Long-time behavior of the mean curvature flow with periodic forcing

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Abstract

We consider the long-time behavior of the mean curvature flow in heterogeneous media with periodic fibrations, modeled as an additive driving force. Under appropriate assumptions on the forcing term, we show existence of generalized traveling waves with maximal speed of propagation, and we prove the convergence of solutions to the forced mean curvature flow to these generalized waves.

1 Introduction

We are interested in the long-time behavior of the mean curvature flow in a periodic heterogeneous medium. The evolution law can be written as a forced mean curvature flow

$$v = \kappa - g$$

where v denotes the inward normal velocity of the evolving hypersurface, κ its mean curvature (with the convention that κ is positive on convex sets) and g is a periodic forcing term. In our model, we assume that the hypersurfaces are graphs with respect to a fixed hyperplane and that the forcing term g does not depend on the variable orthogonal to such hyperplane (fibered medium). Under these assumptions the evolving hypersurface coincides with the graph of the solution to the Cauchy problem

$$\begin{cases} u_t = \sqrt{1+|Du|^2}\,\mathrm{div}\left(\dfrac{Du}{\sqrt{1+|Du|^2}}\right) + g\sqrt{1+|Du|^2} & \text{in } (0,+\infty)\times\mathbb{R}^n \\ u(0,\cdot) = u_0 & \text{in } \mathbb{R}^n. \end{cases} \tag{1}$$

We are particularly interested in the asymptotic behavior as $t \to +\infty$ of solutions to (1), where the initial data u_0 and the forcing term g are assumed to be Lipschitz continuous and $\mathbb{Z}^n$-periodic.
The expected result is that, under appropriate assumptions on g, there exists a unique constant $c \in \mathbb{R}$ and a periodic function ψ such that

$$u(t,y) - ct - \psi(y) \to 0, \qquad \text{as } t \to +\infty, \text{ uniformly in } \mathbb{R}^n.$$

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This is a result on the asymptotic stability of special solutions to (1), called traveling wave solutions, which are of the form $\psi + ct$. The constant c and the function ψ are respectively the propagation speed and the profile of the wave.
The first question we address in Section 3 of this paper is about existence of traveling wave solutions to (1). We provide a construction of such solutions using a variational approach developed in [23] (see also [24]). In particular, our solutions are critical points of appropriate functionals, which are exponentially weighted area functionals with a volume term, depending on the speed of propagation c. Exploiting this variational structure, we show existence of traveling waves under rather weak assumptions on the forcing term g, i.e.

$$\exists\, A \subseteq (0,1)^n \text{ s.t. } \int_A g(y)\,dy \;>\; \mathrm{Per}(A, \mathbb{T}^n)$$

where $\mathrm{Per}(A, \mathbb{T}^n)$ is the periodic perimeter of A (see Section 2). Notice that, if $\int_{(0,1)^n} g > 0$, then the previous condition holds true by taking $A = (0,1)^n$.
As our solutions are in general not globally defined, we call them *generalized traveling waves*. In Propositions 3.7 and 3.10 we discuss the regularity of these solutions and of their support. Moreover, in Section 3.1 we list some stronger conditions on the forcing term, involving only the oscillation and the norm of g, under which we show existence of classical (i.e. globally defined) traveling waves (see Proposition 3.15).
We point out that the variational method selects the *fastest* traveling waves for (1) which are bounded above, in particular it is uniquely defined the speed of propagation $\overline{c}$ of such waves and it holds $\overline{c} \geq \int_{(0,1)^n} g$ (see Corollary 3.2).
We recall that the problem of existence of classical traveling waves for the forced mean curvature flow has already been considered in the literature, under different assumptions on the forcing term [20, 15, 11]. We also mention [22], where the authors construct V-shaped traveling waves in the whole space for a constant forcing term (see also [26, 9, 8] for similar results in the planar case). The construction of the traveling fronts in these papers relies mainly on maximum principle type arguments, while we use here a variational approach.
The second question of interest is about the convergence, as $t \to +\infty$, of the solution to (1) to a traveling wave solution. We point out that the long-time behavior of solutions of parabolic problems using viscosity solutions type arguments has been extensively considered in the literature: see [25] and [7] for the case of semilinear and quasilinear parabolic problems in periodic environments, [14] where the author considers uniformly parabolic operators in bounded domains with Neumann boundary conditions, and [6] for the case of viscous Hamilton-Jacobi equations in bounded domains with Dirichlet boundary conditions. However, none of these results applies to mean curvature type equations such as (1).
In Section 4 we prove a convergence result under the assumption that there exists a global traveling wave solution. In particular, in Corollary 4.9 we show that the solution $u(t,y)$ to (1) satisfies

$$u(t,y) - \overline{c}t \to \psi(y) \qquad \text{in } \mathcal{C}^{1+\alpha}(\mathbb{R}^n), \text{ as } t \to +\infty,$$

where $\psi + \overline{c}t$ is the traveling wave, which in this case is unique up to an additive constant.
In the general case, we obtain a weaker convergence result. First, in Proposition 4.6 we describe the asymptotic behavior as $t \to +\infty$ of the maximum of the function $u(t,\cdot)$. Namely,

letting $Q := (0,1)^n$, we show that there exists a constant $K > 0$ such that

$$\min_Q u_0 + \bar{c}t \le \max_Q u(t,y) \le \bar{c}t + K + \frac{\log(1+t)}{\bar{c}}.$$

Then, in Theorem 4.7 we show that, along a subsequence $t_n \to +\infty$,

$$u(t_n, y) - \max_Q u(t_n, \cdot) \longrightarrow \begin{cases} \psi(y) & \text{locally in } \mathcal{C}^{1+\alpha}(E) \\ -\infty & \text{locally uniformly in } Q \setminus \overline{E} \end{cases}$$

for all $\alpha \in (0,1)$, where $\psi + \bar{c}t$ is a generalized traveling wave supported in $E \subset Q$.
We point out that the proof of the convergence result, as well as the proof of existence of generalized waves, essentially uses variational methods, rather than maximum principle based argumets.

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2 Notation and preliminary results

We refer to [2] for a general introduction to functions of bounded variation and sets of finite perimeter. Letting $Q := (0,1)^n$, it is a classical result that any $u \in BV(Q)$ admits a trace u^Q on ∂Q (see e.g. [2, Thm. 3.87]). Let $\partial_0 Q := \partial Q \cap \{y : \prod_{i=1}^n y_i = 0\}$ and let $\sigma : \partial_0 Q \to \partial Q$ be the function $\sigma(y) := y + \sum_{i=1}^n \lambda_i(y) e_i$, where $\lambda_i(y) = 1$ if $y_i = 0$ and $\lambda_i(y) = 0$ otherwise. We consider the space $BV_{\text{per}}(Q)$ of functions which have periodic bounded variation in Q, where the periodic total variation of $u \in BV(Q)$ is defined as

$$|Du|_{\text{per}}(Q) := |Du|(Q) + \int_{\partial_0 Q} |u^Q(y) - u^Q(\sigma(y))| \, d\mathcal{H}^{n-1}(y). \tag{2}$$

The space $BV_{\text{per}}(Q)$ is the space $BV(Q)$ endowed with the norm

$$\|u\|_{BV_{\text{per}}(Q)} := \|u\|_{L^1(Q)} + |Du|_{\text{per}}(Q).$$

Observe that $BV_{\text{per}}(Q)$ coincides with $BV(\mathbb{T}^n)$, where $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$ is the n-dimensional torus. For every $E \subseteq Q$ we define the periodic perimeter of E as

$$\text{Per}(E, \mathbb{T}^n) := |D\chi_E|_{\text{per}}(Q) \tag{3}$$

where χ_E is the characteristic function of E. We recall the isoperimetric inequality [2]:

Proposition 2.1. *There exists $C_n > 0$ such that*

$$\text{Per}(E, \mathbb{T}^n) \ge C_n |E|^{\frac{n-1}{n}} \tag{4}$$

for all $E \subseteq Q$ of finite perimeter and such that $|E| \le 1/2$.

Remark 2.2. Notice that $C_1 = 2$.

In this paper we always make the following regularity assumption on the initial datum and on the forcing term:

$$u_0,\, g \text{ are Lipschitz continuous and } [0,1]^n\text{-periodic.} \tag{5}$$

Using the comparison principle [4] and (5), we get that there exists a unique continuous solution u to (1) with periodic boundary conditions. Moreover, this solution is locally Lipschitz continuous [13, 16] and hence smooth for all positive times, due to the regularity theory for parabolic problems.

