EXISTENCE OF EULERIAN SOLUTIONS TO THE SEMIGEOSTROPHIC EQUATIONS IN PHYSICAL SPACE: THE 2-DIMENSIONAL PERIODIC CASE

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Abstract. In this paper we use new regularity and stability estimates for Alexandrov solutions to Monge-Ampère equations, recently estabilished by De Philippis and Figalli [14], to provide global in time existence of distributional solutions to the semigeostrophic equations on the 2-dimensional torus, under very mild assumptions on the initial data. A link with Lagrangian solutions is also discussed.

1. Introduction

The semigeostrophic equations are a simple model used in meteorology to describe large scale atmospheric flows. As explained for instance in [6, Section 2.2] and [18, Section 1.1] (see also [11] for a more complete exposition), the semigeostrophic equations can be derived from the 3-d incompressible Euler equations, with Boussinesq and hydrostatic approximations, subject to a strong Coriolis force. Since for large scale atmospheric flows the Coriolis force dominates the advection term, the flow is mostly bi-dimensional. For this reason, the study of the semigeostrophic equations in 2-d or 3-d is pretty similar, and in order to simplify our presentation we focus here on the 2-dimentional periodic case, though we expect that our results could be extended to three dimensions.

The semigeostrophic system on the 2-dimensional torus $\mathbb{T}^2$ is given by

$$\begin{cases} \partial_t u^g_t(x) + \big(u_t(x)\cdot\nabla\big)u^g_t(x) + \nabla p_t(x) = -Ju_t(x) & (x,t)\in\mathbb{T}^2\times(0,\infty) \\ u^g_t(x) = J\nabla p_t(x) & (x,t)\in\mathbb{T}^2\times[0,\infty) \\ \nabla\cdot u_t(x) = 0 & (x,t)\in\mathbb{T}^2\times[0,\infty) \\ p_0(x) = p^0(x) & x\in\mathbb{T}^2. \end{cases} \tag{1.1}$$

Here p^0 is the initial datum, J is the rotation matrix given by

$$J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and the functions u_t and p_t represent respectively the *velocity* and the *pressure*, while u^g_t is the so-called *semi-geostrophic* wind.[1] Clearly the pressure is defined up to a (time-dependent) additive constant. In the sequel we are going to identify functions (and measures) defined on the torus $\mathbb{T}^2$ with $\mathbb{Z}^2$-periodic functions defined on $\mathbb{R}^2$.

Substituting the relation $u^g_t = J\nabla p_t$ into the equation, the system (1.1) can be rewritten as

$$\begin{cases} \partial_t J\nabla p_t + J\nabla^2 p_t u_t + \nabla p_t + Ju_t = 0 \\ \nabla\cdot u_t = 0 \\ p_0 = p^0 \end{cases} \tag{1.2}$$

[1]Note that we are using the notation $u_t,\ p_t,\ u^g_t$ to denote the functions $u(t,\cdot),\ p(t,\cdot),\ u^g(t,\cdot)$

with u_t and p_t periodic.

Energetic considerations (see [11, Section 3.2]) show that it is natural to assume that p_t is (-1)-convex, i.e., the function $P_t(x) := p_t(x) + |x|^2/2$ is convex on $\mathbb{R}^2$. If we denote with $\mathscr{L}_{\mathbb{T}^2}$ the (normalized) Lebesgue measure on the torus, then formally[2] $\rho_t := (\nabla P_t)_\sharp \mathscr{L}_{\mathbb{T}^2}$ satisfies the following *dual problem* (see the Appendix):

$$\begin{cases} \partial_t \rho_t + \nabla \cdot (U_t \rho_t) = 0 \\ U_t(x) = J(x - \nabla P_t^*(x)) \\ \rho_t = (\nabla P_t)_\sharp \mathscr{L}_{\mathbb{T}^2} \\ P_0(x) = p^0(x) + |x|^2/2. \end{cases} \tag{1.3}$$

Here P_t^* is the convex conjugate of P_t, namely

$$P_t^*(y) := \sup_{x \in \mathbb{R}^2} \big(y \cdot x - P_t(x)\big).$$

Notice that, since $P_t(x) - |x|^2/2$ is periodic,

$$\nabla P_t(x+h) = \nabla P_t(x) + h \qquad \forall x \in \mathbb{R}^2, \;\; h \in \mathbb{Z}^2. \tag{1.4}$$

Hence ∇P_t can be viewed as a map from $\mathbb{T}^2$ to $\mathbb{T}^2$ and ρ_t is a well defined measure on $\mathbb{T}^2$. One can also verify easily that the inverse map ∇P_t^* satisfies (1.4) as well. Accordingly, we shall understand (1.3) as a PDE on $\mathbb{T}^2$, i.e., using test functions which are $\mathbb{Z}^2$-periodic in space.

The dual problem (1.3) is nowadays pretty well understood. In particular, Benamou and Brenier proved in [6] existence of weak solutions to (1.3), see Theorem 3.1 below. On the contrary, much less is known about the original system (1.1). Formally, given a solution ρ_t of (1.3) and defining P_t^* through the relation $\rho_t = (\nabla P_t)_\sharp \mathscr{L}_{\mathbb{T}^2}$ (namely the optimal transport map from ρ_t to $\mathscr{L}_{\mathbb{T}^2}$, see Theorem 2.1) the pair (p_t, u_t) given by[3]

$$\begin{cases} p_t(x) := P_t(x) - |x|^2/2 \\ u_t(x) := [\partial_t \nabla P_t^*](\nabla P_t(x)) + [\nabla^2 P_t^*](\nabla P_t(x)) J(\nabla P_t(x) - x) \end{cases} \tag{1.5}$$

solves (1.2). However, being P_t^* just a convex function, *a priori* $\nabla^2 P_t^*$ is just a matrix-valued measure, thus as pointed out in [12] it is not clear the meaning to give to the previous equation.

In this paper we prove that (1.5) is a well defined velocity field, and that the couple (p_t, u_t) is a solution of (1.1) in a distributional sense. In order to carry out our analysis, a fundamental tool is a recent result for solutions of the Monge-Ampère equation, proved by the third and fourth author in [14], showing $L \log^k L$ regularity on $\nabla^2 P_t^*$ (see Theorem 2.2(ii) below).

Thanks to this result, we can easily show that the second term appearing in the definition of the velocity u_t in (1.5) is a well defined L^1 function (see the proof of Theorem 1.2). Moreover, following some ideas developed in [17] we can show that the first term is also L^1, thus giving a meaning to u_t (see Proposition 3.3). At this point we can prove that the pair (p_t, u_t) is actually a distributional

[2] Given a measure μ on $\mathbb{T}^2$ and a Borel map $f : \mathbb{T}^2 \to \mathbb{T}^2$, we define the measure $f_\sharp \mu$ through the relation

$$\int_{\mathbb{T}^2} h(y)\, d\, f_\sharp \mu(y) = \int_{\mathbb{T}^2} h(f(x))\, d\mu(x)$$

[3] Because of the many compositions involved in this paper, we use the notation $[\partial_t f](g)$ (resp. $[\nabla f](g)$) to denote the composition $(\partial_t f) \circ g$ (resp. $(\nabla f) \circ g$), avoiding the ambiguous notation $\partial_t f(g)$ (resp. $\nabla f(g)$)

solution of system (1.2). Let us recall, following [12], the proper definition of *weak Eulerian solution* of (1.2).

Definition 1.1. Let $p : \mathbb{T}^2 \times (0,\infty) \to \mathbb{R}$ and $u : \mathbb{T}^2 \times (0,\infty) \to \mathbb{R}^2$. We say that (p,u) is a *weak Eulerian solution* of (1.2) if:

- $|u| \in L^\infty((0,\infty), L^1(\mathbb{T}^2))$, $p \in L^\infty((0,\infty), W^{1,\infty}(\mathbb{T}^2))$, and $p_t(x) + |x|^2/2$ is convex for any $t \geq 0$;
- For every $\phi \in C_c^\infty(\mathbb{T}^2 \times [0,\infty))$, it holds

$$\int_0^\infty \int_{\mathbb{T}^2} J\nabla p_t(x)\Big\{\partial_t \phi_t(x) + u_t(x)\cdot\nabla\phi_t(x)\Big\} - \Big\{\nabla p_t(x) + Ju_t(x)\Big\}\phi_t(x)\,dx\,dt + \int_{\mathbb{T}^2} J\nabla p_0(x)\phi_0(x)\,dx = 0; \tag{1.6}$$

- For a.e. $t \in (0,\infty)$ it holds

$$\int_{\mathbb{T}^2} \nabla\psi(x)\cdot u_t(x)\,dx = 0 \qquad \text{for all } \psi \in C^\infty(\mathbb{T}^2). \tag{1.7}$$

We can now state our main result.

