x), F(x)\rangle$$

we have

$$\begin{aligned}\int_H D_\gamma^* F\, ud\gamma &= \int_H D_\mu^* F\, ud\gamma - \int_H \langle F, D\log\rho\rangle ud\gamma \\ &= \int_H D_\mu^* F\, u\rho d\mu - \int_H \langle F, D\rho\rangle ud\mu = \int_H \rho\langle F, dD_\mu u\rangle\end{aligned}$$

for every $F \in \mathscr{D}(D_\gamma^*)$, therefore $u \in BV(H;\gamma)$ and $D_\gamma u = \rho D_\mu u$.

Finally, notice that the inclusion $BV(H,\gamma) \subset BV(H,\mu)$ holds only if ρ is bounded away from 0, which is equivalent to saying that U is globally bounded, a case which is not interesting. In fact, the equality $D_\gamma u = \rho D_\mu u$ shows that not even the inclusion $W^{1,1}(H;\gamma) \subset W^{1,1}(H;\mu)$ can be true in general.

Example 2.11 (The perimeter of a halfspace)**.** Let us recall that a Borel set $B \subset H$ is said to have finite perimeter if $1l_B \in BV(H,\gamma)$. We show that the halfspace $B = \{x \in H : \langle x,h\rangle > c\}$ has finite perimeter for any $c \in \mathbb{R}$ and $h \in H$. Clearly, we have $1\!\!1_B(x) = 1\!\!1_{(c,\infty)}(\langle x,h\rangle)$. Let

$$m_t(x) = \left\langle x, e^{tA}h\right\rangle, \quad \sigma_t^2 = \frac{1}{2}\left\langle (-A)^{-1}\left(I - e^{-2tA}\right)h, h\right\rangle,$$

and

$$n(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \quad z \in \mathbb{R}.$$

Then

$$\begin{aligned} P_t \mathbb{1}_B(x) &= \int_H \mathbb{1}_{(c,\infty)} \left(\left\langle e^{tA}x + y, h \right\rangle \right) \mu_t(dy) = \int_{\mathbb{R}} \mathbb{1}_{(c,\infty)} \left(m_t(x) + \sigma_t z \right) n(dz) \\ &= \int_{\frac{c-m_t(x)}{\sigma_t}}^{\infty} n(z) dz \end{aligned}$$

and therefore

$$DP_t \mathbb{1}_B(x) = \frac{1}{\sigma_t} n \left(\frac{c - m_t(x)}{\sigma_t} \right) e^{-tA} h.$$

Then

$$\begin{aligned} \int_H |DP_t \mathbb{1}_B(x)| \, \mu(dx) &= \left| e^{-tA} h \right| \int_{\mathbb{R}} \frac{1}{\sigma_t} n \left(\frac{c - \beta_t z}{\sigma_t} \right) n(z) dz \\ &= \left| e^{-tA} h \right| \int_{\mathbb{R}} \frac{1}{\sigma_t} n \left(\frac{z - \alpha_t}{b_t} \right) \frac{1}{\sqrt{2\pi}} e^{-c^2/2\left(\sigma_t^2 + \beta_t^2\right)} \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\sigma_t^2 + \beta_t^2}} e^{-c^2/2\left(\sigma_t^2 + \beta_t^2\right)} \left| e^{-tA} h \right| \end{aligned}$$

where

$$\beta_t^2 = \frac{1}{2} \left\langle (-A)^{-1} e^{2tA} h, h \right\rangle, \quad b_t^2 = \frac{\sigma_t^2}{\sigma_t^2 + \beta_t^2}, \quad \alpha_t = \frac{\beta_t c}{\sigma_t^2 + \beta_t^2}.$$

Then a result from [1] yields $I_B \in BV(H, \mu)$ and

$$\left| D_\mu \mathbb{1}_B \right| (H) = \frac{|h|}{|(-A)^{-1/2} h| \sqrt{\pi}} e^{-c^2/\left|(-A)^{-1/2} h\right|^2}.$$

Finally,

$$D_\mu \mathbb{1}_B = \delta_{\{\langle x, h \rangle = c\}} h.$$

Invoking Proposition 2.10 we obtain

$$D_\gamma \mathbb{1}_B = \rho \delta_{\{\langle x, h \rangle = c\}} h.$$

3 The semigroup P_t in $BV(H,\gamma)$

In this section we show that for any $u \in BV(H,\gamma)$ and any $t > 0$ we have $P_t u \in W^{1,1}(H,\gamma)$. For this we need two main ingredients. The first one is the regularizing power of the semigroup P_t, that is the fact that P_t is Strong Feller.

