Tightness-Concentration Principles and Compactness for Evolution Problems in Banach Spaces

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Abstract

Compactness in the space $L^p(0,T;B)$, B being a separable Banach space, has been deeply investigated by J.P. Aubin (1963), J.L. Lions (1961,1969), J. Simon (1987), and, more recently, by J.M. Rakotoson and R. Temam (2001), who have provided various criteria for relative compactness, which turn out to be crucial tools in the existence proof of solutions to many abstract time dependent problems related to evolutionary PDE's. In the present paper, the problem is examined in view of Young measure theory: exploiting the underlying principles of "tightness" and "concentration", new necessary and sufficient conditions for compactness are given, unifying some of the previous contributions and showing that the Aubin-Lions condition is not only sufficient but also necessary for compactness. Furthermore, the related issue of compactness with respect to convergence in measure is studied and a general criterion is proved.

Key words: Evolution problems, strong compactness in L^p spaces, compactness in measure, Young measures, tightness, concentration.

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1 Introduction and main results.

Let us consider a *bounded* family $\mathcal{U}$ of functions in $L^p(0,T;B)$, where B is a separable Banach space and $1 \le p < \infty$.

Compactness in finite dimension: a "strong concentration condition". When B is of *finite* dimension, then the celebrated theorem of Riesz-Fréchet-Kolmogorov (see e.g. [4, Thm. IV.26]) says that $\mathcal{U}$ is totally bounded in $L^p(0,T;B)$ if and only if

$$\lim_{h\downarrow 0} \int_0^{T-h} \|u(t+h)-u(t)\|_B^p dt = 0 \quad \text{uniformly for } u \in \mathcal{U}, \tag{1.1a}$$

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i.e. there exists a (non-decreasing, concave) modulus of continuity $\omega : [0, +\infty) \to [0, +\infty)$ with $\lim_{h\downarrow 0} \omega(h) = 0$ such that

$$\Big(\int_0^{T-h} \|u(t+h) - u(t)\|_B^p dt\Big)^{1/p} \le \omega(h) \quad \forall\, h \in (0, T),\ u \in \mathcal{U}. \tag{1.1b}$$

Usually, in several evolution problems, (1.1b) is provided by exhibiting a uniform estimate in $W^{1,p}(0, T; B)$ for the functions in $\mathcal{U}$ (in that case, $\omega(h) = h \sup_{u\in\mathcal{U}} \|\frac{d}{dt}u\|_{L^p(0,T;B)}$), or even in a Sobolev-Besov space of fractional order, see e.g. [11], [12], [5]. Adopting a terminology which will appear more clear in the sequel, we will refer to (1.1a,b) as the "*strong concentration condition*".

Aubin-Lions Theorem in infinite dimension: a first "tightness condition". If the dimension of B is not finite, then condition (1.1a) is no longer sufficient to ensure the relative compactness and some extra condition should be imposed on $\mathcal{U}$: roughly speaking, the idea is that the values of the functions $u \in \mathcal{U}$ should belong, in a suitable integral sense, to some compact set of B, a property which we will call "*tightness*", following a probabilistic terminology. A general sufficient condition, which played a crucial role in the so called "compactness method" for nonlinear evolution problems [11], was given by J.L. Lions [10, Ch. IV, §4] (for Hilbert spaces) and J.P. Aubin [1] (initially for reflexive Banach spaces) by assuming that there exists another Banach space $A \subset B$ such that

$$\text{the inclusion } A \subset B \text{ is compact, } \mathcal{U} \text{ is bounded in } L^p(0, T; A). \tag{1.2}$$

Remark 1.1. It should be remarked that, once (1.2) holds, in (1.1a) the norm of B could be replaced by the weaker norm of any Banach space C in which B is continuously contained.

It would be interesting to know if Aubin-Lions criterion is also necessary for the compactness of $\mathcal{U} \subset L^p(0, T; B)$: since it is easy to see that a compact set $\mathcal{U}$ satisfies (1.1a), one has to find a suitable Banach space $A \subset B$ such that the functions of $\mathcal{U}$ take their value in A and (1.2) holds. We shall give an affirmative reply to this question; first we recall the "integral" approach by Simon.

The integral characterization by Simon. J.Simon [18] provided a complete characterization of the compact sets in $L^p(0, T; B)$, showing that it is necessary and sufficient for $\mathcal{U}$ to satisfy (1.1a) and

$$\Big\{\int_0^t u(s)ds\ : u \in \mathcal{U}\Big\} \quad \text{is relatively compact in } B \quad \forall t \in (0, T). \tag{1.3}$$

Simon's proof is a clever combination of Ascoli-Arzelà compactness Theorem in $C^0([0, T]; B)$ and of an approximation argument by convolution, which inherits the integral compactness property of (1.3); convexity of the norm and linearity of (1.3) play an important role. It is easy to see that (1.2) is stronger than (1.3); on the other hand, (1.2) seems easier to handle in many applications (see e.g. [11]), where it follows directly from *a priori* estimates involving the values of $u \in \mathcal{U}$ instead of their time integrals.

A characterization of compactness through a more general "tightness condition". So it is natural to wonder whether (1.3) can be replaced by another (necessary and sufficient) condition closer to (1.2). In order to understand in what direction we can generalize (1.2), let us rephrase it in a slightly different form: we introduce the functional $\mathscr{F}_{p,A} : B \to [0, +\infty]$

$$\mathscr{F}_{p,A}(v) := \begin{cases} \|v\|_A^p & \text{if } v \in A, \\ +\infty & \text{if } v \in B \setminus A. \end{cases} \tag{1.4}$$

It is easy to see that the sublevels of $\mathscr{F}_{p,A}$

$$\{v \in B \, : \mathscr{F}_{p,A}(v) \leq c\}, \quad c \in [0, +\infty), \quad \text{are compact in } B, \tag{1.5}$$

and therefore $\mathscr{F}_{p,A}$ is lower semicontinuous. (1.2) is then equivalent to

$$\sup_{u \in \mathcal{U}} \int_0^T \mathscr{F}_{p,A}(u(t))dt < +\infty. \tag{1.6}$$

A natural idea is to replace $\mathscr{F}_{p,A}$ by a general integrand $\mathscr{F}$ with compact sublevels; more generally, we can consider a *coercive normal integrand* $\mathscr{F}$: $(0, T) \times B \to [0, +\infty]$ explicitly depending on the time t also. If $\mathcal{L}$ and $\mathcal{B} = \mathcal{B}(B)$ denote the σ-algebras of the Lebesgue-measurable subsets of $(0, T)$ and of the Borel subsets of B respectively, we recall that $\mathscr{F}$ is a *normal integrand* if

$$\mathscr{F} : (0, T) \times B \to [0, +\infty] \quad \text{is } \mathcal{L} \otimes \mathcal{B}\text{-measurable}, \tag{1.7a}$$

$$\text{the maps } v \mapsto \mathscr{F}_t(v) := \mathscr{F}(t, v) \text{ are l.s.c. for a.e. } t \in (0, T); \tag{1.7b}$$

$\mathscr{F}$ is also *coercive* if

$$\{v \in B : \mathscr{F}_t(v) \leq c\} \text{ are compact for any } c \geq 0 \text{ and for a.e. } t \in (0, T). \tag{1.7c}$$

These conditions were introduced by E.J.Balder [2] in developing a Young measure framework for studying lower semicontinuity in optimal control problems; following [2, §2], we introduce the following notion:

Definition 1.2 (Tightness). *We say that $\mathcal{U}$ is* tight *w.r.t. a normal coercive integrand $\mathscr{F}$ satisfying* (1.7*a, b, c*) *if*

$$S := \sup_{u \in \mathcal{U}} \int_0^T \mathscr{F}(t, u(t))dt < +\infty. \tag{1.7d}$$

We say that $\mathcal{U}$ is tight in B if there exists a normal coercive integrand $\mathscr{F}$ for which (1.7d) *holds.*

If we compare (1.7a,b,c,d) with the assumptions in (1.4, 1.5, 1.6), we can see that there are no more convexity or homogeneity type constraints as those yielded by the norm functional: actually, the link with (1.3) is no longer directly available.

Remark 1.3. If the functions $u \in \mathcal{U}$ are a.e. valued in a Banach space $A \subset B$ with compact embedding and if for a nonnegative l.s.c. function $G : [0, +\infty] \to [0, +\infty]$

$$\sup_{u \in \mathcal{U}} \int_0^T G\big(\|u(t)\|_A\big) < +\infty, \quad \text{with } \lim_{s \uparrow +\infty} G(s) = +\infty, \tag{1.8}$$

then it is easy to see that $\mathcal{U}$ is tight. Choosing, e.g., $G(s) := s^q$, $0 < q < +\infty$, we see that boundedness in $L^q(0, T; A)$ always implies tightness.

Remark 1.4 (Measure-theoretic formulation of tightness). Definition 1.2 is strictly related to Prohorov's Compactness Theorem for probability measures (see Theorem 2.8): we only remark that, at least in the case of a functional $\mathscr{F}$ *independent of* t, (1.7d) yields

$$\forall\,\varepsilon>0\quad\exists\,K_\varepsilon\subset\subset B:\quad\big|\{t\in(0,T):u(t)\notin K_\varepsilon\}\big|\le\varepsilon\quad\forall\,u\in\mathcal{U}. \tag{1.9}$$

In fact, we can choose

$$K_\varepsilon:=\{v\in B:\mathscr{F}(v)\le S/\varepsilon\}, \tag{1.10}$$

which is compact by (1.7c). Thus, definition 1.2 ensures that the function $u\in\mathcal{U}$ are uniformly compact-valued in a measure-theoretic sense. Conversely, if (1.9) holds, then it is possible to find a coercive integrand $\mathscr{F}:B\to[0,+\infty]$ such that (1.7d) holds. Let us comment that (1.7d) could be easier to check than (1.9) in many applications, where it could be directly obtained by proving integral *a priori* estimates on the elements of $\mathcal{U}$.

Our first result concerns a characterization of relative compact subsets in $L^p(0,T;B)$ in which (1.3) can be replaced by (1.7d):

Theorem 1 ("Compactness=strong concentration+tightness"). *A bounded family $\mathcal{U}\subset L^p(0,T;B)$ is relatively compact if and only if* (1.1a) *holds and $\mathcal{U}$ satisfies (1.7d) for a normal coercive integrand $\mathscr{F}$. Moreover, if $\mathcal{U}$ is relatively compact in $L^p(0,T;B)$, it is possible to choose an integrand*

$$\mathscr{F}\ \text{independent}\ \textit{of}\ t\textit{,}\ \text{convex}\textit{, and satisfying}$$
$$\lim_{\|v\|_B\uparrow+\infty}\frac{\mathscr{F}(v)}{\|v\|_B^p}=+\infty,\qquad\mathscr{F}(v)\ge\mathscr{F}_{p,A}(v)\quad\forall\,v\in B, \tag{1.11}$$

where A is a Banach space compactly embedded in B and $\mathscr{F}_{p,A}$ is defined by (1.4)*.*

Remark 1.5. Let us rephrase the last statement: if $\mathcal{U}\subset L^p(0,T;B)$ is relatively compact, then (1.1a,b) hold and there exists a Banach space A compactly embedded in B such that the functions of $\mathcal{U}$ take their values (up to a negligible set) in A and

$$\sup_{u\in\mathcal{U}}\int_0^T\|u(t)\|_A^p\,dt<+\infty. \tag{1.12}$$

This proves exactly the converse of the Aubin-Lions theorem.

The proof of Theorem 1 relies on the fundamental compactness and lower semicontinuity result of parametrized (Young) measure theory: the idea is to reduce the problem of compactness in $L^p(0,T;B)$ to the problem of compactness with respect to convergence in measure, via some extra uniform integrability estimate, as suggested by Proposition 1.7 below. In this approach, the "tightness" condition (1.7d) (which, as we said before, generalizes (1.2)) allows us to extract from every sequence in $\mathcal{U}$ a convergent subsequence in the (very weak) sense of Young measures; (1.1a) will provide a further "concentration" property for the limiting Young measure and the uniform p-integrability estimate [1]

$$\lim_{|J|\downarrow0}\sup_{u\in\mathcal{U}}\int_J\|u(t)\|_B^p\,dt=0. \tag{1.13}$$

[1] where the limit is obviously restricted to Lebesgue measurable subsets J of $(0,T)$ and $|J|$ denotes their Lebesgue measure.

The combined effect of concentration and p-uniform integrability yields strong convergence in $L^p(0,T;B)$.

