Minimization problems for average distance functionals

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Abstract

The paper is concerned with problems of finding minima among the set of all compact connected subsets $\Sigma \subset \Omega$ ($\Omega \subset \mathbb{R}^n$ being a given compact ambient set) of the functional

$$I(\Sigma) := \int_\Omega A(\operatorname{dist}(x,\Sigma))\, d\varphi(x)$$

subject to some penalization on the length $\mathcal{H}^1(\Sigma)$ of Σ, where $A\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a given nondecreasing function and $\operatorname{dist}(x,\Sigma)$ stands for the distance between x and Σ defined with the use of the geodesic distance d relative to Ω in the usual way, i.e.

$$\operatorname{dist}(x,\Sigma) := \inf_{z\in\Sigma} d(x,z).$$

We provide various applications which are sources of similar problems, and give some results concerning qualitative properties of minimizers in the case Ω is a convex set (with d a Euclidean distance).

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be some compact set and assume φ to be a Borel measure over Ω. Given a nondecreasing function $A\colon \mathbb{R}^+ \to \mathbb{R}^+$, we may define an average distance functional on the set of all nonempty subsets Σ of Ω by setting

$$I(\Sigma) := \int_\Omega A(\operatorname{dist}(x,\Sigma))\, d\varphi(x),$$

where $\operatorname{dist}(x,\Sigma)$ stands for the distance between x and Σ defined through the geodesic distance d relative to Ω in the usual way, i.e.

$$\operatorname{dist}(x,\Sigma) := \inf_{z\in\Sigma} d(x,z).$$

In this paper we are interested in the problems of minimizing the average distance functional I with some penalization or constraint over the set of admissible Σ. Usually in applications one either considers the class of admissible Σ to be the class of sets having Hausdorff dimension zero, i.e. discrete sets (in this case the number of points $\#\Sigma$ is penalized) or to be the class of closed connected sets having Hausdorff dimension one (in which case the one-dimensional Hausdorff measure $\mathcal{H}^1(\Sigma)$ is usually penalized). The most well-studied problem is the former one (i.e. optimization over discrete sets), which is usually referred to as *optimal location* problem (for a recent survey on the subject see [8]) and has numerous applications especially in economics and urban planning. In this paper we concentrate our attention therefore on the latter case. Namely, we define the class of admissible Σ as the class of all closed connected subsets of Ω, and consider the problem of minimizing

$$F(\Sigma) := I(\Sigma) + H(\mathcal{H}^1(\Sigma))$$

over all admissible Σ, where the penalization $H\colon \mathbb{R}^+ \to \bar{\mathbb{R}}^+ := [0, +\infty]$ is some given nondecreasing function satisfying the coercivity condition

$$H(t) \to \infty \text{ when } t \to +\infty.$$

In particular, the choice

$$H(l) = H_\alpha(l) := \begin{cases} 0, & \text{if } l \leq \alpha, \\ +\infty, & \text{otherwise,} \end{cases}$$

where $\alpha > 0$ is a parameter, corresponds to the problem of minimizing the average distance functional I over the class of closed connected subsets Σ of Ω with total length $\mathcal{H}^1(\Sigma)$ not exceeding α. The existence of minimizers to the problem introduced under rather natural conditions on problem data (e.g. when A is l.s.c.) is is quite easy to obtain (see e.g. [3] where this result has been obtained for the case $H = H_\alpha$).

The aim of the paper is twofold. First, we provide an overview of the applications which can be sources of the problems of this kind. Next, we study qualitative properties of the minimizers to the problem introduced. We remark that the conditions on A and H we will use in all our results are rather natural and in particular include the case $A(t) = t^p$, $p \geq 1$, and rather general H (e.g., in particular, the case of $H = H_\alpha$).

2. Applications

In this section we consider various applicative sources of minimization problems for average distance functionals of the type I.

2.1. Urban planning problems

Assume a closed bounded set $\Omega \subset \mathbb{R}^n$ to model a city where the density of population is given by a finite Borel measure φ^+. One is aimed at finding an optimal urban transportation network (i.e. the set of metro and/or street traffic routes) that would provide the easiest access of the people to the services, the density of which is supposed to be another given finite Borel measure φ^-. Such a transportation network will be modelled by a closed connected set $\Sigma \subset \Omega$. We suppose that the function $A\colon \mathbb{R}^+ \to \bar{\mathbb{R}}^+$ represents the effective cost (in terms of time, money and/or fatigue) for a single citizen of travelling without using the transportation network (i.e. "by own means"), so that covering a distance t in such a way would have a cost $A(t)$. We consider a simplified model in which the analogous cost of travelling with the use of the transportation network (the price of the ticket) is independent on the length of the route used and moreover is negligible with respect to that of travelling by own means. Hence the latter cost will be assumed zero. Under this assumption the total cost of everyday movement of the city population to their workplaces can be modelled by the functional I_Σ defined on the set of finite Borel measures over $\Omega \times \Omega$ by the relationship

$$I_\Sigma(\gamma) := \int_{\Omega\times\Omega} A(d_\Sigma(x,y))\, d\gamma(x,y),$$

where the distance function d_Σ in Ω is defined by the formula

$$d_\Sigma(x,y) := d(x,y) \wedge (\mathrm{dist}\,(x,\Sigma) + \mathrm{dist}\,(y,\Sigma)).$$

In other words, each resident x to reach a destination y will choose the most convenient route between avoiding the urban transportation network (i.e. "not taking any urban traffic line and travelling by own means"), in which case he has to cover the distance $d(x,y)$, or using the latter, in which case he has to cover the distance $\mathrm{dist}\,(x,\Sigma) + \mathrm{dist}\,(y,\Sigma)$.

The measure γ can be viewed here as the "plan of transportation" (i.e. in a very intuitive way one can think that $\gamma(x,y)$ stands for the amount of people which moves from a point x to a point y) and has to satisfy the restriction

$$(\pi^{\pm}_{\#}\gamma) \llcorner \Omega \setminus \Sigma = \varphi^{\pm} \llcorner \Omega \setminus \Sigma \tag{2.1}$$

where $\pi^{\pm}\colon \Omega \times \Omega \to \Omega$ stand for the projections on the first and the second factor respectively, namely, $\pi^{\pm}(x^+, x^-) := x^{\pm}$. When $\varphi^{\pm}$ are not carried by Σ and $\varphi^+(\Omega) = \varphi^-(\Omega)$, the latter condition simply means that all residents have to reach the service areas (or workplaces) in their everyday movements. If however $\varphi^+(\Omega) \neq \varphi^-(\Omega)$, then some citizens stop on (or are originated from) Σ: for instance, in the extreme case $\varphi^- = 0$ there are no services/workplaces in the urban area and hence the only goal of the residents is to reach the network Σ as quick as possible.

The generalized Monge-Kantorovich optimal transport problem consists in finding a Borel measure γ_{opt} over $\Omega \times \Omega$ (called *optimal transport plan*) minimizing I_Σ over all Borel measures γ in $\Omega \times \Omega$ satisfying (2.1) (called *admissible* measures). It is quite easy to show that such a problem always admits a solution. In fact, let $\{\gamma_\nu\}$ be a minimizing sequence of admissible measures for I_Σ. Letting

$$\gamma'_\nu := \gamma_\nu - \gamma_\nu \llcorner \Sigma \times \Sigma,$$

we get that each γ'_ν is still an admissible measure, while $I_\Sigma(\gamma_\nu) = I_\Sigma(\gamma'_\nu)$ and $\gamma'_\nu(\Sigma \times \Sigma) = 0$, and hence

$$\begin{aligned}
\gamma'_\nu(\Omega \times \Omega) &= \gamma'_\nu((\Omega \setminus \Sigma) \times \Omega) + \gamma'_\nu(\Sigma \times (\Omega \setminus \Sigma)) \\
&\leq \gamma'_\nu((\Omega \setminus \Sigma) \times \Omega) + \gamma'_\nu(\Omega \times (\Omega \setminus \Sigma)) \\
&= \gamma_\nu((\Omega \setminus \Sigma) \times \Omega) + \gamma_\nu(\Omega \times (\Omega \setminus \Sigma)) \\
&= \varphi^+(\Omega \setminus \Sigma) + \varphi^-(\Omega \setminus \Sigma),
\end{aligned}$$

i.e. all γ'_ν have uniformly bounded total mass. Therefore, one may extract a subsequence of γ'_ν (not relabeled), which converges in the $*$-weak sense of measures to some admissible measure γ. Minding that I_Σ is l.s.c. with respect to this convergence, we show thus the existence of a minimizer γ to I_Σ.

More realistic models where different pricing policies for using the transportation network are considered can be found in [4] and [1].

In the sequel we will denote by $MK(\varphi^+,\varphi^-,\Sigma)$ both the above Monge-Kantorovich problem with the Dirichlet constraint Σ and the respective minimum value of the functional I_Σ. We will also omit the reference to Σ and write simply $MK(\varphi^+,\varphi^-)$, if $\Sigma=\emptyset$.

Denote

$$\mathcal{R} := \{(x,y)\in\Omega\times\Omega \,:\, \mathrm{dist}\,(x,\Sigma)+\mathrm{dist}\,(y,\Sigma) > d(x,y)\}$$

and observe that $d_\Sigma(x,y)=d(x,y)$ over $\mathcal{R}$. Set now

$$\varphi_r^\pm := \pi_\#^\pm(\gamma \llcorner \mathcal{R}), \qquad \varphi_s^\pm := \pi_\#^\pm(\gamma \llcorner (\Omega\times\Omega)\setminus\mathcal{R}) \llcorner (\Omega\setminus\Sigma),$$

where γ is an optimal plan for the problem $MK(\varphi^+,\varphi^-,\Sigma)$. We denote further

$$\varphi_s := \varphi_s^+ + \varphi_s^-.$$

Observe that obviously $\mathcal{R}\subset(\Omega\setminus\Sigma)\times(\Omega\setminus\Sigma)$, and hence $\varphi_r^\pm$ are concentrated outside of Σ. Since by construction also $\varphi_s^\pm$ are concentrated outside of Σ, then one has

$$\varphi^\pm \llcorner(\Omega\setminus\Sigma) = \varphi_r^\pm + \varphi_s^\pm.$$

It is worth remarking that both $\varphi^\pm$ and φ_s depend on Σ by construction, and when it will be necessary to underline such a dependence, we will write $\varphi_\Sigma^\pm$ and $\varphi_{s,\Sigma}$ respectively.

One has the following result generalizing the analogous one proven in [5] for the case $A(t)=t$.

Lemma 2.1. *Assume that there exist Borel measures $\tilde\varphi_r^\pm$ and $\tilde\varphi_s^\pm$ over $\Omega\setminus\Sigma$ such that $\tilde\varphi_r^+(\Omega)=\tilde\varphi_r^-(\Omega)$ and $\varphi^\pm \llcorner(\Omega\setminus\Sigma)=\tilde\varphi_r^\pm+\tilde\varphi_s^\pm$. Then*

$$\begin{aligned} MK(\varphi^+,\varphi^-,\Sigma) \le{}& MK(\tilde\varphi_r^+,\tilde\varphi_r^-) + \\ & \int_\Omega A(\mathrm{dist}\,(z,\Sigma))\,d\tilde\varphi_s^+(z) + \int_\Omega A(\mathrm{dist}\,(z,\Sigma))\,d\tilde\varphi_s^-(z). \end{aligned}$$

If $\tilde\varphi_r^\pm=\varphi_r^\pm$ and $\tilde\varphi_s^\pm=\varphi_s^\pm$, while A is superadditive (i.e. $A(u)+A(v)\le A(u+v)$ for every $u,v\in\mathbb{R}^+$), then the above estimate becomes an equality.

