DROPLET MINIMIZERS OF AN ISOPERIMETRIC PROBLEM WITH LONG-RANGE INTERACTIONS

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ABSTRACT. We give a detailed description of the geometry of single droplet patterns in a nonlocal isoperimetric problem. In particular, we focus on the sharp interface limit of the Ohta–Kawasaki free energy for diblock copolymers, regarded as a paradigm for energies with short and a long-range interactions. Exploiting fine properties of the regularity theory for minimal surfaces, we extend previous partial results in different directions and give robust tools for the geometric analysis of more complex patterns.

1. INTRODUCTION

In several physical systems competing short-range attractive and long-range repulsive interactions often lead to the formation of patterns on mesoscopic scales. Roughly speaking, the short-range interactions favor phase-separation on a microscopic scale, while the long-range ones frustrate such an ordering on the scale of the whole sample. When these systems are described in terms of a free energy, such a phenomenon is referred to as *energy-driven* pattern formation. Examples of energy-driven patterns are ubiquitous in physics: among the others we recall ferromagnetic and polymeric systems, type-I superconductor films and Langmuir layers. Even if these systems are driven by different physical laws, they exhibit remarkable similarities in the overall geometry of the patterns (see [32, 45]).

Our principal interest is the description of the geometry of patterns. For this reason we focus here on a model energy which encodes only the main features of pattern-formation. More specifically, in what follows we are interested in the minimization of the following energy functional:

$$F_{\gamma,m}(u) := \int_{\mathbb{R}^n} |Du| + \gamma \int_\Omega \int_\Omega G(x,y)\,\big(u(x)-m\big)\big(u(y)-m\big)\,dx\,dy,$$

where u is the order parameter of a two-phases system confined in $\Omega \subset \mathbb{R}^n$, and γ and m are two nonnegative parameters. The two terms in the energy mimic attractive short-range and repulsive long-range energies between the phases. More precisely, the first term is local, favors minimal interface area and drives the system towards a partition into few pure phases, while the second term involving a Coulomb-like kernel G is non-local and favors a fine mixing of the phases. A detailed description of the energy is given in § 2. The competition between these two terms is expected to induce the formation of highly regular mesoscopic patterns (e.g. spherical spots, cylinders, gyroids, lamellae etc. . .), whose geometry strongly depends on the choice of the parameters γ and m.

1.1. **The Ohta–Kawasaki functional for diblock copolymers.**

The model we consider arises as a simplification of a Ginzburg-Landau functional proposed by Ohta and Kawasaki in their pioneering paper [40] as a possible description of diblock copolymers' (DBC) systems. Even though it is questionable whether such an energy actually describes DBCs (see Choksi and Ren [14], Muratov [38] and Niethammer and Oshita [39]), nevertheless it is a first, and mathematically non-trivial, attempt to capture some of the main features of these systems. For such a reason it deserved over the last twenty years great attention from both the mathematical and the physical community (see e.g. [7, 18, 35, 40, 49]). Under several simplifications, the Otha-Kawasaki functional can be written in the following form:

$$\mathscr{E}_{\varepsilon,\sigma}(u) = \int_\Omega \big(\varepsilon^2|\nabla u|^2 + W(u)\big)\,dx + \sigma\int_\Omega\int_\Omega G(x,y)\,(u(x)-m)\,(u(y)-m)\,dx\,dy, \tag{1}$$

where the order parameter u stems for the volume fraction of one block copolymer and W is a standard double-well potential. Here m is the average of u over the whole sample, namely $m = \fint_\Omega u$, and the kernel G is the fundamental solution to the Neumann problem for the Laplace equation in Ω:

$$-\Delta G(x,\cdot) = \delta_x - \frac{1}{|\Omega|}, \qquad \int_\Omega G(x,y)\,dy = 0. \tag{2}$$

As in the classical Ginzburg–Landau energy, the first contribution to the energy forces a phase separation thanks to the competition between the gradient and the non-convex potential. On the other hand, depending on the strength of the coupling constant σ, the long-range contribution favors a uniform distribution of the order parameter. This term has an entropic origin in the case of DBCs (see [7, 18, 35, 40, 49]), but it can also be considered as the energy due to an electrostatic interaction between charged bodies if the order parameter is assumed to represent a density of charges ([11, 19]).

It is well-known from the results by Modica and Mortola [37, 36] that, when $\varepsilon \ll 1$, the Ginzburg-Landau energy can be approximated in the sense of Γ-conver-gence by a sharp interface energy of the form $\varepsilon\, c_0 \int_\Omega |Du|$, with $c_0 > 0$ a constant and u being a function of bounded variation taking the two values $0,1$ (these values identify the pure phases of the system as the sets $\{x : u(x) = 0\}$ and $\{x : u(x) = 1\}$), $|Du|$ denoting the total variation of the measure Du. Formally, this fact gives the link between the Ohta–Kawasaki energy and the functional $F_{\gamma,m}$ (for $\gamma = \sigma/(c_0\,\varepsilon)$). It is worth pointing out that there exists no rigorous derivation of $F_{\gamma,m}$ from $\mathscr{E}_{\varepsilon,\sigma}$ in the sense of Γ-convergence. Indeed, the presence of possible multiple scales (e.g., the one of the phase separation and that of the pattern formation) could force the Γ-limit to be defined on more complex spaces of Young measures, as it happens in the one dimensional case addressed by Alberti and Müller in [3]. Such a complex multiple-scale behavior is actually observed in physical experiments (see, e.g., [30, 45]). The experiments also suggests that, in some regimes of the parameters γ and m, droplets are expected to be equilibrium configurations. The main open issues in this regards are: (1) the rigorous justification of the observed lattice-type patterns (for example in two dimensions the droplets seem to sit on the Abrikosov lattice) and (2) the description of the geometry of the droplets. Regarding the first issue, we quote the remarkable paper by Alberti, Choksi and Otto [2] in which the authors study the uniform distribution of the energy and of the order parameter of the minimizers of $F_{\gamma,m}$ (see [46] for analogous results in the case of the Ohta–Kawasaki functional $\mathscr{E}_{\varepsilon,\sigma}$). In this paper we contribute to the second question. In particular, we investigate a regime of γ and m leading to the formation of a *single* droplet minimizer, as a first step towards the analysis of multiple droplets patterns.

1.2. **Single droplet minimizers.** A single droplet minimizer can be roughly described as a connected region of one phase surrounded by the other one. For this to happen, the competition between the two terms of the energy has to be unbalanced, with the confining term stronger than the nonlocal one. In order to identify the correct regime, we show here the different contributions to the energy of a single ball. As shown in (20), given a ball

$B_{r_m}(p) \subset \Omega$ of radius r_m centered at p and with average mass m, i.e. $m\,|\Omega| = \omega_n\, r_m^n$ (here $|\Omega|$ stands for the n-dimensional volume of Ω), it holds

$$F_{\gamma,m}(\chi_{B_{r_m}(p)}) = \begin{cases} 2\,\pi\, r_m + \gamma\left(\frac{\pi}{2}\, r_m^4 \log r_m + \left(\pi^2\, g_{r_m}(p) - \frac{3\,\pi}{8}\right) r_m^4\right) & \text{if } n = 2, \\ n\,\omega_n\, r_m^{n-1} + \gamma\left(\frac{2\,\omega_n}{4-n^2}\, r_m^{n+2} + \omega_n^2\, g_{r_m}(p)\, r_m^{2n}\right) & \text{if } n \geq 3, \end{cases}$$

where $g_{r_m}(p)$ is uniformly bounded for p in a compact subset of Ω – see § 2.4. Therefore, for the isoperimetric term to be stronger than the nonlocal one, the regimes to be considered are

$$\gamma r_m^3\,|\log r_m| << 1 \quad \text{for} \quad n = 2,$$
$$\gamma r_m^3 << 1 \quad \text{for} \quad n \geq 3.$$

Note that, if $\gamma \to 0$, the conditions above are trivially satisfied. On the other hand, in the most interesting case of $\gamma \geq C > 0$, one is forced to consider the small volume-fraction regime $r_m << 1$, which we will always assume. Under these scalings we provide a detailed analysis of the minimizers of $F_{\gamma,m}$, showing that a single droplet is a minimizer for $F_{\gamma,m}$. In particular, we prove:

(a) the asymptotic convergence of the minimizers to round spheres in strong norms, providing the rate of convergence;
(b) the asymptotic optimal centering of the droplet in the domain;
(c) the expansion of the energy in terms of the radius r_m;
(d) the nonexistence of exact spherical droplets in domains Ω different from a ball; and, on the other hand, the uniqueness of the minimizer when Ω is a ball (in this case the minimizers is itself a ball centered at the center of Ω).

These results are summarized in the following theorem (see next sections for more details on the notation).

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with C^2 boundary. There exist $\delta_0, r_0 > 0$ (depending on Ω) such that the following holds. Assume $r_m \leq r_0$ and*

$$\gamma r_m^3\,|\log r_m| < \delta_0 \quad \text{if } n = 2 \quad \text{or} \quad \gamma r_m^3 < \delta_0 \quad \text{if } n \geq 3.$$

Then, every minimizer $u_m = \chi_{E_m} \in \mathscr{C}_m$ of $F_{\gamma,m}$ satisfies the following properties:

(i) *E_m is a convex set and there exist $p_m \in \Omega$ and $\varphi_m : \mathbb{S}^{n-1} \to \mathbb{R}$ such that $\partial E_m = \{p_m + (r_m + \varphi_m(x))\,x : x \in \mathbb{S}^{n-1}\}$ and*

$$\|\varphi_m\|_{C^1} \lesssim \gamma r_m^{n+3}; \tag{3}$$

(ii) *p_m is close to the set of harmonic centers $\mathscr{H}$ of Ω, i.e.*

$$\lim_{r_m \to 0} \operatorname{dist}(p_m, \mathscr{H}) = 0;$$

(iii) *the energy of u_m has the following asymptotic expansion:*

$$F_{\gamma,m}(u_m) = \begin{cases} 2\,\pi\, r_m + \frac{\pi\,\gamma}{2}\, r_m^4 \log r_m + \gamma\left(-\frac{1}{8} + \pi^2 \min_\Omega h\right) r_m^4 + O(\gamma r_m^6) & n = 2, \\ n\,\omega_n\, r_m^{n-1} + \frac{2\,\gamma\,\omega_n}{4-n^2}\, r_m^{n+2} + \gamma\, \omega_n^2\, r_m^{2n} \min_\Omega h + O(\gamma r_m^{2n+2}) & n \geq 3, \end{cases} \tag{4}$$

where h is the Robin function relative to Ω;

(iv) *E_m is an exact ball if and only if the domain Ω is itself a ball, i.e. up to translations $\Omega = B_R$ for some $R > 0$, in which case $E_m = B_{r_m}$ is the unique minimizer.*

Remark 1.2. We stress that, for the previous statement to be true, in the definition of $F_{\gamma,m}$ we have taken the total variation of Du in the whole space, thus adding to the energy a contribution which may be considered as accounting for an interaction with the boundary of Ω. Indeed, in the case of multiple droplets patterns, on some mesoscopic scale a single droplet feels a repulsive effect due to all the other droplets acting as a 'virtual repulsive boundary' term. If we removed such a boundary contribution to the confining term, the optimal shape would in fact be an almost half ball located in a point of smallest mean curvature of $\partial\Omega$ and a result similar to the one stated in the previous theorem could be true. An analysis of this issue, though interesting in its own, is not pursued here.

Many of the mathematical challenges in Theorem 1.1 are due to our choice to work in any dimension n (previous results are mostly in dimensions $n = 2,3$), with the standard Coulombian kernel and the natural Neumann boundary condition. To this regard, it is worth comparing our results to analogous ones obtained recently on problem with single droplet minimizers. In [21] Figalli and Maggi consider an anisotropic perimeter perturbed via a *local* potential term. In the regime of small masses, they prove convergence of the minimizers to the associated Wulff shape. The presence of the nonlocal term in $F_{\gamma,m}$ does not allow to deduce our results from [21] and requires new ideas. At the time we proved our result, Knüpfer and Muratov studied in [31] the existence of exact spherical solutions to a nonlocal isoperimetric problem in $\mathbb{R}^2$, where the nonlocal term is a Coulombian-type interaction with kernel $K(x,y) = |x-y|^{-\alpha}$ for some $\alpha \in (0,2)$. It is worth observing that such a choice for the nonlocal term, as well as the absence of natural boundary conditions, make the results in [31] different from our 2-dimensional analogue. In the works by Oshita [41] and Ren and Wei [43], by the use of a careful Lyapunov-Schmidt reduction, the authors construct special confined solutions in dimension $n = 2$ to elliptic systems of the form:

$$\begin{cases} -\Delta v = \chi_E - m & \text{in } \Omega, \\ \nabla v \cdot \nu = 0 & \text{on } \partial\Omega, \\ \gamma v + H_{\partial E} = 0 & \text{on } \partial E, \end{cases} \tag{5}$$

where $H_{\partial E}$ is the mean curvature of the boundary of a set E, and γ and m are suitably choosen (see also [44] for the case of multiple droplets in dimension $n = 3$). This system is the Euler–Lagrange equations of the functional $F_{\gamma,m}$ whenever χ_E is a smooth critical point. Therefore, as a byproduct of Theorem 1.1, we are able to extend these results showing the existence of single droplete solutions to (5) in any space dimension as the (local) minimizers of the associated functional $F_{\gamma,m}$.