The second ingredient is the following commutation formula for DP_t,

$$DP_t\varphi = \widehat{P}_t D\varphi, \quad \varphi \in W^{1,1}(H,\gamma), \tag{3.1}$$

where for any $t > 0$, $\widehat{P}_t$ is the bounded operator from $L^1(H,\gamma;H)$ in itself defined by

$$\widehat{P}_t F(x) = \mathbb{E}[\xi(t,x)^* F(X(t,x))], \quad F \in L^1(H,\gamma;H).$$

In discussing the properties of $\widehat{P}_t$ it is useful to realize that for any $(x,h) \in H \times H$ the process $(X(t,x), \xi(t,x,h))$, with $\xi(t,x,h) = \xi(t,x)h$, is a Markov process because it is the solution of the following stochastic differential equation

$$\begin{cases} dX = (AX - DU(X))dt + dW(t) \\ \\ d\xi = (A\xi - D^2U(X)\xi)dt \\ \\ X(0) = x, \quad \xi(0) = h. \end{cases}$$

We denote by $V_t,\ t \geq 0$, the corresponding transition semigroup

$$V_t\Phi(x,h) = \mathbb{E}[\Phi(X(t,x), \xi(t,x,h))],$$

where $\Phi \in C(H \times H)$ has a linear growth:

$$\sup_{x,h\in H} \frac{|\Phi(x,h)|}{1 + |x| + |h|} < \infty.$$

Hence,

$$V_{s+t} = V_s V_t, \quad s,t \geq 0.$$

Lemma 3.1. *$\widehat{P}_t$ is a C_0-semigroup on $L^2(H,\gamma;H)$ and $\widehat{P}_t F \in C_b(H;H)$ for every $t > 0$. For any $F \in \mathscr{E}_A(H;H)$, where $\mathscr{E}_A(H;H)$ is the linear span of all functions $F : H \to H$ of the form*

$$F(x) = \varphi(x)h, \quad h \in \mathscr{D}(A),\ \varphi \in \mathscr{E}_A(H),$$

the infinitesimal generator $\widehat{N}$ of $\widehat{P}_t$ is given by

$$\widehat{N}F(x) = \widehat{N}(\varphi(\cdot)h)(x) = N\varphi(x)\,h + \varphi(x)(Ah - D^2U(x)h). \tag{3.2}$$

Moreover $\widehat{P}_t$ is symmetric and

$$DP_t\phi = \widehat{P}_t D\phi, \quad \phi \in W^{1,2}(H,\gamma). \tag{3.3}$$

Proof. **Step 1**. $\widehat{P}_t,\ t \geq 0$, is a C_0-semigroup on $L^2(H,\gamma;H)$.
First, we show that $\widehat{P}_t$ is well-defined on $L^2(H,\gamma;H)$. Indeed, let F be of the form $F(x) = \varphi(x)h$ with $\varphi \in L^2(h,\gamma)$ (the linear span of such functions is dense in $L^2(H,\gamma;H)$). Then,

$$|\widehat{P}_tF(x)| = |\mathbb{E}[\xi^*(t,x)h\varphi(X(t,x))]| \leq e^{-\omega t}|h||\mathbb{E}[\varphi(X(t,x))]| = e^{-\omega t}|h|\,|P_t\varphi(x)|,$$

whence

$$\int_H |\widehat{P}_tF(x)|^2\gamma(dx) \leq e^{-2\omega t}|h|^2 \int_H |P_t\varphi(x)|^2\gamma(dx) \leq |h|^2 \int_H |\varphi(x)|^2\gamma(dx).$$