Proof – Let γ stand for an optimal transport plan of $MK(\varphi^+, \varphi^-, \Sigma)$. Denote by $\tilde\gamma_r$ an optimal transport plan for the problem $MK(\tilde\varphi_r^+, \tilde\varphi_r^-)$. Let K: $\Omega \multimap \Sigma$ be the set-valued projection to Σ according to the formula

$$K(x) := \{y \in \Sigma \,:\, d(x,y) = \mathrm{dist}\,(x, \Sigma)\}\,,$$

and let k: $\Omega \to \Sigma$ be an arbitrary Borel measurable selector of the latter. Set now

$$\tilde\gamma^+(e) := \tilde\varphi_s^+(\{x \,:\, (x, k(x)) \in e\}), \qquad \tilde\gamma^-(e) := \tilde\varphi_s^-(\{y \,:\, (k(y), y) \in e\})$$

for all Borel sets $e \subset \Omega \times \Omega$. Clearly, in this case $\tilde\gamma^\pm$ are optimal transport plans for problems $MK(\tilde\varphi_s^\pm, \tilde\psi^\mp)$, where $\tilde\psi^\mp := \pi_\#^\mp \tilde\gamma^\pm$. Due to the optimality of γ and to the fact that the measure $\tilde\gamma_r + \tilde\gamma^+ + \tilde\gamma^-$ satisfies the admissibility condition (2.1), one has

$$\begin{aligned}
MK(\varphi^+, \varphi^-, \Sigma) &= \int_{\Omega\times\Omega} A(d_\Sigma(x,y))\, d\gamma(x,y) \\
&\le \int_{\Omega\times\Omega} A(d_\Sigma(x,y))\, d(\tilde\gamma_r + \tilde\gamma^+ + \tilde\gamma^-)(x,y) \\
&= \int_{\Omega\times\Omega} A(d_\Sigma(x,y))\, d\tilde\gamma_r(x,y) + \\
&\qquad \int_{\Omega\times\Omega} A(d_\Sigma(x,y))\, d\tilde\gamma^+(x,y) + \\
&\qquad\quad \int_{\Omega\times\Omega} A(d_\Sigma(x,y))\, d\tilde\gamma^-(x,y).
\end{aligned}$$

But

$$\int_{\Omega\times\Omega} A(d_\Sigma(x,y))\, d\tilde\gamma_r(x,y) \le \int_{\Omega\times\Omega} A(d(x,y))\, d\tilde\gamma_r(x,y) = MK(\tilde\varphi_r^+, \tilde\varphi_r^-),$$

while

$$\int_{\Omega\times\Omega} A(d_\Sigma(x,y))\, d\tilde\gamma^\pm(x,y) = \int_\Omega A(\mathrm{dist}\,(z, \Sigma))\, d\tilde\varphi_s^\pm(z),$$

thus showing the first part of the statement.

To prove the second part, we note that

$$\begin{aligned}
MK(\varphi^+, \varphi^-, \Sigma) &= I_\Sigma^0(\gamma) + I_\Sigma'(\gamma), \text{ where} \\
I_\Sigma^0(\gamma) &:= \int_{\mathcal{R}} A(d_\Sigma(x,y))\, d\gamma(x,y), \\
I_\Sigma'(\gamma) &:= \int_{(\Omega\times\Omega)\setminus\mathcal{R}} A(d_\Sigma(x,y))\, d\gamma(x,y).
\end{aligned}$$

Since $d_\Sigma = d$ over $\mathcal{R}$, one has

$$I^0_\Sigma(\gamma) = MK(\varphi_r^+, \varphi_r^-).$$

It remains to use the superadditivity of A that gives

$$\begin{aligned}
I'_\Sigma(\gamma) &\geq \int_{\Omega\times\Omega} A(d(x,K(x)))\, d\gamma_{\mathcal{R}^c}(x,y) + \int_{\Omega\times\Omega} A(d(y,K(y)))\, d\gamma_{\mathcal{R}^c}(x,y) \\
&= \int_\Omega A(d(x,K(x)))\, d(\varphi_s^+ + \tilde{\psi}^+)(x) + \\
&\qquad\qquad \int_\Omega A(d(y,K(y)))\, d(\varphi_s^- + \tilde{\psi}^-)(y) \\
&= \int_{\Omega\setminus\Sigma} A(\mathrm{dist}\,(x,\Sigma))\, d\varphi_s^+(x) + \int_{\Omega\setminus\Sigma} A(\mathrm{dist}\,(y,\Sigma))\, d\varphi_s^-(y),
\end{aligned}$$

where $\gamma_{\mathcal{R}^c} := \gamma \llcorner (\Omega\times\Omega\setminus\mathcal{R})$ and $\tilde{\psi}^\pm$ are some measures concentrated on Σ.
$\square$

Now the following simple statement relates the problem of minimization of $MK(\varphi^+, \varphi^-, \cdot)$ to that of the respective average distance functional.

Proposition 2.2. *If A is superadditive and Σ_{opt} is optimal for $MK(\varphi^+, \varphi^-, \cdot)$, then it minimizes the average distance functional*

$$I(\Sigma) := \int_\Omega A(\mathit{dist}\,(x,\Sigma))\, d\varphi_s(x),$$

where $\varphi_s := \varphi_{s,\Sigma_{opt}}$, over all admissible Σ.

Proof – If Σ does not minimize I, i.e. there is an admissible Σ' such that $I(\Sigma') < I(\Sigma)$, then we immediately get a contradiction with the optimality of Σ by means of the following series of estimates

$$\begin{aligned}
MK(\varphi^+, \varphi^-, \Sigma) &= MK(\varphi_r^+, \varphi_r^-) + I(\Sigma) \\
&> MK(\varphi_r^+, \varphi_r^-) + I(\Sigma') \\
&\geq MK(\varphi^+, \varphi^-, \Sigma'),
\end{aligned}$$

the last inequality being valid in view of lemma 2.1. $\square$

2.2. Irrigation problems

Another source for problems related to average distance functionals is given by the so-called *irrigation problems*. Let, for instance, Ω represent a tissue with density of cells given by the measure φ. The problem of minimization of F can be interpreted as finding the best possible system of vessels (which should therefore be connected) under a constraint on the total length, so that the optimality condition is that of being as close as possible "in average" to the cells of the tissue.

The most interesting problem within this framework seems to be as follows. Assume that

$$H(l) = H_\alpha(l)$$

with given parameter $\alpha > 0$ (recall that such a penalization corresponds to just fixing an upper bound on the total length of Σ). For each α denote by F_α the minimum value of F (with such a penalization) over closed connected subsets of Ω. One is interested in the asymptotics of F_α as $\alpha \to +\infty$. For the case $A(t) = t$ and $\varphi << \mathcal{L}^n$ this question has been answered in [3]. To get the more precise information on the asymptotic behaviour of F_α, one has to consider the Γ-limit as $\alpha \to \infty$ of functionals $I(\cdot) + H_\alpha(\mathcal{H}^1(\cdot))$ in the space of probability measures (i.e. the closed connected subsets $\Sigma \subset \Omega$ are immersed in the latter by assigning to each Σ the measure $\mathcal{H}^1 \llcorner \Sigma / \mathcal{H}^1(\Sigma)$). Such a problem for the case $A(t) = t^p$, $p \geq 1$, has been solved in [9].

2.3. Lazy travelling salesman problem

One of the most popular versions of the classical *travelling salesman problem* reads as follows: given k points $\{x_1, \ldots, x_k\} \subset \mathbb{R}^n$ representing the cities the travelling salesman has to visit, find the shortest closed path Σ (the embedding of S^1 into $\mathbb{R}^n$) connecting those points (i.e. the route to take to visit those cities). This problem is clearly of purely combinatorial nature.

We assume now, that as opposed to the classical travelling salesman problem, the lazy salesman wants just to choose a route sufficiently close to the cities he has to visit, rather than passing exactly through the cities. Namely, setting

$$\varphi := \sum_{i=1}^{k} \delta_{x_i},$$

the goal of the lazy travelling salesman is to choose the route Σ being a closed path minimizing $I(\Sigma)$ subject to penalization on the length of Σ. The latter depends on the *path length* $\lambda(\Sigma)$ defined by

$$\lambda(\Sigma) := \inf \left\{ \int_{S^1} |\phi'(t)|\, dt \;:\; \phi \in \mathrm{Lip}(S^1, \mathbb{R}^n), \phi(S^1) = \Sigma \right\},$$

rather than on $\mathcal{H}^1(\Sigma)$. In fact, $\lambda(\Sigma)$ represents the total length of the route the salesman has to cover. Summing up, the *lazy travelling salesman problem* is that of finding a minimum of

$$I(\Sigma) + H(\lambda(\Sigma))$$

over the set of admissible Σ (usually, admissibility of Σ, as in the problems considered before, amounts to just compactness and connectedness, though in some applications one adds some more geometric constraints, like e.g. the requirement that Σ pass through some given points). Problems of this kind for the case $A(t) = t^2$, $H(l) = \sigma l^2$, where $\sigma > 0$, have been considered in [10]. Heuristically, when $\sigma \to 0$, the solutions to the above above problems converge to that of the travelling salesman problem as formulated above. In [10] the existence of optimal Σ for such a problem has been proven as well as some qualitative properties of minimizers in particular cases have been studied. In particular, it has been proven that when φ is a discrete measure as above, then the optimal route Σ is either a line segment or a closed polygonal path without self-intersections. Further, in planar case $n = 2$ the question of when the optimal route is a boundary of a convex set is studied.

2.4. Shape optimization

Suppose that $\Omega \subset \mathbb{R}^n$ stand for the domain where the electric charge distributions φ^+ and φ^- are given. One has to distribute in Ω some given quantity m of conducting material, and also put inside Ω some wires of perfect conductor

Σ of given total length l, in order that the final configuration produce minimum possible Joule heating induced by the current flow. If the distribution of the conducting material (i.e. distribution of conductivity) is given by a Borel measure μ, then the total heating of the respective configuration is given by the formula

$$J(\mu,\Sigma) = -\inf\left\{\frac{1}{2}\int_\Omega |\nabla v|^2\,d\mu - \int_\Omega v\,d\tilde{\varphi} \,:\, v\in C^\infty(\Omega), v=0 \text{ on } \Sigma\right\},$$

where $\tilde{\varphi} := \varphi^+ - \varphi^-$. In [2] it is shown that

$$\inf\{J(\mu,\Sigma) \,:\, \mu(\Omega)=m\} = \frac{1}{2m}\left(MK(\varphi^+,\varphi^-,\Sigma)\right)^2,$$

where $A(t)=t$ in the formulation of $MK(\varphi^+,\varphi^-,\Sigma)$. Therefore, the minimum possible heating is given by

$$\inf\left\{J(\mu,\Sigma) \,:\, \mu(\Omega)=m, \mathcal{H}^1(\Sigma)\le l\right\} = \frac{1}{2m}\left(\inf_{\mathcal{H}^1(\Sigma)\le l} MK(\varphi^+,\varphi^-,\Sigma)\right)^2,$$

and hence we see that the above shape optimization problem is reduced to the urban planning type problem just considered.

3. Qualitative properties of optimal sets

In this section we consider some qualitative properties that every minimizer Σ_{opt} of the average distance functional under a reasonable penalization has to fulfil. We present here some problems together with some conjectures which we believe are true. These conjectures are partially justified by numerical approximations of Σ_{opt} provided in [3] for different particular situations as well as by some rigorous results we provide in the sequel which, though sometimes weaker than our expectations, still induce to think that the conjectures we formulate most probably hold true.

When formulating the problems below we assume that $\Omega\subset\mathbb{R}^n$ is a convex body (i.e. the closure of some convex open set), so that the distance d is just the Euclidean distance, and that the measure $\varphi\neq 0$ is absolutely continuous with respect to the Lebesgue measure.