Theorem 1.2. *Let $p_0 : \mathbb{R}^2 \to \mathbb{R}$ be a $\mathbb{Z}^2$-periodic function such that $p_0(x) + |x|^2/2$ is convex, and assume that the measure $(Id + \nabla p_0)_\sharp \mathscr{L}^2$ is absolutely continuous with respect to $\mathscr{L}^2$ with density ρ_0, namely*

$$(Id + \nabla p_0)_\sharp \mathscr{L}^2 = \rho_0 \mathscr{L}^2.$$

Moreover, let us assume that both ρ_0 and $1/\rho_0$ belong to $L^\infty(\mathbb{R}^2)$.

Let ρ_t be the solution of (1.3) *given by Theorem 3.1 and let $P_t : \mathbb{R}^2 \to \mathbb{R}$ be the (unique up to an additive constant) convex function such that $(\nabla P_t)_\sharp \mathscr{L}^2 = \rho_t \mathscr{L}^2$ and $P_t(x) - |x|^2/2$ is $\mathbb{Z}^2$-periodic, $P_t^* : \mathbb{R}^2 \to \mathbb{R}$ its convex conjugate.*

Then the couple (p_t, u_t) defined in (1.5) *is a weak Eulerian solution of* (1.2)*, in the sense of Definition 1.1.*

Although the vector field u provided by the previous theorem is only L^1, as explained in Section 5 we can associate to it a measure-preserving Lagrangian flow. In particular we recover (in the particular case of the 2-dimensional periodic setting) the result of Cullen and Feldman [12] on the existence of Lagrangian solutions to the semigeostrophic equations in physical space.

The paper is structured as follows: in Section 2 we recall some preliminary results on optimal transport maps on the torus and their regularity. Then, in Section 3 we state the existence result of Benamou and Brenier for solutions to the dual problem (1.3), and we show some important regularity estimates on such solutions, which are used in Section 4 to prove Theorem 1.2. In Section 5 we prove the existence of a "Regular Lagrangian Flow" associated to the vector field u provided by Theorem 1.2. Finally, in Section 6 we list some open problems. For completenes, in the Appendix we show the formal computation used to obtain (1.3) from (1.2).

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2. Optimal transport maps on the torus and their regularity

The following theorem can be found in [10] (see for instance [14, Section 2] for the notion of Alexandrov solution of the Monge-Ampère equation).

Theorem 2.1 (Existence of optimal maps on $\mathbb{T}^2$)**.** *Let μ and ν be $\mathbb{Z}^2$-periodic Radon measures on $\mathbb{R}^2$ such that $\mu([0,1)^2) = \nu([0,1)^2) = 1$ and $\mu = \rho\mathscr{L}^2$ with $\rho > 0$ almost everywhere. Then there exists a unique (up to an additive constant) convex function $P : \mathbb{R}^2 \to \mathbb{R}$ such that $(\nabla P)_\sharp \mu = \nu$ and $P - |x|^2/2$ is $\mathbb{Z}^2$-periodic. Moreover*

$$\nabla P(x+h) = \nabla P(x) + h \qquad \text{for a.e. } x \in \mathbb{R}^2,\ \ \forall\, h \in \mathbb{Z}^2, \tag{2.1}$$

$$|\nabla P(x) - x| \le \operatorname{diam}(\mathbb{T}^2) = \frac{\sqrt{2}}{2} \qquad \text{for a.e. } x \in \mathbb{R}^2. \tag{2.2}$$

In addition, if $\mu = \rho\mathscr{L}^2$, $\nu = \sigma\mathscr{L}^2$, and there exist constants $0 < \lambda \le \Lambda < \infty$ such that $\lambda \le \rho, \sigma \le \Lambda$, then P is a strictly convex Alexandrov solution of

$$\det \nabla^2 P(x) = f(x), \qquad \textit{with } f(x) = \frac{\rho(x)}{\sigma(\nabla P(x))}.$$

Proof. Existence of P follows from [10]. To prove uniqueness we observe that, under our assumption, also $p^*(y) := P^*(y) - |y|^2/2$ is $\mathbb{Z}^2$-periodic. Hence, since

$$P(x) = \sup_{y \in \mathbb{R}^2} x \cdot y - P^*(y),$$

we get that the function $p(x) := P(x) - |x|^2/2$ satisfies

$$\begin{aligned} p(x) &= \sup_{y\in\mathbb{R}^2} \Big(-\frac{|y-x|^2}{2} - P^*(y) + \frac{|y|^2}{2}\Big) \\ &= \sup_{y\in[0,1|^2}\sup_{h\in\mathbb{Z}^2} \Big(-\frac{|y+h-x|^2}{2} - p^*(y+h)\Big) \\ &= \sup_{y\in\mathbb{T}^2} \Big(-\frac{d^2_{\mathbb{T}^2}(x,y)}{2} - p^*(y)\Big), \end{aligned}$$

where $d_{\mathbb{T}^2}$ is the quotient distance on the torus, and we used that $p^*(y)$ is $\mathbb{Z}^2$-periodic. This means that the function p is $d^2_{\mathbb{T}^2}$-convex, and that p^* is its $d^2_{\mathbb{T}^2}$-transform (see [20, Chapter 5]). Hence $\nabla P = Id + \nabla p : \mathbb{T}^2 \to \mathbb{T}^2$ is the unique (μ-a.e.) optimal transport map sending μ onto ν ([19, Theorem 9]), and since $\rho > 0$ almost everywhere this uniquely characterizes P up to an additive constant. Finally, all the other properties of P follow from [10]. □

Combining the previous theorem and the known regularity results for strictly convex Alexandrov solutions of the Monge-Ampère equation (see [7, 8, 9, 10, 14, 16]) we have the following:

Theorem 2.2 (Space regularity of optimal maps on $\mathbb{T}^2$)**.** *Let $\mu = \rho\mathscr{L}^2$, $\nu = \sigma\mathscr{L}^2$, and let P be as in Theorem 2.1 with $\int_{\mathbb{T}^2} P\,dx = 0$. Then:*

(i) *$P \in C^{1,\beta}(\mathbb{T}^2)$ for some $\beta = \beta(\lambda,\Lambda) \in (0,1)$, and there exists a constant $C = C(\lambda,\Lambda)$ such that*

$$\|P\|_{C^{1,\beta}} \le C.$$

(ii) *$P \in W^{2,1}(\mathbb{T}^2)$, and for any $k \in \mathbb{N}$ there exists a constant $C = C(\lambda,\Lambda,k)$ such that*

$$\int_{\mathbb{T}^2} |\nabla^2 P| \log_+^k |\nabla^2 P|\,dx \le C.$$

(iii) *If* ρ, $\sigma \in C^{k,\alpha}(\mathbb{T}^2)$ *for some* $k \in \mathbb{N}$ *and* $\alpha \in (0,1)$, *then* $P \in C^{k+2,\alpha}(\mathbb{T}^2)$ *and there exists a constant* $C = C(\lambda, \Lambda, \|\rho\|_{C^{k,\alpha}}, \|\sigma\|_{C^{k,\alpha}})$ *such that*

$$\|P\|_{C^{k+2,\alpha}} \leq C.$$

Moreover, there exist two positive constants c_1 *and* c_2, *depending only on* λ, Λ, $\|\rho\|_{C^{0,\alpha}}$, *and* $\|\sigma\|_{C^{0,\alpha}}$, *such that*

$$c_1 Id \leq \nabla^2 P(x) \leq c_2 Id \qquad \forall\, x \in \mathbb{T}^2.$$

3. The dual problem and the regularity of the velocity field

In this section we recall some properties of solutions of (1.3), and we show the L^1 integrability of the velocity field u_t defined in (1.5).