Theorem 2.3. *Under assumption* (5)*, problem* (1) *admits a unique solution*

$$u \in \mathcal{C}([0,+\infty)\times Q)\cap C^{1+\frac{\alpha}{2},2+\alpha}((0,T]\times Q)$$

for every $\alpha \in (0,1)$ *and* $T>0$*, with periodic boundary conditions on* ∂Q*. Moreover*

$$u_t \in L^2([0,+\infty)\times Q) \quad \text{and} \quad Du(t,x)\in L^\infty([0,T]\times Q) \quad \text{for every } T>0.$$

We need another condition on the forcing term g, in order to prove existence of generalized traveling wave solutions to (1), namely we assume that

$$\exists A \subseteq Q \text{ such that } \int_A g(y)\,dy \;>\; \mathrm{Per}(A,\mathbb{T}^n). \tag{6}$$

Note that condition (6) implies $\max_Q g > 0$, and is fullfilled for instance if $\int_Q g > 0$.

Remark 2.4. In [5] (see also [12]) we considered a sort of complementary condition to (6). Indeed it is proved that, if g has zero average and there exists $\delta \in (0,1)$ such that

$$\int_A g(y)\,dy < \delta\,\mathrm{Per}(A,\mathbb{T}^n) \qquad \forall A\subseteq Q, \tag{7}$$

then there exists a periodic stationary solution of (1).

We conclude this section by recalling a classical result about the regularity of hypersurfaces of prescribed bounded mean curvature [21, Thm. 4.1], [27, Thm. 1].

Theorem 2.5. *Let* K *be a Caccioppoli set with bounded prescribed mean curvature* $A(x) \in L^\infty$*,* $x \in \partial K$*. Then* $\mathcal{H}^k(\partial K \setminus \partial^\star K) = 0$ *for every* $k > n-8$*, and there exists* $\delta > 0$*, such that for every* $x \in \partial^\star K$ *we get that* $\partial K \cap B(x,\delta) = \partial^\star K \cap B(x,\delta)$ *and* $\partial K \cap B(x,\delta)$ *is a* $\mathcal{C}^{1+\alpha}$ *hypersurface for any* $\alpha \in (0,1)$*. Moreover, letting* $(K_n)_n$ *be a sequence of Caccioppoli sets such that:*

i) every K_n *is a locally minimizer of the functional* $\mathrm{Per}(V)+\int_V A_n(y)dy$*, with* $\|A_n\|_\infty \le A$ *independent of* n*,*

ii) K_n *converges to* K_∞ *locally in the* L^1*-topology,*

and letting $x_n \in \partial K_n$*, with* $x_n \to x_\infty$ *as* $n \to +\infty$*, we have* $x_\infty \in \partial K_\infty$*. If* $x_\infty \in \partial^\star K_\infty$*, then* $x_n \in \partial^\star K_n$ *for all* $n > n_0$*, and the unit outward normal to* $\partial^\star K_n$ *at* x_n *converges to the unit outward normal to* $\partial^\star K_\infty$ *at* x_∞*.*

3 Existence and regularity of generalized traveling waves

We now show existence of special solutions to (1), which we call *generalized traveling waves.* They are solutions of the form $\psi(x) + \bar{c}t$, where the graph of ψ is called the profile of the traveling wave and $\bar{c}$ is called the traveling speed. Observe that to prove the existence of a traveling wave solution it is sufficient to determine $c \in \mathbb{R}$ such that the equation

$$-\mathrm{div}\left(\frac{D\psi}{\sqrt{1+|D\psi|^2}}\right) = g(y) - \frac{c}{\sqrt{1+|D\psi|^2}} \tag{8}$$

admits a $\mathbb{Z}^n$-periodic solution $\psi : \mathbb{R}^n \to \mathbb{R}$. In the following we will show that it is always possible to define a unique traveling speed $\bar{c}$ for the problem under our assumption (6) on the forcing, but in general, the previous equation does not admit a global solution. We will prove that there exists a maximal set $E \subseteq Q$, which is a sufficiently regular domain, and a function $\psi : Q \to [-\infty, +\infty)$ (which is defined up to additive constants) such that $E = \{\psi > -\infty\}$, $\psi \in \mathcal{C}^{2+\alpha}(E)$ and solves

$$-\mathrm{div}\left(\frac{D\psi}{\sqrt{1+|D\psi|^2}}\right) = g(y) - \frac{\bar{c}}{\sqrt{1+|D\psi|^2}}, \qquad \text{in } E \tag{9}$$

with the boundary conditions

$$\psi(x) \to -\infty \quad \text{as } \mathrm{dist}(x, \partial E) \to 0 \qquad \text{for } \mathcal{H}^{n-1} - \text{a.e. } x \in \partial E. \tag{10}$$

Moreover we will show that the solutions we construct satisfy also a stronger boundary condition, more natural in viscosity solutions theory, say

$$\text{for every } \phi \in \mathcal{C}^1_{\mathrm{per}}(\overline{Q}),\ \phi - \psi \text{ achieves its minimum in } E. \tag{11}$$

First of all we note that the equation (8) can be interpreted, for any $c > 0$, as the Euler-Lagrange equation associated to the functional

$$F_c(\psi) = \int_Q e^{c\psi(y)}\left(\sqrt{1+|D\psi(y)|^2} - \frac{g(y)}{c}\right)dy \qquad \psi \in \mathcal{C}^1_{\mathrm{per}}(Q). \tag{12}$$

Using the change of variable $\Psi(y) := \frac{e^{c\psi(y)}}{c}$, we can rewrite the functional F_c as

$$F_c(\psi) = G_c(\Psi) := \int_Q \sqrt{c^2\Psi^2(y) + |D\Psi(y)|^2} - g(y)\Psi(y)dy, \tag{13}$$

which can extended as a lower semicontinuous functional on $BV_{\mathrm{per}}(Q)$, see [2]. Using G_c, we can extend the functional F_c to all measurable functions $\psi : Q \to [-\infty, 0)$ such that $e^{c\psi(y)} \in BV_{\mathrm{per}}(Q)$ (where we use the notation $e^{-\infty} = 0$) by setting

$$F_c(\psi) := G_c\left(\frac{e^{c\psi(y)}}{c}\right). \tag{14}$$

In particular, for all such ψ the following representation formula holds (cfr. [18, Sec. 12]):

$$\begin{aligned} F_c(\psi) &= \sup\left\{\int_Q e^{c\psi(y)}\left(\frac{\mathrm{div}\phi'}{c}+\phi_{n+1}\right)dy : (\phi',\phi_{n+1})\in\mathcal{C}^1_{\mathrm{per}}(Q;\mathbb{R}^{n+1}),\ |\phi'|^2+\phi_{n+1}^2\le 1\right\} \\ &\quad -\int_Q \frac{e^{c\psi(y)}}{c}g(y)\,dy \end{aligned} \tag{15}$$

which can be easily checked on smooth functions, and then extends by relaxation to all ψ such that $e^{c\psi(y)}\in BV_{\mathrm{per}}(Q)$.

Proposition 3.1. *Under the standing assumption* (6) *there exists a unique constant* $\overline{c}>0$, *with* $\int_Q g\le\overline{c}\le\max_Q g$, *such that*

- *if* $0<c<\overline{c}$, *then* $\inf\{G_c(\Psi)\mid\Psi\in BV_{\mathrm{per}}(Q),\ \Psi\ge 0\}=-\infty$,
- *if* $c>\overline{c}$, *then* $\inf\{G_c(\Psi)\mid\Psi\in BV_{\mathrm{per}}(Q),\ \Psi\ge 0\}=0$, *and* $G_c(\Psi)>0$ *for every* $\Psi\not\equiv 0$,
- $\min\{G_{\overline{c}}(\Psi)\mid\Psi\in BV_{\mathrm{per}}(Q),\ \Psi\ge 0\}=0$, *and there exists* $\Psi\not\equiv 0$ *s.t.* $G_{\overline{c}}(\Psi)=0$.