A different point of view: reinforcing weak convergence. The flexibility of the previous argument can be better understood by recalling the approach of *reinforcing weak convergence* recently proposed by RAKOTOSON AND TEMAM [15]: in order to guarantee the strong $L^p(0,T;B)$ convergence of the sequence $\mathcal{U} := \{u^n\}_{n\in\mathbb{N}}$ they assume the Aubin-Lions "tightness" condition (1.2) (in the particular case of $p = 2$ and of two Hilbert spaces A, B) and the uniform integrability (1.13): they replace the "strong concentration condition" (1.1a), by a different one of weaker type

$$\exists\, w-\lim_{n\uparrow+\infty} u^n(t) = u(t) \quad \text{for a.e. } t \in (0,T). \tag{1.14}$$

In this case the combined effect of tightness (1.2) and weak concentration (1.14) provides the convergence in measure: if one assumes (1.13), strong convergence in $L^p(0,T;B)$ can be deduced. Observe that in this framework (1.13) is no more guaranteed by the other two assumptions.

It is then natural to ask if the extra assumptions (Hilbert structure, $p = 2$) could be removed and substituted by the tightness condition (1.7d) and if (1.14) could be weakened, too.

We will propose an answer to this question (see Theorem 1.15 later on) and the proof of (the "sufficiency part" of) Theorem 1 by providing a general characterization of compactness with respect to convergence in measure, which is a subject of independent interest.

Compactness for the convergence in measure.

The topology of convergence in measure. Let us denote by $\mathcal{M}(0,T;B)$ the space of strongly-measurable B-valued functions; we recall that a sequence $\{u^n\}_{n\in\mathbb{N}} \subset \mathcal{M}(0,T;B)$ converges in measure to $u \in \mathcal{M}(0,T;B)$ as $n \uparrow +\infty$ if

$$\lim_{n\uparrow+\infty} \big|\{t \in (0,T) : \|u^n(t) - u(t)\|_B \geq \sigma\}\big| = 0 \quad \forall \sigma > 0. \tag{1.15}$$

It is well known that

$$\text{an a.e. converging sequence } \{u^n\}_{n\in\mathbb{N}} \text{ also converges in measure,} \tag{1.16}$$

whereas from every sequence $\{u^n\}_{n\in\mathbb{N}}$ converging to u in $\mathcal{M}(0,T;B)$ it is always possible to extract a subsequence $\{u^{n_k}\}_{k\in\mathbb{N}}$ a.e. converging to the same limit u.

It can be shown that $\mathcal{M}(0,T;B)$ is an F*-space*, i.e. its topology is induced by a complete metric δ, invariant by translations: e.g. if we set

$$d_B(v,w) := \min\{1, \|v - w\|_B\} \quad \forall\, v, w \in B, \tag{1.17}$$

an admissible metric δ is given by

$$\delta(v,w) := \int_0^T d_B(v(t), w(t))\, dt \quad \forall\, v, w \in \mathcal{M}(0,T;B). \tag{1.18}$$

It is not difficult to show that the bounded distance δ induces the convergence in measure (see [7, III.2, IV.11]).

By the Chebychev inequality, it is easy to see that L^p convergence (resp. compactness) yields convergence (resp. compactness) in $\mathcal{M}(0,T;B)$. On the other hand, the latter notion, though weaker, entails the stronger one if some extra information of uniform integrability type is supplied.

Uniform integrability and weak L^1 compactness. We already recalled that a subset $\mathcal{U} \subset L^p(0,T;B)$ is *p-uniformly integrable* (or simply *uniformly integrable* if $p=1$) if (1.13) holds, or, equivalently, if

$$\forall\,\varepsilon > 0 \quad \exists\,\delta > 0: \quad \forall\,J \subset (0,T) \quad |J| < \delta \;\Rightarrow\; \sup_{u\in\mathcal{U}} \int_J \|u(t)\|_B^p\,dt \le \varepsilon. \tag{1.19}$$

It is easy to see that boundedness in $L^{p+\varepsilon}(0,T;B)$ and the Hölder inequality entail p-uniform integrability for every $\varepsilon > 0$.

Conversely, it is clear that uniform integrability implies boundedness in $L^1(0,T;B)$; furthermore, in the case $B := \mathbb{R}$, uniform integrability is equivalent to weak compactness in the space $L^1(0,T)$, as stated by this fundamental result:

Theorem 1.6 (Dunford-Pettis Criterion). *Let $\mathcal{V} \subset L^1(0,T)$. The following conditions are equivalent:*
1. *$\mathcal{V}$ is (sequentially) weakly relatively compact in $L^1(0,T)$;*
2. *$\mathcal{V}$ is uniformly integrable;*
3. *There exists a positive, convex and super-linearly increasing function $G : [0,+\infty) \to \mathbf{R}$ such that*

$$\lim_{s\to\infty} \frac{G(s)}{s} = +\infty, \quad \sup_{v\in\mathcal{V}} \int_0^T G(|v(t)|)\,dt < \infty. \tag{1.20}$$

(See e.g. [6, Th. 22, 25 Chap.III], [7, Cor. IV.8.11] and [8, Th. 4.21.2] for the proof).

In general, the link between L^p convergence, uniform integrability and convergence in measure is precised in [7, Th.III.3.6]:

Proposition 1.7. *On p-uniformly integrable sets the topologies of $L^p(0,T;B)$ and of $\mathcal{M}(0,T;B)$ coincide. In particular, a set $\mathcal{U} \subset L^p(0,T;B)$ is (relatively) compact in $L^p(0,T;B)$ iff it is p-uniformly integrable and (relatively) compact in $\mathcal{M}(0,T;B)$.*

Remark 1.8 (Strong concentration yields uniform integrability). If $\mathcal{U} \subset L^p(0,T;B)$ is bounded and fulfills (1.1a), $\mathcal{U}$ is also p-uniformly integrable. In fact, from (1.1a) we have

$$\lim_{h\to 0} \sup_{u\in\mathcal{U}} \int_0^{T-h} \Big| \|u(t+h)\|_B - \|u(t)\|_B \Big|^p dt = 0,$$

which implies, by the Riesz-Fréchet-Kolmogorov criterion, that $\{\|u\|_B\}_{u\in\mathcal{U}}$ is relatively compact in the strong topology of $L^p(0,T)$. Then it is easy to see that $\{\|u\|_B^p\}_{u\in\mathcal{U}}$ is relatively sequentially compact in the strong topology of $L^1(0,T)$, hence uniformly integrable by the Dunford-Pettis criterion.

A "weak concentration" condition. We present now our main result on compactness for the convergence in measure; as we briefly said before, we will show that the right assumptions are the tightness condition w.r.t. a normal coercive integrand as in Theorem 1 and a concentration condition substantially weaker than (1.1a): more precisely, we will replace it by

$$\lim_{h\downarrow 0}\int_0^{T-h} g(u(t+h),u(t))\,dt=0,\quad \text{uniformly for } u\in\mathcal{U}, \tag{1.21a}$$

where the function $g: B\times B\to[0,+\infty]$, which plays the same role of the distance induced by the norm $\|\cdot\|_B$, satisfies

$$g: B\times B\to[0,+\infty],\quad g \text{ is lower semicontinuous}, \tag{1.21b}$$

and it should be connected to $\mathscr{F}$ by some sort of *compatibility* conditions; more precisely, denoting by $D(\mathscr{F}_t)$ the proper domain of $\mathscr{F}_t$ (see (1.7b))

$$D(\mathscr{F}_t):=\{v\in B:\mathscr{F}(t,v)<+\infty\},$$

we are assuming that

$$u,v\in D(\mathscr{F}_t),\ g(u,v)=0\quad\Rightarrow\quad u=v\quad\text{for a.e. } t\in(0,T). \tag{1.21c}$$

It is clear that, if g is actually a distance, then (1.21c) is always verified, independently of $\mathscr{F}$.

Theorem 2 (Compactness in measure=tightness+weak concentration). *Let $\mathcal{U}$ be a family of measurable B-valued functions; if there exist a* normal coercive integrand (1.7a,b,c) $\mathscr{F}:(0,T)\times B\to[0,+\infty]$ *and a l.s.c. map $g: B\times B\to[0,+\infty]$ compatible with $\mathscr{F}$ in the sense of* (1.21c)*, such that*

$$\mathcal{U} \textit{ is tight w.r.t. } \mathscr{F}, \textit{ i.e. }\quad S:=\sup_{u\in\mathcal{U}}\int_0^T\mathscr{F}(t,u(t))dt<+\infty, \tag{1.22}$$

and

$$\lim_{h\downarrow 0}\sup_{u\in\mathcal{U}}\int_0^{T-h} g(u(t+h),u(t))\,dt=0, \tag{1.23}$$

then $\mathcal{U}$ is relatively compact in $\mathcal{M}(0,T;B)$.
Conversely, if $\mathcal{U}$ is relatively compact in measure, then $\mathcal{U}$ is tight w.r.t. a normal coercive integrand $\mathscr{F}$ independent of the variable t *and* (1.23) *holds for any bounded continuous (semi-)distance g on B (thus inducing a weaker topology than the strong one).*

Before showing some examples of possible applications of Theorem 2, let us briefly discuss some straightforward extensions of the above results.

Extension 1 (From Banach to Polish (metric) spaces). Convergence in measure and Theorem 2 are of *metric* nature, i.e. the linear structure of B is irrelevant: it is sufficient to substitute each occurrence of terms like $\|v-w\|_B$ by the distance $d_B(v,w)$; therefore, the statement of

Theorem 2 also holds if (B,d_B) is a *complete, separable metric space,* (1.24)

a *Polish space* according to the probabilistic terminology. We will adopt this more general metric point of view in the proofs.

Extension 2 (Unbounded intervals). Theorem 1 can be extended to unbounded intervals, e.g. for $T = +\infty$, simply by adding to (1.1a,b) and (1.7d) a uniform "vanishing integral condition" at infinity as in the scalar case (see e.g. [4, Cor. IV.27]), i.e.

$$\lim_{T\uparrow+\infty} \sup_{u\in\mathcal{U}} \int_T^{+\infty} \|u(t)\|^p \, dt = 0. \tag{1.25}$$

Extension 3 (Dependence on multiple variables). Theorems 1 and 2 can be rephrased in the case of B-valued functions defined in a *bounded open subset* Ω of some euclidean space $\mathbb{R}^d$, simply by modifying the (weak or strong) concentration conditions in an obvious way. For instance, (1.1b) reads

$$\int_{\Omega_{|h|}} \|u(x+h) - u(x)\|_B \, dx \le \omega(|h|) \quad \forall\, h \in \mathbb{R}^d,\ u \in \mathcal{U}, \tag{1.26}$$

where

$$\Omega_{|h|} := \big\{ x \in \Omega : \inf_{y\in\partial\Omega} |x-y| > |h| \big\}.$$

Examples and applications

***Example* 1.** Let B, C be separable Banach spaces, continuously embedded in another Hausdorff topological vector space V; we suppose that the norm of C is l.s.c. with respect to strong B-convergence, i.e.

$$\left.\begin{aligned} v_n \in B\cap C, \quad \lim_{n\uparrow+\infty} \|v_n - v\|_B = 0 \\ \liminf_{n\uparrow+\infty} \|v_n\|_C < +\infty \end{aligned}\right\} \Rightarrow\ v \in C,\ \|v\|_C \le \liminf_{n\uparrow+\infty} \|v_n\|_C. \tag{1.27}$$

The following result is an immediate corollary of Theorem 2, in the same spirit of Remark 1.1.

Theorem 1.9. *If $\mathcal{U} \subset \mathcal{M}(0,T; B\cap C)$ is tight in B (def. 1.2) and satisfies the strong concentration property in C, i.e.*

$$\lim_{h\downarrow 0} \sup_{u\in\mathcal{U}} \int_0^{T-h} \|u(t+h) - u(t)\|_C dt = 0, \tag{1.28}$$

then $\mathcal{U}$ is relatively compact in $\mathcal{M}(0,T;B)$; if $\mathcal{U}$ is also p-uniformly integrable (1.19)*, then it is relatively compact in $L^p(0,T;B)$.*

Example* 2 *(Transversality). Let $L(t) : D(L(t)) \subset B \to C,\ t \in (0,T)$, be a measurable family of (possibly multivalued) *closed operators* between B and another Banach space C.

Theorem 1.10. *Let $\mathcal{U} \subset \mathcal{M}(0,T;B)$ and $\mathcal{V} \subset \mathcal{M}(0,T;C)$ satisfy*

$$\forall\, u \in \mathcal{U} \quad \exists\, v \in \mathcal{V} : \quad v(t) \in L(t)u(t) \ \text{a.e. in } (0,T), \tag{1.29a}$$

$$\mathcal{U} \text{ is tight w.r.t. some coercive normal integrand } \mathscr{F}, \tag{1.29b}$$

$$\mathcal{V} \text{ is relatively compact in } \mathcal{M}(0,T;C). \tag{1.29c}$$

If L and $\mathscr{F}$ fulfil a transversality *condition, i.e. for a.e. $t \in (0,T)$*

$$u_1, u_2 \in D(\mathscr{F}_t), \quad \exists\, w \in L(t)u_1 \cap L(t)u_2 \quad \Rightarrow \quad u_1 = u_2, \tag{1.29d}$$

then $\mathcal{U}$ is relatively compact in $\mathcal{M}(0,T;B)$.