We know by Theorem 2.1 that ρ_t uniquely defines P_t (and so also P_t^*) through the relation $(\nabla P_t)_\sharp \mathscr{L}_{\mathbb{T}^2} = \rho_t$ up to an additive constant. We have the following result (see [6, 12]):

Theorem 3.1 (Existence of solutions of (1.3))**.** *Let* $P_0 : \mathbb{R}^2 \to \mathbb{R}$ *be a convex function such that* $P_0(x) - |x|^2/2$ *is* $\mathbb{Z}^2$*-periodic,* $(\nabla P_0)_\sharp \mathscr{L}_{\mathbb{T}^2} \ll \mathscr{L}^2$, *and the density* ρ_0 *satisfies* $0 < \lambda \leq \rho_0 \leq \Lambda < \infty$. *Then there exist convex functions* $P_t, P_t^* : \mathbb{R}^2 \to \mathbb{R}$, *with* $P_t(x) - |x|^2/2$ *and* $P_t^*(y) - |y|^2/2$ *periodic, uniquely determined up to time-dependent additive constants, such that* $(\nabla P_t)_\sharp \mathscr{L}^2 = \rho_t \mathscr{L}^2$, $(\nabla P_t^*)_\sharp \rho_t = \mathscr{L}_{\mathbb{T}^2}$. *In addition, setting* $U_t(x) = J(x - \nabla P_t^*(x))$, ρ_t *is a distributional solution to* (1.3), *namely*

$$\int \int_{\mathbb{T}^2} \Big\{ \partial_t \varphi_t(x) + \nabla \varphi_t(x) \cdot U_t(x) \Big\} \rho_t(x)\, dx\, dt + \int_{\mathbb{T}^2} \varphi_0(x) \rho_0(x)\, dx = 0 \tag{3.1}$$

for every $\varphi \in C_c^\infty(\mathbb{R}^2 \times [0,\infty))$ $\mathbb{Z}^2$*-periodic in the space variable.*

Finally, the following regularity properties hold:

(i) $\lambda \leq \rho_t \leq \Lambda$*;*
(ii) $\rho_t \mathscr{L}^2 \in C([0,\infty), \mathcal{P}_w(\mathbb{T}^2))$*;*[4]
(iii) $P_t - \fint_{\mathbb{T}^2} P_t$, $P_t^* - \fint_{\mathbb{T}^2} P_t^* \in L^\infty([0,\infty), W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^2)) \cap C([0,\infty), W^{1,r}_{\mathrm{loc}}(\mathbb{R}^2))$ *for every* $r \in [1,\infty)$*;*
(iv) $\|U_t\|_\infty \leq \sqrt{2}/2$.

To be precise, in [6, 12] the proof is given in $\mathbb{R}^3$, but actually it can be rewritten verbatim on the 2-dimensional torus, using the optimal transport maps provided by Theorem 2.1. Observe that, by Theorem 3.1(ii), $t \mapsto \rho_t \mathscr{L}^2$ is weakly continuous, so ρ_t is a well-defined function *for every* $t \geq 0$.

Further regularity properties of ∇P_t and ∇P_t^* with respect to time will be proved in Propositions 3.3 and 3.6.

In the proof of Theorem 1.2 we will need to test with functions which are merely $W^{1,1}$. This is made possible by the following lemma.

Lemma 3.2. *Let* ρ_t *and* P_t *be as in Theorem 3.1. Then* (3.1) *holds for every* $\varphi \in W^{1,1}(\mathbb{T}^2 \times [0,\infty))$ *which is compactly supported in time. (Now* $\varphi_0(x)$ *has to be understood in the sense of traces.)*

Proof. Let $\varphi^n \in C^\infty(\mathbb{T}^2 \times [0,\infty))$ be strongly converging to φ in $W^{1,1}$, so that φ_0^n converges to φ_0 in $L^1(\mathbb{T}^2)$. Taking into account that both ρ_t and U_t are uniformly bounded from above in $\mathbb{T}^2 \times [0,\infty)$, we can apply (3.1) to the test functions φ^n and let $n \to \infty$ to obtain the same formula with φ. □

[4] Here $\mathcal{P}_w(\mathbb{T}^2)$ is the space of probability measures on the torus endowed with the *weak* topology induced by the duality with $C(\mathbb{T}^2)$

The following proposition, which provides the Sobolev regularity of $t \mapsto \nabla P_t^*$, is our main technical tool. Notice that, in order to prove Theorem 1.2, only finiteness of the left hand side in (3.2) would be needed, and the proof of this fact involves only a smoothing argument, the regularity estimates of [14] collected in Theorem 2.2(ii), and the argument of [17, Theorem 5.1]. However, the continuity result in [15] allows to show the validity of the natural *a priori* estimate on the left hand side in (3.2).

Proposition 3.3 (Time regularity of optimal maps). *Let ρ_t and P_t be as in Theorem 3.1. Then $\nabla P_t^* \in W^{1,1}_{\rm loc}(\mathbb{T}^2 \times [0,\infty);\mathbb{R}^2)$, and for every $k \in \mathbb{N}$ there exists a constant $C(k)$ such that, for almost every $t \geq 0$,*

$$\begin{aligned}(3.2)\quad &\int_{\mathbb{T}^2} \rho_t |\partial_t \nabla P_t^*| \log_+^k(|\partial_t \nabla P_t^*|)\, dx \\ &\qquad \leq C(k) \left(\int_{\mathbb{T}^2} \rho_t |\nabla^2 P_t^*| \log_+^{2k}(|\nabla^2 P_t^*|)\, dx + \operatorname*{ess\,sup}_{\mathbb{T}^2} \left(\rho_t |U_t|^2\right) \int_{\mathbb{T}^2} |\nabla^2 P_t^*|\, dx \right).\end{aligned}$$

To prove Proposition 3.3, we need some preliminary results.

Lemma 3.4. *For every $k \in \mathbb{N}$ we have*

$$(3.3)\qquad ab \log_+^k(ab) \leq 2^{k-1} \left[\left(\frac{k}{e}\right)^k + 1 \right] b^2 + 2^{3(k-1)} a^2 \log_+^{2k}(a) \qquad \forall\, (a,b) \in \mathbb{R}^+ \times \mathbb{R}^+.$$

Proof. From the elementary inequalities

$$\log_+(ts) \leq \log_+(t) + \log_+(s), \quad (t+s)^k \leq 2^{k-1}(t^k + s^k), \quad \log_+^k(t) \leq \left(\frac{k}{e}\right)^k t$$

which hold for every $t,\, s > 0$, we infer

$$\begin{aligned} ab \log_+^k(ab) &\leq ab \left[\log_+\left(\frac{b}{a}\right) + 2\log_+(a) \right]^k \\ &\leq 2^{k-1} ab \left[\log_+^k\left(\frac{b}{a}\right) + 2^k \log_+^k(a) \right] \\ &\leq 2^{k-1} \left[\left(\frac{k}{e}\right)^k b^2 + 2^k ab \log_+^k(a) \right] \\ &\leq 2^{k-1} \left[\left(\frac{k}{e}\right)^k b^2 + b^2 + 2^{2(k-1)} a^2 \log_+^{2k}(a) \right], \end{aligned}$$

which proves (3.3). □

Lemma 3.5 (Space-time regularity of transport). *Let $k \in \mathbb{N} \cup \{0\}$, and let $\rho \in C^\infty(\mathbb{T}^2 \times [0,\infty))$ and $U \in C^\infty(\mathbb{T}^2 \times [0,\infty);\mathbb{R}^2)$ satisfy*

$$0 < \lambda \leq \rho_t(x) \leq \Lambda < \infty \qquad \forall\, (x,t) \in \mathbb{T}^2 \times [0,\infty),$$

$$\partial_t \rho_t + \nabla \cdot (U_t \rho_t) = 0 \qquad \text{in } \mathbb{T}^2 \times [0,\infty),$$

and $\int_{\mathbb{T}^2} \rho_t\, dx = 1$ for all $t \geq 0$. Let us consider convex conjugate maps P_t and P_t^* such that $P_t(x) - |x|^2/2$ and $P_t^*(y) - |y|^2/2$ are $\mathbb{Z}^2$-periodic, $(\nabla P_t^*)_\sharp \rho_t = \mathscr{L}_{\mathbb{T}^2}$, $(\nabla P_t)_\sharp \mathscr{L}_{\mathbb{T}^2} = \rho_t$. Then:

(i) *$P_t^* - \fint_{\mathbb{T}^2} P_t^* \in \mathrm{Lip}_{\rm loc}([0,\infty); C^k(\mathbb{T}^2))$ for any $k \in \mathbb{N}$.*

(ii) *The following linearized Monge-Ampère equation holds:*

$$\nabla \cdot \big(\rho_t (\nabla^2 P_t^*)^{-1} \partial_t \nabla P_t^*\big) = -\nabla \cdot (\rho_t U_t). \tag{3.4}$$

Proof. Let us fix $T > 0$. From the regularity theory for the Monge-Ampère equation (see Theorem 2.2) we obtain that $P_t \in C^\infty(\mathbb{R}^2)$, uniformly for $t \in [0,T]$, and there exist universal constants c_1, $c_2 > 0$ such that

$$c_1 Id \le \nabla^2 P_t^*(x) \le c_2 Id \qquad \forall\, (x,t) \in \mathbb{T}^2 \times [0,T]. \tag{3.5}$$

Since ∇P_t^* is the inverse of ∇P_t, by the smoothness of P_t and (3.5) we deduce that $P_t^* \in C^\infty(\mathbb{R}^2)$, uniformly on $[0,T]$.