Proof. As G_c is positively one-homogeneous, it follows that $\inf_{\Psi\in BV_{\mathrm{per}}(Q),\ \Psi\ge 0}G_c(\Psi)$ can be either 0 or $-\infty$. By definition of G_c, if $c>\max_Q g$, then $G_c(\Psi)\ge 0$ for every $\Psi\ge 0$, so that $\inf_{\Psi\ge 0}G_c(\Psi)=0$. On the other hand, take $\Psi=\chi_A$, where χ_A is the characteristic function of the set A appearing in (6). If $A\subset Q$, then by condition (6) there exists $k>1$ such that

$$G_c(\chi_A)=\mathrm{Per}(A,\mathbb{T}^n)+c|A|-\int_A g<-(k-1)\mathrm{Per}(A,\mathbb{T}^n)+c|A|.$$

Then, choosing $0<c<(k-1)\mathrm{Per}(A,\mathbb{T}^n)/|A|$, we obtain that $G_c(\chi_A)<0$, which implies $\inf_{\Psi\ge 0}G_c(\Psi)=-\infty$. Moreover if $\int_Q g>0$, then

$$G_c(\chi_Q)<0 \qquad \text{for every } 0<c<\int_Q g. \tag{16}$$

For $c>0$ we consider the constrained problem

$$\inf\left\{G_c(\Psi)\mid\Psi\in BV_{\mathrm{per}}(Q),\ \Psi\ge 0,\ \int_Q g\Psi=1\right\}. \tag{17}$$

By the direct method of the Calculus of Variations, one can easily show that this problem admits a (possibly nonunique) minimizer Ψ_c [18]. We define the function minimum value as

$$c\mapsto\mu_c:=G_c(\Psi_c)$$

and we claim that this function is continuous and strictly increasing. Notice that, by minimality of Ψ_c, we have

$$\int_Q c\Psi_c dy\le G_c(\Psi_c)+\int_Q g\Psi_c dy=\mu_c+1. \tag{18}$$

The monotonicity of μ_c is due to the fact that $G_c(\Psi_c)$ is increasing as a function of c. To prove the continuity, we follow the same argument as in [24, Prop. 4.1]. For $c_1 < c_2$, we get

$$
\begin{aligned}
0 &< G_{c_2}(\Psi_{c_2}) - G_{c_1}(\Psi_{c_1}) \le G_{c_2}(\Psi_{c_1}) - G_{c_1}(\Psi_{c_1}) \\
&= \int_Q \sqrt{{c_2}^2\Psi_{c_1}^2 + |D\Psi_{c_1}|^2} - \sqrt{{c_1}^2\Psi_{c_1}^2 + |D\Psi_{c_1}|^2} \\
&= (c_2 - c_1) \int_Q \frac{c(y)\Psi_{c_1}^2}{\sqrt{c(y)^2\Psi_{c_1}^2 + |D\Psi_{c_1}|^2}} dy \\
&\le (c_2 - c_1) \int_Q \Psi_{c_1} dy \le \frac{c_2 - c_1}{c_1} (\mu_{c_1} + 1)
\end{aligned}
$$

for some $c(y) \in [c_1, c_2]$, where the last inequality follows from (18). Since the value function is continuous and strictly increasing, it is possible to define $\overline{c} > 0$ as the unique constant for which $\mu_{\overline{c}} = G_{\overline{c}}(\Psi_{\overline{c}}) = 0$. From (16) it follows $\overline{c} \ge \int_Q g$.
Observe that, due to the constraints, $\Psi_{\overline{c}} \not\equiv 0$ and, due to the positive one-homogeneity of G_c, $k\Psi_{\overline{c}}$ is also a minimizers of $G_{\overline{c}}$ for every $k \ge 0$.
Finally, observe that necessarily if $c > \overline{c}$ and $\Psi \not\equiv 0$, then $G_c(\Psi) > 0$. On the contrary, if $G_c(\Psi) = 0$ and $\Psi \not\equiv 0$, then $\int_Q g\Psi = \lambda > 0$. So $\lambda^{-1}\Psi$ would be a minimizer to (17), and $\mu_c = 0$, for $c > \overline{c}$, in contradiction with the monotonicity of the value function. □

Recalling (14), it is immediate to state the analogous result for the functional F_c.

Corollary 3.2. *There exists a unique constant $\overline{c} > 0$ with $\int_Q g \le \overline{c} \le \max_Q g$ such that*

- *if $0 < c < \overline{c}$, then $\inf\{F_c(\psi) \mid e^{c\psi} \in BV_{\rm per}(Q)\} = -\infty$,*
- *if $c > \overline{c}$, then $\inf\{F_c(\psi) \mid e^{c\psi} \in BV_{\rm per}(Q)\} = 0$, and $F_c(\psi) > 0$ for all $\psi \not\equiv -\infty$,*
- *there exists $\psi : Q \to [-\infty, +\infty)$ such that $\psi \not\equiv -\infty$, $e^{\overline{c}\psi} \in BV_{\rm per}(Q)$ and $F_{\overline{c}}(\psi) = 0$.*

Remark 3.3. Notice that Proposition 3.1 and Corollary 3.2, assuring the existence of generalized traveling waves solutions, requires only $g \in L^\infty(Q)$.

We now analyze the regularity of the minima of $F_{\overline{c}}$ (or equivalently of $G_{\overline{c}}$).
We first give a geometric representation of the functional F_c (cfr. [18, Thm. 14.6]). Given $c > 0$ and $\Sigma \subset Q \times \mathbb{R}$ we define a weighted perimeter

$$
\begin{aligned}
\mathrm{Per}_c(\Sigma, \mathbb{T}^n \times \mathbb{R}) := \sup \Big\{ \int_\Sigma e^{cz} \left(\mathrm{div}\phi(y,z) + c\phi_{n+1}(y,z) \right) dydz : \\
\phi \in \mathcal{C}^1_{\rm per}(Q \times \mathbb{R}; \mathbb{R}^{n+1}),\ |\phi|^2 \le 1 \Big\}.
\end{aligned}
\tag{19}
$$

Notice that, for all $\Sigma \subset Q \times \mathbb{R}$ of locally finite perimeter we have

$$
\mathrm{Per}_c(\Sigma, \mathbb{T}^n \times \mathbb{R}) = \int_{\partial^*\Sigma} e^{cz}\, d\mathcal{H}^n + \int_{\mathbb{R}} e^{ct} \int_{\partial_0 Q} |\chi^Q_\Sigma(y) - \chi^Q_\Sigma(\sigma(y))|\, d\mathcal{H}^{n-1}(y)\, dt
$$

where σ is as in (2).

Proposition 3.4. *Let $\psi : Q \to [-\infty, +\infty)$ be such that $e^{c\psi} \in BV_{\rm per}(Q)$. Then*

$$F_c(\psi) = \mathcal{F}_c(\Sigma_\psi) := \mathrm{Per}_c(\Sigma_\psi, \mathbb{T}^n \times \mathbb{R}) - \int_{\Sigma_\psi} e^{cz} g(y)\, dydz \tag{20}$$

where $\Sigma_\psi := \{(y, z) \in Q \times \mathbb{R} \mid z < \psi(y)\}$ is the epigraph of ψ.

Proof. By exploiting formula (15) and the definition of Per_c in (19), it is possible to check that $F_c(\psi) \le \mathcal{F}_c(\Sigma_\psi)$. For the reverse inequality, we observe first of all that (20) holds on smooth functions $\psi \in \mathcal{C}^1_{\rm per}(Q)$ and then the inequality extends to all ψ's by relaxation. For a similar argument see [18, Thm. 14.6]. □

Lemma 3.5. *Let $\psi : Q \to [-\infty, +\infty)$ be a non trivial minimizer of $F_{\bar c}$, then the epigraph Σ_ψ of ψ is a minimizer, under compact perturbations, of the functional $\mathcal{F}_{\bar c}$ defined in* (20)*.*

Proof. We reason as in [18, Thm. 14.9]. Given $F \subset Q \times \mathbb{R}$ such that $\int_F e^{\bar c z} dydz < +\infty$, we consider $\psi_F : Q \to [-\infty, +\infty)$ be such that

$$\frac{e^{\bar c \psi_F(y)}}{\bar c} = \int_{-\infty}^{\psi_F(y)} e^{\bar c z} dz = \int_{F_y} e^{\bar c z} dz \qquad \text{for a.e. } y \in Q,$$

where $F_y := \{z \in \mathbb{R} :\ (y, z) \in F\}$. Observe that, by definition, $e^{\bar c \psi_F} \in BV(Q)$ and

$$\int_F e^{\bar c z} g(y)\, dydz = \int_Q e^{\bar c \psi_F(y)} \frac{g(y)}{\bar c}\, dy. \tag{21}$$

Moreover, by definition of $\mathrm{Per}_{\bar c}$, for all $\phi = (\phi', \phi_{n+1}) \in \mathcal{C}^1_{\rm per}(Q; \mathbb{R}^{n+1})$ we have

$$\mathrm{Per}_{\bar c}(F, \mathbb{T}^n \times \mathbb{R}) \ge \int_F e^{\bar c z} \left(\mathrm{div}\phi' + \bar c \phi_{n+1}\right) dydz = \int_Q e^{\bar c \psi_F} \left(\frac{\mathrm{div}\phi'}{\bar c} + \phi_{n+1}\right) dy. \tag{22}$$

By taking the supremum over all ϕ's in (22), and using the representation formula (15) and (21), we then get

$$\mathcal{F}_{\bar c}(F) \ge F_{\bar c}(\psi_F) \ge F_{\bar c}(\psi) = \mathcal{F}_{\bar c}(\Sigma_\psi)$$

where the last equality follows from Proposition 3.4, thus proving the claim. □

Notice that if Σ is a minimizer of $\mathcal{F}_{\bar c}$, then $\Sigma + (0, z)$ is also a minimizer for all $z \in \mathbb{R}$, that is, the class of minimizers is invariant by vertical shifts. Reasoning as in [18, Prop. 5.14] (see also [1]) one can prove a density estimate for minimizers of $\mathcal{F}_{\bar c}$.