Remark 1.11. Let us recall that L is measurable if the global graph in $(0,T)\times B\times C$

$$H := \Big\{(t,u,v)\in(0,T)\times B\times C : v\in L(t)u\Big\}\quad \text{is } \mathcal{L}\otimes\mathcal{B}(B)\otimes\mathcal{B}(C)\text{ measurable.}$$

In fact we could assume the following weaker condition:

$$\begin{aligned}&\Big\{(t,u,v)\in(0,T)\times B\times K : \mathscr{F}(t,u)\le c,\quad v\in L(t)u\Big\}\\ &\text{is } \mathcal{L}\otimes\mathcal{B}(B)\otimes\mathcal{B}(C)\text{ measurable, } \forall\, c\in[0,+\infty),\ \forall\, K\subset\subset C.\end{aligned}\tag{1.30}$$

The closeness assumption on $L(t)$ could be relaxed, too: in fact it is sufficient for the "restriction" of $L(t)$ on each sublevel of $\mathscr{F}_t$ to be closed, i.e. for a.e. $t\in(0,T)$

$$(u_n,v_n)\to(u,v),\ \sup_n \mathscr{F}_t(u_n)<+\infty,\ v_n\in L(t)u_n \quad\Rightarrow\quad v\in L(t)u. \tag{1.31}$$

Of course, if $L(t)$ is a continuous family of continuous maps the above assumptions are satisfied.
If the operators $L(t)$ are single-valued, then (1.29a,c) take the more readable form

$$\text{the set } \mathcal{V} := \Big\{t\mapsto L(t)u(t)\Big\}_{u\in\mathcal{U}} \text{ is relatively compact in } \mathcal{M}(0,T;C). \tag{1.32}$$

As we shall see in section 4, this result is a simple consequence of Theorem 2 applied in the product space $B\times C$.

Remark 1.12. Theorem 1.10 and its "weak" version 1.21 considerably extend the abstract theory developed in [14] and [17] in order to deal with some non-trivial compactness problems arising in quasi-stationary phase field models. This kind of questions arose from the pioneering paper of Luckhaus [13], who proposed a direct method for solving the Stefan problem with the Gibbs-Thomson law (see also [20]); the original ideas of Luckhaus can thus be seen as an alternative way (with respect to the standard Aubin-Lions approach) to combine tightness and concentration principles to get compactness for evolution problems. Nevertheless, Theorem 2 shows the common root of these apparently different arguments. We refer to [17], [16], for other examples, applications, and discussions of this and related subjects.

***Example* 3.** Here we present an application of Theorem 2 to the framework considered by Rakotoson-Temam in [15].

Definition 1.13. *Let S be a subset of the dual B^* of the separable Banach space B; we say that S* separates *the points of B (or S is a* separating set*) if*

$$v\in B,\quad \langle w^*,v\rangle=0\quad \forall\, w^*\in S\quad\Rightarrow\quad v=0. \tag{1.33}$$

We say that S is a determining set *for (the norm of) B if*

$$\|v\|_B=\sup\Big\{\langle w^*,v\rangle : w^*\in S, \|w^*\|_{B^*}\le 1\Big\}\quad \forall\, v\in B. \tag{1.34}$$

Remark 1.14. Of course, a determining subset is always separating; B^* is determining; if $B = C^*$ is the dual of another Banach space C, then C (or, rather, the image of C in $B^* = C^{**}$) is a determining set. If $B = \mathscr{L}(E,F)$ is the Banach space of all bounded linear operators between the Banach spaces E, F then

$$S := \Big\{ \ell \in \mathscr{L}(E,F)' : \ell(A) = \langle f^*, Ae\rangle \quad \text{for some } e \in E, f^* \in F^* \Big\}$$

is a determining set. $\mathscr{D}(\Omega)$ is a determining set of $L^1(\Omega)$, Ω being an open subset of some Euclidean space. The Hahn-Banach Theorem shows that $S \subset B^*$ is a separating set for B if and only if the vector space $B_0^* := \operatorname{span}(S)$ generated by S is weakly* dense in B^*.

Theorem 1.15. *Let $\mathcal{U} \subset \mathcal{M}(0,T;B)$ be tight (def. 1.2) and let us suppose that*

$$\Big\{ \langle w^*, u\rangle : u \in \mathcal{U} \Big\} \quad \textit{is relatively compact in } \mathcal{M}(0,T), \quad \forall\, w^* \in S, \tag{1.35}$$

being $S \subset B^*$ a separating set as stated in (1.33)*. Then $\mathcal{U}$ is relatively compact in $\mathcal{M}(0,T;B)$. In particular, if $\mathcal{U}$ is uniformly p-integrable, then $\mathcal{U}$ is compact in $L^p(0,T;B)$, too.*

Weak convergence

***Example* 4.** The previous Example 3 shows an interesting link between strong and "weak" convergence in measure: here we want to investigate the latter aspect more carefully .

First of all, since the topology of weak convergence is not metrizable, we would have to extend the notion of convergence in measure a little bit; in the sequel we fix

$$\text{a closed } \textit{determining} \text{ and } \textit{strongly separable} \text{ subspace } B_0^* \subset B^*, \tag{1.36}$$

and we will use the adjective "weak" referring to the weak topology $\sigma(B, B_0^*)$ of B induced by B_0^*.

Definition 1.16. *We denote by $\mathcal{M}_w(0,T;B)$ the space of the function $u : (0,T) \to B$ which are $\sigma(B, B_0^*)$-weakly measurable, i.e.*

$$t \mapsto \langle w^*, u(t)\rangle \quad \textit{is measurable in } (0,T) \quad \forall\, w^* \in B_0^*. \tag{1.37}$$

We say that a sequence $\{u^n\}_{n\in\mathbb{N}}$ $\sigma(B, B_0^*)$-weakly converges in measure to $u \in \mathcal{M}_w(0,T;B)$ if

$$\lim_{n\uparrow+\infty} \langle w^*, u^n\rangle = \langle w^*, u\rangle \quad \textit{in } \mathcal{M}(0,T) \quad \forall\, w^* \in B_0^*.$$

Correspondingly, a set $\mathcal{U} \subset \mathcal{M}_w(0,T;B)$ is $\sigma(B, B_0^*)$-weakly (sequentially) relatively compact in measure if every sequence $\{u^n\}_{n\in\mathbb{N}} \subset \mathcal{U}$ admits a subsequence u^{n_k} weakly convergent in measure.

Remark 1.17. If B is separable and $B_0^* = B^*$, then we are speaking of the usual weak convergence; observe that by Pettis' Theorem there is no difference between weakly and strongly measurable functions.

The case $B = C^*$ and $B_0^* = C$ corresponds to weak* convergence (in this case C should be separable). Different choices are possible: e.g. if $B = C(K)$, K being a compact metric space, the choice

$$B_0^* := \Big\{\phi \in C(K) \mapsto \phi(x),\ x \in K\Big\}$$

induces the pointwise convergence.

Now we obviously extend the notion of tightness to weak topologies: for simplicity, we are limiting our analysis to integrands *independent of time.*

Definition 1.18 (Weak tightness). *We say that an integrand* $\mathscr{F} : B \to [0, +\infty]$ *is* weakly coercive *if for every* $c \geq 0$

$$\{v \in B : \mathscr{F}_t(v) \leq c\} \ \textit{ are } \sigma(B, B_0^*)\textit{-compact for a.e. } t \in (0, T). \tag{1.38}$$

$\mathcal{U}$ *is* weakly tight *(w.r.t. the weakly coercive integrand* $\mathscr{F}$*) if*

$$\sup_{u \in \mathcal{U}} \int_0^T \mathscr{F}(u(t))\, dt < +\infty. \tag{1.39}$$

Remark 1.19. If B is a separable reflexive space and $B_0^* = B^*$, then a functional $\mathscr{F}$ is weakly coercive iff

$$\lim_{\|v\| \uparrow +\infty} \mathscr{F}(v) = +\infty \quad \text{for a.e. } t \in (0, T). \tag{1.40}$$

Correspondingly, $\mathcal{U}$ is weakly tight iff

$$\forall\, \varepsilon > 0 \quad \exists M_\varepsilon > 0 : \quad \big|\{t \in (0, T) : \|u(t)\|_B > M_\varepsilon\}\big| \leq \varepsilon \quad \forall\, u \in \mathcal{U}. \tag{1.41}$$

Theorem 1.20. *Suppose that a subset* $\mathcal{U} \subset \mathcal{M}_w(0, T; B)$ *is* $\sigma(B, B_0^*)$*-weakly tight and*

$$\Big\{\langle w^*, u\rangle : u \in \mathcal{U}\Big\} \quad \textit{is relatively compact in } \mathcal{M}(0, T), \quad \forall\, w^* \in S, \tag{1.42}$$

$S \subset B_0^*$ *being a separating set for* B*; then* $\mathcal{U}$ *is (sequentially)* $\sigma(B, B_0^*)$*-weakly compact in measure.*

Example 5 (Weak transversality and strong compactness). Let us consider the framework of example 2, but now we suppose that $L(t) : D(L(t)) \subset B \to C,\ t \in (0, T)$, is a family of (possibly multivalued) *strongly- weakly closed operators*, for a fixed determining closed and separable subspace C_0^* of C^*: more precisely, for a suitable normal coercive integrand $\mathscr{F} : (0, T) \times B \to [0, +\infty]$ and a $\sigma(B, B_0^*)$-weakly coercive integrand $\mathscr{G} : C \to [0, +\infty]$, for a.e. $t \in (0, T)$

$$\begin{matrix} u_n \to u, & \sup_n \mathscr{F}_t(u_n) < +\infty, \\ v_n \rightharpoonup v, & \sup_n \mathscr{G}(v_n) < +\infty \end{matrix} \quad v_n \in L(t)u_n \quad \Rightarrow \quad v \in L(t)u. \tag{1.43a}$$

L is supposed to be measurable, in the sense that

$$\begin{gathered} \Big\{(t, u, v) \in (0, T) \times B \times C : \mathscr{F}(t, u) \leq c, \mathscr{G}(v) \leq c, \quad v \in L(t)u\Big\} \\ \text{is } \mathcal{L} \otimes \mathcal{B}(B) \otimes \mathcal{B}(C) \text{ measurable, } \forall\, c \in [0, +\infty). \end{gathered} \tag{1.43b}$$

The interest here is that a concentration condition of weak type yields strong compactness.

Theorem 1.21. *Let $\mathcal{U} \subset \mathcal{M}(0,T;B)$ and $\mathcal{V} \subset \mathcal{M}_w(0,T;C)$ satisfy*

$$\forall u \in \mathcal{U} \quad \exists v \in \mathcal{V} : \quad v(t) \in L(t)u(t) \text{ a.e. in } (0,T), \tag{1.43c}$$

$$\mathcal{U} \subset \mathcal{M}(0,T;B) \text{ is tight w.r.t. } \mathscr{F}, \tag{1.43d}$$

$$\mathcal{V} \text{ is weakly tight w.r.t. } \mathscr{G} \text{ and satisfies a condition like (1.42).} \tag{1.43e}$$

If L and $\mathscr{F}$ fulfil the transversality *condition* (1.29d) *then $\mathcal{U}$ is* strongly *relatively compact in $\mathcal{M}(0,T;B)$.*

Here is a typical case of application of the previous Theorem when C is a separable reflexive Banach space:

Corollary 1.22. *Let $\mathscr{F} : B \to [0,+\infty]$ a coercive integrand with* superlinear growth *and $L : D(\mathscr{F}) \to C$ be an* injective *operator* weakly continuous *on each sublevel of $\mathscr{F}$, C being reflexive. Assume that a given sequence $\{u^n\}_{n\in\mathbb{N}}$ of B-valued measurable functions satisfies an* a priori estimate *(independent of n) of the type*

$$\int_0^T \Big(\mathscr{F}(u^n(t)) + \|Lu^n(t)\|_C\Big)\,dt \le M < +\infty \qquad \int_0^T |\langle \tfrac{d}{dt} Lu^n(t), w^*\rangle|\,dt \le C(w^*)$$

where w^* is an arbitrary element of a separating set $S \subset C^*$. Then there exists a subsequence u^{n_k} strongly convergent in $L^1(0,T;B)$.

Plan of the paper.

