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Introduction

This thesis is devoted to the space BD of functions with bounded deformation and to its applications in variational problems. The space BD has been introduced by Matthies, Strang and Christiansen in [150] and has been widely studied in the last decades by various authors, for instance, Anzellotti-Giaquinta [37], Kohn [141], Suquet [186, 187], Kohn-Temam [142], Temam-Strang [192, 193] and Hardt-Kinderleher [137], in connection with the variational formulation of problems arising in the theory of plasticity.

Precisely, let $\Omega \subset \mathbb{R}^3$ be a bounded open set with locally Lipschitz boundary $\partial\Omega$. If $u : \Omega \to \mathbb{R}^3$ represents the displacement field of an elasto-plastic body whose reference configuration is Ω, the variational formulation of the equilibrium under a system of forces f and g involves the following variational problem:

$$(P) \qquad \inf_{u=u_0 \text{ on } \Gamma_D} \left\{ \int_\Omega \varphi\big(\mathcal{E}^D u(x)\big) dx + \frac{k}{2} \int_\Omega |\mathrm{div} u|^2 dx + \int_\Omega (f,u) dx + \int_{\Gamma_N} (g,u) ds \right\}$$

where $\Gamma_N \cup \Gamma_D = \partial\Omega$ and if M_s^0 is the space of symmetric matrices 3×3 with vanishing trace, $\varphi : M_s^0 \to [0, +\infty]$ is a convex function whose conjugate (in the duality theory of Fenchel-Moreau) is

$$\varphi^\star(\sigma) := \begin{cases} \frac{1}{4\mu}|\sigma|^2 & \text{if } \sigma \in K \\ +\infty & \text{otherwise} \end{cases}$$

and

$$K := \{\sigma \in M_s^0 \colon \, |\sigma| \leq k\}.$$

For any matrix A its projection on M_s^0 is given by $A^D := A - \frac{1}{3}(\mathrm{tr}A)I$; k is a positive constant and μ is one of the Lamé constants. This is the Hencky model of plasticity with the Von Mises' criterion. There are others models, for example the Tresca model corresponding to

$$K := \{\sigma \in M_s\colon\ \lambda_M(\sigma) - \lambda_m(\sigma) \leq C\}$$

where C is a fixed positive constant, $\lambda_M(\sigma)$ and $\lambda_m(\sigma)$ denote respectively the maximum and the minimum eigenvalue of σ.

The problem (P) presents formidable difficulties. For example, when plastification occurs on Γ_D, the boundary condition $u = u_0$ will not be satisfied. Also, from the mechanical standpoint the body Ω can exhibit sliding or slippages, these phenomena lead to the existence of internal surfaces of discontinuities for the displacement field. So, this excludes the possibility that the displacement fields u belong to a space of continuous functions or to a Sobolev space.

On the other hand, there is a formal analogy between the problem (P) and the problem of finding graphs of prescribed mean curvature

$$\inf_{\substack{v \in BV(\Omega) \\ v=\psi \text{ } on \text{ } \Gamma_D}} \left\{ \int_\Omega \sqrt{1+|Dv|^2} + \int_\Omega f v dx + \int_{\Gamma_N} h v ds \right\}$$

or more general, problems of the type

$$\inf_{\substack{v \in BV(\Omega) \\ v=\psi \text{ } on \text{ } \Gamma_D}} \left\{ \int_\Omega \phi(Dv) + \int_\Omega f v dx + \int_{\Gamma_N} h v ds \right\}$$

studied for example in Giaquinta-Modica-Souček [130], Giusti [131] with ϕ having a linear growth.

Very often minimization problems are proved to be well posed following Tonelli's direct methods, i.e., for (P) one looks for a function space and a suitable topology on it such that the functional

$$F(u) := \int_\Omega \varphi\Big(\mathcal{E}^D u(x)\Big)dx + \frac{k}{2}\int_\Omega |\mathrm{div} u|^2 dx + \int_\Omega (f,u)dx + \int_{\Gamma_N} (g,u)ds$$

is both lower semicontinuous and coercive. By the analogy above, one would think to the space $BV(\Omega, \mathbb{R}^3)$ of functions of bounded variation. More precisely, one attempt could be to consider the space

$$\mathcal{X}(\Omega) := BV(\Omega, \mathbb{R}^3) \cap \{u \in L^1(\Omega, \mathbb{R}^3)\colon \operatorname{div} u \in L^2(\Omega)\}.$$

Unfortunately, a counter-example given in Ornstein [170] (see Theorem 1.7.8), shows that no Korn's inequality is available in $W^{1,1}(\Omega, \mathbb{R}^3)$. So, the functional F is not coercive in $\mathcal{X}(\Omega)$. According to Matthies-Strang-Christiansen [150], in order to gain coerciveness, the functional F must be extended to account for displacement fields u with the only property that the strains $Eu := (Du + Du^t)/2$ are measures. This leads, in the general n-dimension case, to the introduction of the space $BD(\Omega)$ of functions with bounded deformation, defined by

$$BD(\Omega) := \left\{u \in L^1(\Omega, \mathbb{R}^n)\colon\ Eu = (Du + Du^t)/2 \in \mathcal{M}_b(\Omega, M_s^{n\times n})\right\},$$

where Du is the distributional derivative of u, A^t is the transpose of the matrix A and $\mathcal{M}_b(\Omega, M_s^{n\times n})$ is the space of Radon measures with finite total variation in Ω and taking values in the space of symmetric $n \times n$ matrices. Precisely, plasticity problems are studied in the following space

$$\mathcal{P}(\Omega) := \{u \in BD(\Omega)\colon \operatorname{div} u \in L^2(\Omega)\}.$$

A variety of properties of the space BD like the density of regular functions, the trace theorem and its consequences, the Poincaré type inequality, the embedding theorems established by many authors (see e.g., Temam [191]) leads to good weak formulations of plasticity problems and to the proof of existence of generalized solutions (see for instance, Anzellotti-Giaquinta [37], Suquet [186, 187] and Temam [191]). All those by now classical properties are presented here in the first chapter.

The space BD has also been devised by Anzellotti [35], Giaquinta-Giusti [129], Del Piero [107], Del Piero-Ru Chong [108] as a suitable function space in which one can study the well-posedness of the equilibrium problem related to masonry-like materials. These are described as structures only able to sustain very low tensile stresses. So, they easily exhibit fractures under tensile loads.

According to Di Pasquale [112], Del Piero [107], the mathematical formulation of those problems requires some constraints on the Cauchy stresses, which lead to a constitutive equation for the admissible stresses. The viewpoint taken by Anzellotti in [35], as well as by Giaquinta-Giusti in [129], is to characterize the equilibrium configuration for such materials by minimizing a non-coercive energy functional on a proper subspace of $BD(\Omega)$. Under suitable assumptions on Ω and on the class of external loads, existence theorems are proven in those papers.

Actually, our research on the space BD is motivated by recent results obtained on BV functions, in connection with the theory of *free discontinuity problems*. This terminology, introduced by De Giorgi in [98], indicated a family of variational problems involving both a volume and a surface energy. These model a variety of problems suggested by applied mathematics or mathematical physics, like the image segmentation problem proposed by Mumford & Shah in [164], problems in fracture mechanics (see for example Ambrosio-Braides [16], Griffith [135], Liebowitz [146], Rice [174]), the equilibrium configuration of liquid crystals studied by Ambrosio [11], De Giorgi-Ambrosio [99], Virga [196, 197].

A prototype of free discontinuity problems is the following

$$(P_1) \qquad \min\left\{\int_{\Omega\setminus K} f(x,\nabla u)dx + \lambda\mathcal{H}^{n-1}(K)\right\}$$

where Ω is a bounded open subset of $\mathbb{R}^n$, K varies in the class of closed subsets of $\bar{\Omega}$, u varies in $\mathcal{C}^1(\Omega\setminus K)$, λ is a fixed positive parameter and f is a Borel function. (P_1) is called a free discontinuity problem because the set K can be thought as (a set containing) the set of discontinuity points of u.

The application of the direct methods of the calculus of variation to the problem (P_1) presents many difficulties; for example, there is no topology on closed subsets ensuring at the same time the coercivity and the lower semicontinuity of the functional

$$F(u,K) := \int_{\Omega\setminus K} f(x,\nabla u)dx + \lambda\mathcal{H}^{n-1}(K).$$

So, there is a necessity of providing a weak formulation of the problem (P_1). To handle the difficulty described above, De Giorgi and Ambrosio introduced in [99] a class of discontinuous functions and proposed to replace the free surface K by the set of *discontinuity points* of these functions. The function space which seems more suitable for this kind of problems is the space $BV(\Omega,\mathbb{R}^k)$ of $\mathbb{R}^k$-valued *functions of bounded variation* in Ω. For such a function u, it is possible to define in a measure theoretic sense, the set of discontinuity points S_u, the orientation ν_u and both traces u^+ and u^- on the two sides of S_u. Moreover, the function u has an *approximate differential* ∇u almost everywhere in Ω. However, some BV functions (like the Cantor-Vitali function) have a pointwise behaviour non compatible with problems of the type (P_1). So, a proper subspace of *special functions of bounded variation*, $SBV(\Omega,\mathbb{R}^k)$ is introduced in [99] and for such a function u, the objects S_u, ν_u, u^+, u^- and ∇u describe completely the measure distributional derivative Du. Precisely,

$$Du = \nabla u\mathcal{L}^n\llcorner\Omega + (u^+ - u^-)\otimes\nu_u\mathcal{H}^{n-1}\llcorner S_u$$

where $\mathcal{L}^n$ and $\mathcal{H}^{n-1}$ denotes respectively the Lebesgue and the $(n-1)$-dimensional Hausdorff measures. In general, for any $u\in BV(\Omega,\mathbb{R}^k)$ we have the decomposition

$$Du = \nabla u\mathcal{L}^n\llcorner\Omega + (u^+ - u^-)\otimes\nu_u\mathcal{H}^{n-1}\llcorner S_u + Cu$$

where Cu, called the *Cantor part* of Du, is a measure singular with respect to $\mathcal{L}^n$ which does not charge sets which are σ-finite with respect to $\mathcal{H}^{n-1}$.

The problem (P_1) admits in $SBV(\Omega,\mathbb{R}^k)$ the following weak formulation

$$(P_2) \qquad \min\left\{\int_\Omega f(x,\nabla u)dx + \lambda\mathcal{H}^{n-1}(S_u)\right\}.$$

In the framework of this theory, general functionals of the type

$$(P') \qquad \int_\Omega f\Big(x, u(x), \nabla u(x)\Big)dx + \int_{S_u} g\Big(x, u^+(x), u^-(x), \nu_u(x)\Big) d\mathcal{H}^{n-1}(x)$$

have been considered, with g being some Borel function. Under some suitable growth and convexity assumptions on f and g, lower semicontinuity theorems and existence results for problems involving functionals of the type (P') have been proved by Ambrosio [9, 10, 11, 12], Ambrosio-Braides [16, 17]; in some cases it has been proved that any weak solution in SBV induces a minimizing pair (u, K) for the original problem (see for instance, De Giorgi-Carriero-Leaci [100] and Dal Maso-Morel-Solimini [86]).

Concerning fracture mechanics problems, if we deal with a linear elastic body, the function f is a quadratic form in ∇u depending only on $\mathcal{E}u = (\nabla u + \nabla u^t)/2$. So, the analogue of (P_1) is

$$(P_1') \qquad \min\left\{\int_{\Omega\setminus K} f(x, \mathcal{E}u(x))dx + \lambda\mathcal{H}^{n-1}(K)\right\}$$

where K and u represent respectively the fracture zone and the displacement field. The weak formulation of (P_1') in $SBV(\Omega, \mathbb{R}^n)$ will lead to

$$(P_2') \qquad \inf_{v\in SBV(\Omega,\mathbb{R}^n)}\left\{\int_\Omega f(x, \mathcal{E}u(x))dx + \lambda\mathcal{H}^{n-1}(S_u)\right\}$$

or in general,

$$(P_2'') \qquad \inf_{v\in SBV(\Omega,\mathbb{R}^n)}\left\{\int_\Omega f(x, \mathcal{E}u(x))dx + \int_{S_u} g\Big(x, u^+(x), u^-(x), \nu_u(x)\Big)d\mathcal{H}^{n-1}(x)\right\}$$

where $\mathcal{E}u$ indicates the symmetric part of the approximate gradient ∇u of u. However, the functionals in (P_2') or (P_2'') are not coercive in $SBV(\Omega, \mathbb{R}^n)$ since there is no control on the antisymmetric part of ∇u.

In order to provide suitable weak formulations of the problem (P_1') in the space $BD(\Omega)$, following the strategy of De Giorgi-Ambrosio [99], it has been necessary to study the fine properties of BD functions. Kohn [141], Bellettini-Coscia [45], Bellettini-Coscia-Dal Maso [46, 47], Hajłasz [136], Ambrosio-Coscia-Dal Maso [21] and Ebobisse [114] established various properties of this type. The second chapter of this thesis is a complete and self contained survey on this topic. For instance, we present the structure theorem 2.5.1 by Ambrosio-Coscia-Dal Maso [21], allowing to construct BD functions from suitable one-dimensional sections, we introduce in analogy with BV the space $SBD(\Omega)$ of *special functions with bounded deformation* and finally we conclude the chapter with a compactness criterion (Theorem 2.7.1) in $SBD(\Omega)$ proved by Bellettini-Coscia-Dal Maso [46, 47].

After, we study in Chapter III the local behaviour of BD functions, presenting two results related to their *approximate differentiability*. This argument was discussed for the first time by Hajłasz [136] and by Ambrosio-Coscia-Dal Maso [21]. We propose in the second part of this chapter another way of deducing the approximate differentiability, by a quantitative Lusin type approximation of BD functions by Lipschitz ones.

The chapter IV is devoted to the study of some functionals defined in $BD(\Omega)$ or in some subspaces of it. Using both classical and fine properties of BD functions presented in the previous chapters, we study first the lower semicontinuity of functionals with linear growth of the form

$$F(u) := \int_\Omega f\Big(\mathcal{E}u(x)\Big)dx$$

defined in the space $LD(\Omega) := \{u \in BD(\Omega)\colon\ Eu << \mathcal{L}^n\}$ under an interesting weak convexity property of f analogous, in this context, to the Morrey quasi-convexity. In the second part we present some lower semicontinuity and existence theorems in $SBD(\Omega)$ proved by Bellettini-Coscia-Dal Maso [46, 47]. Finally, we prove a new lower semicontinuity theorem for a class of functionals with linear growth defined in all the space $BD(\Omega)$.

We conclude this thesis by studying in Chapter V a homogenization problem involving a particular subspace of $BD(\Omega)$. Analogously to the results of the paper

[32], we study here the asymptotic behaviour of functionals of the type

$$F_\varepsilon(u,\Omega) := \int_\Omega f\Big(\frac{x}{\varepsilon}, \frac{dEu}{d\mu_\varepsilon}\Big)\, d\mu_\varepsilon$$

modelling, according to Bouchitté-Buttazzo-Seppecher [57] and Ambrosio-Buttazzo-Fonseca [20], some energies of lower dimensional or multi-dimensional structures. Using the theory of Γ-convergence we prove a homogenization theorem, obtaining an integral representation of the homogenized energy in the spaces $W^{1,p}(\Omega,\mathbb{R}^n)$ and $BD(\Omega)$, depending on the growth conditions on f.

Chapter 1

The Space $BD(\Omega)$: Classical Properties

1.1 Introduction

This chapter deals essentially with the preliminaries on the classical properties of functions with bounded deformation. Most of those properties have been established for example by Matthies-Christiansen-Strang [150], Anzellotti-Giaquinta [37], Moreau [157], Paris [172], Suquet [186, 187], Kohn [141], Temam-Strang [192, 193], Temam [191].

After introducing some notations and some fundamental tools on the space of Radon measures, we will give the structure properties of functions with bounded variation related to free discontinuity problems (see e.g., Ambrosio [9, 10, 11, 12], De Giorgi-Ambrosio [99], De Giorgi-Carriero-Leaci [100] and Alberti-Mantegazza [8]). The main results in this chapter will be the trace theorem and the Poincaré type inequalities for functions with bounded deformation and their consequences. Some of the results here will be given without proofs.

1.2 Preliminaries and notations

In all what follows, we will denote by $M^{n\times m}$ the space of $n\times m$ matrices, $M_s^{n\times n}$ the subspace of $n\times n$ symmetric matrices. Let A^t denote the transpose matrix of $A\in M^{n\times m}$. $(\cdot,\cdot)$ and $\|\cdot\|$ will respectively denote the euclidean scalar product and norm in $\mathbb{R}^n$ and in $M^{n\times n}$. $\|\cdot\|_p$ will be the norm in the space L^p for any $1\le p\le +\infty$. For any $(a,b)\in \mathbb{R}^n\times\mathbb{R}^n$, $a\otimes b$ and $a\odot b = 1/2(a\otimes b+b\otimes a)$ are respectively the tensor and symmetric products of a and b. We will denote by $B_\rho(x)$ the open ball of center x and radius $\rho>0$. $\mathcal{L}^n$ and $\mathcal{H}^k$ are respectively the Lebesgue measure in $\mathbb{R}^n$ and the k-dimensional Hausdorff measure. w_k denotes the k-dimensional Hausdorff measure of the unit ball. For any $A\subset\mathbb{R}^n$, $|A|$ will indicate the measure of A.

1.2.1 The space of Radon measures

Let Ω be an open subset of $\mathbb{R}^n$ and $\mathcal{B}(\Omega)$ the σ-algebra of Borel subsets of Ω. For any Borel measure μ on Ω with values in $\mathbb{R}^m$, we will indicate by $|\mu|$ the total variation of μ. For any $B\in\mathcal{B}(\Omega)$ $(\mu\llcorner F)(B):=\mu(F\cap B)$ defines the measure restriction of μ on the Borel subset F of Ω. For any function $f:\Omega\to\mathbb{R}^m$ μ-integrable, the measure $f\mu$ is defined by $(f\mu)(B):=\int_B f d\mu$ for every $B\in\mathcal{B}(\Omega)$. We will indicate by $\mathcal{M}_b(\Omega,\mathbb{R}^m)$ the space of $\mathbb{R}^m$-valued Radon measures on Ω with finite total variation. $\mathcal{M}_b(\Omega,\mathbb{R}^m)$ is a Banach space when equipped with the norm $\|\mu\|_b := |\mu|(\Omega)$. It is well known that $\mathcal{M}_b(\Omega,\mathbb{R}^m)$ is the dual of $\mathcal{C}_0(\Omega,\mathbb{R}^m)$, closure with respect to the uniform convergence of the space $\mathcal{C}_c(\Omega,\mathbb{R}^m)$ of continuous functions with compact support in Ω. $\mathcal{C}_0(\Omega,\mathbb{R}^m)$ is the space of $\mathbb{R}^m$-valued continuous functions in $\bar\Omega$ vanishing at the boundary. Following the duality above, a weak* topology is defined on $\mathcal{M}_b(\Omega,\mathbb{R}^m)$ by the so-called vague convergence of measures: we will say that a sequence of $\{\mu_j\}_j$ in $\mathcal{M}_b(\Omega,\mathbb{R}^m)$ converges weakly (or vaguely) to $\mu \in \mathcal{M}_b(\Omega, \mathbb{R}^m)$ if, for any $\varphi \in \mathcal{C}_0(\Omega, \mathbb{R}^m)$,

$$\lim_{j \to +\infty} \int_\Omega \varphi(x) d\mu_j(x) = \int_\Omega \varphi(x) d\mu(x).$$

Another interesting topology on $\mathcal{M}_b(\Omega, \mathbb{R}^m)$ is the one associated with the family of seminorms

$$|\int_\Omega \varphi d\mu|, \quad \varphi \in \mathcal{C}_0(\Omega, \mathbb{R}^m)$$

which defines the weak* topology and the pseudometric

$$d(\mu, \nu) = ||\mu|(\Omega) - |\nu|(\Omega)|.$$

This defines a topology intermediate between the weak* topology and the norm topology.

For more results about weak convergence of measures and its consequences we refer to Evans-Gariepy [118], Federer [119], Temam [191], and Ambrosio-Fusco-Pallara [26].

1.2.2 Rectifiable sets

Definition 1.2.1 *A subset K of $\mathbb{R}^n$ is called countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable (briefly rectifiable) if*

$$K = N \cup \bigcup_{i \in \mathbb{N}} K_i,$$

where $\mathcal{H}^{n-1}(N) = 0$ and $\{K_i\}$ is a sequence of compact sets such that K_i is contained in a $\mathcal{C}^1$ hypersurface Γ_i for any $i \in \mathbb{N}$.

For every rectifiable set K there exists a Borel function $\nu_K : K \to \mathbf{S}^{n-1}$ such that $\nu_K(x) = \nu_{\Gamma_i}(x)$ for $\mathcal{H}^{n-1}$-a.e. $x \in K_i$, where for any $i \in \mathbb{N}$, $\nu_{\Gamma_i}(x)$ denotes the unit vector normal to Γ_i at $x \in \Gamma_i$. Since for any pair of hypersurfaces Γ_i and Γ_j, $\nu_{\Gamma_i}(x) = \pm\nu_{\Gamma_j}(x)$ for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma_i \cap \Gamma_j$, then ν_K is well-defined. For $\mathcal{H}^{n-1}$-a.e. $x \in K$, the $(n-1)$-dimensional subspace $(\nu_K(x))^\perp$ is called the approximate

tangent space of K at x and is denoted by $\mathrm{Tan}^{(n-1)}(K,x)$.
For the general theory of rectifiable sets we refer to Federer [119], Simon [181], Morgan [159].

1.2.3 Lebesgue points, discontinuity points, jump points

Let $u \in L^1_{loc}(\Omega, \mathbb{R}^m)$ and $x \in \Omega$ be given.
(i) x is a *Lebesgue point* of u if there exist $\tilde{u}(x) \in \mathbb{R}^m$ such that

$$\lim_{\rho \to 0^+} \fint_{B_\rho(x)} |u(y) - \tilde{u}(x)| dy = 0. \tag{1.2.1}$$

The vector $\tilde{u}(x)$ is called the *approximate limit* of u at the point x. In general for any Borel function $v : \Omega \to \mathbb{R}^m$, $l \in \mathbb{R}^m$ is the approximate limit of v at x and we write $l = \mathrm{ap}\ \lim\limits_{y \to x} v(y)$, if for every $\varepsilon > 0$

$$\lim_{\rho \to 0^+} \frac{1}{\rho^n} |\{y \in B_\rho(x) \colon\ |v(y) - l| \geq \varepsilon\}| = 0.$$

Ω_u denotes the set of Lebesgue points of u and $S_u = \Omega \backslash \Omega_u$ is called the *discontinuity set* of u. By the Lebesgue's differentiation theorem, the set S_u is $\mathcal{L}^n$-negligible and the function $\tilde{u} : \Omega_u \to \mathbb{R}^m$ called the *Lebesgue representative* of u coincides with u $\mathcal{L}^n$ -almost everywhere in Ω_u.
(ii) We say that the function u has *one-sided Lebesgue limits* $u^+(x)$ and $u^-(x)$ at x, with respect to a suitable direction $\nu_u(x) \in \mathbf{S}^{n-1} := \{\xi \in \mathbb{R}^n \colon\ |\xi| = 1\}$, if

$$\lim_{\rho \to 0^+} \fint_{B^\pm_\rho(x, \nu_u(x))} |u(y) - u^\pm(x)| dy = 0, \tag{1.2.2}$$

where we have set $B^\pm_\rho(x, \nu_u(x)) := \{y \in B_\rho(x) \colon\ (y - x, \pm\nu_u(x)) > 0\}$. In this case the functions $u_\rho(y) := u(x + \rho y)$ converge in $L^1(B_1(0), \mathbb{R}^m)$ to the function

$$u_0(y) := \begin{cases} u^+(x) & \text{if } (y, \nu_u(x)) > 0 \\ u^-(x) & \text{if } (y, \nu_u(x)) < 0. \end{cases} \tag{1.2.3}$$

- If $u^+(x) \neq u^-(x)$ then the triplet $(u^+(x), u^-(x), \nu_u(x))$ is uniquely determined up to a change of sign of $\nu_u(x)$ and a simultaneous permutation of $u^+(x)$ and $u^-(x)$. In that case we say that x is a *jump point* of u and J_u indicates the set of jump points of u.

- If $u^+(x) = u^-(x)$ then $x \in \Omega_u$ and $\tilde{u}(x) = u^+(x) = u^-(x)$.

1.3 Functions of bounded variation

Definition 1.3.1 *We say that a function $u : \Omega \to \mathbb{R}^m$ is of bounded variation and we write $u \in BV(\Omega, \mathbb{R}^m)$ if $u \in L^1(\Omega, \mathbb{R}^n)$ and its distributional gradient Du is an $M^{n\times m}$-valued Radon measure in Ω with finite total variation $|Du|$.*

We denote $BV(\Omega) = BV(\Omega, \mathbb{R})$. The discontinuity set S_u of $u \in BV(\Omega, \mathbb{R}^m)$ is countably $(\mathcal{H}^{n-1}, n-1)$ rectifiable. Moreover, for $\mathcal{H}^{n-1}$-almost every $x \in S_u$, the function u has one-sided Lebesgue limits $u^+(x) \neq u^-(x)$ with respect to the approximate unit normal $\nu_{S_u}(x)$. Therefore, for $\mathcal{H}^{n-1}$-a.e. $x \in S_u$, $\nu_u(x) = \nu_{S_u}(x)$ and x is a jump point of u. That is $\mathcal{H}^{n-1}(S_u \setminus J_u) = 0$.
If $u \in BV(\Omega, \mathbb{R}^m)$, then the Radon-Nikodym decomposition of the measure Du gives

$$Du = D^a u + D^s u = \nabla u \mathcal{L}^n + D^s u,$$

where $D^a u$ and $D^s u$ are respectively the absolute continuous and the singular parts of Du with respect to the Lebesgue measure. The density ∇u of $D^a u$ with respect to $\mathcal{L}^n$ is interpreted as the *approximate differential* of u, i.e.,

$$\lim_{\rho\to 0^+} \fint_{B_\rho(x)} \frac{|u(y) - \tilde{u}(x) - (\nabla u(x), y - x)|}{|y - x|} dy \tag{1.3.1}$$

for $\mathcal{L}^{n-1}$-almost every $x \in \Omega$. The singular part $D^s u$ can be split into two mutually singular measures as

$$D^s u = D^j u + D^c u = (u^+ - u^-) \otimes \nu_u \mathcal{H}^{n-1} \llcorner S_u + D^c u, \tag{1.3.2}$$

where $D^j u = D^s u \llcorner S_u = Du \llcorner S_u$ is the $(n-1)$-dimensional part also called the *jump part* of Du and $D^c u = D^s u \llcorner \Omega_u$ is the *Cantor part* of Du with the property

$$|Du|(B) = 0 \quad \text{for any } B \in \mathcal{B}(\Omega) \quad \mathcal{H}^{n-1}\ \sigma\text{-finite}\ .$$

Definition 1.3.2 *We say that u is a special function of bounded variation in Ω with values in $\mathbb{R}^m$, and we write $u \in SBV(\Omega, \mathbb{R}^m)$, if $u \in BV(\Omega, \mathbb{R}^m)$ and $D^c u = 0$ in Ω.*

We will write $SBV(\Omega) = SBV(\Omega, \mathbb{R})$. For a general exposition on function with bounded variation we refer to Giusti [132], Ziemer [202], Evans-Gariepy [118], Ambrosio-Fusco-Pallara [26]; for the properties of the space $SBV(\Omega, \mathbb{R}^m)$, see for example, Ambrosio [9, 10, 11, 12], Pallara [171], Ambrosio-Fusco-Pallara [26]. Here we have just recalled some definitions and results that will be used in the sequel.

1.3.1 Structure theorem for BV functions

Before giving the theorem of reconstruction of BV functions from their one-dimensional sections, we will first present the space of one-dimensional BV functions and its equivalent definitions.

Pointwise variation and distributional derivative

Let $A \subset \mathbb{R}$ be open, $E \in \mathcal{B}(A)$ such that $\mathcal{L}^1(A \setminus E) = 0$ and $u : E \to \mathbb{R}^m$ integrable. Then u induces a distribution on A, whose derivative is denoted by Du. Let $I \subset A$ be an open interval, the *pointwise variation* $Vu(I)$ of u on I is defined by

$$Vu(I) := \sup \left\{ \sum_{i=1}^{k-1} |u(t_{i+1}) - u(t_i)| \, : \, t_1 < t_2 < \cdots < t_k,\ t_i \in E \cap I \right\}. \tag{1.3.3}$$

If $Vu(I) < +\infty$, then Du can be represented by a bounded measure on I, whose total variation satisfies

$$|Du|(I) \leq Vu(I). \tag{1.3.4}$$

Precisely

$$|Du|(I) = \min\{Vv(I)\colon\ v = u\ \mathcal{L}^1 - \text{a.e. in } I\}.$$

Moreover, if $Vu(I) < +\infty$ for every open interval $I \subset A$, then it is possible to extend Vu to a non-negative Radon measure defined on $\mathcal{B}(A)$, which is still denoted by Vu. In fact, we define for every open subset U of A

$$Vu(U) := \sum_{I \in \mathcal{CC}(U)} Vu(I) \tag{1.3.5}$$

where $\mathcal{CC}(U)$ denotes the set of connected components of U. It turns out that the set function $U \to Vu(U)$ is increasing, subadditive, additive on disjoint open sets, and inner regular, i.e., for any open set $O \subset A$

$$Vu(O) = \sup\{Vu(U)\colon\ U \text{ open},\ U \subset\subset O\}.$$

Then following a well-known result by De Giorgi-Letta [103] (see, e.g., Dal Maso [83, Theorem 14.23]),

$$Vu(B) := \inf\{Vu(U)\colon\ U \text{ open},\ B \subset U \subset A\} \quad \forall B \in \mathcal{B}(A).$$

defines a Radon measure on A which coincides with the function defined by (1.3.3) on all open intervals $I \subset A$. If $Vu(I) < +\infty$ for every open interval $I \subset A$, then there exists an extension v of u, defined on A, such that $Vu = Vv$ as measures on A. Therefore, applying the classical theory of functions of bounded variation on bounded intervals to v on each open interval $I \subset A$, we obtain that u has left and right limits respectively

$$u^+(t) = \lim_{\rho \to 0^+} \frac{1}{\rho} \int_t^{t+\rho} u(s)ds \quad \text{and} \quad u^-(t) = \lim_{\rho \to 0^+} \frac{1}{\rho} \int_{t-\rho}^{t} u(s)ds$$

at every $t \in A$ and these limits coincide except on the countable set

$$A_u := \{t \in A\colon\ Du(\{t\}) \neq 0\}.$$

Moreover, the classical derivative

$$u'(t) = \lim_{\substack{s\to t\\ s\in E}} \frac{u(s)-u(t)}{s-t}$$

exists for $\mathcal{L}^1$-almost every $t \in E$ and coincides with the density of the absolutely continuous part of the measure Du with respect to $\mathcal{L}^1$. So, the distributional derivative of a SBV function of one variable can be recovered throught the left and right limits and the classical derivative of its representatives. If $E = A$ and $u \in \mathcal{C}^1(A, \mathbb{R}^m)$ then one can easily verify that

$$Vu = |Du| = |u'|\mathcal{L}^1 \tag{1.3.6}$$

as measures on A.

One-dimensional sections of a set

For every $y, \xi \in \mathbb{R}^n$, $\xi \neq 0$ and $B \subset \mathbb{R}^n$ we define

$$\begin{aligned} \pi_\xi &:= \{y \in \mathbb{R}^n\colon\ (y,\xi) = 0\},\\ B_y^\xi &:= \{t \in \mathbb{R}\colon\ y + t\xi \in B\},\\ B^\xi &:= \{y \in \pi_\xi\colon\ B_y^\xi \neq \emptyset\}, \end{aligned} \tag{1.3.7}$$

respectively as the hyperplane orthogonal to ξ and passing through the origin, the one-dimensional section of B on the straight line passing through y in the direction ξ and the image of B by the orthogonal projection p_ξ onto π_ξ.

Measures obtained by integration

Let $\Omega_1 \subset \mathbb{R}^p$, $\Omega_2 \subset \mathbb{R}^q$ be two open sets and μ be a finite positive Radon measure on Ω_1. Let ν be the mapping which associates to every $x \in \Omega_1$ a Radon measure ν_x on Ω_2, such that $x \to \nu_x(A)$ is a Borel function for any open subset $A \subset \Omega_2$ and

$$\int_{\Omega_1} |\nu_x|(\Omega_2)d\mu(x) < +\infty.$$

Then we can define by integration with respect to μ a Radon measure on $\Omega_1 \times \Omega_2$ denoted by

$$\int_{\Omega_1} \nu_x d\mu(x)$$

and characterized by

$$\int_{\Omega_1 \times \Omega_2} f(x,y) d\left(\int_{\Omega_1} \nu_x d\mu(x)\right)(x,y) = \int_{\Omega_1}\int_{\Omega_2} f(x,y) d\nu_x(y) d\mu(x)$$

for any bounded Borel function $f : \Omega_1 \times \Omega_2 \to \mathbb{R}$. By Aumann's measurable selection theorem (see Castaing-Valadier [77, Theorem III.30]), the total variation of that measure is

$$\left|\int_{\Omega_1} \nu_x d\mu(x)\right| = \int_{\Omega_1} |\nu_x| d\mu(x). \tag{1.3.8}$$

In particular, we construct on the open subset $\Omega \subset \mathbb{R}^n$ a measure by integration with respect to the $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$. precisely, if for any $y \in \Omega^\xi$,

$$\nu_y : \mathcal{B}(\Omega^\xi_y) \to \mathbb{R}$$

is a Radon measure, the map $y \in \Omega^\xi \to \int_{\Omega^\xi_y} \varphi(t) d\nu_y(t)$ is a Borel function for any $\varphi \in \mathcal{C}_c(\mathbb{R})$ and

$$\int_{\Omega^\xi} |\nu_y|(\Omega^\xi_y) d\mathcal{H}^{n-1}(y) < +\infty,$$

then the Radon measure denoted by

$$\int_{\Omega^\xi} \nu_y d\mathcal{H}^{n-1}(y)$$

is defined on Ω.

Structure theorem for BV functions

Since BV functions of one variable are more manageable through their nice pointwise behaviour, in general in a BV setting, one-dimensional variational problems are more easier to study than their corresponding n-dimensional versions.

The following theorem on the reconstruction of BV functions from their one-dimensional sections is useful to reduce some problems in n-dimensional domains to one-dimensional problems.

Theorem 1.3.3 *Let $u : \Omega \to \mathbb{R}$, $\xi \in \mathbb{R}^n$, $\xi \neq 0$ and $y \in \Omega^\xi$. Let $u_{y,\xi} : \Omega^\xi_y \to \mathbb{R}$ be the function defined by $u_{y,\xi}(t) := u(y+t\xi)$ for any $t \in \Omega^\xi_y$. If for some orthonormal basis $\{\xi_1, \xi_2, \dots, \xi_n\}$ of $\mathbb{R}^n$*

$$u_{y,\xi_i} \in BV(\Omega^{\xi_i}_y) \text{ for } \mathcal{H}^{n-1}\text{-a.e. } y \in \Omega^{\xi_i} \text{ and}$$

$$\int_{\Omega^{\xi_i}} |Du_{y,\xi_i}|(\Omega^{\xi_i}_y) d\mathcal{H}^{n-1}(y) < +\infty$$

for any $i \in \{1, \dots, n\}$, then $u \in BV(\Omega)$.
Conversely, if $u \in BV(\Omega)$ then for $\mathcal{H}^{n-1}$-a.e. $y \in \Omega^\xi$, $u_{y,\xi} \in BV(\Omega^\xi_y)$ and

$$\int_{\Omega^\xi} |Du_{y,\xi}|(\Omega^\xi_y) d\mathcal{H}^{n-1}(y) = |D_\xi u|(\Omega) < +\infty.$$

Moreover,

$$\int_{\Omega^\xi} D^a u_{y,\xi} d\mathcal{H}^{n-1}(y) = (\nabla u, \xi)\mathcal{L}^n, \tag{1.3.9}$$

$$\int_{\Omega^\xi} D^j u_{y,\xi} d\mathcal{H}^{n-1}(y) = (D^j u, \xi), \tag{1.3.10}$$

$$\int_{\Omega^\xi} D^c u_{y,\xi} d\mathcal{H}^{n-1}(y) = (D^c u, \xi), \tag{1.3.11}$$

and for $\mathcal{H}^{n-1}$-a.e. $y \in \Omega^\xi$

$$\nabla u_{y,\xi}(t) = (\nabla u(y+t\xi), \xi) \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in \Omega^\xi_y, \tag{1.3.12}$$

$$S_{u_{y,\xi}} = \{t \in \Omega^\xi_y \colon y + t\xi \in S_u\}, \tag{1.3.13}$$

$$(u_{y,\xi})^+(t) = u^+(y+t\xi) \text{ and } (u_{y,\xi})^- = u^-(y+t\xi) \quad \forall t \in \Omega^\xi_y. \tag{1.3.14}$$

The structure theorem 1.3.3 is the main ingredient used by Ambrosio in [10, 13] for the proof of the following compactness criterion in $SBV(\Omega)$.

1.3.2 The compactness theorem in $SBV(\Omega)$

Let $\varphi : [0,+\infty[\to [0,+\infty]$, $\theta :]0,+\infty[\to [0,+\infty]$ be two functions, respectively convex and concave, such that

$$\lim_{t\to+\infty}\frac{\varphi(t)}{t}=+\infty, \quad \varphi \text{ is nondecreasing,} \tag{1.3.15}$$

$$\theta(+\infty)=\lim_{t\to+\infty}\theta(t),\ \lim_{t\to 0^+}\frac{\theta(t)}{t}=+\infty, \quad \theta \text{ is nondecreasing.} \tag{1.3.16}$$

Theorem 1.3.4 *Let Ω be a bounded open subset of $\mathbb{R}^n$. Let $\{u_h\}_{h\in\mathbf{N}}$ be a sequence in $SBV(\Omega,\mathbb{R}^m)$. Assume that*

$$\sup_{h\in\mathbb{N}}\left\{\int_\Omega \varphi(|\nabla u_h|)dx+\int_{S_{u_h}}\theta(|u_h^+-u_h^-|)d\mathcal{H}^{n-1}+\|u_h\|_\infty\right\}<+\infty.$$

Then there exists a subsequence $\{u_{h_k}\}_{k\in\mathbf{N}}$ and a function $u\in SBV(\Omega,\mathbb{R}^m)$ such that:

$$u_{h_k}\longrightarrow u \text{ strongly in } L^1(\Omega,\mathbb{R}^m),$$

$$\nabla u_{h_k}\rightharpoonup \nabla u \text{ weakly in } L^1(\Omega,M^{n\times m}),$$

$$\int_{S_u}\theta(|u^+-u^-|)d\mathcal{H}^{n-1}\le\liminf_{k\to+\infty}\int_{S_{u_{h_k}}}\theta(|u_{h_k}^+-u_{h_k}^-|)d\mathcal{H}^{n-1}.$$

1.4 The space $BD(\Omega)$ of functions with bounded deformation: classical properties

Before presenting the space $BD(\Omega)$ and its topological properties, we will first give some useful characterizations of the distributions which are deformation tensors.

1.4.1 Characterization of distributions which are deformation tensors

Given a vectorial distribution $u \in \mathcal{D}'(\Omega, \mathbb{R}^n)$, we denote by

$$Eu := (Du + D^t u)/2$$

the $M_s^{n\times n}$-valued distribution called the deformation tensor of u, where Du indicates the derivative of u. The subspace

$$\mathcal{R} := \{u \in \mathcal{D}'(\Omega, \mathbb{R}^n):\ Eu = 0\}$$

of infinitesimal rigid motion has the following useful characterization.

Lemma 1.4.1 *Let Ω be a connected open subset of $\mathbb{R}^n$. Then*

$$\mathcal{R} = \{u(x) = Ax + b;\ b \in \mathbb{R}^n \text{ and } A \text{ is an } n \times n \text{ skew-symmetric matrix}\}.$$

Proof. Let $u \in \mathcal{R}$. We set $w_{ij}(u) = \frac{1}{2}(D_j u_i - D_i u_j)$ and note that

$$\begin{aligned} D_k w_{ij} &= \frac{1}{2}(D_{kj} u_i - D_{ik} u_j) \\ &= \frac{1}{2}(D_{jk} u_i + D_{ij} u_k) - \frac{1}{2}(D_{ij} u_k + D_{ik} u_j) \\ &= D_j E_{ik} u - D_i E_{jk} u = 0 \quad \text{since } u \in \mathcal{R}. \end{aligned}$$

Therefore $D_k w_{ij}(u) = 0$ for all i, j, k. It follows that $D_{ik} u_j = D_k w_{ij}(u) = 0$, $\forall i, j, k$. Since Ω is connected, the functions u_j are affine, i.e., $u(x) = Bx + b$, where B is an

$n \times n$ matrix. Since $Eu = 0$, the matrix B is only reduced to its skew-symmetric part. $\square$

The following theorem proved by Moreau [157] in its general formulation (see also Temam-Strang [193]) characterizes those distributions which are tensors of deformation.

Theorem 1.4.2 *Let Ω be an open connected subset of $\mathbb{R}^n$. Let $\{E_{ij}\}_{i,j}$, be a matrix of distributions such that $E_{ij} = E_{ji}$. Then the following facts are equivalent:*

(i) *There exists a distribution $u \in \mathcal{D}'(\Omega, \mathbb{R}^n)$ such that $E_{ij} = E_{ij}u \quad \forall i, j$.*

(ii) *For any $\{\phi_{ij}\}_{i,j} \in (\mathcal{C}_c^\infty(\Omega))^{n^2}$ such that* $\mathrm{div}\phi = 0$, $\sum_{i,j=1}^{n} \langle E_{ij}, \phi_{ij} \rangle = 0$, *where $\langle \cdot, \cdot \rangle$ denotes the symbol of duality between $\mathcal{D}'(\Omega)$ and $\mathcal{C}_c^\infty(\Omega)$.*

For the reader's convenience we repeat here the proof of this theorem by Moreau [157]. In all the rest of this paragraph we shall adopt the Einstein summation convention in which repeated indices imply summation over the index range. The proof of Theorem 1.4.2 will be based on the following two lemmas.

Lemma 1.4.3 *If $\{E_{ij}\}_{ij} \subset \mathcal{D}'(\Omega, M_s^{n\times n})$ verifies the property (ii) of theorem 1.4.2, then*

$$\langle E_{ij,k} - E_{jk,i}, v_j \rangle = 0 \tag{1.4.1}$$

for any $v \in \mathcal{C}_c^\infty(\Omega, \mathbb{R}^n)$ with $\mathrm{div} v = 0$.

Proof. We want to prove that

$$\langle E_{ij}, v_{j,k} \rangle - \langle E_{jk}, v_{j,i} \rangle = 0. \tag{1.4.2}$$

Let us first remark that

$$\langle E_{ij}, v_{j,k} \rangle = \langle E_{lm}, \delta_i^l v_{m,k} \rangle$$

and

$$\langle E_{jk}, v_{j,i} \rangle = \langle E_{lm}, \delta_k^m v_{l,i} \rangle;$$

where δ_i^j denotes the Kronecker symbol. Since $\{E_{lm}\}$ is symmetric we have

$$\begin{aligned}\langle E_{ij}, v_{j,k}\rangle - \langle E_{jk}, v_{j,i}\rangle &= \langle E_{lm}, \delta_i^l v_{m,k} - \delta_k^m v_{l,i}\rangle \\ &= \langle E_{lm}, \mathrm{Sym}(\delta_i^l v_{m,k} - \delta_k^m v_{l,i})_{lm}\rangle \\ &= \frac{1}{2}\langle E_{lm}, S_{lm}\rangle\end{aligned}$$

where $\mathrm{Sym}(A) := (A + A^t)/2$ denotes the symmetric part of the matrix A and

$$S_{lm} := \delta_i^l v_{m,k} + \delta_i^m v_{l,k} - \delta_k^m v_{l,i} - \delta_k^l v_{m,i} \quad \text{for } k, i \text{ fixed.}$$

The matrix $\{S_{lm}\}_{lm} \subset \mathcal{C}_c^\infty(\Omega, M_s^{n\times n})$ is divergence free. In fact,

$$\begin{aligned}(\mathrm{div} S)_l = S_{lm,m} &= \delta_i^l v_{m,km} + \delta_i^m v_{l,km} - \delta_k^m v_{l,im} - \delta_k^l v_{m,im} \\ &= \delta_i^l (v_{m,m})_k + v_{l,ki} - v_{l,ik} - \delta_k^l (v_{m,m})_i \\ &= \delta_i^l (\mathrm{div} v)_k - \delta_k^l (\mathrm{div} v)_i = 0.\end{aligned}$$

Therefore $\langle E_{lm}, S_{lm}\rangle = 0$ by the property (ii) of Theorem 1.4.2. $\square$

Lemma 1.4.4 *If $\{E_{ij}\}_{i,j} \subset \mathcal{D}'(\Omega, M_s^{n\times n})$ verifies the property (ii) of Theorem 1.4.2, then*

$$\langle x_j(E_{ij,k} + E_{ik,j} - E_{kj,i}), v_k\rangle = 0 \tag{1.4.3}$$

for any $v \in \mathcal{C}_c^\infty(\Omega, \mathbb{R}^n)$ with $\mathrm{div} v = 0$, where x_j denotes the j^{th} component of the generic element x of $\mathbb{R}^n$ in the chosen orthonormal basis.

Proof. The identity (1.4.3) is equivalent to

$$\langle E_{ij}, (x_j v_k)_{,k}\rangle + \langle E_{ik}, (x_j v_k)_{,j}\rangle - \langle E_{kj}, (x_j v_k)_{,i}\rangle = 0. \tag{1.4.4}$$

Since $\langle E_{ij}, v_j\rangle = \langle E_{ki}, v_k\rangle = 0$ and $\{E_{lm}\}$ is symmetric, the first member of (1.4.4) is equal to

$$\langle E_{ij}, \delta_j^k v_k + x_j v_{k,k}\rangle + \langle E_{ik}, \delta_j^j v_k + x_j v_{k,j}\rangle - \langle E_{kj}, \delta_j^i v_k + x_j v_{k,i}\rangle$$

$$
\begin{aligned}
&= \langle E_{ij}, v_j\rangle + \langle E_{ik}, nv_k + x_j v_{k,j}\rangle - \langle E_{ki}, v_k\rangle - \langle E_{kj}, x_j v_{k,i}\rangle \\
&= \langle E_{ik}, nv_k + x_j v_{k,j}\rangle - \langle E_{kj}, x_j v_{k,i}\rangle \\
&= \langle E_{lm}, \delta_l^i \delta_m^k (nv_k + x_j v_{k,j}) - \delta_l^k \delta_m^j x_j v_{k,i}\rangle \\
&= \langle E_{lm}, \mathrm{Sym}(\delta_l^i \delta_m^k (nv_k + x_j v_{k,j}) - \delta_l^k \delta_m^j x_j v_{k,i})_{lm}\rangle \\
&= \frac{1}{2}\langle E_{lm}, S_{lm}\rangle
\end{aligned}
$$

where

$$S_{lm} := n\delta_l^i v_m + \delta_l^i x_j v_{m,j} - x_m v_{l,i} + n\delta_m^i v_l + \delta_m^i x_j v_{l,j} - x_l v_{m,i}.$$

The matrix $\{S_{lm}\}_{l,m} \subset \mathcal{C}_c^\infty(\Omega, M_s^{n\times n})$ is divergence free. In fact,

$$
\begin{aligned}
S_{lm,m} = {} & n\delta_l^i v_{m,m} + \delta_l^i \delta_j^m v_{m,j} + \delta_l^i x_j v_{m,jm} - nv_{l,i} - x_m v_{l,im} \\
& + n\delta_m^i v_{l,m} + \delta_m^i \delta_j^m v_{l,j} + \delta_m^i x_j v_{l,jm} - \delta_l^m v_{m,i} - x_l v_{m,im}.
\end{aligned}
$$

Since

$$v_{m,m} = \delta_j^m v_{m,j} = v_{m,jm} = v_{m,im} = 0,$$

then

$$S_{lm,m} = nv_{l,i} + v_{l,i} + x_j v_{l,ji} - nv_{l,i} - x_m v_{l,im} - v_{l,i} = 0.$$

Therefore $\langle E_{lm}, S_{lm}\rangle = 0$ by the point (ii) of Theorem 1.4.2 and this achieves the proof of the lemma. $\square$

Proof of Theorem 1.4.2. We only need to prove the implication $(ii) \Rightarrow (i)$. If $\{E_{ij}\}$ verifies (ii) then by Lemma 1.4.3 and by De Rham's theorem characterizing distributions which are gradients, we have for any i, k the existence of a distribution F_{ik} defined up to an additive constant such that

$$F_{ik,j} = E_{ij,k} - E_{jk,i}$$

On the other hand $(F_{ik} + F_{ki})_{,j} = E_{ij,k} - E_{jk,i} + E_{kj,i} - E_{ji,k} = 0$. Then $F_{ik} + F_{ki}$ is a constant distribution in Ω. We can always assume that the constants obtained in the construction of F_{ik} are such that $F_{ik}+F_{ki}=0$. Let us prove that there exists $u\in\mathcal{D}'(\Omega,\mathrm{I\!R}^n)$ such that $u_{i,k}=E_{ik}+F_{ik}$. We remark that

$$E_{ik}+F_{ik}=(x_j(E_{ij}+F_{ij}))_{,k}-x_j(E_{ij,k}+F_{ij,k}). \tag{1.4.5}$$

Hence we only need to prove that the terms $x_j(E_{ij,k}+F_{ij,k})$ are gradient components of an element of $\mathcal{D}'(\Omega,\mathrm{I\!R}^n)$. Following the De Rham's theorem it is sufficient to prove that

$$\langle x_j(E_{ij,k}+F_{ij,k}),v_k\rangle=0$$

For any $v\in\mathcal{C}_c^\infty(\Omega,\mathrm{I\!R}^n)$ with $\mathrm{div}v=0$. This follows from Lemma 1.4.4 since

$$E_{ij,k}+F_{ij,k}=E_{ij,k}+E_{ik,j}-E_{kj,i}.$$

Then $x_j(E_{ij,k}+F_{ij,k})=w_{i,k}$ for some $w\in\mathcal{D}'(\Omega,\mathrm{I\!R}^n)$. Therefore setting

$$u_i=x_j(E_{ij}+F_{ij})+w_i$$

we obtain $E_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i})$ and this achieves the proof of Theorem 1.4.2. □

Corollary 1.4.5 *Let $\{u_h\}_h\subset\mathcal{D}'(\Omega,\mathrm{I\!R}^n)$. If $E_{ij}u_h$ converges in $\mathcal{D}'(\Omega)$ to E_{ij}, then there exists $u\in\mathcal{D}'(\Omega,\mathrm{I\!R}^n)$ such that $E_{ij}u=E_{ij}\quad\forall i,j$.*

The following proposition on the regularity of the distributions whose tensor of deformation is a Radon measure with finite total variation is similar to the characterization by Miranda [153] of a distribution whose gradient is a measure.