Lemma 3.6. *There exist constants $\lambda, r_0 > 0$, depending only on n and $\|g\|_\infty$, such that for all minimizers Σ of $\mathcal{F}_{\bar c}$, $x \in \Sigma$ and $r \in (0, r_0)$ the following density estimate holds:*

$$|\Sigma \cap B_r(x)| \ge \lambda\, r^{n+1}. \tag{23}$$

Proposition 3.7. *Let $\psi : Q \to [-\infty, +\infty)$ be a non trivial minimizer of $F_{\bar c}$. Then $\Gamma_\psi := \partial \Sigma_\psi$ is a $\mathcal{C}^{2+\alpha}$ hypersurface for all $\alpha < 1$, out of a closed singular set $S_\psi \subset \Gamma_\psi$ of Hausdorff dimension at most $n - 7$. Moreover, letting $E_\psi := \Pi_{\mathbb{R}^n}(\Gamma_\psi \setminus S_\psi)$ the projection onto $\mathbb{R}^n$ of $\Gamma_\psi \setminus S_\psi$, we have that*

1. E_ψ *is a open set and* $E_\psi = \mathrm{int}(\overline{E_\psi}) = \mathrm{int}(\Pi_{\mathbb{R}^n}\Gamma_\psi)$,

2. $\psi \equiv -\infty$ *a.e. on* $Q \setminus E_\psi$,

3. $\psi \in \mathcal{C}^{2+\alpha}_{\mathrm{loc}}(E_\psi)$ *for all* $\alpha < 1$,

4. ψ *solves* (9) *in* E_ψ *with boundary conditions* (10).

Finally, letting $\tilde{\psi}$ *another minimizer of* $F_{\bar{c}}$*, for every connected component* E_i *of* E_ψ *there exists* $k_i \in \mathbb{R}$ *such that* $\tilde{\psi} = \psi + k_i$.

Proof. By Lemma 3.5 Σ_ψ is a minimizer of $\mathcal{F}_{\bar{c}}$ under compact perturbations. Classical results about regularity of minimal surfaces with prescribed curvature [21, 1] then imply that Γ_ψ is $\mathcal{C}^{2+\alpha}$ for all $\alpha < 1$, out of a closed singular set S_ψ of Hausdorff dimension at most $n-7$. Recalling that g is Lipschitz continuous and $\mathrm{Per}_{\bar{c}}(\Sigma_\psi, \mathbb{T}^n \times \mathbb{R}) < +\infty$, we can reason as in [18, p. 168 and Prop. 14.11] (see also [19]) to obtain that $\nu_{n+1} \neq 0$ on $\Gamma_\psi \setminus S_\psi$, where $\nu = (\nu_1, \ldots, \nu_{n+1})$ denotes the exterior unit normal to Σ_ψ. Reasoning as in [18, Thm. 14.13] it then follows $E_\psi = \mathrm{int}(\overline{E_\psi}) = \mathrm{int}(\Pi_{\mathbb{R}^n}\Gamma_\psi)$ and $\psi \in \mathcal{C}^{2+\alpha}_{\mathrm{loc}}(E_\psi)$. From the density estimate (23) we can derive that $\psi \le C$ for some $C > 0$, using the same argument as in Thm 14.10, [18]. So, this implies that ψ solves (9) in E_ψ with boundary conditions (10).
To prove the last assertion we notice that, letting $\tilde{\psi}$ be another minimum of $F_{\bar{c}}$, by convexity we have $F_{\bar{c}}(\lambda\psi + (1-\lambda)\tilde{\psi}) = 0$ for every $\lambda \in [0,1]$. By definition of $F_{\bar{c}}$ we then get

$$0 = F_{\bar{c}}\left(\lambda\psi + (1-\lambda)\tilde{\psi}\right) = \lambda F_{\bar{c}}(\psi) + (1-\lambda)F_{\bar{c}}\left(\tilde{\psi}\right)$$

if and only if

$$\psi\, D\tilde{\psi} = \tilde{\psi}\, D\psi \qquad \text{on } E_\psi \cap E_{\tilde{\psi}},$$

which implies the assertion. □

Remark 3.8. Integrating (9) on E_ψ and using (10) we obtain

$$\mathrm{Per}(E_\psi, \mathbb{T}^n) = -\int_{E_\psi} \mathrm{div}\left(\frac{D\psi}{\sqrt{1+|D\psi|^2}}\right) dy = \int_{E_\psi}\left(g(y) - \frac{\bar{c}}{\sqrt{1+|D\psi(y)|^2}}\right) dy, \tag{24}$$

which implies that E_ψ has finite perimeter.

Corollary 3.9. *Let* ψ *as in Proposition 3.7. Then* ψ *satisfies the boundary conditions* (11) *on* ∂E_ψ.

Proof. Let $\phi \in \mathcal{C}^1_{\mathrm{per}}(Q)$. By Proposition 3.7, $\min_{\overline{Q}}(\phi - \psi) = \min_{\overline{E}_\psi}(\phi - \psi)$. Assume by contradiction that $\phi - \psi$ attains its minimum at $y_0 \in \partial E_\psi$. Without loss of generality, we can assume that $z_0 := \phi(y_0) = \psi(y_0)$ and that $\phi(y) - \psi(y) > 0$ for every $y \neq y_0$. Again by Proposition 3.7, we have $x_0 := (y_0, z_0) \in S_\psi$, where S_ψ is the singular set of Γ_ψ.
Let us now blow-up the sets Σ_ψ and the subgraph Σ_ϕ of ϕ around x_0 If we let

$$\begin{aligned} \Sigma^s_\psi &:= \{x \in \mathbb{R}^{n+1} |\, sx \in \Sigma_\psi - x_0\} \\ \Sigma^s_\phi &:= \{x \in \mathbb{R}^{n+1} |\, sx \in \Sigma_\phi - x_0\}, \end{aligned}$$

by standard arguments of the theory of minimal surfaces [18, Chapter 9], one can prove that along a subsequence $s_i \to 0$, $\Sigma_\phi^{s_i}$ converges to a half-space $H \subset \mathbb{R}^{n+1}$, and $\Sigma_\psi^{s_i}$ converges to a minimal cone C. From the inclusion $\Sigma_\psi \subseteq \Sigma_\phi$ it follows $C \subseteq H$, but this implies that $C = H$, thus leading to a contradiction since the cone C is singular. □

We now define the maximal support E for minima of the functional $F_{\overline{c}}$, and study the regularity of such set.