Putting $\Phi(x,h) = \langle G(x),h\rangle$ for $G \in C_b(H;H)$ we find that

$$V_t\Phi(x,h) = \mathbb{E}[\langle G(X(t,x)),\xi(t,x,h)\rangle]$$

and $\langle \widehat{P}_tG(x),h\rangle = V_t\Phi(x,h)$, and therefore

$$\begin{aligned}\langle \widehat{P}_{t+s}G(x),h\rangle &= V_{t+s}\Phi(x,h) = V_t\left(V_s\Phi\right)(x,h) = \mathbb{E}[\left(V_s\Phi\right)(X(t,x),\xi(t,x,h))]\\ &= \int_{H\times H} V_s\Phi(y,z)\mathbb{P}^{X(t,x),\xi(t,x,h)}(dy,dz)\\ &= \int_{H\times H} \mathbb{E}[\langle G\left(X(s,y)\right),\xi(s,x,z)\rangle]\mathbb{P}^{X(t,x),\xi(t,x,h)}(dy,dz),\end{aligned}$$

where $\mathbb{P}^{X(t,x),\xi(t,x,h)}$ stands for the joint distribution of $(X(t,x),\xi(t,x,h))$ on $H\times H$. Thereby

$$\begin{aligned}\langle \widehat{P}_{t+s}G(x),h\rangle &= \int_{H\times H} \langle \widehat{P}_sG(y),z\rangle \mathbb{P}^{X(t,x),\xi(t,x,h)}(dy,dz)\\ &= \mathbb{E}[\langle \widehat{P}_sG(X(t,x)),\xi(t,x,h)\rangle] = \langle \widehat{P}_t\widehat{P}_sG(x),h\rangle.\end{aligned}$$

This completes the proof of the semigroup property for $\widehat{P}_t,\ t \geq 0$. Since, in view of (1.6), $|\widehat{P}_tF| \leq |F|$ and $t \mapsto \xi^*(t,x)F(X(t,x))$ is continuous, the semigroup $\widehat{P}_t,\ t \geq 0$, extends to a semigroup on $L^2(H,\gamma;H)$ and the strong continuity follows by a standard argument.

Step 2. Proof of (3.2) and symmetry of the semigroup.
For $F = \varphi\, h \in \mathscr{E}_A(H;H)$ we consider

$$\frac{\widehat{P}_tF(x) - F(x)}{t} = \frac{\mathbb{E}[\left(\xi^*(t,x) - I\right)F(X(t,x))]}{t} + \frac{\mathbb{E}[F(X(t,x))] - F(x)}{t}.$$

Then, using the Itô formula and the equation satisfied by $\xi(t,x)$ we find that

$$\lim_{t\to 0}\frac{1}{t}\left(\widehat{P}_tF(x) - F(x)\right) = N\varphi(x)h + \varphi(x)Ah - \varphi(x)D^2U(x)h.$$

Taking into account that N is symmetric in $L^2(H,\gamma)$, it is easily seen that

$$\langle \widehat{N}F, G\rangle = \langle \widehat{N}G, F\rangle, \quad F, G \in \mathscr{E}_A(H;H),$$

and since $\mathscr{E}_A(H;H)$ is dense in $L^2(H,\gamma;H)$, it follows that

$$\langle \widehat{P}_t F, G\rangle = \langle \widehat{P}_t G, F\rangle, \quad F, G \in L^2(H,\gamma;H).$$

Finally, using the definition of the semigroup $\widehat{P}_t$, $t \geq 0$, we find easily that (3.3) holds. □

Lemma 3.2. *Assume that $F \in \mathscr{D}(D^*_\infty)$ and $t > 0$. Then $\widehat{P}_t F \in \mathscr{D}(D^*_\infty)$ and we have*

$$D^*_\infty \widehat{P}_t F(x) = P_t D^*_\infty F(x), \quad x \in H. \tag{3.4}$$

Proof. We have to show that for some constant C and all $\varphi \in W^{1,1}(H,\gamma)$, we have

$$K := \left| \int_H \langle D\varphi(x), \widehat{P}_t F(x)\rangle \gamma(dx) \right| \leq C \int_H |\varphi(x)|\gamma(dx). \tag{3.5}$$

We have in fact, thanks to (3.1),

$$K = \left| \int_H \langle \widehat{P}_t D\varphi(x), F(x)\rangle \gamma(dx) \right| = \left| \int_H \langle D P_t \varphi(x), F(x)\rangle \gamma(dx) \right|.$$

On the other hand, since $F \in \mathscr{D}(D^*_\infty)$ there exists $C > 0$ such that

$$K \leq C \int_H |P_t\varphi(x)|\gamma(dx) \leq C \int_H |\varphi(x)|\gamma(dx).$$

So, (3.5) holds and (3.4) follows. □

In order to prove that $P_t u \in W^{1,1}(H,\gamma)$ for $u \in BV(H,\gamma)$ we need to know that the class $\mathscr{D}$ is invariant under $\widehat{P}_t$. This is proved in the next lemma; we stress that this is the only point where we need C^3 regularity for U.