1.3. **An approach via regularity theory.** The reason why most of the previous results are two dimensional is partially due to the fact that the isoperimetric confinement in the plane is strong enough to allow non-parametric techniques. In higher dimensions, on the contrary, having small perimeter does not even imply boundedness (e.g., consider a very thin tube). One of the main contributions of this paper is to provide robust arguments to overcome this difficulty. To this aim, we exploit a combination of two facts in the regularity theory of minimal surfaces: the uniform regularity properties of minimizers and the use of the optimal quantitative isoperimetric inequality. To our knowledge, this is the first time that the sharp exponent 2 in the quantitative isoperimetric inequality has an essential role, and we are aware of only one case where the uniform regularity property is exploited similarly in a recent paper by Acerbi, Fusco and Morini [1], in which the authors study local minimizers for the functional $F_{\gamma,m}$ via second variations.

The techniques developed in this paper may also be applied to several other related models which have been considered in the last years. Indeed, the arguments exploited here do not rely strongly on the form of the isoperimetric term neither on that of the nonlocal one, but rather on energy scaling and regularity properties of minimizers. For example, the ideas we develope may be useful to address the challenging problem of multiple droplets minimizers in its full generality (partial results are proved in [12, 13] by Choksi and Peletier for a regime of finitely many droplets, and in [38] by Muratov for multiple droplets (asymptotically infinite) in two dimensions for a slightly different nonlocal interaction). Among the possible extension of our results, we mention the cases of:

(1) multiple droplets patterns;
(2) models presenting different Coulombian-type kernels G;
(3) droplets minimizers for the Ohta–Kawasaki functional (1);
(4) anisotropic perimeter functionals;
(5) *nonlocal perimeters*, as those considered by Carlen et al. in [10].

1.4. **Plan of the paper.** The paper is organized as follows. In § 2 we fix the notation and recall some known preliminary results which will be used in the proof of Theorem 1.1. In § 3 we prove a quantitative Lipschitz continuity of the nonlocal part of the energy, deriving the first regularity conclusions such as the almost minimality of the minimizers. Then, in the short section § 4 we show how this observation leads to the main result of this paper in the simpler case of periodic boundary conditions. The proof of the general case is given in § 5. In § 6 we discuss the existence of perfectly spherical solutions, showing how the regularity plays a role also in the study of the stability. The final Appendix is devoted to the proof of some estimates on the Green function used through the paper.

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2. Notation and preliminaries

In what follows $\Omega \subset \mathbb{R}^n$ is a bounded open set with C^2 boundary $\partial\Omega$. For given constants $m \in (0,1)$ and $\gamma > 0$, we consider the sharp interface limit of the Ohta–Kawasaki functional $F_{\gamma,m}$ which can be written in the following way:

$$F_{\gamma,m}(u) := \int_{\mathbb{R}^n} |Du| + \gamma \int_\Omega \int_\Omega G(x,y)\,\big(u(x)-m\big)\big(u(y)-m\big)\,dx\,dy. \tag{6}$$

The order parameter u belongs to the class $\mathscr{C}_m(\Omega)$ (often we will simply write $\mathscr{C}_m$) of functions with bounded variation taking values in $\{0,1\}$, whose average in Ω is m and which are constantly 0 outside Ω:

$$\mathscr{C}_m := \left\{ u \in BV(\mathbb{R}^n,\{0,1\}) \,:\, \fint_\Omega u = m, \quad u|_{\mathbb{R}^n\setminus\Omega} = 0 \right\}. \tag{7}$$

As already noticed in the introduction, we stress that the total variation of Du is computed in the whole $\mathbb{R}^n$, thus accounting also for possible concentration of this measure (interfaces of the physical system) on the boundary of Ω. In the second term in (6), G denotes the Green function of the Laplacian with Neumann boundary conditions on $\partial\Omega$.

Denoting by ν the exterior normal to $\partial\Omega$ and by $|A|$ the n-dimensional Lebesgue measure of the set A, G is defined by the following boundary value problem: for every $x \in \Omega$,

$$\begin{cases} -\Delta G(x,\cdot) = \delta_x - \frac{1}{|\Omega|} & \text{in } \Omega, \\ \nabla G(x,\cdot)\cdot\nu = 0 & \text{on } \partial\Omega, \\ \int_\Omega G(x,y)\,dy = 0. \end{cases} \tag{8}$$

In place of the average m, we often make use of the parameter r_m corresponding to the radius of a ball whose volume fraction in Ω is m, i.e.

$$\omega_n\, r_m^n := m\,|\Omega|.$$

Moreover, we often identify $u \in \mathscr{C}_m$ with the set of finite perimeter E such that $u = \chi_E$ (see [5, 26]). Accordingly, we write the energy $F_{\gamma,m}$ of χ_E as depending on E in the following way:

$$F_{\gamma,m}(E) := \mathrm{Per}(E) + \gamma \mathrm{NL}(E),$$

where $\mathrm{Per}(E) = \int_{\mathbb{R}^n} |D\chi_E|$ is the perimeter of E in $\mathbb{R}^n$ and NL is the nonlocal part of the energy. Note that, thanks to $\int_\Omega G(x,y)\,dy = 0$ for every $x \in \Omega$, we may rewrite the nonlocal term as

$$\mathrm{NL}(E) := \int_\Omega \int_\Omega G(x,y)\,\chi_E(x)\,\chi_E(y)\,dx\,dy. \tag{9}$$

Finally, we fix the following convention regarding the constants we use in the formula. Every time we use the letter C for a constant, this is assumed to be positive and depending only on the dimension of the space n and the domain Ω. When possible, we will use the simbols $a \lesssim b$, $a \gtrsim b$ and $a \simeq b$ for $a \le Cb$, $a \ge Cb$ and $C^{-1}b \le a \le Cb$, respectively. When we need to keep track of the constants, we number them accordingly.

2.1. **Robin function and harmonic centers.** Here we recall some facts about the Green function G. First of all its symmetry $G(x,y) = G(y,x)$ for all $x \neq y \in \Omega$. Next let Γ be the fundamental solution of the Laplacian, i.e.

$$\Gamma(t) := \begin{cases} \dfrac{\log t}{2\pi} & \text{if } n = 2, \\ \dfrac{t^{2-n}}{n(2-n)\omega_n} & \text{if } n \ge 3, \end{cases} \tag{10}$$

and define the regular part R of the Green function in (8) as

$$R(x,y) := G(x,y) + \Gamma(|x-y|).$$

Even if, in principle, R is not defined in $y = x$, nevertheless, for every $x \in \Omega$, $R(x,\cdot)$ solves the following boundary value problem:

$$\begin{cases} \Delta R(x,\cdot) = \frac{1}{|\Omega|} & \text{in } \Omega, \\ \nabla R(x,\cdot)\cdot\nu = \nabla\Gamma(|x-\cdot|)\cdot\nu & \text{on } \partial\Omega. \end{cases} \tag{11}$$

This implies that $R(x,\cdot)$ is an analytic function in the whole Ω and we can consider its extension to $y = x$:

$$h(x) := R(x,x).$$

The function h is called the Robin function. As it can be easily seen from (11) h turns out to be analytic in Ω.

Several estimates on the regular part of the Green function and on the Robin function will play an important role in the identification of the concentration points for the minimizers of $F_{\gamma,m}$. The following facts are used in the proofs: there exists r_0 depending only on Ω such that, for all $r \le r_0$,

$$|R(x,y)| \simeq |\Gamma(r)| \quad \forall\, x,y : \mathrm{dist}(x,\partial\Omega)+\mathrm{dist}(y,\partial\Omega) \simeq r,\ |x-y| \lesssim r. \tag{12}$$

Moreover, from (12) we deduce also:

$$|G(x,y)| \lesssim -\Gamma(|x-y|)+1, \quad \forall\, x,y \in \Omega \tag{13}$$

$$h(x) \simeq \big|\Gamma\big(\mathrm{dist}(x,\partial\Omega)\big)\big|, \quad \forall\, x \in \Omega\setminus\Omega_{r_0}, \tag{14}$$

where, for every $r>0$, we denote by Ω_r the complement in Ω of the closed r-neighborhood of $\partial\Omega$:

$$\Omega_r := \{x\in\Omega\,:\,\mathrm{dist}(x,\partial\Omega) > r\}. \tag{15}$$

These estimates are well-known in the case of Dirichlet boundary conditions (see for example [23]). Since we are not able to point out a reference for the Neumann boundary conditions, we give a proof in the Appendix A. From the regularity of h and (14), it follows that h is bounded from below. In particular, since h is analytic and blows up on the boundary of Ω (hence, in particular it has no constant directions), it follows that the set of minimum points of h is an analytical variety compactly supported in Ω: we denote this set by $\mathscr{H}$ and call it the *harmonic centers* of Ω.

2.2. **The quantitative isoperimetric inequality.** The classical isoperimetric inequality states that the perimeter of any measurable set E is bigger than the perimeter of a ball B_E having the same volume as E, with equality only in the case E is itself a ball. Quantitative versions of this inequality, also called *Bonnesen-type inequalities* [42], have been widely studied (see, for instance, [28, 29]). The following, called sharp quantitative isoperimetric inequality, has been proved in [16, 22, 24].

Proposition 2.1 (Sharp quantitative isoperimetric inequality)**.** *There exists a dimensional constant $C=C(n)>0$ such that for every measurable set $E\subset\mathbb{R}^n$ of finite measure with $0<|E|<+\infty$, it holds*

$$C\,\alpha(E)^2 \le \frac{\mathrm{Per}(E)-\mathrm{Per}(B_E)}{\mathrm{Per}(B_E)}, \tag{16}$$

where $\alpha(E)$ is the Frankel asymmetry of E,

$$\alpha(E) := \inf\left\{\frac{|E\triangle(x+B_E)|}{|B_E|},\ x\in\mathbb{R}^n\right\}.$$

Here, $V\triangle W=(V\setminus W)\cup(W\setminus V)$ is the symmetric difference between V and W. For any given $E\subset\mathbb{R}^n$ measurable set of positive and finite measure, we say that B_E^{opt} is an *optimal ball* for E if $|B_E^{opt}|=|E|$ and

$$\frac{|E\triangle B_E^{opt}|}{|B_E^{opt}|} = \alpha(E).$$

The center of an optimal ball will also be referred to as an *optimal center*. In general the optimal ball may not be unique. However, as proven in [17, Lemma 6.4] by an elementary application of the Brunn-Minkowsky inequality, whenever E is a strictly convex set the optimal ball is actually unique. Finally, we observe that, denoting by r the radius of B_E, (16) scales in r as follows:

$$|E\triangle B_E^{opt}|^2 \lesssim r^{n+1}\left(\mathrm{Per}(E)-\mathrm{Per}(B_E^{opt})\right). \tag{17}$$

2.3. **First variations.** The first variations of $F_{\gamma,m}$ have been computed for regular sets by Muratov [38] in dimension 2 and 3, and then in any dimension by Choksi and Sternberg [15]. Given a critical point E of $F_{\gamma,m}$ and $x \in \partial E$ a regular point of its boundary, the Euler–Lagrange equation of $F_{\gamma,m}$ in a neighborhood of x is given by:

$$H_{\partial E} + 4\gamma v = c, \tag{18}$$

where $H_{\partial E}$ denotes the scalar mean curvature of ∂E (namely, $H_{\partial E} = \operatorname{div} \nu_E$, with ν_E the outer normal to ∂E), $c \in \mathbb{R}$ is a constant coming from a Lagrange multiplier and v is the solution of the following boundary value problem:

$$\begin{cases} -\Delta v = \chi_E - m & \text{in } \Omega, \\ \nabla v \cdot \nu = 0 & \text{on } \partial\Omega, \\ \int_\Omega v = 0. \end{cases} \tag{19}$$

Since $\|\chi_E - m\|_{L^\infty} \le 1$, it follows that $v \in C^{1,\alpha}$ for every $\alpha \in (0,1)$. Therefore, from standard elliptic estimates for the quasilinear mean curvature operator (see [25]), (18) implies that, for every critical point E, ∂E is $C^{3,\alpha}$ for every $\alpha \in (0,1)$ in a neighborhood of a regular point. As shown in the next sections, every minimizer of $F_{\gamma,m}$ is regular except a singular set of Hausdorff dimension at most $n-8$ (in particular, the singular set is empty in the physical dimensions $n = 2,3$).