As we mentioned before, the proof of Theorem 2 relies on some basic results of infinite-dimensional Young measures theory. In order to make this paper more readable and almost self-contained (at least for the statements of the main properties we will use), we will present in the next section a brief summary of this theory and of the related measure-theoretic results; of course, the expert reader may skip this part without difficulty.

The proofs of Theorems 1 and 2 are developed in Section 3; the last section 4 contains the proofs of the applications of Theorem 2 and it relies on the statement of Theorem 2 only, being therefore independent of sections 2, 3.

2 Preliminary results.

Notation. In the sequel, (B, d_B) is a *complete, separable metric space* (Polish space): in particular, a separable Banach space with the distance induced by its norm. $\mathcal{L}$ and $\mathcal{B} = \mathcal{B}(B)$ denote the σ-algebras of the Lebesgue measurable subsets of $(0,T)$ and of the Borel subsets of B, respectively, $|J|$ is the Lebesgue measure of a set J in $\mathcal{L}$. The set of all Borel probability measures on B is denoted by $\mathcal{P}(B)$.

$\mathcal{L} \otimes \mathcal{B}$ is the usual product σ-algebra in $(0,T) \times B$. Recall that if a real function φ defined on $(0,T) \times B$ is $\mathcal{L} \otimes \mathcal{B}$-measurable, then each of its partial mappings $t \mapsto \varphi(t,v)$, $v \mapsto \varphi(t,v)$ is measurable on the corresponding factor space. Conversely, if $\varphi : (0,T) \times B \to \mathbb{R}$ is a Carathéodory map [3], i.e. it satisfies

$$\begin{aligned} &t \mapsto \varphi(t,v) \text{ is } \mathcal{L}\text{-measurable } \forall\, v \in B \\ &v \mapsto \varphi(t,v) \text{ is continuous for a.e. } t \in (0,T) \end{aligned} \tag{2.1}$$

then φ is $\mathcal{L}\otimes\mathcal{B}$-measurable. As we already said in the previous section, a *positive normal integrand* is a

$$\begin{gathered}\mathcal{L}\otimes\mathcal{B}\text{-measurable map } \varphi:(0,T)\times B\to[0,+\infty] \text{ such that}\\ \varphi(t,\cdot) \text{ is l.s.c. on } B \text{ for a.e. } t\in(0,T).\end{gathered} \tag{2.2}$$

We will denote by $C^b(B)$ the Banach space of continuous and bounded real functions defined on B.

Parametrized measures.

Definition 2.1 (Parametrized measures). *A* parametrized measure *is a family* $\boldsymbol{\nu} := \{\nu_t\}_{t\in(0,T)}$ *of probability measures in* $\mathcal{P}(B)$*, such that one of the following two (equivalent) conditions holds*

$$t\in(0,T)\mapsto\nu_t(D) \quad \textit{is} \quad \mathcal{L}\textit{-measurable} \quad \forall D\in\mathcal{B}; \tag{2.3a}$$

$$t\in(0,T)\mapsto\int_B\phi(\xi)\,d\nu_t(\xi) \quad \textit{is} \quad \mathcal{L}\textit{-measurable} \quad \forall\phi\in C^b(B). \tag{2.3b}$$

We denote by $\mathcal{Y}(0,T;B)$ *the set of all parametrized measures.*

The following is a (enhanced) version of Fubini's Theorem, adapted to families of parametrized measures [6, p. 20-II]

Theorem 2.2. *Let* $\boldsymbol{\nu}=\{\nu_t\}_{t\in(0,T)}$ *be a parametrized measure in* B*; there exists one and only one measure (which we still denote by* ν*) on* $\mathcal{L}\otimes\mathcal{B}$ *such that*

$$\nu(I\times A)=\int_I\nu_t(A)\,dt \quad \forall\, I\in\mathcal{L},\ A\in\mathcal{B}, \tag{2.4}$$

in particular

$$\nu(I\times B)=|I| \quad \forall\, I\in\mathcal{L}. \tag{2.5}$$

Moreover, for every $\mathcal{L}\otimes\mathcal{B}$*-measurable function* $\varphi:(0,T)\times B\to[0,+\infty]$ *the function*

$$t\mapsto\int_B\varphi(t,\xi)d\nu_t(\xi) \quad \textit{is } \mathcal{L}\textit{-measurable}, \tag{2.6}$$

and the following extension of Fubini's formula holds:

$$\int_{(0,T)\times B}\varphi(t,\xi)\,d\nu(t,\xi)=\int_0^T\Big(\int_B\varphi(t,\xi)d\nu_t(\xi)\Big)\,dt. \tag{2.7}$$

Remark 2.3. To each measurable function $u\in\mathcal{M}(0,T;B)$ is uniquely associated the parametrized measure $\{\delta_{u(t)}\}_{t\in(0,T)}$, where for every $w\in B$ we denote by δ_w the usual Dirac's measure concentrated on $\{w\}$

$$\delta_w(A):=\begin{cases}1 & \text{if } w\in A,\\ 0 & \text{if } w\notin A\end{cases} \quad \forall\, A\subset B. \tag{2.8}$$

Conversely, a parametrized measure $\boldsymbol{\nu}=\{\nu_t\}_{t\in(0,T)}$ is associated to a measurable function u if

$$\text{the support } \operatorname{supp}(\nu_t) \text{ is a singleton } \{u(t)\} \text{ for a.e. } t\in(0,T). \tag{2.9}$$

Narrow convergence of parametrized measures. The identification of a parametrized measure $\{\nu_t\}_{t\in(0,T)}$ with the measure $\boldsymbol{\nu}$ on $E := (0,T)\times B$ given by (2.4) allows to introduce a topology on $\mathcal{Y}(0,T;B)$ simply by considering the topology of the narrow[2] convergence of measures in $E = (0,T)\times B$; we reproduce both definitions below.

Definition 2.4. *Let E be a Polish (i.e. separable, complete) metric space and let $\{\nu^n\}_{n\in\mathbb{N}}, \nu$ be finite Borel measures on E. We say that ν^n* narrowly converges *to ν if*

$$\lim_{n\uparrow+\infty}\int_E \phi(\xi)\,d\nu^n(\xi) = \int_E \phi(\xi)\,d\nu(\xi) \quad \forall \phi\in C^b(E). \tag{2.10}$$

In particular, when $E = (0,T)\times B$, we say that a sequence of parametrized measures $\boldsymbol{\nu}^n = \{\nu^n_t\}_{t\in(0,T)} \in \mathcal{Y}(0,T;B)$ narrowly converges *to $\boldsymbol{\nu} = \{\nu_t\}_{t\in(0,T)}$ if $\forall\varphi\in C^b((0,T)\times B)$ (see (2.7))*

$$\lim_{n\uparrow+\infty}\int_{(0,T)\times B}\varphi(t,\xi)d\nu^n(t,\xi) = \int_{(0,T)\times B}\varphi(t,\xi)d\nu(t,\xi). \tag{2.11}$$

Remark 2.5. In the literature of control theory and Young measures, the space of bounded continuous real functions on $(0,T)\times B$ is often replaced by the space (of Carathéodory integrands) $L^1(0,T;C^b(B))$. Since these two topologies coincide on parametrized measures, (as it is possible to verify using, e.g., the approximation results contained in [3]), we prefer the previous simpler definition, based on continuous test functions.

Remark 2.6. Of course, when $\boldsymbol{\nu}^n = \{\delta_{u^n(t)}\}_{t\in(0,T)}$ are the Young measures associated to the sequence $u^n\in\mathcal{M}(0,T;B)$ as in Remark 2.3, then we write $u^n\to\boldsymbol{\nu}$ in $\mathcal{Y}(0,T;B)$ and (2.11) simply means that for every function $\varphi\in C^b((0,T)\times B)$

$$\lim_{n\uparrow+\infty}\int_0^T\varphi(t,u_n(t))\,dt = \int_0^T\Big(\int_B\varphi(t,\xi)\,d\nu_t(\xi)\Big)\,dt. \tag{2.12}$$

Observe that (2.12) implies that for every $\phi\in C^b(B)$

$$\phi(u^n)\rightharpoonup^*\int_B\phi(\xi)\,d\nu_t(\xi) \quad \text{in } L^\infty(0,T). \tag{2.13}$$

It is possible to show that this property is in fact equivalent to definition (2.4).

Lower semicontinuous and normal integrands. The following property of narrow convergence plays a basic role for the application of Young measures theory (see [19, Thm. 7]).

Proposition 2.7. *Let E be a Polish space and let ν^n,ν be finite measures such that $\nu^n\to\nu$ narrowly. If $\phi : E\to(-\infty,+\infty]$ is l.s.c. bounded from below, then*

$$\liminf_{n\uparrow+\infty}\int_E\phi(\xi)\,d\nu^n(\xi) \ge \int_E\phi(\xi)\,d\nu(\xi). \tag{2.14}$$

[2] Sometimes, the ambiguous term of "weak convergence" is also used.

Analogously, if $E = (0,T) \times B$ *with* $\boldsymbol{\nu}^n, \boldsymbol{\nu} \in \mathcal{Y}(0,T;B)$ *and* $\boldsymbol{\nu}^n \to \boldsymbol{\nu}$ *narrowly, then for every nonnegative* normal integrand $\varphi : (0,T) \times B \to [0,+\infty]$ *(see (2.2))*

$$\liminf_{n\uparrow+\infty} \int_{(0,T)\times B} \varphi(t,\xi) d\nu^n(t,\xi) \geq \int_{(0,T)\times B} \varphi(t,\xi) d\nu(t,\xi). \tag{2.15}$$

Compactness. Let us first recall the fundamental Prohorov's compactness result for a tight family of measures [6, 72-II]

Theorem 2.8 (Prohorov). *Let* E *be a Polish space and let* $\mathcal{V}$ *be a family of finite Borel measures on* E *such that* $\nu(E)$ *is independent of* $\nu \in \mathcal{V}$*.* $\mathcal{V}$ *is relatively (sequentially) compact w.r.t. the narrow convergence iff* $\mathcal{V}$ *is tight, i.e.*

$$\forall\, \varepsilon > 0 \quad \exists\, K_\varepsilon \subset\subset E : \quad \nu(E \setminus K_\varepsilon) \leq \varepsilon \quad \forall\, \nu \in \mathcal{V}. \tag{2.16}$$

Now we can state the related compactness result [2, Thm.1], which allows to associate (at least one) parametrized measure to each tight sequence $u^n \in \mathcal{M}(0,T;B)$.

Theorem 2.9 (Balder). *Let* $u^n \in \mathcal{M}(0,T;B)$ *be tight w.r.t. a normal coercive integrand* (1.7a,b,c,d)*. Then there exists a subsequence* u^{n_k} *and a parametrized measure* $\boldsymbol{\nu} = \{\nu_t\}_{t\in(0,T)} \in \mathcal{Y}(0,T;B)$*, which we call a* Young measure associated to u^n*, such that* $u^{n_k} \to \boldsymbol{\nu}$ *as* $k \uparrow \infty$ *in the sense of* (2.12)*; in particular, for every nonnegative normal integrand* $\varphi : (0,T) \times B \to [0,+\infty]$

$$\liminf_{k\to\infty} \int_0^T \varphi(t, u^{n_k}(t))\, dt \geq \int_0^T \left(\int_B \varphi(t,\xi) d\nu_t(\xi) \right) dt. \tag{2.17}$$

The link with the convergence in measure.

Lemma 2.10. *Let* $u^n, u \in \mathcal{M}(0,T;B)$*,* $n \in \mathbb{N}$*, and let* $\boldsymbol{\nu}^n, \boldsymbol{\nu}$ *be the associated parametrized measures. Then*

$$u^n \to u \text{ in measure} \quad \Leftrightarrow \quad \boldsymbol{\nu}^n \to \boldsymbol{\nu} \quad \text{narrowly}. \tag{2.18}$$

In particular, if $\boldsymbol{\nu} = \{\nu_t\}_{t\in(0,T)}$ *is the* narrow limit *of a sequence* $\{u^n\}_{n\in\mathbb{N}} \subset \mathcal{M}(0,T;B)$*, then* u^n *is convergent in measure iff* ν_t *is concentrated on a singleton for a.e.* $t \in (0,T)$*.*

Remark 2.11. The statement above implies that if $\mathcal{U} \subset \mathcal{M}(0,T;B)$ is relatively compact in measure, the set of the associated Young measures $\mathcal{V}$ is narrowly (sequentially) compact in $E = (0,T) \times B$, which entails, (here we refer to Theorem 2.8) that $\mathcal{V}$ is tight in the classical probabilistic sense, i.e. it satisfies (2.16), which is equivalent to (1.9) in terms of $\mathcal{U}$.

Tensor products of parametrized measures. The following fiber-product Lemma will be useful (see [19, Th. 13]).