Proposition 1.4.6 *Let $\Omega\subset\mathrm{I\!R}^n$ be open connected and $u\in\mathcal{D}'(\Omega,\mathrm{I\!R}^n)$ such that $E_{ij}(u)\in\mathcal{M}_b(\Omega)$ $\forall i,j$. Then $u\in L^1_{loc}(\Omega,\mathrm{I\!R}^n)$. Moreover, if Ω is bounded with locally Lipschitz boundary, then $u\in L^1(\Omega,\mathrm{I\!R}^n)$.*

For the proof of this proposition we refer for example to Temam-Strang [193], Temam [191] and Paris [172].

In the sequel Ω will be an open connected subset of $\mathbb{R}^n$. Let $u \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ and $\varphi \in \mathcal{C}_c^\infty(\Omega)$. Then

$$\langle E_{ij}u, \varphi\rangle = -\frac{1}{2}\int_\Omega (u_i \frac{\partial \varphi}{\partial x_j} + u_j \frac{\partial \varphi}{\partial x_i})dx.$$

For any open subset A of Ω we set

$$\int_A |Eu| := \sup\left\{\int_A (u, \mathrm{div}\varphi)dx\colon\ \varphi \in \mathcal{C}_c^\infty(\Omega, \mathbb{R}^{n^2}),\ \mathrm{supt}(\varphi) \subset A \text{ and } \sum \varphi_{ij}^2 \leq 1\right\}.$$

Then Eu is an $M_s^{n\times n}$-valued Radon measure in Ω if and only if

$$\int_A |Eu| < +\infty \quad \text{any open set } A \subset\subset \Omega.$$

In that case the application

$$B \to |Eu|(B) := \inf\left\{\int_A |Eu|\colon\ A \text{ open},\ B \subset A \subset \Omega\right\} \quad B \in \mathcal{B}(\Omega)$$

defines the total variation of Eu.

1.4.2 The space $BD(\Omega)$

Definition 1.4.7 *We say that $u : \Omega \to \mathbb{R}^n$ is a function with bounded deformation in Ω, and we write $u \in BD(\Omega)$ if $u \in L^1(\Omega, \mathbb{R}^n)$ and $Eu \in \mathcal{M}_b(\Omega, M_s^{n\times n})$.*

For every $\xi = (\xi^1, \cdots, \xi^n) \in \mathbb{R}^n$, let D_ξ be the distributional derivative in the direction ξ defined by $D_\xi v = (Dv, \xi)$. For every $u : \Omega \to \mathbb{R}^n$ we define $u^\xi : \Omega \to \mathbb{R}$ by $u^\xi(x) := (u(x), \xi)$.

Theorem 1.4.8 (i) *Let u be in $BD(\Omega)$. Then $D_\xi u^\xi$ is a bounded Radon measure for every $\xi \in \mathbb{R}^n$ and $D_\xi u^\xi = (Eu\xi, \xi)$.*

(ii) *Conversely, let $\xi_1 \cdots, \xi_n$ be a basis of $\mathbb{R}^n$ and $u \in L^1(\Omega, \mathbb{R}^n)$, then the function u is in $BD(\Omega)$ if $D_\xi u^\xi$ is a bounded Radon measure on Ω for every ξ of the form $\xi_i + \xi_j \quad i, j = 1, \cdots, n$.*

1.4.3 The topologies in $BD(\Omega)$

$BD(\Omega)$ is a Banach space when equipped with the norm

$$\|u\|_{BD(\Omega)} := \|u\|_{L^1(\Omega,\mathbb{R}^n)} + \|Eu\|_b = \|u\|_1 + |Eu|(\Omega). \tag{1.4.6}$$

The space $\mathcal{C}^\infty(\bar{\Omega},\mathbb{R}^n)$ is not dense in $BD(\Omega)$ with respect to the norm topology. In other words the $BD(\Omega)$-strong closure of $\mathcal{C}^\infty(\bar{\Omega},\mathbb{R}^n)$, denoted by $LD(\Omega)$ is a proper subset of $BD(\Omega)$.
Two other topologies are defined on $BD(\Omega)$. The first one called weak topology is defined by the following family of norm and semi-norms

. $\|u\|_{L^1(\Omega,\mathbb{R}^n)}$

. $|\langle E_{ij}u,\varphi\rangle| \quad \varphi \in \mathcal{C}_c^\infty(\Omega) \quad i,j = 1,\cdots,n.$

Then a sequence $\{u_h\}_{h\in\mathbb{N}}$ converges weakly to u in $BD(\Omega)$ if and only if

$$\begin{cases} u_h \longrightarrow u \text{ in } L^1(\Omega,\mathbb{R}^n) \quad \text{and} \\ E_{ij}u_h \overset{\star}{\rightharpoonup} E_{ij}u \text{ in } \mathcal{M}_b(\Omega) \quad \text{for any } i,j = 1,\cdots,n. \end{cases}$$

The second topology called *intermediate topology* is the one induced by the following distance

$$d(u,v) := \|u-v\|_{L^1(\Omega,\mathbb{R}^n)} + \big|\,|Eu|(\Omega) - |Ev|(\Omega)\big|.$$

It is clear that this topology is intermediate between the weak topology and the topology of the norm on $BD(\Omega)$.

The weak topology defined above is actually a weak* topology since following a result by Matthies-Strang-Christiansen [150] (see also Temam-Strang [193]), the space $BD(\Omega)$ is up to an isomorphism the dual of some function space.

Theorem 1.4.9 *Assume that Ω is bounded with locally Lipschitz boundary $\partial\Omega$. Let*

$$\mathcal{X} := \{f = (g,h) \,:\, g \in \mathcal{C}_c(\Omega, \mathbb{R}^n),\ h \in \mathcal{C}_c(\Omega, M_s^{n\times n})\}.$$

Then the space $BD(\Omega)$ is continuously isomorphic to the dual of $\mathcal{X}/\mathcal{X}_0$, where

$$\mathcal{X}_0 := \{(g,h) \in \mathcal{X} \,:\, g = div\, h\}$$

and $\mathcal{X}/\mathcal{X}_0$ denotes the quotient space of $\mathcal{X}$ by $\mathcal{X}_0$.

Proof. Let $\mathcal{Y} := \{f^* = (g^*, h^*) \,:\, g^* \in \mathcal{M}_b(\Omega, \mathbb{R}^n),\ h^* \in \mathcal{M}_b(\Omega, M_s^{n\times n})\}$. It's clear that $\mathcal{X}/\mathcal{X}_0$ is continuously isomorphic to the annihilator

$$\mathcal{Y}_0 := \{f^* \in \mathcal{Y} : \langle f^*, f\rangle = 0\}.$$

Then it only remains to prove that $BD(\Omega)$ is continuously isomorphic to $\mathcal{Y}_0$. If we consider $f = (0,h) \in \mathcal{X}_0$, then $f^* \in \mathcal{Y}_0$ satisfies

$$\sum_{i,j=1}^{n} \langle h_{ij}^*, h_{ij}\rangle = 0 \qquad \text{if } h_{ij} = h_{ji} \text{ and } \operatorname{div} h = 0.$$

Then following Theorem 1.4.2, there exists $v \in \mathcal{D}'(\Omega, \mathbb{R}^n)$ such that $h_{ij}^* = E_{ij}v$. In general, the condition $\langle f^*, f\rangle = 0$ for $f^* \in \mathcal{Y}_0$ and $f \in \mathcal{X}_0$ gives successively

$$\langle g^*, \operatorname{div} h\rangle + \langle Ev, h\rangle = 0,$$

$$\langle E(v - g^*), h\rangle = 0.$$

Therefore, $v - g^*$ is a rigid motion r. Setting $u := v - r = g^* \in \mathcal{M}_b(\Omega, \mathbb{R}^n)$, then the distribution $E_{ij}u = h_{ij}^* \in \mathcal{M}_b(\Omega)$. Therefore, following Proposition 1.4.6, $u \in BD(\Omega)$ and the map from $f^* \in \mathcal{Y}_0$ to u such that $f^* = (u, Eu)$ provides the desired continuous isomorphism. $\square$

1.5 Approximation in $BD(\Omega)$

The following approximation theorem is similar to the one by Anzellotti-Giaquinta [36] for BV functions (see also Ambrosio-Mortola-Tortorelli [27] for the vectorial case).

Theorem 1.5.1 *The space $\mathcal{C}^\infty(\bar{\Omega}, \mathbb{R}^n)$ is dense in $BD(\Omega)$ with respect to the intermediate topology.*

For the proof of Theorem 1.5.1 we refer the reader to Temam [191].

1.6 The trace operator for BD functions

Theorem 1.6.1 *Let $\Omega \subset \mathbb{R}^n$ be bounded open with locally Lipschitz boundary Γ. Then there exists a surjective bounded linear operator*

$$\gamma : BD(\Omega) \longrightarrow L^1(\Gamma, \mathbb{R}^n)$$

such that

(i) $\gamma(u) = u|_\Gamma$ *for any* $u \in BD(\Omega) \cap \mathcal{C}(\bar{\Omega}, \mathbb{R}^n)$.

(ii) $\displaystyle\lim_{\rho \to 0^+} \frac{1}{\rho^n} \int_{B_\rho(x) \cap \Omega} |u(y) - \gamma(u)(x)| dy = 0$ *for* $\mathcal{H}^{n-1} - a.e.\ x \in \Gamma$.

(iii) (*generalized Green's formula*)*: For any* $\varphi \in \mathcal{C}^1(\Omega, M_s^{n \times n}) \cap \mathcal{C}(\bar{\Omega}, M_s^{n \times n})$,

$$\int_\Omega (u, div\ \varphi) dx + \int_\Omega \varphi dEu = \int_\Gamma (\varphi, \gamma(u) \odot \nu) d\mathcal{H}^{n-1},$$

where ν denotes the outer unit normal vector of Γ.

Moreover, the operator γ is still continuous when $BD(\Omega)$ is equipped with the intermediate topology.

1.6.1 Consequences of the trace theorem

Theorem 1.6.2 *Let Ω be a Lipschitz bounded open subset of $\mathbb{R}^n$ and u a function in $L^1(\mathbb{R}^n, \mathbb{R}^n) \cap BD(\mathbb{R}^n \setminus \partial\Omega)$, then $u \in BD(\mathbb{R}^n)$. Moreover, if u^-, u^+ are respectively the traces of $u|_\Omega$ and $u|_{\mathbb{R}^n \setminus \bar{\Omega}}$ then*

(i) $Eu(\partial\Omega) = \int_{\partial\Omega} (u^+ - u^-) \odot \nu d\mathcal{H}^{n-1}$.

(ii) $|Eu|(\partial\Omega) = \int_{\partial\Omega} |(u^+ - u^-) \odot \nu| d\mathcal{H}^{n-1}$.

(iii) *There exists a positive constant $C(\Omega)$ such that*

$$\left\| u^+ - u^- \right\|_{L^1(\partial\Omega, \mathbb{R}^n)} \leq C(\Omega) |Eu|(\partial\Omega),$$

where ν is the field of outward unit vectors normal to $\partial\Omega$.

Proof. (i) Write the generalized Green's formula for $u|_\Omega$ and $u|_{\mathbb{R}^n \setminus \bar{\Omega}}$ with $\varphi \equiv 1$ and sum.
(ii) We write the generalized Green's formula for $u|_\Omega$, $u|_{\mathbb{R}^n \setminus \bar{\Omega}}$ and for

$$\{\varphi_{ij}\} \subset \mathcal{C}_c^\infty(\mathbb{R}^n, M^{n \times n}) \quad \text{with} \quad \sum_{i,j=1}^{n} \varphi_{ij}^2 \leq 1.$$

In fact,

$$\int_\Omega \varphi dEu + \int_\Omega (u, \operatorname{div}\varphi) dx = \int_{\partial\Omega} (\varphi, u^+ \odot \nu) d\mathcal{H}^{n-1}$$

$$\int_{\mathbb{R}^n \setminus \bar{\Omega}} \varphi dEu + \int_{\mathbb{R}^n \setminus \Omega} (u, \operatorname{div}\varphi) dx = -\int_{\partial\Omega} (\varphi, u^- \odot \nu) d\mathcal{H}^{n-1}$$

and summing we obtain

$$\int_{\mathbb{R}^n} (u, \operatorname{div}\varphi) dx = -\int_\Omega \varphi dEu - \int_{\mathbb{R}^n \setminus \bar{\Omega}} \varphi dEu - \int_{\partial\Omega} (\varphi, (u^+ - u^-) \odot \nu) d\mathcal{H}^{n-1}. \tag{1.6.1}$$

Then it follows that Eu is a Radon measure in $\mathbb{R}^n$ with

$$|Eu|(\mathbb{R}^n) \leq |Eu|(\Omega) + |Eu|(\mathbb{R}^n \setminus \bar{\Omega}) + \int_{\partial\Omega} |(u^+ - u^-) \odot \nu| d\mathcal{H}^{n-1}.$$

Therefore $u \in BD(\mathbb{R}^n)$ and $|Eu|(\partial\Omega) \le \int_{\partial\Omega} |(u^+ - u^-) \odot \nu| d\mathcal{H}^{n-1}$. In order to prove the reverse inequality, we consider for any $h > 0$ the open set

$$U_h := \{x \in \mathbb{R}^n;\ d(x, \partial\Omega) < \frac{1}{h}\}$$

and $\{\varphi_{ij}^h\} \subset \mathcal{C}_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$ with $\sum_{i,j=1}^n (\varphi_{ij}^h)^2 \le 1$, $\operatorname{spt}(\varphi_{ij}^h \subset U_h$ and

$$\varphi_{ij}^h \to \frac{(u^+ - u^-) \odot \nu_{ij}}{|(u^+ - u^-) \odot \nu|} \quad \text{in } L^1(\partial\Omega).$$

Then we write (1.6.1) for φ^h and obtain

$$\begin{aligned}
&\int_{\partial\Omega} (\varphi^h, (u^+ - u^-) \odot \nu) d\mathcal{H}^{n-1} \\
&= \int_{U_h} \varphi^h dEu + \int_{\Omega \cap U_h} \varphi^h dEu + \int_{(\mathbb{R}^n \setminus \bar{\Omega}) \cap U_h} \varphi^h dEu \\
&\le |Eu|(U_h) + |Eu|(\Omega \cap U_h) + |Eu|(U_h \cap (\mathbb{R}^n \setminus \bar{\Omega})).
\end{aligned}$$

Now letting h goes to $+\infty$ we obtain

$$\int_{\partial\Omega} |(u^+ - u^-) \odot \nu| d\mathcal{H}^{n-1} \le |Eu|(\partial\Omega)$$

and this proves (ii).

(iii) We prove (iii) working locally. Let $\Gamma \subset \partial\Omega$ be a Lipschitz graph over $\xi^\perp$ for some $\xi \in \mathbf{S}^{n-1}$. It is easy to see that $|(\alpha, \nu)|$ is bounded away from 0 if $\alpha \in \mathbf{S}^{n-1}$ is closed enough to ξ. Now if we fix α closed to ξ and we apply the generalized Green's formula for $\varphi = w(\alpha \otimes \alpha)$ with $w \in \mathcal{C}_c^1(\mathbb{R}^n)$, we obtain

$$\int_\Gamma w(\alpha \otimes \alpha) dEu = \int_\Gamma w(u^+ - u^-, \alpha)(\nu, \alpha) d\mathcal{H}^{n-1}.$$

Therefore the identity above holds by approximation for any bounded Borel function. Taking

$$w = \begin{cases} \dfrac{(u^+ - u^-, \alpha)(\nu, \alpha)}{|(u^+ - u^-, \alpha)(\nu, \alpha)|} & \text{if } (u^+ - u^-, \alpha)(\nu, \alpha) \ne 0 \\ 0 & \text{otherwise ,} \end{cases}$$

we obtain $\int_\Gamma |(u^+ - u^-, \alpha)(\nu, \alpha)| d\mathcal{H}^{n-1} \leq |Eu|(\Gamma)$. Now letting α range over a linear independent set of vectors we get

$$\int_\Gamma |u^+ - u^-| d\mathcal{H}^{n-1} \leq C_1(\Omega)|Eu|(\Gamma).$$

Since $\partial\Omega$ can be covered by a finite number of such Γ, the point (iii) follows. □

Corollary 1.6.3 *Let Ω be as in the theorem. Then the map $u \to u_*$ with*

$$u_* := \begin{cases} u & \text{in } \Omega \\ 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases}$$

is continuous from $BD(\Omega)$ to $BD(\mathbb{R}^n)$.

Proof. $u_* \in L^1(\mathbb{R}^n, \mathbb{R}^n) \cap BD(\mathbb{R}^n \setminus \partial\Omega)$ with $(u_*)^+ = 0$ and $(u_*)^- = \gamma(u)$. Then by Theorem 1.6.2 $u_* \in BD(\mathbb{R}^n)$ and

$$|Eu_*|(\partial\Omega) \leq \|\gamma(u)\|_{L^1(\partial\Omega, \mathbb{R}^n)} \leq C(\Omega) \|u\|_{BD(\Omega)}.$$

Therefore $\|u_*\|_{BD(\mathbb{R}^n)} \leq C(\Omega) \|u\|_{BD(\Omega)}$. □

Corollary 1.6.4 *Let Ω be as in the theorem. Then any $u \in BD(\Omega)$ may be extended to some $\bar{u} \in BD(\mathbb{R}^n)$ such that*

$$|E\bar{u}|(\bar{\Omega}) = |Eu|(\Omega).$$

Proof. We apply the Gagliardo's theorem (see Gagliardo [128]) to the trace $\gamma(u)$ of u on $\partial\Omega$ to obtain a function $v \in W^{1,1}(\mathbb{R}^n \setminus \bar{\Omega}, \mathbb{R}^n)$ whose trace on $\partial\Omega$ is equal to $\gamma(u)$. Let now

$$\bar{u} := \begin{cases} u & \text{in } \Omega \\ v & \text{in } \mathbb{R}^n \setminus \bar{\Omega}. \end{cases}$$

Then by theorem 1.6.2 $\bar{u} \in BD(\mathbb{R}^n)$ and $|E\bar{u}|(\partial\Omega) = 0$. Therefore, we finally get $|E\bar{u}|(\bar{\Omega}) = |E\bar{u}|(\Omega) + |E\bar{u}|(\partial\Omega) = |Eu|(\Omega)$. □

Remark 1.6.5 From the theorems 1.6.1 (ii) and 1.6.2, one can easily see that the lower and upper traces of $u \in BD(\mathbb{R}^n)$ along a Lipschitz hypersurface $\Gamma \subset \mathbb{R}^n$ are one-sided Lebesgue limits of u at $\mathcal{H}^{n-1}$-a.e. point of Γ. This can obviously be extended to any countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set M obtaining the following generalization of the point (ii) in Theorem 1.6.1

$$Eu \llcorner M := (u^+ - u^-) \odot \nu_M \mathcal{H}^{n-1} \llcorner M \tag{1.6.2}$$

where ν_M indicates the approximate unit normal of M. In other words the Borel measurable approximate unit normal ν_M of M coincides up to a sign with the direction defining the one-sided Lebesgue limits of u $\mathcal{H}^{n-1}$ -almost everywhere in M. The identity (1.6.2) will be used in the next chapter in order to characterize what we will call the *Jump part* of the measure Eu.

1.7 A Poincaré type inequality for BD functions

This paragraph deals with an approximation theorem which allows to understand the local behaviour of BD functions.

Theorem 1.7.1 *Let Ω be a Lipschitz bounded connected open subset of $\mathbb{R}^n$. Then there exists a linear continuous map $R : BD(\Omega) \to \mathcal{R}$ which leaves fixed the elements of $\mathcal{R}$ and such that*

$$\|u - Ru\|_{\frac{n}{n-1}} \leq C(\Omega)|Eu|(\Omega) \quad \textit{for any } u \in BD(\Omega). \tag{1.7.1}$$

This theorem is a linearization of a result by Kohn [141] on integral estimates of strain functions in non linear elasticity. Precisely, in nonlinear elasticity with Ω corresponding to the reference configuration of some elastic body, the right stretch tensor is defined by

$$G_u(x) := (Du^t(x)Du(x))^{1/2}$$

for a deformation $u : \Omega \to \mathbb{R}^n$ is introduced in order to study the local behaviour of the deformations of the body Ω; G_u^2 is called the right Cauchy-Green tensor. The tensor defined by $E_u(x) := G_u(x) - I$ measures how much the deformation u differs from a rigid motion γ_u. In fact, if $u \in \mathcal{C}^1(\Omega, \mathbb{R}^n)$, $E_u(x) = 0$ implies that the $Du(x)$ is an orthogonal matrix for any $x \in \Omega$ and therefore u is a rigid motion on each connected component of Ω.

Since various linearizations of elasticity problems are based on the assumption that E_u is "small", one would like to study the stability of the dependence of $u - \gamma_u$ on E_u. John [140] gave for the first time an answer to that question by proving the following proposition. Let

$$I_{\Omega,\delta} := \{u : \Omega \to \mathbb{R}^n \colon\ u \text{ is a } \mathcal{C}^1 \text{ diffeomorphism and } \|E_u\|_\infty \leq \delta\}.$$

Proposition 1.7.2 *For each $p > 1$ there exist positive constants $\delta = \delta(n,p)$ and $C = C(n,p)$ such that whenever Ω is a cube in $\mathbb{R}^n$ and $u \in I_{\Omega,\delta}$, then for some orthogonal matrix g_u,*

(1.7.2)
$$\|Du - g_u\|_p \leq C \|E_u\|_p .$$

Remark 1.7.3 (1) From well-known Sobolev inequalities, one can easily see that whenever Ω is a cube of side h and $u \in I_{\Omega,\delta}$, there exists a rigid motion γ_u of $\mathbb{R}^n$ such that

(i) If $1 < p < n$ then $\|u - \gamma_u\|_q \leq C'(\Omega, n, p) \|E_u\|_p$ with $q = \frac{np}{n-p}$.

(ii) If $p > n$ then $\|u - \gamma_u\|_\infty \leq C''(\Omega, n, p) h^\nu \|E_u\|_p$ with $\nu = 1 - \frac{n}{p}$.

(2) The class $I_{\Omega,\delta}$ in which the proposition is proved is more restrictive since in many problems of non linear elasticity (for example in hyperelasticity) it is difficult to obtain such a uniform bound for E_u. Very often there are bounds on the density energy which only lead to integral norm bounds for E_u in L^p spaces with $p < +\infty$. So it is important to look for an extension of John's result in which L^p-norms with $p < +\infty$ are involved.

(3) Note also that John proved its result only in the case $p > 1$.

Kohn [141] was able to extend John's result and obtained integral inequalities with the tensor E_u replaced by a different scalar measure $e(u)$ of the non linear strain. $e(u)$ is defined in the following way: let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of the matrix $G_u(x)$. Set

$$\begin{aligned}
e_1(u) &:= (\lambda_n - 1)_+ \\
e_2(u) &:= (\lambda_2 \cdots \lambda_n - 1)_+ \\
e_3(u) &:= |\det G_u(x) - 1| \\
e(u) &:= e_1(u) + e_2(u) + e_3(u),
\end{aligned}$$

where, for any real number t, $t_+ = \max(t, 0)$. $e_1(u)$ and $e_2(u)$ respectively measure the maximum amount of stretching that Du does to arc lengths and hypersurfaces while $e_3(u)$ measures how Du stretches or shrinks volumes. One can easily see that $e(u) = 0$ if and only if $E_u = 0$. So, when their small, $e(u)$ and E_u are comparable. Using $e(u)$ as a measure of non linear strains, Kohn proved the following results.

Proposition 1.7.4 *Let $p \geq (p \neq n)$, γ a rigid motion in $\mathbb{R}^n$ and $u \in W^{1,p}_{loc}(\mathbb{R}^n, \mathbb{R}^n)$. Suppose moreover that $u = \gamma$ out of a bounded open set Ω of diameter h. Then*

(i) $\|u - \gamma\|_q \leq C(\Omega, n, p) \|e_1(u)\|_p$ *if* $1 \leq p < n$ *and* $q = \frac{np}{n-p}$,

(ii) $\|u - \gamma\|_\infty \leq C(\Omega, n, p) h^\nu \|e_1(u)\|_p$ *if* $p > n$ *and* $\nu = 1 - \frac{n}{p}$.

Theorem 1.7.5 *Let Ω be a bounded Lipschitz open subset of $\mathbb{R}^n$. $(n \geq 2)$, $p \geq 1$ $(p \neq n)$. Then there is a positive constant $C(\Omega, p)$ such that for any bi-lipschitzian map $u : \Omega \to \mathbb{R}^n$, there exists a rigid motion γ_u satisfying*

(i) $\|u - \gamma\|_q + \|u - \gamma\|_{L^p(\partial\Omega, \mathbb{R}^n)} \leq C(\Omega, p) \|e(u)\|_p$ *if* $1 \leq p < n$ *and* $q = \frac{np}{n-p}$,

(ii) $\|u - \gamma\|_\infty \leq C(\Omega, p) \|e(u)\|_p$ *if* $p > n$.

The Poincaré type inequality for functions in $BD(\Omega)$ will then follow from a linearization of the result above in the case $p = 1$. The technique of linearization is the following. Let $u : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth function with compact support. For each $s \in \mathbb{R}$, let $v_s : \mathbb{R}^n \to \mathbb{R}^n$ be $v_s(x) := x + su(x)$. Since $Dv_s = I + sDu$ converges uniformly to I as $s \to 0$, following Theorem I of F. John [140] there exists some $s_0 > 0$ depending on u such that v_s is a diffeomorphism whenever $|s| \leq s_0$. For such s we compute

$$\begin{aligned} G_{v_s} = (Dv_s^t Dv_s)^{1/2} &= I + \frac{1}{2}s(Du^t + Du) + O(s^2) \\ &= I + sEu + O(s^2), \end{aligned}$$

where $Eu = (Du^t + Du)/2$ is the linearized strain. Therefore

$$\int_\Omega |e(v_s)| dx \leq C(\Omega)|s||Eu|(\Omega) + O(s^2) \quad \text{for any open } \Omega \subset \mathbb{R}^n. \tag{1.7.3}$$

1.7.1 The Poincaré type inequality in $BD(\mathbb{R}^n)$

Proposition 1.7.6 *There exists a positive constant $C(n)$ such that*

$$\|u\|_{\frac{n}{n-1}} \leq C(n)|Eu|(\mathbb{R}^n) \quad \textit{for any } u \in BD(\mathbb{R}^n). \tag{1.7.4}$$

Proof. *Step 1*: We take $u \in \mathcal{C}_c^1(\mathbb{R}^n, \mathbb{R}^n)$. Applying proposition 1.7.4 to the function $v_s(x) = x + su(x)$ we obtain

$$\begin{aligned} |s|\, \|u\|_{\frac{n}{n-1}} = \|v_s(x) - x\|_{\frac{n}{n-1}} &\leq C(n)\, \|e(v_s)\|_1 \\ &\leq C(n)|s||Eu|(\mathbb{R}^n) + O(s^2). \end{aligned}$$

Then dividing by $|s|$ and letting s going to 0 we obtain (1.7.4).

Step 2: Suppose $u \in BD(\mathbb{R}^n)$ with a compact support. Then (1.7.4) easily follows by regularization.

Step 3: Let $u \in BD(\mathbb{R}^n)$. For each $k \in \mathbb{N} \setminus \{0\}$ we take $\varphi_k \in \mathcal{C}_c^1(\mathbb{R}^n)$ such that

$$0 \leq \varphi_k \leq 1, \quad \varphi_k = 1 \text{ in } B_k(0), \quad \varphi_k = 0 \text{ in } \mathbb{R}^n \setminus B_{k+1} \quad \text{and} \quad |D\varphi_k| \leq 2.$$

Therefore $\varphi_k u \to u$ in $BD(\mathbb{R}^n)$ as k goes to $+\infty$ and (1.7.4) easily follows from

$$\|\varphi_k u\|_{\frac{n}{n-1}} \leq C(n)|E(\varphi_k u)|(\mathbb{R}^n)$$

by letting k going to $+\infty$. □

Remark 1.7.7 Matthies, Strang and Christiansen used (1.7.4) and the result of Ornstein [170] to prove in [150] that the inclusion $BV(\Omega, \mathbb{R}^n) \subset BD(\Omega)$ is strict. In fact, if we take Ω to be the unit square in $\mathbb{R}^n$ and we suppose that the equality $BV(\Omega, \mathbb{R}^n) = BD(\Omega)$ holds, then by the closed graph theorem there would exist a constant $C(n)$ such that

$$\begin{aligned} \|u\|_{BV(\Omega,\mathbb{R}^n)} &\leq C(n)\, \|u\|_{BD(\Omega)} \\ &\leq C(n) \left\{ \|u\|_{\frac{n}{n-1}} + |Eu|(\Omega) \right\} \end{aligned}$$

for every $u \in BD(\Omega)$. Now using (1.7.4) we obtain for every $u \in \mathcal{C}_c^\infty(\Omega, \mathbb{R}^n)$

$$\|u\|_{BV(\Omega,\mathbb{R}^n)} \leq C(n)|Eu|(\Omega).$$

Therefore

$$\int_\Omega \sum_{i,j=1}^n \left|\frac{\partial u^i}{\partial x_j}\right| dx \leq C(n) \int_\Omega \sum_{i,j=1}^n \left|\frac{\partial u^i}{\partial x_j} + \frac{\partial u^j}{\partial x_i}\right| dx. \tag{1.7.5}$$

For every $f \in \mathcal{C}_c^\infty(\Omega)$, set

$$u^1 = \frac{\partial f}{\partial x_1} \quad \text{and} \quad u^i = -\frac{\partial f}{\partial x_i}\ , \quad 2 \leq i \leq n.$$

Then (1.7.5) implies that

$$\int_\Omega \left|\frac{\partial^2 f}{\partial x_1 \partial x_2}\right| dx \leq C(n) \int_\Omega \sum_{\substack{i,j=1 \\ \{i,j\}\neq\{1,2\}}}^n \left|\frac{\partial^2 f}{\partial x_i \partial x_j}\right| dx. \tag{1.7.6}$$

This is not possible since Ornstein [170] has constructed a family of smooth functions compactly supported in the unit square and for which the inequality (1.7.6) does not hold.

Precisely the following theorem has been proved in [170].

Theorem 1.7.8 *Let Ω be the unit square in $\mathbb{R}^n$. Let $\{B, D_1, \cdots, D_l\}$ be a set of linearly independent linear homogeneous differential operators of degree m. For any $k > 0$ there exists a function f_k in $\mathcal{C}_c^\infty(\Omega)$ such that*

$$\int_\Omega |Bf_k(x)|dx > k \quad \text{and} \quad \int_\Omega |D_i f_k(x)|dx < 1, \quad 1 \leq i \leq l.$$

Then applying this theorem to the following family of second order linear operators

$$B = \frac{\partial^2}{\partial x_1 \partial x_2} \quad \text{and} \quad D_{ij} = \frac{\partial^2}{\partial x_i \partial x_j} \quad \{i,j\} \neq \{1,2\}$$

we get a contradiction to (1.7.6). Therefore $BV(\Omega, \mathbb{R}^n)$ is strictly embedded in $BD(\Omega)$.

From the Ornstein's theorem above, an explicit construction of a BD function which does belong to $BV(\Omega, \mathbb{R}^n)$ has been given in Bellettini-Coscia-Dal Maso [46, 47], Ambrosio-Coscia-Dal Maso [21]. We refer the reader to Proposition 2.5.6 of the next chapter.

1.7.2 The Poincaré type inequality for Lipschitz bounded open subsets of $\mathbb{R}^n$

Let us recall that $\mathcal{R}$ will always denote the subspace of infinitesimal rigid motions. We will need the following lemma.

Lemma 1.7.9 *Let Ω be a Lipschitz bounded open subset of $\mathbb{R}^n$. Then there is a continuous linear function $T_0 : L^1(\Omega, \mathbb{R}^n) \to \mathcal{R}$ which leaves fixed the elements of $\mathcal{R}$ and such that*

$$T_0(\alpha)(x) := \frac{1}{2}(A - A^t)x + b$$

whenever $\alpha(x) := Ax + b$ is any affine map.

Proof. Since $\mathcal{R}$ is a closed subspace of $L^1(\Omega, \mathbb{R}^n)$, the Hahn-Banach theorem will then provide a continuous linear extension $T_0 : L^1(\Omega, \mathbb{R}^n) \to \mathcal{R}$ of the identity map $\mathrm{id} : \mathcal{R} \to \mathcal{R}$.
However, an explicit formula for a suitable function T_0 is given in the following way. Let $x_0 \in \Omega$, $0 < r_0 < d(x_0, \partial\Omega)$ and $\{e_j\}$ be an orthonormal basis with respect to which all the coordinates are given. For any $u \in L^1(\Omega, \mathbb{R}^n)$ set

$$\sigma^i(u) := \fint_{B_{r_0}(x_0)} u^i(y)dy$$

and

$$\rho^{ij}(u) := \frac{n+1}{w_{n-1}r_0^{n+1}} \int_{B_{r_0}(x_0)\cap\{y\in\Omega:\ (y-x_0,e_j)>0\}} [u^i(y) - \sigma^i(u)]dy.$$

Then the map T_0 defined by $T_0(u)(x)^i := \frac{1}{2}\sum\limits_{j=1}^{n}[\rho^{ij}(u) - \rho^{ji}(u)](x - x_0)^j + \sigma^i(u)$ for any $x \in \Omega$ is linear continuous from $L^1(\Omega, \mathbb{R}^n)$ to $\mathcal{R}$ and one can easily verify that $T_0(\alpha)(x) = \frac{1}{2}(A - A^t)(x - x_0) + b$ whenever $\alpha(x) = A(x - x_0) + b$. $\square$

Now using the linearization technique of the preceding paragraph we obtain a Poincaré type inequality for the operator T_0.

Proposition 1.7.10 *Let Ω be a Lipschitz bounded open subset of $\mathbb{R}^n$, T_0 be the operator of Lemma 1.7.9. Then there is a positive constant $C(\Omega, T_0)$ such that*

$$\|u - T_0(u)\|_{\frac{n}{n-1}} \leq C(\Omega, T_0)|Eu|(\Omega) \quad \text{for any } u \in \mathcal{C}_c^\infty(\mathbb{R}^n, \mathbb{R}^n).$$

Proof. Without loss of generality we may suppose $T_0(u) = 0$. We will denote $C(\Omega, T_0)$ all the positive constants depending only on Ω and T_0. For sufficiently small s we apply Theorem 1.7.5 to $v_s(x) = x + su(x)$ (taking $p = 1$) with u in $\mathcal{C}_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$. Then there are rigid motions γ_s of $\mathbb{R}^n$ such that

$$\|v_s - \gamma_s\|_{\frac{n}{n-1}} \leq C(\Omega)|s|\,|Eu|(\Omega) + O(s^2). \tag{1.7.7}$$

Let $\gamma_s(x) := G_s x + d_s$ where G_s is an orthogonal matrix and $d_s \in \mathbb{R}^n$. $T_0(u) = 0$ implies $T_0(v_s) = T_0(id) = 0$. Therefore $T_0(v_s - \gamma_s) = -T_0(\gamma_s)$. By the continuity of the operator T_0 and the inequality (1.7.7), we obtain

$$\begin{aligned} \frac{1}{2}\left\|G_s - G_s^t\right\| + \|d_s\| &= \|T_0(v_s - \gamma_s\| \leq C(\Omega, T_0)\,\|v_s - \gamma_s\|_1 \\ &\leq C(\Omega, T_0)\,\|v_s - \gamma_s\|_{\frac{n}{n-1}} \\ &\leq C(\Omega, T_0)|s|\,|Eu|(\Omega) + O(s^2). \end{aligned}$$

Therefore

$$\left\|G_s - G_s^t\right\| + \|d_s\| \leq C(\Omega, T_0)|s|\,|Eu|(\Omega) + O(s^2). \tag{$*$}$$

One can easily verify that $\|G_s - I\| \leq 1$ for $|s|$ small enough, so that

$$\|G_s - I\| \leq \left\|G_s - G_s^t\right\|.$$

for such s. From this last inequality and $(*)$ we obtain

$$\|G_s - I\| + \|d_s\| \leq C(\Omega, T_0)|s|\,|Eu|(\Omega) + O(s^2)$$

so that

$$\|\gamma_s - I\|_{\frac{n}{n-1}} \leq C(\Omega, T_0)|s|\,|Eu|(\Omega) + O(s^2).$$

Combining this inequality with (1.7.7) we obtain

$$\|v_s - I\|_{\frac{n}{n-1}} \leq C(\Omega, T_0)|s|\,|Eu|(\Omega) + O(s^2).$$

Dividing now both members by $|s|$ and letting $s \to 0$ we obtain

$$\|u\|_{\frac{n}{n-1}} \leq C(\Omega, T_0)|Eu|(\Omega)$$

and this achieves the proof. $\square$

We are now able to prove the Poincaré type theorem. Let us first recall it for the reader's convenience.

Theorem 1.7.11 *Let Ω be a bounded connected open subset of $\mathbb{R}^n$ with a Lipschitz boundary $\partial\Omega$. Then there exists a linear continuous map $R : BD(\Omega) \to \mathcal{R}$ which leaves fixed the elements of $\mathcal{R}$ and such that*

$$\|u - Ru\|_{\frac{n}{n-1}} \leq C(\Omega)|Eu|(\Omega) \quad \textit{for any } u \in BD(\Omega). \tag{1.7.8}$$

Proof. We take R to be the restriction of T_0 to $BD(\Omega)$. Let $u \in BD(\Omega)$. By Corollary 1.6.4 there is $\bar{u} \in BD(\mathbb{R}^n)$ such that $|E\bar{u}|(\bar{\Omega}) = |Eu|(\Omega)$.

Step 1: Suppose $\bar{u} = 0$ off a compact set and apply proposition 1.7.10 to the regularizing sequence of $\bar{u}$ we obtain in the limit

$$\|u - Ru\|_{\frac{n}{n-1}} \leq C(\Omega, T_0)|E\bar{u}|(\bar{\Omega}) = C(\Omega, T_0)|Eu|(\Omega). \tag{$**$}$$

Step 2: In general since Ω is bounded, we take an integer k large enough such that $\Omega \subset\subset B_k(0)$. Let $\{\varphi_k\} \subset \mathcal{C}_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$ with $\varphi_k = 1$ in $B_k(0)$ and $\text{spt}(\varphi_k) \subset B_{k+1}$. Applying step 1 to the function $\varphi_k \bar{u}$ and letting $k \to +\infty$ we obtain $(**)$ in the limit. $\square$

Remark 1.7.12 The Poincaré type inequality (1.7.8) remains true for any other linear continuous map $T : BD(\Omega) \to \mathcal{R}$ which leaves fixed the elements of $\mathcal{R}$. In fact

$$\|u - Tu\|_{\frac{n}{n-1}} \leq \|u - Ru\|_{\frac{n}{n-1}} + \|T(u - Ru)\|_{\frac{n}{n-1}}.$$

Since T is continuous,

$$\|T(u-Ru)\|_{\frac{n}{n-1}} \leq C(\Omega,T)\,\|u-Ru\|_{BD(\Omega)}\,.$$

On the other hand

$$\begin{aligned}\|u-Ru\|_{BD(\Omega)} &= \|u-Ru\|_1 + |Eu|(\Omega)\\ &\leq |\Omega|^{1/n}\,\|u-Ru\|_{\frac{n}{n-1}} + |Eu|(\Omega)\\ &\leq (|\Omega|^{1/n}C(\Omega,T_0)+1)|Eu|(\Omega)\end{aligned}$$

where the last inequality follows by (1.7.8). Therefore

$$\|u-Tu\|_{\frac{n}{n-1}} \leq C(\Omega,T_0,T)|Eu|(\Omega).$$

Remark 1.7.13 As a consequence of the Poincaré type inequalities for BD functions, we have

$$BD(\Omega)\subset L^{\frac{n}{n-1}}(\Omega,\mathbb{R}^n)$$

with continuous injection for any bounded Lipschitz open subset Ω of $\mathbb{R}^n$ (and also for $\Omega=\mathbb{R}^n$) by (1.7.8) and (1.7.4).

Another consequence of the Poincaré type inequalities for BD functions is the following compact injection theorem whose proof can be found for example in Kohn [141] and Temam [191].

Theorem 1.7.14 *Let Ω be a bounded Lipschitz open subset of $\mathbb{R}^n$. The injection of $BD(\Omega)$ in $L^p(\Omega,\mathbb{R}^n)$ is compact, for all $1\leq p<n^\star=n/(n-1)$.*

1.7.3 The Poincaré type inequality for BD functions on small balls

In order to study the local regularity properties of BD functions e.g., the characterization of the set of discontinuity points which are not in the jump set, R. Kohn [141] used the Poincaré type inequality on small balls about any x in Ω and studied how the corresponding infinitesimal rigid motions change as the radius tend to zero. A precise representation in terms of singular integrals of some particular rigid motions has been given. Ambrosio-Coscia-Dal Maso [21] and Ebobisse [114] have used as we will see in the third chapter, this representation in order to prove the approximate differentiability of BD functions and also to obtain some useful weak-L^1 estimates for the approximate differential. Up to translations it is sufficient to work only with balls centered at the origin.

For any $\rho > 0$ we define $T_\rho : BD(B_\rho(0)) \to \mathcal{R}$ by $T_\rho(u)(x) := A_\rho(u)x + b_\rho(u)$ where

$$A_\rho(u) := \frac{1}{2w_n\rho^n}\int_{\mathbf{S}^{n-1}(0,\rho)}[\gamma(u)\otimes\nu - \nu\otimes\gamma(u)]d\mathcal{H}^{n-1}$$

and

$$b_\rho(u) := \frac{1}{nw_n\rho^{n-1}}\int_{\mathbf{S}^{n-1}(0,\rho)}\gamma(u)d\mathcal{H}^{n-1};$$

ν denotes the outward unit normal to $\mathbf{S}^{n-1}(0,\rho)$ and $\gamma(u)$ is the trace of u on $\mathbf{S}^{n-1}(0,\rho)$.

One can easily see that for each $\rho > 0$ the map T_ρ is linear, continuous and leaves fixed the elements of $\mathcal{R}$. Its continuity is a consequence of the trace theorem. Hence, by Theorem 1.7.11 there exists a positive constant $C(n)$ independent of ρ such that

$$\|u - T_\rho(u)\|_{L^{\frac{n}{n-1}}(B_\rho(0),\mathbb{R}^n)} \leq C(n)|Eu|(B_\rho(0)) \tag{1.7.9}$$

whenever $u \in BD(\mathbb{R}^n)$. The constant $C(n)$ is independent of ρ by the following rescaling argument: let $u \in BD(\mathbb{R}^n)$ and $\rho > 0$. We define $v(x) := \frac{1}{\rho}u(\rho x)$ for any $x \in B_1(0)$. It follows that $A_1(v) = A_\rho(u)$, $b_1(v) = \frac{1}{\rho}b_\rho(u)$ and therefore

$T_1(v)(x) = \frac{1}{\rho} T_\rho(u)(\rho x)$ for any $x \in B_1(0)$. Since

$$\|v - T_1(v)\|_{L^{\frac{n}{n-1}}(B_1(0),\mathbb{R}^n)} = \rho^{-n} \|u - T_\rho(u)\|_{L^{\frac{n}{n-1}}(B_\rho(0),\mathbb{R}^n)}$$

and

$$|Ev|(B_1(0)) = \rho^{-n} |Eu|(B_\rho(0)),$$

(1.7.9) will then follow from Remark 1.7.12 applied to the function $v \in BD(B_1(0))$.

The following theorem relies on the representations of A_ρ and b_ρ in terms of singular integrals. As we said before, those representations will allow to easily study the dependence of T_ρ as $\rho \to 0^+$ and also to obtain some useful weak-L^1 estimates.