2.4. **Asymptotic energy of balls.** Here we give an asymptotic expansion of the energy of small round balls in Ω. Let Ω_r be defined as in (15). By the regularity assumption on $\partial\Omega$, there exists $r_0 > 0$ such that, for every $r \le r_0$ and $p \in \Omega_r$, the ball $B_r(p) \in \mathscr{C}_{\omega_n r^n}$. By a direct computation, it follows that

$$\begin{aligned} F_{\gamma,\omega_n r^n}(B_r(p)) &= \mathrm{Per}(B_r(p)) + \gamma \mathrm{NL}(B_r(p)) \\ &= n\,\omega_n\, r^{n-1} + \gamma \int_{B_r(p)} \int_{B_r(p)} \Gamma(|x-y|)\,dx\,dy + \\ &\quad + \gamma \int_{B_r(p)} \int_{B_r(p)} R(x,y)\,dx\,dy \\ &= \begin{cases} 2\pi r + \gamma \left(\frac{\pi}{2} r^4 \log r + \left(\pi^2 g_r(p) - \frac{3\pi}{8} \right) r^4 \right) & \text{if } n = 2, \\ n\,\omega_n\, r^{n-1} + \frac{2\gamma\omega_n}{4-n^2} r^{2n} + \gamma g_r(p) (\omega_n r^n)^2 & \text{if } n \ge 3, \end{cases} \end{aligned} \tag{20}$$

where $g_r : \Omega_r \to \mathbb{R}$ is given by

$$g_r(p) := \fint_{B_r(p)} \fint_{B_r(p)} R(x,y)\,dx\,dy. \tag{21}$$

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with C^2 boundary. Then, there exists $r_0 > 0$ such that, for all $r \le r_0/2$,*

$$\|g_r - h\|_{L^\infty(\Omega_{r_0})} \simeq r^2, \tag{22}$$

and, for every $p \in \Omega \setminus \Omega_{r_0}$,

$$g_s(p) \gtrsim |\Gamma(\mathrm{dist}(p,\partial\Omega))| \quad \forall\, s \le \mathrm{dist}(p,\partial\Omega). \tag{23}$$

Proof. To show (22), let r_0 be as in (12) and note that, since $R|_{\Omega_{r_0}\times\Omega_{r_0}}$ is analytic, we have

$$\begin{aligned} g_r(p)-h(p) &= \fint_{B_r}\fint_{B_r}\left(R(x+p,y+p)-R(p,p)\right)dx\,dy \\ &= \fint_{B_r}\fint_{B_r}\left(DR(p,p)(x,y)+\langle D^2R(p,p)\,(x,y),(x,y)\rangle\right)dx\,dy+o(r^2) \\ &= r^2\fint_{B_1}\fint_{B_1}\langle D^2R(p,p)\,(x,y),(x,y)\rangle\,dx\,dy+o(r^2), \end{aligned} \tag{24}$$

where in the last equality we used that the linear term integrates to 0. By the linearity of the integral and of the scalar product, it follows that

$$\begin{aligned} g_r(p)-h(p) = \sum_{i,j}&\Big(\partial_{x_i}\partial_{x_j}R(p,p)A_{x_ix_j}+2\,\partial_{x_i}\partial_{y_j}R(p,p)A_{x_iy_j}+\partial_{y_i}\partial_{y_j}R(p,p)A_{y_iy_j}\Big) \\ &+o(r^2) \end{aligned} \tag{25}$$

where

$$A_{x_ix_i}=A_{y_iy_i}=\mu:=\fint_{B_1}x_1^2\,dx \quad\text{and}\quad A_{x_ix_j}=A_{x_iy_j}=A_{y_iy_j}=0.$$

Using the simmetry $R(x,y)=R(y,x)$, we infer from (25) that

$$\begin{aligned} g_r(p)-h(p) =&\mu\,\mathrm{Tr}\left(D^2R(p,p)\right)r^2+o(r^2)=2\,\mu\,\Delta_yR(p,p)\,r^2+o(r^2) \\ =&\frac{2\,\mu\,r^2}{|\Omega|}+o(r^2), \end{aligned}$$

thus leading to (22). In order to show (23), it sufficies to notice that

$$g_s(p)\gtrsim\fint_{B_{\frac{s}{2}}(p)}\fint_{B_{\frac{s}{2}}(p)}R(x,y)\,dx\,dy\overset{(12)}{\gtrsim}\Gamma(\mathrm{dist}(p,\partial\Omega)).$$

□

3. REGULARITY OF MINIMIZERS

In this section we prove the Lipschitz continuity of the nonlocal term, from which we derive the uniform regularity properties of the minimizers in the relevant regimes.

3.1. **Lipschitz continuity of NL.**

Proofs of the Lipschitz continuity of NL already appeared in the literature (see, for instance, [1, 38, 48]). For our purposes, a more careful quantitative estimate of the Lipschitz constant is necessary.

Proposition 3.1. *For every $\chi_{E_m},\chi_{G_m}\in\mathscr{C}_m$, setting $w=\Gamma*\chi_{G_m}$, it holds*

$$\mathrm{NL}(G_m)-\mathrm{NL}(E_m)\lesssim\left(\|w\|_{L^\infty(E_m\triangle G_m)}+|G_m|\right)|E_m\triangle G_m|. \tag{26}$$

Proof. We start from (9) to get

$$
\begin{aligned}
\mathrm{NL}(G_m)-\mathrm{NL}(E_m) &= \int_\Omega\int_\Omega G(x,y)\big(\chi_{G_m}(x)\,\chi_{G_m}(y)-\chi_{E_m}(x)\,\chi_{E_m}(y)\big)\,dx\,dy\\
&= \int_\Omega\int_\Omega G(x,y)\chi_{G_m}(x)\,\big(\chi_{G_m}(y)-\chi_{E_m}(y)\big)\,dx\,dy+\\
&\quad+\int_\Omega\int_\Omega G(x,y)\,\chi_{E_m}(y)\big(\chi_{G_m}(x)-\chi_{E_m}(x)\big)\,dx\,dy\\
&= 2\int_\Omega\int_\Omega G(x,y)\,\chi_{G_m}(x)\,\big(\chi_{G_m}(y)-\chi_{E_m}(y)\big)\,dx\,dy-\\
&\quad-\int_\Omega\int_\Omega G(x,y)\,\big(\chi_{G_m}(y)-\chi_{E_m}(y)\big)\cdot\\
&\qquad\cdot\big(\chi_{G_m}(x)-\chi_{E_m}(x)\big)\,dx\,dy, \qquad (27)
\end{aligned}
$$

where in the last equality we used the symmetry $G(x,y)=G(y,x)$. Since

$$
\int_\Omega\int_\Omega G(x,y)\,\big(\chi_{G_m}(y)-\chi_{E_m}(y)\big)\cdot\big(\chi_{G_m}(x)-\chi_{E_m}(x)\big)\,dx\,dy=\int_\Omega|\nabla z(x)|^2\,dx\geq 0,
$$

where z solves

$$
\begin{cases}-\Delta z=\chi_{G_m}-\chi_{E_m} & \text{in } \Omega,\\ \nabla z\cdot\nu=0 & \text{on } \partial\Omega,\end{cases}
$$

we deduce:

$$
\begin{aligned}
\mathrm{NL}(G_m)-\mathrm{NL}(E_m) &\leq 2\int_\Omega\int_\Omega G(x,y)\,\chi_{G_m}(x)\,\big(\chi_{G_m}(y)-\chi_{E_m}(y)\big)\,dx\,dy \qquad (28)\\
&\overset{(13)}{\lesssim}\int_\Omega\int_\Omega(-\Gamma(|x-y|)+1)\,\chi_{G_m}(x)\,\big|\chi_{G_m}(y)-\chi_{E_m}(y)\big|dx\,dy\\
&=\int_\Omega(|G_m|-w(y))\big|\chi_{G_m}(y)-\chi_{E_m}(y)\big|\,dy\\
&\lesssim(\|w\|_{L^\infty(E_m\triangle G_m)}+|G_m|)\,|E_m\triangle G_m|. \qquad\square
\end{aligned}
$$

A straightforward consequence of Proposition 3.1 is that, if E_m is a minimizer of $F_{\gamma,m}$, then

$$
\begin{aligned}
\mathrm{Per}(E_m)-\mathrm{Per}(G_m) &\leq \gamma\big(\mathrm{NL}(G_m)-\mathrm{NL}(E_m)\big)\\
&\lesssim\gamma(\|w\|_{L^\infty(E_m\triangle G_m)}+|G_m|)\,|E_m\triangle G_m|. \qquad (29)
\end{aligned}
$$

By the radial monotonicity of Γ, it holds

$$
\|\Gamma*\chi_{G_m}\|_{L^\infty}\leq\|\Gamma*\chi_{B_{r_m}}\|_{L^\infty}=\begin{cases}\frac{r_m^2}{2}\left(\frac12-\log r_m\right) & \text{if } n=2,\\ \frac{r_m^2}{2\,(n-2)} & \text{if } n\geq 3,\end{cases} \qquad (30)
$$

for every G_m with $|G_m|=|B_{r_m}|$. As a consequence, for r_m sufficiently small, we have

$$
\|w\|_{L^\infty}+|G_m|\lesssim\|\Gamma*\chi_{B_{r_m}}\|_{L^\infty}+r_m^n\lesssim\begin{cases}\frac{r_m^2}{2}\left(\frac12-\log r_m\right) & \text{if } n=2,\\ \frac{r_m^2}{2\,(n-2)} & \text{if } n\geq 3.\end{cases} \qquad (31)
$$

Here we have used the direct computations:

$$
\Gamma*\chi_{B_r}(x)=\begin{cases}\frac{|x|^2}{4}+\frac{r^2}{2}\,(\log r-1) & \text{if } |x|\leq r,\\ \frac{r^2}{2}\left(\log|x|-\frac12\right) & \text{if } |x|>r,\end{cases} \qquad \text{if } n=2, \qquad (32)
$$

$$\Gamma * \chi_{B_r}(x) = \begin{cases} \frac{|x|^2}{2n} + \frac{r^2}{2(2-n)} & \text{if } |x| \le r, \\ \frac{r^n}{n(2-n)|x|^{n-2}} & \text{if } |x| > r, \end{cases} \qquad \text{if } n \ge 3. \tag{33}$$

As for r_m small enough $\chi_{B_{r_m}(p)} \in \mathscr{C}_m$ for some $p \in \Omega$, it follows by the previous two estimates, with $G_m = B_{r_m}(p)$, that

$$\mathrm{Per}(E_m) - \mathrm{Per}(B_{r_m}(p)) \lesssim \begin{cases} \gamma r_m^2 |\log r_m| |E_m \triangle B_{r_m}(p)| & \text{if } n = 2, \\ \gamma r_m^2 |E_m \triangle B_{r_m}(p)| & \text{if } n \ge 3. \end{cases} \tag{34}$$

By the quantitative isoperimetric inequality (17), there exists an optimal isoperimetric ball $B_{E_m}^{opt}$ for E_m such that

$$|E_m \triangle B_{E_m}^{opt}|^2 \lesssim r_m^{n+1} \left(\mathrm{Per}(E_m) - \mathrm{Per}(B_{E_m}^{opt})\right). \tag{35}$$

In the case $\chi_{B_{E_m}^{opt}} \in \mathscr{C}_m$, gathering together (35) and (34), we have that

$$|E_m \triangle B_{E_m}^{opt}| \lesssim \begin{cases} \gamma r_m^{n+3} |\log r_m| & \text{if } n = 2, \\ \gamma r_m^{n+3} & \text{if } n \ge 3. \end{cases} \tag{36}$$

3.2. **Volume constraint.** In order to deduce uniform regularity properties for minimizers, it is convenient to get rid of the volume constraint. To this purpose we use a penalization argument. We first rescale our sets: set p_m for the barycenter of E_m and

$$H_m := \frac{E_m - p_m}{r_m} \subset \Omega_m := \frac{\Omega - p_m}{r_m}.$$

We note that H_m is a minimizer of $F_{\gamma r_m^3, m}$ in $\mathscr{C}_m(\Omega_m)$. The following lemma shows that, if H_m is sufficiently close to a given $H \subset \mathbb{R}^n$ well-contained in Ω_m, the volume constraint can be dropped.

Lemma 3.2. *Let $m_0 > 0$ be a given constant and $H_m \subset \Omega_m$ be as above and $\gamma r_m^3 \lesssim 1$ for $m \in (0, m_0)$. Let $H \subset \Omega_m$ be a set of finite perimeter such that* $\mathrm{dist}(H, \partial\Omega_m) \ge 1$ *for every $m \in (0, m_0)$. Then, there exists $\Lambda > 0$ with this property: for every $m \in (0, m_0)$, if $|H \triangle H_m| \le \Lambda^{-1}$, then H_m is a minimizer of $G_{\Lambda,m}$,*

$$G_{\Lambda,m}(E) := F_{\gamma r_m^3, m}(E) + \Lambda \big||E| - \omega_n\big|,$$

in the class of all sets E with $|E \triangle H| \le 2\Lambda^{-1}$.

The proof of the lemma follows from a simple adaptation of the computations in [20, Section 2] (see also [1, Proposition 2.7]). We give here only the necessary modifications.

Proof. The proof is by contradiction. Assume that there exist $\Lambda_h \to +\infty$ with this property: there exist $m_h \in (0, m_0)$, H_h minimizers of $F_h := F_{\gamma r_{m_h}^3, m_h}$ and E_h such that:

(a) $|H_h \triangle H| \le \Lambda_h^{-1}$;
(b) $|E_h \triangle H| \le 2\Lambda_h^{-1}$;
(c) $|E_h| < |H_h| = \omega_n$ (the case $|E_h| > |H_h|$ is analogous);
(d) $G_h(E_h) := G_{\Lambda_h, m_h}(E_h) < F_h(H_h)$.