Proposition 3.10. *There exists a set $E = \cup_{i=1}^k E_i \subseteq Q$, where E_i are connected components, such that the support of every minimum ψ of $F_{\overline{c}}$ is given by the union of some connected components of E.*
In particular, if E is connected, then there exists a unique nontrivial minimizer ψ of $F_{\overline{c}}$, up to an additive constant.
Moreover, there exists a closed set $S \subset \partial E$ such that $\partial E \setminus S$ is a $\mathcal{C}^{2+\alpha}$ hypersurface, with $\mathcal{H}^\gamma(S) = 0$ for every $\gamma > n - 8$, and satisfies the geometric equation

$$\kappa = g \qquad \text{on } \partial E \setminus S. \tag{25}$$

Proof. Let ψ_1, ψ_2 be two minima of the functional $F_{\overline{c}}$ and E_1, E_2 be the respective supports. By Proposition 3.7, if E_1^i and E_2^j are connected components respectively of E_1 and E_2 then either $E_1^i \cap E_2^j = \emptyset$ or $E_1^i = E_2^j$. In this case there exists a constant k such that $\psi_1 = \psi_2 + k$ on $E_1^i = E_2^j$. We then define E as the union of all the connected components of the supports of the minima of the functional $F_{\overline{c}}$.
We claim that the connected components of E are finite. Fix E_i connected component of E and ψ_i solution to (9) with support E_i. From (24) we obtain that $\mathrm{Per}(E_i, \mathbb{T}^n) \leq \max_Q g|E_i|$. This, combined with the isoperimetric inequality (4), gives that $|E_i| \geq (C_n / \max_Q g)^n$, which implies our claim.
If E is connected, the uniqueness up to addition of constants of the minimizers is a consequence of Proposition 3.7.
We now show the regularity of ∂E. Let $\psi \geq 0$ be a minimizer of $F_{\overline{c}}$ and assume without loss of generality that $E = E_\psi$. Since $\psi_\lambda = \psi + \lambda$ is also a minimizer for all $\lambda \in \mathbb{R}$, from the proof of Proposition 3.7 we know that the subgraphs $\Sigma_\lambda = \{(y, z) \in Q \times \mathbb{R} \mid z < \psi_\lambda(y)\}$ (locally) minimize the functional $\mathcal{F}_{\overline{c}}$ defined in (20), for all $\lambda \in \mathbb{R}$. In particular, since $\Sigma_\lambda \to E \times \mathbb{R}$ locally in the L^1-topology, as $\lambda \to +\infty$, by compactness of quasi minimizers of the area functional [1] we have that $E \times \mathbb{R}$ is also a minimizer of $\mathcal{F}_{\overline{c}}$ under compact perturbations. The thesis then follows by classical regularity theory for minimal surfaces with prescribed curvature [21, 1].

□

Remark 3.11. When $n = 1$, (25) reduces to

$$g = 0 \qquad \text{on } \partial E.$$

In particular, $E \neq Q$ necessarily implies $\min_Q g \leq 0$.

Remark 3.12. Let $\psi : E \to \mathbb{R}$ be a minimizer of $F_{\overline{c}}$ with maximal support, as in the proof of Proposition 3.10, and let $\Psi = \frac{e^{\overline{c}\psi}}{\overline{c}}$ be the corresponding minimizer of $G_{\overline{c}}$. Since $G_{\overline{c}}$ is a convex functional on $L^2(Q)$, by the general theory of subdifferentials in [10, 3] there exist a vector field $\xi_\Psi = \xi : Q \to \mathbb{R}^n$, with $|\xi| \le 1$ and $\mathrm{div}(\xi) \in L^2(Q)$, and a function $h_\Psi = h : Q \to \mathbb{R}$, with $0 \le h \le 1$, such that

$$\int_Q (-\mathrm{div}\,\xi(y) + \bar{c}h(y) - g(y))\,(w - \Psi)dy \ge 0 \qquad \text{in } Q, \tag{26}$$

for all $w \in BV_{\mathrm{per}}(Q)$ such that $w \ge 0$. Moreover, for all $y \in E_\psi$,

$$\begin{aligned} h(y) &= \frac{\bar{c}\Psi(y)}{\sqrt{\bar{c}^2\Psi^2(y) + |D\Psi(y)|^2}} \\ \xi(y) &= \frac{D\Psi(y)}{\sqrt{\bar{c}^2\Psi^2(y) + |D\Psi(y)|^2}}. \end{aligned}$$

If we apply inequality (26) to $w = \Psi + \chi_F$, where $F \subseteq Q$ is a set of finite perimeter, we obtain

$$\mathrm{Per}(F, Q) + \int_F (\bar{c}h(y) - g(y))\,dy \ge 0\,. \tag{27}$$

In particular, (24) and (27) imply that E is a minimum for the functional

$$\mathcal{G}(F) = \mathrm{Per}(F, \mathbb{T}^n) + \int_V (\bar{c}h(y) - g(y))\,dy \qquad F \subseteq Q.$$

Remark 3.13. We observe that, if ψ a solution to (8) such that $e^{c\psi} \in BV_{\mathrm{per}}(Q)$ for some $c > 0$, which by regularity amounts to say that ψ is bounded from above, then necessarily $F_c(\psi) = 0$ so that $c \le \bar{c}$ (see Corollary 3.2). Moreover, if $c < \bar{c}$, the support of ψ is strictly smaller than Q. This means that our variational method selects the *fastest* traveling wave solutions to (1) which are bounded from above [23].
However, there might exist other traveling wave solutions with $c > \bar{c}$, which are not in $BV_{\mathrm{per}}(Q)$ (see for instance [22]).

3.1 Existence of classical traveling waves

In this subsection we state some condition on the forcing term g under which equation (9) admits a bounded solution ψ in Q. This problem can be restated as following: find sufficient conditions on g, under which the maximal support E defined in Proposition 3.10 coincides with Q.

Remark 3.14. Observe that a first necessary condition on g, under which equation (9), with $\bar{c} > 0$, admits a bounded solution ψ in Q is that $\int_Q g > 0$. In fact, if $\int_Q g = 0$ and ψ is a bounded solution to (8), then $c = 0$. In [5] we show that condition (7) is sufficient to get the existence of a bounded smooth solution to (8) on Q with $c = 0$. Proposition 3.7 shows that this condition is essentially optimal for the existence of *stationary wave solutions*.

We consider a solution ψ to (9) with boundary conditions (10) and maximal support E. Let $\Psi = \frac{e^{\bar{c}\psi}}{\bar{c}}$. We recall that by (24)

$$\mathrm{Per}(E, \mathbb{T}^n) = \int_E (g(y) - \bar{c}h(y))\,dy \le \max_Q g\,|E|, \tag{28}$$

where $h = h_\Psi$ is the function defined in (26), In Remark 3.12. Since by (27)

$$\int_Q \bar{c}h(y) - g(y)dy \geq 0,$$

we also have

$$\mathrm{Per}(E, \mathbb{T}^n) \leq \int_{Q\setminus E} \bar{c}h(y) - g(y)dy. \tag{29}$$

From inequality (29), recalling $0 \leq h \leq 1$ and that $\int_Q g \leq \bar{c} \leq \max_Q g$, it follows

$$\mathrm{Per}(E, \mathbb{T}^n) \leq \left(\max_Q g - \min_Q g\right) |Q \setminus E|. \tag{30}$$

Assume now $|Q \setminus E| > 0$. Recalling the isoperimetric inequality (4), from (28) and (30) we get

$$\left(\max_Q g - \min_Q g\right) \frac{1}{2^{\frac{1}{n}}} \geq \left(\max_Q g - \min_Q g\right) |Q \setminus E|^{\frac{1}{n}} \geq C_n \qquad \text{or} \qquad |Q \setminus E| > \frac{1}{2}.$$

In particular, if

$$\max_Q g - \min_Q g < C_n\, 2^{\frac{1}{n}} \tag{31}$$

we necessarily have $|E| \leq 1/2$ and, from (28),

$$\max_Q g \geq \frac{\mathrm{Per}(E, \mathbb{T}^n)}{|E|} \geq C_n |E|^{-\frac{1}{n}} \geq C_n 2^{\frac{1}{n}}. \tag{32}$$

If $\min_Q g \leq 0$, then (31) implies that , in contradiction with (32).
If $\min_Q g > 0$, from (32) we get

$$\frac{1}{2} \geq |E| \geq \left(\frac{C_n}{\max_Q g}\right)^n.$$

From (30) it then follows

$$\begin{aligned}
\left(\max_Q g - \min_Q g\right)\left(1 - \left(\frac{C_n}{\max_Q g}\right)^n\right) &\geq \left(\max_Q g - \min_Q g\right)(1 - |E|) \\
&\geq C_n |E|^{\frac{n-1}{n}} \\
&\geq C_n \left(\frac{C_n}{\max_Q g}\right)^{n-1}.
\end{aligned}$$

So if $\min_Q g$, we necessarily have $E = Q$ if either $\max_Q g < C_n\, 2^{\frac{1}{n}}$ or $\max_Q g \geq C_n\, 2^{\frac{1}{n}}$ and $\max_Q g - \min_Q g < C_n \left(\frac{C_n}{\max_Q g}\right)^{n-1} \left(1 - \left(\frac{C_n}{\max_Q g}\right)^n\right)^{-1}$,
Collecting the previous results above and recalling Remark 3.11 we get the following proposition.