Given an $n \times n$ matrix of measures $\{\mu_{lm}\}$ on $\mathbb{R}^n$, a fourth-order tensor $\{T_{ij}^{lm}(x)\}$, a third-order tensor $\{H_i^{lm}(x)\}$ and a Borel set $B \subset \mathbb{R}^n$, we introduce the short notations

$$A = \int_B T d\mu \quad \text{for} \quad A_{ij} = \sum_{l,m=1}^{n} \int_B T_{ij}^{lm}(x) d\mu_{lm}(x)$$

and

$$b = \int_B H d\mu \quad \text{for} \quad b^i = \sum_{l,m=1}^{n} \int_B H_i^{lm}(x) d\mu_{lm}(x).$$

Let $\Gamma(x) = \{\Gamma_{ij}^{lm}(x)\}$ and $\bigwedge(x) = \{\bigwedge_i^{lm}(x)\}$ be respectively the fourth and third-order tensors defined by

$$\Gamma_{ij}^{ij}(x) = \Gamma_{ij}^{ji}(x) = x_j^2 - x_i^2,$$

$$\Gamma_{ij}^{ii}(x) = -\Gamma_{ij}^{jj}(x) = 2x_i x_j \qquad \text{if } i \neq j,$$

$$\Gamma_{ij}^{ir}(x) = \Gamma_{ij}^{ri}(x) = x_j x_r \qquad \text{if } i \neq j \text{ and } r \notin \{i,j\},$$

$$\Gamma_{ij}^{jr}(x) = \Gamma_{ij}^{rj}(x) = -x_i x_r \qquad \text{if } i \neq j \text{ and } r \notin \{i,j\},$$

$$\Gamma_{ij}^{lm}(x) = 0 \qquad \text{if } i = j \text{ or } \{l,m\} \cap \{i,j\} = \emptyset.$$

and

$$\begin{aligned} &\Lambda_i^{ri}(x) = \Lambda_i^{ir}(x) = x_r && \text{if } 1 \le r \le n, \\ &\Lambda_i^{rr}(x) = -x_i && \text{if } r \ne i, \\ &\Lambda_i^{lm}(x) = 0 && \text{in all other cases .} \end{aligned}$$

One can easily verify that for every i, j, l, m,

$$\Gamma_{ij}^{lm}(x) = \Gamma_{ij}^{ml}(x) = -\Gamma_{ji}^{lm}(x), \qquad \int_{\mathbf{S}^{n-1}} \Gamma_{ij}^{lm} d\mathcal{H}^{n-1} = 0. \tag{1.7.10}$$

Theorem 1.7.15 *Let $u \in BD(\mathbb{R}^n)$ and $\rho > 0$. For any $x \in \mathbb{R}^n$*

$$A_\rho(u)(x) = -\frac{1}{2w_n}\int_{\{|y-x|\ge\rho\}} \frac{\Gamma(y-x)}{|y-x|^{n+2}} dEu(y), \tag{1.7.11}$$

$$b_\rho(u)(x) = \frac{1}{nw_n}\int_{\{|y-x|\ge\rho\}} \frac{\Lambda(y-x)}{|y-x|^{n}} dEu(y). \tag{1.7.12}$$

Proof. It is not restrictive to prove the theorem in the case $x = 0$. Let us first show that the matrix $A_\rho(u)$ verifies (1.7.11). To this aim, we fix two indices i, j with $i \ne j$, and we let $\tau > \rho$. Since

$$\sum_{m=1}^n D_m \Psi_{lm}(y) = 0 \quad \textit{and} \quad \sum_{m=1}^n \Psi_{lm}(y)\frac{y_m}{|y|} = \begin{cases} \dfrac{y_i}{2w_n|y|^{n+1}} & \text{if } l = i, \\[2ex] \dfrac{y_i}{2w_n|y|^{n+1}} & \text{if } l = j, \\[2ex] 0 & \text{if } l \notin \{i, j\}, \end{cases}$$

applying the generalized Green formula of Theorem 1.6.1 (iii) in $B_\tau(0) \setminus B_\rho(0)$ to the symmetric matrix-valued function

$$\Psi_{lm}(y) = \Gamma_{ij}^{lm}(y)/(2w_n|y|^{n+2})$$

we obtain

$$\int_{B_\tau(0)\setminus B_\rho(0)} \Psi(y) dEu(y) = -\int_{B_\tau(0)\setminus B_\rho(0)} (\operatorname{div}\Psi, u)dy + \int_{\partial(B_\tau(0)\setminus B_\rho(0))} (\gamma(u)\odot\nu, \Psi) d\mathcal{H}^{n-1}$$

where ν denotes the outward unit normal to $\partial(B_\tau(0) \setminus B_\rho(0))$. Then

$$\int_{B_\tau(0)\setminus B_\rho(0)} \Psi(y)dEu(y) = \int_{\partial B_\tau(0)} (\gamma(u) \odot \nu, \Psi)d\mathcal{H}^{n-1} - \int_{\partial B_\rho(0)} (\gamma(u) \odot \nu, \Psi)d\mathcal{H}^{n-1}.$$

Let $r > 0$. Taking into account the fact that $\Psi_{lm} = 0$ for any $l \notin \{i, j\}$, we compute the integral

$$\int_{\partial B_r(0)} (\gamma(u) \odot \nu, \Psi)d\mathcal{H}^{n-1}$$

$$= \sum_{l,m=1}^{n} \int_{\partial B_r(0)} \gamma(u)_l \nu_m \Psi_{lm}(y)d\mathcal{H}^{n-1}(y)$$

$$= \sum_{m=1}^{n} \left\{ \int_{\partial B_r(0)} (\gamma(u)_i \nu_m \Psi_{im}(y) + \gamma(u)_j \nu_m \Psi_{jm}(y))d\mathcal{H}^{n-1}(y) \right\}$$

$$= \int_{\partial B_r(0)} \gamma(u)_i (\sum_{m=1}^{n} \Psi_{im} \frac{y_m}{|y|})d\mathcal{H}^{n-1}(y) + \int_{\partial B_r(0)} \gamma(u)_j (\sum_{m=1}^{n} \Psi_{jm} \frac{y_m}{|y|})d\mathcal{H}^{n-1}(y)$$

$$= \int_{\partial B_r(0)} \gamma(u)_i \frac{y_j}{2w_n |y|^{n+1}} d\mathcal{H}^{n-1}(y) - \int_{\partial B_r(0)} \gamma(u)_j \frac{y_i}{2w_n |y|^{n+1}} d\mathcal{H}^{n-1}(y)$$

$$= \frac{1}{2w_n r^n} \int_{\partial B_r(0)} (\gamma(u) \otimes \nu - \nu \otimes \gamma(u))d\mathcal{H}^{n-1}(y).$$

Therefore

$$\int_{B_\tau(0)\setminus B_\rho(0)} \Psi(y)dEu(y) + A_\rho(u) = \frac{1}{2w_n \tau^n} \int_{\partial B_\tau(0)} (\gamma(u) \otimes \nu - \nu \otimes \gamma(u))d\mathcal{H}^{n-1}(y). \tag{1.7.13}$$

Since

$$\int_0^{+\infty} \int_{\partial B_r(0)} |\gamma(u)|d\mathcal{H}^{n-1}(y)dr = \int_{\mathbb{R}^n} |u|dx < +\infty,$$

then

$$\lim_{r\to+\infty} \int_{\partial B_r(0)} |\gamma(u)|d\mathcal{H}^{n-1}(y) = 0$$

and therefore

$$\lim_{\tau\to+\infty}\frac{1}{\tau^n}\int_{\partial B_\tau(0)}(\gamma(u)\otimes\nu-\nu\otimes\gamma(u))d\mathcal{H}^{n-1}(y)=0.$$

So, letting $\tau\to+\infty$ in (1.7.13) we obtain

$$A_\rho(u)=-\int_{\{|y|\geq\rho\}}\frac{\Gamma(y)}{2w_n|y|^{n+2}}dEu(y).$$

The identity (1.7.12) is proved in a similar way. $\square$

Chapter 2

Fine Properties of BD Functions

2.1 Introduction

E. De Giorgi and L. Ambrosio defined in their pioneering work [99] weak formulations of *free discontinuity* problems which led to the introduction of the space $SBV(\Omega)$ of special functions of bounded variation on Ω. In order to establish existence and regularity results (see e.g., Ambrosio [11], De Giorgi-Carriero-Leaci [100], Ambrosio-pallara [29], Ambrosio-Fusco-Pallara [24, 25, 26], Bonnet [49, 50], David [87, 88] and David-Semmes [89, 90]) for those weak formulations, more properties of functions of bounded variation have been given for instance by De Giorgi-Ambrosio [99], Ambrosio [9, 10, 11, 13], Ambrosio-Dal Maso [22], Alberti [6], Braides-Coscia [67], De Giorgi-carriero-Leaci [100] and Pallara [171].

However, in some applications, for example in fracture mechanics problems where one deals with bulk parts energy which are functions of the symmetrized gradient Eu, the correct function space is not BV any more, nor its offspring SBV, but the space BD of functions with bounded deformation. So, analogously to the space BV there is a necessity to provide fine properties of functions with bounded deformation in order to define weak formulations of free discontinuity problems on that space.

The purpose of this chapter is to present the fine properties of BD functions obtained first by Kohn [141], and recently by Bellettini-Coscia [45], Ambrosio-Coscia-Dal Maso [21] and Bellettini-Coscia-Dal Maso [46, 47]. For instance, the rectifiability of the jump set for BD functions is proved; analogously to the measure Du for a BV function, the measure Eu of a BD function u is split into three main parts each one having a precise interpretation. A subspace of $BD(\Omega)$ called the space of *special functions with bounded deformation* and denoted by $SBD(\Omega)$ is introduced. A structure theorem which allows to characterize BD functions using suitable one-dimensional sections is proved. Finally, a compactness criterion similar to Theorem 1.3.4 of the previous chapter is given and as we will see in the next chapter will be used in order to prove the lower semicontinuity of some functionals defined in $SBD(\Omega)$.

Let us first begin with some notations that will be used along this chapter. If $\mu = \{\mu_{ij}\}$ is an $M_s^{n\times n}$-valued Radon measure on Ω and $\xi = (\xi^1, \cdots, \xi^n) \in \mathbb{R}^n$, then the real-valued Radon measure $(\mu\xi, \xi)$ is defined by

$$(\mu\xi, \xi) := \sum_{i,j=1}^{n} \xi^i \xi^j \mu_{ij}, \tag{2.1.1}$$

so that $(\mu\xi, \xi)(B) = (\mu(B)\xi, \xi)$ for every $B \in \mathcal{B}(\Omega)$. One can easily check that for any basis $\{\xi_1, \ldots, \xi_n\}$ of $\mathbb{R}^n$ there exists a positive constant C, depending on the basis, such that

$$|\mu| \le C \sum_{i,j=1}^{n} |(\mu(\xi_i + \xi_j), \xi_i + \xi_j)|. \tag{2.1.2}$$

2.2 The sections of BD functions

2.2.1 One-dimensional sections

We will use the following notations involving the one-dimensional sections of sets defined in the previous chapter. Given a function $u : B \to \mathbb{R}^n$, defined on a subset B of $\mathbb{R}^n$, for every $y, \xi \in \mathbb{R}^n$, $\xi \neq 0$, the function $u^\xi_y : B^\xi_y \to \mathbb{R}$ called the one-dimensional section of u is defined by

$$u^\xi_y(t) := u^\xi(y + t\xi) = (u(y + t\xi), \xi) \quad \forall t \in B^\xi_y. \tag{2.2.1}$$

When $\xi = e_i$, the i^{th} vector of the canonical basis, we write u^i_y instead of $u^{e_i}_y$. If $u \in L^1(\Omega, \mathbb{R}^n)$, then the one-dimensional section $\tilde{u}^\xi_y$ of the Lebesgue representative $\tilde{u}$ of u is defined for every $t \in (\Omega_u)^\xi_y$ and the corresponding pointwise variation $V\tilde{u}^\xi_y$ is evaluated using only points $t \in (\Omega_u)^\xi_y$. The following theorem characterizes BD functions from their one-dimensional sections.

Theorem 2.2.1 *Let $u \in BD(\Omega)$ and let $\xi \in \mathbb{R}^n$ with $\xi \neq 0$. Then the following two conditions hold for $\mathcal{H}^{n-1}$-almost every $y \in \Omega^\xi$:*

(i) *$\tilde{u}^\xi_y$ is defined and coincides with u^ξ_y $\mathcal{L}^1$-almost everywhere in Ω^ξ_y;*

(ii) *$u^\xi_y \in BV(\Omega^\xi_y)$ and the measures $|Du^\xi_y|$ and $V\tilde{u}^\xi_y$ coincide on Ω^ξ_y.*

Moreover,

$$(Eu\xi, \xi) = \int_{\Omega^\xi} Du^\xi_y d\mathcal{H}^{n-1}(y), \quad |(Eu\xi, \xi)| = \int_{\Omega^\xi} |Du^\xi_y| d\mathcal{H}^{n-1}(y) \tag{2.2.2}$$

as measures in Ω.
Conversely, let $u \in L^1(\Omega, \mathbb{R}^n)$ and let $\{\xi_1, \ldots, \xi_n\}$ be a basis of $\mathbb{R}^n$. If we assume that for every ξ of the form $\xi_i + \xi_j$,

$$\begin{aligned} &u^\xi_y \in BV(\Omega^\xi_y) \quad \text{for } \mathcal{H}^{n-1}\text{ -a.e. } y \in \Omega^\xi, \\ &\int_{\Omega^\xi} |Du^\xi_y|(\Omega^\xi_y) d\mathcal{H}^{n-1}(y) < +\infty. \end{aligned} \tag{2.2.3}$$

Then $u \in BD(\Omega)$.

Proof. Since all the properties are local, by the corollary 1.6.3 of the previous chapter, we may assume that $\Omega = \mathbb{R}^n$. Let $u \in BD(\mathbb{R}^n)$; the condition (i) easily follows from Fubini's theorem since $\mathcal{L}^n(S_u)=0$ and $u=\tilde{u}$ $\mathcal{L}^n$ -almost everywhere in $\mathbb{R}^n$.

Let $u_\varepsilon := u\star\psi_\varepsilon$ be a regularization of u; since $(Eu_\varepsilon\xi,\xi)=(Eu\xi,\xi)\star\psi_\varepsilon$, Fubini's theorem and (1.3.6) of chapter one imply

$$\begin{aligned}|Eu\xi,\xi)|(\mathbb{R}^n) &\geq |(Eu_\varepsilon\xi,\xi)|(\mathbb{R}^n)\\ &= \int_{\pi_\xi} V\left[(u_\varepsilon)_y^\xi\right](\mathbb{R})d\mathcal{H}^{n-1}(y).\end{aligned}$$

Now we pass to the limit as $\varepsilon\to 0^+$ using the fact that u_ε converges pointwise to $\tilde{u}$ at every Lebesgue point of u and noticing that $v\to Vv(\mathbb{R})$ is lower semicontinuous with respect to the pointwise convergence. By Fatou's lemma we get

$$|(Eu\xi,\xi)|(\mathbb{R}^n)\geq\int_{\pi_\xi} V\tilde{u}_y^\xi(\mathbb{R})d\mathcal{H}^{n-1}(y). \tag{2.2.4}$$

By (1.3.4), we infer that $u_y^\xi\in BV(\mathbb{R})$ for $\mathcal{H}^{n-1}$-almost every $y\in\pi_\xi$ and

$$\int_{\pi_\xi}|Du_y^\xi|(\mathbb{R})d\mathcal{H}^{n-1}(y)<+\infty.$$

Let $\varphi\in\mathcal{C}_c^1(\mathbb{R}^n)$, we set $\varphi_{y,\xi}(t):=\varphi(y+t\xi)$ for every $y,\xi\in\mathbb{R}^n$ and for every $t\in\mathbb{R}$. By Fubini's theorem we get

$$\begin{aligned}\int_{\mathbb{R}^n}\varphi d(Eu\xi,\xi) &= -\int_{\mathbb{R}^n}u^\xi D_\xi\varphi dx\\ &= -\int_{\pi_\xi}\left[\int_{\mathbb{R}}u_y^\xi(t)D\varphi_{y,\xi}(t)\right]d\mathcal{H}^{n-1}(y) \qquad (2.2.5)\\ &= \int_{\pi_\xi}\left[\int_{\mathbb{R}}\varphi_{y,\xi}(t)d(Du_y^\xi)(t)\right]d\mathcal{H}^{n-1}(y).\end{aligned}$$

Then the first equality in (2.2.2) follows from a density argument.

The second equality is a consequence of the total variation formula for measures obtained by integration. The condition (ii) easily follows from (2.2.2), (2.2.4) and

(1.3.4).

Conversely, if (2.2.3) is satisfied for every $\xi = \xi_i + \xi_j$, then from (2.2.5) there exists a positive constant $C(\xi)$ such that

$$\int_{\mathbb{R}^n} u^\xi D_\xi \varphi dx \leq C(\xi) \|\varphi\|_{L^\infty(\mathbb{R}^n)} \quad \forall \varphi \in \mathcal{C}_c^1(\mathbb{R}^n).$$

This implies that $D_\xi u^\xi$ is a bounded Radon measure on $\mathbb{R}^n$ and by property (ii) of Theorem 1.4.8, $u \in BD(\mathbb{R}^n)$. □

Remark 2.2.2 The second equality of (2.2.2) implies that

$$|(Eu\xi, \xi)|(B) = 0 \text{ for every } B \in \mathcal{B}(\Omega) \text{ with } \mathcal{H}^{n-1}(B) = 0.$$

Then taking into account (2.1.2) we get

$$\mathcal{H}^{n-1}(B) = 0 \Longrightarrow |Eu|(B) = 0 \tag{2.2.6}$$

for every $B \in \mathcal{B}(\Omega)$. More generally,

$$\begin{aligned} &|Eu|(B) = 0 \text{ for every } B \in \mathcal{B}(\Omega) \text{ such that} \\ &\mathcal{H}^{n-1}(p_\xi(B)) = 0 \text{ for } \mathcal{H}^{n-1} \text{ -almost every } \xi \in \mathbf{S}^{n-1}. \end{aligned} \tag{2.2.7}$$

(2.2.7) has also been proved by Kohn [141, Proposition 4.16 and Lemma 4.2] using a relationship obtained between the symmetric gradient Eu and the "level sets" of u - i.e., the flat currents $\partial(E^n \llcorner \{x \in \Omega : \ (u(x), \xi) \geq t\})$. It is a kind of Fleming-Rishel's co-area formula for BD functions representing one of the links between the analysis of BD functions and geometric measure theory.

2.2.2 BD functions on linear subspaces of $\mathbb{R}^n$

The theorem 2.2.1 is extended in the next proposition to a more general situation. Precisely, one can define the space $BD(V)$ for every linear subspace V of $\mathbb{R}^n$ by introducing a suitable orthogonal coordinate system on V. By virtue of point (ii) in Theorem 1.4.8, this definition will not depend on the choice of the coordinate system on V.

Proposition 2.2.3 *Let V be a linear subspace of $\mathbb{R}^n$ of dimension k, let $V^\perp$ be the orthogonal subspace, and let p_V be the orthogonal projection onto V. Given a function $u \in BD(\mathbb{R}^n)$, for every $z \in V^\perp$ let $v_z : V \to V$ be the function defined by*

$$v_z(y) := p_V u(y+z).$$

Then $v_z \in BD(V)$ for $\mathcal{H}^{n-k}$ -almost every $z \in V^\perp$ and for every $B \in \mathcal{B}(\mathbb{R}^n)$ and every $\xi \in V$,

$$(Eu\xi, \xi)(B) = \int_{V^\perp} (Ev_z\xi, \xi)(B_z) d\mathcal{H}^{n-k}(z), \tag{2.2.8}$$

$$|(Eu\xi, \xi)|(B) = \int_{V^\perp} |(Ev_z\xi, \xi)|(B_z) d\mathcal{H}^{n-k}(z), \tag{2.2.9}$$

where $B_z = \{y \in V\colon\ y + z \in B\}$.

Proof. Let $\{\xi_1, \ldots, \xi_k\}$ be a basis of V and let ξ be a vector of the form $\xi_i + \xi_j$. By (ii) of theorem 1.4.8, it is enough to prove that $D_\xi(v_z)^\xi$ is a Radon measure on V for $\mathcal{H}^{n-k}$ -almost every $z \in V^\perp$.
Let us define for any $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, $\xi \in V$ and $A \subset \mathbb{R}^n$

$$\int_A |D_\xi f| := \sup\left\{\int_A f(x) D_\xi \varphi(x) dx,\ \varphi \in \mathcal{C}^1_c(A)\ \ \|\varphi\|_\infty \le 1\right\}.$$

One can easily verify that

$$\int_A |D_\xi f| = \int_A |(\nabla f(x), \xi)| dx$$

if $f \in \mathcal{C}^1(A)$ and that $f \to \int_A |D_\xi f|$ is strongly lower semicontinuous in $L^1_{\text{loc}}(A)$. Since the function $f_z(y) := f(y,z) \in L^1_{\text{loc}}(A_z)$ for $\mathcal{H}^{n-k}$-almost every $z \in V^\perp$, then we can consider

$$\int_{A_z} |D_\xi f_z| := \sup\left\{\int_{A_z} f(y,z) D_\xi \varphi(y) dy;\ \varphi \in \mathcal{C}^1_c(A_z),\ \|\varphi\|_\infty \le 1\right\} \tag{2.2.10}$$

where the convention $\int_{A_z} |D_\xi f_z| = 0$ whenever $A_z = \emptyset$ has been adopted. It follows by regularization that the function $z \to \int_{A_z} |D_\xi f|$ is $\mathcal{H}^{n-k}$-measurable.

Now, let $u_\varepsilon := u \star \psi_\varepsilon$ be a regularization of u. Since $(Eu_\varepsilon\xi,\xi) = (Eu\xi,\xi)\star\psi_\varepsilon$, then

$$\begin{aligned}
\int_A |D_\xi(u^\xi)| &\geq \int_A |(Eu_\varepsilon\xi,\xi)| \\
&= \int_A |(\nabla(u_\varepsilon)^\xi,\xi)|dx \\
&= \int_{V^\perp}\left[\int_{A_z} |(\nabla(v_\varepsilon)_z^\xi,\xi)|d\mathcal{H}^k(y)\right] d\mathcal{H}^{n-k}(z)
\end{aligned}$$

where $(v_\varepsilon)_z(y) := p_V u_\varepsilon(y+z)$ and the last equality follows from Fubini's theorem. Since $(v_\varepsilon)_z^\xi \to (v_z)^\xi$ in $L^1_{loc}(A_z)$ for $\mathcal{H}^{n-k}$-almost every $z \in V^\perp$, we get from Fatou's lemma and the lower semicontinuity of (2.2.10) in $L^1_{\mathrm{loc}}(A_z)$

$$\begin{aligned}
\int_A |D_\xi(u^\xi)| &\geq \liminf_{\varepsilon\to 0^+}\int_{V^\perp}\left[\int_{A_z} |(\nabla(v_\varepsilon)_z^\xi,\xi)|d\mathcal{H}^k(y)\right] d\mathcal{H}^{n-k}(z) \\
&\geq \int_{V^\perp}\left[\liminf_{\varepsilon\to 0^+}\int_{A_z} |(\nabla(v_\varepsilon)_z^\xi,\xi)|d\mathcal{H}^k(y)\right] d\mathcal{H}^{n-k}(z) \\
&\geq \int_{V^\perp}\int_{A_z} |D_\xi(v_z)^\xi| d\mathcal{H}^{n-k}(z).
\end{aligned}$$

So that

$$\int_{V^\perp}\int_{A_z} |D_\xi(v_z)^\xi| d\mathcal{H}^{n-k}(z) \leq |(Eu\xi,\xi)|(A) < +\infty.$$

Now, let $\varphi \in \mathcal{C}^1(A)$. From Fubini's theorem we get

$$\begin{aligned}
\int_A u^\xi D_\xi\varphi(x)dx &= \int_{V^\perp}\left[\int_{A_z} (v_z)^\xi D_\xi\varphi(y,z) d\mathcal{H}^k(y)\right] d\mathcal{H}^{n-k}(z) \\
&\leq \int_{V^\perp}\int_{A_z} |D_\xi(v_z)^\xi| d\mathcal{H}^{n-k}(z).
\end{aligned}$$

So that we have the equality

$$|(Eu\xi,\xi)|(A) = \int_{V^\perp}\int_{A_z} |D_\xi(v_z)^\xi| d\mathcal{H}^{n-k}(z). \tag{2.2.11}$$

This implies that for $\mathcal{H}^{n-k}$-almost every $z \in V^\perp$, $D_\xi(v_z)^\xi$ is a Radon measure on V; the equality (2.2.11) can be extended to any $B \in \mathcal{B}(\mathbb{R}^n)$ and this gives (2.2.9).

(2.2.8) is obtained as in the proof of the first equality of (2.2.2). In fact, given $\varphi \in \mathcal{C}_c^1(\mathbb{R}^n)$ we get

$$\begin{aligned}
\int_{\mathbb{R}^n} \varphi d(Eu\xi, \xi) &= \int_{\mathbb{R}^n} \varphi D_\xi u^\xi = -\int_{\mathbb{R}^n} u^\xi D_\xi \varphi dx \\
&= -\int_{V^\perp} \left(\int_V v_z^\xi(y) D_\xi \varphi_z(y) d\mathcal{H}^k(y) \right) d\mathcal{H}^{n-k}(z) \\
&= \int_{V^\perp} \left(\int_V \varphi_z D_\xi (v_z)^\xi \right) d\mathcal{H}^{n-k}(z).
\end{aligned}$$

Then (2.2.8) follows by a density argument. □

2.3 The rectifiability of the jump set of a BD function

Let $u \in BD(\Omega)$. In order to prove the rectifiability of the Jump set J_u of u defined in the paragraph 1.2.3 of the previous chapter, Kohn has introduced in [141] the set Θ_u of points in Ω where the $(n-1)$-dimensional upper density of $|Eu|$ is positive, i.e.,

$$\Theta_u := \left\{ x \in \Omega\colon \limsup_{\rho \to 0^+} \frac{|Eu|(B_\rho(x))}{\rho^{n-1}} > 0 \right\}. \tag{2.3.1}$$

Lemma 2.3.1 *For every $u \in BD(\Omega)$, the jump set J_u of u is contained in Θ_u.*

Proof. Let x be a jump point of u; using the lower semicontinuity of the map

$$v \to |Ev|(B_1(0))$$

with respect to the $L^1(B_1(0), \mathbb{R}^n)$-norm convergence, we obtain from (1.2.3) and (1.6.2) that

$$\liminf_{\rho \to 0^+} \frac{|Eu|(B_\rho(x))}{\rho^{n-1}} = \liminf_{\rho \to 0^+} |Eu_\rho|(B_1(0)) \geq |Eu_0|(B_1(0)).$$

Since

$$\begin{aligned} |Eu_0|(B_1(0)) &= w_{n-1}\left\|(u^+(x)-u^-(x))\odot\nu(x)\right\| \\ &\geq \frac{w_{n-1}}{\sqrt{2}}\left\|u^+(x)-u^-(x)\right\| > 0, \end{aligned}$$

we get $x \in \Theta_u$. □

The rectifiability of the jump set J_u is proved in the following theorem.

Theorem 2.3.2 *Let $u \in BD(\Omega)$, then the Jump set J_u of u is a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable Borel set. Moreover, $\mathcal{H}^{n-1}(\Theta_u \setminus J_u) = 0$.*

Proof. It is enough to prove that Θ_u is countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable and that $\mathcal{H}^{n-1}(\Theta_u \setminus J_u) = 0$. The rectifiability of the set Θ_u has been proved by Kohn in [141] (see also Ambrosio-Coscia-Dal Maso [21]). We repeat the proof here for the reader's convenience. For any $\varepsilon > 0$, We only need to prove the same property for the Borel set

$$\Theta_u^\varepsilon := \left\{x \in \Omega\colon\ \limsup_{\rho\to 0^+}\frac{|Eu|(B_\rho(x))}{\rho^{n-1}} > \varepsilon\right\}.$$

By covering theorems (see e.g., Federer [119, Theorem 2.10.19], Evans-Gariepy [118]), we have

$$\varepsilon\mathcal{H}^{n-1}(B) \leq w_{n-1}|Eu|(B) \quad \forall B \in \mathcal{B}(\Omega),\ B \subset \Theta_u^\varepsilon. \tag{2.3.2}$$

In particular, $\mathcal{H}^{n-1}(\Theta_u^\varepsilon) < +\infty$ and by virtue of the Federer's structure theorem (see Federer [119, Theorem 3.3.13 and Theorem 2.10.19]) there exists a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable Borel set $R^\varepsilon \subset \Theta_u^\varepsilon$ such that

$$\mathcal{H}^{n-1}(p_\xi(\Theta_u^\varepsilon \setminus R^\varepsilon)) = 0 \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } \xi \in \mathbf{S}^{n-1}.$$

Then from (2.2.7) of Remark 2.2.2 we deduce that $|Eu|(\Theta_u^\varepsilon \setminus R^\varepsilon) = 0$. Choosing now $B = \Theta_u^\varepsilon \setminus R^\varepsilon$ in (2.3.2) we get $\mathcal{H}^{n-1}(\Theta_u^\varepsilon \setminus R^\varepsilon) = 0$ and this proves that Θ_u^ε is rectifiable.

From the rectifiability of Θ_u and (1.6.2) we deduce that the one-sided Lebesgue limits $u^{\pm}(x)$ of u at x are well defined for $\mathcal{H}^{n-1}$-almost every $x \in \Theta_u$. For every $\varepsilon > 0$ the Borel set

$$B^{\varepsilon} = \{x \in \Theta_u^{\varepsilon}\colon\ u^+(x) = u^-(x)\}$$

verifies from (2.3.2) and (1.6.2) $\varepsilon\mathcal{H}^{n-1}(B^{\varepsilon}) \leq w_{n-1}|Eu|(B^{\varepsilon}) = 0$. Therefore

$$\mathcal{H}^{n-1}(\{x \in \Theta_u\colon\ u^+(x) = u^-(x)\}) = 0.$$

That is $\mathcal{H}^{n-1}(\Theta_u \setminus J_u) = 0$. $\square$

Remark 2.3.3 Following Remark 1.6.2 of Chapter I, if ν_u denotes the Borel field of directions with respect to which the one-sided Lebesgue limits of u are defined, then $\nu_u(x)$ is an approximate unit normal to J_u for $\mathcal{H}^{n-1}$-almost every $x \in J_u$. Therefore, from (1.6.2) we get

$$Eu \llcorner J_u = (u^+ - u^-) \odot \nu_u \mathcal{H}^{n-1} \llcorner J_u.$$

2.4 The decomposition of the measure Eu

Given $u \in BD(\Omega)$. The Radon-Nikodym decomposition of the measure Eu gives

$$Eu = E^a u + E^s u$$

where $E^a u = \mathcal{E}u\mathcal{L}^n \llcorner \Omega$ and $E^s u$ are respectively the absolutely continuous part with density $\mathcal{E}u$ and the singular part of Eu with respect to $\mathcal{L}^n$. The following closed subspace of $BD(\Omega)$

$$LD(\Omega) := \{u \in BD(\Omega)\colon\ E^s u = 0\}$$

introduced and studied in Temam [191] plays the same role in $BD(\Omega)$ as $W^{1,1}(\Omega, \mathbb{R}^n)$ does in $BV(\Omega, \mathbb{R}^n)$. As in the remark 1.7.7, one can easily see that

$$W^{1,1}(\Omega, \mathbb{R}^n) \subset LD(\Omega)$$

with strict inclusion. The space $LD(\Omega)$ will be used in Chapter IV in order to study the lower semicontinuity of some "regular" functionals defined in $BD(\Omega)$, i.e., functionals depending only on the absolutely continuous part $E^a u$ of Eu.

In Ambrosio-Coscia-Dal Maso [21], the measure $E^s u$ has been split into two mutually singular parts. That is, analogously to what has been adopted to BV functions,

$$E^s u = E^j u + E^c u.$$

$E^j u := E^s u \llcorner J_u$ is called the *Jump part* of Eu and is characterized following the remark 2.3.3 and $\mathcal{H}^{n-1}(\Theta_u \setminus J_u) = 0$ by

$$E^j u = (u^+ - u^-) \odot \nu_u \mathcal{H}^{n-1} \llcorner J_u = (u^+ - u^-) \odot \nu_u \mathcal{H}^{n-1} \llcorner \Theta_u.$$

$E^c u = E^s u \llcorner (\Omega \setminus J_u)$ is called the *Cantor part* of Eu and has the following property.

Proposition 2.4.1 *For every $u \in BD(\Omega)$ the measure $E^c u$ vanishes on any Borel set $B \subset \Omega$ which is σ-finite with respect to $\mathcal{H}^{n-1}$.*

Proof. Since $\Theta_u \setminus J_u$ is $\mathcal{H}^{n-1}$-negligible, by (2.2.7) of Remark 2.2.2 we only need to prove that

$$|E^c u|(B) = 0 \text{ for any } B \in \mathcal{B}(\Omega) \text{ with } B \cap \Theta_u = \emptyset \text{ and } \mathcal{H}^{n-1}(B) < +\infty.$$

For such a Borel set B, the $(n-1)$-dimensional density of $|E^c u|$ at any its point is zero. Then following covering theorems (see, e.g., Federer [119, Theorem 2.10.19]) we get $|E^c u|(B) \leq \varepsilon \mathcal{H}^{n-1}(B)$ for any $\varepsilon > 0$; hence by the arbitrariness of ε we obtain $|E^c u|(B) = 0$. $\square$

For any $u \in BV(\Omega, \mathbb{R}^n)$ the density of the absolutely continuous part of the measure Du, as we have seen in Chapter I, is interpreted as the approximate differential of u. For any $u \in BD(\Omega)$, a similar result holds as shown by Bellettini-Coscia-Dal Maso [46] (see also Ambrosio-Coscia-Dal Maso [21]), for the density $\mathcal{E}u$ of the absolutely continuous part of Eu using an appropriate terminology.

Definition 2.4.2 *We say that a symmetric linear map $L : \mathbb{R}^n \to \mathbb{R}^n$ is the approximate symmetric differential of a function $u : \Omega \to \mathbb{R}^n$ at a point $x \in \Omega$ if*

$$\lim_{\rho\to 0^+} \fint_{B_\rho(x)} \frac{|(u(y)-u(x)-L(y-x), y-x)|}{|y-x|^2} dy = 0. \tag{2.4.1}$$

One can easily verify that the approximate symmetric differential of u at x is unique.

Theorem 2.4.3 *Let $u \in BD(\Omega)$. For $\mathcal{L}^n$-almost every $x \in \Omega$,*

$$\lim_{\rho\to 0^+} \fint_{B_\rho(x)} \frac{|(u(y)-u(x)-\mathcal{E}u(x)(y-x), y-x)|}{|y-x|^2} dy = 0. \tag{2.4.2}$$

Proof. We give here the proof by Ambrosio-Coscia-Dal Maso [21].
Let $B_\rho(x) \subset \Omega$, $\tau \in (0,1)$ and $v \in \mathcal{C}^1(B_\rho(x), \mathbf{R}^n)$. Using the identity

$$(v(y) - v(x+\tau(y-x)), y-x) = \int_\tau^1 (\mathcal{E}v(x+t(y-x))(y-x), y-x)dt$$

and Fubini's theorem, we obtain

$$\int_{B_\rho(x)} \frac{|v(y)-v(x+\tau(y-x)), y-x)|}{|y-x|^2} dy \le \int_\tau^1 \frac{|Ev|(B_\rho(x))}{t^n} dt. \tag{2.4.3}$$

Now, take $v \in BD(\Omega)$ and let $v_\varepsilon = v \star \psi_\varepsilon$ be the corresponding regularization. After writing (2.4.3) for v_ε, we pass to the limit as $\varepsilon \to 0^+$ using the fact that

$|Ev_\varepsilon|(B_{t\rho}(x))$ converges to $|Ev|(B_{t\rho}(x))$ for any $t \in (0,1)$ such that

$$|Ev|(\partial B_{t\rho}(x)) = 0.$$

Then we obtain (2.4.3) also for $v \in BD(\Omega)$. If x is a Lebesgue point of v, letting $\tau \to 0^+$ and dividing both sides by ρ^n we get

$$\frac{1}{\rho^n}\int_{B_\rho(x)} \frac{|(v(y)-v(x), y-x)|}{|y-x|^2}dy \le \int_0^1 \frac{|Ev|(B_{t\rho}(x))}{(t\rho)^n}dt. \tag{2.4.4}$$

Let us now fix a Lebesgue point x for u and $\mathcal{E}u$ and such that $|E^s u|$ has n-dimensional density zero at x; applying (2.4.4) to the function

$$v(y) := u(y) - \mathcal{E}u(x)(y-x)$$

we obtain

$$\frac{1}{\rho^n}\int_{B_\rho(x)} \frac{|(u(y)-u(x)-\mathcal{E}u(x)(y-x), y-x)|}{|y-x|^2}dy$$

$$\le \sup_{t\in(0,1)} \left[\frac{|E^s u|(B_{t\rho}(x))}{(t\rho)^n} + \frac{1}{(t\rho)^n}\int_{B_{t\rho}(x)} |\mathcal{E}u(y)-\mathcal{E}u(x)|dy\right].$$

By the choice of x, both terms in the right-hand side vanish as $\rho \to 0^+$. □

To summarize, we have seen that for any $u \in BD(\Omega)$, the measure Eu is split into three main parts. That is

$$Eu = E^a u + E^j u + E^c u = \mathcal{E}u\mathcal{L}^n \llcorner \Omega + (u^+ - u^-) \odot \nu_u \mathcal{H}^{n-1} \llcorner J_u + E^c u.$$

2.5 The structure theorem for BD functions

The main result of this paragraph is the theorem 2.5.1 which allows not only to construct any $u \in BD(\Omega)$ from its one-dimensional sections as in Theorem 2.2.2, but also to reproduce each of the three parts of the measure Eu from their corresponding one-dimensional section. In other words, Theorem 2.5.1 below is an analogous of Theorem 1.3.3.

Before giving the statement of the structure theorem, let us introduce the following notations. For any $\xi \in \mathbb{R}^n$, we define

$$J_u^\xi := \{x \in J_u \colon\ (u^+(x) - u^-(x), \xi) \neq 0\}.$$

Since

$$\mathcal{H}^{n-1}(\{\xi \in \mathbf{S}^{n-1} \colon\ (u^+(x) - u^-(x), \xi) = 0\})$$

for every $x \in J_u$, by Fubini's theorem we have

$$\mathcal{H}^{n-1}(J_u \setminus J_u^\xi) = 0 \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } \xi \in \mathbf{S}^{n-1}. \tag{2.5.1}$$

Theorem 2.5.1 (**Structure theorem for** BD **functions**)
Let $u \in BD(\Omega)$ and let $\xi \in \mathbb{R}^n$ with $\xi \neq 0$. Then

(i) $(E^a u\xi, \xi) = \displaystyle\int_{\Omega^\xi} D^a u_y^\xi d\mathcal{H}^{n-1}(y), \quad |E^a u\xi, \xi)| = \int_{\Omega^\xi} |D^a u_y^\xi| d\mathcal{H}^{n-1}(y).$

(ii) *For $\mathcal{H}^{n-1}$-almost every $y \in \Omega^\xi$, the functions u_y^ξ and $\tilde{u}_y^\xi$ belong to $BV(\Omega_y^\xi)$ and coincide $\mathcal{L}^1$-almost everywhere on Ω_y^ξ, the measures $|Du_y^\xi|$ and $V\tilde{u}_y^\xi$ coincide on Ω_y^ξ, and $(\mathcal{E}u(y+t\xi)\xi, \xi) = \nabla u_y^\xi(t) = (\tilde{u}_y^\xi)'(t)$ for $\mathcal{L}^1$-almost every $t \in \Omega_y^\xi$.*

(iii) $(E^j u\xi, \xi) = \displaystyle\int_{\Omega^\xi} D^j u_y^\xi d\mathcal{H}^{n-1}(y), \quad |(E^j u\xi, \xi)| = \int_{\Omega^\xi} |D^j u_y^\xi| d\mathcal{H}^{n-1}(y).$

(iv) *For $\mathcal{H}^{n-1}$ -almost every $y \in \Omega^\xi$, we have $\left(J_u^\xi\right)_y^\xi = J_{u_y^\xi}$ and*

$$(u^+(y+t\xi), \xi) = \left(u_y^\xi\right)^+(t) = \lim_{s \to t^+} \tilde{u}_y^\xi(s),$$

$$(u^-(y+t\xi),\xi) = \left(u_y^\xi\right)^-(t) = \lim_{s\to t^-} \tilde{u}_y^\xi(s)$$

for every $t \in \left(J_u^\xi\right)_y^\xi$, *where the normals to* J_u *and* $J_{u_y^\xi}$ *are oriented so that* $(\nu_u,\xi) \geq 0$ *and* $\nu_{u_y^\xi} = 1$.

(v) $(E^c u\xi,\xi) = \int_{\Omega^\xi} D^c u_y^\xi d\mathcal{H}^{n-1}(y), \quad |(E^c u\xi,\xi)| = \int_{\Omega^\xi} |D^c u_y^\xi| d\mathcal{H}^{n-1}(y).$

Remark 2.5.2 From the structure theorem above one can easily see that a function $u \in L^1(\Omega,\mathbb{R}^n)$ belongs to $LD(\Omega)$ if and only if for any $\xi \in \mathbf{S}^{n-1}$ the following two conditions hold:

(a) $u_y^\xi \in W^{1,1}(\Omega_y^\xi)$ for $\mathcal{H}^{n-1}$-almost every $y \in \Omega^\xi$,

(b) $\int_{\Omega^\xi} \left[\int_{\Omega_y^\xi} |(u_y^\xi)'(t)|dt\right] d\mathcal{H}^{n-1}(y) < +\infty.$

2.5.1 The space $SBD(\Omega)$ of special functions with bounded deformation

Before beginning the proof of the structure theorem, let us present here a proper subspace of $BD(\Omega)$ introduced first by Bellettini, Coscia and Dal Maso in [46] and also studied by Ambrosio-Coscia-Dal Maso [21] under another definition. It is the class $SBD(\Omega)$ of *special functions with bounded deformation.* The first definition of $SBD(\Omega)$ given in [46] is the following.

Definition 2.5.3 (Bellettini-Coscia-Dal Maso [46])
A function $u \in BD(\Omega)$ *belongs to* $SBD(\Omega)$ *if for* $\mathcal{H}^{n-1}$*-almost every* $\xi \in \mathbf{S}^{n-1}$ *we have*

$$u_y^\xi \in SBV(\Omega_y^\xi) \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } y \in \Omega^\xi.$$

As we have seen, for any $u \in BD(\Omega)$ the Cantor part $E^c u$ does not have a precise representation. Motivated by the structure theorem above Ambrosio, Coscia and Dal Maso have introduced in [21] another definition of the class $SBD(\Omega)$ and they were able to prove that their definition is equivalent to the previous one.

Definition 2.5.4 (Ambrosio-Coscia-Dal Maso [21])
A function $u \in BD(\Omega)$ is in $SBD(\Omega)$ if the Cantor part $E^c u$ of the measure Eu is zero, i.e., $|E^s u|(\Omega \setminus J_u) = 0$.

Proposition 2.5.5 *Let $u \in BD(\Omega)$ and let $\{\xi_1, \dots, \xi_n\}$ be a basis of $\mathbb{R}^n$. Then the following three conditions are equivalent:*

(i) *$u \in SBD(\Omega)$ following Definition 2.5.4.*

(ii) *For every $\xi = \xi_i + \xi_j$ with $1 \le i, j \le n$, we have $u_y^\xi \in SBV(\Omega_y^\xi)$ for $\mathcal{H}^{n-1}$-almost every $y \in \Omega^\xi$.*

(iii) *The measure $|E^s u|$ is concentrated on a Borel set $B \subset \Omega$ which is σ-finite with respect to $\mathcal{H}^{n-1}$.*

The proof of this proposition easily follows from (v) of the structure theorem 2.5.1, from (2.1.2) and the rectifiability of J_u.

In the following proposition proved in [46, 47] we show some relationships between the spaces $BD(\Omega)$, $SBD(\Omega)$, $BV(\Omega, \mathbb{R}^n)$ and $SBV(\Omega, \mathbb{R}^n)$.

Proposition 2.5.6 *We have $BV(\Omega, \mathbb{R}^n) \cap SBD(\Omega) = SBV(\Omega, \mathbb{R}^n)$. Moreover, the inclusions $SBV(\Omega, \mathbb{R}^n) \subset SBD(\Omega) \subset BD(\Omega)$ are strict.*

Proof. The inclusion $SBV(\Omega, \mathbb{R}^n) \subset BV(\Omega, \mathbb{R}^n) \cap SBD(\Omega)$ is obvious. To prove the opposite inclusion, let us fix $u \in BV(\Omega, \mathbb{R}^n) \cap SBD(\Omega)$. Therefore $E^c u = 0$. Let $D^c u$ be the Cantor part of Du and let Ψ the density of $D^c u$ with respect to its variation $|D^c u|$. Since $u \in BV(\Omega, \mathbb{R}^n)$, the measure $E^c u$ is the symmetric part of $D^c u$. This means that the symmetric part of the matrix $\Psi(x)$ vanishes for $|D^c u|$-almost every $x \in \Omega$. On the other hand, following a result by Alberti [7] $\Psi(x)$ is a rank-one matrix for $|D^c u|$-almost every $x \in \Omega$. Therefore $\Psi = 0$ $|D^c u|$-almost everywhere in Ω and this implies $D^c u = 0$, so that $u \in SBV(\Omega, \mathbb{R}^n)$.

Since it is known that there exists a function u in $BV(\Omega, \mathbb{R}^n)$ which does not belong to $SBV(\Omega, \mathbb{R}^n)$, the identity $BV(\Omega, \mathbb{R}^n) \cap SBD(\Omega) = SBV(\Omega, \mathbb{R}^n)$ implies $u \notin SBD(\Omega)$, and so the inclusion $SBD(\Omega) \subset BD(\Omega)$ is strict.
In order to prove that the inclusion $SBV(\Omega, \mathbb{R}^n) \subset SBD(\Omega)$ is strict, it is enough to construct a function $u \in SBD(\Omega)$ such that $u \notin BV(\Omega, \mathbb{R}^n)$. The following example has been given in [21]. Let $B = B_1(0)$. Using an example of Ornstein [170] we can construct a sequence $\{\varphi_h\} \in \mathcal{C}_c^\infty(B, \mathbb{R}^n)$ which does not satisfy Korn's inequality for the L^1 norm, i.e., if we normalize $\int_B |\nabla \varphi_h| dx$, we have

$$2^h \int_B |\mathcal{E}\varphi_h| dx \leq \int_B |\nabla \varphi_h| dx = 1. \tag{2.5.2}$$

In particular, by Poincaré inequality, the integrals $\int_B |\varphi_h| dx$ are bounded by a constant C independent of h. Let us now consider a sequence of pairwise disjoint balls $B_h := B_{r_h}(x_h)$ contained in B such that $\sum_{h=1}^\infty r_h < +\infty$ and whose centers x_h converges to 0. Let us consider the function $u \in \mathcal{C}^\infty(\mathbb{R}^n \setminus \{0\}, \mathbb{R}^n)$ with compact support in B and defined by

$$u(x) := \begin{cases} r_h^{1-n} \varphi_h(r_h^{-1}(x - x_h)) & \text{if } x \in B_h, \\ 0 & \text{otherwise.} \end{cases}$$

As $\sum_h r_h < +\infty$, we have $u \in L^1(\Omega, \mathbb{R}^n)$ and, by a change of variables we have

$$\begin{aligned} \int_B |\nabla u| dx &\geq \sum_{h=1}^{+\infty} r_h^{1-n} \int_{B_h} |\nabla(\varphi_h(r^{-1}(x - x_h)))| dx \\ &= \sum_{h=1}^{\infty} \int_B |\nabla \varphi_h| dx = +\infty, \end{aligned}$$

while by (2.5.2), $\int_B |\mathcal{E}u| dx \leq \sum_{h=1}^{\infty} 2^{-h} < +\infty$, so that $u \in LD(\Omega) \subset SBD(\Omega)$, but $u \notin BV(\Omega, \mathbb{R}^n)$. $\square$

2.5.2 An integral-geometric measure related to BD functions

Another consequence of the structure theorem is the introduction in [46] of an integral-geometric measure associated with the jump set of the one-dimensional sections of BD functions. The first compactness criterion in the space $SBD(\Omega)$ has been proved in [46] using that measure.

Definition 2.5.7 *For any $u \in BD(\Omega)$ we define the non-negative Borel measure λ_u on Ω as*

$$\lambda_u(B) := k_n \int_{\mathbf{S}^{n-1}} \lambda_u^\xi(B) d\mathcal{H}^{n-1}(\xi) \quad \forall B \in \mathcal{B}(\Omega).$$

where $k_n := (2w_{n-1})^{-1}$ and for any $\xi \in \mathbf{S}^{n-1}$,

$$\lambda_u^\xi(B) := \int_{\Omega^\xi} \mathcal{H}^0(J_{u_y^\xi} \cap B_y^\xi) d\mathcal{H}^{n-1}(y) \quad \forall B \in \mathcal{B}(\Omega).$$

The following theorem characterizing the integral-geometric measure λ_u as the restriction to J_u of $\mathcal{H}^{n-1}$ has allowed Bellettini, Coscia and Dal Maso to prove in [47] their compactness criterion in $SBD(\Omega)$ in a more realistic form.

Theorem 2.5.8 *For every $u \in BD(\Omega)$ and any $\xi \in \mathbf{S}^{n-1}$,*

$$\lambda_u^\xi(B) = \int_{J_u^\xi \cap B} |(\nu_u, \xi)| d\mathcal{H}^{n-1}(y) \quad B \in \mathcal{B}(\Omega) \tag{2.5.3}$$

where ν is the approximate unit normal to J_u. Moreover $\lambda_u = \mathcal{H}^{n-1} \llcorner J_u$.

Proof. From the equality $J_{u_y^\xi} = \left(J_u^\xi\right)_y^\xi$ in (iv) of the structure theorem, we get

$$\begin{aligned} \lambda_u^\xi(B) &= \int_{\Omega^\xi} \mathcal{H}^0(\{t \in \mathbb{R}\colon\, y + t\xi \in B \cap J_u^\xi\}) d\mathcal{H}^{n-1}(y) \\ &= \mathcal{H}^{n-1}\Big(p_\xi(B \cap J_u^\xi)\Big) \end{aligned} \tag{2.5.4}$$

where the last equality follows from the area formula for the map $p_\xi : J_u \to \pi_\xi$. Since $|(\nu_u, \xi)|$ is the area element of p_ξ, we get 2.5.3. The equality $\lambda_u = \mathcal{H}^{n-1} \llcorner J_u$

will then follow integrating the two members of (2.5.3) on $\mathbf{S}^{n-1}$ and using the formula

$$\int_{\mathbf{S}^{n-1}} |(\nu,\xi)| d\mathcal{H}^{n-1}(\xi) = k_n^{-1} = 2w_{n-1}$$

for any $\nu \in \mathbf{S}^{n-1}$. $\square$

2.5.3 The proof of the structure theorem 2.5.1

Since all the arguments here are local, we can assume $\Omega = \mathbb{R}^n$. The crucial point in the proof of this theorem is the use of an induction process on the dimension n in order to prove that the measure $\int_{\pi_\xi} D^j u_y^\xi d\mathcal{H}^{n-1}(y)$ is concentrated on J_u for $u \in BD(\mathbb{R}^n)$. The main ingredient in the passage from the lower dimensions to n is the following theorem which states that under some assumptions, for any $\eta \in \mathbf{S}^{n-1}$, points of $(n-1)$-dimensional density zero for $|Eu|$ are Lebesgue points of the restriction of $p_\eta u$ to π_η.