Since $E_h \to H$ in $L^1(\mathbb{R}^n)$, as in [1, Proposition 2.7], one can show the existence of suitable deformations $\tilde{E}_h$ and constants $\sigma_h > 0$ satisfying the following: $|\tilde{E}_h| = |H_h| = \omega_n$,

$\mathrm{dist}(\tilde{E}_h, H) < 1$ (in particular, $\tilde{E}_h \subset \Omega_m$) and

$$|H_h| - |E_h| \geq c_1(n)\,\sigma_h, \tag{37}$$

$$\mathrm{Per}(\tilde{E}_h) \leq \mathrm{Per}(E_h)\,(1 + c_2(n)\,\sigma_h), \tag{38}$$

$$|\tilde{E}_h \triangle E_h| \leq c_3(n)\,\sigma_h\,\mathrm{Per}(E_h), \tag{39}$$

where $c_1, c_2, c_3 > 0$ are dimensional constants. Hence, we infer that, for h sufficiently large,

$$\begin{aligned} F_h(\tilde{E}_h) &= G_h(\tilde{E}_h) \\ &\leq G_h(E_h) + \big[c_2(n)\,\sigma_h\,\mathrm{Per}(E_h) + C\,|E_h \triangle \tilde{E}_h| - \Lambda_h\,||E_h| - \omega_n|\big] \\ &\overset{(d),(37)-(39)}{<} F_h(H_h) + \sigma_h\,\big[c_2\,\mathrm{Per}(E_h) + C\,c_3\,\mathrm{Per}(E_h) - c_1\,\Lambda_h\big] \\ &< F_h(H_h), \end{aligned}$$

where we used the Lipschitz continuity of the nonlocal term with $\gamma r_m^3 \lesssim 1$, $\Lambda_h \to +\infty$ and the uniform bound on $\mathrm{Per}(E_h)$ implied by (d):

$$\mathrm{Per}(E_h) < F_h(H_h) \leq F_{\gamma r_{m_h}^3, m_h}(B_1) < +\infty \quad \forall\, h \in \mathbb{N}.$$

This contradicts the minimizing property of H_h and proves the lemma. □

3.3. **Λ-minimizers.** It follows from Lemma 3.2 that the sets H_m are uniform strong Λ-minimizer of the perimeter according to the following definition.

Definition 3.3. Let $\Omega \subset \mathbb{R}^n$ be open. A set of finite perimeter $E \subset \Omega$ is a *strong* Λ-*minimizer* of the perimeter in Ω at scale $\eta > 0$ if

$$P(E) \leq P(F) + \Lambda |E \triangle F|, \quad \forall\, E \triangle F \subset\subset \Omega,\ |E \triangle F| \leq \eta. \tag{40}$$

This notion of almost minimality has been widely studied in the regularity theory for minimal surfaces. By the theory developed in [4, 8, 27, 50], strong Λ-minimizers have regularity estimates which are uniform in the parameters Λ and η. More precisely, recall the notation $D\chi_E$ for the vector valued measure given by the distributional derivative of the BV function χ_E; then, for every $\alpha \in (0,1)$, there exists a constant $\varepsilon_0 = \varepsilon_0(n, \alpha, \Lambda, \eta)$ such that

$$\mathrm{Exc}(E, B_r(x)) := r^{1-n}\Big(|D\chi_E|(B_r(x)) - |D\chi_E(B_r(x))|\Big) \leq \varepsilon_0 \\ \Longrightarrow \partial E \cap B_{\frac{r}{2}}(x) \in C^{1,\alpha}.$$

Since the quantity Exc is continuous under L^1 convergence of Λ-minimizers, uniform regularity estimates can be inferred for Λ-minimizers in a neighborhood of a given smooth set.

Proposition 3.4. *Let $\Lambda, \eta > 0$ be given constants and let $F \subset \Omega$ be a set with smooth boundary and* $\mathrm{dist}(F, \partial\Omega) \geq 1$. *Then, for every $\alpha \in (0,1)$, there exist constants $\eta_0 = \eta_0(n, \alpha, \Lambda, \eta, F) > 0$, $R = R(n, \Lambda, \eta, F) > 0$, $c = c(n) > 0$ and a modulus of continuity $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ with this property: for every $E \subset \Omega$ strong Λ-minimizers at scale η,*

(i) *if $|E \triangle F| \leq \eta_0$, then ∂E can be parametrized on ∂F by a function $\varphi : \partial F \to \mathbb{R}$,*

$$\partial E = \big\{x + \varphi(x)\,\nu_F(x) \,:\, x \in \partial F\big\},$$

with $\|\varphi\|_{C^{1,\alpha}} \leq \omega(|E \triangle F|)$;

(ii) *for all $x \in E$ and $0 < r < R$ with $B_r(x) \subset \Omega$, it holds*

$$c(n)\, r^n \leq |E \cap B_r(x)|. \tag{41}$$

Although never stated in this form, Proposition 3.4 is a direct consequence of the already known regularity theory (in particular, see [50, Theorem 1.9 and Proposition 3.4]). Note, however, that we will apply Proposition 3.4 always in the case $F = B_1$.

3.4. **Higher regularity.** Thanks to Proposition 3.4, the first variations in § 2.3 can be used to improve the regularity of the minimizers of $F_{\gamma,m}$.

Proposition 3.5. *Let E_m be a minimizer of $F_{\gamma,m}$ and let H_m, p_m and Ω_m be as in § 3.2. Then, for every $\alpha \in (0,1)$, there exists $\eta > 0$ such that, if $\gamma r_m^3 \lesssim 1$, $|H_m \triangle B_1| \leq \eta$ and $\mathrm{dist}(B_1, \partial\Omega_m) \geq 1$, then H_m can be parametrized on ∂B_1,*

$$\partial H_m = \big\{(1 + \varphi_m(x))\, x : x \in \partial B_1\big\},$$

and $\|\varphi_m\|_{C^{3,\alpha}} \leq \bar\omega(|H_m \triangle B_1|)$ for a given modulus of continuity $\bar\omega$.

Proof. The existence of a parametrization φ_m is guaranteed by Proposition 3.4 (i), under the hypothesis that η is chosen sufficiently small. We need only to show that the Euler–Lagrange equation for $F_{\gamma,m}$, namely

$$H_{\partial H_m}(x + \varphi_m(x)\, x) + 4\, \gamma\, r_m^3\, w_m(x + \varphi_m(x)\, x) = \lambda_m, \tag{42}$$

allows actually to infer the higher regularity claimed in the statement. Here $\lambda_m \in \mathbb{R}$ is a Lagrange multiplier and w_m solves the boundary value problem:

$$\begin{cases} -\Delta w_m = \chi_{H_m} - m & \text{in } \Omega_m, \\ \nabla w_m \cdot \nu = 0 & \text{on } \partial\Omega_m, \\ \int_{\Omega_m} w_m = 0. \end{cases} \tag{43}$$

It suffices to prove $\sup_m \|\varphi_m\|_{C^{3,\alpha'}} \leq C$ for every $\alpha' \in (0,1)$. Indeed, since we have that $\|\varphi_m\|_{C^{1,\alpha}} \leq \omega(|H_m \triangle B_1|)$, where ω is the modulus of continuity in Proposition 3.4, by compactness in the $C^{3,\alpha}$ norm for $\alpha < \alpha'$ we would as well deduce that $\|\varphi_m\|_{C^{3,\alpha}} \to 0$ as $|H_m \triangle B_1| \to 0$.

To show this, we consider separately the two terms in (42). For what concerns λ_m we recall that, by Lemma 3.2 there exists $\Theta > 0$ such that H_m minimize $G_{\Theta,m}$ locally in a neighborhood of B_1. This allows us to compute the first variations of $G_{\Theta,m}$. Since the penalization term $\Theta\, ||E| - \omega_n|$ is not differentiable, we have to distinguish between the variations increasing the volume and those decreasing it. Let $\psi \in C^\infty(\partial B_1)$ and K_ε be the competitor such that

$$\partial K_\varepsilon := \big\{x + (\varphi_m(x) + \varepsilon\, \psi(x))\, x : x \in \partial B_1\big\}.$$

The volume of K_ε is given by

$$|K_\varepsilon| = n^{-1} \int_{\partial B_1} (1 + \varphi_m + \varepsilon\, \psi)^n\, d\mathscr{H}^{n-1},$$

hence, it follows that $|K_\varepsilon| > \omega_n$ or $|K_\varepsilon| < \omega_n$ for small $\varepsilon > 0$ if $\psi > 0$ or $\psi < 0$, respectively. The minimizing property of H_m implies the following variational inequality to hold true:

$$\frac{dG_{\Theta,m}(K_\varepsilon)}{d\varepsilon}\Big|_{\varepsilon = 0^+} \geq 0.$$

In turns this leads to (with analogous computations for the first variations of $F_{\gamma,m}$ as in [15])

$$\int_{\partial B_1} \Big(H_{\partial H_m}(x+\varphi_m(x)x)+4\gamma r_m^3 w_m(x+\varphi_m(x)x)+\Theta\Big)\,\psi(x)\geq 0 \quad \text{if } \psi>0, \tag{44}$$

$$\int_{\partial B_1} \Big(H_{\partial H_m}(x+\varphi_m(x)x)+4\gamma r_m^3 w_m(x+\varphi_m(x)x)-\Theta\Big)\,\psi(x)\geq 0 \quad \text{if } \psi<0. \tag{45}$$

Since ψ in (44) and (45) is an arbitrary positive and negative function respectively, we deduce a uniform bound on the Lagrange multipliers λ_m:

$$|\lambda_m|\leq\Theta \quad \forall m>0. \tag{46}$$

For what concerns w_m, by an analogous computation as in (30) using $|G|\lesssim|\Gamma|+1$ and the radial monotonicity of Γ, we deduce that $\|w_m\|_{L^\infty}\leq C$. Moreover, since $\|\chi_{H_m}-\chi_{B_1}\|_{L^p}\lesssim\eta$ for every $p>n$, the Sobolev embeddings and the Gagliardo–Nirenberg interpolation inequality lead to uniform $W^{2,p}$ bounds and, hence, $C^{1,\alpha'}$ bounds on w_m for every $\alpha'\in(0,1)$ (see [9, Chapter 9]). Therefore, since φ_m has also uniform $C^{1,\alpha'}$ bounds, the non-parametric theory for the mean curvature-type equation (42) (see [33, Chapter 3] or [26, Appendix C]) finally yelds the desired uniform $C^{3,\alpha'}$ estimates for φ_m. □

4. Periodic boundary conditions: $\Omega=\mathbb{T}^n$

Here we show the proof of our main result in a technically simpler case, namely for periodic boundary conditions. Indeed, in this case one discards the interactions with the boundary and the optimal centering of the asymptotic droplet, and the proof is a direct consequence of the regularity arguments developed in the previous section.

4.1. **Notation and statement.** Let $\mathbb{T}^n$ be the n-dimensional torus obtained as the quotient of $\mathbb{R}^n$ via the $\mathbb{Z}^n$ lattice or, equivalently, $\mathbb{T}^n:=[0,1]^n$ with the identification of opposite faces. We consider functions

$$u\in BV(\mathbb{T}^n;\{0,1\}) \quad \text{with} \quad \fint_{\mathbb{T}^n} u=m.$$

As usual such functions u can be identified with measurable sets $E\subseteq\mathbb{R}^n$ invariant under the action of $\mathbb{Z}^n$ and such that $|E\cap[0,1]^n|=m$. The confining term of the energy is then given by the perimeter of E in the torus:

$$\mathrm{Per}(E,\mathbb{T}^n):=\int_{[0,1)^n}|D\chi_E|;$$

and the nonlocal term by:

$$\mathrm{NL}(E):=\int_{[0,1]^n}\int_{[0,1]^n} G(x,y)\,\chi_E(x)\,\chi_E(y)\,dx\,dy,$$

where G is the Green function for the Laplacian in $\mathbb{T}^n$, i.e.

$$\begin{cases}-\Delta G(x,\cdot)=\delta_x-1 & \text{in } \mathbb{T}^n,\\ \int_\Omega G(x,y)\,dy=0.\end{cases} \tag{47}$$

By the invariance of the torus under translations, we can write with a sligth abuse of notation $G(x,y)=G(|x-y|)$. In the case of periodic boundary conditions, Theorem 1.1 reduces to a statement regarding the shape of the minimizers E_m and the asymptotic behavior of the energy.

Theorem 4.1. *There exists $\delta_0 > 0$ such that the following holds. Assume $r_m < 1$ and*

$$\gamma r_m^3 |\log r_m| < \delta_0 \quad \textit{if} \quad n = 2 \quad \textit{or} \quad \gamma r_m^3 < \delta_0 \quad \textit{if} \quad n \geq 3.$$

Then, every minimizer $E_m \subset \mathbb{T}^n$ of $F_{\gamma,m}$ is, up to a translation, a convex set such that

$$\partial E_m = \{(1 + \psi_m(x))\, r_m\, x : x \in \mathbb{S}^{n-1}\},$$

for some $\psi_m : \mathbb{S}^{n-1} \to \mathbb{R}$ with

$$\|\psi_m\|_{C^1} \lesssim \gamma r_m^{n+3}, \tag{48}$$

and its energy has the following asymptotic expansion:

$$F_{\gamma,m}(\chi_{E_m}) = \begin{cases} 2\pi r_m + \frac{\pi\gamma}{2} r_m^4 \log r_m + \gamma\left(-\frac{1}{8} + \pi^2 h(0)\right) r_m^4 + O(\gamma r_m^6) & \textit{if } n = 2, \\ n\,\omega_n r_m^{n-1} + \frac{2\gamma\omega_n}{4-n^2} r_m^{n+2} + \gamma\,\omega_n^2 r_m^{2n} h(0) + O(\gamma r_m^{2n+2}) & \textit{if } n \geq 3, \end{cases} \tag{49}$$

where h is the Robin function associated to G.