Lemma 3.3. *If $F \in \mathscr{D}$ then $\widehat{P}_t F$ belongs to $\mathscr{D}$ for every $t \geq 0$.*

Proof. Let $X(t,x)$ be the solution of Problem (1.1) and, for $h \in H$, let ξ^h be the directional derivative of X as in Section 1. By Theorem [10, 7.3.6], the function $H \ni x \mapsto X(t,x)$ is twice differentiable along all directions. Let us denote by ζ its second derivative and, for every $h \in H$, set $\zeta^h = \zeta(t,x)(h,h)$. Then, ζ^h solves the problem

$$\begin{cases} \dfrac{d}{dt}\zeta^h(t,x) &= A\zeta^h(t,x) - D^2U(X(t,x))\zeta^h(t,x) \\ & \quad -D^3U(X(t,x))(\xi^h(t,x), \xi^h(t,x)) \\ \zeta^h(0,x) &= 0. \end{cases} \tag{3.6}$$

Notice that for $h = e_k$ we have

$$\zeta^{e_k}(t,x) = \int_0^t e^{(t-s)A}\zeta^{e_k}(t,x)ds - \int_0^t e^{(t-s)A} D^3U(X(s,x))(\xi^{e_k}(s,x), \xi^{e_k}(s,x))ds,$$

whence, setting

$$T(t,x) = \sum_{k=1}^{\infty} \zeta^{e_k}(t,x),$$

we get, summing on k,

$$T(t,x) = \int_0^t e^{(t-s)A}T(s,x)ds - \int_0^t \mathrm{Tr}\,[\xi^*(s,x)e^{(t-s)A}D^3U(X(s,x))\xi(s,x)]ds.$$

Notice that by (1.5) the last integral is meaningful.

Fix now $F \in \mathscr{D}$, and set $G(t,x) = (\widehat{P}_t F)(t,x)$. We claim that

$$\mathrm{Tr}\,[DG(t,x)] = \mathbb{E}\,[\mathrm{Tr}\,[\xi^*(t,x)DF(X(t,x))\xi(t,x)]] + \mathbb{E}[\langle F(X(t,x)), T(t,x)\rangle], \quad (3.7)$$

whence condition (a) in Definition 2.6 holds for G. In order to prove (3.7), write

$$\langle G(t,x), e_k\rangle = \mathbb{E}[\langle F(X(t,x)), \xi^{e_k}(t,x)\rangle],$$

from which

$$D_k\langle G(t,x), e_k\rangle = \mathbb{E}[\langle DF(X(t,x))\xi(t,x)e_k, \xi(t,x)e_k\rangle] + \mathbb{E}[\langle F(X(t,x)), \zeta^{e_k}(t,x)\rangle].$$

Since

$$\langle DF(X(t,x))\xi(t,x)e_k, \xi(t,x)e_k\rangle = \langle \xi^*(t,x)DF(X(t,x))\xi(t,x)e_k, e_k\rangle,$$

identity (3.7) follows summing up over k.

Condition (b) in Definition 2.6 follows from the equality

$$\langle Q^{-1}x, G(t,x)\rangle = \mathbb{E}[\langle Q\xi(t,x)Q^{-1}x, Q^{-1}F(X(t,x))\rangle], \quad (3.8)$$

which we are going to prove. First, let us check that the term $Q\xi(t,x)Q^{-1}x$ is meaningful. For, setting

$$v^z(t,x) = Q\xi(t,x)Q^{-1}z = Q\xi^{Q^{-1}z}(t,x), \quad (3.9)$$

we have

$$v^z(t,x) = e^{tA}z - \int_0^t e^{(t-s)A}D^2U(X(s,x))v^z(s,x)ds. \quad (3.10)$$

Since by (3.9) we have $Q\xi(t,x)Q^{-1}x = v^x(t,x)$, it is enough to notice that equation (3.10) has a unique solution. Then, we have

$$\langle x, Q^{-1}G(t,x)\rangle = \mathbb{E}[\langle x, Q^{-1}\xi(t,x)^*F(X(t,x))\rangle] = \mathbb{E}[\langle \xi(t,x)Q^{-1}x, F(X(t,x))\rangle],$$