Lemma 2.12. *Let* $\boldsymbol{\nu}^1 = \{\nu^1_s\}_{s\in(0,T)}$ *and* $\boldsymbol{\nu}^2 = \{\nu^2_t\}_{t\in(0,T)} \in \mathcal{Y}(0,T;B)$ *be parametrized measures on* B*; then the formula*

$$\boldsymbol{\nu} = \boldsymbol{\nu}^1 \otimes \boldsymbol{\nu}^2, \quad \nu_{(s,t)} := \nu^1_s \otimes \nu^2_t \quad \forall (s,t) \in Q := (0,T) \times (0,T), \tag{2.19}$$

defines a parametrized measure in $\mathcal{Y}(Q; B \times B)$. *Moreover, if the sequences* $\{\boldsymbol{\nu}^{i,n}\}_{n\in\mathbb{N}} \subset \mathcal{Y}(0,T;B)$ *narrowly converge to* $\boldsymbol{\nu}^i$ *as* $n \uparrow +\infty$ *for* $i = 1, 2$, *then*

$$\boldsymbol{\nu}^n = \boldsymbol{\nu}^{1,n} \otimes \boldsymbol{\nu}^{2,n} \to \boldsymbol{\nu} = \boldsymbol{\nu}^1 \otimes \boldsymbol{\nu}^2 \quad \textit{narrowly as } n \uparrow +\infty. \tag{2.20}$$

3 Proofs of the main theorems

Proof of Theorem 2: sufficiency.

Since $\mathcal{M}(0,T;B)$ is metrizable, we can equivalently consider sequential compactness; the tightness hypothesis and Theorem 2.9 allow us to extract from every sequence $\{u^n\}_{n\in\mathbb{N}} \subset \mathcal{U}$ a subsequence u^{n_k} with generalized limit $\boldsymbol{\nu} = \{\nu_t\}_{t\in(0,T)}$. Invoking Lemma 2.10, in order to prove that u^{n_k} converges in measure, we simply have to show that $\{\nu_t\}_{t\in(0,T)}$ *is concentrated in a point mass* for a.e. $t \in (0,T)$.

Here are the main points to show this concentration property:

1. First of all, we show that the support of ν_t lies in the effective domain $\mathcal{D}(\mathscr{F}_t)$ of $\mathscr{F}$,
$$D(\mathscr{F}_t) := \{v \in B : \mathscr{F}(t,v) < +\infty\}, \tag{3.1}$$
i.e.
$$\nu_t\big(B \setminus D(\mathscr{F}_t)\big) = 0 \quad \text{for a.e. } t \in (0,T). \tag{3.2}$$

2. Starting from the weak concentration property (1.21a) and *doubling* variables we will end up with an integrated limiting form of (1.21a)
$$\lim_{h\downarrow 0} \frac{1}{h} \int_0^h \int_0^{T-\sigma} \Big(\iint_{B\times B} g(v,w)\, d\nu_{t+\sigma}(v) \otimes d\nu_t(w) \Big)\, dt\, d\sigma = 0. \tag{3.3}$$

3. Then we will pass to the limit in (3.3) obtaining
$$\int_0^T \Big(\iint_{B\times B} g(v,w)\, d\nu_t(v)\, d\nu_t(w) \Big) = 0. \tag{3.4}$$

4. Finally, we combine (3.2), (3.4) and the compatibility condition (1.21c) to see that
$$\operatorname{supp} \nu_t \otimes \nu_t \quad \text{is a singleton} \quad \text{for a.e. } t \in (0,T). \tag{3.5}$$
This property entails the analogous one for ν_t, concluding the proof.

Claim 1. We first apply (2.17) to the functional $\mathscr{F}$ yielding the tightness, so that:

$$\int_0^T \left(\int_B \mathscr{F}(t,v) d\nu_t(v) \right) dt \le \liminf_{k\to\infty} \int_0^T \mathscr{F}(t, u^{n_k}(t)) dt \le S < +\infty.$$

Then

$$\int_B \mathscr{F}(t,v) d\nu_t(v) < +\infty \quad \text{for a.e. } t \in (0,T),$$

which implies that

$$\mathscr{F}(t,v) < +\infty \quad \text{for } \nu_t\text{-a.e. } v \in B, \quad \text{for a.e. } t \in (0,T),$$

i.e. (3.2).

Claim 2. In order to show (3.3), let us set

$$\tau(h) := \sup_{0\le\sigma\le h}\sup_n \int_0^{T-\sigma} g(u^n(t+\sigma),u^n(t))dt, \quad \text{with} \quad \lim_{h\downarrow 0}\tau(h)=0,$$

thanks to (1.23). Of course,

$$\frac{1}{h}\int_0^h \left(\int_0^{T-\sigma} g(u^n(t+\sigma),u^n(t))dt\right) d\sigma \le \tau(h) \quad \forall\, n\in\mathbb{N}; \tag{3.6}$$

on the other hand, Fubini's Theorem yields

$$\frac{1}{h}\int_0^h \left(\int_0^{T-\sigma} g(u^n(t+\sigma),u^n(t))dt\right) d\sigma = \frac{1}{h}\iint_{Q(h)} g(u^n(s),u^n(t))dsdt$$

where $Q(h)$ is the strip

$$Q(h) := \big\{(s,t)\in(0,T)\times(0,T) : t\le s\le t+h\big\}. \tag{3.7}$$

Since by Lemma 2.12 the couple $(u^{n_k}(s),u^{n_k}(t))$ narrowly converges to $\{\nu_s\otimes\nu_t\}_{s,t\in Q_0}$, we can apply (2.15) to the normal integrand

$$G(s,t,v,w) := \chi_{Q(h)}(s,t)g(v,w)$$

obtaining

$$\begin{aligned}&\frac{1}{h}\iint_{Q(h)}\left(\iint_{B\times B} g(v,w)d(\nu_s\otimes\nu_t)(v,w)\right) ds\,dt\\ &\quad\le \liminf_{k\uparrow+\infty}\frac{1}{h}\iint_{Q(h)} g(u^{n_k}(s),u^{n_k}(t))\,ds\,dt \le \tau(h).\end{aligned}$$

A reverse application of Fubini's theorem yields (3.3).

Claim 3. (3.4) follows immediately, if we show that

$$\begin{aligned}\liminf_{h\downarrow 0}\frac{1}{h}\int_0^h\int_0^{T-\sigma}\Big(\iint_{B\times B} g(v,w)\,d\nu_{t+\sigma}(v)\otimes\,d\nu_t(w)\Big)\,dt\,d\sigma&\\ \ge\int_0^T\Big(\iint_{B\times B} g(v,w)\,d\nu_t(v)\,d\nu_t(w)\Big).&\end{aligned} \tag{3.8}$$

We fix $\varepsilon>0$ and observe that Fatou's Lemma yields

$$\begin{aligned}&\liminf_{h\downarrow 0}\frac{1}{h}\int_0^h\int_0^{T-\sigma}\Big(\iint_{B\times B} g(v,w)\,d\nu_{t+\sigma}(v)\otimes\,d\nu_t(w)\Big)\,dt\,d\sigma\\ &\ge\liminf_{h\downarrow 0}\frac{1}{h}\int_0^h\int_0^{T-\varepsilon}\Big(\iint_{B\times B} g(v,w)\,d\nu_{t+\sigma}(v)\otimes\,d\nu_t(w)\Big)\,dt\,d\sigma\\ &\ge\liminf_{h\downarrow 0}\int_0^1\int_0^{T-\varepsilon}\Big(\iint_{B\times B} g(v,w)\,d\nu_{t+h\sigma}(v)\otimes\,d\nu_t(w)\Big)\,dt\,d\sigma\\ &\quad\ge\int_0^1\Big(\liminf_{h\downarrow 0}\int_0^{T-\varepsilon}\Big(\iint_{B\times B} g(v,w)\,d\nu_{t+h\sigma}(v)\otimes\,d\nu_t(w)\Big)\,dt\,\Big)d\sigma\end{aligned}$$

We observe that, again by Fubini's Theorem,

$$
\int_0^{T-\varepsilon} \Big(\iint_{B\times B} g(v,w)\, d\nu_{t+h\sigma}(v) \otimes\, d\nu_t(w) \Big)\, dt =
$$
$$
\int_0^{T-\varepsilon} \Big(\int_B \Big(\int_B g(v,w)\, d\nu_t(w) \Big) d\nu_{t+h\sigma}(v) \Big)\, dt =
$$
$$
\int_0^{T-\varepsilon} \Big(\int_B \varphi(t,v) d\nu_{t+h\sigma}(v) \Big)\, dt
$$

where

$$
\varphi(t,v) := \int_B g(v,w) d\nu_t(w), \tag{3.9}
$$

is a nonnegative normal integrand, in view of Lemma 3.2 below. Thus, by Lemma 3.1 and Fatou's Lemma we get

$$
\liminf_{h\downarrow 0} \int_0^{T-\varepsilon} \Big(\int_B \varphi(t,v)\, d\nu_{t+h\sigma}(v) \Big)\, dt \geq \int_0^{T-\varepsilon} \liminf_{h\downarrow 0} \Big(\int_B \varphi(t,v)\, d\nu_{t+h\sigma}(v) \Big)\, dt
$$
$$
\geq \int_0^{T-\varepsilon} \Big(\int_B \varphi(t,v)\, d\nu_t(v) \Big)\, dt = \int_0^{T-\varepsilon} \Big(\iint_{B\times B} g(v,w)\, d\nu_t(v) \otimes\, d\nu_t(w) \Big)\, dt
$$

Since $\varepsilon > 0$ is arbitrary, the Monotone Convergence Theorem yields (3.8).

Claim 4. Finally, we are showing that for a.e. $t \in (0,T)$ ν_t is a Dirac mass. First of all, (3.4) reads that

$$
\begin{aligned}
&\int_{B\times B} g(\xi,\eta) d(\nu_t \otimes \nu_t)(\xi,\eta) = 0 \quad \text{for a.e. } t \in (0,T), \text{ i.e.} \\
&(\nu_t \otimes \nu_t)\Big\{ (\xi,\eta) \in B \times B : g(\xi,\eta) > 0 \Big\} = 0 \quad \text{for a.e. } t \in (0,T).
\end{aligned} \tag{3.10}
$$

The point here is that, since ν_t is concentrated on $\mathcal{D}(\mathscr{F}_t)$ and g is "non-degenerate" w.r.t. $\mathcal{D}(\mathscr{F}_t)$ for a.e. $t \in (0,T)$ (see 1.21c), $\nu_t \otimes \nu_t$ is concentrated on the diagonal set Δ of $B \times B$, which allows us to conclude that the support of ν_t is a singleton. As a matter of fact, let us suppose, by contradiction, that $\operatorname{supp}(\nu_t)$, which is obviously non-empty, contains two distinct elements $x_1 \neq x_2$; then there exist two *disjoint* open neighborhoods $N_1 \ni x_1$, $N_2 \ni x_2$ such that that $\nu_t(N_i) > 0$ for $i = 1,2$. Hence

$$
(\nu_t \otimes \nu_t)(N_1 \times N_2) = \nu_t(N_1)\, \nu_t(N_2) > 0
$$

On the other hand,

$$
(\nu_t \otimes \nu_t)(N_1 \times N_2) = (\nu_t \otimes \nu_t)(N_1 \times N_2 \cap \Delta) = (\nu_t \otimes \nu_t)(\emptyset) = 0.
$$

which is absurd.

Lemma 3.1. *Let $\boldsymbol{\nu} := \{\nu_t\}_{t\in[0,T]}$ be a parametrized measure on $(0,T) \times B$, and, for $h > 0$, let $\boldsymbol{\nu}^h$ be the measure with disintegration $\{\nu_{t+h}\}_{t\in(0,T-h)}$, i.e.*

$$
\nu^h(J \times A) = \int_J \nu_{t+h}(A)\, dt \quad \forall J \subset (0,T-h),\ J \in \mathcal{L}, A \in \mathcal{B}.
$$

Then $\boldsymbol{\nu}^h \to \boldsymbol{\nu}$ narrowly as $h \downarrow 0$ on every interval $(0, T')$, $T' < T$. In particular, for every nonnegative normal integrand φ

$$\liminf_{h\downarrow 0} \int_0^{T'} \int_B \varphi(t,\xi) d\nu_{t+h}(\xi)\, dt \geq \int_0^{T'} \int_B \varphi(t,\xi) d\nu_t(\xi)\, dt. \tag{3.11}$$

Proof. For a fixed $F \in C^b((0,T') \times B)$ we have

$$\lim_{h\downarrow 0} \int_{(0,T')\times B} F(t,\xi) d\nu^h(t,\xi) = \lim_{h\downarrow 0} \int_0^{T'} \left(\int_B F(t,\xi) d\nu_{t+h}(\xi) \right) dt$$
$$= \lim_{h\downarrow 0} \int_h^{T'+h} \left(\int_B F(t-h,\xi) d\nu_t(\xi) \right) dt = \int_0^{T'} \left(\int_B F(t,\xi) d\nu_t(\xi) \right) dt,$$

where the last limit easily follows from the continuity of F and the Lebesgue Dominated Convergence Theorem. □

Lemma 3.2. *Under the assumptions of Theorem 2, the function φ defined by (3.9) is a normal integrand.*

Proof. Let us first consider the case of a *bounded continuous* function g; then φ is a Carathéodory integrand (see (2.1)), since it is measurable w.r.t. t (by definition of parametrized measure, (2.3b)) and continuous w.r.t. v (as it is immediately checked by the Lebesgue Dominated Convergence Theorem); in particular φ is $\mathcal{L} \times \mathcal{B}$-measurable.