4.2. **Improved perimeter estimate.** Due to the translation invariance, we may assume that for a given minimizer E_m the optimal ball $B_{E_m}^{opt} = B_{r_m}$ is centered at the origin. Therefore, from (36) we infer for $H_m = E_m / r_m$ that

$$|H_m \triangle B_1| \lesssim \begin{cases} \gamma r_m^3 |\log r_m| < \delta_0 & \text{if } n = 2, \\ \gamma r_m^3 < \delta_0 & \text{if } n \geq 3. \end{cases}$$

If δ_0 is chosen sufficiently small, by Lemma 3.2, the sets H_m are minimizer of some penalized functional $G_{\Lambda,m}$ and, hence, are Λ-minimizers of the perimeter. By Proposition 3.4, H_m can be parametrized by the graph of a function φ_m on ∂B_1 satisfying

$$\|\varphi_m\|_{L^\infty(\partial B_1)} \lesssim |H_m \triangle B_1|,$$

thus implying that E_m can be parametrized on ∂B_{r_m} by ψ_m with

$$\|\psi_m\|_{L^\infty(\partial B_{r_m})} \lesssim \frac{|E_m \triangle B_{r_m}|}{r_m^{n-1}}. \tag{50}$$

These observations lead to the following proposition which is a consequence of an improved estimate for the Lipschitz constant of the nonlocal part of the energy.

Proposition 4.2. *There exists $\delta_0 > 0$ such that, if $\gamma r_m^3 |\log r_m| < \delta_0$ in the case $n = 2$ or if $\gamma r_m^3 < \delta_0$ in the case $n \geq 3$, and E_m is a minimizer of $F_{\gamma,m}$, then*

$$\mathrm{Per}(E_m) - \mathrm{Per}(B_{E_m}^{opt}) \lesssim \gamma \frac{|E_m \triangle B_{E_m}^{opt}|^2}{r_m^{n-2}} + \gamma r_m^{n+1} |E_m \triangle B_{E_m}^{opt}|. \tag{51}$$

Proof. Recalling (28) in Proposition 3.1, and assuming as above $B_{r_m} = B_{E_m}^{opt}$, we have that

$$\begin{aligned} \mathrm{NL}(B_{r_m}) - \mathrm{NL}(E_m) &\lesssim \int_\Omega \int_\Omega G(x,y)\, \chi_{B_{r_m}}(x) \left(\chi_{B_{r_m}}(y) - \chi_{E_m}(y)\right) dx\, dy \\ &= \int_\Omega \int_\Omega \Big(\Gamma(|x-y|)\, \chi_{B_{r_m}}(x) - \alpha\Big) \left(\chi_{B_{r_m}}(y) - \chi_{E_m}(y)\right) dx\, dy + \\ &\quad + \int_\Omega \int_\Omega R(x,y)\, \chi_{B_{r_m}}(x) \left(\chi_{B_{r_m}}(y) - \chi_{E_m}(y)\right) dx\, dy, \end{aligned}$$

where we used $\int \chi_{B_{r_m}} - \int \chi_{E_m} = 0$ and we set

$$\alpha := \begin{cases} \frac{r_m^2}{2}\left(\log r_m - \frac{1}{2}\right) & \text{if } n = 2, \\ \frac{r_m^2}{2(2-n)} & \text{if } n \geq 3. \end{cases}$$

By the direct computation of $w = \Gamma * \chi_{B_{r_m}}$ in (32) and (33) (in particular, $|\nabla w| \lesssim r_m$ in a neighborhood of ∂B_{r_m}), we get,

$$\|w-\alpha\|_{L^\infty(E_m\triangle B_{r_m})} \lesssim r_m\,\|\psi_m\|_{L^\infty(\partial B_{r_m})} \overset{(50)}{\lesssim} \frac{|E_m\triangle B_{r_m}|}{r_m^{n-2}}.$$

Moreover, again using $\int \chi_{B_{r_m}} - \int \chi_{E_m} = 0$,

$$\begin{aligned}
&\int_\Omega\int_\Omega R(x,y)\,\chi_{B_{r_m}}(x)\,\big(\chi_{B_{r_m}}(y)-\chi_{E_m}(y)\big)\,dx\,dy\\
&\qquad = R(0,0)\int_\Omega\int_\Omega \chi_{B_{r_m}}(x)\,\big(\chi_{B_{r_m}}(y)-\chi_{E_m}(y)\big)\,dx\,dy + O(r_m^{n+1})|E_m\triangle B_{r_m}|\\
&\qquad \simeq r_m^{n+1}\,|E_m\triangle B_m|.
\end{aligned}$$

Hence, gathering together the previous estimates, by the minimality of E_m, it follows:

$$\begin{aligned}
\mathrm{Per}(E_m)-\mathrm{Per}(B_{r_m}) &\le \gamma\mathrm{NL}(B_{r_m})-\gamma\mathrm{NL}(E_m)\\
&\simeq \gamma\|w-\alpha\|_{L^\infty(E_m\triangle B_{r_m})}\,|E_m\triangle B_{r_m}| + \gamma r_m^{n+1}|E_m\triangle B_{r_m}|\\
&\simeq \gamma\,\frac{|E_m\triangle B_{r_m}|^2}{r_m^{n-2}} + \gamma r_m^{n+1}\,|E_m\triangle B_{r_m}|. \qquad \square
\end{aligned}$$

4.3. **Proof of Theorem 4.1.** In order to prove (48) we use the quantitative isoperimetric inequality and the improved estimate in Proposition 4.2 to infer that

$$\begin{aligned}
|E_m\triangle B_{r_m}|^2 &\overset{(17)}{\lesssim} r_m^{n+1}\big(\mathrm{Per}(E_m)-\mathrm{Per}(B_{r_m})\big)\\
&\overset{(51)}{\lesssim} \gamma r_m^3\,|E_m\triangle B_{r_m}|^2 + \gamma r_m^{2n+2}\,|E_m\triangle B_{r_m}|.
\end{aligned}$$

This implies $|E_m\triangle B_{r_m}| \lesssim \gamma r_m^{2n+2}$, that is, by (50),

$$\|\psi_m\|_{L^\infty(\partial B_1)} \lesssim \gamma r_m^{n+3}. \tag{52}$$

From the $C^{3,\alpha}$ regularity of ψ_m proved in Proposition 3.5, the convexity of E_m and (48) follows. Similarly, by comparing the energy of E_m with that of B_{r_m}, using Proposition 3.1, Proposition 4.2 and Lemma 2.2, (49) follows:

$$\begin{aligned}
F_{\gamma,m}(E_m) &= F_{\gamma,m}(B_{r_m}) + O(\gamma r_m^{3n+3})\\
&= \begin{cases} 2\,\pi\, r_m + \frac{\pi\gamma}{2}\, r_m^4\,\log r_m + \gamma\big(-\frac{1}{8}+\pi^2\,h(0)\big)\,r_m^4 + O(\gamma r_m^6) & \text{if } n=2,\\ n\,\omega_n\,r_m^{n-1} + \frac{2\gamma\omega_n}{4-n^2}\,r_m^{n+2} + \gamma\,\omega_n^2\,r_m^{2n}\,h(0) + O(\gamma r_m^{2n+2}) & \text{if } n\ge 3.\end{cases}
\end{aligned}$$

$\square$

5. Strong convergence to round spheres

In this section we prove Theorem 1.1 (i), (ii) and (iii). We remark that, in this general case, before we may argue as in the proof of Theorem 4.1, we need to show that the minimizers of $F_{\gamma,m}$ are well-contained in Ω. This is crucial in order to apply the regularity results in § 3, which hold under the hypothesis of being at a fixed distance from the boundary. The proof is based on the analysis of the nonlocal energy of a minimizer when it gets close to $\partial\Omega$. To this extent a key role will be played by the estimates (12), (13) and (14).

5.1. **Localization of minimizers.** We prove that the minimizers of $F_{\gamma,m}$ are well-contained in Ω.

Proposition 5.1. *There exist $\delta_0, r_0 > 0$ such that the following holds. Assume $r_m \le r_0/2$, $\gamma r_m^3 |\log r_m| < \delta_0$ if $n = 2$ or $\gamma r_m^3 < \delta_0$ if $n \ge 3$. Then, every minimizer E_m of $F_{\gamma,m}$ satisfies*

$$E_m \subset B_{2r_m}(q) \quad \text{for some } q \in \Omega_{r_0}. \tag{53}$$

Proof. We prove the result in the case $n \ge 3$, the case $n = 2$ being similar up to minor changes. The proof consists of three steps.

STEP 1. *If δ_0 and r_0 are sufficiently small, then there exists a ball $B_m := B_{r_m}(p_m) \subset \Omega$ such that*

$$|B_m \triangle E_m| \lesssim \delta_0^{\frac{1}{n+1}} r_m^n \quad \text{and} \quad \operatorname{dist}(p_m, \partial\Omega) \gtrsim \delta_0^{-\frac{1}{(n+1)(n-2)}} r_m. \tag{54}$$

For any ball of radius $B_{r_m}(p) \subset \Omega$ (note that such a ball exists if r_0 is choosen sufficiently small), by (34) it holds

$$\operatorname{Per}(E_m) - \operatorname{Per}(B_{r_m}(p)) \lesssim \gamma r_m^2 |E_m \triangle B_{r_m}(p)| \lesssim \gamma r_m^{n+2}. \tag{55}$$

On the other hand, by the quantitative isoperimetric inequality, there exists an optimal ball $B^{opt}_{E_m} \subset \mathbb{R}^n$ such that

$$|E_m \triangle B^{opt}_{E_m}|^2 \overset{(35)}{\lesssim} r_m^{n+1} \left(\operatorname{Per}(E_m) - \operatorname{Per}(B^{opt}_{E_m})\right) \overset{(55)}{\lesssim} \gamma r_m^{2n+3}. \tag{56}$$

Note that $B^{opt}_{E_m}$ may not be contained in Ω. Nevertheless, since $B^{opt}_{E_m} \setminus \Omega \subset B^{opt}_{E_m} \triangle E_m$, it follows from (56) that

$$|B^{opt}_{E_m} \setminus \Omega| \lesssim (\gamma r_m^3)^{1/2} r_m^n \lesssim \delta_0^{1/2} r_m^n.$$

We can now use a simple geometric argument proved in Lemma 5.3 below to deduce the existence of a vector $v \in \mathbb{R}^n$ such that

$$|v| \simeq \delta_0^{1/(n+1)} r_m \quad \text{and} \quad B_m := B^{opt}_{E_m} + v \subset \Omega.$$

Setting p_m as the center of B_m, namely $B_m = B_{r_m}(p_m)$, this amounts to say that $p_m \in \Omega_{r_m}$. We claim that B_m satisfies (54). Note that, since the measure of the symmetric difference between two balls is linear with the distance of the centers, we infer the first conclusion in (54), namely

$$|E_m \triangle B_m| \le |E_m \triangle B^{opt}_{E_m}| + |B^{opt}_{E_m} \triangle B_m| \lesssim \left(\delta_0^{1/2} + \delta_0^{1/(n+1)}\right) r_m^n \lesssim \delta_0^{1/(n+1)} r_m^n. \tag{57}$$

On the other hand, appealing to the minimality of E_m (now $\chi_{B_m} \in \mathscr{C}_m$) and using (20), we get:

$$\begin{aligned}
\gamma(\omega_n r_m^n)^2 g_{r_m}(p_m) - \gamma(\omega_n r_m^n)^2 \min_{p \in \Omega_{r_m}} g_{r_m}(p) &= F_{\gamma,m}(B_m) - \min_{p \in \Omega_{r_m}} F_{\gamma,m}(B_{r_m}(p)) \\
&\le F_{\gamma,m}(B_m) - F_{\gamma,m}(E_m) \\
&\le \gamma \mathrm{NL}(B_m) - \gamma \mathrm{NL}(E_m) \\
&\lesssim \gamma \delta_0^{1/(n+1)} r_m^{n+2},
\end{aligned} \tag{58}$$

where in the last inequality we have used Proposition 3.1 and (31). Then, by Lemma 2.2 and (10) we obtain the second inequality in (54):

$$\operatorname{dist}(p_m, \partial\Omega)^{2-n} \lesssim \delta_0^{1/(n+1)} r_m^{2-n}.$$

STEP 2. *The whole E_m is well-contained in Ω, i.e.*

$$E_m \subset B_{2r_m}(p_m). \tag{59}$$

With the notation as in § 3.2, by (36) we have that $|H_m \triangle B_1| \lesssim \gamma r_m^3 \leq \delta_0$. Then, using Lemma 3.2, the sequence of sets H_m turns out to be a sequence of uniform Λ-minimizer of the perimeter in Ω_m. Moreover, by (54), if δ_0 is small enough, we have that

$$\mathrm{dist}(0, \partial\Omega_m) \gtrsim \delta_0^{-\frac{1}{(n+1)(n-2)}} \geq 4. \tag{60}$$

As a consequence, we are in position to use the density estimate (41) in Proposition 3.4 according to which there exists $R > 0$ (without loss of generality we assume $R < 1$) such that, for every $x \in H_m \cap (B_3 \setminus B_2)$,

$$c(n)\,R^n \leq |H_m \cap B_R(x)| \leq |H_m \cap B_1(x)|.$$

Therefore, since for every $x \in H_m \cap (B_3 \setminus B_2)$ it holds $B_1(x) \cap B_1 = \emptyset$, we get:

$$c(n)\,R^n \leq |H_m \cap B_1(x)| \leq |H_m \triangle B_1| \overset{(54)}{\lesssim} \delta_0^{1/(n+1)}.$$

