A VISCOSITY-DRIVEN CRACK EVOLUTION

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Abstract. We present a model of crack growth in brittle materials which couples dissipative effects on the crack tip and viscous effects. We consider the 2-dimensional antiplane case with pre-assigned crack path, and firstly prove an existence result for a rate-dependent evolution problem by means of time-discretization. The next goal is to describe the rate-independent evolution as limit of the rate-dependent ones when the dissipative and viscous effects vanish. The rate-independent evolution satisfies a Griffith's criterion for the crack growth, but, in general, it does not fulfil a global minimality condition; its fracture set may exhibit jump discontinuities with respect to time. Under suitable regularity assumptions, the quasi-static crack growth is described by solving a finite-dimensional problem.

Keywords: vanishing viscosity, viscoelasticity, rate-dependent evolution, rate-independent evolution, quasi-static crack evolution, Griffith's criterion.

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Contents

1. Introduction

The importance in applications of understanding and predicting how cracks develop in fractured elastic materials has driven the attention on the mathematical side of the issue. Already at the beginning of the 20^{th} century, Griffith [9] proposed an energetic model, based on the idea that the crack grows if the release of stored energy is larger than the energy dissipated by the crack creation.

In this paper we study the problem of crack growth in the antiplane case with a prescribed crack path $\varGamma = \{\gamma(\sigma) : \sigma \in [0, L]\}$ in a domain $\Omega \subset \mathbb{R}^2$. With the same setting, Francfort & Marigo [8] proposed a variational model for an evolution $(s(t), u(t))$, where $s(t)$ and $u(t)$ are the crack tip position and the displacement respectively, driven by a monotone increasing boundary loading $w(t)$ (i.e. $u(t) = w(t)$ on $\partial_D\Omega \subset \partial\Omega$) and satisfying the following conditions at any instant $t \in [0, T]$:

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(1) global minimality:
$$E_d(u(t)) + E_s(s(t)) \leq E_d(v) + E_s(\sigma)$$
for any $\sigma \geq s(t-)$ and $v \in H^1(\Omega \setminus \Gamma(\sigma))$ with $v = w(t)$ on $\partial_D\Omega$;

(2) energy conservation:
$$E_d(u(t-)) + E_s(s(t-)) = E_d(u(t+)) + E_s(s(t+))$$

where $E_d(v)$ is the bulk energy associated to the displacement v and $E_s(\sigma)$ is the energy for the crack $\Gamma(\sigma) = \gamma([0,\sigma])$ of length σ. They show that, if $s(t)$ is sufficiently regular, then a Griffith's criterion holds true:
$$\begin{aligned} &\dot{s}(t) \geq 0 \\ &\mathcal{G}(s(t), w(t)) - \mathfrak{c} \leq 0 \\ &[\mathcal{G}(s(t), w(t)) - \mathfrak{c}]\, \dot{s}(t) = 0, \end{aligned} \tag{1.1}$$
having denoted by $\mathcal{G}(\sigma, \psi)$ the energy release rate for a crack of length σ and a boundary loading ψ, and by $\mathfrak{c}$ the material toughness.

The model obtained through the selection principle (1)-(2) generates solutions $s(\cdot)$, called *globally stable quasi-static evolutions*, which show jumps (i.e. discontinuities) with respect to time that are not motivated by the mechanics of the system. In particular, the global minimality condition forces the solution to jump between potential wells, without taking into account potential barriers.

Griffith's criterion is anyway reasonable from the physical point of view. We propose a selection principle for the evolution different than [8], in order to still fulfil the three conditions in (1.1) and in the meantime avoid unjustified jumps in time of the fracture. It is important to underline that discontinuities in time of the crack term are never physically acceptable, since they represent sudden increases of the fracture. In our model, we will interpret those discontinuities as limits of fast growing cracks.

Firstly we consider a *rate-dependent* model which takes into account dissipations due to the crack tip velocity and viscous effects (viscoelasticity). In mathematical terms, we study the problem
$$\begin{cases} \mathfrak{a}\Delta u(t) + \mathfrak{b}\Delta \dot{u}(t) = 0 & \text{in } \Omega \setminus \Gamma(s(t)) \\ u(t) = w(t) & \text{on } \partial_D\Omega \\ \dot{s}(t) \geq 0 \\ -\mathcal{G}(s(t), \mathfrak{a}w(t) + \mathfrak{b}\dot{w}(t)) + \mathfrak{c}s(t) + \mathfrak{d}\dot{s}(t) \geq 0 \\ [-\mathcal{G}(s(t), \mathfrak{a}w(t) + \mathfrak{b}\dot{w}(t)) + \mathfrak{c}s(t) + \mathfrak{d}\dot{s}(t)]\, \dot{s}(t) = 0 \end{cases} \tag{1.2}$$
under proper initial conditions $s(0) = s_0$ and $u(0) = u_0$. The dimensional constants $\mathfrak{a}$ and $\mathfrak{b}$ are the Young modulus and the coefficient of viscosity, respectively, while $\mathfrak{d}$ is a dissipation constant. The existence of a solution $(s(\cdot), u(\cdot))$ to (1.2) is proved by time discretization and minimization of energy functionals, whose form and physical meaning are explained later on (see Section 3); at that moment the selection principle, different than (1)-(2), is made explicit. The selected solution is such that the map $t \mapsto \nabla u(t)$ is continuous from an interval $[0,T]$ to $L^2(\Omega;\mathbb{R}^2)$ and differentiable a.e., with $t \mapsto \nabla\dot{u}(t)$ belonging to $L^2(0,T;L^2(\Omega;\mathbb{R}^2))$, while $t \mapsto s(t)$ is continuous.

Our next step is to replace $\mathfrak{b}$ and $\mathfrak{d}$ in (1.2) by $\varepsilon\mathfrak{b}$ and $\nu\mathfrak{d}$ for positive adimensional parameters ε and ν, and to study the behaviour of the solutions, labelled $s^{\varepsilon,\nu}$ and $u^{\varepsilon,\nu}$, as ε, ν vanish. It turns out that, up to subsequences, $(s^{\varepsilon,\nu}, u^{\varepsilon,\nu})$ converges (in proper functional spaces) to a couple (s, u) for which Griffith's criterion (1.1) is satisfied whenever s is sufficiently regular (and a weaker principle holds true otherwise), and u is characterized by the minimality condition
$$\begin{cases} \mathfrak{a}\Delta u(t) = 0 & \text{in } \Omega \setminus \Gamma(s(t)) \\ \frac{\partial u(t)}{\partial \mathbf{n}} = 0 & \text{on } \Gamma(s(t)) \\ u(t) = w(t) & \text{on } \partial_D\Omega. \end{cases}$$

The couple (s,u) is called *vanishing viscosity evolution.* It corresponds to the one found in [11, 18] in case of monotone increasing boundary loadings (i.e. $w(t)=tw_0$), as pointed out at the end of the paper. Here we do not make that assumption on the loading w and we pay the price of this choice by taking a loading sufficiently regular with respect to the time variable.

Finally, assuming the energy release rate $\mathcal{G}$ to be regular enough, we carry out a 1-dimensional analysis, similar to the one in [14]. It comes out unexpectedly that, in this situation, the dissipation due to the crack tip velocity is the leading actor of our selection principle, while the viscosity plays a noninfluential role. We describe the behaviour of the solution s in terms of level sets of $\mathcal{G}$ and propose an algorithmic procedure to detect the solution obtained by means of our selection principle.

The paper is organized in the following way. In Section 2 we introduce the principal notations and definitions of the paper. Section 3 is devoted to the study of the time-incremental problem. In Section 4 we recall the definition of the energy release rate and its main properties. In Sections 5 and 7 we deal with the time-continuous evolutions; in the first one we consider the evolutions subject to viscous and dissipative effects, while in the second we study the behaviour as these effects vanish. In Section 6 we discuss the role of a parameter introduced in Section 3, which is related to the viscoelasticity assumptions in the model. Finally, in Section 8 we present an algorithmic way to detect the solutions previously described, and an example showing the different behaviour of the globally stable quasi-static evolutions and the vanishing viscosity evolutions.

2. Notations and setting of the problem

In this section we describe the setting of the problem and introduce the basic definitions and the notations that we will use throughout the paper.

Let $\Omega\subset\mathbb{R}^2$ be a connected bounded open set with Lipschitz boundary $\partial\Omega$. Let $\varGamma$ be a $C^{1,1}$ simple curve and $\gamma:[0,L]\to\overline{\Omega}$ be its parametrization by arc length, where $L:=\mathcal{H}^1(\varGamma)$. We assume the following geometrical landscape:

- $\varGamma\cap\partial\Omega=\{\gamma(0),\gamma(L)\}$;
- $\Omega\backslash\varGamma=\Omega^1\cup\Omega^2$, where Ω^1 and Ω^2 are non-empty connected open sets with Lipschitz boundary, and $\Omega^1\cap\Omega^2=\emptyset$;
- $\partial\Omega=\partial_D\Omega\cup\partial_N\Omega$, where $\partial_D\Omega\cap\partial_N\Omega=\emptyset$, $\partial_D\Omega$ is relatively open in $\partial\Omega$ with $\mathcal{H}^1(\partial_D\Omega)>0$, and $\partial_D\Omega\cap\partial\Omega^1\neq\emptyset\neq\partial_D\Omega\cap\partial\Omega^2$.

In other words, we assume $\varGamma$ to split the domain in two connected subdomains, with the Dirichlet boundary laid on the boundary of both subdomains.

For every $\sigma\in(0,L]$, set

$$\varGamma(\sigma):=\{\gamma(\bar\sigma)\,:\,0\le\bar\sigma\le\sigma\}$$

and

$$\Omega_\sigma:=\Omega\setminus\varGamma(\sigma).$$

By the regularity assumption on Ω,Ω^1 and Ω^2, the trace operators $tr:H^1(\Omega)\to H^{1/2}(\partial\Omega)$ and $tr_i:H^1(\Omega^i)\to H^{1/2}(\partial\Omega^i)$, $i=1,2$, are well defined. In particular, for every $v\in H^1(\Omega\setminus\varGamma)$ we define its jump function across $\varGamma$, $[v]\in H^{1/2}(\varGamma)$

$$[v]=tr_1(v)|_\varGamma-tr_2(v)|_\varGamma.$$

The functional space $H^1(\Omega_\sigma)$ corresponds to the set

$$\left\{u\in H^1(\Omega\setminus\varGamma):[u]=0\text{ on }\varGamma\setminus\varGamma(\sigma)\right\}.$$

This fact allows us to work in a fixed Hilbert space, i.e. $H^1(\Omega\setminus\varGamma)$, and to check the condition on the jump $[u]$ to establish if $u\in H^1(\Omega\setminus\varGamma)$ belongs to one of the smaller spaces $H^1(\Omega_\sigma)\subset H^1(\Omega\setminus\varGamma)$.

We point out that we will write u instead of $tr(u)$ whenever it is clear from the contest that we are referring to the trace of the function u.

By the notations $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ we understand the inner product and the norm in the Hilbert spaces $L^2(\Omega\setminus\Gamma)$ or $L^2(\Omega\setminus\Gamma;\mathbb{R}^2)$. In any other case or for the sake of clarity, we will specify the space the norm refers to, for example $\|\cdot\|_{H^1(\Omega\setminus\Gamma)}$.

Fix $s_0\in(0,L)$. For any boundary loading $\psi\in H^1(\Omega_{s_0})$ and crack $\Gamma(\sigma)$, $\sigma\in[0,L]$, define the set of *admissible displacements*

$$AD(\psi,\sigma):=\big\{v\in H^1(\Omega_\sigma)\ :\ v=\psi \text{ on } \partial_D\Omega\big\}$$

where the last equality is in the sense of traces.

We study the evolution problem in a fixed time interval $[0,T]$. When dealing with an element $u\in H^1(0,T;H^1(\Omega\setminus\Gamma))$, we always assume u to be the *continuous* representative (with respect to the time variable) of its class. Therefore it makes sense to consider the pointwise value $u(t)$ for every $t\in[0,T]$. On the Dirichlet part of the boundary, $\partial_D\Omega$, we prescribe a time-dependent boundary displacement which, at each instant $t\in[0,T]$, is given by the value of (the trace of) a function $w\in C^2([0,T];H^1(\Omega_{s_0}))$ at t.

The initial configuration is the couple (u_0,s_0) where $u_0\in H^1(\Omega_{s_0})$ is solution to

$$\begin{cases} \mathfrak{a}\Delta u_0=0 & \text{in } \Omega_{s_0}\\ \frac{\partial u_0}{\partial \mathbf{n}}=0 & \text{on } \Gamma(s_0)\\ u_0=w(0) & \text{on } \partial_D\Omega. \end{cases} \tag{2.1}$$

In our computations we will need to slightly “move” the crack tip forward or backward along Γ while keeping Ω invariant. This can be done thanks to the regularity assumptions on Γ: fixed $\sigma\in(0,L)$, it is possible to construct a neighbourhood $\omega\subset\subset\Omega$ of $\gamma(\sigma)$ and a $C^{1,1}$ vector field $\eta_\theta:\mathbb{R}^2\to\mathbb{R}^2$, with $\theta\in\mathbb{R}$, such that η_θ is the identity map in $\mathbb{R}^2\setminus\omega$, $\eta_\theta(\Gamma)\subset\Gamma$ and

$$\eta_\theta(\Gamma(\sigma))=\Gamma(\sigma+\theta) \tag{2.2}$$

for $|\theta|$ sufficiently small. Even though η_θ depends on σ, there is no need to make it explicit since it will be clear from the contest which fixed σ it refers to.

Given an open interval $I\subset\mathbb{R}$ and a function $f:I\to\mathbb{R}$, for every $t\in I$ we denote

$$f(t-):=\sup_{\substack{\tau<t\\ \tau\in I}} f(\tau)\qquad f(t+):=\inf_{\substack{\tau>t\\ \tau\in I}} f(\tau).$$

Finally, the constant C may vary also within the same proof and is independent of all the parameters, unless we explicitly write the dependence. It might happen that C is a dimensional constant.

3. The time-incremental problem

This section is devoted to the definition of the time-incremental problems and the properties of their solutions. In particular we establish some crucial *a priori* estimates to obtain existence of solutions to the problem (1.2) by means of compactness arguments. Throughout the section, the dimensional parameters $\mathfrak{a}>0$ and $\mathfrak{b}>0$ are fixed.

Fix a time-step $\tau\in(0,T)$. For any $u,v\in H^1(\Omega\setminus\Gamma)$ define the functionals (dependent on τ)

$$E(u,v):=\frac{1}{2}\mathfrak{a}\|\nabla u\|^2+\frac{\mathfrak{b}}{2\tau}\|\nabla u-\nabla v\|^2$$

and

$$\mathcal{E}(u,v):=\frac{1}{2\mathfrak{a}}\|\mathfrak{a}\nabla u+\frac{\mathfrak{b}}{\tau}(\nabla u-\nabla v)\|^2.$$

By a simple computation it can be seen that, for any fixed $v\in H^1(\Omega\setminus\Gamma)$, the functionals $E(\cdot,v),\mathcal{E}(\cdot,v)\,:\,H^1(\Omega\setminus\Gamma)\to\mathbb{R}$ have the same Fréchet differential up to a multiplicative constant. Actually it is

$$\mathcal{E}(u,v) = \frac{1}{\mathfrak{a}}\left(\mathfrak{a} + \frac{\mathfrak{b}}{\tau}\right) E(u,v) - \frac{\mathfrak{b}}{2\tau}\|\nabla v\|^2. \tag{3.1}$$

Consequently, having fixed $\sigma \in [s_0, L)$, $v \in H^1(\Omega_\sigma)$, $\psi \in H^1(\Omega_{s_0})$ and $\tau \in (0,T)$, the following relation holds true

$$U = \operatorname{argmin}\{\mathcal{E}(u,v) : \, u \in AD(\psi,\sigma)\} \Leftrightarrow U = \operatorname{argmin}\{E(u,v) : \, u \in AD(\psi,\sigma)\}. \tag{3.2}$$

The functional E represents a discretized version of the stored elastic energy plus a viscoplastic friction term, energy which should have the form

$$\mathfrak{a}\|\nabla u(t)\|^2 + \mathfrak{b}\int_0^t \|\nabla \dot{u}(\xi)\|^2 \, d\xi$$

for an evolution $u \in H^1(0,T; H^1(\Omega \setminus \Gamma))$ of the displacement field. Fixed σ and v, when we minimize $E(\cdot, v)$ (or, equivalently, $\mathcal{E}(\cdot, v)$) we are penalizing the L^2 distance of the gradients of the two functions u and v, i.e. in the discrete-time evolution below we penalize large variations of the deformation gradient with respect to time.

By the algebraic equivalence (3.1), the functional $\mathcal{E}$ provides an equivalent way to select minima, even though it does not have a proper interpretation as energy. Nevertheless, it plays an important part in finding estimates.

Concerning the energy dissipated by the crack creation, according to Griffith's model it is proportional to the crack length; in our model, we add one more term taking into account the rate of crack increase. As for the viscoelastic part, in the time-incremental problem below the fracture energy shows two dimensional positive constants $\mathfrak{c}$ and $\mathfrak{d}$ and, for any fixed $\bar{\sigma} \in [s_0, L)$ and every $\sigma \in [\bar{\sigma}, L]$, it has the form

$$\mathfrak{a}\left(\mathfrak{a} + \frac{\mathfrak{b}}{\tau}\right)^{-1}\left(\mathfrak{c}\sigma + \frac{\mathfrak{d}}{2\tau}(\sigma - \bar{\sigma})^2\right).$$

In order to avoid a trivial solution, we really have to consider the time-step dependent adimensional quantity

$$\mathfrak{a}\left(\mathfrak{a} + \frac{\mathfrak{b}}{\tau}\right)^{-1} = \mathfrak{a}\,\frac{\tau}{\mathfrak{a}\tau + \mathfrak{b}}.$$

The value

$$\tau_{\mathfrak{b}} := \frac{\tau}{\mathfrak{a}\tau + \mathfrak{b}}$$

can be interpreted as a characteristic time of the viscoelastic material; in Section 6 we describe the consequences of neglecting the parameter $\mathfrak{a}\tau_{\mathfrak{b}}$. Let us observe that, if it were $\mathfrak{b} = 0$, then $\mathfrak{a}\tau_0 = 1$.

Throughout the paper, $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$ are fixed.

We define the *time-incremental problem with time-step* $\tau \in (0,T)$ in the following way: let $N_\tau \in \mathbb{N}$ be such that $T - \tau < \tau N_\tau \leq T$. Set

- $u_0^\tau := u_0$, $s_0^\tau := s_0$;
- for any $1 \leq i \leq N_\tau$ and $\sigma \geq s_0$, let $u_i^{\tau,\sigma}$ be the unique solution to

$$\min\left\{E(u, u_{i-1}^\tau) : \, u \in AD(w(i\tau), \sigma)\right\} \tag{3.3}$$

  and

$$s_i^\tau \in \operatorname{argmin}\left\{E(u_i^{\tau,\sigma}, u_{i-1}^\tau) + \mathfrak{a}\left(\mathfrak{a} + \frac{\mathfrak{b}}{\tau}\right)^{-1}\left(\mathfrak{c}\sigma + \frac{\mathfrak{d}}{2\tau}(\sigma - s_{i-1}^\tau)^2\right) : \, s_{i-1}^\tau \leq \sigma \leq L\right\} \tag{3.4}$$

  and set $u_i^\tau := u_i^{\tau, s_i^\tau}$.

We introduce the piecewise-constant and piecewise-affine interpolants for both the u_i^τ and s_i^τ:

- $u^\tau, \tilde{u}^\tau : [0,T] \to H^1(\Omega \setminus \Gamma)$ as
$$u^\tau(t) := u_i^\tau$$
$$\tilde{u}^\tau(t) := u_i^\tau + \frac{t - i\tau}{\tau}(u_{i+1}^\tau - u_i^\tau)$$
for $i\tau \le t < (i+1)\tau, i = 0, \dots, N_\tau - 1$, and
$$u^\tau(t) := u_{N_\tau}^\tau \qquad \tilde{u}^\tau(t) := u_{N_\tau}^\tau$$
for $iN_\tau \le t \le T$;
- $s^\tau, \tilde{s}^\tau : [0,T] \to [s_0, L]$ as
$$s^\tau(t) := s_i^\tau$$
$$\tilde{s}^\tau(t) := s_i^\tau + \frac{t - i\tau}{\tau}(s_{i+1}^\tau - s_i^\tau)$$
for $i\tau \le t < (i+1)\tau$, $i = 0, \dots, N_\tau - 1$, and
$$s^\tau(t) := s_{N_\tau}^\tau \qquad \tilde{s}^\tau(t) := s_{N_\tau}^\tau$$
for $iN_\tau \le t \le T$.