The general case of a l.s.c. function g can be easily recovered, since (by a well known approximation result) we can find a non decreasing sequence of bounded continuous functions $g_n : B \times B \to [0,\infty)$ such that

$$g(\xi,\eta) = \lim_{n\uparrow+\infty} g_n(\xi,\eta) = \sup_{n\in\mathbb{N}} g_n(\xi,\eta) \quad \forall (\xi,\eta) \in B \times B. \tag{3.12}$$

By the monotone convergence theorem, $\forall (t,\xi) \in (0,T) \times B$

$$\varphi(t,\xi) = \int_B g(\xi,\eta) d\nu_t(\eta) = \lim_{n\uparrow+\infty} \int_B g_n(\xi,\eta) d\nu_t(\eta) = \sup_{n\in\mathbb{N}} \varphi_n(t,\xi)$$

which gives the desired conclusion, since φ_n is $\mathcal{L}\times\mathcal{B}$-measurable and continuous w.r.t. ξ for a.e. $t \in (0,T)$. □

Remark 3.3 (*Proof of Theorem 1: sufficiency*). One of the implications of Theorem 1 follows directly from Theorem 2: if $\mathcal{U}$ satisfies (1.1a) then, taking account of Remark 1.8, $\mathcal{U}$ is uniformly p-integrable and, by Proposition 1.7, it is sufficient to prove its relative compactness in $\mathcal{M}(0,T;B)$, which is supplied by Theorem 2.

Proof Theorem 2: necessity

Let $\mathcal{U} \subset \mathcal{M}(0,T;B)$ be a relatively compact subset. By Remark 2.11 we know that there exists an increasing sequence of compact sets $\{K_n\}_{n=0}^{+\infty}$ of B such that

$$\big|\{t \in (0,T) : u(t) \notin K_n\}\big| \leq 2^{-n} \quad \forall\, u \in \mathcal{U}. \tag{3.13}$$

We define

$$\mathscr{F}(v) := \min\big\{n \in \mathbb{N} : v \in K_n\big\}, \quad \text{with} \quad \mathscr{F}(v) = +\infty \text{ if } v \notin \bigcup_{n\in\mathbb{N}} K_n. \tag{3.14}$$

$\mathscr{F}$ is l.s.c. and coercive, since its level subsets are exactly the compact sets K_n; moreover

$$\mathscr{F}(v) = \sum_{n=0}^{+\infty} \chi_{K'_n}(v) = \sum_{n=1}^{+\infty} n\chi_{K_n\setminus K_{n-1}}(v), \quad \text{where} \quad K'_n := B \setminus K_{n-1}.$$

Finally, if $u \in \mathcal{U}$ then

$$\int_0^T \mathscr{F}(u(t))\,dt = \sum_{n=0}^{+\infty} \big|K'_n\big| \le \sum_{n=0}^{+\infty} 2^{-n} < +\infty,$$

i.e. (1.22) is satisfied.

Let us now fix a continuous bounded distance $g : B \times B \to [0, +\infty)$: we want to show that (1.23) holds. As in (1.18) we set

$$\delta_g(v, w) := \int_0^T g(v(t), w(t))\,dt \quad \forall\, v, w \in \mathcal{M}(0, T; B) \tag{3.15}$$

and it is easy to see that δ_g is a continuous distance in $\mathcal{M}(0, T; B)$; in particular, $\mathcal{U}$ is relatively compact with respect to this metric. Let us fix $v_0 \in B$; for a given $v \in \mathcal{M}(0, T; B)$ and $h \in (0, T)$ let us set

$$\begin{cases} v^h(t) := v(t + h) & \text{if } t \in (0, T - h); \\ v^h(t) := v_0 & \text{if } t \in (T - h, T) \end{cases} \tag{3.16}$$

and

$$\omega(v; h) := \delta_g(v^h, v) = \int_0^{T-h} g(v(t + h), v(t))\,dt + \int_{T-h}^T g(v_0, v(t))\,dt. \tag{3.17}$$

An easy variant of the well known property for integrable functions (obtained via approximation by continuous functions) shows that

$$h \mapsto \omega(v; h) \quad \text{is continuous in } [0, T) \quad \forall\, v \in \mathcal{M}(0, T; B). \tag{3.18}$$

Moreover,

$$|\omega(v_1; h) - \omega(v_2; h)| \le 2\delta_g(v_1, v_2) \quad \forall\, v_1, v_2 \in \mathcal{M}(0, T; B), \tag{3.19}$$

i.e. the functions $v \mapsto \omega(v; h)$ are uniformly equicontinuous in $\mathcal{M}(0, T; B)$. Since $\mathcal{U}$ is relatively compact and they pointwise converge to 0 as $h \downarrow 0$, Ascoli-Arzelà's Theorem yields uniform convergence and we can conclude that

$$\lim_{h\downarrow 0} \sup_{u\in\mathcal{U}} \omega(u; h) = 0.$$

Remark 3.4. By checking the previous construction (3.13)-(3.14), when B is a Banach space we can always find a coercive integrand $\mathscr{F}$ satisfying the following additional properties:

$$\mathscr{F}(v) \geq \mathscr{F}(0) = 0, \quad \mathscr{F}(-v) = \mathscr{F}(v) \quad \forall\, v \in B, \tag{3.20}$$

$$\{v \in B : \mathscr{F}(v) \leq c\} \quad \text{is convex } \forall\, c \geq 0. \tag{3.21}$$

In fact, if K_n is a family of compact sets as in (3.13), we can consider the new family

$$\hat{K}_n := \text{closed convex hull of } \Big(K_n \cup (-K_n)\Big); \tag{3.22}$$

$\hat{K}_n$ is still compact and satisfy (3.13): thus we can define $\mathscr{F}$ as in (3.14) starting from $\hat{K}_n$.

Proof of Theorem 1: necessity.

We are now supposing that B is a (separable) Banach space. First of all we recall a useful application of the Lebesgue Dominated Convergence theorem [9, p. 22]

Lemma 3.5. *Suppose that, for $n, k \in \mathbb{N}$, we are given $Z_{n,k}, Z_k \in B, z_{n,k}, z_k \in [0, +\infty)$ with*

$$\lim_{n\uparrow+\infty} Z_{n,k} = Z_k \quad \textit{strongly in } B, \qquad \lim_{n\uparrow+\infty} z_{n,k} = z_k, \qquad \|Z_{n,k}\| \leq z_{n,k}.$$

If

$$\lim_{n\uparrow+\infty} \sum_{k=1}^{+\infty} z_{n,k} = \sum_{k=1}^{+\infty} z_k < +\infty \quad \textit{then} \quad \lim_{n\uparrow+\infty} \sum_{k=1}^{+\infty} Z_{n,k} = \sum_{k=1}^{+\infty} Z_k. \tag{3.23}$$

We need the following auxiliary result, which is well known when $\boldsymbol{\zeta}$ maps B in a finite dimensional space [2].

Theorem 3.6. *Let B be a separable Banach space, $\hat{\mathscr{F}} : B \to [0, +\infty]$ be a coercive integrand (i.e. with compact sublevels) such that*

$$\lim_{\|v\|\uparrow+\infty} \frac{\hat{\mathscr{F}}(v)}{\|v\|} = +\infty. \tag{3.24}$$

If $\mathcal{S} \subset \mathcal{P}(B)$ is a family of probability measures with

$$S := \sup_{\mu\in\mathcal{S}} \int_B \hat{\mathscr{F}}(v)\, d\mu(v) < +\infty, \tag{3.25}$$

then $\mathcal{S}$ is narrowly (sequentially) compact. For every continuous function $\boldsymbol{\zeta} : B \to B$ such that

$$\sup_B \frac{\|\boldsymbol{\zeta}(v)\|}{1 + \|v\|} < +\infty, \tag{3.26}$$

the map

$$\mu \in \mathcal{S} \mapsto Z(\mu) := \int_B \boldsymbol{\zeta}(v)\, d\mu(v) \tag{3.27}$$

is strongly *continuous w.r.t. the narrow convergence in $\mathcal{S}$.*

Proof. Since $\hat{\mathscr{F}}$ has compact sublevels, $\mathcal{S}$ is tight according to (2.16) by the Chebychev inequality: compactness of $\mathcal{S}$ then follows by Prohorov's Theorem 2.8. (3.24) and (3.26) yield

$$\mu \in \mathcal{S} \quad \Rightarrow \quad \int_B \|\boldsymbol{\zeta}(v)\| \, d\mu(v) < +\infty,$$

so that (the Bochner integral) (3.27) makes sense.

Claim 1: if $\phi : B \to \mathbb{R}$ is continuous and at most linearly increasing then

$$\mu \mapsto \int_B \phi(z) \, d\mu(z) \quad \textit{is continuous in } \mathcal{S}.$$

In particular, Z is weakly continuous. We fix a sequence $\mu_n \in \mathcal{S}$ narrowly converging to $\mu \in \mathcal{S}$ and we define for $\varepsilon > 0$

$$\phi^\varepsilon(v) := \phi(v) + \varepsilon\hat{\mathscr{F}}(v), \quad \phi_\varepsilon(v) := \phi(v) - \varepsilon\hat{\mathscr{F}}(v).$$

Thanks to (3.24), since ϕ is linearly increasing, ϕ_ε is u.s.c. and bounded from above, ϕ^ε is l.s.c. and bounded from below. It follows from Proposition 2.7 that

$$\begin{aligned}\liminf_{n\uparrow+\infty} \int_B \phi(v) \, d\mu_n(v) &\geq \liminf_{n\uparrow+\infty} \int_B \phi^\varepsilon(v) \, d\mu_n(v) - \varepsilon S \\ &\geq \int_B \phi^\varepsilon(v) \, d\mu(v) - \varepsilon S \geq \int_B \phi(v) \, d\mu(v) - \varepsilon S,\end{aligned}$$

and analogously

$$\begin{aligned}\limsup_{n\uparrow+\infty} \int_B \phi(v) \, d\mu_n(v) &\leq \limsup_{n\uparrow+\infty} \int_B \phi_\varepsilon(v) \, d\mu_n(v) + \varepsilon S \\ &\leq \int_B \phi_\varepsilon(v) \, d\mu(v) + \varepsilon S \leq \int_B \phi(v) \, d\mu(v) + \varepsilon S..\end{aligned}$$

Being ε arbitrary, we conclude. Finally, choosing $\phi(v) := \langle w^*, \boldsymbol{\zeta}(v)\rangle$, $w^* \in B^*$, we deduce that Z is weakly continuous.

Claim 2: if $\boldsymbol{\zeta}$ is bounded then Z is strongly continuous. Let us suppose that

$$\sup_{v\in B} \|\boldsymbol{\zeta}(v)\| = C < +\infty;$$

we want to show that $Z(\mathcal{S})$ is (strongly) relatively compact in B; by the previous Claim, this would entail the strong continuity of Z.

Being B complete, it will suffice to show that $Z(\mathcal{S})$ is totally bounded: we fix $\varepsilon > 0$ and we choose

$$K_\varepsilon \subset\subset B : \quad \mu(B \setminus K_\varepsilon) \leq \varepsilon/2C \quad \forall\, \mu \in \mathcal{S}.$$

Of course

$$\left\| \int_B \boldsymbol{\zeta}(v) \, d\mu(v) - \int_{K_\varepsilon} \boldsymbol{\zeta}(v) \, d\mu(v) \right\|_B \leq \varepsilon/2 \quad \forall\, \mu \in \mathcal{S}. \tag{3.28}$$

On the other hand, if H_ε is the closed convex hull of $\boldsymbol{\zeta}(K_\varepsilon)$, H_ε is compact and it is well known that

$$\int_{K_\varepsilon} \boldsymbol{\zeta}(v)\, d\mu(v) \in H_\varepsilon \quad \forall\, \mu \in \mathcal{S}. \tag{3.29}$$

Thus we can cover H_ε by a finite collection of balls $\{B_{\varepsilon/2}(x_i)\}_{i=1,\dots N}$ of radius $\varepsilon/2$; it follows from (3.28) that $\{B_\varepsilon(x_i)\}_{i=1,\dots N}$ is a finite covering of $Z(\mathcal{S})$; being ε arbitrary, we conclude.