Now, to prove (i), we need to investigate the time regularity of $P_t^* - \fint_{\mathbb{T}^2} P_t^*$. Moreover, up to adding a time dependent constant to P_t, we can assume without loss of generality that $\int_{\mathbb{T}^2} P_t^* = 0$ for all t. By the condition $(\nabla P_t^*)_\sharp \rho_t = \mathscr{L}_{\mathbb{T}^2}$ we get that for any $0 \le s,t \le T$ and $x \in \mathbb{R}^2$ it holds

$$\begin{aligned}\frac{\rho_s(x) - \rho_t(x)}{s-t} &= \frac{\det(\nabla^2 P_s^*(x)) - \det(\nabla^2 P_t^*(x))}{s-t} \\ &= \sum_{i,j=1}^2 \left(\int_0^1 \frac{\partial \det}{\partial \xi_{ij}}(\tau \nabla^2 P_s^*(x) + (1-\tau)\nabla^2 P_t^*(x))\, d\tau\right) \frac{\partial_{ij} P_s^*(x) - \partial_{ij} P_t^*(x)}{s-t}.\end{aligned} \tag{3.6}$$

Given a 2×2 matrix $A = (\xi_{ij})_{i,j=1,2}$, we denote by $M(A)$ the cofactor matrix of A. We recall that

$$\frac{\partial \det(A)}{\partial \xi_{ij}} = M_{ij}(A), \tag{3.7}$$

and if A is invertible then $M(A)$ satisfies the identity

$$M(A) = \det(A)\, A^{-1}. \tag{3.8}$$

Moreover, if A is symmetric and satisfies $c_1 Id \le A \le c_2 Id$ for some positive constants c_1, c_2, then

$$\frac{c_1^2}{c_2} Id \le M(A) \le \frac{c_2^2}{c_1} Id. \tag{3.9}$$

Hence, from (3.6), (3.7), (3.5), and (3.9), it follows that

$$\frac{\rho_s - \rho_t}{s-t} = \sum_{i,j=1}^2 \left(\int_0^1 M_{ij}(\tau \nabla^2 P_s^* + (1-\tau)\nabla^2 P_t^*)\, d\tau\right) \partial_{ij}\left(\frac{P_s^* - P_t^*}{s-t}\right), \tag{3.10}$$

with

$$\frac{c_1^2}{c_2} Id \le \int_0^1 M_{ij}(\tau \nabla^2 P_s^* + (1-\tau)\nabla^2 P_t^*)\, d\tau \le \frac{c_2^2}{c_1} Id$$

Since $\nabla^2 P_t^*$ is smooth in space, uniformly on $[0,T]$, by classical elliptic regularity theory[5] it follows that for any $k \in \mathbb{N}$ and $\alpha \in (0,1)$ there exists a constant $C := C(\|(\rho_s - \rho_t)/(s-t)\|_{C^{k,\alpha}(\mathbb{T}^2 \times [0,T])})$ such that

$$\left\| \frac{P_s^*(x) - P_t^*(x)}{s-t} \right\|_{C^{k+2,\alpha}(\mathbb{T}^2)} \le C.$$

[5]Note that equation (3.6) is well defined on $\mathbb{T}^2$ since $P_t^* - P_s^*$ is $\mathbb{Z}^2$-periodic. We also observe that $P_t^* - P_s^*$ has average zero on $\mathbb{T}^2$.

This proves point (i) in the statement. To prove the second part, we let $s \to t$ in (3.10) to obtain

$$\partial_t \rho_t = \sum_{i,j=1}^{2} M_{ij}(\nabla^2 P_t^*(x))\, \partial_t \partial_{ij} P_t^*(x). \tag{3.11}$$

Taking into account the continuity equation and the well-known divergence-free property of the cofactor matrix

$$\sum_i \partial_i M_{ij}(\nabla^2 {P_t}^*(x)) = 0, \qquad j = 1, 2,$$

we can rewrite (3.11) as

$$-\nabla \cdot (U_t \rho_t) = \sum_{i,j=1}^{2} \partial_i \big(M_{ij}(\nabla^2 P_t^*(x))\, \partial_t \partial_j P_t^*(x) \big).$$

Hence, using (3.8) and the Monge-Ampère equation $\det(\nabla^2 P_t^*) = \rho_t$, we finally get (3.4). □

Proof of Proposition 3.3. We closely follow the proof of [17, Theorem 5.1], and we split the proof in two parts. In the first step we assume that

$$\rho_t \in C^\infty(\mathbb{T}^2 \times \mathbb{R}),\ U_t \in C^\infty(\mathbb{T}^2 \times \mathbb{R}; \mathbb{R}^2)\,, \tag{3.12}$$

$$0 < \lambda \le \rho_t \le \Lambda < \infty\,, \tag{3.13}$$

$$\partial_t \rho_t + \nabla \cdot (U_t \rho_t) = 0\,, \tag{3.14}$$

$$(\nabla P_t)_\sharp \mathscr{L}_{\mathbb{T}^2} = \rho_t \mathscr{L}_{\mathbb{T}^2}, \tag{3.15}$$

and we prove that (3.2) holds for every $t \ge 0$. In the second step we prove the general case through an approximation argument.

Step 1: The regular case. Let us assume that the regularity assumptions (3.12), (3.13), (3.14), (3.15) hold. Moreover, up to adding a time dependent constant to P_t, we can assume without loss of generality that $\int_{\mathbb{T}^2} P_t^* = 0$ for all $t \ge 0$, so that by Lemma 3.5 we have $\partial_t P_t^* \in C^\infty(\mathbb{T}^2)$. Fix $t \ge 0$. Multiplying (3.4) by $\partial_t P_t^*$ and integrating by parts, we get

$$\begin{aligned} \int_{\mathbb{T}^2} \rho_t |(\nabla^2 P_t^*)^{-1/2} \partial_t \nabla P_t^*|^2\, dx &= \int_{\mathbb{T}^2} \rho_t \partial_t \nabla P_t^* \cdot (\nabla^2 P_t^*)^{-1} \partial_t \nabla P_t^*\, dx \\ &= -\int_{\mathbb{T}^2} \rho_t \partial_t \nabla P_t^* \cdot U_t\, dx. \end{aligned} \tag{3.16}$$

(Since the matrix $\nabla^2 {P_t}^*(x)$ is nonnegative, both its square root and the square root of its inverse are well-defined.) From Cauchy-Schwartz inequality it follows that the right-hand side of (3.16) can be rewritten and estimated with

$$\begin{aligned} &-\int_{\mathbb{T}^2} \rho_t \partial_t \nabla P_t^* \cdot (\nabla^2 P_t^*)^{-1/2} (\nabla^2 P_t^*)^{1/2} U_t\, dx \\ &\le \left(\int_{\mathbb{T}^2} \rho_t |(\nabla^2 P_t^*)^{-1/2} \partial_t \nabla P_t^*|^2\, dx \right)^{1/2} \left(\int_{\mathbb{T}^2} \rho_t |(\nabla^2 P_t^*)^{1/2} U_t|^2\, dx \right)^{1/2}. \end{aligned} \tag{3.17}$$

Moreover, the second factor in the right-hand side of (3.17) can be estimated with

$$\int_{\mathbb{T}^2} \rho_t U_t \cdot \nabla^2 P_t^* U_t\, dx \le \max_{\mathbb{T}^2} \left(\rho_t |U_t|^2 \right) \int_{\mathbb{T}^2} |\nabla^2 P_t^*|\, dx. \tag{3.18}$$