By definition through (3.3), u^τ and $\tilde{u}^\tau$ satisfy the variational equation
$$\mathfrak{a}\langle \nabla u^\tau(t), \nabla\varphi\rangle + \mathfrak{b}\langle \nabla \dot{\tilde{u}}^\tau(t-\tau), \nabla\varphi\rangle = 0 \tag{3.5}$$
for every $\varphi \in H^1\left(\Omega_{s^\tau(t)}\right)$ with $\varphi = 0$ on $\partial_D\Omega$, and $t \in [\tau, N_\tau\tau]$.

Remark 3.1. By the equivalence (3.2), the minimum problem (3.3)-(3.4) is equivalent to the following one:

- $u_0^\tau := u_0$, $s_0^\tau := s_0$;
- for any $1 \le i \le N_\tau$ and $\sigma \ge s_0$, let $u_i^{\tau,\sigma}$ be the unique solution to
$$\min\left\{\mathcal{E}(u, u_{i-1}^\tau) : \ u \in AD(w_i^\tau, \sigma)\right\}$$
and
$$s_i^\tau \in \operatorname{argmin}\left\{\mathcal{E}(u_i^{\tau,\sigma}, u_{i-1}^\tau) + \mathfrak{c}\sigma + \frac{\mathfrak{d}}{2\tau}(\sigma - s_{i-1}^\tau)^2 : \ s_{i-1}^\tau \le \sigma \le L\right\};$$
then set $u_i^\tau := u_i^{\tau, s_i^\tau}$.

For convenience in the discussions below, consider the discretized version of the boundary loading: for every $\tau \in (0,T)$ and $i = 0, \dots, N_\tau$, let
$$w_i^\tau := w(i\tau)$$
and let w^τ be the piecewise-constant interpolant of the w_i^τ. Then, being $w \in C^2([0,T]; H^1(\Omega_{s_0}))$, we obtain
$$w_{i+1}^\tau - w_i^\tau = \int_{i\tau}^{(i+1)\tau} \dot{w}(\tau)\, d\tau$$
$$\nabla w_{i+1}^\tau - \nabla w_i^\tau = \int_{i\tau}^{(i+1)\tau} \nabla\dot{w}(\xi)\, d\xi,$$
where the integrals are Bochner integrals (see [1]).

In the following lemmas, we establish some estimates for the families of the displacements $\{u^\tau\}$ and of the crack tips $\{s^\tau\}$. These results will be useful in order to apply compactness arguments.

Lemma 3.2. *There exists a non-negative function* $\rho(\tau) \to 0$ *as* $\tau \to 0^+$ *such that for every* $0 \le i < j \le N_\tau$

$$\frac{1}{2}\mathfrak{a}\|\nabla u_j^\tau\|^2 + \frac{\mathfrak{b}}{2\tau}\sum_{h=i}^{j-1}\|\nabla u_{h+1}^\tau - \nabla u_h^\tau\|^2 + \mathfrak{a}\left(\mathfrak{a}+\frac{\mathfrak{b}}{\tau}\right)^{-1}\left(\mathfrak{c}s_j^\tau + \frac{\mathfrak{d}}{2\tau}\sum_{h=i}^{j-1}|s_{h+1}^\tau - s_h^\tau|^2\right)$$
$$\le \frac{1}{2}\mathfrak{a}\|\nabla u_i^\tau\|^2 + \mathfrak{a}\left(\mathfrak{a}+\frac{\mathfrak{b}}{\tau}\right)^{-1}\mathfrak{c}s_i^\tau + \mathfrak{a}\int_{i\tau}^{j\tau}\langle\nabla u^\tau(\xi), \nabla\dot{w}(\xi)\rangle\, d\xi$$
$$+ \frac{\mathfrak{b}}{2}\int_{i\tau}^{j\tau}\|\nabla\dot{w}(\xi)\|^2\, d\xi + \rho(\tau).$$

Proof. Taking $\varphi = u_h^\tau + w_{h+1}^\tau - w_h^\tau \in AD(w_{h+1}^\tau, s_h^\tau)$ as test function in (3.4) (with $i = h+1$), we have

$$\frac{1}{2}\mathfrak{a}\|\nabla u_{h+1}^\tau\|^2 + \frac{\mathfrak{b}}{2\tau}\|\nabla u_{h+1}^\tau - \nabla u_h^\tau\|^2 + \mathfrak{a}\left(\mathfrak{a}+\frac{\mathfrak{b}}{\tau}\right)^{-1}\left(\mathfrak{c}s_{h+1}^\tau + \frac{\mathfrak{d}}{2\tau}|s_{h+1}^\tau - s_h^\tau|^2\right)$$
$$\le \frac{1}{2}\mathfrak{a}\|\nabla u_h^\tau + \nabla w_{h+1}^\tau - \nabla w_h^\tau\|^2 + \frac{\mathfrak{b}}{2\tau}\|\nabla w_{h+1}^\tau - \nabla w_h^\tau\|^2 + \mathfrak{a}\left(\mathfrak{a}+\frac{\mathfrak{b}}{\tau}\right)^{-1}\mathfrak{c}s_h^\tau$$
$$\le \frac{1}{2}\mathfrak{a}\|\nabla u_h^\tau\|^2 + \mathfrak{a}\int_{h\tau}^{(h+1)\tau}\langle\nabla u^\tau(\xi), \nabla\dot{w}(\xi)\rangle\, d\xi + \frac{1}{2}\mathfrak{a}\|\nabla w_{h+1}^\tau - \nabla w_h^\tau\|^2$$
$$+ \frac{\mathfrak{b}}{2\tau}\left(\int_{h\tau}^{(h+1)\tau}\|\nabla\dot{w}(\xi)\|\, d\xi\right)^2 + \mathfrak{a}\left(\mathfrak{a}+\frac{\mathfrak{b}}{\tau}\right)^{-1}\mathfrak{c}s_h^\tau$$
$$\le \frac{1}{2}\mathfrak{a}\|\nabla u_h^\tau\|^2 + \mathfrak{a}\int_{h\tau}^{(h+1)\tau}\langle\nabla u^\tau(\xi), \nabla\dot{w}(\xi)\rangle\, d\xi$$
$$+ \frac{1}{2}\mathfrak{a}\left(\max_{0\le k<N_\tau}\int_{k\tau}^{(k+1)\tau}\|\nabla\dot{w}(\xi)\|\, d\xi\right)\int_{h\tau}^{(h+1)\tau}\|\nabla\dot{w}(\xi)\|\, d\xi$$
$$+ \frac{\mathfrak{b}}{2}\int_{h\tau}^{(h+1)\tau}\|\nabla\dot{w}(\xi)\|^2\, d\xi + \mathfrak{a}\left(\mathfrak{a}+\frac{\mathfrak{b}}{\tau}\right)^{-1}\mathfrak{c}s_h^\tau.$$

Iterating over $h = i, \dots, j-1$ and defining

$$\rho(\tau) := \frac{1}{2}\mathfrak{a}T\tau\left(\max_{0\le\xi\le T}\|\nabla\dot{w}(\xi)\|\right)^2$$

the proof is complete. □

Lemma 3.3. *There exists a constant* $C > 0$, *independent of* $\mathfrak{b}, \mathfrak{d}, \tau, t$, *such that the following estimates hold true for every* $\tau \in (0, T)$, $t \in [0, T]$, $j = 1, \dots, N_\tau$

$$\|u^\tau(t)\|_{H^1(\Omega\setminus\Gamma)} \le C \tag{3.6}$$
$$\|\tilde{u}^\tau(t)\|_{H^1(\Omega\setminus\Gamma)} \le C \tag{3.7}$$
$$\mathfrak{b}\int_0^{j\tau}\|\nabla\dot{\tilde{u}}^\tau(\xi)\|^2\, d\xi = \frac{\mathfrak{b}}{\tau}\sum_{h=0}^{j-1}\|\nabla u_{h-1}^\tau - \nabla u_h^\tau\|^2 \le C \tag{3.8}$$
$$\|\tilde{u}^\tau\|_{L^2(0,T;H^1(\Omega\setminus\Gamma))} \le CT^{1/2} \tag{3.9}$$
$$\mathfrak{b}\|\dot{\tilde{u}}^\tau\|^2_{L^2(0,T;H^1(\Omega\setminus\Gamma))} \le C. \tag{3.10}$$

Proof. Fix $t \in [0,T]$ and let $j := j(t) \in 0,\dots,N_\tau - 1$ be such that it satisfies $j\tau \le t < (j+1)\tau$. By the inequality in Lemma 3.2 for $i=0$ we obtain

$$\begin{aligned}&\frac{1}{2}\mathfrak{a}\|\nabla u_j^\tau\|^2 + \frac{\mathfrak{b}}{2}\int_0^{j\tau} \|\nabla \dot{\tilde{u}}^\tau(\xi)\|^2\, d\xi \\ \le&\frac{1}{2}\mathfrak{a}\|\nabla u_0\|^2 + \mathfrak{a}\int_0^{j\tau} \langle \nabla u^\tau(\xi), \nabla \dot{w}(\xi)\rangle\, d\xi + \frac{\mathfrak{b}}{2}\int_0^{j\tau} \|\nabla \dot{w}(\xi)\|^2\, d\xi + \rho(\tau).\end{aligned} \tag{3.11}$$

Hölder's inequality and (3.11) imply

$$\mathfrak{a}\|\nabla u^\tau(t)\|^2 \le C + 2\mathfrak{b}\left(\int_0^t \|\nabla u^\tau(\xi)\|^2\, d\xi\right)^{1/2}\left(\int_0^t \|\nabla \dot{w}(\xi)\|^2\, d\xi\right)^{1/2},$$

where $C>0$ is independent of $\mathfrak{b}, \mathfrak{d}, \tau, t$. By a refined version of the Gronwall lemma (Lemma 4.1.8 in [2]), we deduce that for every $t \in [0,T]$

$$\left(\int_0^t \|\nabla u^\tau(\xi)\|^2\, d\xi\right)^{1/2} \le (TC)^{1/2} + 2T\|\nabla \dot{w}\|_{L^2(0,T;L^2(\Omega_{s_0};\mathbb{R}^2))}.$$

The last two inequalities imply that $\nabla u^\tau(t)$ is bounded in $L^2(\Omega\setminus\Gamma;\mathbb{R}^2)$ uniformly with respect to $\mathfrak{b}, \mathfrak{d}, \tau, t$. Using the Poincaré inequality we obtain (3.6) and (3.7). Then, considering (3.11), the estimates (3.8) and (3.9) follow. Finally, using the Poincaré inequality for $\dot{\tilde{u}}$

$$\|\dot{\tilde{u}}\|_{L^2(0,T;L^2(\Omega\setminus\Gamma))} \le C\left(\|\nabla\dot{\tilde{u}}\|_{L^2(0,T;L^2(\Omega\setminus\Gamma;\mathbb{R}^2))} + \|w\|_{C^2([0,T];H^1(\Omega_{s_0}))}\right)$$

and (3.8), we obtain (3.10). □

Set

$$\begin{aligned} z_0^\tau &:= \mathfrak{a}w(0) + \mathfrak{b}\dot{w}(0)\\ z_i^\tau &:= \mathfrak{a}w_i^\tau + \frac{\mathfrak{b}}{\tau}\left(w_i^\tau - w_{i-1}^\tau\right) \quad \text{for } 1 \le i \le N_\tau\end{aligned}$$

and call $\tilde{u}_0$ the solution to

$$\begin{cases} \Delta\tilde{u}_0 = 0 & \text{in } \Omega_{s_0}\\ \tilde{u}_0 = \dot{w}(0) & \text{on } \partial_D\Omega\\ \frac{\partial\tilde{u}_0}{\partial\mathbf{n}} = 0 & \text{on } \Gamma(s_0).\end{cases}$$

For $\tau \in (0,T)$ define the incremental problem

- $v_0^\tau := \mathfrak{a}u_0 + \mathfrak{b}\tilde{u}_0$, $\sigma_0^\tau := s_0$;
- for any $1 \le i \le N_\tau$ and $\sigma \ge s_0$, let $v_i^{\tau,\sigma}$ be the unique solution to

$$\min\left\{\|\nabla v\|^2 : v \in AD(z_i^\tau, \sigma)\right\} \tag{3.12}$$

  and

$$\sigma_i^\tau \in \operatorname{argmin}\left\{\frac{1}{\mathfrak{a}}\|\nabla v\|^2 + \mathfrak{c}\sigma + \frac{\mathfrak{d}}{2\tau}(\sigma - s_{i-1}^\tau)^2 : \sigma_{i-1}^\tau \le \sigma \le L\right\} \tag{3.13}$$

  and set $v_i^\tau := v_i^{\tau, s_i^\tau}$.

It is easy to check that for $i=1$ it is

$$v_1^{\tau,\sigma} := \mathfrak{a}u_1^{\tau,\sigma} + \frac{\mathfrak{b}}{\tau}\left(u_1^{\tau,\sigma} - u_0^\tau\right)$$

and

$$\mathcal{E}(u_1^{\tau,\sigma}, u_0^\tau) = \frac{1}{2\mathfrak{a}}\|\nabla v_1^{\tau,\sigma}\|^2$$

for every $\sigma \in [s_0, L]$, so that we can assume $\sigma_1^\tau = s_1^\tau$. Iterating this argument, we suppose $\sigma_i^\tau = s_i^\tau$ for $1 \le i \le N_\tau$ and, consequently,

$$\begin{aligned} v_0^\tau &= \mathfrak{a}u_0 + \mathfrak{b}\tilde{u}_0\\ v_i^\tau &= \mathfrak{a}u_i^\tau + \tfrac{\mathfrak{b}}{\tau}\left(u_i^\tau - u_{i-1}^\tau\right) \quad \text{for } 1 \le i \le N_\tau\end{aligned} \tag{3.14}$$

so that (3.3)-(3.4) and (3.12)-(3.13) provide the same evolution (up to the relation (3.14) between u_i^τ and v_i^τ), and in addition

$$\mathcal{E}(u_i^{\tau,\sigma}, u_{i-1}^\tau) = \frac{1}{2\mathfrak{a}}\|\nabla v_i^{\tau,\sigma}\|^2 \tag{3.15}$$

holds for every $1 \le i \le N_\tau$ and $\sigma \ge s_{i-1}^\tau$.

By the minimality of v_i^τ, we obtain estimates for the crack tip evolution s^τ as well:

Lemma 3.4. *There exists a non-negative function $\tilde\rho(\tau) \to 0$ as $\tau \to 0^+$ such that for every $0 \le i < j \le N_\tau$*

$$\begin{aligned}
&\frac{1}{2\mathfrak{a}}\|v_j^\tau\|^2 + \mathfrak{c}s_j^\tau + \frac{\mathfrak{d}}{2}\tau\sum_{h=i}^{j-1}\left(\frac{s_{h+1}^\tau - s_h^\tau}{\tau}\right)^2 \\
\le&\frac{1}{2\mathfrak{a}}\|v_i^\tau\|^2 + \mathfrak{c}s_i^\tau + \frac{1}{\mathfrak{a}}\sum_{h=i}^{j-1}\langle\nabla v_h^\tau, \nabla z_{h+1}^\tau - \nabla z_h^\tau\rangle + \tilde\rho(\tau).
\end{aligned} \tag{3.16}$$

Proof. Taking $\varphi = v_{h-1}^\tau + z_h^\tau - z_{h-1}^\tau \in AD(z_h^\tau, s_{h-1}^\tau)$ as test function in (3.13), it is

$$\begin{aligned}
\frac{1}{2\mathfrak{a}}\|\nabla v_h^\tau\|^2 + \mathfrak{c}s_h^\tau + \frac{\mathfrak{d}}{2\tau}(s_h^\tau - s_{h-1}^\tau) &\le \frac{1}{2\mathfrak{a}}\|\nabla v_{h-1}^\tau + \nabla z_h^\tau - \nabla z_{h-1}^\tau\|^2 + \mathfrak{c}s_{h-1}^\tau \\
&\le\frac{1}{2\mathfrak{a}}\|\nabla v_{h-1}^\tau\|^2 + \frac{1}{\mathfrak{a}}\langle\nabla v_{h-1}^\tau, \nabla z_h^\tau - \nabla z_{h-1}^\tau\rangle + \frac{1}{2\mathfrak{a}}\|\nabla z_h^\tau - \nabla z_{h-1}^\tau\|^2 + \mathfrak{c}s_{h-1}^\tau.
\end{aligned}$$

Arguing similarly as in the proof of Lemma 3.2 and using the assumption $w \in C^2([0,T]; H^1(\Omega_{s_0}))$, it is

$$\sum_{h=1}^{N_\tau}\|\nabla z_h^\tau - \nabla z_{h-1}^\tau\|^2 \le \tilde\rho(\tau) \to 0 \quad \text{as } \tau \to 0$$

with $\tilde\rho$ dependent only on $\mathfrak{a}$, τ and $\|w\|_{C^2([0,T];H^1(\Omega_{s_0}))}$.

Iterating the inequality above for $i \le h \le j-1$, we obtain the thesis. □

The term

$$\sum_{h=i}^{j}\langle\nabla v_{h-1}^\tau, \nabla z_h^\tau - \nabla z_{h-1}^\tau\rangle$$

in (3.16) has the explicit form

$$\mathfrak{a}\int_{i\tau}^{j\tau}\langle\nabla v^\tau(\xi), \nabla\dot w(\xi)\rangle\,d\xi + \mathfrak{b}\int_{i\tau}^{j\tau}\langle\nabla v^\tau(\xi), \frac{\nabla w^\tau(\xi) - 2\nabla w^\tau(\xi-\tau) + \nabla w^\tau(\xi-2\tau)}{\tau^2}\rangle\,d\xi$$

where we call v^τ the piecewise constant interpolant of the v_i^τ and, with abuse of notations, $w^\tau(0) - w^\tau(-\tau) := \dot w(0)$ in case $i = 0$. As $w \in C^2([0,T]; H^1(\Omega_{s_0}))$, for every $t \in (0,T)$ the difference quotient

$$\frac{\nabla w^\tau(t) - 2\nabla w^\tau(t-\tau) + \nabla w^\tau(t-2\tau)}{\tau^2}$$

converges strongly in $L^2(\Omega_{s_0}; \mathbb{R}^2)$ to $\nabla\ddot w(t)$ as $\tau \to 0$, uniformly with respect to t.

Lemma 3.5. *There exists a constant $C > 0$, independent of $\mathfrak{b}, \mathfrak{d}, \tau, t$, such that for every $\tau \in (0,T)$, $t \in [0,T]$ the following estimates are satisfied:*

$$\|v^\tau(t)\|_{H^1(\Omega\setminus\Gamma)} \le C \tag{3.17}$$

$$\mathfrak{d}\|\dot s^\tau\|_{L^2(0,T)}^2 \le C. \tag{3.18}$$

Proof. Taking $i=0$ in Lemma 3.4 and using Hölder's inequality, it is

$$\begin{aligned}&\frac{1}{2\mathfrak{a}}\|\nabla v^\tau(t)\|^2+\mathfrak{c}s^\tau(t)+\frac{1}{2}\mathfrak{d}\int_0^t|\dot{s}^\tau(\xi)|^2\,d\xi \\ \le&\frac{1}{2\mathfrak{a}}\|\nabla v_0\|^2+\mathfrak{c}s_0+\left(\|\dot{w}\|_{L^\infty}+\frac{\mathfrak{b}}{\mathfrak{a}}\|\ddot{w}\|_{L^\infty}\right)\left(\int_0^t\|\nabla v^\tau(t)\|^2\right)^{1/2}+\tilde{\rho}(\tau).\end{aligned} \tag{3.19}$$

Arguing similarly to Lemma 3.3, we obtain

$$\|\nabla v^\tau(t)\|\le C$$

for every t. Using the Poincaré inequality, since $v_i^\tau=z_i^\tau$ on $\partial_D\Omega$, estimate (3.17) follows. Then (3.18) is consequence of (3.19) and (3.17). □

4. Energy release rate

In order to achieve a complete description of the evolution of the system, we look for a differential constraint for the crack tip evolution s. To the purpose, in this section we introduce the *energy release rate* functional. Its existence under suitable regularity conditions (which are satisfied in our case) has been studied and proved in some papers (see for example [12, 13]). Besides we introduce an artificial quantity playing the role of the energy release rate at the level of the time-incremental solutions (s^τ,u^τ) defined in Section 3, in order to establish a sort of time-incremental version of the Griffith's criterion (1.1).