Claim 3: Z is (strongly) continuous for every $\boldsymbol{\zeta}$ satisfying (3.26)*, i.e. if $\mu_n \in \mathcal{S}$ is narrowly converging to $\mu \in \mathcal{S}$ then*

$$\lim_{n\uparrow+\infty} Z(\mu_n) = Z(\mu) \quad \text{strongly in } B. \tag{3.30}$$

Let us fix a continuous partition of the unity $\{\phi_k(x)\}_{k=1}^{+\infty}$ on $[0,+\infty)$ such that

$$\operatorname{supp}\phi_k \subset [k-1,k+1], \quad \phi_k(x) \geq 0, \quad \sum_{k=1}^{+\infty} \phi_k(x) = 1 \quad \forall\, x \in [0,+\infty). \tag{3.31}$$

We set

$$\boldsymbol{\zeta}_k(v) := \phi_k(\|v\|)\boldsymbol{\zeta}(v), \quad \text{so that } \boldsymbol{\zeta}(v) = \sum_{k=1}^{+\infty} \boldsymbol{\zeta}_k(v), \quad \boldsymbol{\zeta}_k \text{ is bounded on } B.$$

Therefore

$$Z(\mu_n) = \sum_{k=1}^{+\infty} Z_k(\mu_n), \quad \text{with} \quad Z_k(\mu_n) := \int_B \boldsymbol{\zeta}_k(v)\, d\mu_n(v) \quad \forall\, n \in \mathbb{N}.$$

Since by the previous Claim

$$\lim_{n\uparrow+\infty} Z_k(\mu_n) = Z_k(\mu),$$

we simply have to show that it is possible to invert the order of the limit and the summation in the formula

$$\lim_{n\uparrow+\infty} \sum_{k=1}^{+\infty} Z_k(\mu_n).$$

We apply Lemma 3.5, choosing $Z_{n,k} := Z_k(\mu_n)$ and

$$z_{n,k} := \int_B \phi_k(\|v\|)\|\boldsymbol{\zeta}(v)\|\, d\mu_n(v), \quad z_k := \int_B \phi_k(\|v\|)\|\boldsymbol{\zeta}(v)\|\, d\mu(v).$$

Of course $\lim_{n\uparrow+\infty} z_{n,k} = z_k$ and, by Claim 1, in view of (3.26) as well, we have

$$\lim_{n\uparrow+\infty} \sum_{k=1}^{+\infty} z_{n,k} = \lim_{n\uparrow+\infty} \int_B \|\boldsymbol{\zeta}(v)\|\, d\mu_n(v) = \int_B \|\boldsymbol{\zeta}(v)\|\, d\mu(v) = \sum_{k=1}^{+\infty} z_k.$$

□

The following lemma is a standard result of convex analysis:

Lemma 3.7. *Let $p \in [1, +\infty)$ be a fixed exponent and suppose that $G : [0, +\infty) \to [0, +\infty)$ is a l.s.c. function satisfying the p-growth condition*

$$\lim_{s\uparrow+\infty} \frac{G(s)}{s^p} = \lim_{s\uparrow+\infty} \frac{G^{1/p}(s)}{s} = +\infty. \tag{3.32}$$

There exists a convex super-linearly *increasing function $\hat{G} : [0, +\infty) \to [0, +\infty)$ such that*

$$\lim_{s\uparrow+\infty} \frac{\hat{G}(s)}{s} = +\infty, \quad G(s) \geq \big(\hat{G}(s)\big)^p \quad \forall\, s \in [0, +\infty). \tag{3.33}$$

Proof. Condition (3.32) is equivalent to

$$-G_p^*(\alpha) := \inf_{s\geq 0} (G^{1/p}(s) - \alpha s) > -\infty \quad \forall\, \alpha \geq 0.$$

Observe that, by definition,

$$\alpha s - G_p^*(\alpha) \leq G^{1/p}(s) \quad \forall\, \alpha, s \geq 0; \tag{3.34}$$

thus we define

$$\hat{G}(s) = \sup_{\alpha\geq 0} \big(\alpha s - G_p^*(\alpha)\big)$$

which is clearly convex, $\big(\hat{G}(s)\big)^p \leq G(s)$ by (3.34), and

$$\liminf_{s\uparrow+\infty} \frac{\hat{G}(s)}{s} \geq \liminf_{s\uparrow+\infty} \frac{\alpha s - G_p^*(\alpha)}{s} \geq \alpha \quad \forall\, \alpha \geq 0,$$

so that (3.33) holds. □

Now we can conclude the ***proof of Theorem 1***. Let us suppose that $\mathcal{U}$ is a relatively compact subset of $L^p(0, T; B)$. Since (1.1a) is well known, we must show that there exists a Banach space A compactly embedded in B and a *convex* functional $\mathscr{F}$ satisfying (1.11) such that $\mathcal{U}$ is tight w.r.t. $\mathscr{F}$.

Claim 1: $\mathcal{U}$ is tight w.r.t. a coercive integrand $\hat{\mathscr{F}}^p : B \to [0, +\infty]$ such that

$$\begin{gathered} 1 \leq \hat{\mathscr{F}}(0) = \min_{v\in B} \hat{\mathscr{F}}(v) < +\infty, \quad \hat{\mathscr{F}}(-v) = \hat{\mathscr{F}}(v) \quad \forall\, v \in B, \\ \hat{\mathscr{F}}(v) \geq \|v\|_B \quad \forall\, v \in B, \qquad \lim_{\|v\|\uparrow+\infty} \frac{\hat{\mathscr{F}}(v)}{\|v\|} = +\infty. \end{gathered} \tag{3.35}$$

Theorem 2 and Remark 3.4 provide an integrand $\mathscr{G} : B \to [0, +\infty]$ with compact sublevels such that

$$\mathscr{G}(0) = 0, \quad \mathscr{G}(-v) = \mathscr{G}(v) \quad \forall\, v \in B, \quad \sup_{u\in\mathcal{U}} \int_0^T \mathscr{G}^p(u(t))\, dt < +\infty.$$

Moreover, since

$$\Big\{ \|u\|^p : u \in \mathcal{U} \Big\} \quad \text{is compact in } L^1(0, T),$$

the Dunford-Pettis Theorem 1.6 and Lemma 3.7 yield an increasing convex function $\hat G : [0,+\infty) \to [0,+\infty)$ such that

$$\hat G(s) \ge s, \quad \lim_{s\uparrow+\infty} \frac{\hat G(s)}{s} = +\infty, \quad \sup_{u\in\mathcal U} \int_0^T \hat G^p\big(\|u(t)\|_B\big)\,dt < +\infty. \tag{3.36}$$

The functional

$$\hat{\mathscr F}(v) := \mathscr G(v) + \hat G\big(\|v\|_B\big) + 1 \tag{3.37}$$

is coercive and satisfies the required condition (3.35).

Claim 2: $\mathcal U$ is tight w.r.t. $\mathscr F^p$, where $\mathscr F :=$ l.s.c. convex envelope of $\hat{\mathscr F}$, is coercive, convex, and satisfies (3.35)*.*
More precisely we set

$$\mathscr F(v) := \inf\Big\{ \int_B \hat{\mathscr F}(w)\,d\mu(w) : \mu\in\mathcal P(B), \quad \int_B w\,d\mu(w) = v\Big\}. \tag{3.38}$$

Observe that, whenever $\mathscr F(v) < +\infty$, the infimum in (3.38) is in fact a minimum, thanks to Theorem 3.6 applied to $\boldsymbol\zeta(w) := w$. It is then easy to show that $\mathscr F$ is convex, symmetric with respect to the origin, and, by Jensen inequality,

$$\hat G(\|v\|_B) \le \mathscr F(v) \le \hat{\mathscr F}(v) \quad \forall\, v\in B,$$

so that $\mathscr F$ satisfies (3.35) and $\mathcal U$ is tight w.r.t. $\mathscr F$ as well. It remains to show that the sublevels of $\mathscr F$ are strongly compact. Let us fix $c\in[0,+\infty)$ and let us set

$$\mathcal S := \Big\{\mu\in\mathcal P(B) : \int_B \hat{\mathscr F}(w)\,d\mu(w) \le c\Big\}, \quad Z(\mu) := \int_B w\,d\mu(w).$$

Since

$$\big\{v\in B : \mathscr F(w) \le c\big\} = Z(\mathcal S),$$

being $\mathcal S$ compact and Z strongly continuous, we conclude.

Claim 3: let us set

$$\|v\|_A := \inf_{\lambda>0} \frac{1}{\lambda}\mathscr F(\lambda v), \quad A := \big\{v\in B : \|v\|_A < +\infty\big\}. \tag{3.39}$$

Then A is a normed vector space with norm $\|\cdot\|_A$ continuously embedded in B and $\mathscr F_{p,A} \le \mathscr F$.
Let us first observe that if $v\in A\setminus\{0\}$, the coercivity property of $\mathscr F$ ensures that the infimum in (3.39) is attained. $\|\cdot\|_A$ is symmetric (being $\mathscr F$ symmetric) and homogeneous of degree one:

$$\|\alpha v\|_A = \min_{\lambda>0}\frac{1}{\lambda}\mathscr F(\alpha\lambda v) = \alpha\,\min_{\lambda>0}\frac{1}{(\alpha\lambda)}\mathscr F(\alpha\lambda v) = \alpha\,\|v\|_A \quad \forall\,\alpha>0;$$

moreover, since $0\in D(\mathscr F)$ and $\mathscr F(v)\ge\|v\|_B$

$$\|0\|_A = 0, \quad \|v\|_A \ge \|v\|_B \quad \forall\, v\in B. \tag{3.40}$$

Let us check the sub-additivity of $\|\cdot\|_A$: we are given $v_1, v_2\in A\setminus\{0\}$, $\lambda_1,\lambda_2>0$ with

$$\|v_i\|_A = \frac{1}{\lambda_i}\mathscr F(\lambda_i v_i) \quad i=1,2;$$

then, for

$$\lambda^{-1} = \lambda_1^{-1} + \lambda_2^{-1}, \quad \text{so that } \frac{\lambda}{\lambda_1} + \frac{\lambda}{\lambda_2} = 1,$$

we have

$$\begin{aligned}\|v_1 + v_2\|_A &\le \frac{1}{\lambda}\mathscr{F}(\lambda(v_1+v_2)) = \frac{1}{\lambda}\mathscr{F}(\frac{\lambda}{\lambda_1}\lambda_1 v_1 + \frac{\lambda}{\lambda_2}\lambda_2 v_2)\\ &\le \frac{1}{\lambda}\Big(\frac{\lambda}{\lambda_1}\mathscr{F}(\lambda_1 v_1) + \frac{\lambda}{\lambda_2}\mathscr{F}(\lambda_2 v_2)\Big) = \|v_1\|_A + \|v_2\|_A.\end{aligned}$$

It follows that $\big(A, \|\cdot\|_A\big)$ is a normed vector space.

Claim 4: the unit ball of A is compact in B; in particular the map $v \mapsto \|v\|_A$ is l.s.c. in B.
Let us take a sequence $v_n \in A \setminus \{0\}$ and $\lambda_n \in [1, +\infty)$ such that

$$\|v_n\|_A = \frac{1}{\lambda_n}\mathscr{F}(\lambda_n v_n) \le 1 \tag{3.41}$$

We can always extract a subsequence (still denoted by v_n, λ_n) such that

$$\exists \lim_{n\uparrow+\infty} \lambda_n = \lambda \in [1, +\infty].$$

We have now to distinguish two cases: if $\lambda < +\infty$ then

$$\sup_n \mathscr{F}(\lambda_n v_n) < +\infty \tag{3.42}$$

and therefore there exists subsequences v_{n_k}, λ_{n_k} such that

$$\lim_{k\uparrow+\infty} \lambda_{n_k} v_{n_k} = v \quad \text{in } B, \text{ so that} \quad \exists \lim_{k\uparrow+\infty} v_{n_k} = v/\lambda.$$

Moreover, by the lower semicontinuity of $\mathscr{F}$,

$$\|v\|_A \le \frac{1}{\lambda}\mathscr{F}(\lambda v) \le \liminf_{k\uparrow+\infty} \frac{1}{\lambda_{n_k}}\mathscr{F}(\lambda_{n_k} v_{n_k}) \le 1.$$

If $\lambda = +\infty$ let us call $\ell := \liminf_{n\uparrow+\infty} \|v_n\|_B$; if $\ell > 0$, then

$$\ell = \liminf_{n\uparrow+\infty} \frac{\|\lambda_n v_n\|_B}{\mathscr{F}(\lambda_n v_n)} \frac{\mathscr{F}(\lambda_n v_n)}{\lambda_n} \le \liminf_{n\uparrow+\infty} \frac{\|\lambda_n v_n\|_B}{\mathscr{F}(\lambda_n v_n)} \le \limsup_{\|w\|_B\uparrow+\infty} \frac{\|w\|_B}{\mathscr{F}(w)} = 0,$$

which is absurd; therefore $\ell = 0$, i.e. there exists a subsequence v_{n_k} converging to $v = 0$ in B.