Problem 3.1 (Regularity). *Study the regularity properties of the solutions Σ_{opt}. We actually expect that Σ_{opt} are piecewise smooth, i.e. made of a finite number of smooth curves connected through a finite number of singular points.*

Problem 3.2 (Absence of loops). *Study the topological properties of the solutions Σ_{opt}. In some cases we expect that Σ_{opt} do not form closed loops in Ω. When the space dimension n is equal to 2 this can be expressed by saying that $\mathbb{R}^2 \setminus \Sigma_{opt}$ is connected.*

Problem 3.3 (Triple points). *Study the nature of the singular points mentioned in Problem 3.1. We expect that they can only be triple points, that is points where three curves meet, with angles of 120 degrees.*

In what follows we assume the following conditions on the function A:

(α_1) A is Lipschitz continuous over $[0, \operatorname{diam}\Omega]$, i.e. there is a $\Lambda > 0$ such that

$$|A(x) - A(y)| \leq \Lambda |x - y|$$

whenever $x, y \in [0, \operatorname{diam}\Omega]$;

(α_2) for every $c > 0$ there is a $\lambda = \lambda(c) > 0$ such that

$$|A(x) - A(y)| \geq \lambda |x - y|$$

whenever $\{x, y\} \subset [c, \operatorname{diam}\Omega]$. In particular, A is injective (i.e. strictly increasing since A is supposed to be nondecreasing) over $[0, \operatorname{diam}\Omega]$.

It is worth mentioning that the above conditions are surely satisfied by the functions $A(t) := t^p$, $p \geq 1$.

3.1. Some auxiliary lemmata

In the sequel we will denote by $k\colon\ \Omega \to \Sigma_{opt}$ the projection map to the optimal set Σ defined a.e., and set $\psi := k_{\#}\varphi$.

We will further extensively use the following lemma generalizing the analogous result from [5].

Lemma 3.4. *Let φ be a nonzero finite Borel measure over Ω, $\varphi << \mathcal{L}^n$, while $\Sigma_\nu \subset \Omega$ be a sequence of closed sets. Assume that $\Sigma := \cap_\nu \Sigma_\nu \neq \emptyset$. Let T' stand for the set of $y \in \Omega$ such that there is a $\nu_0 = \nu_0(y) \in \mathbb{N}$ with*

$$0 < dist\,(y, \Sigma) < dist\,(x, \Sigma_\nu \setminus \Sigma)$$

for all $\nu \geq \nu_0$, and assume that $\varphi(T') > 0$. Then under the condition (α_2) for every $\varepsilon > 0$ there is a closed segment $I_\varepsilon \subset \Omega$ with $\mathcal{H}^1(I_\varepsilon) = \varepsilon$ such that $\Sigma \cap I_\varepsilon \neq \emptyset$ and

$$\int_\Omega A(dist\,(x, \Sigma_{\nu,\varepsilon}))\, d\varphi(x) \leq \int_\Omega A(dist\,(x, \Sigma_\nu))\, d\varphi(x) - L(\varepsilon),$$

where $\Sigma_{\nu,\varepsilon} := \Sigma_\nu \cup I_\varepsilon$ and $L(\varepsilon) > 0$, while there is an $\varepsilon_0 > 0$ independent of ν such that

$$L(\varepsilon) \geq C\varepsilon^{(n+1)/2}$$

for some constant $C > 0$ independent of both ν and ε and for all $\varepsilon < \varepsilon_0$.

Proof – Let $\Omega'_\nu \subset \Omega$ stand for the set of points admitting a unique projection to Σ_ν. The respective projection map will be denoted $k^\nu\colon \Omega'_\nu \to \Sigma_\nu$. Clearly, since Ω'_ν is the set of differentiability points of the distance function $u(\cdot) := \mathrm{dist}\,(\cdot, \Sigma_\nu)$, then $\mathcal{L}^n(\Omega'_\nu) = \mathcal{L}^n(\Omega)$. Our assumption $\varphi << \mathcal{L}^n$ allows hence to consider, without loss of generality, φ to be restricted to $\Omega' := \cap_\nu \Omega'_\nu$.

Further, letting

$$\Sigma^c := \{y \in \Omega \,:\, \mathrm{dist}\,(y, \Sigma) < c\},$$

one observes that $\varphi(\Omega \setminus \Sigma^c) > 0$ for some $c > 0$. We fix such a $c > 0$ and denote then $\varphi_c := \varphi \llcorner \Omega \setminus \Sigma^c$, $\varphi'_c := \varphi \llcorner \Sigma^c$.

Consider an arbitrary $x \in T' \cap \Omega'$. For every $\nu \geq \nu_0 = \nu_0(x)$ one has $P := k^\nu(x) \in \Sigma$. Let

$$R^\nu_x := \{z \in \Omega' \,:\, k^\nu(z) = P\},$$

i.e. R^ν_x stands for the maximum line segment passing through x and satisfying $k^\nu(R^\nu_x) = P$. Let $l_\nu > 0$ stand for the length of passing R^ν_x. According to the conditions of the lemma being proven, $R^\nu_x \subset R^{\nu'}_x$ once $\nu' \geq \nu \geq \nu_0$. Then

$R_x^\nu \uparrow R_x$ in the set-theoretic sense, where $R_x \subset \Omega'$ is some segment of positive length l ending on Σ. Clearly, $l_\nu \uparrow l$. Let $O_\nu \notin \Sigma$ stand for the upper endpoint of R_x^ν. Then

$$\Sigma_\nu \cap B_{l_\nu}(O_\nu) = \emptyset.$$

We assume without loss of generality the origin of the coordinate system to coincide with O_ν, and, moreover, the coordinate axis x_n to be directed along R_x upwards, so that the lower endpoint P of R_x has coordinates $P = (0, \ldots, 0, -l_\nu)$. Let $P_\varepsilon := (0, \ldots, 0, -l_\nu + \varepsilon l_\nu)$, $I_\varepsilon := [P, P_\varepsilon]$ stand for the closed segment with endpoints A and P_ε and $\Sigma_{\nu,\varepsilon} := \Sigma_\nu \cup I_\varepsilon$. For each $\nu \in \mathbb{N}$ and $\varepsilon < l_\nu$ consider the set

$$\begin{aligned}&\Lambda_{\nu,\varepsilon} := \\ &\qquad \left\{z = (Y, x_n) \in B_{l_\nu}(0) \,:\, x_n \leq 0,\, d\left(z, P_\varepsilon\right) - \text{dist}\ \left(z, \partial B_{l_\nu}(0)\right) \leq -\varepsilon l_\nu/4\right\}.\end{aligned}$$

In the proof of lemma 5.2 from [5] it has been shown that, up to a correct choice of x, one has

$$\varphi_c\left(\Lambda_{\nu,\varepsilon}\right) \geq C\varepsilon^{(n-1)/2} \tag{3.1}$$

for all sufficiently small (independently of ν) $\varepsilon > 0$ and for some constant $C > 0$ (independent both of ν and ε). Then, taking $\nu_1 = \nu_1(x) \geq \nu_0 \in \mathbb{N}$ such that

$$l_\nu \geq l/2 \text{ for all } \nu \geq \nu_1$$

one gets for $z \in \Lambda_{\nu,\varepsilon}$ the estimate

$$\text{dist}\left(z, \Sigma_{\nu,\varepsilon}\right) \leq d(z, P_\varepsilon) \leq \text{dist}\left(z, \partial B_{l_\nu}(0)\right) - \varepsilon l_\nu/4 \leq \text{dist}\left(z, \Sigma\right) - \varepsilon l/8$$

for all $\nu \geq \nu_1$. Hence, minding (α_2), one arrives at the estimate

$$\begin{aligned}\int_\Omega A(\text{dist}\ (z, \Sigma_{\nu,\varepsilon}))\, d\varphi_c(z) - \int_\Omega A(\text{dist}\ (z, \Sigma))\, d\varphi_c(z) \quad &\leq \quad -\lambda\varepsilon l\varphi\left(\mathbf{\Lambda}_{\nu,\varepsilon}\right)/8 \\ &\leq \quad -C\varepsilon^{(n+1)/2},\end{aligned}$$

where $\lambda := \lambda(c/2)$. On the other hand, one has

$$\int_\Omega A(\text{dist}\ (z, \Sigma_{\nu,\varepsilon}))\, d\varphi'_c(z) \leq \int_\Omega A(\text{dist}\ (z, \Sigma)) d\varphi'_c(z)$$

and hence adding the above two estimates and minding that $\varphi = \varphi_c + \varphi'_c$, we conclude the proof. $\square$

Yet another lemma of this ilk will be also useful in the sequel.

Lemma 3.5. *Assume that φ be a nonzero finite Borel measure over Ω, $\varphi << \mathcal{L}^n$, while $\Sigma_\nu \subset \Omega$ be a sequence of closed sets such that $\Sigma := \cap_\nu \Sigma_\nu \neq \emptyset$. Let for some $x \in \Sigma$ there exist a Borel set $T'_x \subset \Omega$ with $\varphi(T'_x) > 0$ such that there is a $\nu_0 \in \mathbb{N}$ for which every $y \in T'_x$ satisfies*

$$0 < \mathit{dist}\,(y, \Sigma) = d(y, x) < \mathit{dist}\,(y, \Sigma_\nu \setminus \Sigma)$$

whenever $\nu \geq \nu_0$.

Then under the condition (α_2) for each $\nu \in \mathbb{N}$ and for each sufficiently small $\varepsilon > 0$ there is a closed connected set $\Sigma_{\nu,\varepsilon} \supset \Sigma_\nu$ such that $\mathcal{H}^1(\Sigma_{\nu,\varepsilon}) \leq \mathcal{H}^1(\Sigma_\nu) + \varepsilon$ and

$$\int_\Omega A(\mathit{dist}\,(x, \Sigma_{\nu,\varepsilon}))\, d\varphi(x) \leq \int_\Omega A(\mathit{dist}\,(x, \Sigma_\nu))\, d\varphi(x) - C\varphi(T'_x)\varepsilon,$$

for some constant $C > 0$ independent of both ν and ε.

Proof – With the help of lemma 3.6 below, we can construct for every given $\delta > 0$, such a connected set Γ'_δ in the boundary $\partial Q^n_\delta(x)$ of the n-dimensional cube $Q^n_\delta(x)$ with sidelength 4δ centered at x, that

$$\mathcal{H}^1(\Gamma'_\delta) \leq C\delta, \text{ while } \mathrm{dist}\,(y, \Gamma'_\delta) \leq \delta$$

for all $y \in \partial Q^n_\delta(x)$, where $C > 0$ depends only on n. Further, let Γ_δ stand for the union of Γ'_δ with a segment joining the latter to x. One has that Γ_δ is closed and connected, while $\mathcal{H}^1(\Gamma_\delta) \leq C'\delta$, where $C' > 0$ depends only on n.

Let now $\varepsilon := C'\delta$ and set

$$\Sigma_{\nu,\varepsilon} := \Sigma_\nu \cup \Gamma_\delta.$$

Clearly, $\Sigma_{\nu,\varepsilon}$ is still closed, connected and $\mathcal{H}^1(\Sigma_{\nu,\varepsilon}) \leq \mathcal{H}^1(\Sigma_\nu) + \varepsilon$. As in the proof of lemma 3.4, set

$$\Sigma^c := \{x \in \Omega \,:\, \mathrm{dist}\,(x, \Sigma) \leq c\}$$

and observe that there is a $c > 0$ such that $\varphi(T'_x \setminus \Sigma^c) \geq \varphi(T'_x)/2$ (in fact, $\varphi(T'_x \cap \Sigma^c) \to \varphi(T'_x \cap \Sigma) = 0$ as $c \downarrow 0$ in view of our assumption $\varphi << \mathcal{L}^n$). Fixing such a $c > 0$, we denote then $\varphi_c := \varphi \llcorner \Omega \setminus \Sigma^c$, $\varphi'_c := \varphi \llcorner \Sigma^c$.