For every $\sigma\in[s_0,L]$ and $g\in H^{1/2}(\partial_D\Omega)$, let $v(\sigma,g)\in H^1(\Omega_\sigma)$ be the solution to the problem

$$\begin{cases}\Delta v=0 & \text{in } \Omega_\sigma\\ v=g & \text{on } \partial_D\Omega\\ \frac{\partial v}{\partial\mathbf{n}}=0 & \text{on } \Gamma(\sigma)\end{cases} \tag{4.1}$$

and set

$$\mathcal{F}(\sigma,g):=\frac{1}{2}\|\nabla v(\sigma,g)\|^2.$$

The energy release rate is defined as the derivative of $\mathcal{F}$ with respect to its "geometric" variable σ. The result below states the regularity properties for the functional $\mathcal{F}$ and introduces the energy release rate by means of the definition (4.2).

Proposition 4.1. *The functional $\mathcal{F}$ is continuous from $[s_0,L]\times H^{1/2}(\partial_D\Omega)$ to $\mathbb{R}$. For any fixed $g\in H^{1/2}(\partial_D\Omega)$, the map $\sigma\mapsto\mathcal{F}(\sigma,g)$ is differentiable at every $\sigma\in[s_0,L)$. Set*

$$\mathcal{G}(\sigma,g):=-\frac{\partial\mathcal{F}(\sigma,g)}{\partial\sigma}, \tag{4.2}$$

it is continuous in $[s_0,L)\times H^{1/2}(\partial_D\Omega)$.

We do not prove the proposition above, since it is a well known result. Let us recall the main tools and some references. To obtain the continuity of $\mathcal{F}$ it is sufficient to apply [6, Theorem 5.1], which moves the issue of the convergence of $\nabla v(\sigma,g)$ to check the convergence of the boundary data and crack variables. The differentiability of $\mathcal{F}$ is proved in [13] under weaker assumptions than ours, and therein the following explicit formula for $\mathcal{G}$ is also provided:

$$\mathcal{G}(\sigma,g)=-\langle\nabla v(\sigma,g),\nabla\lambda^\sigma\,\nabla v(\sigma,g)\rangle+\frac{1}{2}\langle\nabla v(\sigma,g),\operatorname{div}(\lambda^\sigma)\,\nabla v(\sigma,g)\rangle \tag{4.3}$$

where $v(\sigma,g)$ is defined through (4.1), λ^σ is a Lipschitz vector field such that $\operatorname{supp}(\lambda^\sigma)\subset\Omega$, $\lambda^\sigma(\gamma(\bar\sigma))=\zeta^\sigma(\gamma(\bar\sigma))\dot\gamma(\bar\sigma)$ for every $\bar\sigma\in[0,L]$, where ζ^σ is a cut-off function, equal to one in a neighbourhood of $\gamma(\sigma)$. If we fix $s_1\in(s_0,L)$, then we can assume λ^σ to be fixed for any $\sigma\in[s_0,s_1]$, i.e. it is $\lambda^\sigma=\lambda$ for every $\sigma\in[s_0,s_1]$ and λ is a Lipschitz vector field such that $\operatorname{supp}(\lambda^\sigma)\subset\Omega$ and $\lambda(\gamma(\bar\sigma))=\zeta(\gamma(\bar\sigma))\dot\gamma(\bar\sigma)$, where ζ is a cut-off function, equal to one in a neighbourhood of $\gamma([s_0,s_1])$.

In the following, exploiting the continuity of the trace operator we consider the space $H^1(\Omega_{s_0})$ instead of $H^{1/2}(\partial_D\Omega)$: we assume $\mathcal{F}$ to be defined on $[s_0,L]\times H^1(\Omega_{s_0})$ and, with abuse of notation, we identify every $g\in H^1(\Omega_{s_0})$ with its trace on $\partial_D\Omega$, so that Proposition 4.2 still holds true for the functional $\mathcal{F}:[s_0,L]\times H^1(\Omega_{s_0})\to\mathbb{R}$.

Proposition 4.2. *Let $s_1\in(s_0,L)$ be fixed. Then $\mathcal{G}(\sigma,\cdot):H^1(\Omega_{s_0})\to\mathbb{R}$ is Lipschitz continuous, uniformly in $\sigma\in[s_0,s_1]$.*

Proof. Fix $\sigma\in[s_0,s_1]$. For $i=1,2$, let $g_i\in H^1(\Omega_{s_0})$ and $v(\sigma,g_i)\in H^1(\Omega_\sigma)$ be the solution to (4.1) with $g=tr(g_i)$, and write

$$v(\sigma,g_i)=\tilde v(\sigma,g_i)+g_i.$$

Then, for every $\varphi\in H^1(\Omega_\sigma)$ with $\varphi=0$ on $\partial_D\Omega$, it is

$$0=\langle\nabla v(\sigma,g_i),\nabla\varphi\rangle=\langle\nabla\tilde v(\sigma,g_i),\nabla\varphi\rangle+\langle\nabla g_i,\nabla\varphi\rangle,$$

i.e.

$$\langle\nabla\tilde v(\sigma,g_i),\nabla\varphi\rangle=-\langle\nabla g_i,\nabla\varphi\rangle \tag{4.4}$$

for any φ as before.

Note that, in particular $\tilde v(\sigma,g_i)\in H^1(\Omega_\sigma)$ with $\tilde v(\sigma,g_i)=0$ on $\partial_D\Omega$.

Considering (4.4) and applying Hölder's inequality,

$$\begin{aligned}\|\nabla\tilde v(\sigma,g_1)-\nabla\tilde v(\sigma,g_2)\|^2 =&\langle\nabla\tilde v(\sigma,g_1)-\nabla\tilde v(\sigma,g_2),\nabla\tilde v(\sigma,g_1)-\nabla\tilde v(\sigma,g_2)\rangle\\ =&-\langle\nabla g_1-\nabla g_2,\nabla\tilde v(\sigma,g_1)-\nabla\tilde v(\sigma,g_2)\rangle\\ \le&\|\nabla g_1-\nabla g_2\|\;\|\nabla\tilde v(\sigma,g_1)-\nabla\tilde v(\sigma,g_2)\|,\end{aligned}$$

so that $\|\nabla\tilde v(\sigma,g_1)-\nabla\tilde v(\sigma,g_2)\|\le\|\nabla g_1-\nabla g_2\|$. Therefore

$$\|\nabla v(\sigma,g_1)-\nabla v(\sigma,g_2)\|\le\|\nabla\tilde v(\sigma,g_1)-\nabla\tilde v(\sigma,g_2)\|+\|\nabla g_1-\nabla g_2\|\le 2\|\nabla g_1-\nabla g_2\|.$$

In the expression (4.3) for $\mathcal{G}$, we can assume $\lambda^\sigma=\lambda$ for every $\sigma\in[s_0,s_1]$. Then, by (4.3) and the above inequality, we obtain

$$|\mathcal{G}(\sigma,g_1)-\mathcal{G}(\sigma,g_1)|\le C\|\nabla g_1-\nabla g_2\|\le C\|g_1-g_2\|_{H^1(\Omega_{s_0})}$$

with $C\ge\mathrm{Lip}(\lambda)$, where $\mathrm{Lip}(\lambda)$ is the Lipschitz constant of λ. □

At this point we want to establish a Griffith's criterion for the time-incremental problems. Firstly we introduce the energy release rate for each time-step τ. By definition of $v_i^{\tau,\sigma}$, at every fixed $i\in\{0,\dots,N_\tau\}$ it satisfies

$$\begin{cases}\Delta v_i^{\tau,\sigma}=0 & \text{in } \Omega_\sigma\\ v_i^{\tau,\sigma}=z_i^\tau & \text{on } \partial_D\Omega\\ \frac{\partial v_i^{\tau,\sigma}}{\partial\mathbf{n}}=0 & \text{on } \Gamma(\sigma).\end{cases}$$

Applying Proposition 4.1 with $g=z_i^\tau$ and having in mind the equality (3.15), the function

$$\sigma\in[s_{i-1}^\tau,L]\mapsto\mathcal{E}(u_i^{\tau,\sigma},u_{i-1}^\tau)=\mathcal{F}(\sigma,z_i^\tau)$$

is differentiable at every $\sigma\in[s_{i-1}^\tau,L)$. For every $\tau\in(0,T)$ and $t\in[0,T]$ such that $s^\tau(t)<L$ we define

$$G(\tau,t):=\mathcal{G}(s^\tau(t),z^\tau(t))=-\left[\frac{d}{d\sigma}\mathcal{E}(u_{i_\tau}^{\tau,\sigma},u^\tau(t-\tau))\right]_{\sigma=s^\tau(t)}, \tag{4.5}$$

with $i_\tau:=i_\tau(t)$ such that $i_\tau\tau\le t<(i_\tau+1)\tau$.

The second step is to establish an energetic growth condition for the fracture term: it is the mathematical translation of the fact that the crack increases if and only if the release of stored energy is larger than the energy dissipated by the crack creation. At each fixed i, by minimality of (s_i^τ, u_i^τ) it is

$$\mathcal{E}(u_i^\tau, u_{i-1}^\tau) + \mathfrak{c} s_i^\tau + \frac{\mathfrak{d}}{2\tau}(s_i^\tau - s_{i-1}^\tau) \leq \mathcal{E}(u_i^{\tau,\sigma}, u_{i-1}^{\tau,\sigma}) + \mathfrak{c}\sigma + \frac{\mathfrak{d}}{2\tau}(\sigma - s_{i-1}^\tau)$$

for every $\sigma \in [s_{i-1}^\tau, L]$. If $s_i^\tau < L$, then for every $\sigma \in (s_i^\tau, L]$ we have

$$-\frac{\mathcal{E}(u_i^\tau, u_{i-1}^\tau) - \mathcal{E}(u_i^{\tau,\sigma}, u_{i-1}^{\tau,\sigma})}{\sigma - s_i^\tau} \leq \mathfrak{c} + \frac{\mathfrak{d}}{2\tau}(\sigma + s_i^\tau - 2s_{i-1}^\tau);$$

if in addition $s_i^\tau > s_{i-1}^\tau$, then for every $\sigma \in [s_{i-1}^\tau, s_i^\tau)$ we have

$$-\frac{\mathcal{E}(u_i^\tau, u_{i-1}^\tau) - \mathcal{E}(u_i^{\tau,\sigma}, u_{i-1}^{\tau,\sigma})}{\sigma - s_i^\tau} \geq \mathfrak{c} + \frac{\mathfrak{d}}{2\tau}(\sigma + s_i^\tau - 2s_{i-1}^\tau).$$

The above inequalities and the differentiability property described in (4.5) provide the Griffith's criterion we were looking for: for every $\tau \in (0,T)$ and every $1 \leq i \leq N_\tau$ such that $s_i^\tau < L$, the following conditions are satisfied:

$$\begin{aligned}
&s_i^\tau \geq s_{i-1}^\tau \\
&G(\tau, i\tau) \leq \mathfrak{c} + \mathfrak{d}\left(\frac{s_i^\tau - s_{i-1}^\tau}{\tau}\right) \\
&\left[-G(\tau, i\tau) + \mathfrak{c} + \mathfrak{d}\left(\frac{s_i^\tau - s_{i-1}^\tau}{\tau}\right)\right]\left(s_i^\tau - s_{i-1}^\tau\right) = 0.
\end{aligned}$$

We introduce the concept of *failure time*, important from now on.

Definition 4.3. Let $s : [0,T] \to [s_0, L]$ be a non-decreasing function with $s(0) = s_0$. The instant

$$T_f(s) := \sup\{t \in [0,T] : \, s(t) < L\}$$

is called *failure time* for s.

Actually, in the analysis above we proved the following fact:

Proposition 4.4 (Time-incremental Griffith's criterion)**.** *For every $\tau \in (0,T)$ and every $t \in [0, T_f(\tilde{s}^\tau))$ the following conditions hold true:*

$$\dot{\tilde{s}}^\tau(t) \geq 0 \tag{4.6}$$

$$G(\tau, t) \leq \mathfrak{c} + \mathfrak{d}\dot{\tilde{s}}^\tau(t) \tag{4.7}$$

$$\left[-G(\tau, t) + \mathfrak{c} + \mathfrak{d}\dot{\tilde{s}}^\tau(t)\right]\dot{\tilde{s}}^\tau(t) = 0. \tag{4.8}$$

5. The irreversible viscoelastic evolution

The goal of the section is to describe the fracture problem with continuous time variable. We investigate the behaviour of the sequence of time-incremental solutions (s^τ, u^τ) as the time-step τ decreases to 0.

Definition 5.1. For any $s_0 \in (0, L)$, $w \in C^2([0,T]; H^1(\Omega_{s_0}))$ and u_0 satisfying (2.1), an *irreversible viscoelastic evolution* is a couple

$$(s, u) : [0,T] \to [s_0, L] \times H^1(\Omega \setminus \Gamma)$$

such that $(s(0), u(0)) = (s_0, u_0)$, $s \in H^1(0,T)$ is non-decreasing and

(1) $u \in H^1(0,T; H^1(\Omega \setminus \Gamma))$ and $u(t) \in H^1(\Omega_{s(t)})$ for every $t \in [0,T]$;
(2) $u(t) = w(t)$ on $\partial_D\Omega$ for every $t \in [0,T]$;
(3) for a.e. $t \in (0,T)$, for every $\varphi \in H^1(\Omega_{s(t)})$ with $\varphi = 0$ on $\partial_D\Omega$,

$$\mathfrak{a}\langle \nabla u(t), \nabla\varphi\rangle + \mathfrak{b}\langle \nabla\dot{u}(t), \nabla\varphi\rangle = 0;$$

(4) *Griffith's criterion*: for every $t \in [0, T_f(s))$ the following conditions hold true:

$$\dot{s}(t) \geq 0 \tag{5.1}$$

$$\mathcal{G}(s(t), \mathfrak{a}w(t) + \mathfrak{b}\dot{w}(t)) \leq \mathfrak{c} + \mathfrak{d}\dot{s}(t) \tag{5.2}$$

$$[-\mathcal{G}(s(t), \mathfrak{a}w(t) + \mathfrak{b}\dot{w}(t)) + \mathfrak{c} + \mathfrak{d}\dot{s}(t)]\, \dot{s}(t) = 0. \tag{5.3}$$

The requirements in the definition above can be rephrased in physical terms. The monotonicity of s means that the crack does not heal, while (1) affirmes that the jump set of the displacement $u(t)$ is contained in $\Gamma(s(t))$. Conditions (1)-(3) tell us that u is a weak solution to the problem

$$\begin{cases} \mathfrak{a}\Delta u(t) + \mathfrak{b}\Delta \dot{u}(t) = 0 & \text{in } \Omega_{s(t)} \\ \mathfrak{a}\frac{\partial u(t)}{\partial \mathbf{n}} + \mathfrak{b}\frac{\partial \dot{u}(t)}{\partial \mathbf{n}} = 0 & \text{on } \Gamma(s(t)) \\ u(t) = w(t) & \text{on } \partial_D \Omega \\ u(0) = u_0 \\ s(0) = s_0. \end{cases}$$

Roughly speaking, u behaves almost as an elastic body in the uncracked domain $\Omega_{s(t)}$ (except for the viscous term $\mathfrak{b}\Delta\dot{u}$). Finally, (4) expresses a further relation between u and s, and provides an energetic criterion for the crack evolution.

The main result of the section is the following existence theorem, which will be proven combining several lemmas.

Theorem 5.2. *For any $s_0 \in (0, L)$, $w \in C^2([0,T]; H^1(\Omega_{s_0}))$ and u_0 satisfying (2.1), there exists an irreversible viscoelastic evolution.*

Consider the time-incremental evolutions (s^τ, u^τ), for $\tau \in (0, T)$. The estimate (3.10) assures the existence of a map $u \in H^1(0, T, H^1(\Omega \setminus \Gamma))$ such that

$$\tilde{u}^\tau \rightharpoonup u \tag{5.4}$$

weakly in $H^1(0, T, H^1(\Omega \setminus \Gamma))$ as $\tau \to 0^+$ along a suitable subsequence.

Remark 5.3. When we write $\tau \to 0$ we refer to the subsequence selected in (5.4), or to a further subsequence of it.

Concerning the crack tip evolution, by monotonicity of s^τ and Helly's theorem, we find a further subsequence of $(s^\tau)_{\tau \in (0,T)}$ and a function $s : [0, T] \to [s_0, L]$ such that

$$s^\tau(t) \to s(t) \tag{5.5}$$

for every $t \in [0, T]$, as $\tau \to 0^+$. The function s is non-decreasing, since by pointwise convergence it inherites the monotonicity of the functions s^τ.

Below we investigate how u and s are mutually related, since so far we do not have any information about the jump set of u. Furthermore, we obtain a regularity estimate for s, since by (3.18) and the fact that $\|s^\tau\|_\infty < L$ we expect it to belong to $H^1(0, T)$ as well.

Lemma 5.4. *Up to subsequences, $u^\tau(t) \rightharpoonup u(t)$ weakly in $H^1(\Omega \setminus \Gamma)$ for every $t \in [0, T]$.*

Proof. The set

$$B_C := \left\{ v \in H^1(\Omega \setminus \Gamma) \,:\, \|v\|_{H^1(\Omega \setminus \Gamma)} \leq C \right\}.$$

is a compact subset of $L^2(\Omega)$. The estimate (3.7) implies that $\tilde{u}^\tau(t) \in B_C$ for every $t \in [0, t]$, while by (3.10) it is

$$\|\tilde{u}^\tau(t_1) - \tilde{u}^\tau(t_2)\| \leq C(\mathfrak{b})\sqrt{|t_1 - t_2|}$$

for every $t_1, t_2 \in [0, T]$, where $C(\mathfrak{b})$ only depends on $\mathfrak{b}$.

By a refined version of the Ascoli-Arzelá theorem (see [2, Proposition 3.3.1]), there exists $\hat{u} : [0, T] \to B_C$ continuous such that, up to subsequences, for every $t \in [0, T]$

$$\tilde{u}^\tau(t) \to \hat{u}(t) \tag{5.6}$$

strongly in $L^2(\Omega\setminus\Gamma)$ when $\tau\to 0^+$; since (5.4) holds, $\hat u(t)=u(t)$ for a.e. t. In particular the equality is true for every $t\in[0,T]$, since we are considering the continuous representative of u in $H^1(0,T;H^1(\Omega\setminus\Gamma))$.

Fix $t\in[0,T]$. For every τ, set $0\le i\le N_\tau$ such that $i\tau\le t<(i+1)\tau$. We have

$$\|\tilde u^\tau(t)-u^\tau(t)\|_{H^1(\Omega\setminus\Gamma)}=\left(\frac{t-i\tau}{\tau}\right)\|u^\tau_{i+1}-u^\tau_i\|_{H^1(\Omega\setminus\Gamma)}\le\|u^\tau_{i+1}-u^\tau_i\|_{H^1(\Omega\setminus\Gamma)}. \tag{5.7}$$

By properties of the trace operator and regularity of w, we obtain

$$\begin{aligned}\|tr(w^\tau_{i+1}-w^\tau_i)\|_{L^2(\partial_D\Omega)}\le&C\|w^\tau_{i+1}-w^\tau_i\|_{H^1(\Omega_{s_0})}=C\left\|\int_{i\tau}^{(i+1)\tau}\dot w(\xi)\,d\xi\right\|_{H^1(\Omega_{s_0})}\\ &\le\int_{i\tau}^{(i+1)\tau}\|\dot w(\xi)\|_{H^1(\Omega_{s_0})}\,d\xi\le CM\tau\end{aligned}$$

with $M:=\max_{\xi\in[0,T]}\|\dot w(\xi)\|_{H^1(\Omega_{s_0})}$. This estimate, together with the Poincaré inequality and (3.8), implies

$$\begin{aligned}\|u^\tau_{i+1}-u^\tau_i\|\le&C\left(\|\nabla u^\tau_{i+1}-\nabla u^\tau_i\|+\|tr(u^\tau_{i+1}-u^\tau_i)\|\right)\\ =&C\left(\|\nabla u^\tau_{i+1}-\nabla u^\tau_i\|+\|tr(w^\tau_{i+1}-w^\tau_i)\|\right)\le C\tau,\end{aligned}$$

so that by (5.7) and (3.8) we deduce

$$\|\tilde u^\tau(t)-u^\tau(t)\|_{H^1(\Omega\setminus\Gamma)}\le\|u^\tau_{i+1}-u^\tau_i\|_{H^1(\Omega\setminus\Gamma)}\le C\tau$$

for the fixed t, with C dependent on $\mathfrak{b}$ but not on t. Therefore

$$\sup_{t\in[0,T]}\|\tilde u^\tau(t)-u^\tau(t)\|_{H^1(\Omega\setminus\Gamma)}\to 0 \tag{5.8}$$

as $\tau\to 0$. Since $u^\tau(t)\in B_C$, we conclude by means of (5.6) and (5.8). □

Lemma 5.5. *It results $u(t)\in H^1(\Omega_{s(t)})$ for every $t\in[0,T]$.*

Proof. Fix $t\in[0,T]$. If $s(t)=L$, then the claim is automatically satisfied since $u(t)\in H^1(\Omega\setminus\Gamma)$ for every t.