Hence, from (3.16), (3.17), and (3.18) it follows that

$$\int_{\mathbb{T}^2} \rho_t |(\nabla^2 P_t^*)^{-1/2} \partial_t \nabla P_t^*|^2 \, dx \le \max_{\mathbb{T}^2} \left(\rho_t |U_t|^2\right) \int_{\mathbb{T}^2} |\nabla^2 P_t^*| \, dx. \tag{3.19}$$

We now apply Lemma 3.4 with $a = |(\nabla^2 P_t^*)^{1/2}|$ and $b = |(\nabla^2 P_t^*)^{-1/2} \partial_t \nabla P_t{}^*(x)|$ to deduce the existence of a constant $C(k)$ such that

$$\begin{aligned} |\partial_t \nabla P_t^*| \log_+^k(|\partial_t \nabla P_t^*|) &\le C(k) \left(|(\nabla^2 P_t^*)^{1/2}|^2 \log_+^{2k}(|(\nabla^2 P_t^*)^{1/2}|^2) + |(\nabla^2 P_t^*)^{-1/2} \partial_t \nabla P_t^*|^2 \right) \\ &= C(k) \left(|\nabla^2 P_t^*| \log_+^{2k}(|\nabla^2 P_t^*|) + |(\nabla^2 P_t^*)^{-1/2} \partial_t \nabla P_t^*|^2 \right). \end{aligned}$$

Integrating the above inequality over $\mathbb{T}^2$ and using (3.19), we finally obtain

$$\begin{aligned} &\int_{\mathbb{T}^2} \rho_t |\partial_t \nabla P_t^*| \log_+^k(|\partial_t \nabla P_t^*|) \, dx \\ &\le C(k) \left(\int_{\mathbb{T}^2} \rho_t |\nabla^2 P_t^*| \log_+^{2k}(|\nabla^2 P_t^*|) \, dx + \int_{\mathbb{T}^2} \rho_t |(\nabla^2 P_t^*)^{-1/2} \partial_t \nabla P_t^*|^2 \, dx \right) \\ &\le C(k) \left(\int_{\mathbb{T}^2} \rho_t |\nabla^2 P_t^*| \log_+^{2k}(|\nabla^2 P_t^*|) \, dx + \max_{\mathbb{T}^2} \left(\rho_t |U_t|^2\right) \int_{\mathbb{T}^2} |\nabla^2 P_t^*| \, dx \right), \end{aligned} \tag{3.20}$$

which proves (3.2).

Step 2: The approximation argument. First of all, we extend the functions ρ_t and U_t for $t \le 0$ by setting $\rho_t = \rho_0$ and $U_t = 0$ for every $t < 0$. We notice that, with this definition, ρ_t solves the continuity equation with velocity U_t on $\mathbb{R}^2 \times \mathbb{R}$.

Fix now $\sigma_1 \in C_c^\infty(\mathbb{R}^2)$, $\sigma_2 \in C_c^\infty(\mathbb{R})$, define the family of mollifiers $(\sigma^n)_{n \in \mathbb{N}}$ as $\sigma^n(x,t) := n^3 \sigma_1(nx) \sigma_2(nt)$, and set

$$\rho^n := \rho * \sigma^n, \qquad U^n(x) := \frac{(\rho U) * \sigma^n}{\rho * \sigma^n}.$$

Since $\lambda \le \rho \le \Lambda$ then

$$\lambda \le \rho^n \le \Lambda.$$

Therefore both ρ^n and U^n are well defined and satisfy (3.12), (3.13), (3.14). Moreover for every $t > 0$ the function ρ_t^n is $\mathbb{Z}^2$-periodic and it is a probability density when restricted to $(0,1)^2$ (once again we are identifying periodic functions with functions defined on the torus). Let P_t^n be the only convex function such that $(\nabla P_t^n)_\sharp \mathscr{L}^2 = \rho_t^n$ and its its convex conjugate P_t^{n*} satisfies $\int_{\mathbb{T}^2} P_t^{n*} = 0$ for all $t \ge 0$. Since $\rho_t^n \to \rho_t$ in $L^1(\mathbb{T}^2)$ for any $t > 0$ (recall that, by Theorem 3.1(ii), ρ_t is weakly continuous in time), from standard stability results for Alexandrov solutions of Monge-Ampère (see for instance [15]) it follows that

$$\nabla P_t^{n*} \to \nabla P_t^* \qquad \text{in } L^1(\mathbb{T}^2) \tag{3.21}$$

for any $t > 0$. Moreover, by Theorems 2.1 and 2.2(ii), for every $k \in \mathbb{N}$ there exists a constant $C := C(\lambda, \Lambda, k)$ such that

$$\int_{\mathbb{T}^2} \rho_t^n |\nabla^2 P_t^{n*}| \log_+^k(|\nabla^2 P_t^{n*}|) \, dx \le C,$$

and by the stability theorem in the Sobolev topology estabilished in [15, Theorem 1.3] it follows that

$$\int_{\mathbb{T}^2} \rho_t^n |\nabla^2 P_t^{n*}| \log_+^k(|\nabla^2 P_t^{n*}|) \, dx \to \int_{\mathbb{T}^2} \rho_t |\nabla^2 P_t^*| \log_+^k(|\nabla^2 P_t^*|) \, dx, \tag{3.22}$$

$$\int_{\mathbb{T}^2} |\nabla^2 P_t^{n*}| \, dx \to \int_{\mathbb{T}^2} |\nabla^2 P_t^*| \, dx. \tag{3.23}$$

Finally, since the function $(w, t) \mapsto F(w, t) = |w|^2/t$ is convex on $\mathbb{R}^2 \times (0, \infty)$, by Jensen inequality we get

$$\|\rho^n |U^n|^2\|_\infty = \|F(\rho^n U^n, \rho^n)\|_\infty \le \|\rho |U|^2\|_\infty. \tag{3.24}$$

Let us fix $T > 0$ and $\phi \in C_c^\infty((0, T))$ nonnegative. From the previous steps and Dunford-Pettis Theorem, it is clear that $\phi(t)\rho_t^n \partial_t \nabla P_t^{n*}$ weakly converge to $\phi(t)\rho_t \partial_t \nabla P_t^*$ in $L^1(\mathbb{T}^2 \times (0, T))$. Moreover, since the function $w \mapsto |w| \log_+^k(|w|/r)$ is convex for every $r \in (0, \infty)$ we can apply Ioffe lower semicontinuity theorem [1, Theorem 5.8] to the functions $\phi(t)\rho_t^n \partial_t \nabla P_t^{n*}$ and $\phi(t)\rho_t^n$ to infer

$$\int_0^T \phi(t) \int_{\mathbb{T}^2} \rho_t |\partial_t \nabla P_t^*| \log_+^k(|\partial_t \nabla P_t^*|) \, dx \, dt \le \liminf_{n \to \infty} \int_0^T \phi(t) \int_{\mathbb{T}^2} \rho_t^n |\partial_t \nabla P_t^{n*}| \log_+^k(|\partial_t \nabla P_t^{n*}|) \, dx \, dt. \tag{3.25}$$

By Step 1 we can apply (3.2) to ρ_t^n, U_t^n. Taking (3.22), (3.23), (3.24) and (3.25) into account, by Lebesgue dominated convergence theorem we obtain

$$\begin{aligned} &\int_0^T \phi(t) \int_{\mathbb{T}^2} \rho_t |\partial_t \nabla P_t^*| \log_+^k(|\partial_t \nabla P_t^*|) \, dx \, dt \\ &\qquad \le C(k) \int_0^T \phi(t) \left(\int_{\mathbb{T}^2} \rho_t |\nabla^2 P_t^*| \log_+^{2k}(|\nabla^2 P_t^*|) \, dx + \operatorname*{ess\,sup}_{\mathbb{T}^2} \left(\rho_t |U_t|^2\right) \int_{\mathbb{T}^2} |\nabla^2 P_t^*| \, dx \right) dt. \end{aligned}$$

Since this holds for every $\phi \in C_c^\infty((0, T))$ nonnegative, we obtain the desired result. □

It is clear from the proof of Proposition 3.3 that the particular coupling between the velocity field U_t and the transport map P_t is not used. Actually, using Theorem 2.2(ii) and [15, Theorem 1.3], and arguing again as in the proof of [17, Theorem 5.1], the following more general statement holds (compare with [17, Theorem 5.1, Equations (27) and (29)]):

Proposition 3.6. *Let ρ_t and v_t be such that $0 < \lambda \le \rho_t \le \Lambda < \infty$, $v_t \in L^\infty_{\mathrm{loc}}(\mathbb{T}^2 \times [0, \infty), \mathbb{R}^2)$, and*

$$\partial_t \rho_t + \nabla \cdot (v_t \rho_t) = 0.$$

Assume that $\int_{\mathbb{T}^2} \rho_t \, dx = 1$ for all $t \ge 0$, let P_t be a convex function such that

$$(\nabla P_t)_\sharp \mathscr{L}_{\mathbb{T}^2} = \rho_t \mathscr{L}_{\mathbb{T}^2},$$

and denote by P_t^* its convex conjugate.