Proposition 3.15. *Assume that* $\int_Q g > 0$. *Then equation* (9) *admits a bounded solution* ψ *in* Q *if one of the following conditions is verified.*

- $\min_Q g \le 0$ *and* $\max_Q g - \min_Q g < C_n 2^{1/n}$*;*

- $g > 0$ *on* Q *and* $\max_Q g < C_n 2^{1/n}$*;*

- $g > 0$ *on* Q*,* $\max_Q g \ge C_n 2^{1/n}$ *and* $\max_Q g - \min_Q g < \max_Q g \left(\left(\frac{\max_Q g}{C_n}\right)^n - 1\right)^{-1}$*;*

- $n = 1$ *and* $g > 0$ *on* Q

where C_n *is the isoperimetric constant appearing in* (4) *(and* $C_1 = 2$*).*

Remark 3.16. Observe that the assumptions in the previous Proposition assure the existence of classical traveling wave solutions to (1), i.e. solutions of the form $\bar{c}t + \psi(x)$, where ψ is a smooth, $\mathbb{Z}^n$-periodic solution to (9).

Remark 3.17. In [20] Lions and Souganidis showed that (9) admits a (periodic) solution over all Q if g does not change sign and satifies the condition

$$\exists \theta \in (0,1) \;\; \text{s.t.} \; \min_{x \in Q} \left(\theta g^2(x) - (n-1)^2 |Dg(x)|\right) > 0.$$

In [11] Cardaliaguet, Lions and Souganidis proved that, when $n = 1$ and $\int_0^1 g(y)dy > 0$, the following condition implies the solvability of the cell problem:

$$0 \le \int_0^1 g(y)dy - \min_{z \in [0,1]} g(z) < 2. \tag{33}$$

4 Stability and long-time behavior

If u is a solution to (1), then $w(t,y) = u(t,y) - \bar{c}t$ is a solution to

$$w_t = \operatorname{tr}\left[\left(\mathbf{I} - \frac{Dw \otimes Dw}{1 + |Dw|^2}\right) D^2 w\right] + g\sqrt{1 + |Dw|^2} - \bar{c} \qquad \text{in } (0,+\infty) \times Q \tag{34}$$

with periodic boundary conditions and initial datum $w(0,y) = u_0(y)$. Note that w is the unique solution to (34), and it is also a classical solution, see Theorem 2.3. Standard comparison gives that $(\min g - \bar{c})t - \|u_0\|_\infty \le w(t,x) \le (\max g - \bar{c})t + \|u_0\|_\infty$ for every $t \ge 0$, $x \in \mathbb{R}^n$. Moreover, under the assumption (6), w is bounded (from below) uniformly in t.

Lemma 4.1. *Let* w *be the solution to* (34) *and* ψ *be any solution to* (9)*, then*

$$w(t,y) - \psi(y) \ge \min_Q (u_0 - \psi) \qquad \forall\, t \ge 0,\; y \in Q. \tag{35}$$

Moreover, if there exists a solution ψ *to* (9) *in* Q*, then there exists a constant* M*, depending only on* $\|u_0\|_\infty$ *such that* $|w(t,x)| \le M$ *for every* $t \ge 0$ *and* $y \in Q$*.*

Proof. We fix a ψ solution to (9), and let $E = E_\psi$ (see Proposition 3.7). We recall that by Corollary 3.9, ψ satisfies the boundary conditions (11) on ∂E_ψ.
We shall prove that

$$m(t) := \min_{x \in Q}(w(t,x) - \psi(x))$$

is nondecreasing in t. Obviously this is sufficient to prove that $\min_{x\in\overline{E}}(w(t,x)-\psi(x))$ is nondecreasing in t. We fix $s \geq 0$ and observe that $w(t+s,x)$ is the solution to

$$v_t(t,x) = \operatorname{tr}\left[\left(\mathbf{I} - \frac{Dv \otimes Dv}{1+|Dv|^2}\right) D^2 v\right] + g\,(x)\,\sqrt{1+|Dv|^2} - \overline{c} \quad \text{in } (0,+\infty)\times E$$

with initial datum $v(0,x) = w(s,x)$, and with boundary conditions $v(t,x) = w(t+s,x)$ on ∂E for all $t \geq 0$. Notice that $\psi(y) + \min_{\hat{y}\in Q}(w(s,\hat{y}) - \psi(\hat{y}))$ is a regular (stationary) subsolution to the same problem. Moreover by Corollary 3.9 we have that $w(t+s,x) - [\psi(x) + \min_{y\in Q}(w(s,y) - \psi(y))]$ can attain its minima only in the interior of E. So we can apply comparison principle arguments (see [4]) to conclude that $w(t+s,x) - \psi(x) \geq \min_{y\in Q}(w(s,y) - \psi(y))$ for every $t \geq 0$ and $x \in Q$.
Finally, if there exists a solution ψ to (9) in the whole Q, then $\psi(x) + \|u_0\|_\infty + \|\psi\|_\infty$ and $\psi(x) - \|u_0\|_\infty - \|\psi\|_\infty$ are, respectively, a supersolution and a subsolution to (1) and we conclude by the standard comparison principle. □

Remark 4.2. Note that if there is a solution to (9) in the whole Q, a similar argument gives that

$$M(t) := \max_{x\in Q}(w(t,x) - \psi(x))$$

is nonincreasing in t.

Lemma 4.3. *Let w be a solution to* (34). *Then for all $\tau > 0$ there exists a constant $C > 0$, depending on u_0, g and τ, such that $\|w_t\| \leq C$ for all $t \geq \tau$.*

Proof. Recalling Theorem 2.3, we define

$$C := \left\|\operatorname{tr}\left[\left(\mathbf{I} - \frac{Dw(\tau,\cdot)\otimes Dw(\tau,\cdot)}{1+|Dw(\tau,\cdot)|^2}\right) D^2 w(\tau,\cdot)\right] + g(x)\sqrt{1+|Dw(\tau,\cdot)|^2} - \overline{c}\right\|_{L^\infty(Q)} < +\infty$$

Then $S(t,x) = Ct + w(t,\cdot)$ is a supersolution to (34) and $s(t,x) = -Ct + w(t,\cdot)$ is a subsolution for all $t > \tau$. Then by comparison [4] we obtain $-Ct \leq w(t,x) - w(\tau,x) \leq Ct$. Moreover, for every fixed $s > \tau$, we get that $w(t,x) + \sup_x |w(s,x) - w(\tau,x)|$ and $w(t,x) - \sup_x |w(s,x) - w(\tau,x)|$ are respectively a supersolution and a subsolution to (34) with initial data $w(s,x)$. So, again by comparison, and recalling the previous estimate, for every $\tau \leq s \leq t$ we obtain

$$-Cs \leq w(t+s,x) - w(t,x) \leq Cs.$$

□

The estimate in Lemma 4.3 imlies that, for all $t > 0$ the function $w(t,\cdot)$ satisfies in the viscosity sense

$$-C - g(x) \leq \operatorname{div}\left(\frac{Dw(t,x)}{\sqrt{1+|Dw(t,x)|^2}}\right) \leq C + \overline{c} - g(x) \text{ in } \mathbb{R}^n \ \forall t \geq \tau. \tag{36}$$

So, this gives in particular that the curvature of the graph of $w(t,\cdot)$ is uniformly bounded with respect to $t \in [\tau,+\infty)$.

Proposition 4.4. *Let $\Gamma_w(t) \subset Q \times \mathbb{R}$ be the graph of $w(t,\cdot)$. Then, for all $\tau > 0$, $\Gamma_w(t)$ are hypersurfaces of class $\mathcal{C}^{1+\alpha}$, for all $\alpha \in (0,1)$, uniformly in $t \in [\tau, +\infty)$.*

Proof. Assume by contradiction the statement to be false. Then we can find $(x_n, t_n) \in Q \times [0, +\infty)$ such that, for all $\rho > 0$, the hypersurfaces $\Gamma_w(t_n) \cap B_\rho(x_n, t_n)$ are not uniformly $\mathcal{C}^{1+\alpha}$. Letting $\widetilde{w}_n(x) := w(x, t_n) - w(x_n, t_n)$, from (36) we have that

$$-\mathrm{div}\left(\frac{D\widetilde{w}_n(x)}{\sqrt{1+|D\widetilde{w}_n(x)|^2}}\right) = h_n(x), \tag{37}$$

with $\|h_n\|_\infty \leq \widetilde{C}$ for some $\widetilde{C}$ independent of n. As a consequence $\widetilde{w}_n$ is a minimizer of the prescribed curvature functional

$$\int_Q \left(\sqrt{1+|Du|^2} - h_n u\right) dy.$$

By the compactness theorem for quasi minimizers of the perimeter [1] the graphs $\Gamma_{\widetilde{w}_n}$ of $\widetilde{w}_n$ converge locally in the L^1-topology, up to a subsequence, to a limit hypersurface Γ_∞ of class $\mathcal{C}^{1+\alpha}$. We can also assume that $x_n \to x$ for some $x \in Q$, and let ν_∞ be the normal vector to Γ_∞ at $(x, 0)$. However, by Theorem 2.5 there exists $\rho > 0$ such that $\Gamma_{\widetilde{w}_n} \cap B_\rho(x, 0)$ and $\Gamma_\infty \cap B_\rho(x, 0)$ can all be written as graphs in the direction given by ν_∞. Therefore, by elliptic regularity for minimizers of the prescribed curvature functional [21], the sets $\Gamma_{\widetilde{w}_n} \cap B_\rho(x, 0)$ are uniformly of class $\mathcal{C}^{1+\alpha}$ for all $\alpha \in (0,1)$, thus leading to a contradiction. □

The following lemma that will be useful in the following.