Theorem 2.5.9 *Let $u \in BD(\mathbb{R}^n)$, $x \notin \Theta_u$, $\eta \in \mathbf{S}^{n-1}$. Assume that $\mathcal{H}^{n-1}$-almost every point of $x + \pi_\eta$ is a Lebesgue point of u in $\mathbb{R}^n$. If we further assume that the function $v : \pi_\eta \to \pi_\eta$ defined by $v(y) := p_\eta \tilde{u}(x+y)$ has one-sided Lebesgue limits at 0 in π_η with respect to a suitable direction $\nu \in \mathbf{S}^{n-1} \cap \pi_\eta$, then 0 is a Lebesgue point of v in π.*

Proof. Up to a rotation and a translation we can always assume that $\eta = e_n$ and $x = 0$. If we denote $\pi = \pi_{e_n}$, the function $v : \pi \to \pi$ is defined by $v(y) = (\tilde{u}^1, \cdots, \tilde{u}^{n-1}, 0)$. Up to a rotation in π it is not restrictive to assume also that $\nu^1 = (\nu, e_1) > 0$ and that the one-sided Lebesgue limits v_0^+ and v_0^- of v in π satisfy $2|(v_0^+ - v_0^-, e_1)| \geq |v_0^+ - v_0^-|$. We claim that $v_0^+ = v_0^-$.
Let

$$Y := \left\{ \rho \geq 0 \colon \begin{array}{l} y + \rho e_1 \text{ and } y - \rho e_1 \text{ are Lebesgue} \\ \text{points of } u \text{ for } \mathcal{H}^{n-1}\text{-a.e. } y \in \pi \end{array} \right\}.$$

By Fubini's theorem and the fact that $\mathcal{L}^n$-a.e. $x \in \mathbb{R}^n$ are Lebesgue points of u we have $\mathcal{L}^1(\mathbb{R}_+ \setminus Y) = 0$. By our assumption we also have that $0 \in Y$.

For every $\rho \geq 0$, let C_ρ be the n-dimensional closed cube in $\mathbb{R}^n$ with centre 0 and sides parallel to the axes with length 2ρ. We set

$$Q_\rho := C_\rho \cap \pi, \quad Q_\rho^\pm := \{y \in Q_\rho \colon (y, \pm\nu) \geq 0\}$$

and $A_\rho := \pi \cap \overline{B_{\nu^1\rho}(0)}$ so that $A_\rho + \rho e_1 \subset Q_{2\rho}^+$ and $A_\rho - \rho e_1 \subset Q_{2\rho}^-$.
By the assumptions we have that

$$\lim_{\rho\to 0^+} \frac{1}{\rho^{n-1}} |Eu|(C_\rho) = 0 \tag{2.5.5}$$

and

$$\lim_{\rho\to 0^+} \frac{1}{\rho^{n-1}} \int_{Q_\rho^\pm} |v(y) - v_0^\pm| d\mathcal{H}^{n-1}(y) = 0. \tag{2.5.6}$$

Since $|v_0^+ - v_0^-| \leq 2|(v_0^+ - v_0^-, e_1)|$, for every $\rho > 0$ we have

$$\begin{aligned} |v_0^+ - v_0^-| \;&\leq\; 2\Big(|(v(y+\rho e_1) - v_0^+, e_1)| \\ &\qquad + |(v(y-\rho e_1) - v_0^-, e_1)| \\ &\qquad\quad + |(v(y+\rho e_1) - v(y-\rho e_1), e_1)|\Big) \\ &\leq\; 2\Big(|v(y+\rho e_1) - v_0^+| + |v(y-\rho e_1) - v_0^-| \\ &\qquad\quad + |v^1(y+\rho e_1) - v^1(y-\rho e_1)|\Big). \end{aligned} \tag{2.5.7}$$

Now integrating the two members of (2.5.7) in A_ρ with respect to $\mathcal{H}^{n-1}$ we obtain

$$\begin{aligned} |v_0^+ - v_0^-| \;&\leq\; \frac{2k}{\rho^{n-1}} \Big[\int_{A_\rho} |v(y+\rho e_1) - v_0^+| d\mathcal{H}^{n-1}(y) \\ &\qquad + \int_{A_\rho} |v(y-\rho e_1) - v_0^-| d\mathcal{H}^{n-1}(y) \\ &\qquad\quad + \int_{A_\rho} |v^1(y+\rho e_1) - v^1(y-\rho e_1)| d\mathcal{H}^{n-1}(y)\Big], \end{aligned}$$

where $k = (w_{n-1}(\nu^1)^{n-1})^{-1}$.

Then by changes of variables we get

$$
\begin{aligned}
(2.5.8) \qquad |v_0^+ - v_0^-| \le\; & \frac{2k}{\rho^{n-1}}\Big[\int_{A_\rho+\rho e_1} |v(y) - v_0^+|\,d\mathcal{H}^{n-1}(y) \\
&+ \int_{A_\rho-\rho e_1} |v(y) - v_0^-|\,d\mathcal{H}^{n-1}(y) \\
&+ \int_{A_\rho} |v^1(y+\rho e_1) - v^1(y-\rho e_1)|\,d\mathcal{H}^{n-1}(y)\Big] \\
\le\; & \frac{2k}{\rho^{n-1}}\Big[\int_{Q_{2\rho}^+} |v(y) - v_0^+|\,d\mathcal{H}^{n-1}(y) \\
&+ \int_{Q_{2\rho}^-} |v(y) - v_0^-|\,d\mathcal{H}^{n-1}(y)\Big] \\
&+ \frac{2k}{\rho^{n-1}}\Big[\int_{A_\rho} |v^1(y+\rho e_1) - v^1(y-\rho e_1)|\,d\mathcal{H}^{n-1}(y)\Big].
\end{aligned}
$$

By (2.5.6) the first two terms of (2.5.8) vanishes as $\rho \to 0^+$. So, to achieve the proof of the theorem, we only need to show that also the last term vanishes as $\rho \to 0^+$. Let us observe that for any $\rho \in Y$ with $\rho > 0$ the term

$$2\Big[v^1(y+\rho e_1) - v^1(y-\rho e_1)\Big]$$

can be written as

$$
\begin{aligned}
2\Big[v^1(y+\rho e_1) - v^1(y-\rho e_1)\Big] =\; & \Big[(\tilde u^1 - \tilde u^n)(y-\rho e_n) - (\tilde u^1 - \tilde u^n)(y-\rho e_1)\Big] \\
-\; & \Big[(\tilde u^1 - \tilde u^n)(y+\rho e_n) - (\tilde u^1 - \tilde u^n)^1(y+\rho e_1)\Big] \\
(2.5.9) \qquad +\; & \Big[(\tilde u^1 + \tilde u^n)(y-\rho e_n) - (\tilde u^1 + \tilde u^n)^1(y-\rho e_1)\Big] \\
-\; & \Big[(\tilde u^1 + \tilde u^n)(y-\rho e_n) - (\tilde u^1 + \tilde u^n)^1(y+\rho e_1)\Big] \\
+\; & \Big[\tilde u^n(y-\rho e_n) - \tilde u^n(y+\rho e_n)\Big]
\end{aligned}
$$

for $\mathcal{H}^{n-1}$-almost every $y \in \pi$. Since the last term in (2.5.9) is an increment of the function $\tilde u_n$ in the direction e_n, by the point (ii) of theorem 2.2.1 we get

$$|\tilde u^n(y-\rho e_n) - \tilde u^n(y+\rho e_n)| \le V\tilde u^n_y([-\rho,\rho]) = |Du^n_y|([-\rho,\rho])$$

for $\mathcal{H}^{n-1}$-almost every $y \in A_\rho$. Then integrating the two members on A_ρ and using the second equality of (2.2.2) we obtain

$$\int_{A_\rho} |\tilde{u}^n(y-\rho e_n) - \tilde{u}^n(y+\rho e_n)| d\mathcal{H}^{n-1}(y) \leq |(Eue_n, e_n)|(C_\rho) \leq |Eu|(C_\rho)$$

for any $\rho \in Y$ with $\rho > 0$. Analogously the other terms of (2.5.9) are increments of the function $\tilde{u}^1 - \tilde{u}^n$ in the direction $e_1 - e_n$ or of the function $\tilde{u}^1 + \tilde{u}^n$ in the direction $e_1 + e_n$. Let us estimate for example the term

$$\int_{A_\rho} |(\tilde{u}^1 + \tilde{u}^n)(y+\rho e_n) - (\tilde{u}^1 + \tilde{u}^n)(y - \rho e_1)| d\mathcal{H}^{n-1}(y).$$

Set $\xi = e_1 + e_n, \quad p(y,\rho) = p_\xi(y - \rho e_1) \quad$ and $\quad \Sigma_\rho = p_\xi(A_\rho - \rho e_1)$; we have

$$y + \rho e_n = p(y,\rho) + y_1 \xi \quad \text{and} \quad y - \rho e_1 = p(y,\rho) + (y_1 - \rho)\xi.$$

Then

$$\begin{aligned}
&\int_{A_\rho} |(\tilde{u}^1 + \tilde{u}^n)(y+\rho e_n) - (\tilde{u}^1 + \tilde{u}^n)(y - \rho e_1)| d\mathcal{H}^{n-1}(y) \\
&\quad = \int_{A_\rho} |\tilde{u}^\xi_{p(y,\rho)}(y_1) - \tilde{u}^\xi_{p(y,\rho)}(y_1 - \rho)| d\mathcal{H}^{n-1}(y) \\
&\quad = \int_{A_\rho - \rho e_1} |\tilde{u}^\xi_{p_\xi y}(y_1 + \rho) - \tilde{u}^\xi_{p_\xi y}(y_1)| d\mathcal{H}^{n-1}(y) \\
&\quad \leq \int_{A_\rho - \rho e_1} V\tilde{u}^\xi_{p_\xi y}([(-\nu^1 + 1)\rho, (\nu^1 + 1)\rho]) d\mathcal{H}^{n-1}(y) \\
&\quad \leq \sqrt{2} \int_{\Sigma_\rho} |Du^\xi_{p_\xi y}|([(-\nu^1 + 1)\rho, (\nu^1 + 1)\rho]) d\mathcal{H}^{n-1}(y) \\
&\quad \leq \sqrt{2} |(Eu\xi, \xi)|(C_{2\rho}) \leq 2\sqrt{2}|Eu|(C_{2\rho}).
\end{aligned}$$

Analogously, we obtain the estimates of the other terms. Hence for some positive constant C we finally get

$$\int_{A_\rho} |v^1(y+\rho e_1) - v^1(y - \rho e_1)| d\mathcal{H}^{n-1}(y) \leq C|Eu|(C_{2\rho})$$

for any $\rho > 0$ with $\rho > 0$. Thus by (2.5.6) the last term at the right-hand side of (2.5.8) also vanishes as $\rho \to 0^+$; hence the equality $v_0^+ = v_0^-$ follows from (2.5.8) and this achieves the proof of the theorem. □

The proof of Theorem 2.5.1. We assume that $\Omega = \mathbb{R}^n$. Let $u \in BD(\mathbb{R}^n)$ and let $\xi \in \mathbb{R}^n$ with $\xi \neq 0$. Let

$$Y := \left\{ y \in \pi_\xi \colon \begin{array}{l} u_y^\xi \in BV(\mathbb{R}),\ \tilde{u}_y^\xi = u_y^\xi\ \mathcal{L}^1\text{-almost everywhere} \\ \text{in } \mathbb{R} \text{ and } V\tilde{u}_y^\xi = |Du_y^\xi| \text{ as measures on } \mathbb{R} \end{array} \right\}.$$

We have $\mathcal{H}^{n-1}(\pi_\xi \setminus Y) = 0$ by Theorem 2.2.1. From the identity

$$(Eu\xi, \xi) = \int_{\pi_\xi} Du_y^\xi d\mathcal{H}^{n-1}(y)$$

proved in Theorem 2.2.1, we get that

$$(E^a u\xi, \xi) - \int_{\pi_\xi} D^a u_y^\xi d\mathcal{H}^{n-1}(y) = \int_{\pi_\xi} D^s u_y^\xi d\mathcal{H}^{n-1}(y) - (E^s u\xi, \xi).$$

Since the left-hand side is absolutely continuous with respect to $\mathcal{L}^n$, while the right-hand side is singular, necessary both members are 0 and then the first equality in (i) is proved. The second equality follows from the formula (1.3.8) in Chapter I giving the total variation of a measure defined by integration.
Let us recall that for $y \in Y$ we have $D^a u_y^\xi = \nabla u_y^\xi \mathcal{L}^1 = \left(\tilde{u}_y^\xi\right)' \mathcal{L}^1$. Then we get

$$(E^a u\xi, \xi) = \int_{\pi_\xi} \nabla u_y^\xi \mathcal{L}^1 d\mathcal{H}^{n-1}(y) = \int_{\pi_\xi} \left(\tilde{u}_y^\xi\right)' \mathcal{L}^1 d\mathcal{H}^{n-1}(y).$$

Now since the density of $E^a u$ is $\mathcal{E}u$, the point (ii) follows by Fubini's theorem.
The decompositions in jump and Cantor parts of the measures $E^s u$ and $D^s u_y^\xi$ yield to the equality

$$(E^j u\xi, \xi) - \int_{\pi_\xi} D^j u_y^\xi d\mathcal{H}^{n-1}(y) = \int_{\pi_\xi} D^c u_y^\xi d\mathcal{H}^{n-1}(y) - (E^c u\xi, \xi). \tag{2.5.10}$$

Now we claim that both sides in (2.5.10) are 0. By definition $|E^c u|(J_u) = 0$. Using the fact that the measures $D^c u_y^\xi$ are non-atomic and the result of Federer [119, Corollary 2.10.11] which states that for $\mathcal{H}^{n-1}$ -almost every $y \in \pi_\xi$, the line $y + \xi\mathbb{R}$ meets any set B which is σ-finite with respect to $\mathcal{H}^{n-1}$ in an at most countable set, we can show that the restriction to J_u of the right-hand side of (2.5.10) is identically equal to 0. So, to prove the assertion, we only need to show that the left-hand side is concentrated on J_u.

To this aim it is sufficient to prove that the measure $\int_{\pi_\xi} D^j u_y^\xi d\mathcal{H}^{n-1}(y)$ is concentrated on J_u. This will be done by induction on $n \geq 2$ using Theorem 2.5.9.

The case $n = 2$ is trivial since the measure $\mu := \int_{\pi_\xi} D^j u_y^\xi d\mathcal{H}^{n-1}(y)$ is concentrated on the set of points of the form $y + t\xi$ with $y \in Y$ and $t \in J_{u_y^\xi}$. By Theorem 2.5.9, these points belong to Θ_u; otherwise such t will be Lebesgue's points of u_y^ξ. Hence μ is concentrated on Θ_u. Since by definition, the measure μ vanishes on $\mathcal{H}^{n-1}$-negligible sets and $\mathcal{H}^{n-1}(\Theta_u \setminus J_u) = 0$, then μ is concentrated on J_u.

Now we assume that the statement is valid for $n - 1 \geq 2$ and we want to prove it for n. Let us choose $\eta \in \mathbf{S}^{n-1}$ orthogonal to ξ, so that the hyperplane π_η contains 0 and ξ. For any $s \in \mathbb{R}$ we consider the function $v_s : \pi_\eta \to \pi_\eta$ defined by $v_s(z) := p_\eta u(z + s\eta)$. By Proposition 2.2.3 we have $v_s \in BD(\pi_\eta)$ for $\mathcal{L}^1$-almost every $s \in \mathbb{R}$. Moreover, since $\xi \in \pi_\eta$, for any $w \in \pi_{\eta\xi} := \pi_\eta \cap \pi_\xi$ and any $s, t \in \mathbb{R}$ we have

$$(v_s)_w^\xi(t) = (v_s(w + t\xi), \xi) = (p_\eta u(w + s\eta + t\xi), \xi) = u_{w+s\eta}^\xi(t).$$

On the other hand, if for any $B \in \mathcal{B}(\Omega)$ and $s \in \mathbb{R}$ we set

$$B_s := \{z \in \pi_\eta\colon\ z + s\eta \in B\},$$

then using the fact that $\xi \in \pi_\eta$ we also get $(B_s)_w^\xi = B_{w+s\eta}^\xi$ for any $w \in \pi_{\eta\xi}$. Fubini's theorem then yields to the equality

$$\text{(2.5.11)} \quad \begin{aligned} \int_{\pi_\xi} D^j u_y^\xi(B_y^\xi) d\mathcal{H}^{n-1}(y) &= \int_{\mathbb{R}} \Big[\int_{\pi_{\eta\xi}} D^j u_{w+s\eta}^\xi(B_{w+s\eta}^\xi) d\mathcal{H}^{n-2}(w)\Big] ds \\ &= \int_{\mathbb{R}} \Big[\int_{\pi_{\eta\xi}} D^j (v_s)_w^\xi\big((B_s)_w^\xi\big) d\mathcal{H}^{n-2}(w)\Big] ds. \end{aligned}$$

Let

$$S := \{s \in \mathbb{R}\colon\ v_s \in BD(\pi_\eta),\ v_s(z) = p_\eta \tilde{u}(z + s\eta) \text{ for } \mathcal{H}^{n-1}\text{-a.e. } z \in \pi_\eta\}.$$

Fubini's theorem implies $\mathcal{L}^1(\mathbb{R} \setminus S) = 0$. By the induction assumption, for any $s \in S$ the measure

$$\mu_s(B) := \int_{\pi_\eta \xi} D^j(v_s)^\xi_w \left((B_s)^\xi_w\right) d\mathcal{H}^{n-2}(w) \quad B \in \mathcal{B}(\pi_\eta).$$

is concentrated on J_{v_s}. Therefore the measure μ is concentrated on the set

$$Y' := \{x \in \mathbb{R}^n\colon\ x = z + s\eta,\ z \in J_{v_s} \text{ and } s \in S\}.$$

This set is contained in Θ_u. Otherwise by Theorem 2.5.9 for any point $x = z+s\eta \in Y'$ such z would be a Lebesgue point of the function v_s and this is impossible since $z \in J_{v_s}$ by definition. Therefore, exactly as in the case $n = 2$ we deduce that μ is concentrated on J_u. Then this proves that the two members of (2.5.10) are identically equal to 0 so that the first equalities of points (iii) and (v) are proved. The second equalities follow from (1.3.8) of Chapter I. Let us now prove the point (iv). Applying the area formula for rectifiable sets to the map $p_\xi : J_u \to \pi_\xi$ whose jacobian is $(\nu_u, \xi) \geq 0$ we obtain

$$\begin{aligned}
(E^j u\xi, \xi)(B) &= \int_{B \cap J_u^\xi} (u^+(x) - u^-(x), \xi)(\nu_u(x), \xi) d\mathcal{H}^{n-1}(x) \\
&= \int_{\pi_\xi} \Big[\sum_{t \in (B \cap J_u^\xi)^\xi_y} (u^+(y + t\xi) - u^-(y + t\xi), \xi) \Big] d\mathcal{H}^{n-1}(y)
\end{aligned}$$

for any Borel set $B \subset \mathbb{R}^n$. Hence, from (iii) we get the equality of measures

$$D^j u^\xi_y = \sum_{t \in (J_u^\xi)^\xi_y} (u^+(y + t\xi) - u^-(y + t\xi), \xi)\delta_t \tag{2.5.12}$$

for $\mathcal{H}^{n-1}$-almost every $y \in Y$, where δ_t denotes the Dirac measure. On the other hand, the measure $D^j u^\xi_y$ is characterized by the equality

$$D^j u^\xi_y = \sum_{t \in J_{u^\xi_y}} \Big[\left(u^\xi_y\right)^+(t) - \left(u^\xi_y\right)^-(t) \Big] \delta_t \tag{2.5.13}$$

for every $y \in Y$ (recalling that the sets J_u and J_u^ξ have been oriented in such a way that $\nu_{u_y^\xi} = 1$). The equalities

$$\left(J_u^\xi\right)_y^\xi = J_{u_y^\xi}, \tag{2.5.14}$$

$$(u^+(y+t\xi) - u^-(y+t\xi), \xi) = \left(u_y^\xi\right)^+(t) - \left(u_y^\xi\right)^-(t) \quad \forall t \in \left(J_u^\xi\right)_y^\xi. \tag{2.5.15}$$

are then obtained from (2.5.13) for any $y \in Y$ satisfying (2.5.12).

We prove the full property (iv) in three main steps using a localization argument. Precisely, from the rectifiability of J_u, we can say that this set is covered $\mathcal{H}^{n-1}$-almost everywhere by a countable union of boundaries of $\mathcal{C}^1$ bounded open sets $\{\Omega_k\}_k \subset \mathbb{R}^n$. That is

$$J_u \subset \bigcup_{k \in \mathbb{N}} \partial\Omega_k \quad \text{and} \quad \mathcal{H}^{n-1}\Big(\bigcup_{k \in \mathbb{N}} \partial\Omega_k \setminus J_u\Big) = 0.$$

For any $k \in \mathbb{N}$, let us define the function $v^k : \mathbb{R}^n \to \mathbb{R}^n$ by

$$v^k(x) := \begin{cases} u(x) & \text{if } x \in \Omega_k \\ 0 & \text{otherwise.} \end{cases}$$

From Corollary 1.6.3, we have $v^k \in BD(\mathbb{R}^n)$. We choose the sign of ν_{v^k} so that $(\nu_{v^k}, \xi) \geq 0$ in J_{v^k}. In order to simplify the notations we will write $v = v^k$. Let us divide the boundary $\partial\Omega_k$ in three disjoint parts which are

$$\partial_\xi^\pm \Omega_k := \{x \in \partial\Omega_k : \ \pm (\nu_{\Omega_k}(x), \xi) > 0\} \text{ and } \partial_\xi^0 \Omega_k := \{x \in \partial\Omega_k : (\nu_{\Omega_k}(x), \xi) = 0\},$$

where ν_{Ω_k} denotes the outward unit normal to $\partial\Omega_k$. Since $(\nu_u, \xi) \geq 0$, by the definition of the approximate unit normal we have

$$\nu_u(x) = \pm\nu_{\Omega_k}(x) \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in J_u \cap \partial_\xi^\pm \Omega_k. \tag{2.5.16}$$

Step 1: Let us prove that for $\mathcal{H}^{n-1}$-almost every $y \in \pi_\xi$ we have

$$(u^-(y+t\xi), \xi) = (u_y^\xi)^-(t) \quad \forall t \in (J_u^\xi \cap \partial_\xi^+ \Omega_k)_y^\xi. \tag{2.5.17}$$

If $x \in J_u^\xi \cap \partial_\xi^+ \Omega_k$ satisfies $\nu_u(x) = \nu_{\Omega_k}(x)$, then as a consequence of the trace theorem, the function v has one-sided Lebesgue limits with respect to $\nu_u(x)$ and

$$v^+(x) = 0 \quad , \quad v^-(x) = u^-(x). \tag{2.5.18}$$

Let $y \in Y$ and $t \in (J_u^\xi \cap \partial_\xi^+ \Omega_k)_y^\xi$. From the fact that $(\nu_{\Omega_k}(y+t\xi), \xi) > 0$, one can see that the functions u_y^ξ and v_y^ξ have one-sided Lebesgue limits with respect to $\nu = 1$ and

$$\left(v_y^\xi\right)^+(t) = 0, \qquad \left(v_y^\xi\right)^-(t) = \left(u_y^\xi\right)^-(t). \tag{2.5.19}$$

Let us now divide $J_u^\xi \cap \partial_\xi^+ \Omega_k$ into the following two disjoint sets:

$$M := \{x \in J_u^\xi \cap \partial_\xi^+ \Omega_k \colon (u^-(x), \xi) \neq 0\}, \quad N := \{x \in J_u^\xi \cap \partial_\xi^+ \Omega_k \colon (u^-(x), \xi) = 0\}.$$

If we apply (2.5.14) to the function v we obtain that $J_{v_y^\xi} = (J_v^\xi)_y^\xi$ for $\mathcal{H}^{n-1}$-almost every $y \in \pi_\xi$. From (2.5.18) and (2.5.16) one can easily see that $N_y^\xi \cap (J_v^\xi)_y^\xi = \emptyset$ for $\mathcal{H}^{n-1}$-almost every $y \in \pi_\xi$. Therefore $\left(v_y^\xi\right)^+ = \left(v_y^\xi\right)^-$ in N_y^ξ for $\mathcal{H}^{n-1}$-almost every $y \in \pi_\xi$. By (2.5.19) this implies that $\left(u_y^\xi\right)^- = 0$ in N_y^ξ and consequently, for $\mathcal{H}^{n-1}$-almost every $y \in \pi_\xi$ we have

$$(u^-(y+t\xi), \xi) = 0 = \left(u_y^\xi\right)^-(t) \quad t \in N_y^\xi. \tag{2.5.20}$$

On the other hand recalling that $(\nu_v, \xi) \geq 0$, from the equalities (2.5.19) and (2.5.16) we have for $\mathcal{H}^{n-1}$-almost every $y \in \pi_\xi$

$$M_y^\xi \subset \left(J_v^\xi\right)_y^\xi \quad \text{and} \quad \nu_v(y+t\xi) = \nu_u(y+t\xi) \quad \text{for every } t \in M_y^\xi.$$

Applying (2.5.15) to the function v and taking into account (2.5.19) and (2.5.16), for $\mathcal{H}^{n-1}$-almost every $y \in \pi_\xi$ we obtain

$$(u^-(y+t\xi), \xi) = \left(u_y^\xi\right)^-(t) \quad \forall t \in M_y^\xi. \tag{2.5.21}$$

The equality (2.5.17) follows then from (2.5.20) and (2.5.21).

Step 2: From (2.5.17) and (2.5.15) we get

$$(u^+(y+t\xi),\xi) = \left(u_y^\xi\right)^+(t), \quad (u^-(y+t\xi),\xi) = \left(u_y^\xi\right)^-(t) \tag{2.5.22}$$

for $\mathcal{H}^{n-1}$-almost every $y \in \pi_\xi$ and for every $t \in (J_u^\xi \cap \partial_\xi^+\Omega_k)_y^\xi$. In the same way we can also prove that (2.5.22) holds for $\mathcal{H}^{n-1}$-almost every $y \in \pi_\xi$ and for every $t \in (J_u^\xi \cap \partial_\xi^-\Omega_k)_y^\xi$.

Step 3: Let us now consider the third part $\partial_\xi^0\Omega_k$ of $\partial\Omega_k$. By the area formula,

$$\mathcal{H}^{n-1}(p_\xi(J_u^\xi \cap \partial_\xi^0\Omega_k)) = \int_{J_u^\xi\cap\partial_\xi^0\Omega_k} |(\nu_{\Omega_k}(x),\xi)|d\mathcal{H}^{n-1}(x) = 0.$$

Then (2.5.22) holds for $\mathcal{H}^{n-1}$-almost every $y \in \pi_\xi$ and for every $t \in (J_u^\xi \cap \partial\Omega_k)_y^\xi$. Since J_u is covered $\mathcal{H}^{n-1}$-almost everywhere by the union of $\partial\Omega_k$, we get that (2.5.22) holds for $\mathcal{H}^{n-1}$-almost every $y \in \pi_\xi$ and for every $t \in (J_u^\xi)_y^\xi$.

We conclude the proof of the structure theorem by observing that the last equalities in (iv) of theorem 2.5.1 follow from the fact that for any $y \in Y$, $\tilde{u}_y^\xi$ and u_y^ξ coincide $\mathcal{L}^1$-almost everywhere in $\mathbb{R}$ and $\tilde{u}_y^\xi$ has pointwise bounded variation. □

2.6 Discontinuity points which are not in the jump set

It is known that for any BV function u, the set $S_u \setminus J_u$ is $\mathcal{H}^{n-1}$-negligible; This property is not known for BD functions. Kohn [141] proved that the set $S_u \setminus J_u$ whenever $u \in BD(\Omega)$, has Hausdorff dimension at most $n-1$, i.e.,

$$\mathcal{H}^{n-1+\varepsilon}(S_u \setminus J_u) = 0 \quad \text{for any } \varepsilon > 0.$$

Ambrosio, Coscia and Dal Maso have proved the following theorem in [21].

Theorem 2.6.1 *If $u, v \in BD(\Omega)$, then $|Ev|(S_u \setminus J_u) = 0$. In other words, $|Ev|$-almost every point $x \in \Omega$ is either a Lebesgue point or a Jump point of u.*

Before beginning the proof of Theorem 2.6.1, let us first examine known assumptions under which the measure $|Ev|$, for any $v \in BD(\Omega)$ vanishes.

Remark 2.6.2

(i) From theorem 2.6.1 one can easily see that for any $u \in BD(\Omega)$, the set $S_u \setminus J_u$ is purely $(\mathcal{H}^{n-1}, n-1)$-unrectifiable, i.e., $\mathcal{H}^{n-1}((S_u \setminus J_u) \cap M) = 0$ for any $\mathcal{C}^1$ hypersurface $M \subset \mathbb{R}^n$. In fact, for such M we have

$$\begin{aligned} 0 &= |Eu|((S_u \setminus J_u) \cap M) \\ &= \int_{(S_u \setminus J_u) \cap M} |(u^+ - u^-) \odot \nu_u| d\mathcal{H}^{n-1} \\ &\geq \frac{1}{\sqrt{2}} \int_{(S_u \setminus J_u) \cap M} |u^+ - u^-| d\mathcal{H}^{n-1}. \end{aligned}$$

If $\mathcal{H}^{n-1}((S_u \setminus J_u) \cap M) > 0$, then $\mathcal{H}^{n-1}$-almost every $x \in (S_u \setminus J_u) \cap M$ will be a Lebesgue point of u and this will contradicts the fact that $x \in S_u$. Therefore $\mathcal{H}^{n-1}((S_u \setminus J_u) \cap M) = 0$.

(ii) For any $u \in BV(\Omega, \mathbb{R}^k)$, it is known by the co-area formula that the measure $|Du|$ vanishes on purely $(\mathcal{H}^{n-1}, n-1)$-unrectifiable Borel set while the measure $|Eu|$ whenever $u \in BD(\Omega)$ vanishes on purely $(\mathcal{H}^{n-1}, n-1)$-unrectifiable Borel sets which are σ-finite with respect to $\mathcal{H}^{n-1}$. In fact, for such a Borel set B we have from the decomposition of the measure Eu

$$|Eu|(B) = |E^j u|(B) = |Eu|(B \cap J_u) = \int_{B \cap J_u} |(u^+ - u^-) \odot \nu_u| d\mathcal{H}^{n-1}.$$

Since J_u is countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable, we get $\mathcal{H}^{n-1}(B \cap J_u) = 0$ and then $|Eu|(B) = 0$.

Let us observe that we cannot use the argument of Remark 2.6.2 (ii) in the proof of Theorem 2.6.1. In fact, first of all it is not known whether the set $S_u \setminus J_u$ for any $u \in BD(\Omega)$ is σ-finite with respect to the measure $\mathcal{H}^{n-1}$ and the extension of the argument in Remark 2.6.2 (ii) to a more general class of rectifiable sets is still an open problem. However, Ambrosio, Coscia and Dal Maso were able to prove Theorem 2.6.1 by induction on n. The induction step being based essentially on the following refinement of Theorem 2.5.9

Theorem 2.6.3 *Under the assumptions of Theorem 2.5.9, x is a Lebesgue point of $p_\eta u$ in $\mathbb{R}^n$.*

In the proof of Theorem 2.6.3 we will need the following lemma whose proof has been first given by Kohn [141] and after by Ambrosio-Coscia-Dal Maso [21] under some variants.

Lemma 2.6.4 *Let $u \in BD(\Omega)$. Then*

$$\lim_{\rho\to 0^+} \min_{b\in\mathbb{R}^n} \frac{1}{\rho^n} \int_{B_\rho(x)} |u(y) - b|dy = 0 \tag{2.6.1}$$

for every $x \in \Omega \setminus \Theta_u$.

Proof. For the reader's convenience we reproduce here the proof by Ambrosio-Coscia-Dal Maso [21]. We can always assume by the locality of the statement that $\Omega = \mathbb{R}^n$ and $x = 0$. The Poincaré type inequality for BD functions on small balls gives the following inequality:

$$\int_{B_\rho(0)} |u(y) - b_\rho(u) - A_\rho(u)y|dy \le C(n)\rho|Eu|(B_\rho(0)).$$

This implies

$$\int_{B_\rho(0)} |u - b_\rho(u)|dy \le w_n\rho^{n+1}|A_\rho(u)| + C(n)\rho|Eu|(B_\rho(0)).$$

Then

$$\min_{b\in\mathbb{R}^n} \frac{1}{\rho^n} \int_{B_\rho(0)} |u(y) - b|dy \le w_n\rho|A_\rho(u)| + C(n)\frac{|Eu|(B_\rho(0))}{\rho^{n-1}}.$$

Since $0 \notin \Theta_u$, then

$$\lim_{\rho\to 0^+} \frac{|Eu|(B_\rho(0))}{\rho^{n-1}} = 0.$$

So, to conclude the proof of the lemma, we only need to prove that $\rho A_\rho(u)$ converges to 0 as $\rho \to 0^+$.

Let us fix $\tau > 0$ and let $\rho \in (0,\tau)$. From the equality (1.7.13) at the end of the previous chapter, i.e.,

$$A_\rho(u) = -\int_{\{\tau\geq|y|\geq\rho\}} \frac{\Gamma(y)}{2w_n|y|^{n+2}} dEu(y) + \frac{1}{2w_n\tau^n}\int_{\partial B_\tau(0)} (\gamma(u)\otimes\nu - \nu\otimes\gamma(u))\, d\mathcal{H}^{n-1}(y).$$

Then, it follows that

$$\rho|A_\rho(u)| \leq C\rho\left(\int_{\{\tau\geq|y|\geq\rho\}} \frac{1}{2w_n|y|^n} d|Eu|(y) + \frac{1}{w_n\tau^n}\int_{\partial B_\tau(0)} |\gamma(u)|\, d\mathcal{H}^{n-1}(y)\right).$$

Therefore,

$$\limsup_{\rho\to0^+} \rho|A_\rho(u)| \leq \limsup_{\rho\to0^+} C\rho\int_{\{\tau\geq|y|\geq\rho\}} \frac{1}{2w_n|y|^n} d|Eu|(y). \tag{2.6.2}$$

It suffices to prove that the right-hand side of (2.6.2) goes to 0 as $\tau \to 0^+$. Setting

$$\mu = |Eu| \llcorner \{\tau > |y| \geq \rho\}$$

and using the change of variables $t = \dfrac{1}{s^n}$ we get

$$\begin{aligned}
\int_{\{\tau\geq|y|\geq\rho\}} \frac{1}{|y|^n} d|Eu|(y) &= \int_{\mathbb{R}^n} \frac{1}{|y|^n} d\mu(y) = \int_0^{+\infty} \mu(\{\frac{1}{|y|^n}\})dt \\
&= n\int_0^{+\infty} \frac{\mu(\{|y|<s\})}{s^{n+1}} ds = n\int_\rho^\tau \frac{|Eu|(\{s>|y|\geq\rho\})}{s^{n+1}} ds \\
&+ \frac{|Eu|(\{\tau>|y|\geq\rho\})}{\tau^n}.
\end{aligned}$$

From this we get the estimate

$$\limsup_{\rho\to0^+} \rho\int_{\{\tau>|y|\geq\rho\}} \frac{1}{|y|^n} d|Eu|(y) \leq n \sup_{t\in(0,\tau)} \frac{|Eu|(\{|y|<t\})}{t^{n-1}}. \tag{2.6.3}$$

Since $0 \notin \Theta_u$, the right-hand side of (2.6.3) vanishes as $\tau \to 0^+$. $\square$

Proof of Theorem 2.6.3. We use the notations adopted in the proof of Theorem 2.5.9 assuming that $x = 0$ and $\pi = \pi_{e_n}$. From Theorem 2.5.9, we know that 0 is a Lebesgue point of the function $v(y) := (\tilde{u}^1(y), \cdots, \tilde{u}^{n-1}(y), 0)$ in π. In order to prove that 0 is a Lebesgue point of $(u^1(x), \cdots, u^{n-1}(x), 0)$ in $\mathbb{R}^n$, we only need to prove that the first component v_0^1 of the Lebesgue representative in π of v at 0 coincide with $\tilde{u}^1(0)$, the Lebesgue representative of u^1 at 0. The other components will be obtained by a similar argument.

Since $0 \notin \Theta_u$, by Lemma 2.6.4 the quantity

$$w(\varepsilon) := \frac{1}{\varepsilon^n} \int_{Q_\varepsilon} \Big[\int_{-\varepsilon}^{\varepsilon} |u^1(y + te_n) - b_\varepsilon| dt \Big] d\mathcal{H}^{n-1}(y)$$

for some $b_\varepsilon \in \mathbb{R}$ vanishes as $\varepsilon \to 0^+$. Since

$$\frac{1}{\varepsilon^n} \int_{C_\varepsilon} |u^1(x) - v_0^1| dx \le w(\varepsilon) + |b_\varepsilon - v_0^1|,$$

we have only to prove that b_ε converges to v_0^1 in $\mathbb{R}$ as $\varepsilon \to 0^+$. We have

$$w(\varepsilon) = \frac{1}{\varepsilon} \int_0^{\varepsilon} \Big(\frac{1}{\varepsilon^{n-1}} \int_{Q_\varepsilon} |\tilde{u}^1(y + te_n) - b_\varepsilon| + |\tilde{u}^1(y - te_n) - b_\varepsilon| d\mathcal{H}^{n-1}(y) \Big) dt.$$

Then

$$2w(\varepsilon) \ge \frac{2}{\varepsilon} \int_{\frac{1}{2}\varepsilon}^{\varepsilon} \Big(\frac{1}{\varepsilon^{n-1}} \int_{Q_\varepsilon} |\tilde{u}^1(y + te_n) - b_\varepsilon| + |\tilde{u}^1(y - te_n) - b_\varepsilon| d\mathcal{H}^{n-1}(y) \Big) dt.$$

Therefore, for any $\varepsilon > 0$ there exists $\rho(\varepsilon) \in (\frac{1}{2}\varepsilon, \varepsilon) \cap Y$ (where the set Y has been introduced in the proof of Therorem 2.5.9) such that

$$2w(\varepsilon) \ge \frac{1}{\varepsilon^{n-1}} \int_{Q_\varepsilon} |\tilde{u}^1(y + \rho e_n) - b_\varepsilon| + |\tilde{u}^1(y - \rho e_n) - b_\varepsilon| d\mathcal{H}^{n-1}(y).$$

On the other hand we write

$$\begin{aligned} &\tilde{u}^1(y + \rho e_n) + \tilde{u}^1(y - \rho e_n) - 2v_0^1 \\ &= \Big[(\tilde{u}^1 + \tilde{u}^n)(y + \rho e_n) - (\tilde{u}^1 + \tilde{u}^n)(y - \rho e_1) \Big] \\ &+ \Big[(\tilde{u}^1 - \tilde{u}^n)(y - \rho e_n) - (\tilde{u}^1 - \tilde{u}^n)(y - \rho e_1) \Big] \\ &+ \Big[\tilde{u}^n(y - \rho e_n) - \tilde{u}^n(y + \rho e_n) \Big] + 2 \Big[v^1(y - \rho e_1) - v_0^1 \Big]. \end{aligned} \tag{2.6.4}$$

Now integrating both sides of (2.6.4) on Q_ε and exploiting the same arguments as in the proof of Theorem 2.5.9 we get

$$\int_{Q_\varepsilon} |\tilde{u}^1(y+\rho e_n) + \tilde{u}^1(y-\rho e_n) - 2v_0^1| d\mathcal{H}^{n-1}(y)$$
$$\leq C|Eu|(C_{3\varepsilon}) + 2\int_{Q_\varepsilon} |v^1(y-\rho e_n) - v_0^1| d\mathcal{H}^{n-1}(y).$$

Using the fact that $0 \notin \Theta_u$ and that 0 is a Lebesgue point of v in π we obtain

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^{n-1}} \int_{Q_\varepsilon} |\tilde{u}^1(y+\rho e_n) + \tilde{u}^1(y-\rho e_n) - 2v_0^1| d\mathcal{H}^{n-1}(y) = 0. \tag{2.6.5}$$

From (2.6.5) and

$$\begin{aligned} 2|b_\varepsilon - v_0^1| \ &\leq \ \frac{1}{\varepsilon^{n-1}} \int_{Q_\varepsilon} |\tilde{u}^1(y+\rho e_n) + \tilde{u}^1(y-\rho e_n) - 2v_0^1| d\mathcal{H}^{n-1}(y) \\ &\quad + \frac{1}{\varepsilon^{n-1}} \int_{Q_\varepsilon} |\tilde{u}^1(y+\rho e_n) - b_\varepsilon| + |\tilde{u}^1(y-\rho e_n) - b_\varepsilon| d\mathcal{H}^{n-1}(y) \\ &\leq \ 2w(\varepsilon) + \frac{1}{\varepsilon^{n-1}} \int_{Q_\varepsilon} |\tilde{u}^1(y+\rho e_n) + \tilde{u}^1(y-\rho e_n) - 2v_0^1| d\mathcal{H}^{n-1}(y). \end{aligned}$$

we get $\lim\limits_{\varepsilon\to 0^+} b_\varepsilon = v_0^1$ and this achieves the proof of Theorem 2.6.3. □

Let us now prove Theorem 2.6.1 in the particular case $u = v$.

Proposition 2.6.5 *For every $u \in BD(\Omega)$, $|Eu|(S_u \setminus J_u) = 0$.*

The following lemma will be used in the proof of Proposition 2.6.5.

Lemma 2.6.6 *Let $u \in L^1(\mathbb{R}^n, \mathbb{R}^n)$, let $\mu = \{\mu_{ij}\}$ be a symmetric matrix of Radon measure in $\mathbb{R}^n$. Assume that, for any $\xi \in \mathbb{R}^n$, $|(\mu\xi, \xi)|$-almost every point of $\mathbb{R}^n$ is a Lebesgue point of $u^\xi : \mathbb{R}^n \to \mathbb{R}$ defined by $u^\xi(x) := (u(x), \xi)$. Then $|\mu|$-almost every point of $\mathbb{R}^n$ is a Lebesgue point for u.*

Proof. We have to prove that for any $\eta \in \mathbb{R}^n$, $|\mu|$-almost every point of $\mathbb{R}^n$ is a Lebesgue point of u^η. By the inequality (2.1.2) it is enough to prove that for any choice of $\xi \in \mathbb{R}^n$, $|(\mu\xi, \xi)|$-almost every point of $\mathbb{R}^n$ is a Lebesgue point for u^η. If we apply the assumptions of the Lemma to the vectors ξ, η, $\xi \pm \eta$ and set $\mu_{\xi\xi} := (\mu\xi, \xi)$, $\mu_{\eta\eta} := (\mu\eta, \eta)$, $\mu_{\xi\eta} := (\mu\xi, \eta)$ we obtain the following properties:

(a) $|\mu_{\xi\xi}|$-almost every $x \in \mathbb{R}^n$ is a Lebesgue point for u^ξ.

(b) $|\mu_{\eta\eta}|$-almost every $x \in \mathbb{R}^n$ is a Lebesgue point for u^η.

(c) $|\mu_{\xi\xi} \pm 2\mu_{\xi\eta} + \mu_{\eta\eta}|$-almost every $x \in \mathbb{R}^n$ is a Lebesgue point for $u^\xi \pm u^\eta$.

Set $\lambda := |\mu_{\xi\xi}| + |\mu_{\eta\eta}| + |\mu_{\xi\eta}|$. From the absolute continuity of the measures $\mu_{\xi\xi}$, $\mu_{\xi\eta}$ and $\mu_{\xi\eta}$ with respect to λ, we get the existence of three locally λ-integrable functions h_1, h_2 and h_3 such that

$$\mu_{\xi\xi} = h_1\lambda, \quad \mu_{\eta\eta} = h_2\lambda \quad \text{and} \quad \mu_{\xi\eta} = h_3\lambda.$$

Let us consider the following partition of $\mathbb{R}^n$

$$A_1 := \{x \in \mathbb{R}^n\colon\ h_1(x) + h_2(x) = 0\}; \quad A_2 := \{x \in \mathbb{R}^n\colon\ h_1(x) + h_2(x) \neq 0\}.$$

On A_1 we have $\mu_{\xi\xi} = -\mu_{\eta\eta}$. Hence the total variations are the same on A_1 and (b) implies that $|\mu_{\xi\xi}|$-almost every $x \in A_1$ is a Lebesgue point for u^η. In order to conclude the proof of the lemma we need only to prove that $|\mu_{\xi\xi}|$-almost every $x \in A_2$ is a Lebesgue point for u^η. Set

$$C_1 := \left\{ \begin{array}{ll} x \in A_2\colon & x \text{ is neither a Lebesgue point of } u^\xi + u^\eta \\ & \text{nor a Lebesgue point of } u^\xi - u^\eta \end{array} \right\}.$$

Then by the property (c)

$$0 = |\mu_{\xi\xi} \pm 2\mu_{\xi\eta} + \mu_{\eta\eta}|(C_1) = \int_{C_1} |h_1 + h_2 \pm 2h_3| d\lambda.$$

If $\lambda(C_1) > 0$, then for λ-almost every $x \in C_1$ we have $h_1(x) + h_2(x) \pm 2h_3(x) = 0$; this is impossible since $C_1 \subset A_2$. Therefore $\lambda(C_1) = 0$ and this means that λ-almost every $x \in A_2$ is either a Lebesgue point of $u^\xi + u^\eta$ or a Lebesgue point of $u^\xi - u^\eta$. On the other hand, if x is either a Lebesgue point of $u^\xi + u^\eta$ or of $u^\xi - u^\eta$ and x is a Lebesgue point of u^ξ, then x is also a Lebesgue point of u^η. Hence, from the property (a) and the fact that $|\mu_{\xi\xi}|(C_1) = 0$ we obtain that $|\mu_{\xi\xi}|$-almost every $x \in A_2$ is a Lebesgue point of u^η. This achieves the proof of Lemma 2.6.6. $\square$

The proof of Proposition 2.6.5. Also in this case we may assume that $\Omega = \mathbb{R}^n$ since the argument to be proved is local. Let $u \in BD(\mathbb{R}^n)$. we will proceed as we have said before by induction on n. The statement is trivial if $n = 1$. Assume that it is true for $n-1 \geq 1$ and let us prove it for n.

For any $\xi \in \mathbb{R}^n$, if we assert that $|(Eu\xi, \xi)|$-almost every $x \in \mathbb{R}^n \setminus J_u$ is a Lebesgue point of u^ξ, then by Lemma 2.6.6 above with $\mu = Eu \llcorner (\mathbb{R}^n \setminus J_u)$ we have $|Eu|$-almost every $x \in \mathbb{R}^n \setminus J_u$ is a Lebesgue point of u. So, it will be sufficient to prove that the assertion: $|(Eu\xi, \xi)|$-a.e. $x \in \mathbb{R}^n \setminus J_u$ is a Lebesgue point of u^ξ holds.

Let $\eta \in \mathbf{S}^{n-1}$ be orthogonal to ξ so that 0 and ξ belong to π_η. Let

$$S := \left\{ s \in \mathbb{R}\colon \begin{array}{l} v_s(y) := p_\eta \tilde{u}(y + s\eta) \quad \text{is defined for } \mathcal{H}^{n-1}\text{-a.e. } y \in \pi_\eta \\ \text{and } v_s \in BD(\pi_\eta) \end{array} \right\}.$$

By Proposition 2.2.3 we have $\mathcal{L}^1(\mathbb{R} \setminus S) = 0$. By the induction assumption, for any $s \in S$, $|Ev_s|(S_{v_s} \setminus J_{v_s}) = 0$. Therefore v_s has one-sided (possibly equal) Lebesgue limits v_s^+ and v_s^- in π_η at $|(Ev_s\xi, \xi)|$-almost every point in π_η. On the other hand, we have seen in Proposition 2.2.3 that for any $B \in \mathcal{B}(\mathbb{R}^n)$,

$$(Eu\xi, \xi)(B) = \int_{-\infty}^{+\infty} (Ev_s\xi, \xi)(B_s)\, ds$$

where $B_s = \{y \in \pi_\eta\colon s\eta + y \in B\}$. Therefore the measure $(Eu\xi, \xi)$ is concentrated on the set

$$Y' := \{x = s\eta + y\colon s \in S \text{ and } y \text{ such that } v_s^+ \text{ and } v_s^- \text{ exists at } y\}.$$

From Theorem 2.6.3 any point $x \in Y'$ either is in Θ_u or is a Lebesgue point of $p_\eta u$ in $\mathbb{R}^n$. Since $\xi \in \pi_\eta$, this implies that $|(Eu\xi, \xi)|$-almost every point either belongs to Θ_u or is a Lebesgue point of u^ξ. Since $\mathcal{H}^{n-1}(\Theta_u \setminus J_u) = 0$ then by the second equality (2.2.2) of Theorem 2.2.1, we have $|(Eu\xi, \xi)|(\Theta_u \setminus J_u) = 0$. Therefore, $|(Eu\xi, \xi)|$-almost every point of $\mathbb{R}^n \setminus J_u$ is a Lebesgue point of u^ξ and this achieves the proof of Proposition 2.6.5. $\square$

Using Proposition 2.6.5 we are in a position to prove Theorem 2.6.1 in the general case.