Let assume $s(t)<L$ and let $\alpha\in(0,L-s(t))$. By definition of s through (5.5) and continuity of γ, it is $\Gamma(s^\tau(t))\subset\Gamma(s(t)+\alpha)$ for τ sufficiently small. Since $u^\tau(t)\in H^1(\Omega_{s^\tau(t)})$ for every t, we have $[u^\tau(t)]=0$ on $\Gamma\setminus\Gamma(s(t)+\alpha)$ for τ small enough. By Lemma 5.4 and the compactness of the trace operator, up to a subsequence $u^\tau(t)\to u(t)$ $\mathcal{H}^1$-a.e. on Γ, so that $[u(t)]=0$ on $\Gamma\setminus\Gamma(s(t)+\alpha)$. Being α arbitrary, $[u^\tau(t)]=0$ on $\Gamma\setminus\Gamma(s(t))$, i.e. the thesis holds true. □

Lemma 5.6. *The sequence $(\tilde s^\tau)$ converges to s weakly in $H^1(0,T)$ and pointwise for every $t\in[0,T]$. Moreover, $\mathfrak{d}\|\dot s\|^2_{L^2(0,T)}\le C$.*

Proof. By the estimate (3.18), it is $\sup_{\tau\in(0,T)}\|s^\tau\|_{H^1(0,T)}<C(\mathfrak{d})$ for some constant $C(\mathfrak{d})$ dependent only on $\mathfrak{d}$. We deduce the existence of $\hat s\in H^1(0,T)$ such that (up to subsequences) $\tilde s^\tau\rightharpoonup\hat s$ weakly in $H^1(0,T)$. Let us show that $\hat s=s$.

Fix t and for every τ set $0\le i\le N_\tau$ such that $i\tau\le t<(i+1)\tau$. Then

$$\begin{aligned}0\le\tilde s^\tau(t)-s^\tau(t)=&\frac{t-i\tau}{\tau}\left(s^\tau_{i+1}-s^\tau_i\right)\le\tau\dot{\tilde s}^\tau(t)\\ &=\int_{i\tau}^{(i+1)\tau}\dot{\tilde s}^\tau(\xi)\,d\xi\le\tau^{1/2}\left(\int_{i\tau}^{(i+1)\tau}\left(\dot{\tilde s}^\tau(\xi)\right)^2\,d\xi\right)^{1/2}\le\tau^{1/2}C(\mathfrak{d})\end{aligned}$$

where the last inequality is due to (3.18). Then, considering (5.5), $\tilde s^\tau(t)\to s(t)$ as $\tau\to 0$ for every t and necessarily $\hat s=s$, so that $s\in H^1(0,T)$.

Finally, the estimate for $\dot s$ is a consequence of the weak convergence $\tilde s^\tau\rightharpoonup s$ in $H^1(0,T)$ and (3.18). □

Note that by (5.5) we only knew that s is monotone, while Lemma 5.6 provides the additional information that it is continuous as well: the crack really grows continuously, without the non-physical behaviour of jumps of the fracture set.

At this point we would like to define a Griffith's criterion for the couple (s, u), exploiting the one for the time-incremental solutions obtained in Proposition 4.4.

Lemma 5.7. *It results $s \in C^1((0, T_f(s)) \cup (T_f(s), T))$ and (5.1)-(5.3) hold true for every $t \in [0, T_f(s))$.*

Proof. First of all, we show that $G(\tau, t)$ converges to $\mathcal{G}(s(t), \mathfrak{a}w(t) + \mathfrak{b}\dot{w}(t))$ for every $t \in [0, T_f(s))$ when τ vanishes. Fix $t \in [0, T_f(s))$; for τ small enough it is $s^\tau(t) < L$, so that it is meaningful to consider $G(\tau, t)$. Since $s^\tau(t) \to s(t)$ by (5.5) and $z^\tau(t) \to \mathfrak{a}w(t) + \mathfrak{b}\dot{w}(t)$ in $H^1(\Omega_{s_0})$, the continuity of $\mathcal{G}$ stated in Proposition 4.1 implies

$$G(\tau, t) = \mathcal{G}(s^\tau(t), z^\tau(t)) \to \mathcal{G}(s(t), \mathfrak{a}w(t) + \mathfrak{b}\dot{w}(t))$$

as $\tau \to 0$.

Next we obtain (5.1)-(5.3) for a.e. $t \in [0, T_f(s))$. It is sufficient to prove that (at least for a subsequence of $\tilde{s}^\tau$) $\dot{\tilde{s}}^\tau(t) \to \dot{s}(t)$ for a.e. t and then to pass to the limit in (4.6)-(4.8).

Let $t \in [0, T_f(s))$ be such that $\dot{s}(t)$ exists and consider the sequence $\tilde{s}^\tau$ approximating s. By (4.6), only two situations are possible:

(i) $\dot{\tilde{s}}^\tau(t) > 0$ for any element of the sequence;
(ii) for a subsequence $\tilde{s}^{\tau_j}$ it is $\dot{\tilde{s}}^{\tau_j}(t) = 0$ for every j.

If (i) is the case, then (4.8) forces the equality $\mathfrak{d}\dot{\tilde{s}}^\tau(t) = G(\tau, t) - \mathfrak{c}$ to be satisfied. Since the right-hand side converges (by what previously proved), then we obtain that $\dot{\tilde{s}}^\tau(t) \to \vartheta(t) = \frac{1}{\mathfrak{d}}[\mathcal{G}(s(t), \mathfrak{a}w(t) + \mathfrak{b}\dot{w}(t)) - \mathfrak{c}]$.

If we assume (ii), then by (4.7) it is $G(\tau, t) \leq \mathfrak{c}$ and, as τ vanishes, we get

$$\mathcal{G}(s(t), \mathfrak{a}w(t) + \mathfrak{b}\dot{w}(t)) \leq \mathfrak{c}. \tag{5.9}$$

Call $\tilde{s}^{\tau_k}$ the elements in the (at most countable) set $\{\tilde{s}^\tau\} \setminus \{\tilde{s}^{\tau_j}\}$. If there are finitely many $\tilde{s}^{\tau_k}$, then $\lim \dot{\tilde{s}}^\tau(t) = \lim \dot{\tilde{s}}^{\tau_j} = 0$. If there are infinitely many $\tilde{s}^{\tau_k}$, let us show that $\dot{\tilde{s}}^{\tau_k} \to 0$. Repeating the same argument as for (i), $\dot{\tilde{s}}^{\tau_k} \to \vartheta(t) \geq 0$. Then

$$0 \leq \mathfrak{d}\vartheta(t) = \mathfrak{d} \lim_{\tau_k \to 0} \dot{\tilde{s}}^{\tau_k} = \lim_{\tau_k \to 0} G(\tau_k, t) - \mathfrak{c} = \mathcal{G}(s(t), \mathfrak{a}w(t) + \mathfrak{b}\dot{w}(t)) - \mathfrak{c} \leq 0$$

where the last inequality is due to (5.9). Therefore, if (ii) is the case, then $\dot{\tilde{s}}^\tau(t) \to 0$.

The previous analysis shows that a function $\vartheta : [0, T] \to \mathbb{R}$ is defined such that $\dot{\tilde{s}}^\tau(t)$ converges to $\vartheta(t)$ as $\tau \to 0$, for every $t \in [0, T]$. Furthermore ϑ satisfies the following two relations at every $t \in [0, T]$:

$$\mathcal{G}(s(t), \mathfrak{a}w(t) + \mathfrak{b}\dot{w}(t)) \leq \mathfrak{c} + \mathfrak{d}\vartheta(t)$$
$$[-\mathcal{G}(s(t), \mathfrak{a}w(t) + \mathfrak{b}\dot{w}(t)) + \mathfrak{c} + \mathfrak{d}\vartheta(t)]\, \vartheta(t) = 0.$$

In order to prove (5.2) and (5.3) a.e. in $[0, T]$, it is enough to consider the above relations and to observe that, since $s^\tau \rightharpoonup s$ weakly in $H^1(0, T)$, necessarily it has to be $\dot{s}(t) = \vartheta(t)$ for a.e. $t \in [0, T]$. Instead (5.1) is true a.e. in $[0, T]$ by monotonicity of s.

In order to conclude the proof, observe that (5.2) and (5.3) imply that s solves a.e. in $[0, T_f(s))$ the differential relation

$$\mathfrak{d}\dot{s}(t) = [\mathcal{G}(s(t), \mathfrak{a}w(t) + \mathfrak{b}\dot{w}(t)) - \mathfrak{c}]^+ . \tag{5.10}$$

Since s is continuous in $[0, T]$ and $w \in C^2([0, T]; H^1(\Omega_{s_0}))$, the right-hand side in (5.10) is continuous. Then, being s absolutely continuous, it is

$$s(t) = s_0 + \int_0^t \dot{s}(\xi) d\xi = s_0 + \int_0^t \frac{1}{\mathfrak{d}} [\mathcal{G}(s(\xi), \mathfrak{a}w(\xi) + \mathfrak{b}\dot{w}(\xi)) - \mathfrak{c}]^+ d\xi$$

for every $t \in [0, T_f(s))$. The right-hand side a C^1 function, thus we conclude that $s \in C^1([0, T_f(s)))$ as well; hence (5.1)-(5.3) are satisfied everywhere in $[0, T_f(s))$. Finally, if $T_f(s) < T$, then we have $s \equiv L$ in $[T_f(s), T]$, so that $s \in C^1(T_f(s), L)$. □

Collecting together the above lemmas, we deduce the main result of the section.

Proof of Theorem 5.2. Consider u and s obtained by means of compactness arguments in (5.4) and (5.5), respectively, as limits of u^τ and s^τ as τ decreases to 0.

By construction, we have $u^\tau(0) = u_0$ and $s^\tau(0) = s_0$. Lemma 5.4 implies that $u^\tau(0) \rightharpoonup u(0)$ in $H^1(\Omega \setminus \Gamma)$, so that $u(0) = u_0$; concerning s, we obtain $s(0) = s_0$ by pointwise convergence (5.5). Hence the initial condition is satisfied.

Lemma 5.6 assures the regularity for s, which by construction through Helly's Theorem is non-decreasing as the functions s^τ are.

By (5.4) and Lemma 5.5 condition (1) is satisfied.

Fix $t \in [0, T]$. It is $u^\tau(t) = w_i^\tau$ on $\partial_D\Omega$, where $i\tau \leq t < (i+1)\tau$ for every τ. Combining together Lemma 5.4, the compactness of the trace operator and the fact that $w_i^\tau \to w(t)$ strongly in $H^1(\Omega_{s_0})$, we obtain that $u(t) = w(t)$ on $\partial_D\Omega$, i.e. (2) is verified.

Griffith's criterion (4) is established in Lemma 5.7.

We are left to prove condition (3). Let $t \in (0, T)$ be a Lebesgue point for $\dot{u}$ and $\varphi \in H^1(\Omega_{s(t)})$ with $\varphi = 0$ on $\partial_D\Omega$. If $s(t) < L$, consider the flow η_θ described in (2.2), with $\theta > 0$, and define $\varphi_\theta(\cdot) := \varphi(\eta_\theta(\cdot))$. If $s(t) = L$, we assume $\varphi_\theta \equiv \varphi$. By the properties of η_θ, $\varphi_\theta \in H^1(\Omega_{s(t)-\theta})$ and $\varphi_\theta = 0$ on $\partial_D\Omega$. By pointwise convergence (5.5), for τ sufficiently small it is $s^\tau(t) > s(t) - \theta$; therefore $\varphi_\theta \in H^1(\Omega_{s^\tau(t)})$ and, since s^τ are monotone, we get $\varphi_\theta \in H^1(\Omega_{s^\tau(\xi)})$ for every $\xi \geq t$.

Fix $\delta \in (0, T - t)$. For any $\xi \in [t, T]$ the equality (3.5) holds with φ_θ, and integrating it over $[t, t + \delta]$ we have

$$\int_t^{t+\delta} \left(\mathfrak{a}\langle \nabla u^\tau(\xi), \nabla\varphi_\theta\rangle + \mathfrak{b}\langle \nabla\dot{\tilde{u}}^\tau(\xi - \tau), \nabla\varphi_\theta\rangle \right) d\xi = 0. \tag{5.11}$$

Lemma 5.4 assures that

$$\langle \nabla u^\tau(\xi), \nabla\varphi_\theta\rangle \to \langle \nabla u(\xi), \nabla\varphi_\theta\rangle$$

for every $\xi \in [t, t + \delta]$, while considering (3.6) we deduce the estimate

$$|\langle \nabla u^\tau(\xi), \nabla\varphi_\theta\rangle| \leq \|\nabla u^\tau(\xi)\|\|\varphi_\theta\| \leq C\|\varphi_\theta\|.$$

Then, by the Dominated Convergence Theorem,

$$\int_t^{t+\delta} \langle \nabla u^\tau(\xi), \nabla\varphi_\theta\rangle \, d\xi \to \int_t^{t+\delta} \langle \nabla u(\xi), \nabla\varphi_\theta\rangle \, d\xi.$$

Concerning the other term in (5.11), by (5.4)

$$\int_t^{t+\delta} \langle \nabla\dot{\tilde{u}}^\tau(\xi - \tau), \nabla\varphi_\theta\rangle \, d\xi \to \int_t^{t+\delta} \langle \nabla\dot{u}(\xi), \nabla\varphi_\theta\rangle \, d\xi.$$

Collecting together the two limits above and (5.11), it is

$$\frac{1}{\delta}\int_t^{t+\delta} \left(\mathfrak{a}\langle \nabla u(\xi), \nabla\varphi_\theta\rangle + \mathfrak{b}\langle \nabla\dot{u}(\xi), \nabla\varphi_\theta\rangle \right) d\xi = 0.$$

Since $\varphi_\theta \rightharpoonup \varphi$ in $H^1(\Omega \setminus \Gamma)$ and t is a Lebesgue point for $\dot{u}$, we obtain condition (3) by considering the limits as $\theta \to 0^+$ and $\delta \to 0^+$, in this order. □

Remark 5.8. Consider the inequality in Lemma 3.2 with $i = 0$:

$$\begin{aligned}\frac{1}{2}\mathfrak{a}\|\nabla u^\tau(t)\|^2 + \frac{\mathfrak{b}}{2}\int_0^t \|\nabla\dot{\tilde{u}}^\tau(\xi)\|^2\,d\xi + \mathfrak{a}\left(\mathfrak{a}+\frac{\mathfrak{b}}{\tau}\right)^{-1}\left(\mathfrak{c}s^\tau(t) + \frac{\mathfrak{d}}{2}\int_0^t \|\nabla\dot{\tilde{s}}^\tau(\xi)\|^2\,d\xi\right)\\ \le \frac{1}{2}\mathfrak{a}\|\nabla u_0\|^2 + \mathfrak{a}\left(\mathfrak{a}+\frac{\mathfrak{b}}{\tau}\right)^{-1}\mathfrak{c}s^\tau(t_1) + \mathfrak{a}\int_0^t \langle \nabla u^\tau(\xi), \nabla\dot{w}(\xi)\rangle\,d\xi\\ + \frac{\mathfrak{b}}{2}\int_0^t \|\nabla\dot{w}(\xi)\|^2\,d\xi + \rho.\end{aligned}$$

As τ vanishes, $\left(\mathfrak{a}+\frac{\mathfrak{b}}{\tau}\right)^{-1}$ vanishes as well, while all the other terms converge, so that we find the inequality:

$$\frac{1}{2}\mathfrak{a}\|\nabla u(t)\|^2 + \frac{\mathfrak{b}}{2}\int_0^t \|\nabla\dot{u}(\xi)\|^2\,d\xi \le \frac{1}{2}\mathfrak{a}\|\nabla u_0\|^2 + \mathfrak{a}\int_0^t \langle\nabla u(\xi), \nabla\dot{w}(\xi)\rangle\,d\xi + \frac{\mathfrak{b}}{2}\int_0^t \|\nabla\dot{w}(\xi)\|^2\,d\xi.$$

In the energy balance above there is no longer trace of the crack energy. Without giving an interpretation at this stage, we underline the analogy of this fact with what proved in [5] in the damped case. We only point out that the absence of the fracture term is probably related to the presence of the viscoelastic term, as the analysis in Section 6 seems to suggest.

We conclude the section with some estimates on the irreversible viscoelastic evolution.

Lemma 5.9. *Let (s, u) be given by Theorem 5.2. Then there exists a constant $C > 0$, independent of $\mathfrak{b}, \mathfrak{d} > 0$ (fixed at the beginning) and t, such that for every $t \in [0, T]$ the following estimates hold:*

$$\|u(t)\|_{H^1(\Omega\setminus\Gamma)} \le C \tag{5.12}$$

$$\|u\|_{L^2(0,T;H^1(\Omega\setminus\Gamma))} \le C \tag{5.13}$$

$$\mathfrak{b}\|\nabla\dot{u}\|^2_{L^2(0,T;L^2(\Omega\setminus\Gamma;\mathbb{R}^2))} \le C \tag{5.14}$$

$$\mathfrak{d}\|\dot{s}\|^2_{L^2(0,T)} \le C \tag{5.15}$$

The proof is a straightforward consequence of Lemma 3.2 and (5.4) for what concerns u, and of Lemmas 3.4 and 5.6 for s.

6. A comment on the role of $\tau_{\mathfrak{b}}$

We make clear the role of the parameter

$$\mathfrak{a}\tau_{\mathfrak{b}} = \mathfrak{a}\left(\mathfrak{a}+\frac{\mathfrak{b}}{\tau}\right)^{-1}$$

introduced in Section 3, in order to justify its presence in front of the fracture energy. We do not prove again every statement, since generally the proofs are similar to those in Section 3.

Consider the following time-incremental evolution: for every $\tau \in (0, T)$, let u_0^τ, s_0^τ and $u_i^{\tau,\sigma}$ be defined as in (3.3); for the crack tip s_i^τ, instead of (3.4) we choose

$$s_i^\tau \in \operatorname{argmin}\left\{E(u_i^{\tau,\sigma}, u_{i-1}^\tau) + \mathfrak{c}\sigma + \frac{\mathfrak{d}}{2\tau}(\sigma - s_{i-1}^\tau)^2 : s_{i-1}^\tau \le \sigma \le L\right\}. \tag{6.1}$$

As before, set $u_i^\tau := u_i^{\tau,s_i^\tau}$ and define the interpolant functions $u^\tau, \tilde{u}^\tau, s^\tau, \tilde{s}^\tau$.

Arguing as in Lemma 3.2, for every $0 \le i < j \le N_\tau$

$$\begin{aligned}\frac{1}{2}\mathfrak{a}\|\nabla u_j^\tau\|^2 + \frac{\mathfrak{b}}{2\tau}\sum_{h=i}^{j-1}\|\nabla u_{h+1}^\tau - \nabla u_h^\tau\|^2 + \mathfrak{c}s_j^\tau + \frac{\mathfrak{d}}{2\tau}\sum_{h=i}^{j-1}|s_{h+1}^\tau - s_h^\tau|^2\\ \le \frac{1}{2}\mathfrak{a}\|\nabla u_i^\tau\|^2 + \mathfrak{c}s_i^\tau + \mathfrak{a}\int_{i\tau}^{j\tau}\langle \nabla u^\tau(\xi), \nabla \dot{w}(\xi)\rangle\, d\xi\\ + \frac{\mathfrak{b}}{2}\int_{i\tau}^{j\tau}\|\nabla \dot{w}(\xi)\|^2\, d\xi + \rho.\end{aligned}$$

Then Lemma 3.3 holds true, since in its proof we do not take into account the fracture term. The main difference is that in the current situation the inequality above provides an L^2 estimate for $\dot{\tilde{s}}^\tau$ too:

$$\mathfrak{d}\|\dot{\tilde{s}}^\tau\|_{L^2(0,T)} \le C$$

for any $\tau \in (0,T)$.

Following the steps of Sections 4 and 5, by the Helly selection principle there exists $s : [0,T] \to [s_0, L]$ pointwise limit of a subsequence of the family $\{s^\tau\}_{\tau\in(0,T)}$, and it satisfies

$$\tilde{s}^\tau \rightharpoonup s$$

weakly in $H^1(0,T)$ as $\tau \to 0^+$, as in Lemma 5.6.