Clearly, if δ_0 is small enough, this inequality cannot be satisfied, thus implying $H_m \cap (B_3 \setminus B_2) = \emptyset$. In order to complete the proof of (59), we need to show that $H_m \cap (\Omega_m \setminus B_3) = \emptyset$ as well. To this purpose, we argue by contradiction and show that, in this case, a suitable rescaling of $J_m := H_m \cap B_2$ would have lower energy than H_m. We fix the notation:

$$K_m := H_m \setminus J_m \quad \text{and} \quad L_m := \rho_m\, J_m,$$

with $\rho_m \geq 1$ such that $|L_m| = |H_m|$. Note first the following two observations: by a simple computation on the volumes, it follows that

$$\rho_m - 1 \lesssim |K_m|; \tag{61}$$

consequently, we can estimate $|L_m \triangle J_m|$ in the following way:

$$\begin{aligned} |L_m \triangle J_m| &= \int_{\mathbb{R}^n} |\chi_{J_m}(\rho_m^{-1}x) - \chi_{J_m}(x)|\,dx \\ &\leq \int_{B_3} \int_0^1 |D\chi_{J_m}(sx + (1-s)\,\rho_m^{-1}x)|\,(1-\rho_m^{-1})\,|x|\,ds\,dx \\ &\lesssim (\rho_m - 1)\,\mathrm{Per}(J_m) \lesssim |K_m|, \end{aligned} \tag{62}$$

where, in order to rigourously justify the second inequality without referring to fine properties of functions of bounded variation, it is enough to consider an approximation via smooth functions and to pass to the limit. Recalling that $F_{\gamma,m}(E_m) = r_m^{n-1}\,F_{\gamma r_m^3,m}(H_m)$, we can compare the energies of H_m and J_m as follows:

$$\begin{aligned} F_{\gamma r_m^3,m}(L_m) &= \rho_m^{n-1}\,\mathrm{Per}(J_m) + \gamma r_m^3\,NL(L_m) \\ &\leq \rho_m^{n-1}\,\mathrm{Per}(H_m) - \rho_m^{n-1}\,\mathrm{Per}(K_m) + \gamma r_m^3\,C\,|L_m \triangle H_m| + \gamma r_m^3\,NL(H_m) \\ &\overset{(61)}{\leq} (1 + C\,|K_m|)\,\mathrm{Per}(H_m) - \rho_m^{n-1}\mathrm{Per}(K_m) + \\ &\quad + \gamma r_m^3\,C\left(|L_m \triangle J_m| + |K_m|\right) + \gamma r_m^3\,NL(H_m) \\ &\overset{(62)}{\leq} F_{\gamma r_m^3,m}(H_m) + C\,|K_m| + C\,\delta_0\,|K_m| - C\,|K_m|^{(n-1)/n} \\ &< F_{\gamma r_m^3,m}(H_m), \end{aligned} \tag{63}$$

if δ_0 is sufficiently small because $|K_m| \le \delta_0^{1/(n+1)} < 1$. Clearly, this is a contradiction with the minimality of H_m, thus proving that $H_m \setminus B_2 = \emptyset$ or, after scaling by r_m, that (59) holds true.

STEP 3. *Proof of* (53). We set $E'_m := E_m - p_m$ and, as a consequence of (59), we note that $E'_m \subset B_{2r_m}$. For all $q \in \Omega_{2r_m}$, let us set $E_m(q) := E'_m + q$ (in particular, $E_m(q) \subset \Omega$ and $E_m = E_m(p_m)$). We may write the energy of $E_m(q)$ as

$$F_{\gamma,m}(E_m(q)) = \mathrm{Per}(E'_m) + \gamma \int\int \Gamma(|x-y|)\,\chi_{E'_m}(x)\,\chi_{E'_m}(y)\,dx\,dy + \gamma \int\int R(x+q,y+q)\,\chi_{E'_m}(x)\,\chi_{E'_m}(y)\,dx\,dy. \tag{64}$$

Since E_m minimizes $F_{\gamma,m}$, we have that $F_{\gamma,m}(E_m(p_m)) \le F_{\gamma,m}(E_m(q))$ for every $q \in \Omega_{2r_m}$. By (64) this implies that

$$\int\int R(x+p_m,y+p_m)\,\chi_{E'_m}(x)\,\chi_{E'_m}(y)\,dx\,dy \le \int\int R(x+q,y+q)\,\chi_{E'_m}(x)\,\chi_{E'_m}(y)\,dx\,dy.$$

In view of $E'_m \subset B_{2r_m}$, (12) and the last inequaltiy imply that p_m is contained in a compact subset of Ω. □

Remark 5.2. It follows a posteriori that the optimal balls $B^{opt}_{E_m}$ for E_m are, in fact, well-contained in Ω and (56) holds, i.e.

$$\mathrm{dist}(B^{opt}_{E_m}, \partial\Omega) \gtrsim 1 \quad \text{and} \quad |B^{opt}_{E_m} \triangle E_m| \lesssim \delta_0^{1/2}\, r_m^n. \tag{65}$$

The following is the geometric lemma used in the Step 1 of the proof of Proposition 5.1.

Lemma 5.3. *Let $\Omega \subset \mathbb{R}^n$ be an open set with C^2 boundary. Then, there exist $r_0, h_0 > 0$ with this property: for $r < r_0$, $h \le h_0$ and $p \in \Omega$ such that $|B_r(p) \setminus \Omega| \le h\,r^n$, there exists $v \in \mathbb{R}^n$ with $|v| \lesssim h^{2/(n+1)} r$ such that $B_r(p+v) \subset \Omega$.*

Proof. The main argument in the proof is given by an elementary consideration. Assume first that $r = 1$ and $\partial\Omega \cap B_1(p) \subset \mathbb{R}^{n-1} \times \{0\}$ is flat. If $|B_1(p) \setminus \Omega| \le h$ and $h \le h_0$ is small enough, then $\beta := 1 - |p| \simeq h^{2/(n+1)}$. To see this, one can easily compute the exact expression for β solving the equation

$$h = (n-1)\,\omega_{n-1} \int_0^{\sqrt{2\beta-\beta^2}} \left(\sqrt{1-r^2} - 1 + \beta\right) r^{n-2}\,dr.$$

Alternatively, one can simply notice that $\sqrt{1-|p|^2} \simeq \beta^{1/2}$ and the volume of $B_1(p) \setminus \Omega$ is comparable with that of the cylinder with base $\partial\Omega \cap B_1(p)$ and height β (in fact, the cylinder with half the height and half the radius of the base is contained in $B_1(p) \setminus \Omega$). Hence, $h \simeq \beta^{(n+1)/2}$, from which the conclusion. Clearly, $v = -\beta\, e_n$ fulfills the conclusion of the lemma.

If Ω is not flat, we need to restrict the size of the balls we consider choosing r_0 small enough to have $|A_{\partial\Omega}| \le \varepsilon(n)\, r_0^{-1}$, where $A_{\partial\Omega}$ is the second fundamental form of $\partial\Omega$ and $\varepsilon(n) > 0$ is a dimensional constant to be chosen momentarily. Consider $r \le r_0$ and p as in the statement. By a simple rescaling of the variable by a factor r and a translation, we find $B_1(p')$ and new domain Ω' such that $|B_1(p') \setminus \Omega'| \le h \le h_0$ and

$$\partial\Omega' \cap B_1(p') \subset \left\{(x,y) \in \mathbb{R}^{n-1} \times \mathbb{R} \,:\, -\varepsilon(n)\,|x|^2 \le y \le \varepsilon(n)\,|x|^2\right\}. \tag{66}$$

Note that, by an analogous computation as above (now $\beta := 1-|p'|$), we have that

$$(n-1)\,\omega_{n-1}\int_0^{\sqrt{2\beta-\beta^2}}\Big(\sqrt{1-r^2}-1+\beta-\varepsilon(n)\,r^2\Big)\,r^{n-2}\,dr \le h$$
$$\le (n-1)\,\omega_{n-1}\int_0^{\sqrt{2\beta-\beta^2}}\Big(\sqrt{1-r^2}-1+\beta+\varepsilon(n)\,r^2\Big)\,r^{n-2}\,dr.$$

One can easily compute (or argue by elementary geometric consideration as before) that $h \simeq \beta^{(n+1)/2}$. Note that, setting as before $v' := -\beta\,e_n$, we have that $B_1(p'+v') \subset \Omega'$ because of (66). Hence, scaling back to Ω, the conclusion follows. □

5.2. **Proof of Theorem 1.1: part I.** We are now ready for the proof of Theorem 1.1 (i), (ii) and (iii).

The proof of (i) follows from the same arguments in Theorem 4.1. Indeed, thanks to Proposition 5.1, the minimizers E_m of $F_{\gamma,m}$ are well contained in Ω. By Proposition 3.1 and Lemma 3.2, we know that the E_m are uniform Λ-minimizers. We can, hence, use the regularity in Proposition 3.5 and infer that the sets E_m can be parametrized on a optimal isoperimetric ball $B^{opt}_{E_m}$ by a $C^{3,\alpha}$ regular function. Therefore, we can derive for E_m the improved perimeter estimate as in Proposition 4.2 and use the optimal isoperimetric inequality to conclude (3).

For what concerns (ii), let $q \in \mathscr{H}$ be a generic harmonic center and let p^{opt}_m be the center of the optimal ball for E_m, namely $B^{opt}_{E_m} = B_{r_m}(p^{opt}_m)$. We compare the energy of E_m with that of $B_{r_m}(q)$ and use $|E_m \triangle B^{opt}_{E_m}| \lesssim \gamma\, r_m^{2n+2}$ as shown in the proof of Theorem 4.1 to get:

$$\begin{aligned}\gamma\, r_m^{2n}\, g_{r_m}(p^{opt}_m) - \gamma\, r_m^{2n}\, g_{r_m}(q) &= F_{\gamma,m}(B^{opt}_{E_m}) - F_{\gamma,m}(B_{r_m}(q)) \le F_{\gamma,m}(B^{opt}_{E_m}) - F_{\gamma,m}(E_m)\\ &\le \gamma NL(B^{opt}_{E_m}) - \gamma NL(E_m)\\ &\lesssim \gamma\,\frac{|E_m\triangle B^{opt}_{E_m}|^2}{r_m^{n-2}} + \gamma\, r_m^{n+1}\,|E_m\triangle B^{opt}_{E_m}|\\ &\lesssim \gamma^2\, r_m^{3n+3} \lesssim \gamma\,\delta_0\, r_m^{3n}.\end{aligned} \tag{67}$$

By (22) in Lemma 2.2, this implies that

$$h(p^{opt}_m) - h(q) = g_{r_m}(p^{opt}_m) - g_{r_m}(q) + C\, r_m^2 \lesssim \delta_0\, r_m^n + r_m^2 \lesssim r_m^2.$$

Since the harmonic centers are compactly contained in Ω, from this estimate it follows that p^{opt}_m belongs to some neighborhood of the harmonic centers.

Finally, the proof of (iii) follows as in Theorem 4.1 by comparison with the energy of $B_{r_m}(p^{opt}_m)$. □

6. Stability and exact solutions

In this section we address the problem of the formation of exact spherical droplets, proving assertion (iv) in Theorem 1.1.

6.1. **Non spherical domains: non existence of critical spherical droplets.** In this section we show that if Ω is not itself a ball, the critical points of $F_{\gamma,m}$ cannot be exactly spherical.

Proposition 6.1. *Let $\Omega \subset \mathbb{R}^n$ be a C^2 bounded open set and assume that Ω is not a ball. Then, $\chi_{B_{r_m}(p)}$ with $B_{r_m}(p) \subset \Omega$ is not a critical point of $F_{\gamma,m}$.*

Proof. The proof is a simple consequence of a unique continuation argument. Indeed, we show that if $\chi_{B_{r_m}(p)}$ satisfies the Euler–Lagrange equations (18) and (19), namely

$$\begin{cases} H_{\partial B_{r_m}(p)} + 4\gamma v_m = \lambda_m, \\ -\Delta v_m = \chi_{B_{r_m}(p)} - m & \text{in } \Omega, \\ \nabla v_m \cdot \nu = 0 & \text{on } \partial\Omega, \\ \int_\Omega v_m = 0, \end{cases} \tag{68}$$

then v_m is a radially symmetric function with respect to p, and hence Ω must be a ball. Assume without loss of generality that $p = 0$ and (68) holds, and consider the case $n \geq 3$ (the two dimensional case is analogous). Since $H_{\partial B_{r_m}} \equiv (n-1)/r_m$, it follows from the first equation in (68) that $v_m|_{\partial B_{r_m}} \equiv c_m \in \mathbb{R}$. Thus, from the uniqueness for the Dirichlet problem for the Laplacian, we infer that $v_m|_{B_{r_m}}$ is radially symmetric and:

$$v_m(x) = \frac{(1-m)\,(|x|^2 - r_m^2)}{2n} + c_m, \quad \text{for } |x| \leq r_m. \tag{69}$$

Moreover, in $\Omega \setminus B_{r_m}$, v_m solves the boundary value problem:

$$\begin{cases} \Delta v_m = m & \text{in } \Omega \setminus B_{r_m}, \\ v_m = c_m & \text{on } \partial B_{r_m}, \\ \nabla v_m \cdot \nu_{\partial B_{r_m}} = \frac{(1-m)\,r_m}{n} & \text{on } \partial B_{r_m}. \end{cases} \tag{70}$$

Note that also (70) has a unique solution. Indeed, given v_1, v_2 solving (70), $w = v_1 - v_2$ solves

$$\begin{cases} \Delta w = 0 & \text{in } \Omega \setminus B_{r_m}, \\ w = \nabla w \cdot \nu_{\partial B_{r_m}} = 0 & \text{on } \partial B_{r_m}, \end{cases} \tag{71}$$

which is extended to a harmonic function in Ω setting $w \equiv 0$ in B_{r_m}, thus implying $w \equiv 0$ in $\Omega \setminus B_{r_m}$. By a direct computation, the solution of (70) is given by

$$v_m(x) := -\frac{m\,(|x|^2 - r_m^2)}{2n} + c_m + \frac{r_m^2}{n\,(n-2)} - \frac{r_m^n}{n\,(n-2)\,|x|^{n-2}}.$$

Therefore, since $\nabla v_m \cdot \nu \equiv 0$ on $\partial\Omega$, it follows by the radial symmetry of v_m that Ω is a ball, which contradicts the hypothesis. □

Remark 6.2. In particular, in the case of periodic boundary conditions the exact sphere is never an equilibrium configuration.