(3.8) follows and the thesis is proved because the right hand side is in $C_b(H)$. □

Theorem 3.4. *Let $u \in BV(H,\gamma)$. Then for all $t > 0$ we have $P_t u \in W^{1,1}(H,\gamma)$ and*

$$\limsup_{t\to 0} \int_H |DP_t u| d\gamma \le Du_{TV}(H). \tag{3.11}$$

Proof. Let $u \in BV(H,\gamma)$ and $t > 0$. Then by the definition and applying (2.6) with $F \in \mathscr{D}$ we have

$$\int_H u(x) D_\infty^* F(x)\gamma(dx) = \int_H \langle F(x), Du(dx)\rangle, \quad \forall\, F \in \mathscr{D}.$$

We first prove that $P_t u \in BV(H,\gamma)$. In fact, from (3.4) and Lemma 3.3 it follows that

$$\begin{aligned}
&\int_H (P_t u)(x)\, D_\infty^* F(x)\gamma(dx) = \int_H u(x)\, P_t D_\infty^* F(x)\gamma(dx) \qquad (3.12)\\
&= \int_H u(x) D_\infty^*(\widehat{P}_t F)(x)\gamma(dx) = \int_H \langle \widehat{P}_t F(x), Du(dx)\rangle = \int_H \langle F(x), \widehat{P}_t'(Du)(dx)\rangle.
\end{aligned}$$

This shows that $P_t u \in BV(H,\gamma)$ and $DP_t u = \widehat{P}_t'(Du)$. We claim that

$$\widehat{P}_t'(Du) \ll \gamma. \tag{3.13}$$

Let in fact $B \in \mathscr{B}(H)$ be such that $\gamma(B) = 0$, and take $h \in H$. Write

$$\begin{aligned}
\langle \widehat{P}_t'(Du)(B), h\rangle &= \int_H \langle \mathbb{1}_B(x) h, \widehat{P}_t'(Du)(dx)\rangle = \int_H \langle \widehat{P}_t(\mathbb{1}_B h)(x), Du(dx)\rangle\\
&= \int_H \langle \mathbb{E}[\xi^*(t,x) h \mathbb{1}_B(X(t,x))], Du(dx)\rangle\\
&\le \int_H |\mathbb{E}[\xi^*(t,x) h \mathbb{1}_B(X(t,x))]| Du_{TV}(dx)\\
&\le e^{-\omega t}|h| \int_H \mathbb{E}[\mathbb{1}_B(X(t,x))] Du_{TV}(dx)\\
&= e^{-\omega t}|h| \int_H \pi_t(x,B) Du_{TV}(dx) = 0,
\end{aligned}$$

because $\pi_t(x,\cdot) \ll \gamma$ and $\gamma(B) = 0$.

From (3.13) and (3.12) we deduce that $P_t u \in W^{1,1}(H,\gamma)$. Moreover

$$\int_H |DP_t u(x)|\gamma(dx) = (\widehat{P}_t'(Du))_{TV}(H) \le Du_{TV}(H)$$

and (3.11) follows. □

Remark 3.5. Since by (2.7) the total variation of $u \in BV(H,\gamma)$ is L^1-lower semicontinuous and P_t is strongly continuous, for $u \in BV(H,\gamma)$ we have