The proof of Theorem 2.6.1. Let u, $v \in BD(\Omega)$. By the Radon-Nikodym decomposition of the measure $|Ev|$ with respect to $|Eu|$, there exists a Borel set N such that $|Eu|(N) = 0$ and $|Ev| \llcorner (\Omega \setminus N)$ is absolutely continuous with respect to $|Eu|$. By Proposition 2.6.5 we have $|Eu|((S_u \setminus J_u) \setminus N) = 0$ and this implies that $|Ev|((S_u \setminus J_u) \setminus N) = 0$. So, it only remains to prove that $|Ev|((S_u \setminus J_u) \cap N) = 0$. Since

$$S_u \setminus J_u \subset (S_v \setminus J_v) \cup ((S_u \setminus J_u) \setminus S_v) \cup ((S_u \setminus J_u) \cap J_v)$$

and $|Ev|(S_v \setminus J_v) = 0$ (by Proposition 2.6.5), it is enough to prove that

$$|Ev|(((S_u \setminus J_u) \setminus S_v) \cap N) = 0, \tag{2.6.6}$$

$$|Ev|((S_u \setminus J_u) \cap J_v \cap N) = 0. \tag{2.6.7}$$

Observing that any Lebesgue point of v is a Lebesgue or jump point of u if and only if it is respectively a Lebesgue or Jump point of $u + v$, we have the equality

$$(S_u \setminus J_u) \setminus S_v = (S_{u+v} \setminus J_{u+v}) \setminus S_v.$$

So that by Proposition 2.6.5 $|E(u + v)|(S_u \setminus J_u) \setminus S_v) = 0$. Since Ev and $E(u + v)$ coincide as measures on N, we obtain (2.6.6).

Let $x \in J_v$. If $x \notin S_{u+v}$, then $x \in J_u$; If $x \in J_{u+v}$ and $\nu_{u+v}(x) = \pm\nu_v(x)$, then $x \notin S_u$ or $x \in J_u$. So, if we set

$$A := \{x \in J_v \cap J_{u+v} \colon \nu_{u+v}(x) = \pm\nu_v(x)\}$$

then $A \subset J_v \cup (\Omega \setminus S_u)$. This implies that $(S_u \setminus J_u) \cap J_v \subset J_v \setminus A$. On the other hand,

$$\begin{aligned} J_v \setminus A &= (J_v \cap J_{u+v} \setminus A) \cup (J_v \setminus (J_v \cap J_{u+v})) \\ &= (J_v \cap J_{u+v} \setminus A) \cup (J_v \cap S_{u+v} \setminus (J_v \cap J_{u+v})) \cup ((J_v \setminus S_{u+v}) \setminus (J_v \cap J_{u+v})). \end{aligned}$$

Since $J_v \cap S_{u+v} \setminus (J_v \cap J_{u+v}) = J_v \cap (S_{u+v} \setminus J_{u+v})$ and $J_v \setminus S_{u+v} \subset J_v \cap J_u$ we obtain

$$(S_u \setminus J_u) \cap J_v \subset (J_v \cap J_{u+v} \setminus A) \cup (J_v \cap (S_{u+v} \setminus J_{u+v})) \cup ((J_v \cap J_u) \setminus (J_v \cap J_{u+v})).$$

However, $[(S_u \setminus J_u) \cap J_v] \cap [(J_v \cap J_u) \setminus (J_v \cap J_{u+v})] = \emptyset$ so that we finally get

$$(S_u \setminus J_u) \cap J_v \subset (J_v \cap J_{u+v} \setminus A) \cup (S_{u+v} \setminus J_{u+v}).$$

Since J_u and J_{u+v} are countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable, the Borel set $J_u \cap J_{u+v} \setminus A$ is $\mathcal{H}^{n-1}$-negligible. So, by Proposition 2.6.5 and the fact that the measure $|E(u+v)|$ vanishes on $\mathcal{H}^{n-1}$-negligible set, we get $|E(u+v)|((S_u \setminus J_u) \cap J_v) = 0$. Since Ev and $E(u+v)$ coincide on N as measures, we obtain $|Ev|((S_u \setminus J_u) \cap J_v \cap N) = 0$ and this achieves the proof of Theorem 2.6.1. $\square$

2.7 A compactness criterion in $SBD(\Omega)$

Let Ω be a bounded open subset of $\mathbb{R}^n$. In order to study the lower semicontinuity of some family of functionals defined on the space $SBD(\Omega)$ of special functions with bounded deformation, Bellettini, Coscia and Dal Maso have proved in [47] the following compactness theorem in $SBD(\Omega)$.

Theorem 2.7.1 *Let $\phi : [0,+\infty[\to [0,+\infty[$ be a non-decreasing function such that*

$$\lim_{t\to+\infty} \frac{\phi(t)}{t} = +\infty. \tag{2.7.1}$$

Let $\{u_h\}$ be a sequence in $SBD(\Omega)$ such that

$$\int_\Omega |u_h|dx + |E^j u_h|(\Omega) + \int_\Omega \phi(|\mathcal{E}u_h|)dx + \mathcal{H}^{n-1}(J_{u_h}) \leq K \tag{2.7.2}$$

for some positive constant K independent of h. Then there exists a subsequence, still denoted by $\{u_h\}$ and a function $u \in SBD(\Omega)$ such that

$$u_h \to u \text{ strongly in } L^1_{\text{loc}}(\Omega, \mathbb{R}^n), \tag{2.7.3}$$

$$\mathcal{E}u_h \rightharpoonup \mathcal{E}u \text{ weakly in } L^1(\Omega, M_s^{n\times n}), \tag{2.7.4}$$

$$E^j u_h \rightharpoonup E^j u \text{ weakly } \star \text{ in } \mathcal{M}_b(\Omega, M_s^{n\times n}), \tag{2.7.5}$$

$$\mathcal{H}^{n-1}(J_u) \leq \liminf_{h\to+\infty} \mathcal{H}^{n-1}(J_{u_h}). \tag{2.7.6}$$

Remark 2.7.2

(i) This theorem is the analogue of Ambrosio's compactness theorem in the space $SBV(\Omega)$ (see Theorem 1.3.4 of Chapter one).

(ii) An other version of Theorem 2.7.1 has been first proved in Bellettini-Coscia-Dal Maso [46] with the measure $\mathcal{H}^{n-1} \llcorner J_u$ replaced by the integral-geometric measure λ_u, since at the time when the authors were writing that paper it was not clear whether λ_u and $\mathcal{H}^{n-1} \llcorner J_u$ coincide as measures on Ω.

(iii) Let us observe that Theorem 2.7.1 cannot be obtained directly from the Ambrosio's compactness theorem. In other words, even if the sequence $\{u_h\}$ were taken in $SBV(\Omega, \mathbb{R}^n)$, the assumptions of Theorem 2.7.1 would not be sufficient to obtain the conclusion using the Ambrosio's compactness theorem. In fact, the bound (2.7.2) on $\mathcal{E}u_h$ does not lead to a similar bound for the approximate gradients $\{\nabla u_h\}$ because there is no analogue of Korn's inequality allowing to control $\int_\Omega \phi(|\nabla u_h|)dx$ from (2.7.2) uniformly with respect to h.

(iv) the main ingredient in the proof of Theorem (2.7.1) will be the structure Theorem 2.5.1 proved before. The basic idea is to obtain from the bound (2.7.2), new analogue bounds for the one-dimensional sections of the functions u_h, which will allow to apply the one-dimensional version of the Ambrosio's compactness theorem and to obtain the results by integration.

The proof of Theorem 2.7.1. By (2.7.1) and (2.7.2) the sequence $\{u_h\}$ is bounded in $BD(\Omega)$ so that by the compact injection of $BD(\Omega)$ in $L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ (see Theorem 1.7.14 of Chapter I), there exists a subsequence, still denoted by $\{u_h\}$, which converges in $L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ to a function $u \in BD(\Omega)$. Again by (2.7.1) and (2.7.2) and the Dunford-Pettis criterion we can also assume that $\{\mathcal{E}u_h\}$ converges weakly in $L^1(\Omega, M^{n\times n}_s)$.

We claim that $u \in SBD(\Omega)$ and that (2.7.6) holds. To simplify the notation we set

$$I^\xi_y(u_h) := \int_{\Omega^\xi_y} \phi(|\nabla u^\xi_{h,y}(t)|)dt, \quad H^\xi_y(u_h) := |Du^\xi_{h,y}|(\Omega^\xi_y),$$

where $u^\xi_{h,y} = (u_h)^\xi_y$. By the point (ii) of the structure theorem, using Fubini-tonelli's theorem and the fact that ϕ is non-decreasing, for any $\xi \in \mathbf{S}^{n-1}$ we get

$$\int_{\pi_\xi} I^\xi_y(u_h)d\mathcal{H}^{n-1}(y) = \int_\Omega \phi(|\langle\mathcal{E}u_h(x)\xi, \xi\rangle|)dx \le \int_\Omega \phi(|\mathcal{E}u_h(x)|)dx \le K. \tag{2.7.7}$$

Moreover, by (2.2.2) of Theorem 2.2.1, we have

$$\int_{\pi_\xi} H_y^\xi(u_h)\,d\mathcal{H}^{n-1}(y) = |(Eu_h\xi,\xi)|(\Omega) \le |Eu_h|(\Omega) \le C \tag{2.7.8}$$

where $0 < C < +\infty$ depends on K and also on the function ϕ. Let $\{u_k\} = \{u_{h_k}\}$ be a subsequence such that

$$\lim_{k\to+\infty} \mathcal{H}^{n-1}(J_{u_k}) = \liminf_{h\to+\infty} \mathcal{H}^{n-1}(J_{u_h}) < +\infty. \tag{2.7.9}$$

By the structure theorem 2.5.1, for any $\xi \in \mathbf{S}^{n-1}$, the functions $u_{k,y}^\xi$ belong to $SBV(\Omega_y^\xi)$ for $\mathcal{H}^{n-1}$-almost every $y \in \Omega^\xi$. Let us fix $0 < \varepsilon < 1$. Using the fact that $\mathcal{H}^{n-1} \llcorner J_u = \lambda_u$ we get for any $\xi \in \mathbf{S}^{n-1}$,

$$\limsup_{k\to+\infty} \int_{\pi_\xi} [\varepsilon I_y^\xi(u_k) + \varepsilon H_y^\xi(u_k)]\,d\mathcal{H}^{n-1}(y)$$

$$+ \liminf_{k\to+\infty} \int_{\mathbf{S}^{n-1}} [\int_{\pi_\xi} \mathcal{H}^0(J_{u_{k,y}^\xi})\,d\mathcal{H}^{n-1}(y)]\,d\mathcal{H}^{n-1}(\xi) < +\infty.$$

Applying now Fatou's lemma to the second term above we get

$$\limsup_{k\to+\infty} \int_{\pi_\xi} [\varepsilon I_y^\xi(u_k) + \varepsilon H_y^\xi(u_k)]\,d\mathcal{H}^{n-1}(y)$$

$$+ \int_{\mathbf{S}^{n-1}} \liminf_{k\to+\infty} \int_{\pi_\xi} \mathcal{H}^0(J_{u_{k,y}^\xi})\,d\mathcal{H}^{n-1}(y)\,d\mathcal{H}^{n-1}(\xi) < +\infty.$$

So that we finally obtain

$$\liminf_{k\to+\infty} \int_{\pi_\xi} [\varepsilon I_y^\xi(u_k) + \varepsilon H_y^\xi(u_k) + \mathcal{H}^0(J_{u_{k,y}^\xi})]\,d\mathcal{H}^{n-1}(y) < +\infty.$$

Let us fix $\xi \in \mathbf{S}^{n-1}$ such that the inequality above holds. Let $\{u_m\} = \{u_{k_m}\}$ be such that

$$\lim_{m\to+\infty} \int_{\pi_\xi} [\varepsilon I_y^\xi(u_m) + \varepsilon H_y^\xi(u_m) + \mathcal{H}^0(J_{u_{m,y}^\xi})]\,d\mathcal{H}^{n-1}(y) \tag{2.7.10}$$

$$= \liminf_{k\to+\infty} \int_{\pi_\xi} [\varepsilon I_y^\xi(u_k) + \varepsilon H_y^\xi(u_k) + \mathcal{H}^0(J_{u_{k,y}^\xi})]\,d\mathcal{H}^{n-1}(y) < +\infty.$$

By Fubini-Tonelli's theorem we have

$$\begin{aligned}
&\lim_{m\to+\infty}\int_{\Omega^\xi}\Big[\int_{\Omega^\xi_y}|u^\xi_{m,y}(t)-u^\xi_y(t)|dt\Big]d\mathcal{H}^{n-1}(y)\\
&=\lim_{m\to+\infty}\int_\Omega|(u_m(x)-u(x),\xi)|dx\le\lim_{m\to+\infty}\int_\Omega|u_m(x)-u(x)|dx=0.
\end{aligned}$$

Therefore, up to a subsequence, we have for $\mathcal{H}^{n-1}$-almost every $y\in\Omega^\xi$

$$u^\xi_y\in BV(\Omega^\xi_y)\text{ and }u^\xi_{m,y}\to u^\xi_y\text{ in }L^1(\Omega^\xi_y). \tag{2.7.11}$$

Now, by (2.7.10) and Fatou's lemma, we obtain for $\mathcal{H}^{n-1}$-almost every $y\in\Omega^\xi$

$$\liminf_{m\to+\infty}[\varepsilon I^\xi_y(u_m)+\varepsilon H^\xi_y(u_m)+\mathcal{H}^0(J_{u^\xi_{m,y}})]<+\infty. \tag{2.7.12}$$

Let us fix $y\in\Omega^\xi$ satisfying (2.7.11) and (2.7.12) and such that $u^\xi_{m,y}\in SBV(\Omega^\xi_y)$ for any m. Passing to a further subsequence $\{u_j\}=\{u_{m_j}\}$ (depending on ε, ξ, and y) we can assume that

$$\begin{aligned}
&\lim_{j\to+\infty}[\varepsilon I^\xi_y(u_j)+\varepsilon H^\xi_y(u_j)+\mathcal{H}^0(J_{u^\xi_{j,y}})]\\
&=\liminf_{m\to+\infty}[\varepsilon I^\xi_y(u_m)+\varepsilon H^\xi_y(u_m)+\mathcal{H}^0(J_{u^\xi_{m,y}})]<+\infty.
\end{aligned} \tag{2.7.13}$$

From (2.7.11) and (2.7.13) we deduce that the sequence $\{u^\xi_{j,y}\}$ is bounded in $BV(A)$, hence in $L^\infty(A)$, for any open set $A\subseteq\Omega^\xi_y$ with a finite number of connected components. By (2.7.13) we can apply the one-dimensional version of the Ambrosio's compactness theorem (see [10, Proposition 4.2]) to obtain $u^\xi_y\in SBV(A)$ and

$$\begin{aligned}
\mathcal{H}^0(J_{u^\xi_y}\cap A)&\le\liminf_{j\to+\infty}\mathcal{H}^0(J_{u^\xi_{j,y}})\\
&\le\lim_{j\to+\infty}[\varepsilon I^\xi_y(u_j)+\varepsilon H^\xi_y(u_j)+\mathcal{H}^0(J_{u^\xi_{j,y}})]\\
&=\liminf_{m\to+\infty}[\varepsilon I^\xi_y(u_m)+\varepsilon H^\xi_y(u_m)+\mathcal{H}^0(J_{u^\xi_{m,y}})]<+\infty.
\end{aligned} \tag{2.7.14}$$

As $u_y^\xi \in BV(\Omega_y^\xi)$, from the arbitrariness of A we deduce that $u_y^\xi \in SBV(\Omega_y^\xi)$. Therefore, we have proved that for $\mathcal{H}^{n-1}$-almost every $\xi \in \mathbf{S}^{n-1}$ and $y \in \Omega^\xi$, $u_y^\xi \in SBV(\Omega_y^\xi)$. So that for such ξ,

$$|(E^c u\xi, \xi)| = 0 \tag{2.7.15}$$

by the structure theorem. Hence, there exists a basis $\{\xi_1, \cdots, \xi_n\}$ of $\mathbb{R}^n$ such that (2.7.15) holds for any $\xi := \xi_i + \xi_j$. Then, it follows that $u \in SBD(\Omega)$. Moreover, (2.7.14) yields to

$$\mathcal{H}^0(J_{u_y^\xi}) \le \liminf_{m\to+\infty}[\varepsilon I_y^\xi(u_m) + \varepsilon H_y^\xi(u_m) + \mathcal{H}^0(J_{u_{m,y}^\xi})]. \tag{2.7.16}$$

Integrating the two members over π_ξ and using (2.7.10), (2.7.16), Fatou's lemma and Definition 2.5.7, we get

$$\begin{aligned}\lambda_u^\xi(\Omega) &\le \lim_{m\to+\infty}\int_{\pi_\xi}\Big[\varepsilon I_y^\xi(u_m) + \varepsilon H_y^\xi(u_m) + \mathcal{H}^0(J_{u_{m,y}^\xi})\Big]d\mathcal{H}^{n-1}(y)\\ &= \liminf_{k\to+\infty}\int_{\pi_\xi}\Big[\varepsilon I_y^\xi(u_k) + \varepsilon H_y^\xi(u_k) + \mathcal{H}^0(J_{u_{k,y}^\xi})\Big]d\mathcal{H}^{n-1}(y) < +\infty\end{aligned}$$

for $\mathcal{H}^{n-1}$-almost every $\xi \in \mathbf{S}^{n-1}$. Integrating with respect to ξ and using Theorem 2.5.8 we get

$$\mathcal{H}^{n-1}(J_u) \le c\varepsilon + \lim_{k\to+\infty}\mathcal{H}^{n-1}(J_{u_k}) = c\varepsilon + \liminf_{h\to+\infty}\mathcal{H}^{n-1}(J_{u_h})$$

where $c = k_n\mathcal{H}^{n-1}(\mathbf{S}^{n-1})(K + C)$; k_n being the constant used in the definition of λ_u. Letting $\varepsilon \to 0^+$, we obtain (2.7.6).

Now we claim (2.7.4). Using Fatou's lemma, from (2.7.2) and Theorem 2.5.8, we get

$$\int_{\mathbf{S}^{n-1}}\Big[\liminf_{h\to+\infty}\lambda_{u_h}^\xi(\Omega)\Big]d\mathcal{H}^{n-1}(\xi) < +\infty.$$

Hence there exists a basis $\{\xi_1, \cdots, \xi_n\}$ of $\mathbb{R}^n$ such that $\liminf_{h\to+\infty}\lambda_{u_h}^\xi(\Omega) < +\infty$ for every ξ of the form $\xi_i + \xi_j$. Let us fix such a ξ. Then, there exists a subsequence

$\{u_k\} = \{u_{h_k}\}$ (depending on ξ) such that

$$\lim_{k\to+\infty} \lambda^\xi_{u_k}(\Omega) < +\infty. \tag{2.7.17}$$

Let $w \in L^1(\Omega)$ and set

$$K^\xi_y(u_k, w) := \int_{\Omega^\xi_y} |\nabla u^\xi_y(t) - w(y+t\xi)|dt$$

By Fubini-Tonelli's theorem and the point (ii) of Theorem 2.5.1 we obtain

$$\begin{aligned}\int_{\pi_\xi} K^\xi_y(u_k, w)d\mathcal{H}^{n-1}(y) &= \int_{\pi_\xi}\int_{\Omega^\xi_y} |(\mathcal{E}u_k(y+t\xi)\xi,\xi) - w(y+t\xi)|dtd\mathcal{H}^{n-1}(y)\\ &= \int_\Omega |(\mathcal{E}u_k(x)\xi,\xi) - w(x)|dx.\end{aligned}$$

Then from (2.7.2) we can find a further subsequence $\{u_l\} = \{u_{k_l}\}$ (depending on ξ and w) such that

$$\lim_{l\to+\infty}\int_{\pi_\xi} K^\xi_y(u_l, w)d\mathcal{H}^{n-1}(y) = \liminf_{k\to+\infty}\int_\Omega |(\mathcal{E}u(x)\xi,\xi) - w(x)|dx < +\infty. \tag{2.7.18}$$

Now using the same arguments as in (2.7.10) and (2.7.11) we can construct from $\{u_l\}$ a further subsequence $\{u_m\} = \{u_{l_m}\}$ (depending on ε, w, and ξ) such that (2.7.11) is satisfied for $\mathcal{H}^{n-1}$-almost every $y \in \Omega^\xi$ and

$$\begin{aligned}&\lim_{m\to+\infty}\int_{\pi_\xi}\{K^\xi_y(u_m, w) + \varepsilon[I^\xi_y(u_m) + H^\xi_y(u_m) + \mathcal{H}^0(J_{u^\xi_{m,y}})]\}d\mathcal{H}^{n-1}(y)\\ &= \liminf_{l\to+\infty}\int_{\pi_\xi}\{K^\xi_y(u_l, w) + \varepsilon[I^\xi_y(u_l) + H^\xi_y(u_l) + \mathcal{H}^0(J_{u^\xi_{l,y}})]\}d\mathcal{H}^{n-1}(y)\\ &< +\infty\end{aligned} \tag{2.7.19}$$

where the last inequality is a consequence of (2.7.7) ,(2.7.8) and (2.7.18). Now, using Fatou's lemma, for $\mathcal{H}^{n-1}$-almost every $y \in \Omega^\xi$ we obtain

$$\liminf_{m\to+\infty}\{K^\xi_y(u_m, w) + \varepsilon[I^\xi_y(u_m) + H^\xi_y(u_m) + \mathcal{H}^0(J_{u^\xi_{m,y}})]\} < +\infty. \tag{2.7.20}$$

Let us fix a point $y \in \Omega^\xi$, satisfying (2.7.11), (2.7.20) and such that $u^\xi_{m,y} \in SBV(\Omega^\xi_y)$ for any m. Passing to a further subsequence $\{u_j\} = \{u_{m_j}\}$ (depending on ξ, w, ε, and y) we can assume that

$$\lim_{j\to+\infty}\{K^\xi_y(u_j,w)+\varepsilon[I^\xi_y(u_j)+H^\xi_y(u_j)+\mathcal{H}^0(J_{u^\xi_{j,y}})]\}$$
$$(2.7.21)\qquad =\liminf_{m\to+\infty}\{K^\xi_y(u_m,w)+\varepsilon[I^\xi_y(u_m)+H^\xi_y(u_m)+\mathcal{H}^0(J_{u^\xi_{m,y}})] < +\infty.$$

Consequently, $H^\xi_y(u_j)$ is bounded uniformly with respect to j. This fact, together with (2.7.11), implies that $\{u^\xi_{j,y}\}$ is bounded in $BV_{\rm loc}(\Omega^\xi_y)$, hence in $L^\infty_{\rm loc}(\Omega^\xi_y)$. So, by (2.7.21) we can apply the one-dimensional version of the Ambrosio's compactness theorem (see [10, Proposition 4.2]) to obtain that $\nabla u^\xi_{j,y}$ converges to ∇u^ξ_y weakly in $L^1_{loc}(\Omega^\xi_y)$ as $j\to+\infty$. Thus we have

$$\begin{aligned} K^\xi_y(u,w) &\le \liminf_{j\to+\infty} K^\xi_y(u_j,w) \\ &\le \lim_{j\to+\infty}\{K^\xi_y(u_j,w)+\varepsilon[I^\xi_y(u_j)+H^\xi_y(u_j)+\mathcal{H}^0(J_{u^\xi_{j,y}})]\} \\ &= \liminf_{m\to+\infty}\{K^\xi_y(u_m,w)+\varepsilon[I^\xi_y(u_m)+H^\xi_y(u_m)+\mathcal{H}^0(J_{u^\xi_{m,y}})]\}. \end{aligned}$$

By Fatou's lemma and by (2.7.19) we get

$$\begin{aligned} &\int_{\pi_\xi} K^\xi_y(u,w)\,d\mathcal{H}^{n-1}(y) \\ &\le \liminf_{j\to+\infty}\int_{\pi_\xi} K^\xi_y(u_j,w)\,d\mathcal{H}^{n-1}(y) \\ &\le \liminf_{l\to+\infty}\int_{\pi_\xi}\{K^\xi_y(u_l,w)+\varepsilon[I^\xi_y(u_l)+H^\xi_y(u_l)+\mathcal{H}^0(J_{u^\xi_{l,y}})]\}\,d\mathcal{H}^{n-1}(y). \end{aligned}$$

Using now the point (ii) of Theorem 2.5.1, (2.7.7), (2.7.8) and (2.7.18) we obtain

$$\begin{aligned} &\int_\Omega |(\mathcal{E}u\xi,\xi)(x)-w(x)|\,dx \\ &\le \liminf_{k\to+\infty}\int_\Omega |(\mathcal{E}u_k\xi,\xi)(x)-w(x)|\,dx+\varepsilon[K+C+\lim_{k\to+\infty}\lambda^\xi_{u_k}(\Omega)]. \end{aligned}$$

Letting $\varepsilon \to 0^+$, we obtain

$$\begin{aligned} &\int_\Omega |(\mathcal{E}u\xi,\xi)(x) - w(x)|dx \\ &\le \liminf_{k\to+\infty} \int_\Omega |(\mathcal{E}u_k\xi,\xi)(x) - w(x)|dx \quad \forall w \in L^1(\Omega). \end{aligned} \tag{2.7.22}$$

From this we obtain $(\mathcal{E}u_k\xi,\xi) \rightharpoonup (\mathcal{E}u\xi,\xi)$ weakly in $L^1(\Omega)$ applying the proposition (4.4) of Ambrosio [10]. Since $\{\mathcal{E}u_h\}$ is weakly convergent in $L^1(\Omega, M_s^{n\times n})$, it follows that the whole sequence $\{(\mathcal{E}u_h\xi,\xi)\}$ converges to $(\mathcal{E}u\xi,\xi)$ weakly in $L^1(\Omega)$ for every $\xi = \xi_i + \xi_j$. As $\{\xi_1,\cdots,\xi_n\}$ is a basis of $\mathbb{R}^n$, (2.7.4) follows.
Let us now prove (2.7.5). Since the sequence $\{u_h\}$ is bounded in $BD(\Omega)$ and u_h converges to u in $L^1_{\rm loc}(\Omega,\mathbb{R}^n)$, then Eu_h converges to Eu weakly* in $\mathcal{M}_b(\Omega, M_s^{n\times n})$. Since u_h and u belong to $SBD(\Omega)$ and $\mathcal{E}u_h \rightharpoonup \mathcal{E}u$ weakly in $L^1(\Omega, M_b^{n\times n})$, we deduce that

$$E^j u_h = Eu_h - \mathcal{E}u_h\mathcal{L}^n \llcorner \Omega \rightharpoonup E^j u = Eu - \mathcal{E}u\mathcal{L}^n \llcorner \Omega$$

weakly* in $\mathcal{M}_b(\Omega, M_s^{n\times n})$ and this achieves the proof of the theorem. □

Remark 2.7.3 Motivated by the compactness theorem 2.7.1, we introduce the following subspace of $SBD(\Omega)$. Let $p > 1$. We set

$$SBD^p(\Omega) := \{u \in SBD(\Omega)\colon\ \mathcal{H}^{n-1}(J_u) < +\infty \text{ and } \mathcal{E}u \in L^p(\Omega, M_s^{n\times n})\}.$$

We define in $SBD^p(\Omega)$ the following weak convergence: we say that the sequence $\{u_h\} \subset SBD^p(\Omega)$ weakly converges in $SBD^p(\Omega)$ to u if

$$\begin{cases} u_h \to u \text{ in } L^1(\Omega,\mathbb{R}^n), \\ \sup_h\{|E^j u_h|(\Omega) + \mathcal{H}^{n-1}(J_{u_h})\} < +\infty, \\ \mathcal{E}u_h \rightharpoonup \mathcal{E}u \text{ in } L^p(\Omega, M_s^{n\times n}). \end{cases} \tag{2.7.23}$$

From (2.7.23) we have

$$\sup_h \Big\{ \|u_h\|_1 + |E^j u_h|(\Omega) + \|\mathcal{E} u_h\|_p + \mathcal{H}^{n-1}(J_{u_h}) \Big\} < +\infty \tag{2.7.24}$$

and by the compactness theorem 2.7.1, we get $u \in SBD^p(\Omega)$. Conversely, any sequence verifying (2.7.24) has a subsequence weakly convergent to a function $u \in SBD(\Omega)^p$.

Chapter 3

The Approximate Differentiability of BD Functions

3.1 Introduction

The notion of *symmetric approximate differential* related, by Theorem 2.4.3 to the density $\mathcal{E}u$ of the absolute continuous part of the measure Eu for a BD function u, makes one think about the existence of the full *approximate differential* ∇u of which $\mathcal{E}u$ is the symmetric part. This fact has been confirmed by Hajłasz [136], Ambrosio-Coscia-Dal Maso [21] (and also by Ebobisse [114]) who proved that functions with bounded deformation are approximately differentiable $\mathcal{L}^n$-almost everywhere inside their domain.

In the first part of this chapter, we will present the proof by Ambrosio, Coscia and Dal Maso of the approximate differentiability of BD functions. In the second part, we present the result of Ebobisse [114] on a quantitative Lusin type approximation of BD functions by Lipschitz ones from which one can also deduce the approximate differentiability result.

3.2 The approximate differentiability

Let us first recall the definition of approximate differentiability.

Definition 3.2.1 *Let $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ be a Borel function and $x \in \Omega$. We say that u is approximately differentiable at x, if there exists an $m \times n$ matrix L such that*

$$\operatorname{ap}\lim_{y \to x} \frac{|u(y) - u(x) - L(y - x)|}{|y - x|} = 0, \tag{3.2.1}$$

i.e., the Borel set

$$A_\varepsilon := \left\{ y \in \Omega\colon \frac{|u(y) - u(x) - L(y - x)|}{|y - x|} > \varepsilon \right\}$$

has $\mathcal{L}^n$ density 0 at x, for any $\varepsilon > 0$.

As we have seen in Chapter I, a stronger definition of approximate differentiability is used for BV functions. That is

$$\lim_{\rho \to 0^+} \fint_{B_\rho(x)} \frac{|u(y) - u(x) - L(y - x)|}{|y - x|} dy = 0. \tag{3.2.2}$$

The following proposition gives a sufficient condition for the approximate differentiability, intermediate between the stronger condition (3.2.2) and (3.2.1).

Proposition 3.2.2 *Let $u \in L^1_{loc}(\Omega, \mathbb{R}^m)$ and assume that*

$$\lim_{\rho \to 0^+} \frac{1}{\rho^n} \int_{B_\rho(x_0)} \frac{|u(y) - u(x_0) - L(y - x_0)|}{\rho} dy = 0 \tag{3.2.3}$$

for some $x_0 \in \Omega$ and for some $m \times n$ matrix L. Then the function u is approximately differentiable at x_0 and $\nabla u(x_0) = L$.

Proof. We need to prove that for every $\varepsilon > 0$ the set

$$A_\varepsilon := \left\{ y \in \Omega\colon \frac{|u(y) - u(x_0) - L(y - x_0)|}{|y - x_0|} > \varepsilon \right\}$$

has $\mathcal{L}^n$-density zero at x_0, i.e.,

$$\lim_{\rho\to 0^+} \frac{|A_\varepsilon \cap B_\rho(x_0)|}{|B_\rho(x_0)|} = 0.$$

For any $\rho > 0$ and any $t \in (0,1)$, splitting $B_\rho(x_0)$ into the disjoint sets $B_{t\rho}(x_0)$ and $B_\rho(x_0) \setminus B_{t\rho}(x_0)$ we obtain

$$\frac{1}{\rho^n}|B_\rho(x_0) \cap A_\varepsilon| \le \frac{1}{t\varepsilon\rho^n} \int_{B_\rho(x_0)} \frac{|u(y) - u(x_0) - L(y - x_0)|}{\rho} dy + w_n t^n.$$

Therefore

$$\limsup_{\rho\to 0^+} \frac{|B_\rho(x_0) \cap A_\varepsilon|}{\rho^n} \le w_n t^n$$

and letting $t \to 0^+$ we obtain the statement. □

Let us now state the approximate differentiability theorem.

Theorem 3.2.3 (**Ambrosio-Coscia-Dal Maso** [21])
Let $u \in BD(\Omega)$. Then for $\mathcal{L}^n$-almost every $x \in \Omega$, there exists an $n \times n$ matrix $\nabla u(x)$ such that

$$\lim_{\rho\to 0^+} \frac{1}{\rho^n} \int_{B_\rho(x)} \frac{|u(y) - u(x) - \nabla u(x)(y - x)|}{\rho} dy = 0. \tag{3.2.4}$$

In particular, u is approximately differentiable $\mathcal{L}^n$-almost everywhere in Ω. Moreover, when Ω is either $\mathbb{R}^n$ or a bounded open subset with a Lipschitz continuous boundary, the approximate differential ∇u satisfies the following L^1 weak estimate

$$\mathcal{L}^n(\{x \in \Omega\colon |\nabla u(x)| > t\}) \le \frac{C(\Omega)}{t} \|u\|_{BD(\Omega)} \quad \forall t > 0 \tag{3.2.5}$$

where $C(\Omega)$ is a constant only depending on Ω.

Remark 3.2.4

(i) From (3.2.4) and Theorem 2.4.3 one can easily verify that

$$\mathcal{E}u(x) = (\nabla u(x) + \nabla u^t(x))/2 \quad \text{for } \mathcal{L}^n\text{-almost every } x \in \Omega,$$

i.e., the approximate symmetric differential is exactly the symmetric part of the approximate differential.

(ii) In the proof of Proposition 2.5.6, using the Ornstein's counter-example we have constructed a function $u \in \mathcal{C}^\infty(\mathbb{R}^n \setminus \{0\})$ having a compact support in $B_1(0)$, which verifies $\int_{\mathbb{R}^n} |\nabla u(x)|dx = +\infty$ and $\|u\|_{BD(\mathbb{R}^n)} < +\infty$. This implies that, in general, no strong L^1 estimate can be obtained for the approximate differential of functions with bounded deformation. We only get the weak L^1 estimate (3.2.5).

The main ingredient in the proof of Theorem 3.2.3 will be the explicit version of the Poincaré type inequality in small balls $B_\rho(x)$ (cf. Section 1.7.3). From this inequality we can estimate at $\mathcal{L}^n$-a.e. $x \in \Omega$ the difference quotient of u up to a constant vector $b_\rho(u)(x)$ and an affine map $\alpha_\rho(u)(x)(y-x)$. The approximate differentiability of u will then follow from Federer [119, Theorem 3.1.8]. From Theorem 2.4.3 we will obtain an estimate of the vector $b_\rho(u)(x)$, while the matrix $\alpha_\rho(u)(x)$ will be expressed as a singular integral. So that a weak L^1 estimate of $\alpha_\rho(u)(x)$ will be obtained applying the following well known proposition proved in Ambrosio-Coscia-Dal Maso [21].

Proposition 3.2.5 *Let $K : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ be a 0-homogeneous function, smooth and with mean value zero on the unit sphere $\mathbf{S}^{n-1}$. For any Radon measure μ with finite total variation in $\mathbb{R}^n$, let us define the functions*

$$h_\rho(x) = \int_{|y-x|\geq\rho} \frac{K(y-x)}{|y-x|^n} d\mu(y) \qquad \rho > 0.$$

Then the function $h(x) = \sup_{\rho>0} |h_\rho(x)|$ satisfies the following weak L^1 estimate

$$\mathcal{L}^n(\{x \in \mathbb{R}^n\colon\ h(x) > t\}) \leq \frac{C(n,K)}{t}|\mu|(\mathbb{R}^n). \tag{3.2.6}$$

Remark 3.2.6 If we apply the proposition above to each component of the fourth-order tensor $\Gamma(x)/|x|^2$ defined in subsection 1.7.3 we obtain

$$\mathcal{L}^n(\{x \in \mathbb{R}^n\colon\ A_u(x) > t\}) \leq \frac{C(n)}{t}|Eu|(\mathbb{R}^n) \tag{3.2.7}$$

where $A_u(x) = \sup_{\rho>0} |A_\rho(u)(x)|$.

The proof of Theorem 3.2.3. Since the approximate differentiability is a local property, for the first part of the statement, we can assume with no loss of generality that $\Omega = \mathbb{R}^n$. From Theorem 2.4.3 and Remark 3.2.6, $\mathcal{L}^n$-almost every $x \in \mathbb{R}^n$ satisfies the properties:

(i)
$$\lim_{\rho\to 0^+} \frac{1}{\rho^n} \int_{B_\rho(x)} \frac{|(u(y) - u(x) - \mathcal{E}u(x)(y-x), y-x)|}{\rho|y-x|} dy = 0;$$

(ii) x is a Lebesgue point for $\mathcal{E}u$ and $\widetilde{\mathcal{E}u}(x) = \mathcal{E}u(x)$;

(iii) $|E^s u|$ has n-dimensional density zero at x;

(iv) The matrices $A_\rho(u)(x)$ in section 1.7.3 are equi-bounded with respect to ρ.

Step 1: Let us fix $x \in \mathbb{R}^n$ verifying (i)-(iv). We want to prove that

$$\lim_{\rho\to 0^+} \frac{1}{\rho^n} \int_{B_\rho(x)} \frac{|u(y) - u(x) - \alpha_\rho(u)(x)(y-x)|}{\rho} dy = 0 \tag{3.2.8}$$

for suitable $n \times n$ matrices $\alpha_\rho(u)(x)$ satisfying

$$\sup_{\rho>0} |\alpha_\rho(u)(x)| \le k_n |\mathcal{E}u(x)| + \sup_{\rho>0} |A_\rho(u)(x)|. \tag{3.2.9}$$

To this end we consider the cut-off function $\psi \in \mathcal{C}^1_c(\mathbb{R})$ such that:

$$\psi(t) = 1 \text{ for } |t| \le 1; \quad \psi(t) = 0 \text{ for } |t| \ge 2 \text{ and } |\psi'(t)| \le 2 \text{ for every } t \in \mathbb{R}.$$

Let us define the functions

$$\varphi(y) := \psi(|y-x|), \quad w(y) := \varphi(y)\mathcal{E}u(x)(y-x)$$

and

$$v(y) := u(y) - w(y)$$

for every $y \in \mathbb{R}^n$. Applying to the function v the explicit version of the Poincaré type inequality in paragraph 1.7.3 we obtain for any $\rho > 0$

$$\begin{aligned}(3.2.10)\qquad &\int_{B_\rho(x)} |v(y) - b_\rho(v)(x) - A_\rho(v)(x)(y-x)| \\ &\leq C(n)\rho|Ev|(B_\rho(x)) \\ &\leq C(n)\rho \int_{B_\rho(x)} |\mathcal{E}u(y) - \mathcal{E}u(x)|dy + C(n)\rho|E^s u|(B_\rho(x))\end{aligned}$$

for suitable vectors $b_\rho(v)(x)$ with

$$\begin{aligned}(3.2.11)\qquad A_\rho(v)(x) &= A_\rho(u)(x) + A_\rho(w)(x) \\ &= A_\rho(u)(x) + \frac{1}{2w_n}\int_{\{|y-x|\geq\rho\}} \frac{\Gamma(y-x)}{|y-x|^{n+2}}\mathcal{E}w(y)dy.\end{aligned}$$

Setting $\alpha_\rho(u)(x) := \mathcal{E}u(x) + A_\rho(v)(x)$ and dividing both sides of (3.2.10) by ρ^{n+1} we obtain

$$\begin{aligned}&\frac{1}{\rho^n}\int_{B_\rho(x)} \frac{|u(y) - b_\rho(v)(x) - \alpha_\rho(u)(x)(y-x)|}{\rho}dy \\ &\leq \frac{1}{\rho^n}\int_{B_\rho(x)} \frac{|v(y) - b_\rho(v)(x) - A_\rho(v)(x)(y-x)|}{\rho}dy \\ &\leq C(n)\frac{1}{\rho^n}\int_{B_\rho(x)} |\mathcal{E}u(y) - \mathcal{E}u(x)| + C(n)\frac{|E^s u|(B_\rho(x))}{\rho^n}.\end{aligned}$$

So that by the properties (ii) and (iii) we get

$$(3.2.12)\qquad \lim_{\rho\to 0^+} \frac{1}{\rho^n}\int_{B_\rho(x)} \frac{|u(y) - b_\rho(v)(x) - \alpha_\rho(u)(x)(y-x)|}{\rho}dy = 0$$

From the inequality

$$\begin{aligned}&\int_{B_\rho(x)} \frac{|(b_\rho(v)(x) - u(x), y-x)|}{\rho|y-x|}dy \\ &\leq \int_{B_\rho(x)} \frac{|(u(y) - u(x) - \alpha_\rho(u)(x)(y-x), y-x)|}{\rho|y-x|}dy \\ &+ \int_{B_\rho(x)} \frac{|u(y) - b_\rho(v)(x) - \alpha_\rho(u)(x)(y-x)|}{\rho}dy,\end{aligned}$$

the identity (3.2.12) and the fact that the property (i) is also valid with $\mathcal{E}u$ replaced by $\alpha_\rho(u)(x)$ (since $A_\rho(v)(x)$ is skew-symmetric) we deduce that

$$\lim_{\rho\to 0^+}\frac{1}{\rho^n}\int_{B_\rho(x)}\frac{|(b_\rho(v)(x)-u(x),y-x)|}{\rho|y-x|}dy=0 \tag{3.2.13}$$

which implies that $b_\rho(v)(x)=u(x)+o(\rho)$. Inserting this into (3.2.12) we obtain (3.2.8).

Now, since

$$\nabla w(y)=\psi(|y-x|)\mathcal{E}u(x)+[\mathcal{E}u(x)(y-x)]\otimes\nabla\varphi(y), \tag{3.2.14}$$

and since the function Γ has mean value zero on the unit sphere, the first term in the right-hand side of (3.2.14) gives no contribution to the integral in (3.2.11). Consequently,

$$\begin{aligned}\Big|\frac{1}{2w_n}\int_{\{|y-x|\geq\rho\}}\frac{\Gamma(y-x)}{|y-x|^{n+2}}\mathcal{E}w(y)dy\Big| &\leq C(n)|\mathcal{E}u(x)|\int_{B_2(x)}\frac{dy}{|y-x|^{n-1}}\\ &= 2nw_nC(n)|\mathcal{E}u(x)|.\end{aligned} \tag{3.2.15}$$

The inequality (3.2.9) is then obtained from (3.2.11), (3.2.12), (3.2.15) and the fact that $|\alpha_\rho(u)(x)|\leq|\mathcal{E}u(x)|+|A_\rho(v)(x)|$.

Step 2. Let us now prove that

$$\text{ap}\ \limsup_{y\to x}\frac{|u(y)-u(x)|}{|y-x|}\leq\sup_{\rho>0}|\alpha_\rho(u)(x)|<+\infty$$

for any $x\in\mathbb{R}^n$ satisfying (i)-(iv) and hence (3.2.8) and (3.2.9). In fact , for any such x and any $\varepsilon>0$, we need only to prove that the Borel set

$$A_\varepsilon:=\left\{y\in\mathbb{R}^n\colon\ \frac{|u(y)-u(x)|}{|y-x|}-\sup_{\rho>0}|\alpha_\rho(u)(x)|>\varepsilon\right\}$$

has $\mathcal{L}^n$ density zero at x. To this end, let $t\in(0,1)$.

Then

$$
\begin{aligned}
(3.2.16) \qquad & \frac{1}{\rho^n}\mathcal{L}^n(A_\varepsilon \cap B_\rho(x)) \\
& \le \frac{1}{\rho^n}\mathcal{L}^n(A_\varepsilon \cap (B_\rho(x) \setminus B_{t\rho}(x))) + \frac{1}{\rho^n}\mathcal{L}^n(A_\varepsilon \cap B_{t\rho}(x) \\
& \le \frac{1}{t\varepsilon\rho^n}\int_{B_\rho(x)} \frac{|u(y)-u(x)-\alpha_\rho(u)(x)(y-x)|}{|y-x|}dy + w_n t^n.
\end{aligned}
$$

By (3.2.8) we obtain

$$\limsup_{\rho\to 0^+} \frac{1}{\rho^n}\mathcal{L}^n(A_\varepsilon \cap B_\rho(x)) \le w_n t^n.$$

So that by letting $t \to 0^+$, we get

$$\lim_{\rho\to 0^+} \frac{1}{\rho^n}\mathcal{L}^n(A_\varepsilon \cap B_\rho(x)) = 0.$$

This means that

$$\text{ap}\limsup_{y\to x} \frac{|u(y)-u(x)|}{|y-x|} \le \varepsilon + \sup_{\rho>0}|\alpha_\rho(x)|.$$

So, by the arbitrariness of ε we get the statement of Step 2.

Now from Federer [119, Theorem 3.1.8], we deduce that the function u is $\mathcal{L}^n$-almost everywhere approximately differentiable in $\mathbb{R}^n$ and that ∇u satisfies

$$|\nabla u(x)| \le \sup_{\rho>0}|\alpha_\rho(x)| \le k_n|\mathcal{E}u(x)| + \sup_{\rho>0}|A_\rho(u)(x)|$$

for $\mathcal{L}^n$-a.e. $x \in \mathbb{R}^n$. So that

$$
\begin{aligned}
\mathcal{L}^n(\{x\in\mathbb{R}^n\colon\ |\nabla u(x)| > t\}) \ &\le\ \mathcal{L}^n(\{x\in\mathbb{R}^n\colon\ |\mathcal{E}u(x)| > \frac{t}{2k_n}\}) \\
&+\ \mathcal{L}^n(\{x\in\mathbb{R}^n\colon\ |A_u(x)| > \frac{t}{2}\}) \\
&\le\ \frac{C(n)}{t}|Eu|(\mathbb{R}^n)
\end{aligned}
$$

where $A_u(x)$ has been introduced in the remark 3.2.6. This last inequality implies the weak L^1 estimate (3.2.5).

Step 3: Now let us prove the stronger condition (3.2.4). Let x be an approximate differentiability point where (3.2.8) holds and the matrices $\alpha_\rho(u)(x)$ are equibounded with respect to ρ. Let L be the limit of $\alpha_{\rho_k}(u)(x)$ for some sequence ρ_k converging to 0 as $k \to +\infty$. By (3.2.8) we have

$$\lim_{k\to+\infty} \frac{1}{(\rho_k)^n} \int_{B_{\rho_k}(x)} \frac{|u(y) - u(x) - L(y-x)|}{\rho_k} dy = 0.$$

This means that the sequence of rescaled functions $v_k(y) := [u(x + \rho_k y) - u(x)]/\rho_k$ converges to Ly in $L^1(B_1(0), \mathbb{R}^n)$ strongly. Then it converges in measure to Ly. On the other hand, from the definition of the approximate differentiability we deduce that the sequence $\{v_k\}$ converges in measure to $\nabla u(x)y$; hence $L = \nabla u(x)$. Since the sequence ρ_k is arbitrary, this proves that $\alpha_\rho(u)(x)$ converges to $\nabla u(x)$ as $\rho \to 0^+$ so that (3.2.8) implies (3.2.5) and this achieves the proof of Theorem 3.2.3. $\square$

3.3 A Lusin type approximation of BD functions

In this paragraph, we present the result of Ebobisse [114] from which we can also deduce the approximate differentiability of functions with bounded deformation. In that paper we prove that any function with bounded deformation can be approximated in the sense of Lusin by Lipschitz functions. Precisely, we prove the following theorem:

Theorem 3.3.1 *Let Ω be a Lipschitz bounded open subset of $\mathbb{R}^n$ and $u \in BD(\Omega)$. Then for any $\lambda > 0$, there exists a Lipschitz function $v_\lambda : \Omega \to \mathbb{R}^n$ with $lip(v_\lambda) \leq C(n,\Omega)\lambda$ such that:*

$$\mathcal{L}^n(\{x \in \Omega \,:\, v_\lambda(x) \neq u(x)\}) \leq \frac{C(n,\Omega)}{\lambda} \|u\|_{BD(\Omega)}, \tag{3.3.1}$$

where $C(n,\Omega)$ is a positive constant only depending on n and Ω.

Let us recall that Lusin type results for Sobolev functions have been established by Liu [147], Michael-Ziemer [152] and Acerbi-Fusco [1]. Those results have been successfully applied by Acerbi-Fusco [1] to prove the lower semicontinuity of quasi-convex integral functionals defined in $W^{1,p}(\Omega)$ $(1 \leq p < +\infty)$. Precisely, they proved that it is possible to extract from bounded sequences in $W^{1,p}(\Omega)$, approximating subsequences in the sense of Lusin which are bounded in $W^{1,\infty}(\Omega)$ and for which the $W^{1,\infty}(\Omega)$- weak* lower semicontinuity of quasi-convex integral functionals was already known.