Lemma 4.5. *Let $F_{\overline{c}}(v) = \int_Q e^{\overline{c}v(y)} \left(\sqrt{1+|Dv(y)|^2} - \frac{g(y)}{\overline{c}}\right) dy$ the functional defined in* (12). *Then for every (smooth) solution w to the equation in* (34),

$$0 \leq F_{\overline{c}}(w(t,\cdot)) \leq F_{\overline{c}}(u_0) \qquad \text{for all } t > 0. \tag{38}$$

Proof. For every solution w to (34), using the definition of the functional $F_{\overline{c}}$, we get

$$\begin{aligned} \frac{dF_{\overline{c}}(w(t,\cdot))}{dt} &= \int_Q e^{\overline{c}w} w_t \left[-\mathrm{div}\left(\frac{Dw}{\sqrt{1+|Dw|^2}}\right) - g + \frac{\overline{c}}{\sqrt{1+|Dw|^2}}\right] \\ &= -\int_Q \frac{e^{\overline{c}w} w_t^2}{\sqrt{1+|Dw|^2}} \leq 0. \end{aligned} \tag{39}$$

□

The first result on the asymptotic behavior of the solutions u to (1) is about the convergence of $\frac{u(t,x)}{t}$ as $t \to +\infty$.

Proposition 4.6. *Let u be the solution to* (1) *and E be the maximal support defined in Proposition 3.10. Then*

$$\lim_{t\to+\infty} \frac{\max_{x\in\mathbb{R}^n} u(t,x)}{t} = \overline{c}, \quad \textit{and} \quad \lim_{t\to+\infty} \frac{u(t,x)}{t} = \overline{c} \quad \textit{locally uniformly in } E.$$

Moreover if there exists a bounded solution to (9),

$$\lim_{t\to+\infty}\frac{u(t,x)}{t}=\bar{c}\qquad \textit{uniformly in } \mathbb{R}^n.$$

In particular there exists a constant $C\in\mathbb{R}$ *such that*

$$\min_Q u_0(x)\le M(t):=\max_Q(u(t,x)-\bar{c}t)\le C+\frac{\log(1+t)}{\bar{c}}. \tag{40}$$

Proof. Recall that if the stationary problem (9) has a bounded solution, Lemma 4.1 gives an uniform bound on $u(t,x)-\bar{c}t$, and then we obtain the result.
We observe, recalling Lemma 4.1, that to prove the general statement it is sufficient to prove (40). The lower bound on M is an immediate consequence of Lemma 4.1, just by choosing ψ as the maximal nonpositive solution to (9).
We define $f(t,x):=\frac{2}{\bar{c}}e^{\frac{\bar{c}w(t,x)}{2}}$, so that $f_t^2(t,x)=w_t^2(t,x)e^{\bar{c}w(t,x)}$. Integrating (39) between 0 and T, we obtain

$$C\ge F_{\bar{c}}(u_0)\ge F_{\bar{c}}(u_0)-F_{\bar{c}}(w(T,\cdot)=\int_0^T\int_Q\frac{f_t^2(t,x)}{\sqrt{1+|Dw(t,x)|^2}}dxdt$$

for some constant $C>0$ depending only on u_0 and g.
Let

$$\widetilde{M}(t)=\max_x \fint_{B_\rho(x)} f(t,y)dy,$$

Given a point $\bar{x}(t)$ where either $M(t)$ or $\widetilde{M}(t)$ attain the maximum, thanks to Proposition 4.4 we can choose $\rho<2/\bar{c}$, independent of t, such that $|Dw(x,t)|\le 1$ for every $x\in B_\rho(\bar{x}(t))$.
Notice that

$$\frac{2}{\bar{c}}e^{\frac{\bar{c}(W(t)-\rho)}{2}}\le\widetilde{M}(t)\le\frac{2}{\bar{c}}e^{\frac{\bar{c}W(t)}{2}}\qquad\text{for all } t\ge 0,$$

so that, in order to prove the second inequality in (40), it is enough to show

$$\widetilde{M}(t)\le C(1+\sqrt{t}). \tag{41}$$

Given $t\ge 0$ let $\mathcal{Z}(t)$ be the set of points where $\widetilde{M}(t)$ attains its maximum. Possibly increasing C, and using the fact that $|Dw(x,t)|\le 1$ on $B_\rho(\bar{x}(t))$, from the previous inequality we get

$$\begin{aligned} C &\ge \int_0^T \max_{\bar{x}(t)\in\mathcal{Z}(t)}\fint_{B_\rho(\bar{x}(t))} f_t^2(t,x)dx\,dt \\ &\ge \int_0^T\left(\max_{\bar{x}(t)\in\mathcal{Z}(t)}\fint_{B_\rho(\bar{x}(t))} f_t(t,x)dx\right)^2 dt \\ &= \int_0^T \widetilde{M}'(t)^2dt\ge\frac{1}{T}\left(\int_0^T|\widetilde{M}'(t)|\,dt\right)^2. \end{aligned} \tag{42}$$

From (42) we then have

$$\widetilde{M}(T)\le\widetilde{M}(0)+\int_0^T|\widetilde{M}'(t)|\,dt\le\widetilde{M}(0)+\sqrt{CT},$$

which proves (41). □

We now prove the main convergence result, on the stability of our traveling wave solutions.

Theorem 4.7. *Let $u(t,x)$ be the unique solution to* (1) *with periodic boundary conditions, let $M(t) := \max_Q w(t,y)$, and let*

$$\tilde{w}(t,x) := w(t,x) - M(t) = u(t,x) - \max_{x\in Q}(u(t,x)) \le 0.$$

Then, for any sequence $t_n \to +\infty$ there exists a subsequence t_{n_k} such that, as $k \to +\infty$,

$$w(t_{n_k}, x) \longrightarrow \begin{cases} \overline{\psi}(x) & \text{locally in } \mathcal{C}^{1+\alpha}(E_{\overline{\psi}}) \\ -\infty & \text{locally uniformly in } Q \setminus \overline{E}_{\overline{\psi}} \end{cases} \tag{43}$$

for all $\alpha \in (0,1)$, where $\overline{\psi}$ is a traveling wave solution to (9)*.*

Proof. We let

$$W(t,y) := \frac{e^{\overline{c}w(t,y)}}{\overline{c}}, \qquad \widetilde{W}(t,y) := \frac{e^{\overline{c}\tilde{w}(t,y)}}{\overline{c}} = e^{-\overline{c}M(t)} W(t,y) \le \frac{1}{\overline{c}}.$$

Notice that from (34) it follows that W satisfies the equation

$$W_t = \sqrt{\overline{c}^2 W^2 + |DW|^2}\left(\operatorname{div}\left(\frac{DW}{\sqrt{\overline{c}^2 W^2 + |DW|^2}}\right) + g\right) - \overline{c}^2 W \qquad \text{in } (0,+\infty)\times Q. \tag{44}$$

By (38) and (40), for all $t \ge 0$ we have

$$G_{\overline{c}}(\widetilde{W}(t,\cdot)) = F_{\overline{c}}(\tilde{w}(t,\cdot)) = e^{-\overline{c}M(t)} F_{\overline{c}}(w(t,\cdot)) \le e^{-\overline{c}(\min_Q u_0)} F_{\overline{c}}(u_0).$$