6.2. **Ball domains: uniqueness of a spherical droplet minimizer.** In this section we consider $\Omega = B_R$ for some $R > 0$. In this case we show that the ball B_{r_m} is the unique minimizer of $F_{\gamma,m}$ in the regime of small mass, thus completing the proof of Theorem 1.1. In order to address this problem, here we need to introduce a new ingredient: the stability analysis of the droplet configurations. In particular, we will show that the spherical droplet B_{r_m} is strictly stable, which will turn to imply that it is the unique minimizer of $F_{\gamma,m}$.

Proposition 6.3. *Assume $\Omega = B_R \subset \mathbb{R}^n$, for some $R > 0$. There exists $\delta_0 > 0$ such that, if $\gamma r_m^3 |\log r_m| < \delta_0$ in the case $n = 2$ or if $\gamma r_m^3 < \delta_0$ in the case $n \geq 3$, then B_{r_m} is the unique minimizer of $F_{\gamma,m}$.*

Proof. The proof of the proposition is divided in three steps.

STEP 1. *The minimizers E_m can be parametrized on B_{r_m} for δ_0 small enough.*

To see this, we start noticing that in the case $\Omega = B_R$, due to the spherical symmetry, the origin is the only minimum point of the Robin function. Moreover, $D^2h(0) \gtrsim \mathrm{Id}$. To check this, one can either use the explict formulas for h (see, e.g., [34, Chapter IV 5] in the case $n = 3$, similar formulas hold in every dimension):

$$h(x) = \frac{R\,|x|^{n-3}}{(R^2-|x|^2)^{n-2}} + \frac{1}{R^{n-2}}\log\left(\frac{R^2}{R^2-|x|^2}\right) + \frac{|x|^2}{2\,n\,\omega_n\,R^n} + h(0), \quad \text{if } n \geq 3;$$

or alternatively, one can simply notice that $R(x,0) = \frac{|x|^2}{2\,n\,\omega_n R^n}$, so that $D^2h(0) = D_x^2 R(0,0) = \frac{\mathrm{Id}}{n\,\omega_n R^n}$. From the definition of g_r in (21) and the radial symmetry of h, it is readily verified that g_r also has minimum in the origin and this minimum is not degenerate as well. From (67), we can hence conclude that $|p_m^{opt}|^2 \lesssim \delta_0\, r_m^n$. Note that, in any dimension n, this implies that

$$|p_m^{opt}| \lesssim \delta_0^{1/2}\, r_m. \tag{72}$$

This actually leads straightforwardly to the claim. Indeed, for δ_0 small enough, there exists $s < 1$ such that, for every point $x \in \partial B_{r_m}$, $B_{sr_m}(x) \cap B_{r_m}^{opt}$ is a graph over ∂B_{r_m} with small Lipschitz constant. Since by (i) of Theorem 1.1 the sets E_m are parametrized on $\partial B_{r_m}^{opt}$ with a graph of small C^1-norm, this implies in turns that ∂E_m is a graph on ∂B_{r_m}. Moreover, the $C^{3,\alpha}$ regularily is clearly preserved for this new parametrization.

STEP 2. *We show now that for δ_0 small enough, the ball B_{r_m} is strictly stable.*

By scaling, we can consider the functional

$$F_{\delta,m}(E) = \mathrm{Per}(E) + \delta\,\mathrm{NL}(E),$$

with $\delta = \gamma r_m^3$, and we show that for δ_0 small enough $E = B_1$ is strictly stable.

Let us recall the second variation for $F_{\delta,m}$. Let E be a stationary point and consider vector fields $X \in C_c^1(\Omega,\mathbb{R}^n)$ such that

$$\int_{\partial E} X\cdot \nu_E \, d\mathscr{H}^{n-1} = 0. \tag{73}$$

Following [6, Lemma 2.4], for every such field, there exists $F : \Omega \times (-\varepsilon,\varepsilon) \to \Omega$ such that:

(a) $F(x,0) = x$ for all $x \in \Omega$, $F(x,t) = x$ for all $x \in \partial\Omega$ and $t \in (-\varepsilon,\varepsilon)$;
(b) $E_t := F(E,t)$ satisfies $|E_t| = |E|$ for every $t \in (-\varepsilon,\varepsilon)$;
(c) $\frac{\partial F(x,t)}{\partial t}|_{t=0} = X(x)$ for every $x \in \partial E$.

The stability operator for deformations as above is given by (see [1, 15])

$$\begin{aligned}
F''_{\delta,m}(E)[X] =& \mathrm{Per}''(E)[X] + \delta\,\mathrm{NL}''(E)[X] \\
=& \int_{\partial E} \left(|\nabla_{\partial E}(X\cdot\nu_E)|^2 - |A|^2\,(X\cdot\nu_E)^2\right) d\mathscr{H}^{n-1} \\
&+ 8\,\delta \int_{\partial E}\int_{\partial E} G(x,y)\,(X(x)\cdot\nu_E)\,(X(y)\cdot\nu_E)\,d\mathscr{H}^{n-1}(x)\,d\mathscr{H}^{n-1}(y) \\
&+ 4\,\delta \int_{\partial E} \nabla v\cdot\nu_E\,(X\cdot\nu_E)^2\,d\mathscr{H}^{n-1} \\
=& \mathrm{Per}''(E)[X] + \delta\,\mathrm{NL}_1''(E)[X] + \delta\,\mathrm{NL}_2''(E)[X],
\end{aligned} \tag{74}$$

where $|A|$ is the norm of the second fundamental form of ∂E and v solves (19).

In order to prove the strict stability of B_{r_m} we need to compute (74) on $E = B_{r_m}$ and show the existence of a constant $c_0(n,m,\delta) > 0$ such that

$$F''_{\delta,m}(B_{r_m})[X] \geq c_0\,\|X\cdot\nu_{B_{r_m}}\|^2_{L^2(\partial B_{r_m})}, \tag{75}$$

for every X as in (73). Write $X\cdot\nu=\Pi(X\cdot\nu)+\Pi^{\perp}(X\cdot\nu)$, where $\Pi(X\cdot\nu)$ is the projection of $X\cdot\nu$ on the 0-eigenspace (corresponding to the constant vectors) of the Laplace-Beltrami operator on the sphere and $\Pi^{\perp}(X\cdot\nu)$ is its orthogonal complement. Moreover, set $X_0:=\Pi(X\cdot\nu)\,\nu$ and $X^{\perp}:=\Pi^{\perp}(X\cdot\nu)\,\nu$.

We start by noticing that, by the discreteness of the spectrum of the Laplace–Beltrami, the following inequality holds true: there exists a constant $c_1(n)>0$ such that

$$\mathrm{Per}''(B_1)[X]\geq c_1(n)\|\Pi^{\perp}(X\cdot\nu)\|^2_{L^2(\partial B_1)}. \tag{76}$$

By explicit computation (v_m is given in (69)), it holds

$$\begin{aligned}\mathrm{NL}_2''(B_1)[X]&=\mathrm{NL}_2''(B_1)[X_0]+\mathrm{NL}_2''(B_1)[X^{\perp}]\\&=-4(1-m)\big(\|X_0\cdot\nu\|^2_{L^2(\partial B_1)}+\|X^{\perp}\cdot\nu\|^2_{L^2(\partial B_1)}\big).\end{aligned} \tag{77}$$

Moreover, for $X=X_0$, since the first eigenfunctions of the Laplace–Beltrami operator on the sphere are linear functions, we can compute explicitly $\mathrm{NL}''(B_1)[X_0]$ in the following way:

$$\begin{aligned}\mathrm{NL}''(B_1)[X_0]&=\frac{d^2\mathrm{NL}(B_1(tX_0))}{dt^2}\Big|_{t=0}\\&=\frac{d^2}{dt^2}\int_{B_1}\int_{B_1}\big(\Gamma(|x-y|)+R(x+tX_0,y+tX_0)\big)\,dx\,dy\Big|_{t=0}\\&=\int_{B_1}\int_{B_1}\langle D^2R(x,y)(X_0,X_0),(X_0,X_0)\rangle\,dx\,dy\\&\gtrsim c_2(n)\,|X_0|^2,\end{aligned} \tag{78}$$

where $c_2(n)$ is a dimensional constant. Here we used again that the regular part of the Green function has in the origin the unique non-degenerate minimum.

Next we estimate $\mathrm{NL}_1''[X]$ as follows:

$$\begin{aligned}\mathrm{NL}_1''(B_1)[X]&=8\int_{\partial B_1}\int_{\partial B_1}G(x,y)\Pi(X\cdot\nu)(x)\Pi(X\cdot\nu)(y)\\&\quad+8\int_{\partial B_1}\int_{\partial B_1}G(x,y)\Pi^{\perp}(X\cdot\nu)(x)\Pi^{\perp}(X\cdot\nu)(y)\,d\mathcal{H}^{n-1}\\&\quad+16\int_{\partial B_1}\int_{\partial B_1}G(x,y)\Pi(X\cdot\nu)(x)\Pi^{\perp}(X\cdot\nu)(y)\\&=\mathrm{NL}_1''(B_1)[X_0]+\mathrm{NL}_1''(B_1)[X^{\perp}]\\&\quad+16\int_{\partial B_1}\int_{\partial B_1}G(x,y)\Pi(X\cdot\nu)(x)\Pi^{\perp}(X\cdot\nu)(y)\\&\geq\mathrm{NL}_1''(B_1)[X_0]-a\,\|\Pi(X\cdot\nu)\|^2_{L^2(\partial B_1)}-C_a\|\Pi^{\perp}(X\cdot\nu)\|^2_{L^2(\partial B_1)}.\end{aligned} \tag{79}$$

The estimate (79) follows from: (a) $\mathrm{NL}_1''(B_1)[X^{\perp}]\geq 0$, (b) the estimates on the Riesz potential in [47, chap. 5 Theorem 1] (note that here the domain ∂B_1 has finite measure, the space dimension is $n-1$, $\alpha=1$, $p=\frac{2(n-1)}{n+1}$ and $q=2$) and (c) the following Hölder and Young inequalities with a constant $a>0$ to be fixed soon (below we set $I_1(|\Pi^{\perp}(X\cdot\nu)|)=$

$\int_{\partial B_1} \frac{|\Pi^\perp(X\cdot\nu)(y)|}{|x-y|^{n-2}}$):

$$
\begin{aligned}
\Big|\int_{\partial B_1}\int_{\partial B_1} G(x,y)\Pi(X\cdot\nu)(x)\Pi^\perp(X\cdot\nu)(y)\Big| \\
&\leq C_0 \int_{\partial B_1}\int_{\partial B_1} \frac{|\Pi(X\cdot\nu)(x)||\Pi^\perp(X\cdot\nu)(y)|}{|x-y|^{n-2}} \\
&\leq C\,\|\Pi(X\cdot\nu)\|_{L^2(\partial B_1)}\|I_1(|\Pi^\perp(X\cdot\nu)|)\|_{L^2(\partial B_1)} \\
&\leq a\,\|\Pi(X\cdot\nu)\|^2_{L^2(\partial B_1)} + C_a\|\Pi^\perp(X\cdot\nu)\|^2_{L^2(\partial B_1)}.
\end{aligned}
$$

The existence of $C_0 > 0$ independent of r_m in the first line follows by (13) taking into account that, as already observed, on a ball B_R we have $R(0,0) = 0$ and $D^2_x R(0,0) = \frac{Id}{n\omega_n R^n}$.