In the current framework, the Griffith's criterion equivalent to (4.6)-(4.8) is

$$\begin{aligned}&\dot{\tilde{s}}^\tau(t) \ge 0\\ &G(\tau,t) \le \frac{1}{\mathfrak{a}}\left(\mathfrak{a} + \frac{\mathfrak{b}}{\tau}\right)\left(\mathfrak{c} + \mathfrak{d}\dot{\tilde{s}}^\tau(t)\right)\\ &\left[-G(\tau,t) + \frac{1}{\mathfrak{a}}\left(\mathfrak{a} + \frac{\mathfrak{b}}{\tau}\right)\left(\mathfrak{c} + \mathfrak{d}\dot{\tilde{s}}^\tau(t)\right)\right]\dot{\tilde{s}}^\tau(t) = 0.\end{aligned} \tag{6.2}$$

Fix any $t \in [0, T_f(s))$. Since $s^\tau(t) \to s(t) < L$, we can assume $t \in [0, T_f(s^\tau))$ for τ sufficiently small, so that it makes sense to speak of $G(\tau,t)$ for those τ.

Assume that s is not constant in $[0,T]$. Since $s \in H^1(0,T)$, there exists $t \in (0,T)$ such that $\dot{s}(t)$ exists and $\dot{s}(t) > 0$. Hence we can find two sequences $t_j^1 < t < t_j^2$ converging to t with $s(t_j^1) < s(t_j^2)$. By construction of s, there exists τ_j converging to 0 with $s^{\tau_j}(t_j^1) < s^{\tau_j}(t_j^2)$ for every j, so that $\dot{\tilde{s}}^{\tau_j}(t_j) > 0$ for some $t_j \in (t_j^1, t_j^2)$. By construction, $t_j \to t$, while Lemma A.1 implies $s^{\tau_j}(t_j) \to s(t)$. Therefore, by continuity of $\mathcal{G}$, it is

$$G(\tau_j, t_j) \to \mathcal{G}(s(t), \mathfrak{a}w(t) + \mathfrak{b}\dot{w}(t)).$$

Being $\dot{\tilde{s}}^{\tau_j}(t_j) > 0$, equality (6.2) gives

$$G(\tau_j, t_j) = \frac{1}{\mathfrak{a}}\left(\mathfrak{a} + \frac{\mathfrak{b}}{\tau_j}\right)\left(\mathfrak{c} + \mathfrak{d}\dot{\tilde{s}}^{\tau_j}(t_j)\right) > \frac{1}{\mathfrak{a}}\left(\mathfrak{a} + \frac{\mathfrak{b}}{\tau_j}\right)\mathfrak{c}.$$

As $\tau_j \to 0$ the two relations above imply

$$\mathcal{G}(s(t), \mathfrak{a}w(t) + \mathfrak{b}\dot{w}(t)) = \lim_{j\to+\infty} G(\tau_j, t_j) = +\infty,$$

which is impossible. We have to conclude that s is necessarily constant. In particular $s \equiv s_0$ and, being continuous in $[0,T]$, $T_f(s) = T$.

The above argument shows that, if we consider (6.1) instead of (3.4), then a real crack evolution does not take place since the crack tip stays still, independently of the boundary loading.

7. The rate-independent evolution

In the previous sections we never made explicit the dependence of the time-incremental evolutions and irreversible viscoelastic evolutions on the parameters $\mathfrak{b}$ and $\mathfrak{d}$. As announced in the introduction, let us replace $\mathfrak{b}$ and $\mathfrak{d}$ by $\varepsilon\mathfrak{b}$ and $\nu\mathfrak{d}$ in the previous analysis, for positive adimensional parameters ε and ν. We are now interested in investigating the behaviour of the fracture and the viscoelastic terms as the viscosity coefficient ε and the dissipation coefficient ν vanish.

Unlike for the irreversible viscoelastic evolution, where the fracture has a continuous growth, when ν "disappears" the crack might perform instantaneous increases, even though the boundary loading varies smoothly. Despite this fact, we can recover a weaker Griffith's criterion describing the process. We will see the sudden changes of the fracture as a limit behaviour of fast moving "dissipated" cracks.

From now on, for any $\varepsilon, \nu > 0$ we use the notation $(s^{\varepsilon,\nu,\tau}, u^{\varepsilon,\nu,\tau})$ for the time-incremental evolutions defined in Section 3, and $(s^{\varepsilon,\nu}, u^{\varepsilon,\nu})$ for the irreversible viscoelastic evolutions obtained in Section 5 as limit of $(s^{\varepsilon,\nu,\tau}, u^{\varepsilon,\nu,\tau})$ when $\tau \to 0$.

The main result of the section is Theorem 7.5, which states the existence of a particular class of rate-independent evolutions, defined below.

Definition 7.1. Let $\sigma : [0,T] \to [s_0, L]$. We say that $t \in [0,T]$ is a *non-constancy instant* for σ if for every neighbourhood U of t there exist $t_1, t_2 \in [0,T] \cap U$ such that $\sigma(t_1) \neq \sigma(t_2)$. We say that $t \in [0,T]$ is a *jump instant* for σ if $\sigma(t-) \neq \sigma(t+)$.

Definition 7.2. Let $s_0 \in (0,L)$, $w \in C^2([0,T]; H^1(\Omega_{s_0}))$ and u_0 satisfy (2.1). We call *rate-independent evolution* with initial condition (s_0, u_0) and boundary loading w, a couple

$$(s,u) : [0,T] \to [s_0, L] \times H^1(\Omega \setminus \Gamma)$$

such that $(s(0), u(0)) = (s_0, u_0)$, s is left-continuous and the following conditions hold true:

(1) $u \in L^2(0,T; H^1(\Omega \setminus \Gamma))$ and $u(t) \in H^1(\Omega_{s(t)})$ for a.e. $t \in [0,T]$;
(2) $u(t) = w(t)$ on $\partial_D \Omega$ for a.e. $t \in [0,T]$;
(3) for a.e. $t \in (0,T)$, for every $\varphi \in H^1(\Omega_{s(t)})$ with $\varphi = 0$ on $\partial_D \Omega$,

$$\mathfrak{a}\langle \nabla u(t), \nabla \varphi \rangle = 0;$$

(4) *Griffith's criterion*:
- s is non-decreasing;
- for every $t \in [0, T_f(s))$

$$\mathcal{G}(s(t), \mathfrak{a}w(t)) \leq \mathfrak{c}; \tag{7.1}$$

- *weak activation principle*: if $t \in [0, T_f(s))$ is a non-constancy instant for s, then

$$\mathcal{G}(s(t\pm), \mathfrak{a}w(t)) = \mathfrak{c}; \tag{7.2}$$

- if $t \in [0, T_f(s))$ is a jump instant for s, then

$$\mathcal{G}(\sigma, \mathfrak{a}w(t)) \geq \mathfrak{c} \tag{7.3}$$

  for every $\sigma \in [s(t-), s(t+)]$;
- if $t \in [0, T_f(s))$ and $\mathcal{G}(s(t), \mathfrak{a}w(t)) < \mathfrak{c}$, then s is differentiable at t and $\dot{s}(t) = 0$;

(5) the function $t \mapsto \mathcal{G}(s(t), \mathfrak{a}w(t))$ is continuous in $[0, T_f(s)]$.

The idea of weak activation principle has been suggested in [15] in order to relax the differential formulation of Griffith's criterion (as stated in (1.1)). Let us point out that the two coincide when the solution s is regular enough. We stress the fact that having a differential criterion valid only on $[0,T] \setminus \mathcal{N}$ with $\mathcal{L}^1(\mathcal{N}) = 0$ might make it totally meaningless, since the jump points of s are at most countable and so they might concentrate on $\mathcal{N}$.

Theorem 7.3. *For any* $s_0 \in (0,L)$, $w \in C^2([0,T];H^1(\Omega_{s_0}))$ *and* u_0 *satisfying (2.1), there exists a rate-independent evolution* (s,u).

Theorem 7.3 is consequence of the result that we will state and prove below. We first introduce another class of evolutions that turn out to be rate-independent evolutions.

Definition 7.4. Let $s_0 \in (0,L)$, $w \in C^2([0,T];H^1(\Omega_{s_0}))$ and u_0 satisfy (2.1). We call *vanishing viscosity evolution* with initial condition (s_0,u_0) and boundary loading w, a couple (s,u) for which there exists a sequence $(s^{\varepsilon,\nu},u^{\varepsilon,\nu})_{\varepsilon,\nu}$ of irreversible viscoelastic evolutions satisfying the same initial and boundary data, and such that

$$u^{\varepsilon,\nu} \rightharpoonup u$$

weakly in $L^2(0,T;H^1(\Omega\setminus\Gamma))$ and

$$s^{\varepsilon,\nu}(t) \to s(t)$$

for every $t\in[0,T]$ as $\varepsilon\to 0$ and $\nu\to 0$.

Theorem 7.5. *For any* $s_0 \in (0,L)$, $w \in C^2([0,T];H^1(\Omega_{s_0}))$ *and* u_0 *satisfying (2.1), there exists a vanishing viscosity evolution* (s,u).

Furthermore, any vanishing viscosity evolution is a rate-independent evolution.

Remark 7.6. It is clear that Theorem 7.3 is proved as soon as Theorem 7.5 is. The last is achieved by combining together a number of lemmas.

We will always write $\varepsilon\to 0$ even in case $\varepsilon = 0$. In this situation, it is understood that we are considering the constant null sequence.

We start identifying a couple (s,u) candidate to satisfy Definition 7.4; similarly to Section 5, we use a compactness argument. For every $\nu>0$ and $\varepsilon\ge 0$, consider the irreversible viscoelastic evolutions whose existence is assured by Theorem 5.2. By the estimates in Lemma 5.9, the sequence $(u^{\varepsilon,\nu})_{\varepsilon\ge 0,\nu>0}$ is uniformly bounded in $L^2(0,T;H^1(\Omega\setminus\Gamma))$. Therefore there exists $u\in L^2(0,T;H^1(\Omega\setminus\Gamma))$ such that

$$u^{\varepsilon,\nu} \rightharpoonup u \tag{7.4}$$

weakly in $L^2(0,T;H^1(\Omega\setminus\Gamma))$ as $\varepsilon\to 0$ and $\nu\to 0$ along suitable sequences.

Concerning the crack tip, Theorem 5.2 assures that the functions $s^{\varepsilon,\nu}$ are monotone non-decreasing. Applying Helly's Theorem to the sequence found in (7.4), up to a further subsequence

$$s^{\varepsilon,\nu}(t) \to s(t) \tag{7.5}$$

for every $t\in[0,T]$, for some $s\in BV([0,T])$. The function s is non-decreasing since the functions $s^{\varepsilon,\nu}$ are, and by pointwise convergence $s(t)\in[s_0,L]$ for every $t\in[0,T]$. We can describe more in detail the convergence:

Lemma 7.7. *The sequence* $(s^{\varepsilon,\nu})$ *is monotonically non-increasing with respect to* ν, *i.e.* $s^{\varepsilon,\nu_1}(t)\ge s^{\varepsilon,\nu_2}(t)$ *for every* $t\in[0,T]$ *if* $0<\nu_1<\nu_2$.

As a consequence, s *is left-continuous.*

Proof. Being $s\in C^1((0,T_f(s)\cup(T_f(s),T))$, equality (5.10) holds true for every $t\in[0,T]$ and not only a.e.; thereby $s^{\varepsilon,\nu}$ solves the Cauchy problem

$$\begin{cases} \dot s^{\varepsilon,\nu}(t) = \dfrac{1}{\nu\mathfrak{d}}\left[\mathcal{G}(s^{\varepsilon,\nu}(t),\mathfrak{a}w(t)+\varepsilon\mathfrak{b}\dot w(t))-\mathfrak{c}\right]^+ \\ s^{\varepsilon,\nu}(0)=s_0. \end{cases}$$

If $\nu_1<\nu_2$, then s^{ε,ν_1} verifies the differential inequality

$$\dot s^{\varepsilon,\nu_1}(t) = \frac{1}{\nu_1\mathfrak{d}}\left[\mathcal{G}(s^{\varepsilon,\nu_1}(t),\mathfrak{a}w(t)+\varepsilon\mathfrak{b}\dot w(t))-\mathfrak{c}\right]^+ \ge \frac{1}{\nu_2\mathfrak{d}}\left[\mathcal{G}(s^{\varepsilon,\nu_1}(t),\mathfrak{a}w(t)+\varepsilon\mathfrak{b}\dot w(t))-\mathfrak{c}\right]^+ .$$

By comparison results for differential equations (see [19, Theorem X.8]), we obtain $s^{\varepsilon,\nu_1}(t)\ge s^{\varepsilon,\nu_2}(t)$.

The first part implies that $s(t) \geq s^{\varepsilon,\nu}(t)$ for every $t \in [0,T]$ and every $\varepsilon \geq 0, \nu > 0$. Assume $s(t) - s(t-) > \alpha$ for some $t \in (0,T]$ and $\alpha > 0$; then $s(t) - s(\tau) > \alpha$ for $\tau < t$. For any ε and ν sufficiently small, $s(t) - s^{\varepsilon,\nu}(t) < \frac{\alpha}{2}$, so that $s^{\varepsilon,\nu}(t) - s^{\varepsilon,\nu}(\tau) \geq \frac{\alpha}{2}$ for any $\tau < t$, in contradiction to the continuity of $s^{\varepsilon,\nu}$. □

Lemma 7.8. *Up to subsequences,* $\nu \dot{s}^{\varepsilon,\nu}(t) \to 0$ *as* $\nu \to 0$ *for a.e.* $t \in (0,T)$.

Proof. Lemma 5.6 for the functions $s^{\varepsilon,\nu}$ reads as $\nu \mathfrak{d} \|\dot{s}^{\varepsilon,\nu}\|^2_{L^2(0,T)} \leq C$ with C independent of ν, so that $\nu \dot{s}^{\varepsilon,\nu} \to 0$ strongly in $L^2(0,T)$ as $\nu \to 0$. Then, up to a subsequence, $\nu \dot{s}^{\varepsilon,\nu}(t) \to 0$ for a.e. $t \in (0,T)$. □

Lemma 7.9. *Let* $t \in (0,T)$ *be a jump instant for* s. *Then there exist subsequences (not relabelled)* $\varepsilon, \nu \to 0$ *and* $t^{\varepsilon,\nu} \in (0,T)$ *such that*

(1) $t^{\varepsilon,\nu} \to t$;
(2) $s^{\varepsilon,\nu}(t^{\varepsilon,\nu}) \to s(t-)$;
(3) $\mathcal{G}(s^{\varepsilon,\nu}(t^{\varepsilon,\nu}), \mathfrak{a}w(t^{\varepsilon,\nu}) + \varepsilon \mathfrak{b} \dot{w}(t^{\varepsilon,\nu})) = \mathfrak{c} + \nu \mathfrak{d} \dot{s}^{\varepsilon,\nu}(t^{\varepsilon,\nu})$.

Similarly, there exists $\hat{t}^{\varepsilon,\nu} \in (0,T)$ *such that* (1),(3) *and*

(2') $s^{\varepsilon,\nu}(\hat{t}^{\varepsilon,\nu}) \to s(t+)$

are satisfied.

Proof. Let us discuss only the case $s(t-)$; for the other, $s(t+)$, it is sufficient to argue analogously.

We initially consider the case $\varepsilon > 0$.

Claim: for every $m \in \mathbb{N}$ there exists $\varepsilon(m), \nu(m) > 0$ such that for every $0 < \varepsilon \leq \varepsilon(m), 0 < \nu \leq \nu(m)$ there exists $t^{\varepsilon,\nu}_m$ satisfying

(i) $|t^{\varepsilon,\nu}_m - t| < \frac{1}{m}$;
(ii) $|s^{\varepsilon,\nu}(t^{\varepsilon,\nu}_m) - s(t-)| < \frac{1}{m}$;
(iii) $\mathcal{G}(s^{\varepsilon,\nu}(t^{\varepsilon,\nu}_m), \mathfrak{a}w(t^{\varepsilon,\nu}_m) + \varepsilon \mathfrak{b} \dot{w}(t^{\varepsilon,\nu}_m)) = \mathfrak{c} + \nu \mathfrak{d} \dot{s}^{\varepsilon,\nu}(t^{\varepsilon,\nu}_m)$.

If the claim holds true, then the lemma is proved. In fact, without loss of generality we can assume $\varepsilon(m+1) < \varepsilon(m), \nu(m+1) < \nu(m)$. If we set

$$t^{\varepsilon,\nu} := t^{\varepsilon,\nu}_m \iff \varepsilon(m+1) < \varepsilon \leq \varepsilon(m), \nu(m+1) < \nu \leq \nu(m)$$

then (1), (2), (3) are consequence of (i),(ii),(iii), respectively.

Proof of the claim. Fix $m \in \mathbb{N}$ such that $\frac{1}{m} < T - t$. There exists $\alpha \in (0, \frac{1}{m})$ such that $|s(t-) - s(\tau)| < \frac{1}{3m}$ for every $t - \alpha < \tau < t$. Fixed $\hat{t} \in (t - \frac{\alpha}{2}, t)$, there exist strictly positive constants $\varepsilon_0(m), \nu_0(m)$ such that

$$|s^{\varepsilon,\nu}(\hat{t}) - s(\hat{t})| < \frac{1}{3m}$$

for every $\varepsilon \leq \varepsilon_0(m), \nu \leq \nu_0(m)$. Define

$$\hat{t}^{\varepsilon,\nu}_m := \sup\{\xi \geq \hat{t} : \ s^{\varepsilon,\nu}(\xi) = s^{\varepsilon,\nu}(\hat{t})\}.$$

It is $\hat{t}^{\varepsilon,\nu}_m \geq \hat{t} > \hat{t} - \frac{\alpha}{2} > t - \frac{1}{m}$.

By contradiction, assume that there exists a subsequence $(t_j)_j$ of $(\hat{t}^{\varepsilon,\nu}_m)$ such that $t_j \geq t + \frac{1}{m}$. Then $s^{\varepsilon_j,\nu_j}(\xi) = s^{\varepsilon_j,\nu_j}(\hat{t})$ for every $\xi \in [\hat{t}, t + \frac{1}{m})$; in particular, $s^{\varepsilon_j,\nu_j}(t + \frac{1}{2m}) = s^{\varepsilon_j,\nu_j}(\hat{t})$. Taking the limit as $\varepsilon_j, \nu_j \to 0$, we obtain

$$s(t+) \leq s\left(t + \frac{1}{2m}\right) = s(\hat{t}) \leq s(t-) < s(t+),$$

which is a contradiction. Hence there exists $0 < \varepsilon(m) \leq \varepsilon_0(m), 0 < \nu(m) \leq \nu_0(m)$ such that

$$t - \frac{1}{m} < \hat{t}^{\varepsilon,\nu}_m < t + \frac{1}{m} \tag{7.6}$$

for every $\varepsilon \leq \varepsilon(m), \nu \leq \nu(m)$.

By definition of $\hat t_m^{\varepsilon,\nu}$, (7.6) and continuity of $s^{\varepsilon,\nu}$, for every $\varepsilon \le \varepsilon(m), \nu \le \nu(m)$ there exists $\beta_{\varepsilon,\nu} > 0$ such that

$$\hat t_m^{\varepsilon,\nu} + \beta_{\varepsilon,\nu} < t + \frac{1}{m} \quad \text{and} \quad 0 < s^{\varepsilon,\nu}(\hat t_m^{\varepsilon,\nu} + \beta_{\varepsilon,\nu}) - s^{\varepsilon,\nu}(\hat t_m^{\varepsilon,\nu}) < \frac{1}{3m}.$$

Being $s^{\varepsilon,\nu} \in C^1((0,T_f(s^{\varepsilon,\nu})) \cup (T_f(s^{\varepsilon,\nu}),T))$, necessarily it holds $\dot s^{\varepsilon,\nu}(t_m^{\varepsilon,\nu}) > 0$ for some $t_m^{\varepsilon,\nu} \in (\hat t_m^{\varepsilon,\nu}, \hat t_m^{\varepsilon,\nu} + \beta_{\varepsilon,\nu})$.

By choice of $t_m^{\varepsilon,\nu}$, (i) is satisfied.

By continuity of $s^{\varepsilon,\nu}$, it is $s^{\varepsilon,\nu}(\hat t_m^{\varepsilon,\nu}) = s^{\varepsilon,\nu}(\hat t)$ and we have the chain of inequalities

$$\begin{aligned}
|s^{\varepsilon,\nu}(t_m^{\varepsilon,\nu}) - s(t-)| \le & |s^{\varepsilon,\nu}(t_m^{\varepsilon,\nu}) - s(\hat t)| + |s(\hat t) - s(t-)| \\
\le & s^{\varepsilon,\nu}(t_m^{\varepsilon,\nu}) - s^{\varepsilon,\nu}(\hat t_m^{\varepsilon,\nu}) + |s^{\varepsilon,\nu}(\hat t) - s(\hat t)| + \frac{1}{3m} \\
\le & \frac{1}{3m} + \frac{1}{3m} + \frac{1}{3m} = \frac{1}{m}
\end{aligned}$$

and (ii) is achieved.