Now, if $2\sqrt{n}\delta < c$ (i.e. when $\Gamma_\delta \subset \partial Q_\delta^n(x) \subset \Sigma^c$), we have for every $y \in T'_x \setminus \Sigma^c$ and for every $\nu \geq \nu_0$ that $\mathrm{dist}\,(y, \Sigma_\nu) = d(y,x)$, while

$$\mathrm{dist}\,(y, \Sigma_{\nu,\varepsilon}) \leq \mathrm{dist}\,(y, \Gamma_\delta) \leq d(y,x) - 2\delta + \delta,$$

and therefore

$$\mathrm{dist}\,(y, \Sigma_{\nu,\varepsilon}) \leq \mathrm{dist}\,(y, \Sigma_\nu) - C\varepsilon$$

for some $C > 0$ independent both on ε and on ν. Minding (α_2), we get therefore

$$\begin{aligned}\int_\Omega A(\mathrm{dist}\ (z, \Sigma_{\nu,\varepsilon}))\, d\varphi_c(z) - \int_\Omega A(\mathrm{dist}\ (z, \Sigma_\nu))\, d\varphi_c(z) &\leq -\lambda C\varepsilon\varphi_c\,(T'_x) \\ &\leq -C\varphi\,(T'_x)\,\varepsilon\end{aligned}$$

(where $\lambda = \lambda(c/2)$ and $C > 0$ may change from line to line, but is always independent on ν and ε). On the other hand, one has

$$\int_\Omega A(\mathrm{dist}\ (z, \Sigma_{\nu,\varepsilon}))\, d\varphi'_c(z) \leq \int_\Omega A(\mathrm{dist}\ (z, \Sigma_\nu)) d\varphi'_c(z)$$

and hence adding the above two estimates and minding that $\varphi = \varphi_c + \varphi'_c$, we conclude the proof. □

Lemma 3.6. *For every $j \in \mathbb{N}$ there is a connected set $\Gamma^j \subset Q^n := [0,1]^n$ such that*

$$\mathcal{H}^1(\Gamma^j) \leq n(j+1)^{n-1} \text{ and } \mathrm{dist}\,(y, \Gamma^j) \leq C/j$$

for all $y \in Q^n$, where $C > 0$ depends only on n.

Proof – We define Γ^j to be the set of all $(x_1, \ldots, x_n) \in Q^n$ such that $jx_i \in \mathbb{N}$ for all $i = 1, \ldots, n$ except at most one (cfr. Definition 2.4 from [9]). The verification of the announced properties of Γ^j is straightforward. □

We now recall briefly the following topological notions.

Definition 3.7. *Let Σ be a connected space. Then $x \in \Sigma$ is called noncut point of Σ, if $\Sigma \setminus \{x\}$ is connected. Otherwise, x is called cut point of Σ.*

It is worth reminding that according to the Moore theorem (theorem IV.5 from [7, § 47]), every continuum (i.e. compact connected space) has at least two noncut points.

In the sequel we will use the following lemma proven in [5].

Lemma 3.8. *Let Σ be a locally connected metric continuum consisting of more than one point and $x \in \Sigma$ be a noncut point of Σ. Then there is a sequence of open sets $D_\nu \subset \Sigma$ satisfying*

(i) *$x \in D_\nu$ for all sufficiently large ν;*

(ii) *$\Sigma \setminus D_\nu$ are connected for all ν;*

(iii) *$\operatorname{diam} D_\nu \searrow 0$ as $\nu \to \infty$;*

(iv) *D_ν are connected for all ν.*

Another notion we will use in the sequel is that of an order of a point, as well as that of branching points and of endpoints.

Definition 3.9. *Let Σ be a topological space. We will say that the order of the point $x \in \Sigma$ does not exceed $\mathfrak{n}$, writing*

$$ord_x \Sigma \leq \mathfrak{n},$$

where $\mathfrak{n}$ is a cardinal, if for every $\varepsilon > 0$ there is an open subset $U \subset \Sigma$ such that $x \in U$, $\operatorname{diam}(U) < \varepsilon$ and $\#\partial U \leq \mathfrak{n}$, $\#$ standing for the cardinality of a set.

The order of the point $x \in \Sigma$ is said to be equal $\mathfrak{n}$, written

$$ord_x \Sigma = \mathfrak{n},$$

if $\mathfrak{n}$ is the least cardinal for which $ord_x \Sigma \leq \mathfrak{n}$.

If $\operatorname{ord}_x \Sigma = \mathfrak{n}$, with $\mathfrak{n} \geq 3$, then x will be called a branching point of Σ, while if $\operatorname{ord}_x \Sigma = 1$, then x will be called an endpoint of Σ.

We recall that according to theorem V.1 from [7, § 51], if $\operatorname{ord}_x \Sigma \leq 1$, then x is a noncut point of Σ. In particular, every endpoint of Σ is a noncut point.

3.2. Length and location

We start with the following easy result.

Theorem 3.10. *Let $n > 1$, $\varphi << \mathcal{L}^n$, and suppose that (α_2) is valid together with*

(α_3) $H(\varepsilon) - H(0) = o(\varepsilon^{(n+1)/2})$ *as* $\varepsilon \to 0^+$.

Then for the optimal set Σ_{opt} one has $\mathcal{H}^1(\Sigma_{opt}) > 0$.

Proof – Suppose $\mathcal{H}^1(\Sigma_{opt}) = 0$. Without loss of generality we may assume that Σ_{opt} is nonempty and hence consists of just one point. According to lemma 3.4 applied with $\Sigma_\nu := \Sigma_{opt}$ (note that then $T' = \Omega \setminus \Sigma_{opt}$, and thus the validity of the condition $\varphi(T') > 0$ is provided in this case by the assumptions $\mathcal{L}^n(\Sigma_{opt}) = 0$ and $\varphi << \mathcal{L}^n$, $\varphi(\Omega) > 0$), one can find for each sufficiently small $\varepsilon > 0$ a closed connected $\Sigma_\varepsilon \subset \Omega$ with $\Sigma_{opt} \subset \Sigma_\varepsilon$ and $\mathcal{H}^1(\Sigma_\varepsilon) = \varepsilon$ such that

$$\int_\Omega A(\mathrm{dist}\,(x, \Sigma_\varepsilon))\, d\varphi(x) \le \int_\Omega A(\mathrm{dist}\,(x, \Sigma_{opt}))\, d\varphi(x) - C\varepsilon^{(n+1)/2}$$

for some $C > 0$ independent of ε. But then

$$\begin{aligned} F(\Sigma_\varepsilon) &= \int_\Omega A(\mathrm{dist}\,(z, \Sigma_\varepsilon))\, d\varphi(z) + H(\mathcal{H}^1(\Sigma_\varepsilon)) \\ &\le \int_\Omega A(\mathrm{dist}\,(z, \Sigma_{opt}))\, d\varphi(z) + H(\mathcal{H}^1(\Sigma_{opt})) \\ &\qquad\qquad - C\varepsilon^{(n+1)/2} + o(\varepsilon^{(n+1)/2}) \\ &< F(\Sigma_{opt}) \end{aligned}$$

once $\varepsilon > 0$ is sufficiently small. The latter estimate contradicts the optimality of Σ_{opt}, hence concluding the proof. □

Another statement is concerned with the location of the optimal set Σ_{opt} in Ω.

Proposition 3.11. *If Σ_{opt} is optimal, then so is its projection to $D := \overline{co}\,(\mathrm{supp}\,\varphi)$. Further, if $n > 1$ and (α_2) holds, while either H is strictly increasing or $\varphi << \mathcal{L}^n$, then necessarily $\Sigma_{opt} \subset D$.*

PROOF – Suppose $\Sigma_{opt} \setminus D \neq \emptyset$. Let then Σ' stand for the projection of Σ_{opt} on D. Clearly, Σ' is still closed and connected. We show now that for all $x \in D$ one has

$$\text{dist}\,(x, \Sigma') < \text{dist}\,(x, \Sigma_{opt}) \tag{3.2}$$

whenever $\text{dist}\,(x, \Sigma_{opt}) = d(x, x_0)$ for some $x_0 \in \Sigma_{opt} \setminus D$. In fact, to prove (3.2), suppose the contrary, namely that

$$\text{dist}\,(x, \Sigma') > \text{dist}\,(x, \Sigma_{opt}). \tag{3.3}$$

for such an $x \in D$. Let x_0' stand for the projection of x_0 on D. Since $x_0' \in \Sigma'$, then

$$\text{dist}\,(x, \Sigma') \leq d(x, x_0').$$

But

$$d(x, x_0') < d(x_0, x).$$

The latter inequality follows immediately from the fact that both x_0' and x belong to the same half-space formed by a hyperplane π which is parallel to the supporting hyperplane of D at x_0' and passes through the middle point of the segment $[x_0, x_0']$: in fact, $\pi = \{z \,:\, d(x_0, z) = d(x_0', z)\}$ (see figure 1). We

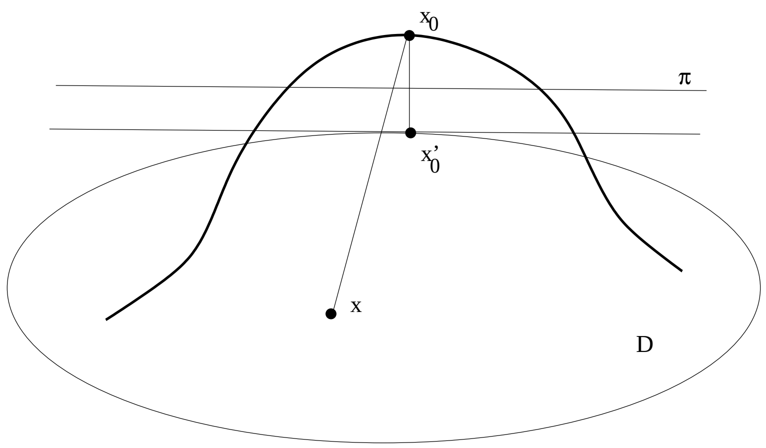

Figure 1: The original set Σ_{opt} (thick line) and the hyperplane π

conclude now

$$\mathrm{dist}\,(x,\Sigma') \leq d(x,x_0') < d(x_0,x) = \mathrm{dist}\,(x,\Sigma_{opt}),$$

contradicting (3.3) and hence proving (3.2).

The estimate (3.2) implies

$$\mathrm{dist}\,(x,\Sigma') \leq \mathrm{dist}\,(x,\Sigma_{opt})$$

for all $x \in \Omega$, and minding that $\mathcal{H}^1(\Sigma') \leq \mathcal{H}^1(\Sigma_{opt})$, we get $F(\Sigma') \leq F(\Sigma_{opt})$, which means that Σ' is optimal.

Suppose now that (α_2) holds. We have to consider two cases. First, suppose that $\Sigma_{opt} \cap D = \emptyset$. Then (3.2) implies $\mathrm{dist}\,(x,\Sigma') < \mathrm{dist}\,(x,\Sigma_{opt})$ for all $x \in D$, and hence $I(\Sigma') < I(\Sigma_{opt})$. Minding that $\mathcal{H}^1(\Sigma') \leq \mathcal{H}^1(\Sigma_{opt})$, we get therefore $F(\Sigma') < F(\Sigma_{opt})$, contradicting the optimality of Σ_{opt}.