The proof of (75) can now be achieved as follows:

$$
\begin{aligned}
F''_{\delta,m}(B_1)[X] &= \mathrm{Per}''(B_1)[X] + \delta\,\mathrm{NL}''_1(B_1)[X] + \delta\,\mathrm{NL}''_2(B_1)[X] \\
&\overset{(76),(79)}{\geq} c_1\|\Pi^\perp(X\cdot\nu)\|^2_{L^2(\partial B_1)} + \delta \mathrm{NL}''_1(B_1)[X_0] - a\,\delta\,\|\Pi(X\cdot\nu)\|^2_{L^2(\partial B_1)} \\
&\qquad - C_a\,\delta\|\Pi^\perp(X\cdot\nu)\|^2_{L^2(\partial B_1)} + \delta\,\mathrm{NL}''_2(B_1)[X_0] + \delta\,\mathrm{NL}''_2(B_1)[X^\perp] \\
&\overset{(77),(78)}{\geq} \Big(c_1 - C_a\,\delta - 4\,(1-m)\,\delta\Big)\|\Pi^\perp(X\cdot\nu)\|^2_{L^2(\partial B_1)} \\
&\qquad + \delta\,c_2\,\|\Pi(X\cdot\nu)\|^2_{L^2(\partial B_1)} - a\,\delta\,\|\Pi(X\cdot\nu)\|^2_{L^2(\partial B_1)} \\
&\geq c\,\|X\cdot\nu\|^2_{L^2(\partial B_1)},
\end{aligned}
$$

as soon as $a < c_2$ and δ is small enough to have $\delta\,(C_a + 4(1-m)) < c_1$.

STEP 3. *B_{r_m} is the unique minimizer.* The conclusion follows from the fact that the minimizers E_m are C^2 close to a strictly stable configuration, namely B_{r_m}, thus implying that actually E_m coincide with B_{r_m}. The proof of this fact, well-known for the area functional, can be achieved by a carefull construction of a flow interpolating ∂E_m and ∂B_{r_m}. Such computations appeared in [1, Theorem 3.9]. In particular, to reduce to this case, let ψ_m be the parametrization of $\partial E_m/r_m$ on ∂B_1, i.e.

$$E_m = \big\{r_m\,x(1+\psi_m(x))\,:\,x\in\partial B_1\big\}.$$

By Proposition 3.5 and (72), for every $\eta > 0$ we can choose δ_0 small enough to have $\|\psi_m\|_{C^{3,\alpha}} \leq \eta$. We are, hence, a small perturbation of the fixed stable configuration B_1 and [1, Theorem 3.9] applies. □

Remark 6.4. The proof of the previous result becomes trivial if the minimizer E_m is such that $B^{opt}_{E_m}$ is centered at the origin,that is $B^{opt}_{E_m} = B_{r_m}$. In this case, it is simple to show that the spherical symmetry of G allows to drop the linear term in (51) and, by the quantitative isoperimetric inequality, we get

$$|B_{r_m}\triangle E_m|^2 \lesssim \gamma\,r_m^3|B_{r_m}\triangle E_m|^2,$$

which clearly implies $E_m = B_{r_m}$ for δ_0 small enough.

APPENDIX A. ON THE BEHAVIOR OF THE FUNCTION R IN A NEIGHBORHOOD OF $\partial\Omega$

In this section we prove the estimates (12) and (13) on the regular part of the Green function:

$$\begin{cases} \Delta R_x = \frac{1}{|\Omega|} & \text{in } \Omega, \\ \nabla R_x \cdot \nu = \nabla \Gamma_x \cdot \nu & \text{on } \partial\Omega, \\ \int_\Omega R_x = \int_\Omega \Gamma_x. \end{cases}$$

We introduce the following notation. Since Ω is assumed to have C^2 regular boundary, for x in a sufficiently small tubolar neighborhood of $\partial\Omega$, there exists a unique point $x_0 \in \partial\Omega$ such that $\mathrm{dist}(x,\partial\Omega) = |x - x_0|$. Hence, we can consider $x^* \in \mathbb{R}^n \setminus \Omega$ such that $x^* - x_0 = x_0 - x$ and set $S_x := R_x + \Gamma_{x^*}$. S_x is also characterized by the following boundary value problem:

$$\begin{cases} \Delta S_x = \frac{1}{|\Omega|} & \text{in } \Omega, \\ \nabla S_x \cdot \nu = (\nabla \Gamma_x + \nabla \Gamma_{x^*}) \cdot \nu & \text{on } \partial\Omega, \\ \int_\Omega S_x = \int_\Omega (\Gamma_x + \Gamma_{x^*}). \end{cases} \tag{80}$$

The main idea behind the estimates are illustrated in the following simple case. Assume that $0 \in \partial\Omega$ and $B_2 \cap \Omega = \{x \in B_2 \,:\, x_n < 0\}$. Then, by an elementary computation, for every $x \in B_1$,

$$\begin{cases} (\nabla \Gamma_x + \nabla \Gamma_{x^*}) \cdot \nu = 0 & \text{on } \partial\Omega \cap B_2, \\ |(\nabla \Gamma_x + \nabla \Gamma_{x^*}) \cdot \nu| \lesssim 1 & \text{on } \partial\Omega \setminus B_2. \end{cases}$$

Therefore, it follows from (80) that $|S_x| \le C$. This in turns implies (12): namely, there exists $r_0 > 0$ such that, for $r \le r_0$ and $x, y \in \Omega \cap B_1$ with $r < -x_n < 2\,r$ and $|x - y| \le r$,

$$|R_x(y)| \simeq |\Gamma_{x^*}(y)| \simeq |\Gamma(r)|.$$

Moreover, since $|\Gamma_{x_*}| \lesssim |\Gamma_x| + 1$ for every $x \in B_1$, (13) follows straightforwardly.

The general case of a C^2 bounded domain Ω can be deduced by a perturbation of the argument above. Let $r_0 > 0$ be such that, for every $x_0 \in \partial\Omega$, $B_{2\,r_0}(x_0) \cap \partial\Omega$ can be written as the graph of a function: namely, up to an affine change of coordinates, we may assume that $x_0 = 0$ and

$$B_{2\,r_0} \cap \Omega = \{(z',t) \,:\, t \le u(z')\},$$

for a given $u : B^{n-1}_{2\,r_0} \subset \mathbb{R}^{n-1} \to \mathbb{R}$ in $C^2(B^{n-1}_{2\,r_0})$ with $u(0) = |\nabla u(0)| = 0$. In particular, for $x = (0,-d)$ it holds $d = \mathrm{dist}(x,\partial\Omega)$. Set $D := B^{n-1}_{2\,r_0} \times [0,1]$ and consider the function $g : D \to \mathbb{R}$ given by

$$g(z',t) := \Big(\nabla \Gamma_x\big((z', t\,u(z'))\big) + \nabla \Gamma_{x^*}\big((z', t\,u(z'))\big)\Big) \cdot \frac{(-t\,\nabla u(z'), 1)}{\sqrt{1 + t^2\,|\nabla u(z')|^2}}.$$

By definition, $g(z',1) = \nabla S_x \cdot \nu|_{\partial\Omega}$ and $g(z',0) = 0$. Writing $z_t = (z', t\,u(z'))$, it holds

$$\begin{aligned} \partial_t\, g(z',t) = {} & u(z') \Big(\frac{\partial}{\partial x_n} \nabla \Gamma_x(z_t) + \frac{\partial}{\partial x_n} \nabla \Gamma_{x^*}(z_t)\Big) \cdot \nu|_{\partial\Omega_t} \\ & - \Big(\nabla \Gamma_x(z_t) + \nabla \Gamma_{x^*}(z_t)\Big) \cdot \frac{\nabla u(z')}{\sqrt{1 + t^2\,|\nabla u(z')|}} \\ & - \frac{t\,|\nabla u(z')|^2}{1 + t^2 |\nabla u(z')|^2} \Big(\nabla \Gamma_x + \nabla \Gamma_{x^*}\Big) \cdot \nu|_{\Omega_t}(z_t). \end{aligned}$$

Since $|u(z')| \le C\,|z'|^2$ and $|\nabla u(z')| \le C\,|z'|$, where $C>0$ depends only on $\|u\|_{C^2}$, one infers that

$$
\begin{aligned}
|\partial_t\, g(z',t)| &\lesssim \Big(|D^2\Gamma_x(z_t)| + |D^2\Gamma_{x^*}(z_t)|\Big)\,|z'|^2 + \Big(|\nabla\Gamma_x| + |\nabla\Gamma_{x^*}|\Big)\,|z'| \\
&\lesssim \frac{|z'|^2}{|x-z_t|^n} + \frac{|z'|}{|x-z_t|^{n-1}}. \qquad (81)
\end{aligned}
$$

Note that, for $|z'| \le r_0$ small enough,

$$
\begin{aligned}
|x-z_t|^2 = |d+tu(z')|^2 + |z'|^2 &\ge \frac{d^2}{2} - |u(z')|^2 + |z'|^2 \ge \frac{d^2}{2} - C\,|z'|^4 + |z'|^2 \\
&\ge \frac{d^2}{2} + \frac{|z'|^2}{2},
\end{aligned}
$$

from which we infer

$$
|\partial_t\, g(z',t)| \lesssim \frac{|z'|^2}{|x-z_t|^n} + \frac{|z'|}{|x-z_t|^{n-1}} \lesssim \frac{|z'|}{(d^2+|z'|^2)^{\frac{n-1}{2}}} =: f(z'). \qquad (82)
$$

It is simple to see that $f \in L^p(B^{n-1}_{2r_0})$ for every $p\in[1,\infty)$ and

$$
\begin{aligned}
\int_{B^{n-1}_{2r_0}} f(z')^p\,dz' &= \int_0^{2r_0} \frac{s^p}{(d^2+s^2)^{\frac{p(n-1)}{2}}}\, s^{n-2}\,ds \\
&= d^{-p\,(n-2)+n-1} \int_0^{\frac{2r_0}{d}} \frac{t^{p+n-2}}{(1+t^2)^{\frac{p(n-1)}{2}}}\,dt \\
&\lesssim \begin{cases} 2\,r_0 & \text{if } n=2,\\ d^{-p\,(n-2)+n-1} \int_0^\infty \frac{t^{p+n-2}}{(1+t^2)^{\frac{p(n-1)}{2}}}\,dt \lesssim d^{-p\,(n-2)+n-1} & \text{if } n\ge 3. \end{cases} \qquad (83)
\end{aligned}
$$

Note that, for $\mathrm{dist}(x,\partial\Omega)\le d_0$ and every $z\in\partial\Omega\cap B_{2r_0}$,

$$
|\nabla S_x(z)\cdot\nu|_{\partial\Omega\cap B_{2r_0}}| = |g(z',1)-g(z',0)| \le \int_0^1 |\partial_t g(z',s)|\,ds.
$$

Therefore, we deduce the following bound on the L^p norm of $\nabla S_x\cdot\nu$:

$$
\begin{aligned}
\|\nabla S_x\cdot\nu\|^p_{L^p(\partial\Omega\cap B_{2r_0})} &\lesssim \int_{\partial\Omega\cap B_{2r_0}} \left(\int_0^1 |\partial_t g(z',s)|\,ds\right)^p dz' + \int_{\partial\Omega\setminus B_{2r_0}} |\nabla S_x\cdot\nu|^p\,dz' \\
&\lesssim \int_{\partial\Omega\cap B_{2r_0}} \int_0^1 |\partial_t g(z',s)|^p\,ds\,dz' + C \\
&\overset{(83)}{\lesssim} \begin{cases} 1 & \text{if } n=2,\\ d^{-p\,(n-2)+n-1} & \text{if } n\ge 3. \end{cases}
\end{aligned}
$$

Setting $\beta=(n-1)/p>0$, by the L^p-regularity theory for (80), we get $\|S_x\|_{W^{1,p}} \lesssim d_0^\beta\,\mathrm{dist}(x,\partial\Omega)^{2-n}$. By the arbitrariness of p and the Sobolev embedding, we finally get

$$
|S_x| \lesssim d_0^\beta\,\mathrm{dist}(x,\partial\Omega)^{2-n}. \qquad (84)
$$

The proofs of (12), (13) and (14) now follows straightforwardly.

A.1. **Proof of** (13)**.** Fix $r_0 \le d_0$ as above. Then, if x belongs to some Ω_{r_0}, then

$$\begin{aligned} |G(x,y)| &\le |\Gamma(x,y)| + |R(x,y)| \\ &\le |\Gamma(x,y)| + |\Gamma(x^*,y)| + |S_x| \\ &\lesssim |\Gamma(x,y)| + 1, \end{aligned}$$

where we have used that $\operatorname{dist}(x,\partial\Omega)^{2-n} \lesssim |\Gamma(x,y)|$ for every $y \in \Omega$ if $n \ge 3$, and $\operatorname{dist}(x,\partial\Omega) \lesssim 1$ in $n = 2$.

A.2. **Proof of** (12)**.** Note that, for r_0 small enough, whenever $r < r_0/2$, $r \le \operatorname{dist}(x,\partial\Omega) \le 2r$ and $|y-x| \le r$, then $|\Gamma_{x^*}(y)| \simeq |\Gamma_x(y)|$. Then, by (84) we may assume that d_0 is sufficiently small that, for $r_0 \le d_0$ and x, y as above, it holds

$$|R_x(y)| \ge |\Gamma_{x^*}(y)| - |S_x(y)| \gtrsim |\Gamma_x(y)| - d_0^{\beta} \operatorname{dist}(x,\partial\Omega)^{2-n} \gtrsim |\Gamma_x(y)|,$$

and

$$|R_x(y)| \le |\Gamma_{x^*}(y)| + |S_x(y)| \lesssim |\Gamma_x(y)| + d_0^{\beta} \operatorname{dist}(x,\partial\Omega)^{2-n} \lesssim |\Gamma_x(y)|.$$

A.3. **Proof of** (14)**.** Straightforward from (12) with $x = y$.

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