By using the notion of maximal function of Radon measures with finite total variation, Ambrosio [12] was able to extend in the space $BV(\Omega,\mathbb{R}^k)$ the Acerbi-Fusco approximating result and proved the $L^1(\Omega,\mathbb{R}^k)$-strong lower semicontinuity of integral functionals defined in $SBV(\Omega,\mathbb{R}^k)$.

The Lusin type approximation of Sobolev functions is also useful to study the regularity properties of minimizers of quasi-convex integral functionals. In fact, a partial regularity result of minimizers of the integral functional $\int_\Omega f(\nabla u(x))dx$ with a twice continuously differentiable and uniformly strictly quasi-convex integrand f was first proved by Evans [117] under a more restrictive assumption on the growth of the second derivative of f. Acerbi-Fusco [2] have singled out the Lusin type approximation of Sobolev functions as a powerful tool to prove the Evans' result without no assumption on the growth of $D^2 f$.

It is well known (see Federer [119, Theorem 3.1.8]) that for any function u approximately differentiable $\mathcal{L}^n$-almost everywhere and any $\varepsilon > 0$ there is a Lipschitz function v that agrees with u on a set whose complement has Lebesgue measure less than ε. Thus, the approximate differentiability of BD functions implies part of Theorem 3.3.1. So, the real interest of this theorem is the quantitative estimate 3.3.1 which could be useful as in the case of Sobolev and SBV spaces, to study the lower semicontinuity of integral functionals involving the symmetric gradient Eu or some of its parts, defined in the spaces $BD(\Omega)$ or $SBD(\Omega)$.

The main ingredients in the proof of Theorem 3.3.1 will be the *maximal function* of the measure Eu, the explicit version of the Poincaré type inequality on small balls and Theorem 2.4.3.

3.3.1 The maximal function of a Radon measure

Let μ be a Radon measure with finite total variation in $\mathbb{R}^n$. We define the maximal function $M(\mu)$ of μ by

$$M(\mu)(x) := \sup_{\rho>0} \frac{|\mu|(B_\rho(x))}{|B_\rho(x)|} \qquad \forall x \in \mathbb{R}^n.$$

By the Besicovitch covering theorem (see Evans-Gariepy [118], Federer [119], Simon [181], etc...), we have the following weak L^1 estimate

$$\mathcal{L}^n(\{x \in \mathbb{R}^n\colon\ M(\mu)(x) > t\}) \leq \frac{C(n)}{t}|\mu|(\mathbb{R}^n).$$

Therefore, for any $u \in BD(\mathbb{R}^n)$, we have

$$\mathcal{L}^n(\{x \in \mathbb{R}^n \colon M(Eu)(x) > t\}) \leq \frac{C(n)}{t}|Eu|(\mathbb{R}^n). \tag{3.3.2}$$

We will need the following abstract lemma in the proof of Theorem 3.3.1.

Lemma 3.3.2 *Let C be a convex subset of $\mathbb{R}^n$ having a positive measure. If we suppose that C generates all the space $\mathbb{R}^n$, then there exists a positive constant α such that*

$$\int_C |(w, y)| dy \geq \alpha |w| \quad \text{for any } w \in \mathbb{R}^n. \tag{3.3.3}$$

Proof. Suppose $w \neq 0$. we have $\alpha = \min_{v \in \mathbf{S}^{n-1}} \int_C |(v, y)| dy > 0$. In fact, $\alpha = 0$ implies $\int_C |(v_0, y)| dy = 0$ for some $v_0 \in \mathbf{S}^{n-1}$. Since C generates all the space $\mathbb{R}^n$, we get $v_0 = 0$ and this is absurd. Therefore

$$\int_C |(\frac{w}{|w|}, y)| dy \geq \alpha$$

and the lemma follows. □

3.3.2 The proof of Theorem 3.3.1

For all what follows, we will denote by $C(n)$, or possibly $C(n, \Omega)$, any positive constant only depending on n, or also depending on Ω.

Step 1: We will first prove the theorem for $\Omega = \mathbb{R}^n$. Let us define for any $\lambda > 0$,

$$E_\lambda = \{x \in \mathbb{R}^n \colon M(Eu)(x) \leq \lambda \ ; \ A_u(x) \leq \lambda\},$$

where

$$A_u(x) = \sup_{\rho > 0} |A_\rho(u)(x)|.$$

From the explicit version of the poincaré type inequality on balls, we have for any $x \in E_\lambda$ and $\rho > 0$,

$$\begin{aligned} \int_{B_\rho(x)} |u(y) - b_\rho(u)(x) - A_\rho(u)(x)(y - x)| dy &\leq C(n)\rho |Eu|(B_\rho(x)) \\ &\leq C(n)\lambda\rho^{n+1}. \end{aligned} \tag{3.3.4}$$

We also have from (2.4.4) in the proof of Theorem 2.4.3

$$\int_{B_\rho(x)} |(u(y)-u(x), y-x)|dy \quad \leq \quad \lambda\rho^{n+2},$$

for any Lebesgue point x of u in E_λ. Now let S_u be the complement of Lebesgue points of u, x and x' two points in $E_\lambda \setminus S_u$ and let $\rho = |x-x'|$. Then

$$\begin{aligned}
&\int_{B_\rho(x)} |(b_\rho(u)(x) + A_\rho(u)(x)(y-x) - u(x), y-x)|dy \\
&\leq \int_{B_\rho(x)} |(u(y) - b_\rho(u)(x) - A_\rho(u)(x)(y-x), y-x)|dy \\
&\qquad + \int_{B_\rho(x)} |(u(y)-u(x), y-x)|dy \\
&\leq C(n)\lambda\rho^{n+2}.
\end{aligned}$$

Therefore

$$\int_{B_\rho(x)} |(b_\rho(u)(x) + A_\rho(u)(x)(y-x) - u(x), y-x)|dy \leq C(n)\lambda\rho^{n+2}$$

and this yields

$$\begin{aligned}
&\int_{B_\rho(x)} |(b_\rho(u)(x) - u(x), y-x)|dy \\
&\leq C(n)\lambda\rho^{n+2} + \int_{B_\rho(x)} |(A_\rho(u)(x)(y-x), y-x)|dy \\
&\leq C(n)\lambda\rho^{n+2}.
\end{aligned} \tag{3.3.5}$$

Then

$$|b_\rho(u)(x) - u(x)| \leq C(n)\lambda\rho \tag{3.3.6}$$

since the left hand side of (3.3.5) is equal to $|b_\rho(u)(x) - u(x)|\rho^{n+1}$ up to a positive multiplicative constant depending only on n. Now from

$$\int_{B_\rho(x')} |(u(y)-u(x'), y-x')|dy \leq \lambda\rho^{n+2} \tag{3.3.7}$$

and the identity

$$(b_\rho(u)(x) - A_\rho(u)(x)(x - x') - u(x'), y - x')$$

$$= (b_\rho(u)(x) + A_\rho(u)(x)(y - x) - u(y), y - x')$$

$$+(u(y) - u(x'), y - x') - (A_\rho(u)(x)(y - x'), y - x'),$$

we obtain

$$\int_{B_\rho(x)\cap B_\rho(x')} |(b_\rho(u)(x) - A_\rho(u)(x)(x - x') - u(x'), y - x')|dy \tag{3.3.8}$$

$$\leq \int_{B_\rho(x)} |(b_\rho(u)(x) + A_\rho(u)(x)(y - x) - u(y), y - x')|dy$$

$$+ \int_{B_\rho(x')} |(u(y) - u(x'), y - x')|dy$$

$$+ \int_{B_\rho(x)\cap B_\rho(x')} |(A_\rho(u)(x)(y - x'), y - x'|dy.$$

From (3.3.4), (3.3.7) one can easily see that the right hand side of (3.3.8) is estimated from above by $C(n)\lambda\rho^{n+2}$. The left hand side can always be estimated from below by

$$\rho^{n+1}|b_\rho(u)(x) - A_\rho(u)(x)(x - x') - u(x')|$$

up to a multiplicative positive constant. In fact, after rescaling, the left hand side of (3.3.8) is equal to

$$\rho^{n+1} \int_{B_1(0)\cap B_1((x-x')/\rho)} |(b_\rho(u)(x) - A_\rho(u)(x)(x - x') - u(x'), y)|dy. \tag{3.3.9}$$

Applying the abstract lemma 3.3.2 with $w = b_\rho(u)(x) - A_\rho(u)(x)(x - x') - u(x')$ and C is the closure of $B_1(0) \cap B_1((x - x')/\rho)$, we obtain the result. Then from (3.3.8), (3.3.9), and Lemma 3.3.2, the following inequality holds

$$|b_\rho(u)(x) - A_\rho(u)(x)(x - x') - u(x')| \leq C(n)\lambda\rho.$$

Therefore

$$|b_\rho(u)(x) - u(x')| \le C(n)\lambda\rho. \tag{3.3.10}$$

So from (3.3.6), (3.3.10) and our choice of ρ we have

$$|u(x) - u(x')| \le C(n)\lambda|x - x'|. \tag{3.3.11}$$

Then we have proved that the function $u|_{E_\lambda \setminus S_u}$ is Lipschitz continuous with its Lipschitz constant less or equal to $C(n)\lambda$. The Kirszbraun's theorem (see Federer [119, Theorem 2.10.43]) will then provide an extension v_λ in $\mathbb{R}^n$ of the function $u|_{E_\lambda \setminus S_u}$ with $\operatorname{lip}(v_\lambda) \le C(n)\lambda$.

Let us now prove (3.3.1) in the case $\Omega = \mathbb{R}^n$. It follows from the weak type estimates (3.2.7) and (3.3.2) that

$$\begin{aligned}
\mathcal{L}^n(\{x \in \mathbb{R}^n\colon\ v_\lambda(x) \ne u(x)\}) &\le \mathcal{L}^n(\mathbb{R}^n \setminus (E_\lambda \setminus S_u)) \\
&\le \mathcal{L}^n(\{x \in \mathbb{R}^n\colon\ M(Eu)(x) > \lambda\}) \\
&\qquad + \mathcal{L}^n(\{x \in \mathbb{R}^n\colon\ A_u(x) > \lambda\}) \\
&\le \frac{C(n)}{\lambda}|Eu|(\mathbb{R}^n).
\end{aligned}$$

Step 2: Now let $u \in BD(\Omega)$, where Ω is a bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$. Then, the extension u_* of u by 0 outside Ω is in $BD(\mathbb{R}^n)$ (see Corollary 1.6.3) and satisfies

$$|Eu_*|(\mathbb{R}^n) \le C(n,\Omega)\,\|u\|_{BD(\Omega)}\,.$$

Applying now what we have done above to u_*, we obtain a Lipschitz function v_λ in $\mathbb{R}^n$ with $\operatorname{lip}(v_\lambda) \le C(n)\lambda$ and

$$\begin{aligned}
\mathcal{L}^n(\{x \in \Omega\colon\ v_\lambda(x) \ne u(x)\}) &\le \mathcal{L}^n(\{x \in \mathbb{R}^n\colon\ v_\lambda(x) \ne u_*(x)\}) \\
&\le \frac{C(n)}{\lambda}|Eu_*|(\mathbb{R}^n) \\
&\le \frac{C(n,\Omega)}{\lambda}\,\|u\|_{BD(\Omega)}\,.
\end{aligned}$$

□

Remark 3.3.3 (i) Let us observe that the Lusin type result proved above does not require all the theorem 2.4.3. But as we have seen, it only needs the inequality (2.4.4) used in the proof of theorem 2.4.3. As we have said before, from Theorem 3.3.1 one can deduce the full *approximate differentiability* of BD-functions. In fact, one can easily see that any density point of the set $\{u = v_\lambda\}$ in which v_λ is differentiable in a classical sense is a point of *approximate differentiability* of the function u.

(ii) Since any Lipschitz function is approximated in the sense of Lusin by C^1 functions, from Theorem 3.3.1 one easily deduces the Lusin type approximation of BD-functions by C^1 ones, with the same quantitative estimate.

Chapter 4

Functionals Defined in $BD(\Omega)$

4.1 Introduction

In these last years a large attention has been devoted to functionals of the type

$$(4.1.1)\qquad F(u) := \int_{\Omega} f(x, u, \nabla u)dx + \int_{S_u \cap \Omega} g(x, u^+, u^-, \nu_u)d\mathcal{H}^{n-1}(x),$$

modelling various phenomena where there is a competion between volume and surface energies. Precisely, those functionals arise in image segmentation theory (e.g., the model of Mumford & Shah) and some fields of mathematical physics as for instance, fracture mechanics and the theory of liquid crystals.

Various existence and approximation results for problems involving those functionals have been obtained in the space $BV(\Omega, \mathbb{R}^k)$ or in $SBV(\Omega, \mathbb{R}^k)$. (See e.g., Ambrosio [11], De Giorgi-Carriero-Leaci [100], Dal Maso-Morel-Solimini [86], Ambrosio-Braides [16, 17], Ambrosio-Tortorelli [30], Chambolle [79], Braides-Dal Maso [68], Gobbino [133]).

Concerning fracture mechanics problems, the study was limited on BV functions only because of the rich structure of this space. Recently, Bellettini, Coscia and Dal Maso have studied in [46, 47] functionals of the type

$$(4.1.2)\qquad G(u) := \int_{\Omega} f(x, u, \mathcal{E}u)dx + \int_{J_u \cap \Omega} g(x, u^+, u^-, \nu_u)d\mathcal{H}^{n-1}(x),$$

for which the correct function space is $BD(\Omega)$ or its offspring $SBD(\Omega)$. By the theory developed in Chapter II, all the elements appearing in (4.1.2) are well defined. The aim of this chapter is to show how some of the properties presented in the previous chapters can be used in order to study the lower semicontinuity and some minimization problems for functionals defined in $BD(\Omega)$.

We will begin this chapter with the result of Ebobisse [115], where we discuss a new weak convexity property under which we study the lower semicontinuity of some "regular" functionals defined in $BD(\Omega)$. In Section 4.3, we present a lower semicontinuity theorem by Bellettini-Coscia-Dal Maso [46, 47] for some functionals of the type (4.1.2) with a superlinear bulk part. Using that theorem two minimization problems are studied in Section 4.4. We end this chapter by proving in the section 4.5 a lower semicontinuity theorem for some linear growth functionals defined in all the space $BD(\Omega)$.

4.2 Symmetric quasi-convexity and lower semicontinuity of functionals defined in $LD(\Omega)$

We study in this paragraph the lower semicontinuity of functionals of the type:

$$F(u) = \int_\Omega f\left(\mathcal{E}u(x)\right) dx, \tag{4.2.1}$$

where f is a Borel function defined in the space $M_s^{n\times n}$ of symmetric $n\times n$ matrices. This kind of functionals arises very often in the framework of the mathematical theory of elasto-plasticity. Precisely, they can model the deformation energy of some elasto-plastic body whose reference configuration is Ω.

When we look for the equilibrium configuration of the body Ω in the absence of loads, we need to minimize the functional (4.2.1) above in some suitable function space. Very often this is done following the so-called direct method of the calculus of variations, i.e. looking for a topology in which the functional (4.2.1) is lower semicontinuous and coercive.

Various lower semicontinuity results for functionals (4.2.1) under the convexity of f have been proved, for example, in the framework of mathematical formulation of perfect plasticity (see for instance Anzellotti-Giaquinta [37, 38], Suquet [186, 187], Temam [191]). We will introduce and study in the first part of this paragraph a weak convexity property of the integrand f related to the lower semicontinuity of the functional (4.2.1) with respect to weak topologies.

Definition 4.2.1 *We will say that a Borel function $f : M_s^{n\times n} \to \mathbb{R}_+$ is symmetric quasi-convex if*

$$\fint_D f\left(A + \mathcal{E}\varphi(x)\right)dx \geq f(A) \tag{4.2.2}$$

for any bounded open subset D of Ω, $A \in M_{sym}^{n\times n}$ and $\varphi \in W_0^{1,\infty}(D, \mathbb{R}^n)$.

This property is similar and also related to the classical *quasi-convexity* in the sense of Morrey [161]. A variety of results about the classical *quasi-convexity* and its connection to the lower semicontinuity of integral functionals in Sobolev spaces and also in the space BV of functions with *bounded variation* can be found in Acerbi-Fusco [1], Ambrosio-Dal Maso [23], Ball [42], Dacorogna [81], Fonseca-Müller [122], Morrey [161] and the references therein. If the classical *quasi-convexity* is more related to functions spaces whose topologies involve gradients (like Sobolev and BV spaces), our property is related to function spaces with topologies involving only the deformation tensor Eu like $BD(\Omega)$.

We will prove in Theorem 4.2.3 and Remark 4.2.4(ii) that the *symmetric quasi-convexity* is a necessary and sufficient condition to the lower semicontinuity of the functional (4.2.1) in the space $LD(\Omega)$ with respect to the $L^1(\Omega, \mathbf{R}^n)$-norm topology.

Following the result of Šverák [189], we will construct an example of non convex function which is *symmetric quasi-convex.*

Let us make some observations about the symmetric quasi-convexity .

Remark 4.2.2 (i) One can easily see that a function f is *symmetric quasi-convex* if and only if $f \circ \pi$ is *quasi-convex* in the sense of Morrey [157], where π denotes the canonical projection of $M^{n\times n}$ on $M_s^{n\times n}$.

(ii) The *symmetric quasi-convexity* is weaker than convexity, i.e. any convex function is *symmetric quasi-convex.*

(iii) If f has a linear growth, by a density argument one can easily verify that the test functions in (4.2.2) could also be taken in the space $LD_0(D)$, closure of $C_c^1(D, \mathbb{R}^n)$ in the $BD(D)$-norm topology.

4.2.1 The lower semicontinuity theorem

For all what follows C, C_1, C_2, C_3 will denote positive constants.

Theorem 4.2.3 *Let $f : M_s^{n\times n} \to [0, +\infty[$ be a Borel function which verifies the following conditions:*

(H1) $C_1 \|A\| - C_2 \leq f(A) \leq C_3(1 + \|A\|)$ *for any $A \in M_s^{n\times n}$;*

(H2) f *is symmetric quasi-convex.*

Then the functional (4.2.1) is lower semicontinuous in the space $LD(\Omega)$ with respect to the $L^1(\Omega, \mathbb{R}^n)$-norm topology.

Remark 4.2.4 (i) From Theorem 4.2.3 we have the lower semicontinuity of the functional (4.2.1) with respect to the following weak convergences in $LD(\Omega)$:

$u_h \rightharpoonup u$ in $LD(\Omega)$ iff $u_h \rightharpoonup u$ in $L^1(\Omega, \mathbb{R}^n)$ and $Eu_h \rightharpoonup Eu$ in $L^1(\Omega, M_s^{n\times n})$.

$u_h \overset{\star}{\rightharpoonup} u$ in $LD(\Omega)$ iff $u_h \to u$ in $L^1(\Omega, \mathbb{R}^n)$ and $Eu_h \overset{\star}{\rightharpoonup} Eu$ in $M_b(\Omega, M_s^{n\times n})$.

(ii) The *symmetric quasi-convexity* of f is a necessary condition for the lower semicontinuity of the functional (4.2.1) in $W^{1,\infty}(\Omega, \mathbb{R}^n)$ with respect to the weak* convergence and also in $W^{1,p}(\Omega, \mathbb{R}^n)$ with respect to the weak convergence, under some additional p-growth assumptions on f. In fact, the weak* (resp. weak) lower semicontinuity of the functional (4.2.1) in $W^{1,\infty}(\Omega, \mathbb{R}^n)$ (resp. $W^{1,p}(\Omega, \mathbb{R}^n)$) implies that $f \circ \pi$ is quasi-convex and hence f is symmetric quasi-convex.

(iii) In many applications, existence theorems for integral functionals with linear growth and with integrands depending only on the tensor Eu involve the space $BD(\Omega)$. So it would be interesting to study, under the *symmetric quasi-convexity* assumption on f, the integral representation of the relaxed functional of (4.2.1) in the space $BD(\Omega)$ with respect to the $L^1(\Omega, \mathbb{R}^n)$-norm topology. That is, for any $u \in BD(\Omega)$,

$$\overline{F}(u) = \inf \left\{ \liminf_{h \to +\infty} F(u_h) \colon \begin{array}{l} u_h \to u \text{ in } L^1_{loc}(\Omega, \mathbb{R}^n), \\ \{u_h\} \subset C^1(\Omega, \mathbb{R}^n) \end{array} \right\}. \tag{4.2.3}$$

In other words one would like to obtain, under the *symmetric quasi-convexity* assumption on the integrand f, a BD-version of the result by Ambrosio-Dal Maso [23], Fonseca-Müller [123]. However, from Theorem 4.2.3 and by regularization we have

$$\overline{F}(u) = \int_\Omega f(\mathcal{E}u(x))dx \quad \text{ for any } u \in LD(\Omega).$$

In fact, from Theorem 4.2.3 we have the inequality

$$\overline{F}(u) \geq \int_\Omega f(\mathcal{E}u(x))dx \quad \text{ for any } u \in LD(\Omega)$$

and by considering the regularizing sequence $u_\varepsilon := u \star \rho_\varepsilon$ we have

$$u_\varepsilon \to u \text{ in } L^1(\Omega, \mathbb{R}^n) \quad \text{and} \quad \mathcal{E}u_\varepsilon \to \mathcal{E}u \text{ in } L^1(\Omega, M^{n\times n}_s).$$

By using the growth assumption (H1) one can show that the function f is continuous. The reverse inequality will then follow from the Lebesgue dominated convergence theorem applied to $\int_\Omega f(\mathcal{E}u_\varepsilon(x))dx$.

Proof of Theorem 4.2.3 : We will divide the proof in three main steps.

Step 1: Under the assumptions of the theorem we show that

$$|\Omega| f(A) \leq \liminf_{h\to+\infty} \int_\Omega f\left(A + \mathcal{E}u_h(x)\right) dx \tag{4.2.4}$$

whenever $A \in M_s^{n\times n}$, Ω is any bounded open subset of $\mathbb{R}^n$ and $\{u_h\}_h \subset LD(\Omega)$ converges to 0 strongly in $L^1(\Omega, \mathbb{R}^n)$. We can assume without loss of generality that:

$$\liminf_{h\to+\infty} \int_\Omega f\left(A + \mathcal{E}u_h(x)\right) dx = \lim_{h\to+\infty} \int_\Omega f\left(A + \mathcal{E}u_h(x)\right) dx < +\infty. \tag{4.2.5}$$

Then, by the growth condition (H1), the sequence $\{\|\mathcal{E}u_h\|_{L^1}\}$ is bounded. So there exists a positive Radon measure $\mu \in M_b(\Omega)$ such that up to a subsequence,

$$\lim_{h\to+\infty} \int_\Omega \varphi(x) \|\mathcal{E}u_h(x)\| dx = \int_\Omega \varphi(x) d\mu(x), \qquad \text{for any } \varphi \in C_0(\Omega). \tag{4.2.6}$$

Let $\{\Omega_k\}_k$ be an increasing sequence of subdomains of Ω and $\{\varphi_k\}_k \subset C_0^1(\Omega)$ such that:

$$\Omega_k \subset\subset \Omega \quad \text{ and } \quad \Omega = \bigcup_{k=1}^{\infty} \Omega_k,$$

$$0 \leq \varphi_k \leq 1, \quad \varphi_k = 1 \text{ in } \Omega_k \quad \text{ and } \quad \varphi_k = 0 \text{ in } \Omega \setminus \bar{\Omega}_{k+1}.$$

Since f is symmetric quasi-convex, we have by Remark 4.2.2 (iii):

$$\begin{aligned} |\Omega| f(A) &\leq \int_\Omega f(A + \mathcal{E}(\varphi_k u_h)(x))dx \\ &= \int_{\Omega\setminus\bar{\Omega}_{k+1}} f(A)dx + \int_{\Omega_{k+1}\setminus\Omega_k} f(A + \mathcal{E}(\varphi_k u_h)(x))dx \\ &\qquad + \int_{\Omega_k} f(A + \mathcal{E}u_h(x))dx. \end{aligned}$$

Then

$$f(A)|\Omega_{k+1}| \leq \int_{\Omega_{k+1}\setminus\Omega_k} f(A+\mathcal{E}(\varphi_k u_h)(x))dx + \int_{\Omega_k} f(A+\mathcal{E}u_h(x))dx.$$

As f is non negative we have

$$\int_\Omega f(A+\mathcal{E}u_h(x))dx - f(A)|\Omega_{k+1}| \geq -\int_{\Omega_{k+1}\setminus\Omega_k} f(A+\mathcal{E}(\varphi_k u_h)(x))dx. \tag{4.2.7}$$

On the other hand

$$\begin{aligned}\int_{\Omega_{k+1}\setminus\Omega_k} f(A+\mathcal{E}(\varphi_k u_h)(x))dx &\leq C_3\int_{\Omega_{k+1}\setminus\Omega_k}(1+\|A+\mathcal{E}(\varphi_k u_h)(x)\|)\,dx \\ &\leq C|\Omega_{k+1}\setminus\Omega_k| + C_3\int_{\Omega_{k+1}\setminus\Omega_k}\varphi_k\,\|\mathcal{E}u_h\|\,dx \\ &+ C_3\int_{\Omega_{k+1}\setminus\Omega_k}|u_h|\,\|\nabla\varphi_k\|\,dx.\end{aligned}$$

Following (4.2.6) and (4.2.7),

$$\begin{aligned}&\lim_{h\to+\infty}\int_\Omega f(A+\mathcal{E}u_h(x))dx - f(A)|\Omega_{k+1}| \\ \geq\ & -C|\Omega_{k+1}\setminus\Omega_k| - C_3\int_\Omega(\varphi_k-\varphi_{k-1})d\mu(x).\end{aligned}$$

Now summing for $k=2,\cdots,i$ and dividing by $i-1$ we obtain

$$\begin{aligned}&\lim_{h\to+\infty}\int_\Omega f(A+\mathcal{E}u_h(x))dx - f(A)\frac{1}{i-1}\sum_{k=2}^{i}|\Omega_{k+1}| \\ &\geq -\frac{C}{i-1}\{|\Omega_{i+1}\setminus\Omega_2| + \int_\Omega(\varphi_i-\varphi_1)d\mu(x)\} \\ &\geq -\frac{C}{i-1}(|\Omega|+\mu(\Omega)).\end{aligned}$$

Then passing to the limit as $i\to+\infty$ we obtain the result of Step 1.
Still assuming (4.2.5) with $A=0$ and $u_h\to u$ in $L^1(\Omega,\mathbb{R}^n)$, then there is a positive Radon measure $\mu\in M_b(\Omega)$ such that:

$$f(\mathcal{E}u_h)\rightharpoonup\mu \quad \text{in} \quad M_b(\Omega) \text{ weakly }^* \text{ as} \quad h\to+\infty.$$

By the Lebesgue-Nikodym decomposition we have:

$$\mu = \mu_a \mathcal{L}^n \llcorner \Omega + \mu_s,$$

where $\mu_s \perp \mathcal{L}^n$ and μ_a is such that:

$$\mu_a(x_0) = \lim_{\rho \to 0^+} \frac{\mu(B(x_0, \rho))}{\mathcal{L}^n(B(x_0, \rho))} < +\infty \tag{4.2.8}$$

for $\mathcal{L}^n$-almost every $x_0 \in \Omega$.

Step 2: If we suppose that

$$\mu_a(x_0) \geq f(\mathcal{E}u(x_0)) \quad \text{for} \quad \mathcal{L}^n - \text{a.e. } x_0 \in \Omega, \tag{4.2.9}$$

then the conclusion of the theorem follows. In fact, let $\{\varphi_k\}_k$ be an increasing sequence of smooth cut-off functions with $0 \leq \varphi_k \leq 1$ and $\sup_k \varphi_k = 1$ in Ω. Then

$$\begin{aligned}
\lim_{h \to +\infty} \int_\Omega f(\mathcal{E}u_h(x))dx &\geq \lim_{h \to +\infty} \int_\Omega \varphi_k(x) f(\mathcal{E}u_h(x))dx \\
&= \int_\Omega \varphi_k(x) d\mu(x) \\
&\geq \int_\Omega \varphi_k(x) d\mu_a(x) \\
&\geq \int_\Omega \varphi_k(x) f(\mathcal{E}u(x))dx.
\end{aligned}$$

Letting $k \to +\infty$, the result follows by the monotone convergence theorem.

Step 3. In this step we prove (4.2.9) using the "blow-up" technique by Fonseca-Müller [122, 123].

Let $B = B(0,1)$ be the open unit ball centered at the origin 0, $B' \subset\subset B$ be a subdomain. Let x_0 be an approximate differentiability point of u such that (4.2.8) holds true and also

$$\mathcal{E}u(x_0) = (\nabla u(x_0) + \nabla u(x_0)^t)/2.$$

Let us now consider the functions u_0, $w_0 : B \to \mathbf{R}^n$ defined by

$$u_0(x) = u(x_0) + \nabla u(x_0)x;$$

$$w_0(x) = \nabla u(x_0)x$$

and

$$\varphi \in C_0(B),\ 0 \leq \varphi \leq 1 \text{ with } \varphi = 1 \text{ in } B'.$$

Then we have

$$\begin{aligned}
\mu_a(x_0) &= \lim_{\rho\to 0^+} \frac{\mu(B(x_0,\rho))}{|B(x_0,\rho)|} \\
&\geq \limsup_{\rho\to 0^+} \frac{1}{|B(x_0,\rho)|} \int_{B(x_0,\rho)} \varphi(\frac{x-x_0}{\rho}) d\mu(x) \\
&= \limsup_{\rho\to 0^+} \lim_{h\to+\infty} \frac{1}{|B(x_0,\rho)|} \int_{B(x_0,\rho)} \varphi(\frac{x-x_0}{\rho}) f(\mathcal{E}u_h(x))dx \\
&= \limsup_{\rho\to 0^+} \lim_{h\to+\infty} \frac{1}{|B|} \int_B \varphi(x) f(\mathcal{E}u_h(x_0+\rho x))dx \\
&\geq \limsup_{\rho\to 0^+} \limsup_{h\to+\infty} \frac{1}{|B|} \int_{B'} f(\mathcal{E}w_{h,\rho}(x))dx,
\end{aligned}$$

where $w_{h,\rho} : B \to \mathbf{R}^n$ are defined by

$$w_{h,\rho}(x) = \frac{1}{\rho}(u_h(x_0+\rho x) - u(x_0)) = \frac{1}{\rho}(u_h(x_0+\rho x) - u_0(\rho x)) + w_0(x).$$

Following (3.2.4) in Theorem 3.2.3 we have

$$\lim_{\rho\to 0^+} \lim_{h\to+\infty} \|w_{h,\rho} - w_0\|_{L^1(B,\mathbf{R}^n)} = 0.$$

Then by a diagonalization argument we obtain a sequence

$$w_k = w_{h_k,\rho_k} \quad \text{with} \quad \rho_k \to 0^+ \quad \text{and} \quad h_k \to +\infty \quad \text{as} \quad k \to +\infty$$

such that $w_k \to w_0$ in $L^1(B, \mathbf{R}^n)$ and

$$\begin{aligned}
\mu_a(x_0) &\geq \limsup_{k\to+\infty} \frac{1}{|B|} \int_{B'} f(\mathcal{E}w_k(x))dx \\
&\geq \frac{1}{|B|} \liminf_{k\to+\infty} \int_{B'} f(\mathcal{E}w_k(x))dx \\
&= \frac{1}{|B|} \liminf_{k\to+\infty} \int_{B'} f(\mathcal{E}u(x_0) + \mathcal{E}(w_k - w_0))dx \\
&\geq \frac{|B'|}{|B|} f(\mathcal{E}u(x_0))
\end{aligned}$$

where the last inequality follows from Step 1. Then

$$\mu_a(x_0) \geq \frac{|B'|}{|B|} f(\mathcal{E}u(x_0)) \qquad \mathcal{L}^n - \text{a.e. } x_0 \in \Omega.$$

By the arbitrariness of B' we obtain (4.2.9) and this achieves the proof of Theorem 4.2.3. $\square$

4.2.2 An example of symmetric quasi-convex function

Let us introduce for a given function $f : M_s^{n\times n} \to [0, +\infty[$, the notion of *symmetric quasi-convex envelope* defined as:

$$SQf(A) = \sup\{g \leq f;\ g \text{ is symmetric quasi-convex}\}. \tag{4.2.10}$$

Following the explicit characterization of the *quasi-convex envelope* by Dacorogna [81], one can show that:

$$SQf(A) = \inf\left\{ \fint_D f(A + \mathcal{E}\varphi(x))dx;\ \varphi \in W_0^{1,\infty}(D, \mathbf{R}^n)\right\}, \tag{4.2.11}$$

for any $A \in M_s^{n\times n}$ and for any bounded open subset D of $\mathbf{R}^n$. One can also verify that the function defined in (4.2.11) is independent of the open subset D.
As we have said in the introduction, we are looking for a non convex function which is *symmetric quasi-convex.*

Theorem 4.2.5 *Let $p \geq 1$, $K = \{I, -I\}$ and $f : M_s^{2\times 2} \to [0, +\infty[$ defined as:*

$$f(A) = d(A, K)^p = \min(\|A - I\|^p, \|A + I\|^p);$$

where I is the identity matrix in $M^{2\times 2}$. Then, $SQf(0) > 0$. In particular, SQf is not a convex function.

Proof: Suppose that $SQf(0) = 0$, then by (4.2.11) there exists for some bounded open subset Ω of $\mathbf{R}^2$ with regular boundary a sequence $\{\varphi_j\} \subset \mathcal{C}_0^\infty(\Omega, \mathbf{R}^2)$ such that

$$\lim_{j\to\infty} \int_\Omega f(\mathcal{E}\varphi_j(x))dx = 0. \tag{4.2.12}$$

Since

$$d(A,K)^p \geq C_p(|A_{11}-A_{22}|^p + |A_{12}|^p),$$

with $C_p = 2^{-\frac{p+2}{2}}$, by (4.2.12) we get

$$\lim_{j\to\infty}\int_\Omega (|\partial_1\varphi_{j1}-\partial_2\varphi_{j2}|^p + |\partial_1\varphi_{j2}+\partial_2\varphi_{j1}|^p)dx = 0. \tag{4.2.13}$$

On the other hand,

$$\begin{cases} \Delta\varphi_{j1} = \partial_1(\partial_1\varphi_{j1}-\partial_2\varphi_{j2}) + \partial_2(\partial_2\varphi_{j1}+\partial_1\varphi_{j2}) \\ \Delta\varphi_{j2} = \partial_1(\partial_2\varphi_{j1}-\partial_1\varphi_{j2}) + \partial_2(\partial_1\varphi_{j1}+\partial_2\varphi_{j2}). \end{cases} \tag{4.2.14}$$

Since the second members of (4.2.14) go to zero in $W^{-1,p}(\Omega,\mathbf{R}^2)$ strongly, using the L^p-strong estimates of the Laplace operator for $p>1$, we deduce that $\{\|D\varphi_j\|\}$ converges strongly to zero in $L^p(\Omega)$ for $p>1$. Then also $\{\|\mathcal{E}\varphi_j\|\}$ strongly converges to zero in $L^p(\Omega)$ and this contradicts (4.2.12).

In the case $p=1$ we have from the L^1-weak estimate of the Laplacian

$$\lim_{j\to\infty} |\{x\in\Omega,\ \|\nabla\varphi_j(x)\|\geq\alpha\}| = 0, \tag{4.2.15}$$

for any $\alpha>0$. Then

$$\lim_{j\to\infty} |\{x\in\Omega,\ \|\mathcal{E}\varphi_j(x)\|\geq\alpha\}| = 0. \tag{4.2.16}$$

On the other hand (4.2.12) implies

$$\lim_{j\to\infty}\int_\Omega \|\mathcal{E}\varphi_j(x) - A_j(x)\|\,dx = 0 \quad \text{for some } A_j:\Omega\to K.$$

Since

$$\int_{\{\|\mathcal{E}\varphi_j(x)\|\geq\alpha\}} \|\mathcal{E}\varphi_j(x)\|\,dx \leq \int_\Omega \|\mathcal{E}\varphi_j(x)-A_j\|\,dx + \|I\|\,|\{\|\mathcal{E}\varphi_j(x)\|\geq\alpha\}|,$$

we have

$$\lim_{j\to\infty}\int_{\{\|\mathcal{E}\varphi_j(x)\|\geq\alpha\}} \|\mathcal{E}\varphi_j(x)\|\,dx = 0. \tag{4.2.17}$$

But from (4.2.16) we have up to a subsequence $\|\mathcal{E}\varphi_j(x)\| \to 0$ for $\mathcal{L}^n$ – a.e. $x \in \Omega$. Then

$$\lim_{j\to\infty} \int_{\{\|\mathcal{E}\varphi_j(x)\|\le\alpha\}} \|\mathcal{E}\varphi_j(x)\|\, dx = 0. \tag{4.2.18}$$

From (4.2.17) and (4.2.18) we deduce that

$$\lim_{j\to\infty} \int_{\Omega} \|\mathcal{E}\varphi_j(x)\|\, dx = 0,$$

and this contradicts (4.2.12). So we conclude that $SQf(0) > 0$.

If the function SQf were convex, then we would have

$$\begin{aligned} SQf(0) &\le \frac{1}{2}(SQf(I) + SQf(-I)) \\ &\le \frac{1}{2}(f(I) + f(-I)) = 0 \end{aligned}$$

and this contradicts the result above. So, the function SQf is an example of a non convex function which is symmetric quasi-convex. □

4.3 The lower semicontinuity of some functionals in $SBD(\Omega)$ with a superlinear bulk part

As a direct consequence of the compactness criterion in $SBD(\Omega)$ of Chapter 2 (Theorem 2.7.1), Bellettini-Coscia-Dal Maso [46, 47] proved the lower semicontinuity of functionals of the form

$$\mathcal{F}(u) := \int_\Omega f(x, \mathcal{E}u)dx + \int_{J_u} g(x, (u^+ - u^-) \odot \nu_u)d\mathcal{H}^{n-1} + \mathcal{H}^{n-1}(J_u), \tag{4.3.1}$$

where $f : \Omega \times M_s^{n\times n} \to [0, +\infty]$ and $g : \Omega \times M_s^{n\times n} \to [0, +\infty]$ satisfy the following properties:

(i) $f(x, \cdot)$ is convex and lower semicontinuous on $M_s^{n\times n}$ for $\mathcal{L}^n$-almost every $x \in \Omega$,

(ii) $f(\cdot, A)$ is measurable for every $A \in M_s^{n\times n}$,

(iii) $f(x,A) \geq \phi(|A|)$ for a non-decreasing function $\phi : [0,+\infty[\to [0,+\infty[$ verifying

$$\lim_{t\to+\infty} \frac{\phi(t)}{t} = +\infty, \tag{4.3.2}$$

(iv) g is lower semicontinuous on $\bar{\Omega} \times M_s^{n\times n}$,

(v) $g(x,\cdot)$ is convex and positively 1-homogeneous on $M_s^{n\times n}$ for every $x \in \bar{\Omega}$.

We have the following lower semicontinuity theorem.

Theorem 4.3.1 *Let*

$$\mathcal{F}_0(u) := \begin{cases} \mathcal{F}(u) & \textit{if } u \in SBD(\Omega) \\ +\infty & \textit{if } u \in L^1_{loc}(\Omega, \mathbb{R}^n) \setminus SBD(\Omega) \end{cases} \tag{4.3.3}$$

and let $\{u_h\}$ be a sequence bounded in $L^\infty(\Omega,\mathbb{R}^n)$ and converging in $L^1_{loc}(\Omega,\mathbb{R}^n)$ to u. Then

$$\mathcal{F}_0(u) \leq \liminf_{h\to+\infty} \mathcal{F}_0(u_h). \tag{4.3.4}$$

Proof. We assume that $\liminf_{h\to+\infty} \mathcal{F}_0(u_h) = \lim_{h\to+\infty} \mathcal{F}_0(u_h) < +\infty$. Then $\{u_h\} \subset SBD(\Omega)$ and

$$\sup_h \left\{ \|u_h\|_1 + \int_\Omega \phi(|\mathcal{E}u_h|)dx + \mathcal{H}^{n-1}(J_{u_h}) \right\} < +\infty.$$

Since $|E^j u_h|(\Omega) \leq 2\,\|u_h\|_\infty \mathcal{H}^{n-1}(J_{u_h})$ and $\{u_h\}$ is bounded in $L^\infty(\Omega,\mathbb{R}^n)$, we have

$$\sup_h \left\{ \|u_h\|_1 + |E^j u_h|(\Omega) + \int_\Omega \phi(|\mathcal{E}u_h|)dx + \mathcal{H}^{n-1}(J_{u_h}) \right\} < +\infty.$$

By the compactness theorem 2.7.1 of Chapter II we have $u \in SBD(\Omega)$ and a subsequence, still denoted by $\{u_h\}$ such that

$$\mathcal{E}u_h \rightharpoonup \mathcal{E}u \text{ weakly in } L^1(\Omega, M_s^{n\times n}), \tag{4.3.5}$$

$$E^j u_h \rightharpoonup E^j u \text{ weakly}^\star \text{ in } \mathcal{M}_b(\Omega, M_s^{n\times n}), \tag{4.3.6}$$

$$\mathcal{H}^{n-1}(J_u) \leq \liminf_{h\to+\infty} \mathcal{H}^{n-1}(J_{u_h}). \tag{4.3.7}$$

By Fatou's lemma, the function

$$M \to \int_\Omega f(x, M(x))dx$$

is lower semicontinuous with respect to the strong convergence in $L^1(\Omega, M_s^{n\times n})$; by convexity, it is also weakly lower semicontinuous. So that from (4.3.5) we get

$$\int_\Omega f(x, \mathcal{E}u)dx \leq \liminf_{h\to+\infty} \int_\Omega f(x, \mathcal{E}u_h)dx. \tag{4.3.8}$$

From (4.3.6) and the Reshetnyak's theorem (see for instance Reshetnyak [173], Ambrosio-Fusco-Pallara [26]) it follows that

$$\int_{J_u} g(x, (u^+ - u^-) \odot \nu_u)d\mathcal{H}^{n-1} \leq \liminf_{h\to+\infty} \int_{J_{u_h}} g(x, (u_h^+ - u_h^-) \odot \nu_{u_h})d\mathcal{H}^{n-1}. \tag{4.3.9}$$

Then (4.3.4) follows from (4.3.7), (4.3.8) and (4.3.9). □

Remark 4.3.2 If we remove the L^∞-boundedness assumption on the sequence $\{u_h\}$ the theorem still holds under the following additional coerciveness assumption on g: there exists a positive constant C such that

$$g(x, A) \geq C|A| \quad \text{for every } (x, A) \in \bar{\Omega} \times M_s^{n\times n}.$$

The functional $\mathcal{F}$ defined in (4.3.1) is related to some variational models arising in fracture mechanics. For example, when one deals with a linearly elastic body which undergoes fractures, the term $\int_\Omega f(x, \mathcal{E}u)dx$ called bulk energy of the body in the "solid" region (or the region where linear elasticity is supposed to hold) is a quadratic form with respect to $\mathcal{E}u$ (e.g.

$$f(x, A) := \frac{\lambda}{2}(\mathrm{Tr}(A))^2 + \mu|A|^2,$$

where $\lambda, \mu > 0$ are the Lamé constants) while the term

$$\int_{J_u} g(x, (u^+ - u^-) \odot \nu_u)d\mathcal{H}^{n-1} + \mathcal{H}^{n-1}(J_u) \tag{4.3.10}$$

represents the energy released by the fracture J_u. The first term in (4.3.10) is the contribution due to the displacement $u^+ - u^-$ while the presence of the second term called the "area" of the fracture follows by Griffith's model in crack propagation. In generale, we have to consider energies of the type

$$\mathcal{F}(u) + \int_\Omega (h, u)dx \quad \text{or} \quad \mathcal{F}(u) + \int_\Omega \varphi(x, u)dx$$

where the additional terms represent the work done by some volume forces.

4.4 Existence results for some minimization problems

In this paragraph, we present two minimization problems in the space $SBD(\Omega)$ proposed by Bellettini, Coscia and Dal Maso in [46, 47].

4.4.1 An obstacle problem in $SBD(\Omega)$

We consider here an obstacle problem with Neumann boundary conditions for the functional

$$\mathcal{F}(u) + \int_\Omega (h, u)dx,$$

where $h : \Omega \to \mathbb{R}^n$ represents the volume force density.

Theorem 4.4.1 *Let f, g be as in Theorem 4.3.1 with the additional coerciveness assumption*

$$g(x, A) \geq C|A| \text{ for every } x \in \bar{\Omega} \text{ and for every } A \in M_s^{n\times n}. \tag{4.4.1}$$

Let $h \in L^1(\Omega, \mathbb{R}^n)$ and let $\{H(x)\}_{x\in\Omega}$ be a uniformly bounded family of closed subsets of $\mathbb{R}^n$. Then the constrained minimum problem

$$(P_1) \qquad \min_{\substack{u \in SBD(\Omega) \\ u(x)\in H(x) \text{ a.e. } x\in\Omega}} \left\{ \mathcal{F}(u) + \int_\Omega (h, u)dx \right\}$$

has solution.

Proof. We assume that the infinimum is finite. Let $\{u_h\}$ be a minimizing sequence for (P_1). Since $\{H(x)\}_{x\in\Omega}$ is uniformly bounded, the sequence $\{u_h\}$ is bounded in $L^\infty(\Omega, \mathbb{R}^n)$. So, by the coerciveness assumption (4.4.1), we have

$$\sup_h \left\{ \|u_h\|_1 + |E^j u_h|(\Omega) + \int_\Omega \phi(|\mathcal{E}u_h|)dx + \mathcal{H}^{n-1}(J_{u_h}) \right\} < +\infty.$$

Applying the compactness theorem 2.7.1, we get a subsequence still denoted by $\{u_h\}$ and a function $u \in SBD(\Omega)$ such that $u_h \to u$ in $L^1_{loc}(\Omega, \mathbb{R}^n)$ and also $\mathcal{L}^n$ a.e. in Ω. From the fact that $H(x)$ is closed for $\mathcal{L}^n$ a.e. $x \in \Omega$ and by the lower semicontinuity theorem 4.3.1 we get that u is a solution of (P_1). □

Remark 4.4.2 (i) In the particular case where $H(x) := \{y \in \mathbb{R}^n\colon\ x + y \in K\}$ for a fixed compact set $K \subset \mathbb{R}^n$ containing Ω, (P_1) represents the problem of looking for the equilibrium configuration of a cracked elastic body contained inside a fixed box K and subject to a volume force with density h. In this case, the constraint

$$u(x) \in H(x) \qquad \mathcal{L}^n\text{-a.e. } x \in \Omega$$

means that up to negligible pieces, all its deformed configurations remain contained inside the box K.

(ii) Let us observe that whenever $g \equiv 0$ and $h \neq 0$, the problem (P_1) would not have solutions in general if we removed the boundedness assumption on $\{H(x)\}$. In fact, if we assume that $f(x,0) \in L^1(\Omega)$ and $h \in L^\infty(\Omega, \mathbb{R}^n)$ with $h \neq 0$, then there exist a closed ball B and a vector $\xi \in \mathbf{S}^{n-1}$ such that $\int_B (h(x), \xi)dx > 0$. Setting $u_k(x) := -k\xi 1_B$, the value of the functional in (P_1) on u_k is

$$\int_\Omega f(x,0)dx + \mathcal{H}^{n-1}(\partial B) - k\int_B (h(x), \xi)dx$$

which tends to $-\infty$ as $k \to +\infty$. So that when $H(x) = \mathbb{R}^n$ we get that the infinimum is equal to $-\infty$ for $h \neq 0$.