It remains thus to consider the case $\Sigma_{opt} \cap D \neq \emptyset$. Then (3.2) implies $\mathrm{dist}\,(x,\Sigma') \leq \mathrm{dist}\,(x,\Sigma_{opt})$ for all $x \in D$, and hence $I(\Sigma') \leq I(\Sigma_{opt})$. On the other hand, in this case $\mathcal{H}^1(\Sigma') < \mathcal{H}^1(\Sigma_{opt})$ in view of lemma 3.12 below (when applying the latter, mind that since $\mathcal{H}^1(\Sigma_{opt}) < +\infty$, then Σ_{opt} admits a Lipschitz continuous parameterization, and hence is in fact a Lipschitz curve).

If H is strictly increasing, we obtain immediately $F(\Sigma') < F(\Sigma_{opt})$ which contradicts the optimality of Σ_{opt} and thus proves $\Sigma_{opt} \subset D$. Otherwise, in view of the relationship $F(\Sigma') \leq F(\Sigma_{opt})$ one has $H(\mathcal{H}^1(\Sigma')) = H(\mathcal{H}^1(\Sigma_{opt}))$. Therefore, if $\varphi << \mathcal{L}^n$, then by lemma 3.4 we are able to construct a new closed connected set $\tilde{\Sigma} \supset \Sigma'$ such that

$$\mathcal{H}^1(\Sigma') < \mathcal{H}^1(\tilde{\Sigma}) < \mathcal{H}^1(\Sigma_{opt}),$$

and hence $H(\mathcal{H}^1(\tilde{\Sigma})) = H(\mathcal{H}^1(\Sigma_{opt}))$, while $I(\tilde{\Sigma}) < I(\Sigma')$. This implies again the contradiction $F(\tilde{\Sigma}) < F(\Sigma_{opt})$, which concludes the proof. □

In the lemma below we slightly abuse the notation identifying the image of a Lipschitz curve with the parameterization of the latter.

Lemma 3.12. *Let $D \subset \mathbb{R}^n$ be a closed convex set, and $k_D\colon \mathbb{R}^n \to D$ stand for the projection map to D. If for a Lipschitz curve γ one has $\gamma \cap D \neq \emptyset$ and $\mathcal{H}^1(\gamma) = \mathcal{H}^1(k_D(\gamma))$, then $\gamma \subset D$.*

PROOF – Without loss of generality we may suppose that γ: $[0,L] \to \mathbb{R}^n$ is parameterized by arclength, so that $|\gamma'(t)| = 1$ for a.e. $t \in [0,L]$. The map $k_D \circ \gamma$: $[0,L] \to \mathbb{R}^n$ is Lipschitz-continuous, hence a.e. differentiable, and, since the Lipschitz constant of k_D does not exceed 1, then $|(k_D \circ \gamma)'(t)| \leq 1$ a.e. Further, since $\mathcal{H}^1(\gamma) = \mathcal{H}^1(k_D(\gamma))$, then $|(k_D \circ \gamma)'(t)| = |\gamma'(t)| = 1$ a.e. over $[0,L]$. Let $t \in [0,L]$ be such that both $\tau := \gamma'(t)$ and $(k_D \circ \gamma)'(t)$ exist and

$$|(k_D \circ \gamma)'(t)| = |\tau| = 1. \tag{3.4}$$

Suppose that $\gamma(t) \notin D$. We show first the existence of a partial derivative $\frac{\partial k_D}{\partial \tau}$ in the point $x := \gamma(t)$. In fact,

$$\begin{array}{rcl} |k_D(x+\delta\tau) - k_D(\gamma(t+\delta))| & \leq & |x + \delta\tau - \gamma(t+\delta)| \\ & = & |\gamma(t) + \delta\gamma'(t) - \gamma(t+\delta)| \\ & = & o(\delta) \end{array}$$

as $\delta \to 0$. Hence

$$\begin{array}{rcl} k_D(x+\delta\tau) - k_D(x) & = & (k_D(\gamma(t+\delta)) - k_D(\gamma(t))) + o(\delta) \\ & = & (k_D \circ \gamma)'(t)\delta + o(\delta) \end{array}$$

as $\delta \to 0$, which means that $\frac{\partial k_D}{\partial \tau}(x)$ exists.

Equation (3.4) implies then that $v_\tau := \frac{\partial k_D}{\partial \tau}(x)$ is parallel to τ, and, moreover,

$$|v_\tau| = |\tau| = 1.$$

We observe now that v_τ is orthogonal to ν, where

$$\nu := \frac{x - k_D(x)}{|x - k_D(x)|}.$$

In fact, otherwise $\cos\alpha := |v_\tau \cdot \nu| > 0$, where α stands for the minimum angle between the line parallel to v_τ and the line parallel to ν. Therefore, letting x'_δ stand for the projection of $x + \delta v_\tau$ on the line $\{x + t\nu \ : \ t \in \mathbb{R}\}$ we get $k_D(x'_\delta) = k_D(x)$, and hence

$$\begin{array}{rcl} |k_D(x+\delta v_\tau) - k_D(x)| & = & |k_D(x+\delta v_\tau) - k_D(x'_\delta)| \\ & \leq & |(x+\delta v_\tau) - x'_\delta| \\ & = & |(x+\delta v_\tau) - x| \sin\alpha \\ & = & |\delta| \sin\alpha. \end{array}$$

On the other hand, we have

$$|k_D(x + \delta v_\tau) - k_D(x)| = |\delta v_\tau + o(\delta)| \geq |\delta| + o(\delta)$$

as $\delta \to 0$, and hence, minding that according to our assumption $\sin\alpha < 1$, we have that for $\delta \neq 0$ small enough

$$|k_D(x + \delta v_\tau) - k_D(x)| > \delta \sin\alpha.$$

This contradiction shows that v_τ is orthogonal to ν.

It remains now to consider the function $f\colon [0, L] \to \mathbb{R}$ defined by

$$f(t) := \operatorname{dist}^2(\gamma(t), D).$$

Since $f'(t) = 2(\gamma(t) - k_D(\gamma(t))) \cdot \gamma'(t)$ a.e. on $[0, L]$, then $f'(t) = 0$. In fact, either $\gamma(t) \in D$, in which case $\gamma(t) = k_D(\gamma(t))$, or $\gamma(t) \notin D$, and then it has been just proven that

$$(\gamma(t) - k_D(\gamma(t))) \cdot \gamma'(t) = \operatorname{dist}(\gamma(t), D)\nu \cdot \tau = \operatorname{dist}(\gamma(t), D)\nu \cdot v_\tau = 0$$

a.e. on $[0, L]$. We conclude now that f is constant, and since it has been supposed that for some t one has $\gamma(t) \in D$, then f is identically zero. This means $\gamma \subset D$. $\square$

According to proposition 3.11, the location of Σ_{opt} is determined by φ rather than by the shape of Ω. Minding this, we will always further consider without loss of generality, possibly enlarging Ω, that

$$\Sigma_{opt} \subset\subset \Omega.$$

The condition $\varphi << \mathcal{L}^n$ is clearly essential both in the statement of theorem 3.10 and of that of proposition 3.11. In fact, let $\varphi := \delta_O$, where O stands for the origin of the coordinate system and $\Omega := B_1(0)$. Let $A(t) = t$ so that (α_2) holds, and

$$H(l) := \begin{cases} 0, & l \leq 1, \\ +\infty, & l > 1, \end{cases}$$

so that also (α_3) holds. Then clearly $\Sigma := \{0\}$ is optimal while $\mathcal{H}^1(\Sigma) = 0$, so that theorem 3.10 does not hold. Moreover, every Σ containing the origin is optimal too, though for all such Σ except $\Sigma = \{O\}$ proposition 3.11 does not hold.

3.3. Regularity

The following mild regularity result can be considered as a very first step towards a solution to problem 3.1. It asserts that the optimal set Σ_{opt} is Ahlfors regular. Recall that a set $\Sigma \subset \mathbb{R}^n$ is called Ahlfors regular, if for every $x \in \Sigma$ and $r < \operatorname{diam} \Sigma$ one has

$$cr \leq \mathcal{H}^1(\Sigma \cap B_r(x)) \leq Cr \tag{3.5}$$

for some constants $c > 0$ and $C > 0$ independent both on x and on $r > 0$. A singleton is considered Ahlfors regular by definition. It is also well-known that Ahlfors regularity of connected one dimensional sets (recall that the optimal set Σ_{opt} is requested to be connected) implies another mild regularity condition called *uniform rectifiability*. The latter condition, which has been extensively studied in [6], provides some nice analytical properties of the set and can be considered as a kind of "quantitative rectifiability" condition as opposed to the classical "qualitative" rectifiability which is one of the basic concepts of the geometric measure theory.

It is worth remarking that to verify Ahlfors regularity of $\Sigma \subset \mathbb{R}^n$, it is enough to show the existence of some $r_0 \in (0, \operatorname{diam} \Sigma)$ (independent of $x \in \Sigma$) such that (3.5) holds just for $r \leq r_0$. In fact, in this case, when $r > r_0$ and $r < \operatorname{diam} \Sigma$, we have

$$\begin{aligned} \frac{\mathcal{H}^1(\Sigma \cap B_r(x))}{r} &= \frac{\mathcal{H}^1(\Sigma \cap B_r(x))}{r_0} \frac{r_0}{r} \\ &\geq \frac{\mathcal{H}^1(\Sigma \cap B_r(x))}{r_0} \frac{r_0}{\operatorname{diam} \Sigma} \\ &\geq \frac{\mathcal{H}^1(\Sigma \cap B_{r_0}(x))}{r_0} \frac{r_0}{\operatorname{diam} \Sigma} \geq c \frac{r_0}{\operatorname{diam} \Sigma}, \end{aligned}$$

and on the other hand,

$$\frac{\mathcal{H}^1(\Sigma \cap B_r(x))}{r} \leq \frac{\mathcal{H}^1(\Sigma \cap B_r(x))}{r_0} \leq \frac{\mathcal{H}^1(\Sigma)}{r_0}.$$

Theorem 3.13. *Let $n = 2$ and $\varphi \in L^p(\Omega)$, where $p > 4/3$. Then under conditions (α_1)-(α_2) the optimal set Σ_{opt} is Ahlfors regular.*

Proof – We may assume $\mathcal{H}^1(\Sigma_{opt}) > 0$ (otherwise Σ_{opt} is a singleton). Since Σ_{opt} is assumed to be connected, then $\mathcal{H}^1(\Sigma_{opt} \cap B_r(x)) \geq r$ for all $x \in \Sigma_{opt}$ and all $r < \operatorname{diam} \Sigma_{opt}/2$. Therefore it is enough to show the existence of an $r_0 > 0$ and a $C > 0$ such that $\mathcal{H}^1(\Sigma_{opt} \cap B_r(x)) \leq Cr$ for all $x \in \Sigma_{opt}$ and for all $r < r_0$. We show that one can take $C = 2\pi + \varepsilon$, where $\varepsilon > 0$ is an arbitrary positive number. Suppose the contrary, namely, the existence of a sequence $\{x_\nu\} \subset \Sigma_{opt}$ and of a sequence $r_\nu \to 0$ such that

$$l_\nu(x) := \mathcal{H}^1(\Sigma_{opt} \cap B_{r_\nu}(x_\nu)) > (2\pi + \varepsilon) r_\nu.$$

Up to choosing a subsequence of $\{x_\nu\}$, we assume $x_\nu \to x \in \Sigma_{opt}$. Further, we will write r instead of r_ν omitting the index ν whenever this cannot cause confusion.