Then ∇P_t and ∇P_t^* belong to $W^{1,1}_{\mathrm{loc}}(\mathbb{T}^2 \times [0, \infty); \mathbb{R}^2)$. Moreover, for every $k \in \mathbb{N}$ there exists a constant $C(k)$ such that, for almost every $t \ge 0$,

$$\begin{aligned} &\int_{\mathbb{T}^2} \rho_t |\partial_t \nabla P_t^*| \log_+^k(|\partial_t \nabla P_t^*|) \, dx \\ &\qquad \le C(k) \left(\int_{\mathbb{T}^2} \rho_t |\nabla^2 P_t^*| \log_+^{2k}(|\nabla^2 P_t^*|) \, dx + \operatorname*{ess\,sup}_{\mathbb{T}^2} \left(\rho_t |v_t|^2\right) \int_{\mathbb{T}^2} |\nabla^2 P_t^*| \, dx \right), \end{aligned} \tag{3.26}$$

$$\begin{aligned} &\int_{\mathbb{T}^2} |\partial_t \nabla P_t| \log_+^k(|\partial_t \nabla P_t|) \, dx \\ &\qquad \le C(k) \left(\int_{\mathbb{T}^2} |\nabla^2 P_t| \log_+^{2k}(|\nabla^2 P_t|) \, dx + \operatorname*{ess\,sup}_{\mathbb{T}^2} \left(\rho_t |v_t|^2\right) \int_{\mathbb{T}^2} |\nabla^2 P_t^*| \, dx \right). \end{aligned} \tag{3.27}$$

Proof. We just give a short sketch of the proof. Equation (3.26) can be proved following the same line of the proof of Proposition 3.3. To prove (3.27) notice that by the approximation argument in the second step of the proof of Proposition 3.3 we can assume that the velocity and the density are smooth and hence, arguing as in Lemma 3.5, we have that P_t, $P_t^* \in \mathrm{Lip}_{\mathrm{loc}}([0,\infty), C^\infty(\mathbb{T}^2))$. Now, changing variables in the the left hand side of (3.19) we get

$$\int_{\mathbb{T}^2} \left| \big([\nabla^2 P_t^*](\nabla P_t)\big)^{-1/2} [\partial_t \nabla P_t^*](\nabla P_t) \right|^2 dx \le \max_{\mathbb{T}^2} \big(\rho_t |v_t|^2\big) \int_{\mathbb{T}^2} |\nabla^2 P_t^*|\, dx. \tag{3.28}$$

Taking into account the identities

$$[\nabla^2 P_t^*](\nabla P_t) = \big(\nabla^2 P_t\big)^{-1} \qquad \text{and} \qquad [\partial_t \nabla P_t^*](\nabla P_t) + [\nabla^2 P_t^*](\nabla P_t) \partial_t \nabla P_t = 0$$

which follow differentiating with respect to time and space $\nabla P_t^* \circ \nabla P_t = Id$, Equation (3.28) becomes

$$\int_{\mathbb{T}^2} |(\nabla^2 P_t)^{-1/2} \partial_t \nabla P_t|^2\, dx \le \max_{\mathbb{T}^2} \big(\rho_t |v_t|^2\big) \int_{\mathbb{T}^2} |\nabla^2 P_t^*|\, dx.$$

At this point the proof of (3.27) is obtained arguing as in Proposition 3.3. □

4. Existence of an Eulerian solution

In this section we prove Theorem 1.2.

Proof of Theorem 1.2. First of all notice that, thanks to Theorem 2.2(i) and Proposition 3.3, it holds $|\nabla^2 P_t^*|$, $|\partial_t \nabla P_t^*| \in L^\infty_{\mathrm{loc}}([0,\infty), L^1(\mathbb{T}^2))$. Moreover, since $(\nabla P_t)_\sharp \mathscr{L}_{\mathbb{T}^2} = \rho_t \mathscr{L}_{\mathbb{T}^2}$, it is immediate to check the function u in (1.5) is well-defined[6] and $|u|$ belongs to $L^\infty_{\mathrm{loc}}([0,\infty), L^1(\mathbb{T}^2))$.

Let $\phi \in C_c^\infty(\mathbb{R}^2 \times [0,\infty))$ be a $\mathbb{Z}^2$-periodic function in space and let us consider the function $\varphi : \mathbb{R}^2 \times [0,\infty) \to \mathbb{R}^2$ given by

$$\varphi_t(y) := J(y - \nabla P_t^*(y)) \phi_t(\nabla P_t^*(y)). \tag{4.1}$$

By Theorem 2.1 and the periodicity of ϕ, $\varphi_t(y)$ is $\mathbb{Z}^2$-periodic in the space variable. Moreover φ_t is compactly supported in time, and Proposition 3.3 implies that $\varphi \in W^{1,1}(\mathbb{R}^2 \times [0,\infty))$. So, by Lemma 3.2, each component of the function $\varphi_t(y)$ is an admissible test function for (3.1). For later use, we write down explicitly the derivatives of φ:

$$\begin{cases} \partial_t \varphi_t(y) = -J[\partial_t \nabla P_t^*](y) \phi_t(\nabla P_t^*(y)) + J(y - \nabla P_t^*(y)) [\partial_t \phi_t](\nabla P_t^*(y)) + \\ \qquad\qquad + J(y - \nabla P_t^*(y)) \big([\nabla \phi_t](P_t^*(y)) \cdot \partial_t \nabla P_t^*(y)\big), \\ \nabla \varphi_t(y) = J(Id - \nabla^2 P_t^*(y)) \phi_t(\nabla P_t^*(y)) + J(y - \nabla P_t^*(y)) \otimes \big([\nabla^T \phi_t](P_t^*(y)) \nabla^2 P_t^*(y)\big). \end{cases} \tag{4.2}$$

Taking into account that $(\nabla P_t)_\sharp \mathscr{L}_{\mathbb{T}^2} = \rho_t \mathscr{L}_{\mathbb{T}^2}$ and that $[\nabla P_t^*](\nabla P_t(x)) = x$ almost everywhere, we can rewrite the boundary term in (3.1) as

$$\int_{\mathbb{T}^2} \varphi_0(y) \rho_0(y)\, dy = \int_{\mathbb{T}^2} J(\nabla P_0(x) - x) \phi_0(x)\, dx = \int_{R^2} J \nabla p_0(x) \phi_0(x)\, dx. \tag{4.3}$$

[6]Note that the composition of $\nabla^2 P_t^*$ with ∇P_t makes sense. Indeed, by the conditions $(\nabla P_t)_\sharp \mathscr{L}_{\mathbb{T}^2} = \rho_t \mathscr{L}_{\mathbb{T}^2} \ll \mathscr{L}_{\mathbb{T}^2}$, if we change the value of $\nabla^2 P_t^*$ in a set of measure zero, also $[\nabla^2 P_t^*](\nabla P_t)$ will change only on a set of measure zero.

In the same way, since $U_t(y) = J(y - \nabla P_t^*(y))$, we can use (4.2) to rewrite the other term as

(4.4)
$$
\begin{aligned}
&\int_0^\infty \int_{\mathbb{T}^2} \Big\{\partial_t\varphi_t(y) + \nabla\varphi_t(y)\cdot U_t(y)\Big\}\rho_t(y)\,dy\,dt \\
&= \int_0^\infty \int_{\mathbb{T}^2} \Big\{ - J[\partial_t \nabla P_t^*](\nabla P_t(x))\phi_t(x) + J(\nabla P_t(x) - x)\partial_t\phi_t(x) \\
&+ J(\nabla P_t(x) - x)\big(\nabla\phi_t(x)\cdot[\partial_t\nabla P_t^*](\nabla P_t(x))\big) \\
&+ \big[J(Id - \nabla^2 P_t^*(\nabla P_t(x)))\phi_t(x) + J(\nabla P_t(x) - x)\otimes\big(\nabla^T\phi_t(x)\nabla^2 P_t^*(\nabla P_t(x))\big)\big]J(\nabla P_t(x) - x)\Big\}\,dx\,dt
\end{aligned}
$$

which, taking into account the formula (1.5) for u, after rearranging the terms turns out to be equal to

$$
\int_0^\infty \int_{\mathbb{T}^2} \big\{J\nabla p_t(x)\big(\partial_t\phi_t(x) + u_t(x)\cdot\nabla\phi_t(x)\big) + \big(-\nabla p_t(x) - Ju_t(x)\big)\phi_t(x)\big\}\ dx\,dt. \tag{4.5}
$$

Hence, combining (4.3), (4.4), (4.5), and (3.1), we obtain the validity of (1.6).