$$Du_{TV}(H) \le \liminf_{t\to 0} \int_H |DP_t u| d\gamma,$$

which, combined with (3.11), gives that

$$Du_{TV}(H) = \lim_{t\to 0} \int_H |DP_t u| d\gamma.$$

4 Sufficient condition for $u \in BV(H,\gamma)$

In order to prove the converse of Theorem 3.4, we first prove that the measure γ admits a disintegration with log-concave fibers. Let us fix some notation. For $k \ge 1$, consider the orthogonal decomposition $H = H_k \oplus H_k^\perp$, where $H_k = \operatorname{span} e_k$. Accordingly, for $y \in H_k^\perp$, we define the sections $B_y = \{s \in H_k : \ (s,y) \in B\}$ for every $B \subset H$, where we have identified $x = s + y$ with the pair (s,y). Denoting by π the orthogonal projection onto $H_k^\perp$, set $\sigma = \pi_\#(\gamma)$. By disintegration (see e.g. [5, Section 10.6]), for σ-a.e. $y \in H_k^\perp$ there is a measure γ_y such that

$$\gamma(B) = \int_{H_k^\perp} \gamma_y(B_y)\sigma(dy), \qquad B \subset H \ \text{ Borel set.}$$

Lemma 4.1. *For σ-a.e. $y \in H_k^\perp$ the measure γ_y is log-concave and non degenerate.*

Proof. We begin by noticing that, for finite Borel measures λ, ν in $H_k^\perp$, $\lambda \ge \nu$ if and only if $\lambda(A) \ge \nu(A)$ for all open convex sets $A \subset H_k^\perp$. In the finite dimensional case this is trivial, as the inequality holds on cubes, hence all open sets and eventually on Borel sets. In the infinite dimensional case we are dealing with, we may consider products of finite dimensional open balls with subspaces and deduce that all finite dimensional projections $\hat\lambda, \hat\nu$ of λ and ν satisfy $\hat\lambda \ge \hat\nu$. Therefore, $\lambda \ge \nu$.

Let now $A \subset H_k^\perp$ be an open convex set, $A_1, A_2 \subset H_k$ open, $t \in]0,1[$. Identifying H with $H_k \otimes H_k^\perp$, the inclusion

$$((1-t)(y+A_1) + t(y+A_2)) \times A \supset (1-t)((y+A_1)\times A) + t((y+A_2)\times A))$$

together with the log-concavity of γ gives

$$\begin{aligned}&\int_A \gamma_y((1-t)(y+A_1) + t(y+A_2))\sigma(dy)\\ &\ge \Big(\int_A (\gamma_y(y+A_1))\sigma(dy)\Big)^{1-t}\Big(\int_A (\gamma_y(y+A_2))\sigma(dy)\Big)^t.\end{aligned}$$

By applying Holder's inequality we get

$$\int_A \gamma_y((1-t)(y+A_1)+t(y+A_2))\sigma(dy) \geq \int_A (\gamma_y(y+A_1))^{1-t}(\gamma_y(y+A_2))^t\sigma(dy).$$

Now, the arbitrariness of the convex open set A gives

$$\gamma_y((1-t)(y+A_1)+t(y+A_2)) \geq (\gamma_y(y+A_1))^{1-t}(\gamma_y(y+A_2))^t \tag{4.1}$$

for σ-a.e. $y \in H_k^\perp$. A priori, the exceptional set depends on t, A_1, A_2, but the separability of H and a simple density argument allow to find a σ-negligible set $N \subset H_k^\perp$ such that (4.1) holds for all $y \in H_k^\perp \setminus N$ and $A_1, A_2 \subset H$ open.

In order to prove that γ_y is non degenerate for a.a. $y \in H_k^\perp$, let us recall that $(\tau_{k,t})_\#(\gamma) \ll \gamma$, where $\tau_{k,t}(x) = x + te_k$ is the translation along e_k. Moreover, since the translation acts orthogonally to $H_k^\perp$, the disintegration of $(\tau_{k,t})_\#(\gamma)$ reads

$$(\tau_{k,t})_\#(\gamma)(B) = \int_{H_k^\perp} \sigma(dy)\gamma_{y,t}(B_y),$$

with the same measure $\sigma = \pi_\#(\gamma)$. Therefore, if the measures γ_y were degenerate for a non-negligible set of y, $(\tau_{k,t})_\#(\gamma)$ could not be absolutely continuous with respect to γ. □

We are now in a position to show the following

Theorem 4.2. *Let $u \in L^1(H,\gamma)$ and assume that $P_t u \in W^{1,1}(H,\gamma)$ for $t > 0$ and that*

$$L := \liminf_{t \to 0} \int_H |DP_t u| d\gamma < +\infty. \tag{4.2}$$

Then, $u \in BV(H,\gamma)$ and $Du_{TV}(H) = L$.