Letting $\Sigma_r := (\Sigma_{opt} \setminus B_r(x_\nu)) \cup \partial B_r(x_\nu)$, one has that the latter set is closed and connected, while

$$\mathcal{H}^1(\Sigma_r) \leq \mathcal{H}^1(\Sigma_{opt}) - \varepsilon r.$$

But then in view of (α_1), one has

$$\int_\Omega A(\operatorname{dist}(x, \Sigma_r))\, d\varphi(x) \leq \int_\Omega A(\operatorname{dist}(x, \Sigma_{opt}))\, d\varphi(x) + 2\Lambda r \varphi(B_r(x_\nu)).$$

We would like to apply now lemma 3.4. For this purpose consider a $\rho > 0$ such that

$$\psi(\Omega) > \psi(B_\rho(x)).$$

Such a ρ exists since otherwise one would have $\psi(\Omega) = \psi(B_\rho(x))$ for all $\rho > 0$, and hence, taking a limit in the latter relationship as $\rho \to 0$, $\psi(\Omega) = \psi(\{x\})$. The latter means $I(\{x\}) = I(\Sigma_{opt})$, and hence $F(\{x\}) \leq F(\Sigma_{opt})$ (since $\mathcal{H}^1(\Sigma_{opt}) > 0 = \mathcal{H}^1(\{x\})$). This is only possible when $H(\mathcal{H}^1(\Sigma_{opt})) = H(0)$ in view of the optimality of Σ_{opt}. But then there would exist according to lemma 3.4 a new closed connected set $\Sigma' \ni x$ such that $I(\Sigma') < I(\Sigma_{opt})$, while $0 < \mathcal{H}^1(\Sigma') < \mathcal{H}^1(\Sigma_{opt})$, and hence $H(\mathcal{H}^1(\Sigma')) = H(\mathcal{H}^1(\Sigma_{opt}))$ implying $F(\Sigma') < F(\Sigma_{opt})$. The latter contradiction with the optimality of Σ_{opt} proves the existence of a $\rho > 0$ satisfying the announced property.

We have now $\Sigma_r \cap B = \Sigma_{opt} \cap B$, where $B := \Omega \setminus B_\rho(x)$, for all sufficiently large ν. What is more, for all $y \in k^{-1}(\Sigma_{opt} \cap B)$ (i.e. y has unique projection on Σ_{opt}, and, moreover, the latter belongs to B) one has

$$\text{dist}\,(y, \Sigma_{opt} \cap B) = \text{dist}\,(y, \Sigma_{opt}) < \text{dist}\,(y, \Sigma_r \setminus B) \tag{3.6}$$

for all $\nu \geq \bar{\nu}$, where $\bar{\nu} = \bar{\nu}(x)$ is some number which may depend on x. In fact, otherwise one would have

$$\text{dist}\,(y, \Sigma_{opt} \cap B) = \text{dist}\,(y, \Sigma_{opt}) \geq \text{dist}\,(y, \Sigma_r \setminus B) \tag{3.7}$$

for a subsequence of ν (not relabeled). The latter can only happen, if there is a $x'_\nu \in \partial B_{r_\nu}(x_\nu)$ such that

$$\text{dist}\,(y, \Sigma_r \setminus B) = d(y, x'_\nu). \tag{3.8}$$

Combining the above two relationships (3.7) and (3.8), one gets

$$\text{dist}\,(y, \Sigma_{opt} \cap B) = \text{dist}\,(y, \Sigma_{opt}) \geq d(y, x'_\nu)$$

and taking then a limit as $\nu \to \infty$ in the resulting estimate one has,

$$\text{dist}\,(y, \Sigma_{opt}) \geq d(y, x),$$

which can only happen when y has a nonunique projection on Σ_{opt}, while this case is ruled out by our construction. Hence, $k^{-1}(\Sigma_{opt} \cap B) \subset T'$ and

$$\varphi(T') \geq \psi(B) = \psi(\Omega) - \psi(B_\rho(x)) > 0.$$

We may use then lemma 3.4 with $\Sigma_\nu := \Sigma_r$ to find a new closed and connected set $\Sigma_{r,\varepsilon} \subset \Omega$ satisfying $\mathcal{H}^1(\Sigma_{r,\varepsilon}) \leq \mathcal{H}^1(\Sigma_{opt})$, such that

$$\int_\Omega A(\text{dist}\,(x, \Sigma_{r,\varepsilon}))\, d\varphi(x) \leq \int_\Omega A(\text{dist}\,(x, \Sigma_r))\, d\varphi(x) - C_\varepsilon r^{3/2},$$

where C_ε stands for a positive constant depending on ε but not on r. Using Hölder inequality one has $\varphi(B_r(x_\nu)) \leq Cr^{2/p'}$ for some $C > 0$. It is now enough to note that with $p > 4/3$ one has $2/p' + 1 > 3/2$ and hence

$$\int_\Omega A(\text{dist}\,(x, \Sigma_{r,\varepsilon}))\, d\varphi(x) < \int_\Omega A(\text{dist}\,(x, \Sigma_{opt}))\, d\varphi(x)$$

for sufficiently small $r > 0$, which contradicts the optimality of Σ_{opt}. $\square$

3.4. Absence of loops

The following result partially solves problem 3.2.

Theorem 3.14. *Let $n = 2$ and $\varphi << \mathcal{L}^2$. Then under the conditions (α_1) and (α_2) the optimal set Σ_{opt} contains no simple closed curve (a homeomorphic image of S^1). The same conclusion holds in arbitrary space dimension n, if there is an $y \in \Sigma_{opt}$, such that $\psi(\{y\}) > 0$.*

In particular, if $n = 2$, then $\mathbb{R}^2 \setminus \Sigma_{opt}$ is connected.

The proof of the above statement uses the following lemma.

Lemma 3.15. *Let $\Sigma \subset \mathbb{R}^n$ be a closed connected set satisfying $\mathcal{H}^1(\Sigma) < \infty$ which contains a simple closed curve. Then the set NC of noncut points of Σ has positive length, i.e. $\mathcal{H}^1(NC) > 0$.*

Proof – Let $S \subset \Sigma$ be a simple closed curve. If $x \in S$ is a cut point for Σ then there is a continuum L_x such that $L_x \cap S = \{x\}$ and $L_x \setminus \{x\} \neq \emptyset$. One has then $L_x \cap L_y = \emptyset$ whenever $x \neq y$. But then $\mathcal{H}^1(L_x) > 0$ for only at most countable number of points x, otherwise one would have $\mathcal{H}^1(\Sigma) = +\infty$. All the other points $x \in S$ are noncut for Σ. $\square$

We may now claim the following assertion.

Proposition 3.16. *Under the conditions of theorem 3.14 one has that $\mathcal{H}^1$-a.e. $x \in \Sigma_{opt}$ is a cut point of Σ_{opt}.*

Proof – Suppose the contrary, namely that the set NC of noncut points of Σ_{opt} has a positive length, i.e. $\mathcal{H}^1(NC) > 0$. We observe that since $\mathcal{H}^1(\Sigma_{opt}) < +\infty$, then Σ_{opt} is a continuous (in fact, even Lipschitz continuous) image of the unit interval, and hence is locally connected in view of the Hahn-Mazurkiewicz-Sierpiński theorem II.2 from [7, § 50]. Hence for each $x \in NC$ we may consider the sequence of sets $\{D_\nu\}$ with the properties announced in lemma 3.8. We denote $\varepsilon_\nu := \operatorname{diam} D_\nu$. Note that $\mathcal{H}^1(D_\nu) \geq \varepsilon_\nu$. For the sake of brevity of the notation, further on we omit the reference to the index ν writing always ε instead of ε_ν. The contradiction will be achieved in two steps.

Step 1. For every $x \in NC$ and $\varepsilon > 0$ let $T(x,\varepsilon) := k^{-1}(B_\varepsilon(x))$. Choose an arbitrary $x \in NC$ and set $\Sigma_\varepsilon(x) := \Sigma_{opt} \setminus D_\nu$. Clearly, according to our assumption, $\Sigma_\varepsilon(x)$ is still closed and connected, and satisfies the length constraint since $\mathcal{H}^1(\Sigma_\varepsilon(x)) \leq \mathcal{H}^1(\Sigma_{opt})$. Since $D_\nu \subset B_\varepsilon(x)$, the following estimate is valid

$$\begin{aligned} I(\Sigma_\varepsilon(x)) = & \int_{\Omega\setminus T(x,\varepsilon)} A(\mathrm{dist}\,(z,\Sigma_\varepsilon(x)))\,d\varphi(z) + \\ & \int_{T(x,\varepsilon)} A(\mathrm{dist}\,(z,\Sigma_\varepsilon(x)))\,d\varphi(z) \\ \leq & \int_{\Omega\setminus T(x,\varepsilon)} A(\mathrm{dist}\,(z,\Sigma_{opt}))\,d\varphi(z) + \\ & \int_{T(x,\varepsilon)} (A(\mathrm{dist}\,(z,\Sigma_{opt})) + \Lambda\varepsilon)\,d\varphi(z) \\ = & \int_\Omega A(\mathrm{dist}\,(z,\Sigma_{opt}))\,d\varphi(z) + \Lambda\varepsilon\varphi(T(x,\varepsilon)). \end{aligned}$$

By definition of ψ one has

$$\varphi(T(x,\varepsilon)) = \psi(B_\varepsilon(x)).$$

But

$$\limsup_{\varepsilon\to 0^+} \frac{\psi(B_\varepsilon(x))}{\varepsilon} < +\infty$$

for $\mathcal{H}^1$-a.e. $x \in NC$, which implies

$$\psi(B_\varepsilon(x)) \leq C\varepsilon + o(\varepsilon)$$

for some $C = C(x) > 0$ depending on x, $\mathcal{H}^1$-a.e. in NC. Summing up, for $\mathcal{H}^1$-a.e. $x \in NC$ one has

$$\int_\Omega A(\mathrm{dist}\,(z,\Sigma_\varepsilon(x)))\,d\varphi(z) \leq \int_\Omega A(\mathrm{dist}\,(z,\Sigma_{opt}))\,d\varphi(z) + C\varepsilon^2. \tag{3.9}$$

Note that the constant C depends on x. Since $\mathcal{H}^1(NC) > 0$ in view of our original assumption, we can choose an $x \in NC$ such that the above estimate holds with some constant $C > 0$.

Step 2. Let

$$l(\varepsilon) := \mathcal{H}^1(\Sigma_{opt}) - \mathcal{H}^1(\Sigma_\varepsilon(x)).$$

One clearly has $l(\varepsilon) \geq \varepsilon$.

Consider first the case $n = 2$. We may apply then lemma 3.4 with $\Sigma_\nu := \Sigma_\varepsilon(x)$ to find a closed connected $\Sigma' \subset \Omega$ satisfying $\mathcal{H}^1(\Sigma') = \mathcal{H}^1(\Sigma_\varepsilon) + \varepsilon$ and (using the fact that $n = 2$)

$$\int_\Omega A(\operatorname{dist}(z, \Sigma'))\, d\varphi(z) \leq \int_\Omega A(\operatorname{dist}(z, \Sigma_\varepsilon(x)))\, d\varphi(z) - C\varepsilon^{3/2}$$

for some $C > 0$ as $\varepsilon > 0$ is small enough. Combining the above estimate with (3.9), we get

$$\int_\Omega A(\operatorname{dist}(z, \Sigma'))\, d\varphi(z) < \int_\Omega A(\operatorname{dist}(z, \Sigma_{opt}))\, d\varphi(z) \tag{3.10}$$

as $\varepsilon > 0$ is small enough, since $\varepsilon^2 = o(\varepsilon^{3/2})$ as $\varepsilon \to 0^+$. This is a contradiction with the optimality of Σ_{opt}.