Now we prove (1.7). Given $\phi \in C_c^\infty(0,\infty)$ and a $\mathbb{Z}^2$-periodic function $\psi \in C^\infty(\mathbb{R}^2)$, let us consider the function $\varphi : \mathbb{R}^2 \times [0,\infty) \to \mathbb{R}$ defined by

$$
\varphi_t(y) := \phi(t)\psi(\nabla P_t{}^*(y)). \tag{4.6}
$$

As in the previous case, we have that φ is $\mathbb{Z}^2$-periodic in the space variable and $\varphi \in W^{1,1}(\mathbb{T}^2\times[0,\infty))$, so we can use φ as a test function in (1.7). Then, identities analogous to (4.2) yield

$$
\begin{aligned}
0 &= \int_0^\infty \int_{\mathbb{T}^2} \{\partial_t\varphi_t(y) + \nabla\varphi_t(y)\cdot U_t(y)\}\,\rho_t(y)\,dy\,dt \\
&= \int_0^\infty \phi'(t)\int_{\mathbb{T}^2}\psi(x)\,dx\,dt \\
&\quad + \int_0^\infty \phi(t)\int_{\mathbb{T}^2}\Big\{\nabla\psi(x)\cdot\partial_t\nabla P_t{}^*(\nabla P_t(x)) + \nabla^T\psi(x)\nabla^2 P_t^*(\nabla P_t(x))J(\nabla P_t(x) - x)\Big\}\,dx\,dt \\
&= \int_0^\infty \phi(t)\int_{\mathbb{T}^2}\nabla\psi(x)\cdot u_t(x)\,dx\,dt.
\end{aligned}
$$

Since ϕ is arbitrary we obtain

$$
\int_{\mathbb{T}^2}\nabla\psi(x)\cdot u_t(x)\,dx = 0 \qquad \text{for a.e. } t>0.
$$

By a standard density argument it follows that the above equation holds outside a negligible set of times independent of the test function ψ, thus proving (1.7). □

5. Existence of a Regular Lagrangian Flow for the semigeostrophic velocity field

We start with the definition of Regular Lagrangian Flow for a given vector field b, inspired by [2, 3]:

Definition 5.1. Given a Borel, locally integrable vector field $b : \mathbb{T}^2\times(0,\infty)\to\mathbb{R}^2$, we say that a Borel function $F : \mathbb{T}^2\times[0,\infty)\to\mathbb{T}^2$ is a *Regular Lagrangian Flow (in short RLF) associated to* b if the following two conditions are satisfied.

(a) For almost every $x\in\mathbb{T}^2$ the map $t\mapsto F_t(x)$ is locally absolutely continuous in $[0,\infty)$ and

$$
F_t(x) = x + \int_0^t b_s(F_s(x))dx \qquad \forall t\in[0,\infty). \tag{5.1}
$$

(b) For every $t \in [0,\infty)$ it holds $(F_t)_\# \mathscr{L}_{\mathbb{T}^2} \le C \mathscr{L}_{\mathbb{T}^2}$, with $C \in [0,\infty)$ independent of t.

A particular class of RLFs is the collection of the measure-preserving ones, where (b) is strengthened to

$$(F_t)_\# \mathscr{L}_{\mathbb{T}^2} = \mathscr{L}_{\mathbb{T}^2} \qquad \forall t \ge 0.$$

Notice that *a priori* the above definition depends on the choice of the representative of b in the Lebesgue equivalence class, since modifications of b in Lebesgue negligible sets could destroy condition (a). However, a simple argument based on Fubini's theorem shows that the combination of (a) and (b) is indeed invariant (see [2, Section 6]): in other words, if $b = \widetilde{b}$ a.e. in $\mathbb{T}^2 \times (0,\infty)$, then every RLF associated to b is also a RLF associated to $\widetilde{b}$.

We show existence of a measure-preserving RLF associated to the vector field u defined by

$$u_t(x) = [\partial_t \nabla P_t^*](\nabla P_t(x)) + [\nabla^2 P_t^*](\nabla P_t(x)) J(\nabla P_t(x) - x), \tag{5.2}$$

where P_t and P_t^* are as in Theorem 1.2. Recall also that, under these assumptions, $|u| \in L^\infty_{\text{loc}}([0,\infty), L^1(\mathbb{T}^2))$.

Existence for weaker notion of Lagrangian flow of the semigeostrophic equations was proved by Cullen and Feldman, see [12, Definition 2.4], but since at that time the results of [14] were not available the velocity could not be defined, not even as a function. Hence, they had to adopt a more indirect definition. We shall prove indeed that their flow is a flow according to Definition 5.1. We discuss the uniqueness issue in the last section.

Theorem 5.2. *Let us assume that the hypotheses of Theorem 1.2 are satisfied, and let P_t and P_t^* be the convex functions such that*

$$(\nabla P_t)_\sharp \mathscr{L}_{\mathbb{T}^2} = \rho_t \mathscr{L}_{\mathbb{T}^2}, \qquad (\nabla P_t^*)_\sharp \rho_t \mathscr{L}_{\mathbb{T}^2} = \mathscr{L}_{\mathbb{T}^2}.$$

Then, for u_t given by (5.2) *there exists a measure-preserving RLF F associated to u_t. Moreover F is invertible in the sense that for all $t \ge 0$ there exist Borel maps F_t^* such that $F_t^*(F_t) = Id$ and $F_t(F_t^*) = Id$ a.e. in $\mathbb{T}^2$.*

Proof. Let us consider the velocity field in the dual variables $U_t(x) = J(x - \nabla P_t^*(x))$. Since P_t^* is convex, $U_t \in BV(\mathbb{T}^2; \mathbb{R}^2)$ uniformly in time (actually, by Theorem 2.2(ii) $U_t \in W^{1,1}(\mathbb{T}^2; \mathbb{R}^2)$). Moreover U_t is divergence-free. Hence, by the theory of Regular Lagrangian Flows associated to BV vector fields [2, 3], there exists a unique[7] measure-preserving RLF $G : \mathbb{T}^2 \times [0,\infty) \to \mathbb{T}^2$ associated to U.

We now define[8]

$$F_t(y) := \nabla P_t^*(G_t(\nabla P_0(y))). \tag{5.3}$$

The validity of property (b) in Definition 5.1 and the invertibility of F follow from the same arguments of [12, Propositions 2.14 and 2.17]. Hence we only have to show that property (a) in Definition 5.1 holds.

Let us define $Q^n := B * \sigma^n$, where B is a Sobolev and uniformly continuous extension of ∇P^* to $\mathbb{T}^2 \times \mathbb{R}$, and σ^n is a standard family of mollifiers in $\mathbb{T}^2 \times \mathbb{R}$. It is well known that $Q^n \to \nabla P^*$

[7]The uniqueness of Regular Lagrangian Flows has to be understood in the following way: if $G_1, G_2 : \mathbb{T}^2 \times [0,\infty) \to \mathbb{T}^2$ are two RLFs associated to U, then the integral curves $G_1(\cdot, x)$ and $G_2(\cdot, x)$ are equal for $\mathscr{L}^2$-a.e. x.