Proof. Using the notation introduced above, since for σ-a.e. y the measure γ_y is non degenerate and log-concave, then (see [6] or [3, Theorem 9.4.11]) there is a convex function v_y such that $\gamma_y(ds) = e^{-v_y(s)}ds$. For any $\varphi \in C_b^1(\mathbb{R})$, setting $f(x) = \varphi(x_k)$, we get

$$\int_H D_k^* f(x)u(x)\gamma(dx) = \lim_{t\to 0}\int_H D_k^* f(x)P_t u(x)\gamma(dx) = \lim_{t\to 0}\int_H f(x)D_k P_t u(x)\gamma(dx)$$
$$= \lim_{t\to 0}\int_{H_k^\perp}\sigma(dy)\int_{H_k}\varphi(s)D_k P_t u(s,y)e^{-v_y(s)}ds. \leq L$$

This proves that the function $u_y(s) = u(y + se_k)$ has (weighted) bounded variation in $\mathbb{R}$ for σ-a.e. $y \in H_k^\perp$, with derivative $D_{\gamma_y} u_y$. Let us check that the measure

$$\nu_k(B) = \int_{H_k^\perp} D_{\gamma_y} u_y(B_y)\sigma(dy)$$

gives the partial derivative $D_k u$ of u. Indeed, for $F \in \mathscr{D}(D_\infty^*)$, setting $f = \langle F, e_k \rangle$, we have

$$\begin{aligned}\int_H D_k^* f(x)u(x)\gamma(dx) &= \int_{H_k^\perp}\int_{H_k} D_k^* f(y + se_k)u(y + se_k)\gamma_y(ds)\sigma(dy)\\ &= \int_{H_k^\perp}\int_{H_k} f(y + se_k)D_{\gamma_y} u_y(ds)\sigma(dy) = \int_H f\nu_k(dx).\end{aligned}$$

Repeating the argument for every k we construct an H-valued measure $D_\gamma u$ such that (2.6) holds. Finally, from (4.2) and Remark 4.3 it follows that $D_\gamma u$ has finite total variation and the equality $Du_T V(H) = L$ holds. □

Remark 4.3. Let us point out that $DP_t u \to Du$ weakly* as vector measures, as $t \to 0$. In fact,

$$\int_H \langle F(x), DP_t u(x)\rangle\gamma(dx) = \int_H D_\infty^* F(x)P_t u(x)\gamma(dx) \to \int_H \langle F(x), Du(dx)\rangle$$

for every $F \in FC_b^1(H; H)$. In order to show that

$$\lim_{t\to 0}\int_H \langle F(x), DP_t u(x)\rangle\gamma(dx) = \int_H \langle F(x), Du(dx)\rangle \tag{4.3}$$

for all $F \in C_b(H; H)$ we may argue componentwise and check the tightness condition presented in [2, Lemma 2.1], whence compactness follows from Prokhorov Theorem. Therefore, taking (3.11) into account, we have only to show that

$$\liminf_{t\to 0}\int_A |D_k P_t u(x)|\gamma(dx) \geq |D_k u|(A), \quad A \subset H \text{ open}, k \in \mathbb{N}. \tag{4.4}$$

Let $A \subset H$ be open, and notice that for every $y \in H_k^\perp$ the section A_y is open as well. Using the disintegration as in theorem 4.2 and the L^1-lower semicontinuity of the total variation on open sets in $\mathbb{R}$, we have

$$\begin{aligned}|D_k u|(A) &= \int_{H_k^\perp} |D_{\gamma_y} u_y|(A_y)\sigma(dy) \leq \int_{H_k^\perp}\liminf_{t\to 0}\int_{A_y} |D_y P_t u(s, y)|\gamma_y(ds)\sigma(dy)\\ &\leq \liminf_{t\to 0}\int_{H_k^\perp}\int_{A_y} |D_y P_t u(s, y)|\gamma_y(ds)\sigma(dy) = \liminf_{t\to 0}\int_A |D_k P_t u(x)|\gamma(dx).\end{aligned}$$

Then the family of measures $(D_k P_t u)$ is relatively compact and, since (4.3) holds on a dense set, the proof is complete.

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