It remains then to consider the case of arbitrary n assuming the existence of an $y \in \Sigma_{opt}$, such that $\psi(\{y\}) > 0$. Note that $y \neq x$ due to our choice of x. Hence, we may apply lemma 3.5 with $\Sigma_\nu := \Sigma_\varepsilon(x)$ and $T'_y := k^{-1}(y) \setminus T(x, \varepsilon_0)$ (for some fixed sufficiently small $\varepsilon_0 > 0$) to find a closed connected $\Sigma' \subset \Omega$ satisfying $\mathcal{H}^1(\Sigma') = \mathcal{H}^1(\Sigma_\varepsilon) + \varepsilon$ and

$$\int_\Omega A(\operatorname{dist}(z, \Sigma'))\, d\varphi(z) \leq \int_\Omega A(\operatorname{dist}(z, \Sigma_\varepsilon(x)))\, d\varphi(z) - C\varepsilon$$

for some $C > 0$ as $\varepsilon > 0$ is small enough. Combining the above estimate with (3.9), we again get (3.10) as $\varepsilon > 0$ is small enough, since $\varepsilon^2 = o(\varepsilon)$ as $\varepsilon \to 0^+$, and this is again a contradiction with the optimality of Σ_{opt}. $\square$

At this point we are able to prove theorem 3.14.

Proof of theorem 3.14: The first part of the statement follows immediately from proposition 3.16 combined with lemma 3.15. We prove now the connectedness of $\mathbb{R}^2 \setminus \Sigma_{opt}$ in the case $n = 2$. In view of theorem II.5 of [7, § 61], if $D \subset \mathbb{R}^2$ is a bounded connected set with locally connected boundary, then there is a simple closed curve $S \subset \partial D$. Then, if $\mathbb{R}^2 \setminus \Sigma_{opt}$ were disconnected, there would exist a bounded connected component D of $\mathbb{R}^2 \setminus \Sigma_{opt}$ such that $\partial D \subset \Sigma_{opt}$, and hence Σ_{opt} would contain a simple closed curve, contrary to what has been just proven. $\square$

3.5. Endpoints and branching points

We observe first the following easy fact.

Lemma 3.17. *Let Σ be a locally connected continuum which contains no simple closed curve. If the set of endpoints of Σ is finite, then so is the set of branching points of Σ. Moreover, in this case for each branching point $x \in \Sigma$ one has that $\mathrm{ord}_\Sigma x$ is finite.*

Proof – Let $k \in \mathbb{N}$ stand for the number of endpoints of Σ. We will say that a point $x \in \Sigma$ has *depth* not exceeding $\mathfrak{n}$, where $\mathfrak{n}$ is a cardinal, if there is an endpoint $l \in \Sigma$ and an arc in Σ connecting x to l such that the number of branching points of Σ on this arc has cardinality $\mathfrak{n}$. Moreover, we will say that the depth of $x \in \Sigma$ is equal to $\mathfrak{n}$, if $\mathfrak{n}$ is the least cardinal for which the depth of x does not exceed $\mathfrak{n}$.

Let us show first that the depth of each point of Σ is at most $k-1$. Consider an arbitrary point $x_0 \in \Sigma$ and an arc γ_0 connecting it to an arbitrary endpoint l_0. Proving the claim amounts to showing that the number of branching points on such an arc is finite. If $x_i \in \gamma_0$ is a branching point, then according to the Menger n-Beinsatz (Claim 8 in [7, § 51.II]), there is an arc γ_i' starting in x_i and intersecting γ_0 only at x_i. Denote x_i' an arbitrary internal point of this arc, and consider any connected component of $\Sigma \setminus \{x_i\}$ containing x_i'. Pick an arbitrary endpoint l_i of the latter. It is clearly also an endpoint of Σ and cannot coincide with l_0 since otherwise Σ would contain a simple closed curve contradicting the assumptions. Moreover, for the same reason when $j \neq i$, then for the same reason $l_j \neq l_i$. Therefore, to each branching point $x_i \in \gamma_0$ corresponds at least one distinct endpoint $l_i \neq l_0$ so that $l_i \neq l_j$ when $i \neq j$, which proves that there can be no more that $k-1$ distinct endpoints on γ_0 (or, in other words, the depth of x_0 doe not exceed $k-1$).

We can represent now the set $\mathcal{B}$ of branching points of Σ in the form

$$\mathcal{B} = \bigcup_{i=0}^{k-1} \mathcal{B}_i,$$

where $\mathcal{B}_i$ stands for the set of branching points of depth i. But clearly, $\#\mathcal{B}_0 \leq k$, while $\#\mathcal{B}_{i+1} \leq \#\mathcal{B}_i$, and hence in the above union all the sets $\mathcal{B}_i$ are finite. Therefore, the set $\mathcal{B}$ is finite, which concludes the proof of the first claim.

Consider now an arbitrary branching point $x \in \Sigma$. We claim that its order cannot exceed k. In fact, otherwise in view of the Menger n-Beinsatz there would exist at least $\mathfrak{n} > k$ closed arcs $\gamma_i \subset \Sigma$, $i = 1, \ldots, \mathfrak{n}$, starting at x and pairwise disjoint outside of x (i.e. $\gamma_i \cap \gamma_j = \{x\}$ for $i \neq j$). Taking an arbitrary internal point x_i of each of the arcs γ_i, we see that each of the connected components of the set $\Sigma \setminus \{x\}$ containing x_i contains necessarily some endpoint l_i which is also an endpoint of Σ. Further, $l_i \neq l_j$ when $i \neq j$ since otherwise Σ would contain a simple closed curve in contradiction with the assumptions. □

We provide now the following general assertion.

Proposition 3.18. *Let $\varphi << \mathcal{L}^n$, and suppose that conditions (α_1)-(α_2) hold. Assume that there is an $y \in \Sigma_{opt}$ such that $\psi(\{y\}) > 0$. Then there is a constant $C > 0$ such that for every noncut point $x \in \Sigma_{opt}$ one has $\psi(\{x\}) \geq C$. In particular, in this case*

(i) the number of noncut points in Σ_{opt} is finite;

(ii) the number of branching points in Σ_{opt} is finite.

Proof – Let $x \in \Sigma_{opt}$ be an arbitrary noncut point, $x \neq y$. We prove

$$\psi(\{x\}) \geq C\psi(\{y\}) \tag{3.11}$$

for some $C > 0$ independent of x. For this purpose let $\{D_\nu\}_{\nu\in\mathbb{N}}$ stand for the decreasing sequence of subsets of Σ_{opt} indicated by lemma 3.8, with $x \in D_\nu$. We also suppose the diameters of D_ν to be small enough so that $y \notin D_\nu$ for all $\nu \in \mathbb{N}$. We set $\varepsilon_\nu := \operatorname{diam} D_\nu$. Note that $\mathcal{H}^1(D_\nu) \geq \varepsilon_\nu$. For the sake of brevity, further on we omit the reference to the index ν writing always ε instead of ε_ν.

Set $\Sigma_\varepsilon^1 := \Sigma_{opt} \setminus D_\nu$. Then

$$\int_\Omega A(\operatorname{dist}(z, \Sigma_\varepsilon^1))\, d\varphi(z) \leq \int_\Omega A(\operatorname{dist}(z, \Sigma_{opt}))\, d\varphi(z) + \Lambda\varepsilon\psi(D_\nu).$$

Note that since $y \notin D_\nu$ for all $\nu \in \mathbb{N}$ such that, then setting $T_y' := k^{-1}(y) \setminus k^{-1}(D_1) = k^{-1}(y)$, we get $\varphi(T_y') = \psi(\{y\})$. We apply therefore lemma 3.5 to find a closed connected $\Sigma_\varepsilon^2 \supset \Sigma_\varepsilon^1$ such that

$$\mathcal{H}^1(\Sigma_\varepsilon^2) \leq \mathcal{H}^1(\Sigma_\varepsilon^1) + \varepsilon \leq \mathcal{H}^1(\Sigma_{opt})$$

and for $\varepsilon > 0$ sufficiently small one has

$$\int_\Omega A(\mathrm{dist}\,(z,\Sigma_\varepsilon^2))\,d\varphi(z) \leq \int_\Omega A(\mathrm{dist}\,(z,\Sigma_\varepsilon^1))\,d\varphi(z) - C\psi(\{y\})\varepsilon$$

for some $C > 0$ independent of ε and of x. Thus

$$\int_\Omega A(\mathrm{dist}\,(z,\Sigma_\varepsilon^2))\,d\varphi(z) \leq \int_\Omega A(\mathrm{dist}\,(z,\Sigma_{opt}))\,d\varphi(z) + \varepsilon\psi(D_\nu) - C\varepsilon\psi(\{y\}).$$

Taking into account that $\varepsilon \to 0$ and $\psi(D_\nu) \to \psi(\{x\})$ as $\nu \to \infty$, one has that the latter inequality would contradict the optimality of Σ_{opt} unless $\psi(\{x\}) \geq C\psi(\{y\})$, which shows (3.11).

Due to arbitrarity of choice of x, one has for every noncut point $x \in \Sigma_{opt}$ that $\psi(\{x\}) > C$ for some constant $C > 0$ (independent of x). It is now immediate to conclude that the number of noncut points is finite since $\psi(\Omega) < +\infty$. The fact that the number of branching points is finite follows from lemma 3.17. $\square$

In [5] it has been shown that the conditions of the above proposition 3.18 are fulfilled when $n = 2$, $A(t) = t$ and $\varphi \in L^p(\Omega)$ with $p > 4/3$.

Another interesting assertion concerns the nature of branching points. In its proof we will use the following notation. For a finite set of points $N \subset \partial B_r(x)$ we call its *centered Steiner connection* the set $St(N,x,r)$ providing the minimum for the length functional

$$\Sigma \mapsto \mathcal{H}^1(\Sigma)$$

among all closed connected $\Sigma \subset \bar{B}_r(x)$ which satisfy $N \cup \{x\} \subset \Sigma$ (the existence of such a set is a well-known fact).

Proposition 3.19. *Under the conditions of proposition 3.18 every branching point of Σ_{opt} is a triple point.*

Proof – Suppose the contrary, namely that there is an $x \in \Sigma_{opt}$ with $\mathrm{ord}_x\Sigma_{opt} > 3$. Then according to the Menger n-Beinsatz, there are at least 4 closed arcs $\gamma_i \subset \Sigma_{opt}$, $i = 1, \ldots, 4$, starting at x and pairwise disjoint outside of x (i.e. $\gamma_i \cap \gamma_j = \{x\}$ for $i \neq j$). We choose an $\varepsilon = \varepsilon_\nu := 1/\nu$, $\nu \in \mathbb{N}$ small enough so that the closed ball $\bar{B}_\varepsilon(x)$ does not contain other noncut or branching points (this is possible since both the former and the latter are finite) and so as to have also

$$\gamma_i \cap \partial B_\varepsilon(x) \neq \emptyset, \qquad i = 1, \ldots, 4.$$

Let each $a_\varepsilon^i \in \gamma_i \cap \partial B_\varepsilon(x)$ stand for the first point where γ_i hits $\partial B_\varepsilon(x)$, i.e. such that for all $y \in (x, a_\varepsilon^i)$ one has $y \in B_\varepsilon(x)$, $i = 1, \ldots, 4$. According to lemma 3.20 below, among these four points there are at least two (without loss of generality, up to renumbering, we assume that the latter are a_ε^1 and a_ε^2) such that the minimum arc of $\partial B_\varepsilon(x)$ has angle $\delta < 2\pi/3$.