In particular,

$$\int_Q \sqrt{\overline{c}^2 \widetilde{W}^2(t,y) + |D\widetilde{W}(t,y)|^2}\, dy = G_{\overline{c}}(\widetilde{W}(t,\cdot)) + \int g(y)\widetilde{W}(t,y)\, dy \le C$$

for all $t \ge 0$, where C depends only on u_0 and g. Hence, up to extracting a subsequence t_{n_k}, $\widetilde{W}(t_{n_k},\cdot) \rightharpoonup W_\infty$ weakly* in $BV_{\mathrm{per}}(Q)$, as $k \to +\infty$. Notice that, as in the previous section, the epigraph of $\tilde{w}(t,\cdot)$ is, for every $t > 0$, a minimizer of the prescribed curvature functional

$$\Sigma \mapsto \mathrm{Per}_c(\Sigma, \mathbb{T}^n \times \mathbb{R}) - \int_\Sigma e^{cz} \tilde{g}_t(y)\, dydz$$

where $\tilde{g}_t$ is an appropriate bounded function, depending on t. It therefore satisfies the lower density bound (23), which implies $W_\infty \not\equiv 0$. We claim that

$$G_{\overline{c}}(W_\infty) = 0. \tag{45}$$

We introduce the modified functional, for $t > 0$,

$$\widetilde{G}_{\overline{c},t}(W) := \int_Q \left(\sqrt{\overline{c}^2 W^2 + |DW|^2} - \tilde{g}_t\, W\right) dy$$

where

$$\tilde{g}_t(y) := g(y) - \frac{W_t(t,y)}{\sqrt{\overline{c}^2 W^2(t,y) + |DW(t,y)|^2}} \in L^\infty(Q), \qquad \|\tilde{g}_t\|_\infty \le C,$$

with C independent of t. Note that from (44) it follows that, at every $t > 0$, $W(t,\cdot)$ is a critical points of the functional $\widetilde{G}_{\overline{c},t}$ and so $\widetilde{G}_{\overline{c},t}(W(t,\cdot)) = 0$. Moreover, also $\widetilde{G}_{\overline{c},t}(\widetilde{W}(t,\cdot)) = 0$. Recalling (39), up to extracting a further subsequence, we can assume that

$$\begin{aligned} \partial_t G_{\overline{c}}(W(t_{n_k},\cdot)) = \partial_t F_{\overline{c}}(w(t_{n_k},\cdot)) &= -\int_Q \frac{e^{\overline{c}w(t_{n_k},y)} w_t^2(t_{n_k},y)}{\sqrt{1+|Dw(t_{n_k},y)|^2}}\, dy \qquad (46)\\ &= -\int_Q \frac{W_t^2(t_{n_k},y)}{\sqrt{\overline{c}^2 W^2(t_{n_k},y) + |DW(t_{n_k},y)|^2}}\, dy \to 0 \end{aligned}$$

as $k \to +\infty$.
Since $G_{\overline{c}}(v) \ge 0$ for every v, to prove the claim (45) it is sufficient to show that $G_{\overline{c}}(W_\infty) \le 0$. We get, using the convexity of $G_{\overline{c}}$ and the definition of the modified functional $\widetilde{G}_{\overline{c},t}$,

$$\begin{aligned} G_{\overline{c}}(W_\infty) &\le \liminf_{k\to+\infty} G_{\overline{c}}(\widetilde{W}(t_{n_k},y)) \\ &= \liminf_{k\to+\infty} \left(\widetilde{G}_{\overline{c},t_{n_k}}(\widetilde{W}(t_{n_k},y)) - \int_Q \frac{\widetilde{W}(t_{n_k},y) W_t(t_{n_k},y)}{\sqrt{\overline{c}^2 W^2(t_{n_k},y) + |DW(t_{n_k},y)|^2}}\, dy \right) \\ &= \liminf_{k\to+\infty} -\int_Q \frac{\widetilde{W}(t_{n_k},y) W_t(t_{n_k},y)}{\sqrt{\overline{c}^2 W^2(t_{n_k},y) + |DW(t_{n_k},y)|^2}}\, dy \end{aligned}$$

since $\widetilde{G}_{\overline{c}}(\widetilde{W}(t_{n_k},y)) = 0$. Using the Hölder inequality, (46) and the definition of $\widetilde{W}$, we obtain

$$\begin{aligned} &\liminf_{k\to+\infty} \int_Q \frac{-\widetilde{W}(t_{n_k},y) W_t(t_{n_k},y)}{\sqrt{\overline{c}^2 W^2(t_{n_k},y) + |DW(t_{n_k},y)|^2}}\, dy \\ \le\ &\liminf_{k\to+\infty} \left(\int_Q \frac{W_t^2(t_{n_k},y)}{\sqrt{\overline{c}^2 W^2(t_{n_k},y) + |DW(t_{n_k},y)|^2}}\, dy \right)^{\frac{1}{2}} \left(\int_Q \frac{e^{-\overline{c}M(t)}}{\overline{c}^2} dy \right)^{\frac{1}{2}} = 0 \end{aligned}$$

which proves our claim. In particular, $\overline{\psi} := \log(\overline{c}W_\infty)/\overline{c} : E_{\overline{\psi}} \to [-\infty, +\infty)$ is a traveling wave solution of (9) with $c = \overline{c}$.
Let us now prove (43). Given $y \in E_{\overline{\psi}}$, by Theorem 2.5 there exists $r > 0$ such that $B_r(y) \subset E_{\overline{\psi}}$ and $\|D\tilde{w}(t_{n_k},y)\|_{L^\infty(B_r(y))}$ is uniformly bounded in k. By standard elliptic regularity [17] it then follows that the functions $\tilde{w}(t_{n_k},\cdot)$ are uniformly bounded in $\mathcal{C}^{1+\alpha}(B_r(y))$ for all $\alpha \in (0,1)$, so that they converge to $\overline{\psi}$ locally in $\mathcal{C}^{1+\alpha}(E_{\overline{\psi}})$.
Fix now $y \in Q \setminus \overline{E}_{\overline{\psi}}$ and take $r > 0$ such that $B_r(y) \subset Q \setminus \overline{E}_{\overline{\psi}}$. Assume by contradiction that there exist $c \in \mathbb{R}$ and $y_k \in B_r(y)$, $k \in \mathbb{N}$, such that $\tilde{w}(t_{n_k}, y_k) \ge c$ for all k. By the density estimate (23) this would imply $\int_Q \widetilde{W}(t_{n_k},y)dy \ge c'$ for some $c' \in \mathbb{R}$, contradicting the fact that $\widetilde{W}(t_{n_k},y) \to W_\infty$ in $L^1(Q)$, with $W_\infty \equiv 0$ in $B_r(y)$. We thus proved (43). □

Remark 4.8. If the functional $F_{\overline{c}}$ admits a unique minimizer $\bar{\psi} : E_{\overline{\psi}} \to \mathbb{R}$ up to an additive constant (for instance if the maximal support E is connected, see Proposition 3.10), then instead of (43) we have

$$\lim_{t\to+\infty} w(t,x) = \begin{cases} \overline{\psi}(x) - \max_{\overline{E_{\overline{\psi}}}} \bar{\psi} & \text{locally in } \mathcal{C}^{1+\alpha}(E_{\overline{\psi}}) \\ -\infty & \text{locally uniformly in } Q \setminus \overline{E_{\overline{\psi}}} \end{cases} \tag{47}$$

for all $\alpha \in (0,1)$.

Corollary 4.9. *Let $u(t,x)$ be the unique solution to* (1) *with periodic boundary conditions, and assume that there exist bounded solutions to* (9) *in Q (see Proposition 3.15). Then*

$$u(t,x) - \overline{c}t \longrightarrow \overline{\psi}(x) \qquad \text{in } \mathcal{C}^{1+\alpha}(Q), \text{ as } t \to +\infty,$$

where $\overline{\psi}$ is a bounded solution to (9)*.*

Proof. By Lemma 4.1 and Remark 4.2, it is enough to prove that $w(t_n,x) \to \overline{\psi}(x)$ uniformly along a subsequence $t_n \to +\infty$. This result can be obtained by repeating the same argument as in the proof of Theorem 4.7. □

Remark 4.10. A straightforward adaptation of the argument in Corollary 4.9 gives that, under assumption (7),

$$u(t,x) \to \psi(x) \qquad \text{in } \mathcal{C}^{1+\alpha}(Q), \text{ as } t \to +\infty,$$

where ψ is a stationary solution of the parabolic equation (1) (whose existence has been shown in [5]).

Remark 4.11. The results of this paper can be easily extended to equation (1) considered on a bounded open set $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary, and with Neumann boundary conditions on $\partial\Omega$.

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