Finally, since $\dot s^{\varepsilon,\nu}(t_m^{\varepsilon,\nu}) > 0$, (iii) is a consequence of (5.3)

In case $\varepsilon = 0$, the previous proof holds true by setting $\varepsilon(m) = 0$ for every m and $t^{0,\nu} := t_m^{0,\nu}$ if and only if $\nu(m+1) < \nu \le \nu(m)$. □

Lemma 7.10. *For every $t \in [0,T_f(s))$ it is $\mathcal{G}(s(t), \mathfrak{a}w(t)) \le \mathfrak{c}$. If $t \in (0,T_f(s))$ is a non-constancy instant for s, then $\mathcal{G}(s(t\pm), \mathfrak{a}w(t)) = \mathfrak{c}$.*

Proof. Without loss of generality, when $t \in [0,T_f(s))$ is fixed we can assume that $t \in [0,T_f(s^{\varepsilon,\nu}))$ for ε,ν small enough, since $s^{\varepsilon,\nu}(t) \to s(t) < L$.

As already noticed in the proof of Lemma 7.7, conditions (5.1)-(5.3) imply that the irreversible viscoelastic evolutions are solutions to the ordinary differential equation

$$\nu\mathfrak{d}\dot s^{\varepsilon,\nu}(t) = [\mathcal{G}(s^{\varepsilon,\nu}(t), \mathfrak{a}w(t) + \varepsilon\mathfrak{b}\dot w(t)) - \mathfrak{c}]^+ \tag{7.7}$$

for $t \in [0,T_f(s^{\varepsilon,\nu}))$. Fix $t \in [0,T_f(s))$ such that $\nu\dot s^{\varepsilon,\nu}(t) \to 0$ when ε and ν vanish. Considering pointwise convergence (7.5) and the continuity properties of w (by assumption) and $\mathcal{G}$ (see Proposition 4.1), from (7.7) we obtain

$$0 = [\mathcal{G}(s(t), \mathfrak{a}w(t)) - \mathfrak{c}]^+ . \tag{7.8}$$

Lemma 7.8 implies that (7.8) holds for a.e. $t \in [0,T_f(s))$; by left continuity of s, the equality is verified everywhere in $[0,T_f(s))$. Finally, (7.8) is equivalent to (7.1), and the first part of the statement is proved.

We point out that inequality (7.1) and continuity of $\mathcal{G}$ imply that

$$\mathcal{G}(s(t+), \mathfrak{a}w(t)) \le \mathfrak{c}$$

for every $t \in [0,T_f(s))$.

Let now $t \in (0,T_f(s))$ be a non-constancy instant for s.

Assume first that t is a jump instant for s and consider the sequence $t^{\varepsilon,\nu}$ defined in Lemma 7.9. Let us prove that

$$\nu\dot s^{\varepsilon,\nu}(t^{\varepsilon,\nu}) \to 0. \tag{7.9}$$

By contradiction, assume that $\nu\dot s^{\varepsilon,\nu}(t^{\varepsilon,\nu}) \to \alpha > 0$. By regularity of w and continuity of $\mathcal{G}$ (Proposition 4.1),

$$0 = -\mathcal{G}(s^{\varepsilon,\nu}(t^{\varepsilon,\nu}), \mathfrak{a}w(t^{\varepsilon,\nu}) + \varepsilon\mathfrak{b}\dot w(t^{\varepsilon,\nu})) + \mathfrak{c} + \nu\mathfrak{d}\dot s^{\varepsilon,\nu}(t^{\varepsilon,\nu}) \to -\mathcal{G}(s(t-), \mathfrak{a}w(t)) + \mathfrak{c} + \mathfrak{d}\alpha,$$

so that $\mathcal{G}(s(t-), \mathfrak{a}w(t)) > \mathfrak{c}$, in contradiction to (7.1) proved above. Regularity of w, continuity of $\mathcal{G}$ (Proposition 4.1) and (7.9) allow to conclude

$$0 = -\mathcal{G}(s^{\varepsilon,\nu}(t^{\varepsilon,\nu}), \mathfrak{a}w(t^{\varepsilon,\nu}) + \varepsilon\mathfrak{b}\dot w(t^{\varepsilon,\nu})) + \mathfrak{c} + \nu\mathfrak{d}\dot s^{\varepsilon,\nu}(t^{\varepsilon,\nu}) \to -\mathcal{G}(s(t), \mathfrak{a}w(t)) + \mathfrak{c},$$

i.e. the thesis, since $s(t) = s(t-)$ by left continuity of s (Lemma 7.7).

Similarly for $s(t+)$, we have

$$\nu\dot{s}^{\varepsilon,\nu}(\hat{t}^{\varepsilon,\nu}) \to 0$$

and we deduce that

$$0 = -\mathcal{G}(s^{\varepsilon,\nu}(\hat{t}\varepsilon,\nu), \mathfrak{a}w(\hat{t}\varepsilon,\nu) + \varepsilon\mathfrak{b}\dot{w}(\hat{t}\varepsilon,\nu)) + \mathfrak{c} + \nu\mathfrak{d}\dot{s}^{\varepsilon,\nu}(\hat{t}\varepsilon,\nu) \to -\mathcal{G}(s(t+), \mathfrak{a}w(t)) + \mathfrak{c}.$$

Assume now that s is continuous at t and fix a neighbourhood U of t. Since s is not constant in U, for ε, ν small enough $s^{\varepsilon,\nu}$ is not constant in U as well, so that $\dot{s}^{\varepsilon,\nu}(t^{\varepsilon,\nu}) > 0$ for some $t^{\varepsilon,\nu} \in U \cap (0, T_f(s^{\varepsilon,\nu}))$. Considering a decreasing sequence of neighbourhoods of t converging to t, we find a sequence $t^{\varepsilon,\nu} \to t$ with $\dot{s}^{\varepsilon,\nu}(t^{\varepsilon,\nu}) > 0$. Since s is continuous at t, Lemma A.1 implies that $s^{\varepsilon,\nu}(t^{\varepsilon,\nu}) \to s(t)$. Arguing as in the previous case, we deduce that $\nu\dot{s}^{\varepsilon,\nu}(t^{\varepsilon,\nu}) \to 0$. Thanks to the regularity assumption on w, continuity of $\mathcal{G}$ (Proposition 4.1) and Lemma 7.8, as before we conclude that

$$0 = -\mathcal{G}(s^{\varepsilon,\nu}(t^{\varepsilon,\nu}), \mathfrak{a}w(t^{\varepsilon,\nu}) + \varepsilon\mathfrak{b}\dot{w}(t^{\varepsilon,\nu})) + \mathfrak{c} + \nu\mathfrak{d}\dot{s}^{\varepsilon,\nu}(t^{\varepsilon,\nu}) \to -\mathcal{G}(s(t), \mathfrak{a}w(t)) + \mathfrak{c}$$

and the thesis is proved since $s(t\pm) = s(t)$. □

Lemma 7.11. *Let $t \in [0, T_f(s))$ be such that*

$$\mathcal{G}(s(t), \mathfrak{a}w(t)) < \mathfrak{c}. \tag{7.10}$$

Then s is differentiable at t and $\dot{s}(t) = 0$.

Proof. By continuity of $\mathcal{G}$ and w, there exist $\eta, \delta_0 > 0$ such that $\mathcal{G}(\sigma, \mathfrak{a}w(\tau)) < \mathfrak{c}$ for $\sigma \in [s(t) - 2\eta, s(t) + 2\eta]$ and $\tau \in [t - \delta_0, t + \delta_0] \cap [0, T]$. Lemma 7.10 and (7.10) imply that s is continuous at t, so that $s(\tau) \in [s(t) - \eta, s(t) + \eta]$ for $\tau \in [t - \delta_1, t + \delta_1] \cap [0, T]$, for some $0 < \delta_1 \le \delta_0$.

By (7.5), it is $s^{\varepsilon,\nu}(t - \delta_1) \ge s(t - \delta_1) - \eta$ and $s^{\varepsilon,\nu}(t + \delta_1) \le s(t + \delta_1) + \eta$ for every ε and ν sufficiently small, say $0 \le \varepsilon < \varepsilon_0$ and $0 < \nu < \nu_0$. Thus we have the chain of inequalities

$$s(t) - 2\eta \le s(t - \delta_1) - \eta \le s^{\varepsilon,\nu}(t - \delta_1) \le s^{\varepsilon,\nu}(\tau) \le s^{\varepsilon,\nu}(t + \delta_1) \le s(t + \delta_1) + \eta \le s(t) + 2\eta$$

for every $\tau \in [t - \delta_1, t + \delta_1] \cap [0, T]$. Consequently

$$\mathcal{G}(s^{\varepsilon,\nu}(\tau), \mathfrak{a}w(\tau)) < \mathfrak{c}$$

for every $\tau \in [t - \delta_1, t + \delta_1] \cap [0, T]$, $0 \le \varepsilon < \varepsilon_0$ and $0 < \nu < \nu_0$. By the regularity of w and $\mathcal{G}$, we further obtain that, for some $0 < \varepsilon_1 \le \varepsilon_0$,

$$\mathcal{G}(s^{\varepsilon,\nu}(\tau), \mathfrak{a}w(\tau) + \varepsilon\mathfrak{b}\dot{w}(\tau)) < \mathfrak{c}$$

for every $\tau \in [t - \delta_1, t + \delta_1] \cap [0, T]$, $0 \le \varepsilon < \varepsilon_1$ and $0 < \nu < \nu_0$.

Then (5.3) implies that

$$s^{\varepsilon,\nu}(\tau) = c^{\varepsilon,\nu} \in [s_0, L]$$

for every $\tau \in [t - \delta_1, t + \delta_1] \cap [0, T]$. We deduce that the limit s is constant on $[t - \delta_1, t + \delta_1] \cap [0, T]$, so that it is differentiable at t and $\dot{s}(t) = 0$. □

Lemma 7.12. *It results $u^{\varepsilon,\nu}(t) \rightharpoonup u(t)$ weakly in $H^1(\Omega \setminus \Gamma)$ for a.e. $t \in [0, T]$. In addition, $u(t) \in H^1(\Omega_{s(t)})$ for a.e. $t \in [0, T]$.*

Proof. By the estimate (5.12), for every $t \in [0, T]$ there exists $\hat{u}(t) \in H^1(\Omega \setminus \Gamma)$ such that $u^{\varepsilon,\nu}(t) \rightharpoonup \hat{u}(t)$ as ε and ν vanish. Then, since (7.4) holds true, it has to be $\hat{u}(t) = u(t)$ for a.e. $t \in [0, T]$.

The second part of the statement can be proved by arguing as in Lemma 5.5 at any t for which the weak convergence $u^{\varepsilon,\nu}(t) \rightharpoonup u(t)$ is satisfied. □

Proof of Theorem 7.5. Consider the limit (s, u) defined in (7.5) and (7.4), respectively. The couple (s, u) is then a vanishing viscosity evolution with initial condition (s_0, u_0) and boundary loading w. The existence part of the theorem is proved.

Consider now any vanishing viscosity evolution with initial condition (s_0, u_0) and boundary loading w, and let $(s^{\varepsilon,\nu}, u^{\varepsilon,\nu})$ be the correspondent sequence of irreversible viscoelastic evolutions. By Theorem 5.2, it is $(s^{\varepsilon,\nu}(0), u^{\varepsilon,\nu}(0)) = (s_0, u_0)$ for every ε, ν, so that pointwise convergence (7.5) implies $s(0) = s_0$. Lemma 7.7 assures the left-continuity for s.

The function u satisfies the boundary condition at a.e. instant since the functions $u^{\varepsilon,\nu}$ do and Lemma 7.12 holds true. Therefore (2) is proved.

In order to obtain (3), argue as in Theorem 5.2 and use the fact that $\varepsilon \nabla \dot{u}^{\varepsilon,\nu}$ converges strongly to 0 in $L^2(0, T; L^2(\Omega \setminus \Gamma; \mathbb{R}^2))$, because of (5.14).

Let now prove the weak version of Griffith's criterion. By construction, s is non-decreasing since the $s^{\varepsilon,\nu}$ are. Inequality (7.1) and the weak activation principle (7.2) are proved in Lemma 7.10. The condition (7.3) on the jump instant can be proved as in [11, Theorem 5.1]. The last requirement has been obtained in Lemma 7.11.

Finally, to show (5) observe that if s is continuous at t, then $\mathcal{G}(s(\cdot), \mathfrak{a}w(\cdot))$ is continuous at t too. If t is a jump instant for s, then

$$\lim_{\substack{\tau \to t \\ \tau < t}} \mathcal{G}(s(\tau), \mathfrak{a}w(\tau)) = \mathcal{G}(s(t-), \mathfrak{a}w(t)) = \mathfrak{c} = \mathcal{G}(s(t+), \mathfrak{a}w(t)) = \lim_{\substack{\tau \to t \\ \tau > t}} \mathcal{G}(s(\tau), \mathfrak{a}w(\tau))$$

where the equalities in the middle are due to (7.2). Therefore $\mathcal{G}(s(\cdot), \mathfrak{a}w(\cdot))$ is continuous at the jump instant of s as well. □

8. 1-dimensional analysis

In this section, taking inspiration by the analyses proposed in [11, 14], we describe the crack tip function of the vanishing viscosity evolutions, with the purpose of clarifying the different behaviour between them and a general rate-independent evolution. In effect, the request of being approximable by irreversible viscoelastic evolutions provides interesting properties. The idea is to obtain a 1-dimensional analysis of a problem that in principle was infinite dimensional, in the sense that its initial setting is in infinite dimensional Sobolev spaces.

At the end of the section, an example shows the different behaviour of the globally stable irreversible evolutions introduced by [8] and the vanishing viscosity ones.

Observe first of all that conditions (4) in Theorem 7.3 show that the $\mathfrak{c}$-level set of $\mathcal{G}$ plays an important role. For convenience, we introduce the function

$$\mathcal{V} : [s_0, L) \times [0, T] \to \mathbb{R}$$

defined as

$$\mathcal{V}(\sigma, t) := \mathcal{G}(\sigma, \mathfrak{a}w(t)) - \mathfrak{c} \tag{8.1}$$

and the sets

$$\begin{aligned}
\mathcal{A}^0 &:= \{(t, \sigma) \in [0, T] \times [s_0, L) \,:\, \mathcal{V}(\sigma, t) = 0\} \\
&= \{(t, \sigma) \in [0, T] \times [s_0, L) \,:\, \mathcal{G}(\sigma, \mathfrak{a}w(t)) - \mathfrak{c} = 0\} \\
\mathcal{A}^+ &:= \{(t, \sigma) \in [0, T] \times [s_0, L) \,:\, \mathcal{V}(\sigma, t) > 0\} \\
&= \{(t, \sigma) \in [0, T] \times [s_0, L) \,:\, \mathcal{G}(\sigma, \mathfrak{a}w(t)) - \mathfrak{c} > 0\} \\
\mathcal{A}^- &:= \{(t, \sigma) \in [0, T] \times [s_0, L) \,:\, \mathcal{V}(\sigma, t) < 0\} \\
&= \{(t, \sigma) \in [0, T] \times [s_0, L) \,:\, \mathcal{G}(\sigma, \mathfrak{a}w(t)) - \mathfrak{c} < 0\}.
\end{aligned}$$

The properties of a rate-independent evolution s, translated in terms of the sets above, are:

- s is non-decreasing and left continuous, with $s(0) = s_0$;
- $(t, s(t)) \in \mathcal{A}^- \cup \mathcal{A}^0$ for every $t \in [0, T]$;

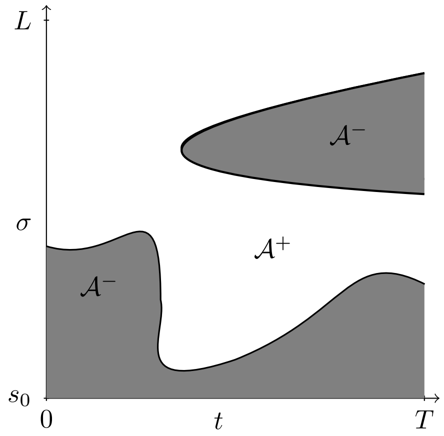


FIGURE 1. The sets $\mathcal{A}^-$, $\mathcal{A}^+$ and $\mathcal{A}^0$ for a sufficiently smooth energy release rate $\mathcal{G}$. $\mathcal{A}^0$ corresponds to the black line separating the gray region $\mathcal{A}^-$ and the white region $\mathcal{A}^+$.

- if t is a non-constancy instant for s, then $(t, s(t\pm)) \in \mathcal{A}^0$;
- if t is a jump instant for s, then $(t, \sigma) \in \mathcal{A}^0 \cup \mathcal{A}^+$ for every $\sigma \in [s(t), s(t+)]$;
- the function $t \mapsto \mathcal{G}(s(t), \mathfrak{a}w(t))$ is continuous.

A priori, a function with this behaviour is not unique, thus we really need to characterize the class of vanishing viscosity evolutions.

It remains open the question whether $s^{\varepsilon,\nu}$ and $s^{0,\nu}$ converge to the same limit for any reasonable $\mathcal{G}$, when ε and ν vanish; the issue arises already at the incremental level. However, if we assume sufficient regularity for $\mathcal{G}$, we are able to give an answer. Let $\mathcal{G}$ be Lipschitz continuous with respect to both its variables. We already obtained a partial result in Proposition 4.2; in order to prove lipschitzianity with respect to the fracture variable σ, we should assume more regularity for the boundary data and the pre-assigned crack path Γ. Proper assumption can be deduced by comparison with the result in [15, Appendix A.3]. In the following we will consider only the evolutions $s^{0,\nu}$, and not the $s^{\varepsilon,\nu}$ with $\varepsilon > 0$, since they both converge to the same limit when $\varepsilon, \nu \to 0$. Indeed, first of all we recall that $s^{\varepsilon,\nu}$ and $s^{0,\nu}$ are solutions to the Cauchy problems

$$\begin{cases} \dot{s}^{\varepsilon,\nu}(t) = \dfrac{1}{\nu\mathfrak{d}} \left[\mathcal{G}(s^{\varepsilon,\nu}(t), \mathfrak{a}w(t) + \varepsilon\mathfrak{b}\dot{w}(t)) - \mathfrak{c}\right]^+ \\ s^{\varepsilon,\nu}(0) = s_0 \end{cases}$$

and

$$\begin{cases} \dot{s}^{0,\nu}(t) = \dfrac{1}{\nu\mathfrak{d}} \mathcal{V}(s^{0,\nu}(t), t)^+ \\ s^{0,\nu}(0) = s_0 \end{cases} \tag{8.2}$$

respectively, where in (8.2) we used the definition (8.1). Being the function $\sigma \in \mathbb{R} \mapsto \sigma^+$ Lipschitz continuous with constant 1 and denoting by K the Lipschitz constant of $\mathcal{G}$, we have

$$
\begin{aligned}
|s^{\varepsilon,\nu}(t)-s^{0,\nu}(t)| \leq & \int_0^t \left|[\mathcal{G}(s^{\varepsilon,\nu}(\tau),\mathfrak{a}w(\tau)+\varepsilon\mathfrak{b}\dot{w}(\tau))-\mathfrak{c}]^+ - \left[\mathcal{G}(s^{0,\nu}(\tau),\mathfrak{a}w(\tau))-\mathfrak{c}\right]^+\right| \, d\tau \\
\leq & \int_0^t \left|\mathcal{G}(s^{\varepsilon,\nu}(\tau),\mathfrak{a}w(\tau)+\varepsilon\mathfrak{b}\dot{w}(\tau))-\mathcal{G}(s^{0,\nu}(\tau),\mathfrak{a}w(\tau))\right| \, d\tau \\
\leq & \int_0^t \left|\mathcal{G}(s^{\varepsilon,\nu}(\tau),\mathfrak{a}w(\tau)+\varepsilon\mathfrak{b}\dot{w}(\tau))-\mathcal{G}(s^{\varepsilon,\nu}(\tau),\mathfrak{a}w(\tau))\right| \, d\tau \\
& + \int_0^t \left|\mathcal{G}(s^{\varepsilon,\nu}(\tau),\mathfrak{a}w(\tau))-\mathcal{G}(s^{0,\nu}(\tau),\mathfrak{a}w(\tau))\right| \, d\tau \\
\leq & \varepsilon\mathfrak{b}K\int_0^t |\dot{w}(\tau)| \, d\tau + K\int_0^t \left|s^{\varepsilon,\nu}(\tau)-s^{0,\nu}(\tau)\right| \, d\tau \\
\leq & \varepsilon\mathfrak{b}CTK + K\int_0^t \left|s^{\varepsilon,\nu}(\tau)-s^{0,\nu}(\tau)\right| \, d\tau.
\end{aligned}
$$

Gronwall Lemma provides the inequality

$$
|s^{\varepsilon,\nu}(t)-s^{0,\nu}(t)| \leq \varepsilon\mathfrak{b}CKTe^{Kt} \leq \varepsilon\mathfrak{b}CKTe^{KT}
$$

uniformly in t and ν, so that the claim is proved.