4.4.2 A Dirichlet boundary value problem in $SBD(\Omega)$

Let us assume that Ω is a bounded open subset of $\mathbb{R}^n$ with Lipschitz continuous boundary, whose outer unit normal is denoted by ν. Given $\psi \in L^1(\partial\Omega, \mathbb{R}^n)$, we introduce for any $u \in BD(\Omega)$ the set

$$\Gamma_u^\psi := \{x \in \partial\Omega\colon\ \gamma(u)(x) \neq \psi(x)\}$$

where $\gamma(u)$ denotes the trace of u on $\partial\Omega$. We consider the following weak formulation of a Dirichlet problem for the functional $\mathcal{F}(u)$ with boundary datum ψ:

$$(P_2) \qquad \min_{u\in SBD(\Omega)} \left\{\mathcal{F}(u) + \int_{\Gamma_u^\psi} g(x, (\psi - \gamma(u)) \odot \nu) d\mathcal{H}^{n-1} + \mathcal{H}^{n-1}(\Gamma_u^\psi)\right\}.$$

We have the following theorem

Theorem 4.4.3 *Let Ω be a bounded open set in $\mathbb{R}^n$ with Lipschitz boundary and let $\psi \in L^1(\partial\Omega, \mathbb{R}^n)$. Let f, g be as in Theorem 4.3.1 with g satisfying the additional condition (4.4.1). Then the minimum problem (P_2) has a solution.*

Proof. Let us fix a bounded open set Ω' such that $\bar{\Omega} \subset \Omega'$. We follow a classical procedure in which the problem (P_2) is equivalent, up to an additive constant, to another problem defined in all Ω' with the Dirichlet condition $\gamma(u) = \psi$ replaced by a new condition holding almost everywhere in $\Omega' \setminus \Omega$. Precisely, by the Gagliardo's trace theorem, (see, e.g., Giusti [132, Theorem 2.16]) there exists a function w in $W^{1,1}(\Omega' \setminus \bar{\Omega}, \mathbb{R}^n)$ with trace ψ on $\partial\Omega$. Since $\mathcal{E}w \in L^1(\Omega' \setminus \bar{\Omega}, M_s^{n\times n})$, there exists a non-decreasing function ϕ satisfying (4.3.2) such that

$$(4.4.2) \qquad \int_{\Omega'\setminus\bar{\Omega}} \phi(|\mathcal{E}w|)dx < +\infty$$

(see, e.g., Dellacherie-Meyers [105, Chapter II, Theorem 22]). We can always assume without loss of generality that ϕ is convex and satisfies also $f(x, A) \geq \phi(|A|)$ for any $x \in \Omega$ and for any $A \in M_s^{n\times n}$. We extend the function f to $\Omega' \times M_s^{n\times n}$ by setting $f(x, A) = \phi(|A|)$. In the same way we also extend g so that the condition

(4.4.1) is satisfied on $\bar{\Omega}' \times M_s^{n\times n}$. For any $u \in BD(\Omega)$ we consider the function $v \in BD(\Omega')$ defined by

$$v(x) := \begin{cases} u(x) & \text{in } \Omega \\ w(x) & \text{in } \Omega' \setminus \Omega. \end{cases}$$

For such a function v we have

$$\int_{\Omega'} f(x, \mathcal{E}v)dx = \int_{\Omega} f(x, \mathcal{E}u)dx + \int_{\Omega'\setminus\Omega} \phi(|\mathcal{E}w|)dx,$$

$$\begin{aligned} \int_{J_v} g(x, (v^+ - v^-) \odot \nu_v) d\mathcal{H}^{n-1} &= \int_{J_u} g(x, (u^+ - u^-) \odot \nu_u) d\mathcal{H}^{n-1} \\ &+ \int_{\Gamma_u^\psi} g(x, (\psi - \gamma(u)) \odot \nu) d\mathcal{H}^{n-1}, \end{aligned}$$

$$\mathcal{H}^{n-1}(J_v) = \mathcal{H}^{n-1}(J_u) + \mathcal{H}^{n-1}(\Gamma_u^\psi).$$

We obtain that u is a a solution of (P$_2$) if and only if v is a solution of the problem

$$(P_3) \qquad \min\left\{ \int_{\Omega'} f(x, \mathcal{E}v)dx + \int_{J_v} g(x, (v^+ - v^-) \odot \nu_v) d\mathcal{H}^{n-1} + \mathcal{H}^{n-1}(J_v) \right\},$$

where the minimum is taken among all functions $v \in SBD(\Omega')$ such that $v = w$ almost everywhere in $\Omega' \setminus \Omega$. So, we only need to prove that the problem (P$_3$) has a solution.

We assume that the infinimum is finite and we consider a minimizing sequence $\{v_h\}$ for (P$_3$). By the growth conditions on f, g, and ϕ we check that the sequence $\{|Ev_h|(\Omega')\}$ is bounded. If we apply the proposition 1.7.6 of Chapter I to the function $v_h - w$ we obtain that the sequence $\{v_h\}$ is bounded in $BD(\Omega')$. By the compactness theorem 2.7.1, we obtain a subsequence still denoted by $\{v_h\}$, and a function $v \in SBD(\Omega')$ such that $v_h \to v$ in $L^1(\Omega', \mathbb{R}^n)$ and $v = w$ almost everywhere in $\Omega' \setminus \Omega$. Since $\{v_h\}$ is a minimizing sequence, by Theorem 4.3.1 and Remark 4.3.2 we get that v is a solution of (P$_3$). $\square$

4.5 A lower semicontinuity theorem for some linear growth functionals defined in $BD(\Omega)$

We begin this section by introducing some useful definitions and notations. We say that a function $\vartheta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is *subadditive* if for all $u, v, w \in \mathbb{R}$ we have

$$\vartheta(u, v) \leq \vartheta(u, w) + \vartheta(w, v).$$

If $f : \mathbb{R}^k \to [0, +\infty]$ is convex, we define the *recession* function $f^\infty : \mathbb{R}^k \to [0, +\infty]$, by

$$f^\infty(p) = \lim_{t \to +\infty} \frac{f(p_0 + tp) - f(p_0)}{t} \qquad \forall p \in \mathbb{R}^k$$

where $p_0 \in \mathbb{R}^k$ is any vector such that $f(p_0) < +\infty$. We remark that f^∞ is a Borel function, which is convex and is positively homogeneous of degree 1. A *modulus of continuity* will be any continuous increasing function $w : [0, +\infty[\to \mathbb{R}$ such that $w(0) = 0$.

For functionals with linear growth defined in $BV(\Omega)$ and having the form

$$F(u) := \int_\Omega f(x, u, \nabla u)dx + \int_{\Omega \setminus S_u} f^\infty(x, \tilde{u}, D^s u) + \int_{S_u \cap \Omega} \vartheta_f(x, u^+, u^-, \nu_u) d\mathcal{H}^{n-1},$$

where f is a lower semicontinuous Caratheodory integrand, convex in the last variable, f^∞ its recession function (with respect to the last variable) and

$$\vartheta_f(x, u, v, \nu) = \Big| \int_u^v f^\infty(x, z, \nu) dz \Big|,$$

lower semicontinuity results have been obtained by Goffman-Serrin [134] for f depending only on the last variable, Giaquinta-Modica-Součeck [130] for $f = f(x, p)$ and Dal Maso [82] for $f = f(x, u, p)$. Dal Maso's result was generalized to BV vector-valued functions by Aviles-Giga [40], Ambrosio-Mortola-Tortorelli [27], Ambrosio-Pallara [28] and Fonseca-Rybka [124] with ϑ_f having a more complex formula involving the solutions of a variational problem. These last results have been refined later for example by Ambrosio-Dal Maso [23], Fonseca-Müller [123] and Braides-Chiado Piat [64] to the case when f is quasiconvex in the last variable.

On the other hand, Bouchitté and Buttazzo have exhibited in [54, 55, 56] a class of non convex functionals defined on measures and for which lower semicontinuity and relaxation results have been obtained. From those results, one can deduce the lower semicontinuity of functionals defined in $BV(a,b)$ of the form

$$F_0(u) = \int_{(a,b)} f(u')dt + \int_{(a,b)\setminus S_u} f^\infty(D^s u) + \sum_{t\in(a,b)\cap S_u} \vartheta(u(t+) - u(t-)),$$

for some subadditive $\vartheta \leq \vartheta_f$.

This last result has been generalized by Bouchitté, Braides and Buttazzo in [53] for radial functionals, by Braides and Coscia in [67] for non-radial functionals or for vector-valued functions under some restrictive hypotheses and also by Braides and De Cicco in [69] for functionals of the type

$$\mathcal{F}(u) := \int_\Omega f(x,u,|\nabla u|)dx + \int_{\Omega\setminus S_u} f^\infty(x,\tilde{u},|D^s u|) + \int_{S_u\cap\Omega} \vartheta(x,u^+,u^-,\nu_u)d\mathcal{H}^{n-1}$$

under some compatibility conditions between f and ϑ that generalize the conditions given in [54, 55], [53].

From the classical properties of functions with bounded deformation and the Reshetnyak lower semicontinuity theorem (see e.g., Ambrosio-Fusco-Pallara [26, Theorem 2.38]) convex functionals with linear growth of the type

$$\int_\Omega (\varphi(\mathcal{E}^D u) + \frac{k}{2}(\mathrm{div} u)^2)dx + \int_\Omega \varphi^\infty\Big(\frac{E_s^D u}{|E_s^D u|}\Big)|E_s^D u|, \tag{4.5.1}$$

where Ω is a bounded open subset of $\mathbb{R}^n$, $A^D = A - \frac{1}{n}\mathrm{Tr}(A)I$ is called deviator of A for any A matrix $n\times n$, have been studied in order to model perfectly plastic bodies under the Von Mises' criterion (see e.g., Anzellotti-Giaquinta [36, 37], Suquet [186, 187], Temam [191]). Precisely, setting

$$\mathcal{P}(\Omega) := \{u \in BD(\Omega)\colon \mathrm{div} u \in L^2(\Omega)\},$$

it has been proved that the functional (4.5.1) is the natural $L^1(\Omega, \mathbb{R}^n)$-lower semicontinuous extension to $\mathcal{P}(\Omega)$ of the strain energy

$$\int_\Omega (\varphi(E^D u) + \frac{k}{2}(\mathrm{div} u)^2) dx$$

of the body Ω.

Braides, Defranceschi and Vitali have proposed in [72] a different approach to problems involving the functional in (4.5.1) by introducing a somehow simpler model. They consider a subspace $\mathcal{S}(\Omega)$ of $SBD(\Omega) \cap \mathcal{P}(\Omega)$ whose elements are allowed to have "singularities" in the form of jump discontinuities along surfaces (the possible slippage surfaces of the material). They proposed in that subspace the following functional

$$(4.5.2) \quad F(u) := \int_\Omega (\varphi(E^D u) + \frac{k}{2}(\mathrm{div} u)^2) dx + c \int_{J_u} |u^+ - u^-| \qquad u \in \mathcal{S}(\Omega)$$

as a more realistic model which, through the presence of the last term, takes into account the possible sliding of the material. By relaxing the functional in (4.5.2) with respect to the $L^1(\Omega, \mathbb{R}^n)$ topology, they obtain

$$\overline{F}(u) = \int_\Omega (\phi(\mathcal{E}^D u) + \frac{k}{2}(\mathrm{div} u)^2) dx + \int_\Omega \phi^\infty\Big(\frac{E_s^D u}{|E_s^D u|}\Big)|E_s^D u| \qquad u \in \mathcal{P}(\Omega)$$

where ϕ is some convex function with linear growth at infinity. They also proved that, whenever the Tresca yielding criterion is adopted, the relaxation procedure described above recovers exactly the functional in (4.5.1).

In the superlinear case, we have seen in Theorem 4.3.1 that the lower semicontinuity is obtained from the compactness theorem 2.7.1 in $SBD(\Omega)$.

In this section we study the lower semicontinuity with respect to the $L^1(\Omega, \mathbb{R}^n)$-norm convergence of some class of linear growth functionals defined in all the space $BD(\Omega)$. Precisely, we study the lower semicontinuity of functionals of the type

$$F(u) := \int_\Omega f(x, \|\mathcal{E}u\|_*) dx + \int_{\Omega \setminus J_u} f^\infty(x, \|E^c u\|_*) + \int_{J_u \cap \Omega} \varphi(x, u^-, u^+, \nu_u) d\mathcal{H}^{n-1}$$

where for any $A \in M_s^{n\times n}$, $\|A\|_* := \sup_{\xi\in \mathbf{S}^{n-1}} |(A\xi,\xi)|$ is the maximum eigenvalue of A. We obtain the following lower semicontinuity theorem.

Theorem 4.5.1 *Let $f : \Omega \times \mathbb{R} \to [0,+\infty[$ and $\vartheta : \Omega \times \mathbb{R} \times \mathbb{R} \to [0,+\infty[$ be two lower semicontinuous functions satisfying the following conditions:*

(i) *for every $x \in \Omega$ the map $\vartheta(x,\cdot,\cdot)$ is continuous, subadditive and*

$$\vartheta(x,u,v) = \vartheta(x,v,u) \quad \textit{for every } u,v \in \mathbb{R},$$

(ii) *for every $x \in \Omega$ the map $p \to f(x,p)$ is convex, even and nondecreasing,*

(iii) *for every $T > 0$ there exists a modulus of continuity w_T such that*

$$\vartheta(x,u,v) \le f^\infty(x,|u-v|) \le \vartheta(x,u,v) + w_T(|u-v|)|u-v|$$

for every $x \in \Omega$, $u,v \in \mathbb{R}$ with $-T \le u < v \le T$

(iv) *the function $x \to f(x,0)$ is bounded,*

(v) *there exists C_1, C_2, C_3 positive constants such that*

$$f(x,p) \ge C_1|p| - C_2 \quad \textit{and} \quad \vartheta(x,u,v) \ge C_3|u-v|.$$

Let $\varphi : \Omega \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbf{S}^{n-1} \to [0,+\infty[$ be defined by

$$\varphi(x,u,v,\nu) := \sup_{\xi\in\mathbf{S}^{n-1}} \Big[\vartheta(x,(u,\xi),(v,\xi))|(\nu,\xi)|\Big].$$

Then the functional $F : BD(\Omega) \to [0,+\infty]$ defined by

$$F(u) := \int_\Omega f(x,\|\mathcal{E}u\|_*)dx + \int_{\Omega\setminus J_u} f^\infty(x,\|E^c u\|_*) + \int_{J_u\cap\Omega} \varphi(x,u^-,u^+,\nu_u)d\mathcal{H}^{n-1} \tag{4.5.3}$$

is lower semicontinuous with respect to the $L^1(\Omega,\mathbb{R}^n)$-convergence.

Remark 4.5.2 (i) From the assumption (i) in the theorem we can easily verify that for any $x \in \Omega$, $\nu \in \mathbf{S}^{n-1}$, the map $\varphi(x, \cdot, \cdot, \nu)$ is also subadditive.

(ii) From the assumption (iii) we obtain

$$\varphi(x, u, v, \nu) \leq f^\infty(x, \|(u-v) \odot \nu\|_*) \leq \varphi(x, u, v, \nu) + w_T(|u-v|)|u-v|$$

for any $x \in \Omega$, $u, v \in \mathbb{R}^n$, with $|u|, |v| \leq T$.

(iii) In the scalar case it has been proved in [69] that the assumptions (i)-(iii) are also necessary. In our case we can say that the convexity of $f(x, \cdot)$ is necessary.

(iv) Let us observe that, if we remove the coercivity assumption (v) on f and ϑ, we only obtain the weaker conclusion that the functional F is sequentially weakly* lower semicontinuous in $BD(\Omega)$. In fact, let $\varepsilon > 0$ and consider the functional

$$F_\varepsilon(u) := F(u) + \varepsilon \|Eu\|_\star (\Omega).$$

Then from Theorem 4.5.1, F_ε is sequentially weakly* lower semicontinuous in $BD(\Omega)$. Now, let $\{u_h\} \subset BD(\Omega)$ weakly* converging to $u \in BD(\Omega)$, and let us verify that $F(u) \leq \liminf_{h \to +\infty} F(u_h)$. Since $u_h \stackrel{\star}{\rightharpoonup} u$ in $BD(\Omega)$, then

$$\|Eu_h\|_\star (\Omega) \leq K$$

for some $K > 0$ independent of h. Therefore,

$$\begin{aligned}
\liminf_{h \to +\infty} F(u_h) &\geq \liminf_{h \to +\infty} F_\varepsilon(u_h) - \varepsilon \limsup_{h \to +\infty} \|Eu_h\|_\star (\Omega) \\
&\geq \liminf_{h \to +\infty} F_\varepsilon(u_h) - \varepsilon K \\
&\geq F_\varepsilon(u) - \varepsilon K \geq F(u) - \varepsilon K
\end{aligned}$$

and the conclusion follows by the arbitrariness of ε.

(v) We will prove the theorem following the arguments used in the papers [53], [69] using the "slicing technique" given by the structure theorem 2.5.1 in Chapter II and which will allow to reduce the lower semicontinuity of F to its known one-dimensional version.

Before beginning the proof of Theorem 4.5.1, let us recall its one dimensional version proved by Braides and De Cicco in [69].

Theorem 4.5.3 *Let $f : (a,b)\times\mathbb{R}\times\mathbb{R} \to [0,+\infty[$ and $\vartheta : (a,b)\times\mathbb{R}\times\mathbb{R} \to [0,+\infty[$ be two lower semicontinuous functions satisfying:*

(i) *for every $t \in (a,b)$ the function $\vartheta(t,\cdot,\cdot)$ is subadditive,*

(ii) *for every $t \in (a,b)$ and $u \in \mathbb{R}$ the function $f(t,u,\cdot)$ is convex,*

(iii) *for every $T > 0$ there exists a modulus of continuity w_T such that*

$$\vartheta(t,u,v) \leq \Big|\int_u^v f^\infty(t,\tau,\frac{v-u}{|u-v|})d\tau\Big| \leq \vartheta(t,u,v) + w_T(|u-v|)|u-v|$$

for all $t \in (a,b)$, $u,v \in \mathbb{R}$ with $|u|$, $|v| \leq T$,

(iv) *the function $(t,u) \to f(t,u,0)$ is bounded.*

Then the functional $F : BV(a,b) \to [0,+\infty[$ defined by

$$F(u) := \int_{(a,b)} f(t,u(t),u'(t))dt + \int_{(a,b)\setminus S_u} f^\infty(t,u(t),D^s u) + \sum_{t\in(a,b)\cap S_u} \vartheta(t,u(t+),u(t-))$$

is lower semicontinuous with respect to the $BV(a,b)$ weak* convergence.

Let us introduce the following notations. Given a direction $\xi \in \mathbf{S}^{n-1}$ we define the functional $F_\xi : BD(\Omega)\times\mathcal{B}(\Omega) \to [0,+\infty]$ associated to F by

$$F_\xi(u,B) := \int_B f(x,|(\mathcal{E}u\xi,\xi)|)dx + \int_{B\setminus J_u} f^\infty(x,|(E^c u\xi,\xi)|) + \int_{B\cap J_u} \vartheta(x,(u^-(x),\xi),(u^+(x),\xi))|(\nu_u(x),\xi)|d\mathcal{H}^{n-1} \tag{4.5.4}$$

and for any $y \in \mathbb{R}^n$ we define $\Phi_y^\xi : BV_{\rm loc}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}) \to [0, +\infty]$ by

$$\Phi_y^\xi(v, I) := \int_I f(y + t\xi, |v'(t)|)dt + \int_{I \setminus S_v} f^\infty(y + t\xi, |D^c v|) + \sum_{t \in I \cap S_v} \vartheta(y + t\xi, v(t-), v(t+)).$$

In the following lemma we prove that the functional F of Theorem 4.5.1 can be approximated by functionals F_ξ.

Lemma 4.5.4 *Let $u \in BD(\Omega)$ with $F(u) < +\infty$. Let $\{\xi_h;\ h \in \mathbb{N}\}$ be a dense subset of $\mathbf{S}^{n-1}$. Then we have*

$$F(u) = \sup \left\{ \sum_{i=1}^k F_{\xi_i}(u, A_i) \colon (A_i)_{i=1,\cdots,k} \text{ disjoint open subsets of } \Omega,\ k \in \mathbb{N} \right\}. \tag{4.5.5}$$

Proof. Let us consider the Radon measure on Ω

$$\mu := \mathcal{L}^n \llcorner \Omega + |E^c u| + \mathcal{H}^{n-1} \llcorner (\Omega \cap J_u)$$

and the non-negative μ-measurable function functions

$$\psi_h(x) := \begin{cases} f(x, |(\mathcal{E}u(x)\xi_h, \xi_h)|) & \mathcal{L}^n\text{-a.e. in } \Omega \\ f^\infty(x, |(\nu_u(x), \xi_h)|) & |E^c u|\text{-a.e. in } \Omega \\ \vartheta(x, (u^-(x), \xi_h), (u^+(x), \xi_h))|(\nu_u(x), \xi_h)| & \mathcal{H}^{n-1}\text{-a.e. in } \Omega \cap J_u, \end{cases}$$

$$\psi(x) := \begin{cases} f(x, \|\mathcal{E}u(x)\|_*) & \mathcal{L}^n\text{-a.e. in } \Omega \\ f^\infty(x, 1) & |E^c u|\text{-a.e. in } \Omega \\ \varphi(x, u^-(x), u^+(x), \nu_u(x)) & \mathcal{H}^{n-1}\text{-a.e. in } \Omega \cap J_u. \end{cases}$$

As f is non-decreasing in the last variable, it is easy to check that $\psi = \sup_h \psi_h$ μ-a.e. on Ω. Moreover, since $F(u) < +\infty$, we have

$$\psi \in L^1_\mu(\Omega) \text{ and } \int_\Omega \psi(x) d\mu(x) = F(u).$$

On the other hand by Beppo-Levi's theorem we get

$$\begin{aligned} \int_\Omega \psi(x) d\mu(x) &= \int_\Omega \sup_h \psi_h(x) d\mu(x) \\ &= \int_\Omega \sup_k \sup_{1 \le i \le k} \psi_i(x) d\mu(x) \\ &= \sup_k \int_\Omega \sum_{i=1}^k \int_{B_i} \psi_i(x) d\mu(x) \end{aligned}$$

where setting

$$C_i := \{x \in \Omega\colon\ \psi_i(x) = \sup_{1\le j\le k} \psi_j(x)\},$$

we take $B_i := C_i \setminus \bigcup_{j<i} C_j$. So that $\{B_i\}$ is a Borel partition of Ω. Therefore

$$\begin{aligned}
&\int_\Omega \psi(x)d\mu(x)\\
&= \sup\left\{\sum_{i=1}^k \int_{B_i} \psi_i(x)d\mu(x)\colon\ (B_i)_{i=1,\cdots,k} \text{ Borel partition of } \Omega,\ k\in\mathbb{N}\right\}\\
&= \sup\left\{\sum_{i=1}^k \int_{K_i} \psi_i(x)d\mu(x)\colon\ (K_i)_{i=1,\cdots,k} \text{ disjoint compact subsets of } \Omega,\ k\in\mathbb{N}\right\}\\
&= \sup\left\{\sum_{i=1}^k \int_{A_i} \psi_i(x)d\mu(x)\colon\ (A_i)_{i=1,\cdots,k} \text{ disjoint open subsets of } \Omega,\ k\in\mathbb{N}\right\}
\end{aligned}$$

where the two last equalities follows from the inner regularity of the measure μ. So that (4.5.5) holds. □

From the lemma 4.5.4 we see that the proof of Theorem 4.5.1 follows immediately from the lower semicontinuity of the functionals $F_\xi(\cdot,\Omega)$. This last result is proved in the following proposition.

Proposition 4.5.5 (i) *Let $u \in BD(\Omega)$. Then the following formula holds*

$$(4.5.6)\qquad F_\xi(u,B) = \int_{B^\xi} \Phi_y^\xi(u_y^\xi, B_y^\xi)d\mathcal{H}^{n-1}(y) \qquad \text{for any } B\in\mathcal{B}(\Omega).$$

(ii) *Assume that (i)-(iv) of Theorem 4.5.1 hold. Then for every open subset A of Ω , the functional $F_\xi(\cdot,A)$ is sequentially lower semicontinuous with respect to the weak* topology of $BD(\Omega)$.*

Proof. We can write for any $B\in\mathcal{B}(\Omega)$

$$\int_{B^\xi} \Phi_y^\xi(u_y^\xi, B_y^\xi)d\mathcal{H}^{n-1}(y) = (I) + (II) + (III)$$

where

$$(I) = \int_{B^\xi}\Big(\int_{B_y^\xi} f(y+t\xi, |(u_y^\xi)'(t)|)dt\Big)d\mathcal{H}^{n-1}(y) = \int_B f(x, |(\mathcal{E}u(x)\xi,\xi)|)dx$$

by Fubini's theorem and the point (iii) of the structure theorem 2.5.1;

$$\begin{aligned}(II) &= \int_{B^\xi}\Big(\int_{B^\xi_y\setminus J_{u^\xi_y}} f^\infty(y+t\xi,|D^c u^\xi_y|)\Big)d\mathcal{H}^{n-1}(y)\\ &= \int_{B^\xi}\Big(\int_{(B\setminus J^\xi_u)^\xi_y} f^\infty(y+t\xi,|D^c u^\xi_y|)\Big)d\mathcal{H}^{n-1}(y)\\ &= \int_{B\setminus J_u} f^\infty(x,|(E^c u\xi,\xi)|)\end{aligned}$$

by using the point (v) of Theorem 2.5.1;

$$\begin{aligned}&(III)\\ &= \int_{B^\xi}\Big(\sum_{t\in B^\xi_y\cap J_{u^\xi_y}} \vartheta(y+t\xi,u^\xi_y(t-),u^\xi_y(t+))\Big)d\mathcal{H}^{n-1}(y)\\ &= \int_{B^\xi}\Big(\sum_{t\in B^\xi_y\cap(J^\xi_u)^\xi_y} \vartheta(y+t\xi,(u^-(y+t\xi),\xi),(u^+(y+t\xi),\xi))\Big)d\mathcal{H}^{n-1}(y)\\ &= \int_{B\cap J_u} \vartheta(x,(u^-(x),\xi),(u^+(x),\xi))|(\nu_u(x),\xi)|d\mathcal{H}^{n-1}(x)\end{aligned}$$

by the point (iii) of Theorem 2.5.1 and the co-area formula.
Therefore, collecting the expressions obtained for (I), (II), and (III), we get

$$\begin{aligned}\int_{B^\xi}\Phi^\xi_y(u^\xi_y,B^\xi_y)d\mathcal{H}^{n-1}(y) &= \int_B f(x,|(\mathcal{E}u(x)\xi,\xi)|)dx + \int_{B\setminus J_u} f^\infty(x,|(E^c u\xi,\xi|)\\ &\quad+\int_{B\cap J_u}\vartheta(x,(u^-(x),\xi),(u^+(x),\xi))|(\nu_u(x),\xi)|d\mathcal{H}^{n-1}(x)\\ &\quad= F_\xi(u,B).\end{aligned}$$

(ii) Now let u_h , $u \in BD(\Omega)$ such that $u_h \to u$ in $L^1(\Omega,\mathbb{R}^n)$. Then there exists a subsequence still denoted by $\{u_h\}$ such that $u^\xi_{h,y} \to u^\xi_y$ in $L^1(\Omega^\xi_y)$ for $\mathcal{H}^{n-1}$-a.e. $y \in \Omega^\xi$. If we suppose that $\liminf\limits_{h\to+\infty} F_\xi(u_h,\Omega) < +\infty$, then by Fatou's lemma and by the identity (4.5.6) we get

$$\int_{\Omega^\xi}\liminf_{h\to+\infty}\Phi^\xi_y((u_h)^\xi_y,\Omega^\xi_y)d\mathcal{H}^{n-1}(y) < +\infty.$$

So, for $\mathcal{H}^{n-1}$-a.e. $y \in \Omega^\xi$ let $\{h_k\}$ be the subsequence (depending a priori on y) such that

$$\liminf_{h\to+\infty} \Phi_y^\xi((u_h)_y^\xi, \Omega_y^\xi) = \lim_{k\to+\infty} \Phi_y^\xi((u_{h_k})_y^\xi, \Omega_y^\xi).$$

Therefore, from the growth assumptions (v) on f and ϑ, we get

$$\limsup_{k\to+\infty} |Du_{h_k,y}^\xi|(\Omega_y^\xi) < +\infty.$$

Therefore, for such y we obtain a subsequence $\{h_{k_p}\}$ such that

$$\sup_p |Du_{h_k,y}^\xi|(\Omega_y^\xi) < +\infty.$$

Moreover, since $u_{h_{k_p},y}^\xi \to u_y^\xi$ in $L^1(\Omega_y^\xi)$, then we have $u_{h_{k_p},y}^\xi \rightharpoonup u_y^\xi$ in $BV(\Omega_y^\xi)$ weak*. Therefore, from Theorem 4.5.3 we get for $\mathcal{H}^{n-1}$-a.e. $y \in \Omega^\xi$

$$\begin{aligned}\liminf_{h\to+\infty} \Phi_y^\xi((u_h)_y^\xi, \Omega_y^\xi) &= \lim_{k\to+\infty} \Phi_y^\xi((u_{h_k})_y^\xi, \Omega_y^\xi) \\ &= \liminf_{p\to+\infty} \Phi_y^\xi((u_{h_{k_p}})_y^\xi, \Omega_y^\xi) \\ &\geq \Phi_y^\xi(u_y^\xi, \Omega_y^\xi).\end{aligned}$$

Hence by Fatou's lemma and (4.5.6) we obtain

$$\begin{aligned}\liminf_{h\to+\infty} F_\xi(u_h, \Omega) &= \liminf_{h\to+\infty} \int_{\Omega^\xi} \Phi_y^\xi((u_h)_y^\xi, \Omega_y^\xi) d\mathcal{H}^{n-1}(y) \\ &\geq \int_{\Omega^\xi} \liminf_{h\to+\infty} \Phi_y^\xi((u_h)_y^\xi, \Omega_y^\xi) d\mathcal{H}^{n-1}(y) \\ &= \int_{\Omega^\xi} \lim_{k\to+\infty} \Phi_y^\xi((u_{h_k})_y^\xi, \Omega_y^\xi) d\mathcal{H}^{n-1}(y) \\ &= \int_{\Omega^\xi} \liminf_{p\to+\infty} \Phi_y^\xi((u_{h_{k_p}})_y^\xi, \Omega_y^\xi) d\mathcal{H}^{n-1}(y) \\ &\geq F_\xi(u, \Omega)\end{aligned}$$

and this achieves the proof of the lemma. □

Chapter 5

Homogenization of Periodic Multi-dimensional Structures

5.1 Introduction

In this chapter we deal with an homogenization problem involving some particular class of functions with bounded deformation. The classical theory of homogenization applied to composite materials like fibred materials, porous media, materials with many small holes or fissures allows to observe how these structures behave as homogeneous materials at a macroscopic scale. The composite materials have been widely studied in the last thirty years. We refer the reader to the papers and books Bensoussan-Lions-Papanicolaou [48], De Giorgi-Spagnolo [104], Spagnolo [183], Tartar [190], Sanchez Palencia [178], Attouch [39], Marcellini [148], Dal Maso [83], Braides [60], Zhikov-Oleinik-Kozlov [201], Nguetseng [167, 168] for periodic structures, Oleinik-Zhikov [169], Braides [63, 64], Braides-Defranceschi [70] for almost periodic structures. In both non linear and non smooth cases the Γ-convergence approach (see De Giorgi-Franzoni [102], Dal Maso [83], Braides-Defranceschi [70]) is the most suited while in the other situations, various other techniques have been introduced; for example the G-convergence by Spagnolo [183] and De Giorgi-Spagnolo [104], the H-convergence by Tartar [190], the compensated compactness by Murat [165] and Murat-Tartar [166], and the two-scale convergence by Nguetseng [167, 168] and Allaire [5].

In these last years there is an interest in media whose microscopic behaviour takes into account lower dimensional or multi-dimensional structures (see e.g., Ansini-Braides-Chiado Piat [32], Bouchitté-Buttazzo-Seppecher [57], Ambrosio-Buttazzo-Fonseca [20]). As a model example we consider the case of perfectly rigid bodies connected with elastic springs studied in [32].

In these problems, we deal with energy functionals concentrated on periodic lower dimensional or multi-dimensional structures and one would study their asymptotic behaviour via Γ-convergence. Following the approach of [20], functionals of the form

$$F_\varepsilon(u,\Omega) := \int_\Omega f\Big(\frac{x}{\varepsilon}, \frac{dDu}{d\mu_\varepsilon}\Big) d\mu_\varepsilon \tag{5.1.1}$$

have been considered in [32], and the problem was set in the framework of Sobolev spaces $W^{1,p}_{\mu_\varepsilon}(\Omega, \mathbb{R}^m)$ with respect to the measures μ_ε; this is the space of functions $u \in L^p(\Omega, \mathbb{R}^n)$, whose distributional derivative is a measure absolutely continuous with respect to μ_ε with p-summable density $\dfrac{dDu}{d\mu_\varepsilon}$. For μ_ε defined by

$$\mu_\varepsilon(B) := \varepsilon^n \mu(\frac{1}{\varepsilon}B)$$

where μ is a fixed 1-periodic finite positive Borel measure and for a Borel function $f : \Omega \times \mathbb{R}^{n\times m} \to [0, +\infty[$, 1-periodic in the first variable, an homogenization theorem for F_ε has been proved under a standard growth condition of order p on f and a notion of p-homogenizability introduced for the measure μ (see [32, Theorem 3.5]).

A "smooth" approach for this problem has been precedently proposed by Zhikov in [200] (see also Braides-Chiado Piat [65]), where the functionals F_ε are defined on $\mathcal{C}^\infty(\Omega, \mathbb{R}^m)$ by

$$F_\varepsilon(u) = \int_\Omega f\Big(\frac{x}{\varepsilon}, \nabla u\Big) d\mu_\varepsilon.$$

In this case an homogenization theorem is obtained whenever $p > n$ and the Γ-limit is computed with respect to L^∞-norm.

However, from the mechanical point of view it is more realistic to consider energy functionals which depend explicitly on the strain tensor Eu. So, we consider here functionals of the type

$$F_\varepsilon(u,\Omega) := \int_\Omega f\Big(\frac{x}{\varepsilon}, \frac{dEu}{d\mu_\varepsilon}\Big) d\mu_\varepsilon. \tag{5.1.2}$$

In this case the problem will be set in a particular class of functions with bounded deformation denoted by $LD^p_{\mu_\varepsilon}(\Omega)$; this is the space of functions $u \in L^p(\Omega, \mathbb{R}^n)$, whose deformation tensor Eu is a measure absolutely continuous with respect to μ_ε with p-summable density $\dfrac{dEu}{d\mu_\varepsilon}$. We would like to study the asymptotic behaviour of the functionals in (5.1.2) as $\varepsilon \to 0^+$ under the same assumptions as in [32] with a modified definition of p-homogenizable measure.

This chapter is organized as follows: in the first section, we give some preliminaries and notations on the Γ-convergence and the notion of Sobolev spaces with respect to Borel measures; in the second section, we present the spaces $LD^p_\mu(\Omega)$ and their properties; in the third section, we introduce the notion of measure p-homogenizable and some examples, while the fourth section is devoted to the statement of the homogenization theorem which is proved in section 5.

5.2 Preliminaries and notations

Let us recall that Ω is a bounded open subset of $\mathbb{R}^n$; we will use standard notations $L^p(\Omega, \mathbb{R}^m)$ and $W^{1,p}(\Omega, \mathbb{R}^m)$ respectively for Lebesgue and Sobolev spaces; p' and $p^\star$ denoting respectively the conjugate and the Sobolev exponent of $p \geq 1$. We denote by $\mathcal{A}(\Omega)$ the family of all open subsets of Ω. The letter C will denote a strictly positive constant independent of the parameters under consideration, whose value may vary from line to line. If E is a subset of $\mathbb{R}^n$ then χ_E is its characteristic function. If $u \in L^1(\Omega, \mathbb{R}^m)$, we denote by $\tilde{u}$ the *precise representative* of u, whose components are defined by

$$\tilde{u}_i(x) := \limsup_{\rho \to 0^+} \fint_{B_\rho(x)} u_i(y)dy, \tag{5.2.1}$$

where $B_\rho(x)$ denotes the open ball of centre x and radius ρ.

5.2.1 The Γ-convergence

Definition 5.2.1 *Let (X,d) be a metric space and let $F, F_h : X \to [-\infty, +\infty]$ be functions. we define*

$$\Gamma(X,d) - \liminf_{h\to+\infty} F_h(u) = \inf \left\{ \liminf_{h\to+\infty} F_h(u_h) \,:\, u_h \to u \ in \ X \right\},$$

and

$$\Gamma(X,d) - \limsup_{h\to+\infty} F_h(u) = \inf \left\{ \limsup_{h\to+\infty} F_h(u_h) \,:\, u_h \to u \ in \ X \right\};$$

if these two quantities coincide their common value will be called the Γ-limit of the sequence $\{F_h\}$ at u and will be denoted by $\Gamma(X,d) - \lim_{h\to+\infty} F_h(u)$ or by $\Gamma(X) - \lim_{h\to+\infty} F_h(u)$ when there is no confusion.

It is easy to check that $l = \Gamma(X) - \lim_{h\to+\infty} F_h(u)$ if and only if

(i) for any sequence $\{u_h\} \subseteq X$ converging to u, we have

$$\liminf_{h\to\infty} F_h(u_h) \geq l;$$

(ii) for any $u \in X$ there exists a sequence $\{u_h\} \subseteq X$ converging to u, such that

$$\limsup_{h\to\infty} F_h(u_h) \leq l.$$

The peculiarity of this type of convergence is its variational character, explained in the following proposition.

Proposition 5.2.2 *Assume that* $\{F_h\}$ Γ*-converges to* F *and that there exists a compact set* $K \subseteq X$ *such that*

$$\inf_{u \in K} F_h(x) = \inf_{u \in X} F_h(u) \qquad \forall h \in \mathbb{N}.$$

Then

(i) $\inf_X F_h$ *converges as* $h \to \infty$ *to* $\min_X F$ *and any limit point of any sequence* $\{u_h\}$ *such that*

$$\lim_{h \to \infty} \Big(F_h(u_h) - \inf_{u \in X} F_h(u) \Big) = 0$$

is a minimizer of F.

(ii) $\{F_h + G\}$ Γ*-converges to* $F + G$ *for any* $G : X \to \mathbb{R}$ *continuous.*

It is easy to see that a sequence $\{F_\varepsilon\}$ Γ-converges to l at u as $\varepsilon \to 0^+$ if for every sequence of positive numbers $\{\varepsilon_h\}$ converging to 0 there exists a subsequence $\{\varepsilon_{h_k}\}$ such that

$$l = \Gamma(X) - \lim_{k \to +\infty} F_{\varepsilon_{h_k}}(u).$$

Let us recall that the Γ-upper and Γ-lower limits defined above are (X, d)-lower semicontinuous. We refer the reader to Dal Maso [83] for an exhaustive treatment of Γ-convergence.

5.2.2 Sobolev spaces with respect to a measure

The following notion of Sobolev space with respect to a measure has been introduced by Ambrosio, Buttazzo and Fonseca in [20] where they studied the lower semicontinuity of integral functionals of the form (5.1.1), f being a quasiconvex integrand independent of the variable x.

Definition 5.2.3 *Let* λ *be a finite Borel positive measure on the open set* $\Omega \subset \mathbb{R}^n$, *and let* $1 \leq p \leq +\infty$. *We define the space*

$$W^{1,p}_\lambda(\Omega, \mathbb{R}^m) := \left\{ u \in L^p(\Omega, \mathbb{R}^m) \colon Du << \lambda, \ \frac{dDu}{d\lambda} \in L^p_\lambda(\Omega, M^{n \times m}) \right\}$$

where $L^p_\lambda(\Omega, \mathbb{R}^N)$ *stands for the usual Lebesgue space of* p*-summable* $\mathbb{R}^N$*-valued functions with respect to* λ*.*

It is easy to see that Sobolev spaces with respect to measures are subspaces of $BV(\Omega, \mathbb{R}^m)$. When $\lambda = \mathcal{L}^n \llcorner \Omega$, we obtain the classical Sobolev space $W^{1,p}(\Omega, \mathbb{R}^m)$. The Sobolev spaces with respect to measures have been devised in [20] as suitable functions spaces in which one could study for example fracture mechanics problems with localized fractures. This would correspond for example to the space $W^{1,p}_\lambda(\Omega, \mathbb{R}^m)$ with $\lambda = \mathcal{L}^n \llcorner \Omega + \mathcal{H}^{n-1} \llcorner M$, where M is a $\mathcal{C}^1$ hypersurface in which the fracture is localized. In this case $W^{1,p}_\lambda(\Omega, \mathbb{R}^m)$ is a subspace of functions $u \in SBV(\Omega, \mathbb{R}^m)$ with $S_u \subset M$ up to a $\mathcal{H}^{n-1}$-negligible subset. Properties of lower semicontinuity and relaxation for functionals defined on Sobolev spaces with respect to a measure have been studied in [20].

5.3 The spaces $LD^p_\lambda(\Omega)$

Definition 5.3.1 *Let* λ *be a finite Borel positive measure on the open set* $\Omega \subset \mathbb{R}^n$*, and let* $1 \le p \le +\infty$*. We define the space*

$$LD^p_\lambda(\Omega) := \left\{ u \in L^p(\Omega, \mathbb{R}^n) \colon Eu << \lambda, \ \frac{dEu}{d\lambda} \in L^p_\lambda(\Omega, M^{n\times n}_s) \right\}$$

When $p > 1$, $\lambda = \mathcal{L}^n \llcorner \Omega$ with Ω having a Lipschitz continuous boundary, then by Korn's inequality, we have

$$LD^p_\lambda(\Omega) = W^{1,p}(\Omega, \mathbb{R}^n)$$

while the case $p = 1$ with $\lambda = \mathcal{L}^n \llcorner \Omega$ gives the space $LD(\Omega)$ introduced in the chapters II and IV.

Proposition 5.3.2 (i) *The spaces $LD^p_{\lambda_1}(\Omega)$ and $LD^p_{\lambda_2}(\Omega)$ coincide whenever $|\lambda_1 - \lambda_2|(\Omega \setminus B) = 0$ and $\mathcal{H}^{n-1}(B) = 0$ for some Borel subset B of Ω.*

(ii) *The measure λ in Definition 5.3.1 can always be assumed concentrated on a Borel set where its $(n-1)$-dimensional upper density is finite.*

Proof. The point (i) easily follows from the fact that BD functions do not charge $\mathcal{H}^{n-1}$-negligible sets.

(ii) We set

$$E := \left\{ x \in \Omega\colon\ \limsup_{\rho\to 0^+} \frac{\lambda(B_\rho(x))}{\rho^{n-1}} = +\infty \right\}. \tag{5.3.1}$$

From covering theorems (see for instance Ziemer [202]), we have $\mathcal{H}^{n-1}(E) = 0$. Set now

$$\nu := \lambda \llcorner (\Omega \setminus E),$$

then $|\lambda - \nu|(\Omega \setminus E) = 0$ and by (i) we have $LD^p_\lambda(\Omega) = LD^p_\nu(\Omega)$. □

In the following theorem we prove a Leibnitz type formula for the densities with respect a measure λ. This formula will be used in the proof of the fundamental estimate.

Theorem 5.3.3 *Let $u \in LD^p_\lambda(\Omega)$, $v \in W^{1,\infty}_\lambda(\Omega)$ be such that $\tilde u \odot \frac{dDv}{d\lambda}$ belongs to $L^p_\lambda(\Omega, M^{n\times n}_s)$. Then $uv \in LD^p_\lambda(\Omega)$ and*

$$\frac{dE(uv)}{d\lambda} = \tilde v \frac{dEu}{d\lambda} + \tilde u \odot \frac{dDv}{d\lambda}. \tag{5.3.2}$$

Proof. The identity (5.3.2) will follow from the formula

$$E(uv) = \tilde v Eu + \tilde u \odot Dv \tag{5.3.3}$$

which will be proved as an identity between measures. Let us observe that (5.3.3) is equivalent to

$$(E(uv)\xi, \xi) = (\tilde v Eu\xi, \xi) + (\tilde u \odot Dv\xi, \xi) \tag{5.3.4}$$

for any $\xi = \xi_i + \xi_j$ where $\{\xi_1, \ldots, \xi_n\}$ is a basis of $\mathbb{R}^n$.
We will establish (5.3.3) following the "slicing" technique which will allow to obtain it from the one-dimensional version. To this aim, let us remark that if I is an open subset of $\mathbb{R}$ and $u_1, u_2 \in BV(I)$, then from the chain rule formula for BV functions (see e.g., Ambrosio-Fusco-Pallara [26, Theorem 3.93 and Example 3.94]) we have $u_1 u_2 \in BV(I)$ and

$$D(u_1 u_2) = \tilde{u}_1 D u_2 + \tilde{u}_2 D u_1. \tag{5.3.5}$$

Therefore, the formula (5.3.3) is always true in dimension one. Now, let us recall some notations used in the chapters I and II in order to define the one-dimensional sections of BV and BD functions:

$$u_y^\xi(t) := (u(y + t\xi), \xi) \quad \text{and} \quad v_{y,\xi}(t) := v(y + t\xi) \quad \text{for any } t \in \Omega_y^\xi \quad \text{and } y \in \Omega^\xi.$$

Since $u_y^\xi, v_{y,\xi} \in BV(\Omega_y^\xi)$ for $\mathcal{H}^{n-1}$-almost every $y \in \Omega^\xi$, then by (5.3.5) we have

$$(uv)_y^\xi = u_y^\xi v_{y,\xi} \in BV(\Omega_y^\xi)$$

and

$$D\Big((uv)_y^\xi\Big) = \tilde{u}_y^\xi D v_{y,\xi} + \tilde{v}_{y,\xi} D u_y^\xi. \tag{5.3.6}$$

for $\mathcal{H}^{n-1}$-almost every $y \in \Omega^\xi$, where we have used from the structure theorems for BD and BV functions, the following identities

$$\tilde{u}_y^\xi = \widetilde{u_y^\xi} \quad \text{and} \quad \tilde{v}_{y,\xi} = \widetilde{v_{y,\xi}}.$$

Note that from the hypotheses of the theorem and by using the second identity of (2.2.2) in Chapter II and Theorem 1.3.3 of Chapter I we get

$$\int_{\Omega^\xi} \left(\int_{\Omega_y^\xi} |\tilde{v}_{y,\xi}| d|D u_y^\xi| \right) d\mathcal{H}^{n-1}(y) = \int_\Omega |\tilde{v}(x)| d|(Eu\xi, \xi)| < +\infty$$

and

$$\int_{\Omega^\xi} \left(\int_{\Omega_y^\xi} |\tilde{u}_y^\xi| d|D v_{y,\xi}| \right) d\mathcal{H}^{n-1}(y) = \int_\Omega |(\tilde{u}, \xi)| d|(Dv, \xi)| < +\infty.$$

Therefore, we get from (5.3.6) that

$$\int_{\Omega^\xi}\left(\int_{\Omega^\xi_y}|D(uv)^\xi_y|(\Omega^\xi_y)\right)d\mathcal{H}^{n-1}(y)<+\infty.$$

So, we have proved that for any $\xi=\xi_i+\xi_j$

$$(uv)^\xi_y\in BV(\Omega^\xi_y)\quad\text{and}\quad\int_{\Omega^\xi}\left(\int_{\Omega^\xi_y}|D(uv)^\xi_y|(\Omega^\xi_y)\right)d\mathcal{H}^{n-1}(y)<+\infty$$

for $\mathcal{H}^{n-1}$-almost every $y\in\Omega^\xi$. Then by Theorem 2.2.1 we have $uv\in BD(\Omega)$. By integrating with respect to the measure $\mathcal{H}^{n-1}$ we obtain from (5.3.6) that

$$\begin{aligned}(E(uv)\xi,\xi)(B) &= \int_{B^\xi}D\Big((uv)^\xi_y\Big)(B^\xi_y)d\mathcal{H}^{n-1}(y)\\ &= \int_{B^\xi}\int_{B^\xi_y}\tilde{u}^\xi_y dDv_{y,\xi}d\mathcal{H}^{n-1}(y)+\int_{B^\xi}\int_{B^\xi_y}\tilde{v}_{y,\xi}dDu^\xi_y d\mathcal{H}^{n-1}(y)\\ &= \int_B(\tilde{u},\xi)d(Dv,\xi)+\int_B\tilde{v}d(Eu\xi,\xi)\end{aligned}$$

for any $B\in\mathcal{B}(\Omega)$. Then we obtain (5.3.4) and hence (5.3.3). □

5.4 p-Homogenizable measures

5.4.1 Definition

In order to obtain a meaningful limit of the functionals F_ε in (5.1.2) as $\varepsilon\to0^+$, we introduce the following notion of p-homogenizable measure.

Definition 5.4.1 *A 1-periodic positive Radon measure μ on $\mathbb{R}^n$ is said to be p-homogenizable if it satisfies the following two properties:*

(i) (*existence of cut-off functions*) *there exist $K>0$ and $\delta>0$ such that for all $\varepsilon>0$ and for all pair U, V of open subsets of $\mathbb{R}^n$ with $U\subset\subset V$ and $\operatorname{dist}(U,\partial V)\geq\delta\varepsilon$, and for all $u\in LD^p_{\mu_\varepsilon}(V)$ there exists $\phi\in W^{1,\infty}_{\mu_\varepsilon}(V)$ with*

$$0\leq\phi\leq1,\ \phi=1\text{ on }U,\ \phi=0\text{ in a neighbourhood of }\partial V,$$

such that

$$\int_V \Big|\frac{dD\phi}{d\mu_\varepsilon} \odot \tilde{u}\Big|^p d\mu_\varepsilon \le \frac{K}{\big(\mathrm{dist}(U, \partial V)\big)^p} \int_{V\setminus U} |u|^p dx. \tag{5.4.1}$$

Such a ϕ will be called a cut-off function between U and V;

(ii) (*existence of periodic test-functions*) *for all $i, j = 1, \cdots, n$, there exist functions z_{ij} in $LD^p_{\mu,\mathrm{loc}}(\mathbb{R}^n)$ such that $x \to z_{ij}(x) - x_j e_i$ is 1-periodic.*

Remark 5.4.2 (a) Let us first observe that our definition above is a modified version of [32, Definition 3.2]. Note that if μ is p-homogenizable in the sense of [32, Definition 3.2] and z_i is the function in $W^{1,p}_{\mu,\mathrm{loc}}(\mathbb{R}^n)$ which satisfies (iii) of that definition then $z_{ij} = z_j e_i$ satisfies the condition (ii) above; but the converse is not true.