Set

$$\Sigma_\varepsilon^1 := \left(\Sigma_{opt} \setminus ((x, a_\varepsilon^1) \cup (x, a_\varepsilon^2))\right) \cup St(\{a_\varepsilon^1, a_\varepsilon^2\}, x, \varepsilon).$$

Note that Σ_ε^1 is still connected, since the arcs erased did non contain any branching point of Σ_{opt}, and is obviously closed. We estimate then

$$\mathcal{H}^1(\Sigma_\varepsilon^1) \leq \mathcal{H}^1(\Sigma_{opt}) - \mathcal{H}^1((x, a_\varepsilon^1)) - \mathcal{H}^1((x, a_\varepsilon^2)) + \mathcal{H}^1(St(\{a_\varepsilon^1, a_\varepsilon^2\}, x, \varepsilon)).$$

Observe now that $\mathcal{H}^1((x, a_\varepsilon^i)) \geq \varepsilon$, $i = 1, 2$, while

$$\mathcal{H}^1(St(\{a_\varepsilon^1, a_\varepsilon^2\}, x, \varepsilon)) = \varepsilon \mathcal{H}^1(St(\{(a_\varepsilon^1 - x)/\varepsilon, (a_\varepsilon^2 - x)/\varepsilon\}, 0, 1)).$$

Lemma 3.21 below implies then

$$\mathcal{H}^1(St(\{(a_\varepsilon^1 - x)/\varepsilon, (a_\varepsilon^2 - x)/\varepsilon\}, 0, 1) \leq 2 - \beta$$

for some constant $\beta > 0$ depending only on δ. Putting the above estimates together, we get

$$\mathcal{H}^1(\Sigma_\varepsilon^1) \leq \mathcal{H}^1(\Sigma_{opt}) - \beta\varepsilon.$$

Moreover, one has

$$\begin{aligned} \int_\Omega A(\mathrm{dist}\,(z, \Sigma_\varepsilon^1))\, d\varphi(z) \leq & \int_\Omega A(\mathrm{dist}\,(z, \Sigma_{opt}))\, d\varphi(z) \\ & + \Lambda\varepsilon\psi(B_\varepsilon(x) \setminus \{x\}). \end{aligned} \tag{3.12}$$

Let y be a noncut point of Σ_{opt}. One has then due to proposition 3.18 that $\psi(\{y\}) > 0$. We observe that there is a $\nu' \in \mathbb{N}$ such that $y \notin B_{\varepsilon'}(x)$, where $\varepsilon' := \varepsilon_{\nu'}$. Then setting $T_y' := k^{-1}(y) \setminus k^{-1}(B_{\varepsilon'}(x))$, one has $T_y' = k^{-1}(y)$

and hence $\varphi(T'_y) = \psi(\{y\})$. We apply therefore lemma 3.5 to find a closed connected $\Sigma^2_\varepsilon \supset \Sigma^1_\varepsilon$ such that

$$\mathcal{H}^1(\Sigma^2_\varepsilon) \leq \mathcal{H}^1(\Sigma^1_\varepsilon) + \varepsilon \leq \mathcal{H}^1(\Sigma_{opt})$$

and for $\varepsilon > 0$ one has

$$\int_\Omega A(\mathrm{dist}\,(z, \Sigma^2_\varepsilon))\, d\varphi(z) \leq \int_\Omega A(\mathrm{dist}\,(z, \Sigma^1_\varepsilon))\, d\varphi(z) - C\psi(\{y\})\varepsilon \tag{3.13}$$

for some $C > 0$. Since $\psi(B_\varepsilon(x) \setminus \{x\}) = o(1)$ as $\varepsilon \to 0$, then combining (3.12) and (3.13), one obtains the estimate

$$\int_\Omega A(\mathrm{dist}\,(z, \Sigma^2_\varepsilon))\, d\varphi(z) < \int_\Omega A(\mathrm{dist}\,(z, \Sigma_{opt}))\, d\varphi(z).$$

This together with $\mathcal{H}^1(\Sigma^2_\varepsilon) \leq \mathcal{H}^1(\Sigma_{opt})$ implies

$$F(\Sigma^2_\varepsilon) < F(\Sigma_{opt})$$

contradicting the optimality of Σ_{opt}. $\square$

The following technical assertions have been used in the above proof.

Lemma 3.20. *Among four vectors in $\mathbb{R}^n$ there are two, the angle between which is strictly less than $2\pi/3$.*

Proof – Note that without loss of generality one may consider $n = 4$ (since four vectors always belong to a four-dimensional subspace of $\mathbb{R}^n$).

We show first that for three vectors in $\mathbb{R}^3$ either there are two, the angle between which is strictly less than $2\pi/3$, or they all belong to some two-dimensional plane. In fact, let a, b and c be some unit vectors in $\mathbb{R}^3$. We consider the coordinate axes to be directed so that $a = (0, 0, 1)$ and $b = (b_1, 0, b_3)$. We suppose also that the angles between a and b as well as between a and c are not less than $2\pi/3$ (otherwise there is nothing to prove). Clearly, if $c_2 = 0$, then all these vectors belong to the same plane $\{x_2 = 0\}$ and hence the angle δ between b and c must be less or equal $2\pi/3$. Otherwise, if $c_2 \neq 0$, then let $c' := (c_1, 0, c_3)$. Since the angle between a and c is supposed to be not less than $2\pi/3$, then $c_3 \le -1/2$. We may also suppose $c_1^2 + c_3^2 \neq 0$, since otherwise c is perpendicular to a contrary to our assumption. Therefore, for the angle α' between a and c' one gets

$$\cos\alpha' = \frac{c_3}{\sqrt{c_1^2 + c_3^2}} < c_3 \le -\frac{1}{2},$$

since $\sqrt{c_1^2 + c_3^2} < 1$, and hence $\alpha' > 2\pi/3$. Therefore, the angle δ' between b and c' is strictly less than $2\pi/3$, which implies $\cos\delta' > -1/2$. On the other hand

$$\cos\delta' = \frac{b_1 c_1 + b_3 c_3}{\sqrt{c_1^2 + c_3^2}} = \frac{\cos\delta}{\sqrt{c_1^2 + c_3^2}}.$$

Supposing $\cos\delta < 0$ (otherwise $\delta \le \pi/2$ and hence there is nothing to prove), we get $\cos\delta > \cos\delta'$ and hence $\cos\delta > -1/2$, which means $\delta < 2\pi/3$, concluding the proof of the claim.

We are able now to prove the statement. Namely, among four vectors in $\mathbb{R}^4$, consider an arbitrary triple (which automatically belongs to a three-dimensional subspace). Then by the claim just proven, either the latter contains a pair of vectors, the angle between which is strictly less than $2\pi/3$ (in which case there is nothing to prove), or it belongs to some two-dimensional plane. In the latter case, the whole four vectors can be considered to belong to $\mathbb{R}^3$. Therefore, again exploiting the above proven claim, if the angle between each pair of them is not less than $2\pi/3$, then they all must belong to the same two-dimensional plane, and hence there will be a pair of them with angle not exceeding $2\pi/3$. This contradiction concludes the proof. □

The lemma below is a reformulation of lemma 8.1 from [5].

Lemma 3.21. *Let $N = \{a, b\} \subset \partial B_1(0)$, while the minimum arc of $\partial B_1(0)$ between a and b has angle $\delta < 2\pi/3$. Then*

$$\mathcal{H}^1(St(N, 0, 1)) \le 2\sin(\delta/2 + \pi/6).$$

In [5] it has been shown also that, at least in some rather weak sense, in the case $n = 2$, $A(t) = t$ and $\varphi \in L^p(\Omega)$ with $p > 4/3$, the branching points are “regular tripods” (i.e. points where three branches meet at the angle of $2\pi/3$).

4. Final remarks and conclusions

The results of the present paper regarding qualitative properties of minimizers Σ_{opt} to average distance functionals, though being quite general, still do not provide the complete picture thus leaving some open problems. We enlist them below.

- The regularity properties we are able to prove are rather weak. In particular, only Ahlfors regularity (hence also uniform rectifiability and some other very mild regularity properties in the sense of [6]) is proven under some rather restrictive conditions $n = 2$ and $\varphi \in L^p(\Omega)$ with $p > 4/3$. Even such a weak regularity result in generic space dimension n or for more general measures is largely open. We also expect, at least for $n = 2$, stronger regularity for the minimizers.

- Every sufficiently strong regularity result requires a statement on finiteness of endpoints (and hence of branching points) as well as on the topological and geometric nature of branching points. Such statements under conditions which involve only problem data has only been proven in [5] in the case $n = 2$, $A(t) = t$ and $\varphi \in L^p(\Omega)$ with $p > 4/3$. The results we provide here (propositions 3.18 and 3.19), though formally more general, contain an assumption on the existence of a point of Σ_{opt} with positive mass influx (i.e. the existence of such an $y \in \Sigma_{opt}$ that $\psi(\{y\}) > 0$). The verification of this condition in terms of problem data in general situations seems to be another interesting question.

- It is worth remarking however that the natural candidates for the points $y \in \Sigma_{opt}$ with positive mass influx (i.e. $\psi(\{y\}) > 0$) are the endpoints of Σ_{opt}. A priori however the latter may not exist when Σ_{opt} is just a closed loop (a homeomorphic image of S^1). Thus the absence of loops in Σ_{opt} is another important property to study. We remark that also in this case, in terms of conditions involving only problem data, this property has been proven for the minimizers to average distance functional only in the case $n = 2$ (theorem 3.14), while the case of generic space dimension has been treated under the assumption on the existence of a point of Σ_{opt} with positive mass influx. The analogous result can be proven by more subtle computations in the case of generic space dimension n for the measures φ absolutely continuous with respect to Lebesgue measure with densities summable with sufficiently high exponents, i.e. $\varphi \in L^p(\Omega)$, where $p \geq \bar{p}(n)$. The case of generic space dimension and general measures φ seems to be substantially open. In particular, it is not clear whether for n sufficiently high and φ just absolutely continuous with respect to the Lebesgue measure, with no extra summability properties, there might exist an optimizer Σ_{opt} containing a closed loop. It is worth remarking that if such a Σ_{opt} exists, then one can expect it not to contain endpoints (since any endpoint is a natural candidate for a point with positive mass influx), and to be sufficiently smooth (since points of nonsmooothness are also natural candidates for points with positive mass influx), and hence to be just a single smooth closed loop.

- Finally, we mention that it seems to be interesting to characterize all those $\Sigma \subset \mathbb{R}^n$ which are optimal sets for the average distance functional with some $\varphi << \mathcal{L}^n$.

References

[1] A. Brancolini and G. Buttazzo. Optimal networks for mass transportation problems, 2003. `http://cvgmt.sns.it/papers/brabut`.

[2] G. Buttazzo and G. Bouchitté. Characterization of optimal shapes and masses through Monge-Kantorovich equation. *J. European Math. Soc.*, 3:139–168, 2001.

[3] G. Buttazzo, E. Oudet, and E. Stepanov. Optimal transportation problems with free Dirichlet regions. In *Progress in Nonlinear Diff. Equations and their Applications*, volume 51, pages 41–65. Birkhäuser, 2002.

[4] G. Buttazzo, A. Pratelli, S. Solimini, and E. Stepanov. Mass transportation and urban planning problems. Forthcoming.

[5] G. Buttazzo and E. Stepanov. Optimal transportation networks as free Dirichlet regions in the Monge-Kantorovich problem. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, II(4):631–678, 2003.

[6] G. David and S. Semmes. *Analysis of and on uniformly rectifiable sets*, volume 38 of *Math. Surveys Monographs*. Amer. Math. Soc., Providence, RI, 1993.

[7] C. Kuratowski. *Topologie*, volume 1. Państwowe Wydawnictwo Naukowe, Warszawa, 1958. in French.

[8] F. Morgan and R. Bolton. Hexagonal economic regions solve the location problem. *Amer. Math. Monthly*, 109(2):165–172, 2001.

[9] S. J. N. Mosconi and P. Tilli. Γ-convergence for the irrigation problem. `http://cvgmt.sns.it/papers/mostil`.

[10] P. Pollack. Lazy travelling salesman problem. Master's thesis, The Technion–Israel Institute of Technology, Haifa, 2002.