We now describe all the possible behaviours of a vanishing viscosity evolution s at the initial instant. The propositions below can be seen as different steps in an algorithmic procedure. Since we need a bit of regularity, we assume $\mathcal{A}^0$ to be a C^1 manifold of dimension 1.

Proposition 8.1. *If there exists $t \in (0,T]$ such that $[0,t)\times\{s_0\} \subset \mathcal{A}^- \cup \mathcal{A}^0$, then $s(t)=s_0$ for $t \in [0,t_0]$, where $t_0 := \sup\left\{t \in (0,T] : \ [0,t)\times\{s_0\} \subset \mathcal{A}^-\cup\mathcal{A}^0\right\}$.*

Proof. By the regularity assumptions on $\mathcal{G}$, the solution to (8.2) is unique. Since the constant function $\bar{s} \equiv s_0$ solves (8.2) in $[0,t_0)$ for every ν, then it results $s^{0,\nu}(t) = s_0$ for $t \in [0,t_0)$. Being s pointwise limit of the $s^{0,\nu}$ and left-continuous, we have the thesis. □

Proposition 8.2. *Assume there exist $\zeta > 0$ and a continuous function $\bar{\sigma} : [0,t_{\bar{\sigma}}] \to [s_0, L]$ such that $\bar{\sigma}$ is increasing, $\bar{\sigma}(0) = s_0$ and*

$$
\{(t,\bar{\sigma}(t)) \ : 0 < t < t_{\bar{\sigma}}\} \subset \mathcal{A}^0 \tag{8.3}
$$

$$
\{(t,\sigma) \ : 0 < t < t_{\bar{\sigma}}, \bar{\sigma}(t)-\zeta < \sigma < \bar{\sigma}(t)\} \subset \mathcal{A}^+ \tag{8.4}
$$

$$
\{(t,\sigma) \ : 0 < t < t_{\bar{\sigma}}, \bar{\sigma}(t) < \sigma < \bar{\sigma}(t)+\zeta\} \subset \mathcal{A}^-. \tag{8.5}
$$

Then $s(t) = \bar{\sigma}(t)$ for every $t \in [0,t_{\bar{\sigma}}]$.

Remark 8.3. If $\mathcal{G}$ is regular enough, then (8.3)-(8.5) imply that

$$
\partial_\sigma \mathcal{G}(\bar{\sigma}(t),\mathfrak{a}w(t)) < 0
$$

for every $t \in (0,t_{\bar{\sigma}})$. Therefore $\mathcal{E}$ is *convex* along $\bar{\sigma}$ and the lemma states that the vanishing viscosity evolution grows *continuously* where $\mathcal{E}$ is convex.

Proof. First we prove that $s^{0,\nu}(t) \leq \bar{\sigma}(t)$ for every $\nu > 0$ and $t \in [0,t_{\bar{\sigma}})$, so that, by pointwise convergence, $s(t) \leq \bar{\sigma}(t)$ for every $t \in [0,t_{\bar{\sigma}}]$.

By contradiction, assume that for some ν_0 there exists $t \in (0,t_{\bar{\sigma}})$ with $s^{0,\nu_0}(t) > \bar{\sigma}(t)$, and define $t_0 := \inf\left\{t \in (0,t_{\bar{\sigma}}) : \ s^{0,\nu_0}(t) > \bar{\sigma}(t)\right\}$. There exists $t_1 \geq t_0$ and $\delta > 0$ such that $s^{0,\nu_0}(t_1) = \bar{\sigma}(t_1)$, $s^{0,\nu_0}(t) > \bar{\sigma}(t)$ and $(t, s^{0,\nu_0}(t)) \in \mathcal{A}^-$ for every $t \in (t_1, t_1+\delta)$ (here we used (8.5)). Then, for $t \in (t_1,t_1+\delta)$, we have

$$
0 < \bar{\sigma}(t)-\bar{\sigma}(t_1) < s^{0,\nu_0}(t)-s^{0,\nu_0}(t_1) = \frac{1}{\nu\mathfrak{d}}\int_{t_1}^t \mathcal{V}\left(s^{0,\nu_0}(\tau),\tau\right)^+ \, d\tau = 0,
$$

which is a contradiction.

So far, we obtained that $s(t) \le \bar\sigma(t)$ for every $t \in [0, t_{\bar\sigma}]$. Defined

$$\bar t := \sup \{t \in [0, t_{\bar\sigma}] : \ s(\tau) = \bar\sigma(\tau) \text{ for every } 0 \le \tau \le t\},$$

the proof is complete if we show that $\bar t = t_{\bar\sigma}$.

By contradiction, assume that $\bar t < t_{\bar\sigma}$. The definition of $\bar t$ implies the existence of $\tilde t \in (\bar t, \bar t + \delta)$ such that $\bar\sigma(\tilde t) - \zeta < s(\tilde t) < \bar\sigma(\tilde t)$. Being s left continuous, for some $\tilde\delta > 0$ so that $\tilde t - \tilde\delta > \bar t$ and some $0 < \eta < \zeta$, the set

$$\mathcal{D} := \Big\{(t, \sigma) \in [0, t_{\bar\sigma}] \times [s_0, L) : \ \tilde t - \tilde\delta \le t \le \tilde t, s(\tilde t - \tilde\delta) - \eta \le \sigma \le s(t)\Big\}$$

satisfies $\mathcal{D} \subset\subset \mathcal{A}^+$. Therefore there exists $C > 0$ such that $\mathcal{V} \ge C$ on $\mathcal{D}$.

By pointwise convergence, there exists $\nu_0 > 0$ with $s^{0,\nu_0}(\tilde t - \tilde\delta) > s(\tilde t - \tilde\delta) - \eta$. Since the convergence of the $s^{0,\nu}$ to s is monotone with respect to ν, the chain of inequalities

$$s(\tilde t - \tilde\delta) - \eta < s^{0,\nu_0}(\tilde t - \tilde\delta) \le s^{0,\nu}(\tilde t - \tilde\delta) \le s^{0,\nu}(t) \le s(t)$$

shows that $(t, s^{0,\nu}(t)) \in \mathcal{D}$ for every $t \in [\tilde t - \tilde\delta, \tilde t]$ and $0 < \nu < \nu_0$. Then

$$s(\tilde t) - s(\tilde t - \tilde\delta) + \eta > s^{0,\nu}(\tilde t) - s^{0,\nu}(\tilde t - \tilde\delta) = \frac{1}{\nu\mathfrak{d}} \int_{\tilde t - \tilde\delta}^{\tilde t} \mathcal{V}(s^{0,\nu}(\tau), \tau)^+ \, d\tau \ge \frac{1}{\nu\mathfrak{d}} \tilde\delta C \to +\infty$$

as $\nu \to 0$, which is impossible.

Since the contradiction is due to the assumption $\bar t < t_{\bar\sigma}$, it must be $\bar t = t_{\bar\sigma}$, i.e. $s(t) = \bar\sigma(t)$ for every $t \in [0, t_{\bar\sigma}]$. □

In the next proposition, we set $\min \emptyset = +\infty$.

Proposition 8.4. *Assume there exists* $t \in (0, T]$ *such that* $(0, t) \times \{s_0\} \subset \mathcal{A}^+$ *and define*

$$\bar s := \min \big\{L, \min \big\{\mathcal{A}^0 \cap (\{0\} \times [s_0, L])\big\}\big\}.$$

Then

(1) *if* $\bar s = s_0$, *then there exists a continuous increasing function*

$$\bar\sigma : [0, t_{\bar\sigma}] \subset [0, T] \to [s_0, L)$$

with $\bar\sigma(0) = s_0$, *such that* $s = \bar\sigma$ *for* $t \in [0, t_{\bar\sigma}]$;

(2) *if* $s_0 < \bar s < L$, *let*

$$\bar\sigma : [0, t_{\bar\sigma}] \subset [0, T] \to [s_0, L]$$

be a monotone continuous function such that $\bar\sigma(0) = \bar s$ *and* $(t, \bar\sigma(t)) \in \mathcal{A}^0$ *for every* $t \in [0, t_{\bar\sigma}]$.

- *If* $\bar\sigma$ *is increasing, then* $s(t) = \bar\sigma(t)$ *for* $t \in (0, t_{\bar\sigma})$ *and* $s(0+) = \bar\sigma(0)$.
- *If* $\bar\sigma$ *is strictly decreasing, then* $s(t) = \bar\sigma(0)$ *for every* $t \in (0, t_0)$, *where*

$$t_0 := \sup \big\{t \in (0, T) : \ (0, t) \times \{\bar\sigma(0)\} \subset \mathcal{A}^- \cup \mathcal{A}^0\big\},$$

and $s(0+) = \sigma_{i_0}(0)$;

(3) *if* $\bar s = L$, *then* $s(t) = L$ *for every* $t \in (0, T]$ *and, consequently,* $s(0+) = L$.

Proof. **Case (1)**: being $\bar s = s_0$ and $(0, t) \times \{s_0\} \subset \mathcal{A}^+$, the regularity of $\mathcal{A}^0$ implies the existence of a branch $\bar\sigma : [0, t_{\bar\sigma}] \subset [0, T] \to [s_0, L]$ of $\mathcal{A}^0$ such that $\bar\sigma(0) = s_0$ and $\bar\sigma$ is increasing. Then the proof is the same as for Proposition 8.2, since the geometry around $\bar\sigma$ is described by (8.3)-(8.4)-(8.5).

Case (2): first of all observe that, around $\bar\sigma$, conditions (8.3)-(8.4)-(8.5) hold true for some $\zeta > 0$ and there exists $\hat t \le t_{\bar\sigma}$ such that

$$\mathcal{B} := \big\{(t, \sigma) : \ 0 \le t \le \hat t, s_0 \le \sigma < \bar\sigma(t)\big\} \subset \mathcal{A}^+.$$

Assume first that $\bar{\sigma}$ is strictly increasing. Arguing similarly to the first part of Proposition 8.2, we obtain that $s^{0,\nu}(t) \le \bar{\sigma}(t)$ for every $t \in [0, t_{\bar{\sigma}}]$, so that also $s \le \bar{\sigma}$ in the same interval. We now want to prove that the equality holds true.

By contradiction, assume there exists $\tilde{t} \in (0, t_{\bar{\sigma}})$ with $s(\tilde{t}) < \bar{\sigma}(\tilde{t})$. Suppose first that $\tilde{t} < \hat{t}$. By left continuity of s (see Lemma 7.7) and $\bar{\sigma}$, there exists a small $\delta > 0$ such that the set

$$\mathcal{D} := \big\{(t, \sigma) : \, t \in [\tilde{t} - \delta, \tilde{t}], s_0 \le \sigma \le s(t)\big\} \subset\subset \mathcal{A}^+,$$

and consequently $\mathcal{V} \ge C$ on $\mathcal{D}$ for some constant $C > 0$. Since $(t, s^{0,\nu}(t)) \in \mathcal{D}$ for every $t \in [\tilde{t} - \delta, \tilde{t}]$ and $\nu > 0$, it is

$$s(\tilde{t}) - s_0 > s^{0,\nu}(\tilde{t}) - s^{0,\nu}(\tilde{t} - \delta) = \frac{1}{\nu\mathfrak{d}} \int_{\tilde{t}-\delta}^{\tilde{t}} \mathcal{V}(s^{0,\nu}(\tau), \tau)^+ \, d\tau \ge \frac{1}{\nu\mathfrak{d}} C\delta \to +\infty$$

as $\nu \to 0$. This is a contradiction; therefore $s(t) = \bar{\sigma}(t)$ for $t \in (0, \hat{t})$ and $\tilde{t} \in [\hat{t}, t_{\bar{\sigma}})$. Arguing as in the second part of Proposition 8.2, we obtain again a contradiction. Hence we conclude that it is $s(t) = \bar{\sigma}(t)$ for every $t \in (0, t_{\bar{\sigma}})$ and $s(0+) = \bar{\sigma}(0)$.

If $\bar{\sigma}$ is strictly decreasing, first of all we show the following facts:

(2.i) there exists ν_0 such that for every $0 < \nu < \nu_0$ there exists $t_\nu \in (0, \hat{t})$ with $s^{0,\nu}(t_\nu) = \bar{\sigma}(t_\nu)$;

(2.ii) the sequence t_ν is monotonically converging to 0 as $\nu \searrow 0$.

By contradiction, assume that for every $\nu > 0$ there exists a smaller index $0 < \tilde{\nu} < \nu$ such that for every $t \in (0, \hat{t})$ it is $s^{0,\tilde{\nu}}(t) < \bar{\sigma}(t)$. (Observe that $s^{0,\tilde{\nu}}$ cannot be larger than $\bar{\sigma}$ in the interval $(0, \hat{t})$, otherwise by continuity they would coincide at some instant since $s^{0,\tilde{\nu}}(0) = s_0 < \bar{\sigma}(0)$). Therefore we obtain $s(t) \le \bar{\sigma}(t)$ for $t \in (0, \hat{t})$. Being s non-decreasing and $\bar{\sigma}$ strictly decreasing, it is $s(t) < \bar{\sigma}(t)$ for $t \in [0, \hat{t} - \delta]$ for some small $\delta > 0$. Consequently, for every $t \in [0, \hat{t} - \delta]$ and $0 < \nu < \nu_0$ it is

$$(t, s^{0,\nu}(t)) \in R := \big\{(t, \sigma) : \, 0 \le t \le \hat{t} - \delta, s_0 \le \sigma \le s(\hat{t} - \delta)\big\} \subset\subset \mathcal{A}^+.$$

Since $\mathcal{V} \ge C > 0$ on R for some constant C, we obtain the contradiction

$$s(\hat{t} - \delta) \ge s^{0,\nu}(\hat{t} - \delta) = s_0 + \frac{1}{\nu\mathfrak{d}} \int_0^{\hat{t}-\delta} \mathcal{V}(s^{0,\nu}(\tau), \tau)^+ \, d\tau \ge s_0 + \frac{1}{\nu\mathfrak{d}} (\hat{t} - \delta) C \to +\infty$$

as $\nu \to 0$. Hence (2.i) is proved.

Concerning (2.ii), firstly we show that, if $\nu_1 < \nu_2$, then $t_{\nu_1} \le t_{\nu_2}$. In fact, if it were $t_{\nu_1} > t_{\nu_2}$, then we would have

$$\bar{\sigma}(t_{\nu_1}) = s^{0,\nu_1}(t_{\nu_1}) \ge s^{0,\nu_1}(t_{\nu_2}) \ge s^{0,\nu_2}(t_{\nu_2}) = \bar{\sigma}(t_{\nu_2}) > \bar{\sigma}(t_{\nu_1}),$$

where the first inequality is due to the monotonicity of the $s^{0,\nu}$ and the second one to the fact that $s^{0,\nu_1} \ge s^{0,\nu_2}$.

Now we prove that $t_\nu \searrow 0$ as $\nu \searrow 0$. By contradiction, assume that $t_\nu \searrow \tilde{t} > 0$. For every $0 < \nu < \nu_0$ (ν_0 selected at step (2.i)) it is

$$s^{0,\nu}(\tilde{t}) \le s^{0,\nu}(t_\nu) = \bar{\sigma}(t_\nu)$$

and, taking the limit as $\nu \to 0$, we get $s(\tilde{t}) \le \bar{\sigma}(\tilde{t})$. Then, by monotonicity of both s and $\bar{\sigma}$, $s(\tilde{t}/2) < \bar{\sigma}(\tilde{t}/2)$. Being $0 < \tilde{t} \le \hat{t}$, for some $C > 0$ it is $\mathcal{V}(\sigma, t) \ge C$ for every $t \in [0, \tilde{t}/2]$ and $\sigma \in [s_0, s(\tilde{t}/2)]$. Repeating the same argument as before,

$$s(\tilde{t}/2) \ge s^{0,\nu}(\tilde{t}/2) = s_0 + \frac{1}{\nu\mathfrak{d}} \int_0^{\tilde{t}/2} \mathcal{V}(s^{0,\nu}(\tau), \tau)^+ \, d\tau \ge s_0 + \frac{1}{\nu\mathfrak{d}} \frac{\tilde{t}}{2} C \to +\infty$$

as $\nu \to 0$, which is a contradiction. Therefore $t_\nu \searrow 0$ as $\nu \searrow 0$, so that (2.ii) is proved as well.

To prove the claim in case (2), observe that the geometry of $\mathcal{A}^-$ in a neighbourhood of $(0,\bar\sigma(0))$ is the following: there exists $\tilde\tau>0$ such that

$$B := \{(t,\sigma) :\ 0<t<\tilde\tau, \bar\sigma(t)<\sigma<\bar\sigma(0)\} \subset \mathcal{A}^-.$$

For ν sufficiently small, $t_\nu<\tilde\tau$ and $s^{0,\nu}$ has the form

$$s^{0,\nu}(t) = \begin{cases} s_0 + \dfrac{1}{\nu\mathfrak{d}}\displaystyle\int_0^t \mathcal{V}(s^{0,\nu}(\tau),\tau)^+\,d\tau & \text{for } t\in[0,t_\nu) \\ s^{0,\nu}(t_\nu) & \text{for } t\in[t_\nu,t_0]. \end{cases}$$

In fact, with an argument similar to that in Proposition 8.1, the unique solution to the Cauchy problem

$$\begin{cases} \dot\varphi(t) = \dfrac{1}{\nu\mathfrak{d}}\mathcal{V}(\varphi(t),t)^+ \\ \varphi(t_\nu) = s^{0,\nu}(t_\nu) \end{cases}$$

is $\varphi\equiv s^{0,\nu}(t_\nu)$ in $[t_\nu,t_0]$. Consider $t<t_0$ and, by contradiction, let $s(t)<\bar\sigma(0)$, so that also $s^{0,\nu}(t)<\bar\sigma(0)$ for every ν. By (2.ii), $t_\nu\to 0$ and $s^{0,\nu}(t_\nu)=\bar\sigma(t_\nu)\to\bar\sigma(0)$. Hence for ν sufficiently small it is

$$t_\nu<t \quad\text{and}\quad s^{0,\nu}(t_\nu)>s^{0,\nu}(t),$$

which contradicts the monotonicity of $s^{0,\nu}$.

Hence $s(t)=\bar\sigma(t)$ for any $0<t<t_0$ and, by consequence, $s(0+)=\bar\sigma(0)$.

Case (3): assuming that $\bar s=L$, there are two possibilities:

(3.i) $\mathcal{A}^0\cap(\{0\}\times[s_0,L])=\emptyset$.

(3.ii) $\mathcal{A}^0\cap(\{0\}\times[s_0,L])=\{(0,L)\}$.

If (3.i) is the case, the set $\mathcal{A}^0\cap([0,T]\times(s_0,L))$ is far away from the σ-axis, so that there exists $\tilde t>0$ such that for any $0<\delta<\tilde t$ the set $[\delta,\tilde t]\times[s_0,L)\subset\mathcal{A}^+$. Fix $t\in(0,\tilde t)$. By contradiction, assume that $s(t)<L$. Since $[t/2,t]\times[s_0,s(t)]\subset\subset\mathcal{A}^+$, by continuity of $\mathcal{V}$ there exists $C:=C(t)>0$ such that $\mathcal{V}(\sigma,\tau)\ge C$ for every $\tau\in[t/2,t]$ and $\sigma\in[s_0,s(t)]$. Then, since $s_0\le s^{0,\nu}(t)\le s(t)$ for every $\nu>0$, we obtain

$$L-s_0>s(t)-s_0\ge s^{0,\nu}(t)-s^{0,\nu}(t/2)=\frac{1}{\nu\mathfrak{d}}\int_{t/2}^t \mathcal{V}(s^{0,\nu}(\tau),\tau)^+\,d\tau\ge\frac{1}{\nu\mathfrak{d}}C\frac{t}{2}\to+\infty$$

as $\nu\to 0$, which is a contradiction. We proved that $s(t)=L$ for every $t\in(0,\tilde t)$. Then $s(t)=L$ for every $t\in(0,T]$ and $s(0+)=L$.