(b) The conditions (i) and (ii) of Definition 5.4.1 will allow to prove both the fundamental estimate and the growth condition necessary and sufficient to obtain the existence and the integral representation of the Γ-limit.

5.4.2 Examples of p-homogenizable measures

The following measures given [32] are also p-homogenizable in our context.

Example 5.4.3 the Lebesgue measure trivially satisfies the properties (i) and (ii) of Definition 5.4.1

Example 5.4.4 (Perfectly-rigid bodies connected with elastic springs)
We take

$$E = \{y \in \mathbb{R}^n : \exists i \in \{1, \ldots, n\} \text{ such that } y_i \in \mathbf{Z}\},$$

that is, the union of all the boundaries of cubes $Q_i = i + (0,1)^n$ with $i \in \mathbf{Z}^n$. E is an $(n-1)$-dimensional set in $\mathbb{R}^n$. We take

$$\mu(B) = \frac{1}{n}\mathcal{H}^{n-1}(B \cap E)$$

for all Borel sets B. For every $\varepsilon > 0$ we have

$$\mu_\varepsilon(B) = \frac{1}{n}\varepsilon\mathcal{H}^{n-1}(B \cap \varepsilon E)\,.$$

So, in this case $LD^p_{\mu_\varepsilon}(\Omega)$ consists of functions which are rigid displacements in the sets $\varepsilon Q_i \cap \Omega$; i.e., $u_i = R_i x + c_i$ on each $\varepsilon Q_i \cap \Omega$ with R_i skew $n \times n$ symmetric matrix and $c_i \in \mathbb{R}^n$. Hence we have

$$\frac{dEu}{d\mu_\varepsilon} = \frac{n}{\varepsilon}\frac{dEu}{d\mathcal{H}^{n-1}} = \frac{n}{\varepsilon}(u_i - u_j) \odot (i - j) \quad \text{ on } \partial(\varepsilon Q_i) \cap \partial(\varepsilon Q_j) \cap \Omega\,.$$

In this case the functionals F_ε take the form

$$\varepsilon \int_{\Omega \cap \varepsilon E} g(\frac{x}{\varepsilon}, \frac{1}{\varepsilon}\frac{dEu}{d\mathcal{H}^{n-1}})d\mathcal{H}^{n-1}$$

up to a normalization factor. Note that if Ω is bounded then $LD^p_{\mu_\varepsilon}(\Omega) = LD^\infty_{\mu_\varepsilon}(\Omega)$ for all p if the number of connected components of each set $\Omega \cap \varepsilon Q_i$ is finite. The measure μ in this case satisfies the conditions (i) and (ii) of Definition 5.4.1 for all $p \geq 1$. In fact, to prove (i) we consider the same cut-off function in Example 3.4 (a) of [32]. Precisely, for $\delta = 5\sqrt{n}$ and for fixed $\varepsilon > 0$ we set

$$U_\varepsilon := \{\varepsilon Q_i \colon \varepsilon Q_i \cap U \neq \emptyset\}$$

and using the notation $[t]$ for the integer part of t, we set

$$\phi(x) := 1 - \Big(\frac{1}{C}\Big[\frac{1}{\varepsilon}\inf\{\|x-y\|_\infty \colon y \in U_\varepsilon\}\Big] \wedge 1\Big)$$

where $\|x-y\|_\infty := \max\limits_{1\leq i\leq n} |x_i - y_i|$ and

$$C = \Big[\frac{1}{\varepsilon}\inf\{\|x-y\|_\infty \colon x \in U_\varepsilon,\, y \in \partial V\}\Big] - 2.$$

One can verify that $|dD\phi/d\mu_\varepsilon| \leq n/(C\varepsilon) \leq c/\mathrm{dist}(U, \partial V)$ for some constant c independent of U and V. Let $u \in LD^p_{\mu_\varepsilon}$. If γ denotes the trace operator of

Chapter I, in this case we can interpret $\gamma(u)$ as trace of Sobolev function defined on each cube Q_i and we have

$$\left(\int_{\partial Q_i} |\gamma(u)|^p d\mathcal{H}^{n-1}\right)^{1/p} \leq C\, \|u\|_{W^{1,p}(Q_i,\mathbb{R}^n)}\,. \tag{5.4.2}$$

So, by a scaling argument and by Korn's inequality we get

$$\begin{aligned}\left(\varepsilon \int_{\partial\varepsilon Q_i} |\gamma(u)|^p d\mathcal{H}^{n-1}\right)^{1/p} &\leq C\left(\left(\int_{\varepsilon Q_i} |u|^p dx\right)^{1/p} + \left(\varepsilon^{2p-n}\int_{\varepsilon Q_i} |Eu|^p dx\right)^{1/p}\right)\\ &= C\left(\int_{\varepsilon Q_i} |u|^p dx\right)^{1/p}.\end{aligned}$$

If $p=1$ we can apply the trace inequality in $LD(Q_i)$ to obtain

$$\begin{aligned}\int_{\partial Q_i} |\gamma(u)| d\mathcal{H}^{n-1} &\leq C\left(\int_{Q_i} |u|dx + |Eu|(Q_i)\right)\\ &= C\int_{Q_i} |u|dx.\end{aligned}$$

So, we get

$$\varepsilon\int_{\partial\varepsilon Q_i} |\gamma(u)| d\mathcal{H}^{n-1} \leq C\int_{\varepsilon Q_i} |u|dx.$$

Hence for all $p \geq 1$

$$\varepsilon\int_{\partial\varepsilon Q_i} |\gamma(u)|^p\, d\mathcal{H}^{n-1} \leq C\int_{\varepsilon Q_i} |u|^p dx.$$

For two cubes, we obtain

$$\begin{aligned}\varepsilon\int_{\partial\varepsilon Q_i\cap\partial\varepsilon Q_j} |\gamma(u)|^p\, d\mathcal{H}^{n-1} &\leq \varepsilon\int_{\partial\varepsilon Q_i\cap\partial\varepsilon Q_j} (|u_i|^p + |u_j|^p)\, d\mathcal{H}^{n-1}\\ &\leq C\int_{\varepsilon Q_i\cup\varepsilon Q_j} |u|^p dx,\end{aligned}$$

so that

$$\begin{aligned}\int_V \left|\frac{dD\phi}{d\mu_\varepsilon}\odot \tilde{u}\right|^p d\mu_\varepsilon &\leq \frac{C^p\varepsilon}{n\mathrm{dist}(U,\partial V)^p}\int_{(V\setminus U)\cap\varepsilon E\cap\mathrm{spt}(D\phi)} |\tilde{u}|^p\, d\mathcal{H}^{n-1}\\ &\leq \frac{C^p}{\mathrm{dist}(U,\partial V)^p}\int_{V\setminus U} |u|^p dx\,.\end{aligned}$$

The proof of (i) is then complete. To verify (ii) we apply the Remark 5.4.2 and the Example 3.4(a) in [32], so we take simply $z_{ij}(x) = [x_j]e_i$.

Example 5.4.5 (Elastic media connected with springs)
Let E be as in the previous example and let

$$\begin{aligned}\mu(B) &= \frac{1}{n+1}\Big(|B| + \mathcal{H}^{n-1}(E\cap B)\Big)\\ \mu_\varepsilon(B) &= \frac{1}{n+1}\Big(|B| + \varepsilon\mathcal{H}^{n-1}((\varepsilon E)\cap B)\Big)\,.\end{aligned}$$

In this case the functions in $LD^p_{\mu_\varepsilon}(\Omega)$ are functions whose restriction to each $\varepsilon Q_i \cap \Omega$ belongs to $W^{1,p}(\varepsilon Q_i\cap\Omega;\mathbb{R}^n)$ when $p>1$ and to $LD(\varepsilon Q_i\cap\Omega)$ when $p=1$, while the difference of the traces on both sides of $\partial(\varepsilon Q_i)\cap\partial(\varepsilon Q_j)\cap\Omega$ is p-summable for every $i,j\in\mathbf{Z}^n$. The functionals F_ε take the form

$$\frac{1}{n+1}\int_\Omega f(\frac{x}{\varepsilon},\mathcal{E}u(x))dx + \frac{\varepsilon}{n+1}\int_{\Omega\cap\varepsilon E} g(\frac{x}{\varepsilon},\frac{1}{\varepsilon}\frac{dEu}{d\mathcal{H}^{n-1}})d\mathcal{H}^{n-1}$$

where the density g is the one of the previous example.
The measure μ in this case is also p-homogenizable for all $p\geq 1$ by the example 3.4(b) in [32].

5.5 The homogenization theorem

Let us first begin by describing the assumptions. Let μ be a non-zero positive Radon measure on $\mathbb{R}^n$ which is 1-periodic, 1.e.,

$$\mu(B+e_i)=\mu(B)$$

for all Borel subsets of $\mathbb{R}^n$ and for all $i=1,\dots,n$, where (e_i) is the canonical basis of $\mathbb{R}^n$. The measure μ will be fixed throughout the chapter. We will assume the normalization

(5.5.1) $$\mu([0,1)^n)=1.$$

For all $\varepsilon>0$ we define the ε-periodic positive Radon measure μ_ε by

(5.5.2) $$\mu_\varepsilon(B)=\varepsilon^n\mu(\frac{1}{\varepsilon}B)$$

for all Borel sets B. By (5.5.1) the sequence $\{\mu_\varepsilon\}$ converges locally weakly* in the sense of measures to the Lebesgue measure as $\varepsilon \to 0^+$.
In the sequel $f : \mathbb{R}^n \times M_s^{n\times n} \to [0, +\infty[$ will be a fixed Borel function 1-periodic in the first variable and satisfying the growth condition of order $p \geq 1$

$$\alpha|A|^p \leq f(x, A) \leq \beta(1 + |A|^p) \tag{5.5.3}$$

for some constant $0 < \alpha \leq \beta$, for all $x \in \mathbb{R}^n$ and $A \in M_s^{n\times n}$.
For every bounded open set Ω, we define the functionals at scale $\varepsilon > 0$ as

$$F_\varepsilon(u, \Omega) := \begin{cases} \displaystyle\int_\Omega f\Big(\frac{x}{\varepsilon}, \frac{dEu}{d\mu_\varepsilon}\Big)d\mu_\varepsilon & \text{if } u \in LD^p_{\mu_\varepsilon}(\Omega) \\ +\infty & \text{otherwise.} \end{cases} \tag{5.5.4}$$

The homogenization theorem for the functionals in (5.5.4) takes the following form

Theorem 5.5.1 *Let μ be a p-homogenizable measure and for every bounded open subset Ω of $\mathbb{R}^n$ let $F_\varepsilon(\cdot, \Omega)$ be defined on $L^p(\Omega, \mathbb{R}^n)$ by (5.5.4). Then the Γ-limit*

$$F_{\text{hom}}(u, \Omega) = \Gamma(L^p) - \lim_{\varepsilon\to 0^+} F_\varepsilon(u, \Omega) \tag{5.5.5}$$

exists for all bounded open subsets Ω with Lipschitz boundary; it is finite in $(W^{1,p}(\Omega))^n$ (and in $BD(\Omega)$ if $p = 1$). $F_{\text{hom}}(\cdot, \Omega)$ can be represented in $W^{1,p}(\Omega, \mathbb{R}^n)$ as

$$F_{\text{hom}}(u, \Omega) = \int_\Omega f_{\text{hom}}(Eu)dx, \tag{5.5.6}$$

where the homogenized integrand f_{hom} satisfies the asymptotic formula

$$f_{\text{hom}}(A) = \lim_{k\to+\infty} \inf \left\{ \frac{1}{k^n} \int_{[0,k)^n} f\Big(x, \frac{dEu}{d\mu}\Big)d\mu \colon \begin{array}{l} u \in LD^p_{\mu,\text{loc}}(\mathbb{R}^n) \textit{ and} \\ u - Ax \textit{ is } k-\textit{periodic} \end{array} \right\}; \tag{5.5.7}$$

(i) *if $p > 1$ then $F_{\text{hom}}(\cdot, \Omega) = +\infty$ in $L^p(\Omega, \mathbb{R}^n) \setminus W^{1,p}(\Omega, \mathbb{R}^n)$;*

(ii) *if $p = 1$ then $F_{\text{hom}}(\cdot, \Omega) = +\infty$ in $L^1(\Omega, \mathbb{R}^n) \setminus BD(\Omega)$.*

Futhermore, if f is convex then the Γ-limit can be represented as

$$F_{\text{hom}}(u,\Omega) = \int_\Omega f_{\text{hom}}\Big(\mathcal{E}u(x)\Big)dx + \int_\Omega f^\infty_{\text{hom}}\Big(\frac{dE^s u}{d|E^s u|}\Big)d|E^s u| \tag{5.5.8}$$

for all $u \in BD(\Omega)$.

Remark 5.5.2 (i) In the formula (5.5.7) we cannot replace the set $[0,k)^n$ by $(0,k)^n$ if μ charges $[0,1)^n \setminus (0,1)^n$.

(ii) If μ does not satisfy the conditions of Definition 5.4.1 then f_{hom} may be equal to $+\infty$ for all non-zero matrices A. As an example, take

$$\mu(B) = \sum_{i\in\mathbf{Z}^n} \lambda(i+B)\,, \tag{5.5.9}$$

where λ is any probability measure with sptλ contained in $(0,1)^n$. Then test-functions u in (5.5.7) must be rigid displacements on a periodic connected component of $\mathbb{R}^n$, and hence we get that $f_{\text{hom}}(A) = +\infty$ if $A \neq 0$.

5.6 The proof of the homogenization theorem

The proof of Theorem 5.5.1 will be obtained as a consequence of the following propositions in which we will adapt the usual compactness and integral representation results in Dal Maso [83] with some minor modifications. The proofs of the fundamental estimates and the growth condition will be similar to the propositions 3.1, 3.2 and 3.3 of [32].

Proposition 5.6.1 (Fundamental Estimate) *For every $\sigma > 0$ there exist ε_σ and $M_\sigma > 0$ such that for all $U, U', V \subset \Omega$ open with $U' \subset U$ and* $\mathrm{dist}(U', V\setminus U) > 0$*, for all $\varepsilon < \varepsilon_\sigma \mathrm{dist}(U', V\setminus U)$ and for all $u \in LD^p_{\mu_\varepsilon}(\Omega)$, $v \in LD^p_{\mu_\varepsilon}(\Omega)$ there exist* *a cut-off function between U' and U, $\phi \in W^{1,p}_{\mu_\varepsilon}(U \cup V)$, such that*

$$
\begin{aligned}
(5.6.1) \quad F_\varepsilon(\phi u + (1-\phi)v, U' \cup V) \;\; &\leq \;\; (1+\sigma)(F_\varepsilon(u,U) + F_\varepsilon(v,V)) \\
&+ \;\; \frac{M_\sigma}{(\mathrm{dist}(U', V \setminus U))^p} \int_{(U \cap V)\setminus U'} |u-v|^p dx \\
&+ \;\; \sigma \mu_\varepsilon((U \cap V) \setminus U').
\end{aligned}
$$

Proof. Let $K, \delta > 0$ be the constants given in the condition (ii) of Definition 5.4.1; let $N \in \mathbb{N}$ such that $N\delta\varepsilon \leq \mathrm{dist}(U', V \setminus U)$. Let

$$U_0 = U'$$

and

$$U_k := \{x \in U : N\mathrm{dist}(x, U') < k\,\mathrm{dist}(U', V \setminus U)\}\,.$$

For each $k = 1, \ldots, N$ let ϕ_k be a cut-off function between U_{k-1} and U_k, satisfying (5.4.1) (note that such a function exists since $\mathrm{dist}(U_{k-1}, \partial U_k) \geq \delta\varepsilon$). Then using the Leibnitz type formula in (5.3.2) we get

$$
\begin{aligned}
&F_\varepsilon(\phi_k u + (1-\phi_k)v, U' \cup V) \\
&= \int_{U' \cup V} f(\frac{x}{\varepsilon}, \tilde{\phi}_k \frac{dEu}{d\mu_\varepsilon} + (1-\tilde{\phi}_k)\frac{dEv}{d\mu_\varepsilon} + (\tilde{u} - \tilde{v}) \odot \frac{dD\phi_k}{d\mu_\varepsilon}) d\mu_\varepsilon \\
&= \int_{(U' \cup V) \cap U_{k-1}} f(\frac{x}{\varepsilon}, \frac{dEu}{d\mu_\varepsilon}) d\mu_\varepsilon + \int_{(U' \cup V) \setminus U_k} f(\frac{x}{\varepsilon}, \frac{dEv}{d\mu_\varepsilon}) d\mu_\varepsilon \\
&+ \int_{(U' \cup V) \cap (U_k \setminus U_{k-1})} f(\frac{x}{\varepsilon}, \tilde{\phi}_k \frac{dEu}{d\mu_\varepsilon} + (1-\tilde{\phi}_k)\frac{dEv}{d\mu_\varepsilon} + (\tilde{u} - \tilde{v}) \odot \frac{dD\phi_k}{d\mu_\varepsilon}) d\mu_\varepsilon \\
&\leq \int_U f(\frac{x}{\varepsilon}, \frac{dEu}{d\mu_\varepsilon}) d\mu_\varepsilon + \int_V f(\frac{x}{\varepsilon}, \frac{dEv}{d\mu_\varepsilon}) d\mu_\varepsilon \\
&\qquad\qquad\qquad + 3^p \beta \int_{V \cap (U_k \setminus U_{k-1})} \Big| (\tilde{u} - \tilde{v}) \odot \frac{dD\phi_k}{d\mu_\varepsilon} \Big|^p d\mu_\varepsilon \\
&\qquad\qquad\qquad\qquad + 3^p \beta \int_{V \cap (U_k \setminus U_{k-1})} \Big(1 + \Big| \frac{dEu}{d\mu_\varepsilon} \Big|^p + \Big| \frac{dEv}{d\mu_\varepsilon} \Big|^p \Big) d\mu_\varepsilon.
\end{aligned}
$$

Then

$$\begin{aligned}
&F_\varepsilon(\phi_k u + (1-\phi_k)v, U'\cup V)\\
&\le F_\varepsilon(u,U) + F_\varepsilon(v,V) + 3^p\beta \int_{V\cap(U-k\setminus U_{k-1})} \Big(1 + \Big|\frac{dEu}{d\mu_\varepsilon}\Big|^p + \Big|\frac{dEv}{d\mu_\varepsilon}\Big|^p\Big) d\mu_\varepsilon\\
&\qquad\qquad + 3^p\beta \frac{K}{(\mathrm{dist}(U_{k-1},\partial U_k))^p} \int_{U_k\setminus U_{k-1}} |u-v|^p dx\\
&\le F_\varepsilon(u,U) + F_\varepsilon(v,V) + 3^p\beta \int_{V\cap(U_k\setminus U_{k-1})} \Big(1 + \Big|\frac{dEu}{d\mu_\varepsilon}\Big|^p + \Big|\frac{dEv}{d\mu_\varepsilon}\Big|^p\Big) d\mu_\varepsilon\\
&\qquad\qquad + 3^p\beta \frac{KN^p}{(\mathrm{dist}(U',V\setminus U))^p} \int_{U_k\setminus U_{k-1}} |u-v|^p dx\,,
\end{aligned}$$

where K is the constant appearing in (5.4.1). Now we choose k such that

$$\begin{aligned}
&\int_{(U_k\setminus U_{k-1})\cap V} (1 + \Big|\frac{dEu}{d\mu_\varepsilon}\Big|^p + \Big|\frac{dEv}{d\mu_\varepsilon}\Big|^p) d\mu_\varepsilon\\
&\qquad + \frac{KN^p}{(\mathrm{dist}(U',V\setminus U))^p} \int_{(U_k\setminus U_{k-1})\cap V} |u-v|^p dx\\
&\le \frac{1}{N}\Big(\int_{(U\cap V)\setminus U'} (1 + \Big|\frac{dEu}{d\mu_\varepsilon}\Big|^p + \Big|\frac{dEv}{d\mu_\varepsilon}\Big|^p) d\mu_\varepsilon\\
&\qquad + \frac{KN^p}{(\mathrm{dist}(U',V\setminus U))^p} \int_{(U\cap V)\setminus U'} |u-v|^p dx\Big).
\end{aligned}$$

Then, by taking into account the growth condition (5.5.3) on f, we obtain for such k that

$$\begin{aligned}
&F_\varepsilon(\phi_k u + (1-\phi_k)v, U'\cup V)\\
\le\ & F_\varepsilon(u,U) + F_\varepsilon(v,V)\\
&\quad + \frac{3^p\beta}{N\alpha}\Big(\int_{(U\cap V)\setminus U'} f(\frac{x}{\varepsilon}, \frac{dEu}{d\mu_\varepsilon}) d\mu_\varepsilon + \int_{(U\cap V)\setminus U'} f(\frac{x}{\varepsilon}, \frac{dEv}{d\mu_\varepsilon}) d\mu_\varepsilon\Big)\\
&\quad + 3^p\beta \frac{KN^{p-1}}{(\mathrm{dist}(U',V\setminus U))^p} \int_{(U\cap V)\setminus U'} |u-v|^p dx\\
&\quad + \frac{3^p\beta}{N}\mu_\varepsilon((U\cap V)\setminus U').
\end{aligned}$$

Then it follows that

$$
\begin{aligned}
& F_\varepsilon(\phi_k u + (1-\phi_k)v, U' \cup V) \\
\leq \; & \Big(1 + \frac{3^p\beta}{N\alpha}\Big)\Big(F_\varepsilon(u,U) + F_\varepsilon(v,V)\Big) \\
& \qquad + 3^p\beta \frac{KN^{p-1}}{(\mathrm{dist}(U', V\setminus U))^p}\int_{(U\cap V)\setminus U'} |u-v|^p dx + \frac{3^p\beta}{N}\mu_\varepsilon((U\cap V)\setminus U')\,.
\end{aligned}
$$

In order that (5.6.1) holds ε_σ and N should verify

$$\frac{3^p\beta}{N\min(1,\alpha)} < \sigma \text{ and } \varepsilon_\sigma < \frac{1}{N\delta}.$$

So that

$$\frac{3^p\beta}{\sigma\min(1,\alpha)} < N < \frac{1}{\varepsilon_\sigma\delta}.$$

Then by choosing ε_σ such that

$$\frac{3^p\beta}{N\min(1,\alpha)} + 1 = \frac{1}{\varepsilon_\sigma\delta}$$

we will obtain an N for which (5.6.1) is satisfied with $M = 3^pK\beta N^{p-1}$. $\square$

Proposition 5.6.2 *For every $A \in M_s^{n\times n}$ there exists $z_A \in LD^p_{\mu,\mathrm{loc}}(\mathbb{R}^n)$ such that $z_A - Ax$ is 1-periodic and satisfies*

$$\int_{[0,1)^n} \Big|\frac{dEz_A}{d\mu}\Big|^p d\mu \leq C|A|^p. \tag{5.6.2}$$

Proof. Define $z_A = \sum_{i,j=1}^n A_{ij}z_{ij}$, where z_{ij} are as in the condition (ii) of Definition 5.4.1. The inequality (5.6.2) easily follows. $\square$

Let us fix an infinitesimal sequence $\{\varepsilon_j\}$. We define

$$F'(u,U) = \Gamma(L^p)\text{-}\liminf_{j\to+\infty} F_{\varepsilon_j}(u,U)$$

$$F''(u,U) = \Gamma(L^p)\text{-}\limsup_{j\to+\infty} F_{\varepsilon_j}(u,U)$$

for all $u \in L^p(\Omega,\mathbb{R}^n)$ and for all open subsets U of Ω.

Proposition 5.6.3 (Growth Condition) *We have for all open subsets U of Ω with $|\partial U| = 0$*

$$F''(u,U) \leq C \int_U (1 + |Eu|^p)dx$$

for all $u \in W^{1,p}(\Omega, \mathbb{R}^n)$ and

$$F''(u,U) \leq C(|U| + |Eu|(U))$$

for all $u \in BD(\Omega)$.

Proof. *Step 1: We claim that $F''(Ax, U) \leq C|U|\Big(1 + \big|\frac{A+A^t}{2}\big|^p\Big)$ for any matrix $A \in M^{n\times n}$ and for any $U \in \mathcal{A}(\Omega)$ with $|\partial U| = 0$.*

Suppose that A is symmetric and let z_A be given by Proposition 5.6.2. So, the sequence $z_A^\varepsilon(x) := \varepsilon z_A(x/\varepsilon)$ converges in $L^p_{\rm loc}(\mathbb{R}^n, \mathbb{R}^n)$ to Ax, and

$$\begin{aligned} F''\Big(Ax, U\Big) &\leq \limsup_{\varepsilon\to 0+} \int_U f\big(\frac{x}{\varepsilon}, \frac{dEz_A^\varepsilon}{d\mu_\varepsilon}\big)d\mu_\varepsilon \\ &\leq \beta \limsup_{\varepsilon\to 0+} \int_U (1 + \big|\frac{dEz_A^\varepsilon}{d\mu_\varepsilon}\big|^p)d\mu_\varepsilon \leq C|\overline{U}|\,(1 + |A|^p). \end{aligned}$$

where the last inequalities follow from the growth condition (5.5.3) and the inequality (5.6.2). Since the functionals $F_\varepsilon(\cdot, U)$ are invariant by rigid motions, one can easily verify that

$$F'(u + Rx + c, U) = F'(u,U) \quad \text{and} \quad F''(u + Rx + c, U) = F''(u,U)$$

for any skew symmetric matrix $R \in M^{n\times n}$ and for any $c \in \mathbb{R}^n$. So that for any $A \in M^{n\times n}$, we have

$$F''(Ax, U) = F''\Big(\frac{A + A^t}{2}x, U\Big)$$

and the step is proved.

Step 2: we claim $F''(u,U) \leq C \int_U (1 + |Eu|^p)dx$ for any piecewise affine function u and for any open set $U \subseteq \Omega$ with $|\partial U| = 0$.

We write $u = \sum_{i=1}^N \chi_{U_i} u_i$, where $U_1, \ldots, U_N$ are disjoint open subsets of U such that $|U \setminus \bigcup_i U_i| = 0$, $|\overline{U}_i| = |U_i|$, and $u_i(x) = A_i x + c_i$ for some $A_i \in M^{n\times n}$ and $c_i \in \mathbb{R}^n$. For each i we set $u_i^\varepsilon(x) = z_{A_i}^\varepsilon(x) + c_i$, as in Step 1. We prove this step by induction. We use the fundamental estimate on $U_1 \cap U_2$ by setting

$$U' = U_2 \quad U = U^\eta =: \{x \in U,\, d(x, U_2) < \eta\} \quad \text{and} \quad V = U_1$$

where $\eta = \eta_\varepsilon$ will be determined later. Let $\sigma = 1$, $u = u_2^\varepsilon$ and $v = u_1^\varepsilon$. Then we obtain a cut-off function $\phi = \phi_\varepsilon$ between U_2 and U_2^η such that

$$\begin{aligned} F_\varepsilon(\phi_\varepsilon u_2^\varepsilon + (1-\phi_\varepsilon)u_1^\varepsilon, U_1 \cup U_2) \;\; &\le \;\; 2(F_\varepsilon(u_1^\varepsilon, U_1) + F_\varepsilon(u_2^\varepsilon, U_2^\eta)) \\ &\qquad + \frac{M}{\eta^p}\int_{U_1 \cap U_2^\eta} |u_2^\varepsilon - u_1^\varepsilon|^p dx + \mu_\varepsilon(U_1 \cap U_2^\eta)\,. \end{aligned}$$

The constant M is the one given by Proposition 5.6.1 with $\sigma = 1$. Since

$$\int_{U \cap U^\eta} |u_2 - u_1|^p dx \le C|Du|_\infty^p \eta^{p+1},$$

then

$$\lim_{\eta\to 0^+} \lim_{\varepsilon\to 0^+} \frac{1}{\eta^p}\int_{U_1 \cup U_2^\eta} |u_2^\varepsilon - u_1^\varepsilon|^p dx = 0.$$

So, for some sequence $\{\eta_\varepsilon\}$ converging to 0 as $\varepsilon \to 0$ we have

$$\lim_{\varepsilon\to 0^+} \frac{1}{\eta_\varepsilon^p}\int_{U_1 \cup U_2^{\eta_\varepsilon}} |u_2^\varepsilon(x) - u_1^\varepsilon|^p dx = 0.$$

If we define $w^\varepsilon := \phi_\varepsilon u_2^\varepsilon + (1-\phi^\varepsilon)u_1^\varepsilon$ we have $w^\varepsilon \to u$ in $L^p(U_1 \cup U_2, \mathbb{R}^n)$. Therefore

$$\begin{aligned} F''(u, U_1 \cup U_2) \;\; &\le \;\; \limsup_{\varepsilon\to 0^+} F_\varepsilon(w^\varepsilon, U_1 \cup U_2) \\ &\le \;\; 2\Big(\limsup_{\varepsilon\to 0^+} F_\varepsilon(u_1^\varepsilon, U_1) + \limsup_{\varepsilon\to 0^+} F_\varepsilon(u_2^\varepsilon, U_2)\Big) \\ &\le \;\; C\int_{U_1 \cup U_2} (1 + |Eu|^p) dx \end{aligned}$$

where the last inequality follows from step 1.
Now we proceed by induction, repeating at each step j the previous argument replacing U_1 by $U \cup \cdots \cup U_{j-1}$, U_2 by U_j, u_1^ε by w_{j-1}^ε constructed at the previous step and u_2^ε by u_j^ε.

Step 3: We conclude the proof of the proposition by using the lower semicontinuity of the functional $F''(\cdot, U)$, the density of piecewise affine functions in $W^{1,p}(\Omega, \mathbb{R}^n)$-strong and in $BD(\Omega)$ with respect to the so-called intermediate topology introduced in Chapter I. $\square$

The proof of the following proposition on the existence and the integral representation of the Γ-limit will follow the lines of [32, Proposition 4.4].

Proposition 5.6.4 *There exists a subsequence of (ε_j) (not relabeled) such that for all open subsets U of Ω there exists the Γ-limit*

$$\Gamma\text{-}\lim_{j \to +\infty} F_{\varepsilon_j}(u, U) = F(u, U)\,,$$

and there exists a function $\varphi : M^{n\times n} \to \mathbb{R}$ such that

$$F(u, U) = \int_U \varphi(Eu)dx$$

for all $u \in W^{1,p}(\Omega, \mathbb{R}^n)$ and all $U \subset \Omega$ with $|\partial U| = 0$. Furthermore, if the function f is supposed to be convex in the second variable, then the functional $F(\cdot, U)$ has the following integral representation

$$F(u, U) = \int_U \varphi\Big(\mathcal{E}u(x)\Big)dx + \int_U \varphi^\infty\Big(\frac{dE^s u}{d|E^s u|}\Big)d|E^s u| \tag{5.6.3}$$

for all $u \in BD(\Omega)$.

Proof. *Step 1:* Let us first prove the existence of the Γ-limit. Following the compactness property of Γ- convergence in spaces which satisfy the second axiom of countability (see Dal Maso [83, Theorem 8.5]) and by a diagonal procedure, we extract a subsequence (not relabeled) such that the Γ-limit

$$\Gamma(L^p)\text{-}\lim_{j \to +\infty} F_{\varepsilon_j}(u, U) = F(u, U) \tag{5.6.4}$$

exists for all $u \in L^p(\Omega, \mathbb{R}^n)$ and for all sets U in the countable family $\mathcal{R}_0$ of all finite unions of open rectangles of Ω with rational vertices. Exactly as in [70, Proposition 24.17] one can see from the propositions 5.6.1 and 5.6.3 that the set functions $F'(u,\cdot)$ and $F''(u,\cdot)$ are inner regular for every $u \in W^{1,p}(\Omega, \mathbb{R}^n)$ and also for every $u \in BD(\Omega)$. That is, for all $U \in \mathcal{A}(\Omega)$ with $|\partial U| = 0$

$$F''(u,U) = \sup\{F''(u,V) : \ V \subset\subset U, \ V \text{open}\},$$

$$F'(u,U) = \sup\{F'(u,V) : \ V \subset\subset U, \ V \text{open}\}.$$

Since the two set functions coincide in $\mathcal{R}_0$ by (5.6.4) and for every $V \subset\subset U \subset \Omega$ there exists $R \in \mathcal{R}_0$ such that $V \subset\subset R \subset\subset U$ we have the existence of the Γ-limit

$$F(u,U) = \Gamma\text{-}\lim_{j\to+\infty} F_{\varepsilon_j}(u,U)$$

for such u and U. So, the Γ-limit exists and it's also finite in $W^{1,p}(\Omega, \mathbb{R}^n)$ and in $BD(\Omega)$.

Step 2: By using Proposition 5.6.1, one can also repeat here the proof of [83, Proposition 18.3] to obtain by the De Giorgi-Letta measure criterion that $F(u,\cdot)$ is the restriction on the family of open sets U with $|\partial U| = 0$ of a Borel measure. From the integral representation theorems in $W^{1,p}(\Omega, \mathbb{R}^n)$ (see e.g. Buttazzo [73, Theorem 4.3.2]) we have the existence of $\varphi : M^{n\times n} \to \mathbb{R}$ such that

$$F(u,U) = \int_U \varphi(Du)dx$$

for all $u \in W^{1,p}(\Omega, \mathbb{R}^n)$ and for all $U \in \mathcal{A}(\Omega)$ with $|\partial U| = 0$. Note that the explicit independance of the function φ in x is due to the translation invariance of the functional F as in [83, Theorem 24.1].

By the invariance of the Γ-limits $F'(\cdot,U)$ and $F''(\cdot,U)$ with respect to rigid perturbations one can easily verify that for any $A \in M^{n\times n}$

$$\varphi(A) = \varphi\Big(\frac{A+A^t}{2}\Big),$$

so that we obtain

$$F(u,U) = \int_U \varphi(Eu)dx.$$

Step 3: The integral representation of the Γ*-limit in* $BD(\Omega)$ *when the function* f *is convex.*

It is well known that also the function φ is convex. Let us verify that the function φ satisfies the estimate $\varphi(A) \geq \alpha|A|$ for every $A \in M_s^{n\times n}$. In fact, the function φ is defined as

$$\varphi(A) = \limsup_{\rho\to 0^+} \frac{F(Ay, B_\rho(x))}{|B_\rho(x)|}$$

So, for $A \in M_s^{n\times n}$ and $u(y) = Ay$, there exists a sequence $u_\varepsilon \in LD^1_{\mu_\varepsilon}(\Omega)$ such that $u_\varepsilon \to u$ in $L^1(\Omega, \mathbb{R}^n)$ and

$$\limsup_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon, B_\rho(x)) \leq F(u, B_\rho(x)).$$

So,

$$\varphi(A) \geq \limsup_{\rho\to 0^+} \limsup_{\varepsilon\to 0^+} \frac{F_\varepsilon(u_\varepsilon, B_\rho(x))}{|B_\rho(x)|}$$

and hence, by the growth condition of f we get

$$\begin{aligned}\varphi(A) &\geq \alpha \limsup_{\rho\to 0^+} \limsup_{\varepsilon\to 0^+} \frac{|Eu_\varepsilon|(B_\rho(x))}{|B_\rho(x)|}\\ &\geq \alpha \limsup_{\rho\to 0^+} \frac{|Eu|(B_\rho(x))}{|B_\rho(x)|}\\ &= \alpha|A|.\end{aligned}$$

Now, we consider the functional defined in $L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ by

$$G(u) := \begin{cases} \int_\Omega \varphi(Eu)dx & \text{if } u \in \mathcal{C}^1(\Omega, \mathbb{R}^n)\\ +\infty & \text{otherwise,}\end{cases} \tag{5.6.5}$$

and we introduce

$$\overline{G}(u, U) := \inf\left\{\liminf_{h\to+\infty} G(u_h, U)\colon\ \{u_h\} \subset \mathcal{C}^1(\Omega, \mathbb{R}^n)\ u_h \to u \text{ in } L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^n)\right\}$$

the relaxed functional of G. Using the estimate (5.6.5) and following the lower semicontinuity and relaxation theorems for functionals of measures (see for instance

Goffman-Serrin [134], Buttazzo [73]), we obtain

$$\overline{G}(u,U) = \int_U \varphi\Big(\mathcal{E}u(x)\Big)dx + \int_U \varphi^\infty\Big(\frac{dE^s u}{d|E^s u|}\Big)d|E^s u|$$

for every $u \in BD(\Omega)$ (see for example Temam [191, section 5]). Therefore, by the lower semicontinuity of the Γ-limit and the fact that $F(\cdot,U) \leq G(\cdot,U)$ in $W^{1,1}(\Omega,\mathbb{R}^n)$, we obtain the inequality

$$F(u,U) \leq \int_U \varphi\Big(\mathcal{E}u(x)\Big)dx + \int_U \varphi^\infty\Big(\frac{dE^s u}{d|E^s u|}\Big)d|E^s u|$$

for any $u \in BD(\Omega)$. The reverse inequality is obtained by convolution. In fact, if $u_k := u \star \rho_k$ is a regularization of $u \in BD(\Omega)$ and

$$U_k := \{x \in U\colon\ d(x,\partial U) > \frac{1}{k}\},$$

then by the convexity of $F(\cdot,U)$, the Jensen inequality and the translation invariance of F, we get

$$\begin{aligned}
\int_{U_k} \varphi(Eu_k)dx &= F(u_k,U_k)\\
&= F\Big(\int_{B_{\frac{1}{k}}(0)} \rho_k(y)u(x-y)dy, U_k\Big)\\
&\leq \int_{B_{\frac{1}{k}}(0)} \rho_k(y)F(u^y,U_k)dy\\
&= \int_{B_{\frac{1}{k}}(0)} \rho_k(y)F(u,\tau_{-y}U_k)dy\\
&\leq F(u,U).
\end{aligned}$$

On the other hand, we also have

$$\lim_{k\to+\infty} F(u_k,U_k) = \overline{G}(u,U).$$

So, we get the inequality $\overline{G}(u,U) \leq F(u,U)$ and this achieves the proof of Proposition 5.6.4. $\square$

Proposition 5.6.5 (Homogenization formula) *For all $A \in M_s^{n\times n}$ there exists the limit in (5.5.7) and $\varphi(A) = f_{\mathrm{hom}}(A)$.*

The proof of this proposition will follow the lines of [32, Proposition 4.4] and also [70, Proposition 24.18].

Proof. In order to simplify the proof we may assume without loss of generality that $\mu\Big([0,1)^n \setminus (0,1)^n\Big) = 0$. Let us define for all $A \in M_s^{n\times n}$ and $k \in \mathbb{N}$,

$$g_k(A) := \inf\left\{\frac{1}{k^n}\int_{[0,k)^n} f\Big(x, \frac{dEu}{d\mu}\Big)d\mu\colon \begin{array}{l} u \in LD^p_{\mu,\mathrm{loc}}(\mathbb{R}^n) \text{ and} \\ u - Ax \text{ is } k-\text{periodic}\end{array}\right\}.$$

Now fix $A \in M_s^{n\times n}$ and let $u \in LD^p_{\mu,\mathrm{loc}}(\mathbb{R}^n)$ with $u - Ax$ k-periodic and having mean value 0 in $(0,k)^n$. Then the sequence $u_j(x) := \varepsilon_j u(x/\varepsilon_j)$ converges to Ax in $L^p_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$, so that we have

$$\begin{aligned}\varphi(A) = F(Ax, (0,1)^n) &\leq \liminf_{j\to+\infty} F_{\varepsilon_j}(u_j, (0,1)^n) \\ &= \frac{1}{k^n}\int_{(0,k)^n} f\Big(x, \frac{dEu}{d\mu}\Big)d\mu\end{aligned}$$

where the last equality follows from the k-periodicity of the function $x \to f\Big(x, \frac{dEu}{d\mu}\Big)$. Hence,

$$\varphi(A) \leq \liminf_{k\to+\infty} g_k(A).$$

Let us now consider the sequence $\{w_j\}$ which converges to Ax in $L^p_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ with $\varphi(A) = \lim_{j\to+\infty} F_{\varepsilon_j}(w_j, (0,1)^n)$. Let $\sigma > 0$ and ε_σ be given by Proposition 5.6.1. Let $T_j = 1/\varepsilon_j$; $u_j(x) := T_j w_j(x/T_j)$ and $K_j = [T_j] + 1$ where $[T_j]$ denotes the integer part of T_j. For j large enough and $N > 4$ we can apply the proposition 5.6.1 with $\varepsilon = 1$; $u = u_j$; $v = z_A$; $U = (0, T_j)^n$; $V = (0, K_j)^n \setminus (2T_j/N, T_j - 2T_j/N)^n$ and $U' = (T_j/N, T_j - T_j/N)^n$. Then we get

$$\begin{aligned}&F_1\Big(\phi u + (1-\phi)v, (0, K_j)^n)\Big) = F_1\Big(\phi u + (1-\phi)v, U' \cup V\Big) \\ &\leq (1+\sigma)\Big(F_1(u, U) + F_1(v, V)\Big) + MN^pT_j^{-p}\int_{(U\cap V)\setminus U'} |u - v|^p dx \\ &\qquad +\sigma\mu\Big((u \cap V) \setminus U'\Big).\end{aligned}$$

Now if we extend the function $\phi u + (1-\phi)v - Ax$ by K_j-periodicity we get

$$\begin{aligned} K_j^n g_{K_j}(A) &\leq F_1\Big(\phi u + (1-\phi)v, (0,K_j)^n)\Big) \\ &\quad (1+\sigma)\Big(F_1(u_j,(0,T_j)^n) + F_1(z_A,V)\Big) \\ &\qquad + MN^pT_j^{-p}\int_{(0,T_j)^n\setminus(T_j/N,T_j-T_j/N)^n} |u_j - z_A|^p dx \\ &\qquad\quad + \sigma\mu\Big((u\cap V)\setminus U'\Big) \\ &\leq (1+\sigma)\Big(T_j^n F_{\varepsilon_j}(w_j,(0,1)^n) + C\frac{K_j^n}{N}\Big(1+|A|^p\Big)\Big) \\ &\qquad + MN^pT_j^n\int_{(0,1)^n} |w_j - z_{A,j}(x)|^p dx + \sigma K_j^n \end{aligned}$$

where $z_{A,j}(x) = T_j z_A(T_j x)$. Since $z_{A,j}$ converges to Ax in $L^p_{\rm loc}(\mathbb{R}^n,\mathbb{R}^n)$ we obtain

$$\lim_{j\to+\infty}\int_{(0,1)^n} |w_j(x) - z_{A,j}(x)|^p dx = 0$$

so that by dividing the estimate above by K_j^n, and letting $j\to\infty$, $\sigma\to 0$ and $N\to\infty$ we get

$$\limsup_{j\to\infty} g_{K_j}(A) \leq \lim_{j\to\infty} F_{\varepsilon_j}(w_j,(0,1)^n) = \varphi(A).$$

Therefore, we have

$$\varphi(A) = \liminf_{k\to\infty} g_k(A) = \lim_{j\to\infty} g_{K_j}(A).$$

So by repeating the reasoning above for a sequence ε_j such that

$$\lim_{j\to\infty} g_{K_j}(A) = \limsup_{k\to\infty} g_k(A)$$

we get $\varphi(A) = \lim\limits_{k\to\infty} g_k(A)$ and this achieves the proof of the proposition. $\square$

We conclude the proof of Theorem 5.5.1 by proving in the next proposition the coerciveness of the Γ-limit in $W^{1,p}(\Omega,\mathbb{R}^n)$ for $p>1$ and also in $BD(\Omega)$ for $p=1$.

Proposition 5.6.6 *We have the following facts:*

(i) $F(u,\Omega) = +\infty$ *if* $u \in L^p(\Omega, \mathbb{R}^n) \setminus W^{1,p}(\Omega, \mathbb{R}^n)$ *whenever* $p > 1$.

(ii) $F(u,\Omega) = +\infty$ *if* $u \in L^1(\Omega, \mathbb{R}^n) \setminus BD(\Omega)$ *whenever* $p = 1$.

Proof. By the growth assumption (5.5.3) on f and a comparison argument, it is enough to prove the proposition for $f(A) = |A|^p$. So, the function φ is convex and also p-homogeneous. Let us first verify that for all $U \subset\subset \Omega$

$$\limsup_{j\to+\infty} \int_U \varphi(Eu_j)dx \le F'(u,\Omega) \tag{5.6.6}$$

where $u_j := u \star \rho_j$ is a regularization of u. In fact, noting that $F(\cdot, R)$ is convex for any $R \in \mathcal{R}_0$, by Jensen's inequality we have

$$F(u_j, R) \le \int_{B_{\frac{1}{j}}(0)} \rho_j(y) F(\tau_y u, R) dy$$

where $(\tau_y u)(x) := u(x-y)$. Now, using the definition of Γ-liminf one can prove that, for R, $R' \in \mathcal{R}_0$ such that $U \subset\subset R \subset\subset R' \subset\subset \Omega$ we have $F(\tau_y u, R \le F'(u, \tau_{-y}R')$ for any $y \in B_{\frac{1}{j}}(0)$, recalling that $\tau_{-y}R' = \{x \in \Omega \colon\ x-y \in R'\}$. Then the inequality (5.6.6) follows.

(i) We suppose $p > 1$. It is sufficient to prove that

$$\varphi(A) \ge C|A|^p \quad \forall A \in M^{n\times n}.$$

In fact, this would imply for any $u \in L^p(\Omega, \mathbb{R}^n) \setminus W^{1,p}(\Omega, \mathbb{R}^n)$ that

$$\begin{aligned} F'(u,\Omega) &\ge \liminf_{j\to+\infty} \int_U \varphi(Eu_j)dx \\ &= \liminf_{j\to+\infty} \int_U \varphi(Du_j)dx \\ &\ge C \liminf_{j\to+\infty} \int_U |Du_j|^p dx \end{aligned}$$

and by the arbitrariness of U, we get $F'(u,\Omega)=+\infty$. If $\varphi(A)\neq 0$ for $A\neq 0$, then by the p-homogeneity of φ we get

$$\varphi(A)=\varphi\Big(\frac{A}{|A|}|A|\Big)\geq C|A|^p$$

where $C=\inf_{|B|=1}\varphi(B)>0$. So we need only to prove that $\varphi(A)\neq 0$ for $A\neq 0$. To this aim, let $A\neq 0$ be a symmetric matrix such that $u_\varepsilon\to Ax$ in $L^p(\Omega,\mathbb{R}^n)$ and $F_\varepsilon\Big(u_\varepsilon,(0,1)^n\Big)\to\varphi(A)$. If $\varphi(A)=0$, then by the Poincaré type inequality for BD functions in Chapter I, we obtain for a rigid motion R_{u_ε}

$$\begin{aligned}\Big(\int_{(0,1)^n}|u_\varepsilon-R_{u_\varepsilon}|^{\frac{n}{n-1}}\Big)^{\frac{n-1}{n}} &\leq C|Eu_\varepsilon|(0,1)^n\\ &\leq C\Big(\mu_\varepsilon((0,1)^n)\Big)^{\frac{p-1}{p}}\Big(\int_{(0,1)^n}\Big|\frac{dEu_\varepsilon}{d\mu_\varepsilon}\Big|^p d\mu_\varepsilon\Big)^{\frac{1}{p}}\\ &= C\Big(\mu_\varepsilon((0,1)^n)\Big)^{\frac{p-1}{p}}\Big(F_\varepsilon\Big(u_\varepsilon,(0,1)^n\Big)\Big)^{\frac{1}{p}}\end{aligned}$$

where the second inequality follows by the Hölder's inequality. Then we obtain that $R_{u_\varepsilon}\to Ax$ in $L^1((0,1)^n,\mathbb{R}^n)$ and this contradicts the fact that A is symmetric.
(ii) if $p=1$, then for $u\in L^1(\Omega,\mathbb{R}^n)\setminus BD(\Omega)$, we have $|Eu|(\Omega)=+\infty$. Let $\Omega'\subset\subset\Omega$, then we get from (5.6.5) and (5.6.6) that

$$F'(u,\Omega)\geq|Eu|(\Omega'),$$

by the arbitrariness of Ω' we obtain $F(u,\Omega)=+\infty$ and this achieves the proof of the proposition. □

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