[8]Observe that the definition of F makes sense. Indeed, by Theorem 2.2(i), both maps ∇P_0 and ∇P_t^* are Hölder continuous in space. Morever, by the weak continuity in time of $t \mapsto \rho_t$ (Theorem 3.1(ii)) and the stability results for Alexandrov solutions of Monge-Ampère, ∇P^* is continuous both in space and time. Finally, since $(\nabla P_0)_\sharp \mathscr{L}_{\mathbb{T}^2} \ll \mathscr{L}_{\mathbb{T}^2}$, if we change the value of G in a set of measure zero, also F will change only on a set of measure zero.

locally uniformly and in the strong topology of $W^{1,1}_{\rm loc}(\mathbb{T}^2\times[0,\infty))$. Thus, using the measure-preserving property of G_t, for all $T>0$ we get

$$
\begin{aligned}
0 &= \lim_{n\to\infty}\int_{\mathbb{T}^2}\int_0^T \Big\{|Q^n_t-\nabla P^*_t|+|\partial_t Q^n_t-\partial_t\nabla P^*_t|+|\nabla Q^n_t-\nabla^2 P^*_t|\Big\}\,dy\,dt.\\
&= \lim_{n\to\infty}\int_{\mathbb{T}^2}\int_0^T \Big\{|Q^n_t(G_t)-\nabla P^*_t(G_t)|+|[\partial_t Q^n_t](G_t)-[\partial_t\nabla P^*_t](G_t)|+|[\nabla Q^n_t](G_t)-[\nabla^2 P^*_t](G_t)|\Big\}\,dx\,dt.
\end{aligned}
$$

Up to a (not re-labeled) subsequence the previous convergence is pointwise in space, namely, for almost every $x\in\mathbb{T}^2$,

$$
\begin{aligned}
\int_0^T \Big\{&|Q^n_t(G_t(x))-\nabla P^*_t(G_t(x))|+|[\partial_t Q^n_t](G_t(x))-[\partial_t\nabla P^*_t](G_t(x))|\\
&+|[\nabla Q^n_t](G_t(x))-[\nabla^2 P^*_t](G_t(x))|\Big\}\,dt\to 0.
\end{aligned}
\tag{5.4}
$$

Hence, since G is a RLF and by assumption

$$
(\nabla P_0)\mathscr{L}_{\mathbb{T}^2}\ll\mathscr{L}_{\mathbb{T}^2},
$$

for almost every y we have that (5.4) holds at $x=\nabla P_0(y)$, and the function $t\mapsto G_t(x)$ is absolutely continuous on $[0,T]$, with derivative given by

$$
\frac{d}{dt}G_t(x)=U_t(G_t(x))=J(G_t(x)-\nabla P^*_t(G_t(x)))\qquad\text{for a.e. } t\in[0,T].
$$

Let us fix such an y. Since Q^n is smooth, the function $Q^n_t(G_t(x))$ is absolutely continuous in $[0,T]$ and its time derivative is given by

$$
\frac{d}{dt}\big(Q^n_t(G_t(x))\big)=[\partial_t Q^n_t](G_t(x))+[\nabla Q^n_t](G_t(x))J(G_t(x)-\nabla P^*_t(G_t(x))).
$$

Hence, since $J(G_t(x)-\nabla P^*_t(G_t(x)))=U(G_t(x))$ is uniformly bounded, from (5.4) we get

$$
\lim_{n\to\infty}\frac{d}{dt}\big(Q^n_t(G_t(x))\big)=[\partial_t\nabla P^*_t](G_t(x))+[\nabla^2 P^*_t](G_t(x))J(G_t(x)-\nabla P^*_t(G_t(x))):=v_t(y)\qquad\text{in } L^1(0,T).
\tag{5.5}
$$

Recalling that

$$
\lim_{n\to\infty}Q^n_t(G_t(x))=\nabla P^*_t(G_t(x))=F_t(y)\qquad\forall\,t\in[0,T],
$$

we infer that $F_t(y)$ is absolutely continuous in $[0,T]$ (being the limit in $W^{1,1}(0,T)$ of absolutely continuous maps). Moreover, by taking the limit as $n\to\infty$ in the identity

$$
Q^n_t(G_t(x))=Q^n_0(G_0(x))+\int_0^t\frac{d}{d\tau}\big(Q^n_\tau(G_\tau(x))\big)\,d\tau,
$$

thanks to (5.5) we get

$$
F_t(y)=F_0(y)+\int_0^t v_\tau(y)\,d\tau.
\tag{5.6}
$$

To obtain (5.1) we only need to show that $v_t(y)=u_t(F_t(y))$, which follows at once from (5.2), (5.3), and (5.5).

□

6. Open problems

In this short section we point out some open problems. The first one is of course uniqueness for the Cauchy problem, both at the level of (1.3) and at the level of (1.2). Let us point out that *a priori* the two problems are not equivalent, because we proved that solutions to (1.3) induce solutions to (1.2), but at the moment the converse implication is only formal (see the Appendix).

Another open question is the uniqueness of the regular Lagrangian flow associated to u. Uniqueness is known, thanks to the results in [2], for the flow G in the dual variables with velocity $U_t(y) = J(y-\nabla P_t^*(y))$; actually, in light of the $L\log^k L$ Sobolev regularity of U, even the quantitative stability results of [13] are by now available for G. We were able in the previous section to prove that flows G_t of U induce flows F_t of u, via the transformation $F_t = \nabla P_t^* \circ G_t \circ \nabla P_0$. However, our proof used the boundedness of U, an information we do not have when we try to reverse the implication, namely that regular Lagrangian flows F of u induce regular Lagrangian flows G of U via the transformation $G = \nabla P_t \circ F_t \circ \nabla P_0^*$. This question could be settled, at least in the class of measure-preserving Lagrangian flows, if the following conjecture had a positive answer:

Conjecture. *Let $f \in W^{1,1}((0,T)\times\mathbb{T}^2;\mathbb{R}^2) \cap C([0,T]\times\mathbb{T}^2;\mathbb{R}^2)$, and let H_t be a measure-preserving Lagrangian flow relative to b. Assume that*

$$[\partial_t f_t](H_t(x)) + [\nabla f_t](H_t(x)) b_t(H_t(x)) \in L^1(0,T) \qquad \text{for a.e. } x \in \mathbb{T}^2. \tag{6.1}$$

Then for a.e. $x \in \mathbb{T}^2$ the map $t \mapsto f_t(H_t(x))$ is absolutely continuous.

In our case, $f = \nabla P$ and H_t is a measure-preserving flow associated to $b = u$; with these choices, the term in (6.1) is equal to $U_t(x)$, so it is even bounded, even though the summands in the expression might be unbounded.

We remark that if we assume that $f \in W^{1,q}$ for some $q > 1$, and that

$$\int_{\mathbb{T}^2}\int_0^T \left|\frac{d}{dt}H_t(x)\right|^p dt\,dx = \int_{\mathbb{T}^2}\int_0^T \big|b_t(x)\big|^p\,dt\,dx < \infty, \qquad p = \frac{q}{q-1},$$

then a simple approximation argument based on convolving f with smooth convolution kernels, as the one used in the proof of Theorem 5.2, provides a positive answer to the above conjecture. (This result can also be seen as a particular case of the general theory of weak gradients and absolute continuity along curves recently developed in [4, 5]. However, if f is not continuous, one needs to replace f with a suitable "precise representative" in its Lebesgue equivalence class.) Observe that, in this latter case, (6.1) is automatically satisfied by Young inequality.

Appendix A. From physical to dual variables

For completeness, we formally show how the dual equation (1.3) is derived from system (1.2). Taking into account the definition of P_t, the identities $J^2 = -Id$, $\nabla p_t(y)+y = \nabla P_t(y)$, $\nabla^2 p_t(y)+Id = \nabla^2 P_t(y)$ and the fact that u_t is divergence-free, for every test function φ we obtain

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{T}^2}\varphi(x)\,d\rho_t(x) &= \frac{d}{dt}\int_{\mathbb{T}^2}\varphi(\nabla P_t(y))\,dy = \int_{\mathbb{T}^2}\nabla\varphi(\nabla P_t(y))\cdot\frac{d}{dt}\nabla p_t(y)\,dy \\
&= -\int_{\mathbb{T}^2}\nabla\varphi(\nabla P_t(y))\cdot\Big\{(\nabla^2 p_t(y)+Id)u_t(y) - J\nabla p_t(y)\Big\}\,dy \\
&= -\int_{\mathbb{T}^2}\nabla\big[\varphi(\nabla P_t(y))\big]\cdot u_t(y)\,dy + \int_{\mathbb{T}^2}\nabla\varphi(\nabla P_t(y))\cdot J(\nabla P_t(y)-y)\,dy \\
&= \int_{\mathbb{T}^2}\nabla\varphi(x)\cdot J(x-\nabla P_t^*(x))\,d\rho_t(x) = \int_{\mathbb{T}^2}\nabla\varphi(x)\cdot U_t(x)\,d\rho_t(x).
\end{aligned}$$

Notice that this formal derivation holds independently of u (only the divergence-free condition of u is needed), and that u does not appear explicitly in (1.3).

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