In case (3.ii), there exists a monotone function $\bar\sigma:[0,T]\to[s_0,L]$ with $\bar\sigma(0)=L$. If $\bar\sigma\equiv L$, then the proof is the same as for (2) in case of an increasing function; if $\bar\sigma$ is strictly decreasing, argue as in (2) in case of a decreasing function. □

Remark 8.5. The above propositions provide a description of the evolution of the crack tip s up to as time $\tilde t\in(0,T]$. If $\tilde t=T$, then the function s is completely determined over $[0,T]$, otherwise we have to proceed with the analysis. We are not going to prove any further result since, up to modifying slightly the statements and the proofs of Propositions 8.1, 8.2, 8.4, the behaviour of the solution has similar characterizations taking $(\tilde t,s(\tilde t))$ as starting point instead of $(0,s_0)$.

Let us assume more regularity for the $\mathfrak{c}$-level set of $\mathcal{G}$, $\mathcal{A}^0$. In addition to being a C^1 manifold of dimension 1, we request that

- $\nabla\mathcal{V}(\sigma,t)\neq 0$ for every $(t,\sigma)\in\mathcal{A}^0$;
- the singular set

$$\mathcal{S}=\{(t,\sigma)\in[0,T]\times[s_0,L)\ :\ \partial_\sigma\mathcal{V}(\sigma,t)=0 \text{ or } \partial_t\mathcal{V}(\sigma,t)=0\}\cap\mathcal{A}^0$$

is finite.

Applying the Implicit Function Theorem, there exists a finite number of curves

$$\sigma_i : (t_{i,1}, t_{i,2}) \subset (0,T) \to [0,L], \quad i = 1, \dots, k,$$

such that

- for every i, σ_i is continuous and strictly monotone;
- for every i, the limits

$$\sigma_i(t_{i,1}) := \lim_{t \to (t_{i,1})^+} \sigma_i(t) \quad \text{and} \quad \sigma_i(t_{i,2}) := \lim_{t \to (t_{i,2})^-} \sigma_i(t)$$

  exist and are finite;
- set

$$\mathcal{A}_i^0 := \{(t, \sigma_i(t)) : t \in (t_{i,1}, t_{i,2})\}$$

  the graph of σ_i, it is $\mathcal{A}^0 = \cup_{i=1}^k \overline{\mathcal{A}_i^0}$, or, equivalently, $\mathcal{A}^0 \setminus \mathcal{S} = \cup_{i=1}^k \mathcal{A}_i^0$.

In addition, every σ_i verifies one of the following inequalities for every $t \in (t_{i,1}, t_{i,2})$:

$$\partial_\sigma \mathcal{G}(\sigma_i(t), \mathfrak{a}w(t)) < 0 \quad \text{or} \quad \partial_\sigma \mathcal{G}(\sigma_i(t), \mathfrak{a}w(t)) > 0.$$

We are describing a geometry similar to the one in Figure 2.

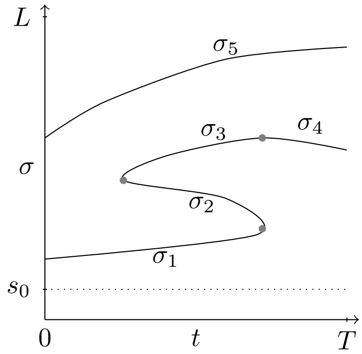


FIGURE 2. Plot of the $\mathfrak{c}$-level set of $\mathcal{G}$, $\mathcal{A}^0$, assuming it is a C^1 manifold of dimension 1, with finite singular set $\mathcal{S}$.

Propositions 8.1, 8.2 and 8.4 provide an "algorithmic" procedure. They can be quickly adapted to the geometry described above, providing a description of the evolution of s up to an instant $\tilde{t} \in (0,T]$. While Proposition 8.1 is still valid, the new statements for Propositions 8.2 and 8.4 are the following ones.

Proposition 8.6. *Assume there exists $i \in \{1, \dots, k\}$ such that σ_i is strictly increasing, $t_{i,1} = 0$, $\sigma_i(0) = s_0$ and $\partial_\sigma \mathcal{G}(\sigma_i(t), \mathfrak{a}w(t)) < 0$ for every t. Then $s(t) = \sigma_i(t)$ for every $t \in [0, t_{i,2}]$.*

Proposition 8.7. *Assume there exists $t \in (0,T]$ such that $(0,t) \times \{s_0\} \subset \mathcal{A}^+$ and define*

$$\bar{s} := \min \{L, \min \{\sigma_i(0) \ : \ 1 \le i \le k \text{ such that } t_{i,1} = 0, \sigma_i(0) \ge s_0, \partial_\sigma \mathcal{G}(\sigma_i(t), \mathfrak{a}w(t)) < 0\}\}.$$

Then

(1) *if $\bar{s} = s_0$, then $s(t) = \sigma_i(t)$ for every $t \in (0, t_{i,2})$, where $i \in \{1 \dots, k\}$ is such that $\sigma_i(0) = s_0$ and $\partial_\sigma \mathcal{G}(\sigma_i(t), \mathfrak{a}w(t)) < 0$;*

(2) *if $s_0 < \bar{s} < L$, set*

$$i_0 := \min \{1 \le i \le k : \ t_{i,1} = 0 \text{ and } \sigma_i(0) > s_0\}.$$

If σ_{i_0} *is strictly increasing, then* $s(t) = \sigma_{i_0}(t)$ *for* $t \in (0, t_{i_0,2})$ *and* $s(0+) = \sigma_{i_0}(0)$.
If σ_{i_0} *is strictly decreasing, then* $s(t) = \sigma_{i_0}(0)$ *for every* $t \in (0, t_0)$, *where*

$$t_0 := \sup\left\{t \in (0,T) :\ (0,t) \times \{\sigma_{i_0}(0)\} \subset \mathcal{A}^- \cup \mathcal{A}^0\right\},$$

and $s(0+) = \sigma_{i_0}(0)$;

(3) *if* $\bar{s} = L$, *then* $s(t) = L$ *for every* $t \in (0,T]$ *and, consequently,* $s(0+) = L$.

Here below, we present an example showing that the fracture evolution selected by the vanishing viscosity construction jumps later than the globally stable evolution obtained in [8]. We recall and use the example in [18, Section 7]. They deal with the antiplane 2-dimensional case with pre-assigned crack path $\Gamma = \gamma([-L, L])$ and monotone increasing loading $w(t, x) = t\psi(x)$ defined on the boundary $\partial\Omega$ of an open bounded set $\Omega \subset \mathbb{R}^2$. When considering the case of linearized elasticity and monotone increasing loadings $w(t, x) = t\psi(x)$, the bulk energy $E_d(\sigma, t)$ has the special form

$$E_d(\sigma, t) = t^2 E(\sigma) \tag{8.6}$$

where

$$E(\sigma) := \min\left\{\|\nabla u\|^2\ : u \in H^1(\Omega \setminus \Gamma_\sigma), u = \psi \text{ on } \partial_D\Omega\right\}$$

is the energy associated to the boundary loading $w(1, x) = \psi(x)$ and the crack $\Gamma_\sigma = \gamma([-L, \sigma])$. The quadratic dependence of E_d on t is due to the linear nature of the problem. The total energy is then given by

$$t^2 E(\sigma) + \sigma \tag{8.7}$$

where $E_s(\sigma) = \sigma$ is the crack energy (for convenience of esposition, we set the material toughness $\mathfrak{c}$ equal to 1).

In [18], they construct a boundary loading ψ and a domain Ω in such a way that the elastic energy functional

$$E : [s_0, L] \subset [-L, L] \to \mathbb{R}$$

is concave on some subinterval of $[s_0, L]$. In particular, for any $\eta > 0$ they consider the domain

$$\Omega^\eta = B_{-2} \cup T^\eta \cup B_2,$$

where B_{-2} and B_2 are the balls of radius 1 and center in $(-2, 0)$ and $(2, 0)$ respectively, $T^\eta = (-2 + \cos\eta, 2 - \cos\eta) \times (-\sin\eta, \sin\eta)$, and a proper boundary loading ψ_η on $\partial\Omega^\eta$. The crack path is $\Gamma = [-3, 3] \times \{0\}$. They assume the body to be fractured at time $t = 0$, with initial crack $[-3, -2] \times \{0\}$, and for $\sigma \in [-2, 3)$ they set

$$E_\eta(\sigma) := \min\left\{\|\nabla u\|^2\ : u \in H^1\left(\Omega^\eta \setminus ([-3, \sigma] \times 0)\right), u = \psi_\eta \text{ on } \partial\Omega^\eta\right\}.$$

In this setting (see the discussion for (8.6)-(8.7)), the total energy at time $t > 0$ for the crack $[-3, \sigma] \times \{0\}$ is

$$t^2 E_\eta(\sigma) + \sigma$$

and the function

$$\sigma \in [-2, 3) \mapsto E_\eta(\sigma) \in \mathbb{R}$$

is C^2. The sets $\mathcal{A}^0, \mathcal{A}^-, \mathcal{A}^+$, defined at the beginning of the section, now take the form

$$\begin{aligned}
\mathcal{A}^0_\eta &= \left\{(t,\sigma) \in [0,T] \times [-2,3]\ : -t^2 E'_\eta(\sigma) = 1\right\}\\
\mathcal{A}^-_\eta &= \left\{(t,\sigma) \in [0,T] \times [-2,3]\ : -t^2 E'_\eta(\sigma) < 1\right\}\\
\mathcal{A}^+_\eta &= \left\{(t,\sigma) \in [0,T] \times [-2,3]\ : -t^2 E'_\eta(\sigma) > 1\right\}.
\end{aligned}$$

In [18] the result on the concavity of E_η is achieved by showing the following three facts:

(i) $\limsup_{\eta\to 0^+} E_\eta(2)$ is finite;
(ii) $\liminf_{\eta\to 0^+} E_\eta(-2) = \infty$;
(iii) $\limsup_{\eta\to 0^+} E'_\eta(-2)$ is finite.

Consequently, along a suitable sequence $\eta_k \to 0^+$, it is

$$E_{\eta_k}(-2) + E'_{\eta_k}(-2)4 > E_{\eta_k}(2),$$

proving that E_{η_k} is necessarily concave in some subinterval of $[-2, 2]$.

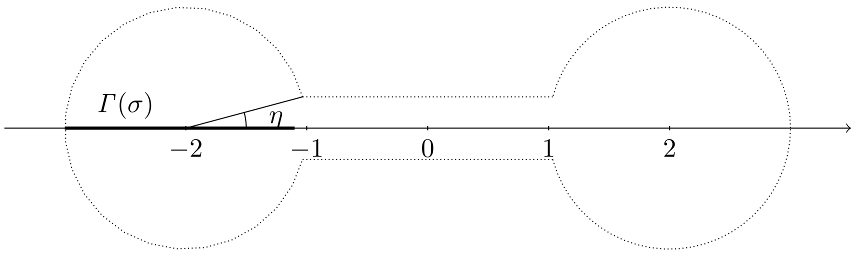


FIGURE 3. The domain Ω^η.

Let us call $E_0(\sigma)$ the elastic energy related to the case where $\Omega = B_{-2}$, the crack set is $[-3, \sigma] \times \{0\}$ for $\sigma \in (-3, -1]$ and the boundary loading is $\sin(\theta/2)$, θ being the angular coordinate between the x-axis and the center $(-2, 0)$ of B_{-2}. In [18], using firstly Irwin's formula relating the energy release rate and the stress intensity factor, and then an integral characterization for the last, they show that for $\sigma \in [-5/2, -3/2]$ it is

$$\limsup_{\eta \to 0^+} E'_\eta(\sigma) = E'_0(\sigma). \tag{8.8}$$

In order to make clear that $[-3, -2]$ is the initial crack, below we write $s_0 = -2$. Considering (i),(ii) and (8.8), take $\eta_0 > 0$ such that for any $0 < \eta < \eta_0$ (belonging to a proper subsequence)

$$E_\eta(s_0) + (E'_0(s_0) - 1)(2 - s_0) > E_\eta(2) \tag{8.9}$$

$$\left|E'_\eta(s_0) - E'_0(s_0)\right| < \frac{1}{2}. \tag{8.10}$$

By (8.10) and continuity of E'_η and E'_0, for any η there exists $s_\eta > s_0$ such that

$$\left|E'_\eta(\sigma) - E'_0(\sigma)\right| < \frac{1}{2} \tag{8.11}$$

for $\sigma \in [s_0, s_\eta]$.

As proved in [17], E_0 is convex in an interval $[s_0, s_1] \subset [s_0, L]$. Without loss of generality, we can assume $s_\eta \le s_1$. From (8.11) and convexity of E_0, we deduce

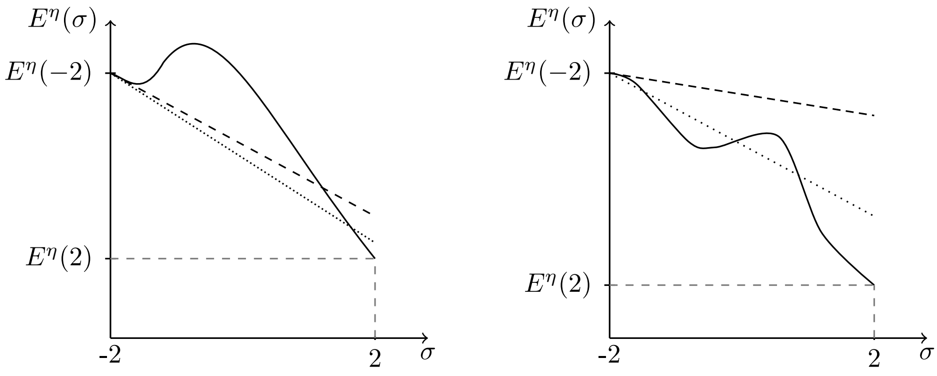
$$E'_\eta(\sigma) > E'_0(\sigma) - \frac{1}{2} \ge E'_0(s_0) - \frac{1}{2}$$

for $\sigma \in [s_0, s_\eta]$, so that the Lagrange theorem implies

$$E_\eta(\sigma) - E_\eta(s_0) = E'_\eta(\xi)(\sigma - s_0) \ge \left(E'_0(s_0) - \frac{1}{2}\right)(\sigma - s_0),$$

where the last inequality is due to the fact that $\xi \in [s_0, \sigma]$. Considering (8.9) too, we obtain

$$\begin{aligned} E_\eta(\sigma) + \left(\frac{1}{2} - E'_0(s_0)\right)\sigma \ge& E_\eta(s_0) + \left(\frac{1}{2} - E'_0(s_0)\right)s_0 \\ >& E_\eta(2) + (1 - E'_0(s_0))(2 - s_0) + \left(\frac{1}{2} - E'_0(s_0)\right)s_0 \\ =& E_\eta(2) + \left(\frac{1}{2} - E'_0(s_0)\right)2 + \frac{1}{2}(2 - s_0). \end{aligned}$$



(A) (B)

FIGURE 4. Plot of two different cases of the function E_η discussed in the example: in the figure (A), E_η is convex in a neighbouhood of -2, while in (B) it is concave. In principle, we do not know which is the situation, nevertheless for η small enough the request (8.9) is satisfied. The dotted line corresponds to the slope $E_0'(-2) - \frac{1}{2}$, while the dashed one is the tangent to E_η at -2, whose slope is larger than $E_0'(-2) - \frac{1}{2}$, according to (8.10).

Defined $t_0 > 0$ by

$$\frac{1}{t_0^2} = \frac{1}{2} - E_0'(s_0),$$

the above inequality becomes

$$E_\eta(\sigma) + \frac{\sigma}{t_0^2} > E_\eta(2) + \frac{2}{t_0^2} + \frac{1}{2}(2 - s_0) = E_\eta(2) + \frac{2}{t_0^2} + 2 \tag{8.12}$$

for $\sigma \in [s_0, s_\eta]$. The map

$$(t, \sigma) \mapsto E_\eta(\sigma) + \frac{\sigma}{t^2} - E_\eta(2) - \frac{2}{t^2}$$

is continuous in a neighbourhood of $\{t_0\} \times [s_0, L)$, thus by (8.12) we obtain

$$E_\eta(\sigma) + \frac{\sigma}{t^2} > E_\eta(2) + \frac{2}{t^2}$$

for every $t \in [t_\eta, t_0]$ and $\sigma \in [s_0, s_\eta]$, for some $t_\eta < t_0$.

Let $s_G : [0, T] \to [s_0, 3]$ be the globally stable quasi-static evolution. Since at each instant it has to satisfy the global minimality condition

$$t^2 E_\eta(s_G(t)) + s_G(t) \leq t^2 E_\eta(\sigma) + \sigma$$

for every $\sigma \geq \sup_{0 \leq t' < t} s_G(t')$, the discussion above shows that $s_G(t) > s_\eta$ for $t \in [t_\eta, T]$.

Consider now $\eta < \eta_0$ such that

$$E_\eta'(s_0) > E_0'(s_0) - \frac{1}{4}.$$

By choice of η_0, what we achieved above still holds true, in particular the result about the globally stable quasi-static evolution s_G. By continuity of E_η', there exists $s_0 < \bar{s}_\eta \leq s_\eta$ for which

$$E_\eta'(\sigma) > E_0'(s_0) - \frac{1}{2}$$

for $\sigma \in [s_0, \bar{s}_\eta]$. Then, when σ belongs to this interval and $t \in (0, t_0]$, it is

$$-E'_\eta(\sigma) < \frac{1}{2} - E'_0(s_0) = \frac{1}{t_0^2} \leq \frac{1}{t^2}.$$

In the formalism previously introduced, it is

$$[0, t_0] \times [s_0, \bar{s}_\eta] \subset \mathcal{A}^-_\eta .$$

Denoted by s_V the vanishing viscosity evolution, the analysis at the beginning of the section (Proposition 8.1) implies that $s_V(t) = s_0$ for $t \in [0, t_0]$.

Summarizing, we have shown the existence of a domain Ω^η and a boundary loading ψ_η for which the globally stable quasi-static evolution performs a crack jump strictly before the vanishing viscosity evolution. In fact, given the initial crack, $[-3, -2] \times \{0\} \subset \overline{\Omega^\eta}$, there exist $s_\eta \in (-2, L)$, $t_0 \in (0, T)$ and $t_\eta \in (0, t_0)$ such that

- the globally stable quasi-static evolution s_G belongs to $[-2, s_\eta]$ for $t \in [0, t_\eta]$ and jumps over s_η at t_η, i.e. $s_G(t) \in [-2, s_\eta]$ for $t \in [0, t_\eta]$ and $s_G(t) > s_\eta$ for $t \in (t_\eta, L]$
- any vanishing viscosity evolution s_V is constant on $[0, t_0]$, with $s_V(t) = -2$.

Appendix A. Technical result

Lemma A.1. *Let $f, f_j : [0, T] \to \mathbb{R}$ be non-decreasing monotone functions such that $f_j(t) \to f(t)$ for every $t \in [0, T]$. Let f be continuous at $\bar{t} \in [0, T]$. Then for every $t_j \to \bar{t}$ it is $f_j(t_j) \to f(\bar{t})$.*

Proof. Fix $\alpha > 0$. By continuity, there exists $\theta > 0$ such that $|f(t) - f(\bar{t})| < \alpha$ for every $|t - \bar{t}| < 2\theta$, $t \in [0, T]$.

Being $t_j \to \bar{t}$, there exists J_0 such that $|t_j - \bar{t}| < \theta$ for every $j > J_0$, so that

$$|f(t_j) - f(\bar{t})| < \alpha$$

for every $j > J_0$. By monotonicity, $f(\bar{t} - \theta) \leq f(t_j) \leq f(\bar{t} + \theta)$ for every $j > J_0$.

Pointwise convergence implies that there exists $J_1 \geq J_0$ such that

$$|f_j(\bar{t} - \theta) - f(\bar{t} - \theta)| < \alpha \quad \text{and} |f_j(\bar{t} + \theta) - f(\bar{t} + \theta)| < \alpha$$

for every $j > J_1$.

By continuity of f and the choice of θ, $|f(\bar{t}) - f(\bar{t} \pm \theta)| < \alpha$. Then by monotonicity and the above inequalities, we obtain

$$f(\bar{t}) - 2\alpha < f(\bar{t} - \theta) - \alpha < f_j(\bar{t} - \theta) \leq f_j(t_j) \leq f_j(\bar{t} + \theta) < f(\bar{t} + \theta) + \alpha < f(\bar{t}) + 2\alpha.$$

for every $j > J_1$. Being α arbitrary, the thesis follows. □

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