Claim 5: A is complete. Let $\{v_n\}_{n\in\mathbb{N}} \in A$ be a Cauchy sequence w.r.t. the norm $\|\cdot\|_A$, i.e.

$$\lim_{n,m\uparrow+\infty} \|v_n - v_m\|_A = 0, \quad \text{which in particular implies} \quad \sup_n \|v_n\|_A < +\infty.$$

Then v_n is a Cauchy sequence in B and therefore it is strongly convergent to some element $v \in B$. Being v_n bounded in A, the lower semicontinuity of $\|\cdot\|_A$ yields $v \in A$ and

$$\limsup_{n\uparrow+\infty} \|v_n - v\|_A \le \limsup_{n\uparrow+\infty} \liminf_{m\uparrow+\infty} \|v_n - v_m\|_A \le \limsup_{n,m\uparrow+\infty} \|v_n - v_m\|_A = 0.$$

4 Proof of the examples

Theorem 1.9 is a direct consequence of Theorem 2; let us consider the second example.

Proof of Theorem 1.10 Since $\mathcal{V}$ is relatively compact in $\mathcal{M}(0,T;C)$, it is tight, i.e. there exists an integrand $\mathscr{G}: C \to [0,+\infty]$ with compact sublevels, such that

$$\sup_{v\in\mathcal{V}} \int_0^T \mathscr{G}(v(t))\,dt < +\infty, \tag{4.1}$$

and

$$\limsup_{h\downarrow 0} \sup_{v\in\mathcal{V}} \int_0^{T-h} d_C(v(t+h), v(t))\,dt = 0. \tag{4.2}$$

We set

$$\mathscr{H}(t,u,v) := \begin{cases} \mathscr{F}(t,u) + \mathscr{G}(v) & \text{if } v \in L(t)u, \\ +\infty & \text{otherwise} \end{cases} \tag{4.3}$$

which is a normal coercive integrand in $(0,T)\times B\times C$ thanks to (1.30) and (1.31). Let us now choose a sequence $\{u_n\}_{n\in\mathbb{N}}$ in $\mathcal{U}$ and a corresponding sequence $\{v_n\}_{n\in\mathbb{N}}$ in $\mathcal{V}$ such that $v_n(t) \in L(t)u_n(t)$ for a.e. $t\in(0,T)$, and observe that the sequence (u_n, v_n) is tight in $B\times C$, since

$$\int_0^T \mathscr{H}(t, u_n(t), v_n(t))\,dt = \int_0^T \mathscr{F}(t, u_n(t))\,dt + \int_0^T \mathscr{G}(v_n(t))\,dt$$

is uniformly bounded w.r.t. n. Moreover, setting

$$g\big((u_1,v_1),(u_2,v_2)\big) := d_C(v_1,v_2) \quad \forall (u_1,v_1),(u_2,v_2) \in B\times C$$

g satisfies (1.21c) with respect to $\mathscr{H}$ and (1.21a) by (4.2). Therefore we can extract a subsequence u_{n_k} converging in $\mathcal{M}(0,T;B)$: we conclude that $\mathcal{U}$ is relatively compact.

Proof of Theorem 1.15 First of all, let us observe that (1.35) holds if w^* belongs to the linear space $\mathrm{span}(S)$ generated by S as well, and, in particular, if w^* belongs the the intersection S_0 of $\mathrm{span}(S)$ with the closed unit ball of B^*. It easy to check that S_0 is a separating set, too.

Since the closed unit ball of B^* endowed with the weak* topology is compact and metrizable, S_0 is totally bounded with respect to the distance which induces the weak* topology and it is also separable.

Thus we can choose a countable set $\tilde{S}_0 := \{w_n^*\}_{n\in\mathbb{N}}$ weakly* dense in S_0: again $\tilde{S}_0$ separates the points of B. We set

$$g(u,v) := \sum_{n=1}^{+\infty} 2^{-n} \min\Big(1, \big|\langle w_n^*, u-v\rangle\big|\Big).$$

g is a continuous and bounded distance on B; moreover, setting

$$\omega_n(h) := \sup_{u\in\mathcal{U}} \int_0^{T-h} \min\big(1, |\langle w_n^*, u(t+h)-u(t)\rangle|\big)\,dt, \tag{4.4}$$

the proof of the necessity statement of Theorem 2 shows that

$$\omega_n(h) \le T, \quad \lim_{h\downarrow 0} \omega_n(h) = 0.$$

Since

$$\int_0^{T-h} g(u(t+h), u(t))\,dt \le \sum_{n=1}^{+\infty} 2^{-n}\omega_n(h) \quad \forall\, u \in \mathcal{U},$$

the Lebesgue Dominated Convergence Theorem for series yields

$$\limsup_{h\downarrow 0} \sup_{u\in\mathcal{U}} \int_0^{T-h} g(u(t+h), u(t))\,dt \le \lim_{h\downarrow 0} \sum_{n=1}^{+\infty} 2^{-n}\omega_n(h) = 0.$$

Applying Theorem 2 again, we conclude.

Remark 4.1. It is easy to check that Theorem 1.15 still holds if we know that and S separates the points of the linear space generated by $D(\mathscr{F})$, where $\mathscr{F}$ is a coercive functional such that

$$\sup_{u\in\mathcal{U}} \int_0^T \mathscr{F}(u(t))\,dt < +\infty.$$

Weak convergence The key ingredient of the proofs of Theorems 1.20 and 1.21 relies in the following result.

Theorem 4.2. *Let B a Banach space, let B_0^* a determining and strongly separable closed subspace of B^*, as in (1.36), and let $S \subset B_0^*$ be a vector space separating the points of B. Then there exists a separable Banach space $\hat{B}$ and a (countable) subset $S_0 \subset S$ such that*

$$B \text{ is continuously and densely embedded in } \hat{B}, \tag{4.5a}$$

$$\text{bounded sets of } B \text{ are totally bounded in } \hat{B}, \tag{4.5b}$$

$$\sigma(B, B_0^*)\text{-weakly compact sets are strongly compact in } \hat{B}, \tag{4.5c}$$

$$S_0 \subset \hat{B}^* \text{ separates the points of } B, \tag{4.5d}$$

$$\mathcal{M}_w(0,T;B) \subset \mathcal{M}(0,T;\hat{B}). \tag{4.5e}$$

Proof. Let us call U the closed unit ball of B and U^* the dual unit ball of B_0^*; we choose a countable subset $D_0 \subset B_0^*$ which is strongly dense in U^* and a countable subset S_0 of S which is strongly dense in $S \cap U^*$: S_0 separates the points of B. Being D_0, S_0 countable, we can order the elements of their union in a sequence, so that

$$D := D_0 \cup S_0 = \{w_m^*\}_{m\in\mathbb{N}}, \quad S_0 = \{w_{m_k}^*\}_{k\in\mathbb{N}} \tag{4.6}$$

for a suitable subsequence $k \mapsto m_k \in \mathbb{N}$; we set

$$\|v\|_{\hat{B}} := \sum_{m=1}^{+\infty} 2^{-m} |\langle w_m^*, v\rangle|. \tag{4.7}$$

It is easy to check that $\|\cdot\|_{\hat{B}}$ is a continuous norm on B: we denote by $\hat{B}$ the completion of B w.r.t. this norm and we identify B with its continuous (dense) image in $\hat{B}$. Observe that

$$u_n, u \in U, \quad \lim_{n\uparrow+\infty} \|u_n - u\|_{\hat{B}} = 0 \quad \Leftrightarrow \quad \lim_{n\uparrow+\infty} \langle w_m^*, u_n\rangle = \langle w_m^*, u\rangle \quad \forall\, m \in \mathbb{N} \tag{4.8}$$

so that the map

$$u \in U \mapsto \boldsymbol{u} := \{\boldsymbol{u}_m\}_{m\in\mathbb{N}} \in [-1,1]^{\mathbb{N}}, \quad \boldsymbol{u}_m := \langle w_m^*, u\rangle$$

is an homeomorphism between U and $\boldsymbol{u}(U)$ with respect to the distance in U induced by $\|\cdot\|_{\hat{B}}$ and the topology of $\boldsymbol{u}(U)$ induced by the compact metric space $[-1,1]^{\mathbb{N}}$. Being the image $\boldsymbol{u}(U)$ relatively compact, U is totally bounded (and thus separable) with respect to $\|\cdot\|_{\hat{B}}$; since $\cup_{n\in\mathbb{N}} nU$ is dense in $\hat{B}$, we obtain that $\hat{B}$ is separable.

(4.8) shows that on B-bounded sets $\|\cdot\|_{\hat{B}}$ is continuous with respect to the $\sigma(B, B_0^*)$-weak topology: therefore, since $\sigma(B, B_0^*)$-weakly compact sets are bounded, they are also strongly compact in $\hat{B}$.

If $w^* = w_{m_k}^* \in S_0$ for some $k \in \mathbb{N}$, then

$$\big|\langle w^*, v\rangle\big| \le 2^{m_k} \|v\|_{\hat{B}} \quad \forall\, v \in B$$

so that w^* can be uniquely extended by continuity to an element of $\hat{B}^*$.

If $u \in \mathcal{M}_w(0,T;B)$ then (4.7) yields

$$t \mapsto \|u(t) - v\|_{\hat{B}} \quad \text{is measurable} \quad \forall\, v \in B;$$

being B dense in $\hat{B}$, then the previous map is measurable even if $v \in \hat{B}$, so that $u \in \mathcal{M}(0,T;\hat{B})$. □

Corollary 4.3. *With the same notation of the previous Theorem 4.2, let us suppose that $\mathcal{U} \subset \mathcal{M}_w(0,T;B)$ is $\sigma(B, B_0^*)$-weakly tight w.r.t. $\mathscr{F}$; then it is (strongly) tight in $\hat{B}$ w.r.t. the functional*

$$\hat{\mathscr{F}}(\hat{v}) := \begin{cases} \mathscr{F}(\hat{v}) & \text{if } \hat{v} \in B, \\ +\infty & \text{if } \hat{v} \in \hat{B} \setminus B. \end{cases} \tag{4.9}$$

If a sequence $u^n \in \mathcal{U}$ converges to u in $\mathcal{M}(0,T;\hat{B})$ as $n \uparrow +\infty$, then $u(t) \in B$ for a.e. t and u^n $\sigma(B, B_0^*)$-weakly converges in measure to u. In particular, $\mathcal{U}$ is $\sigma(B, B_0^*)$-weakly sequentially relatively compact if and only if it is (strongly) relatively compact in $\mathcal{M}(0,T;\hat{B})$.

Proof. The only non trivial part is the assertion about the convergence of the sequence u^n. We know that

$$\sup_n \int_0^T \mathscr{F}(u^n(t))\, dt = \sup_n \int_0^T \hat{\mathscr{F}}(u^n(t))\, dt < +\infty.$$

Observe that the sublevels of $\hat{\mathscr{F}}$ are strongly compact in $\hat{B}$, i.e. $\hat{\mathscr{F}}$ is strongly coercive on $\hat{B}$; therefore $\{u^n\}_{n\in\mathbb{N}}$ is tight in $\hat{B}$. We can find a suitable subsequence

u^{n_k} such that

$$\lim_{k\uparrow+\infty} |u^{n_k}(t) - u(t)| = 0 \quad \text{a.e. in } (0,T), \tag{4.10}$$

$$\int_0^T \hat{\mathscr{F}}(u(t))\,dt \le \liminf_{k\uparrow+\infty} \int_0^T \mathscr{F}(u^{n_k}(t))\,dt < +\infty. \tag{4.11}$$

(4.11) shows that

$$u(t) \in B \quad \text{for a.e. } t \in (0,T),$$

in particular, we can change u on a negligible set, so that its values belong to B. (4.10), the density of D_0 in U^* and the determining property of B_0^* yields

$$t \mapsto \langle w^*, u(t)\rangle, \quad t \mapsto \|u(t)\|_B \quad \text{are measurable } \forall\, w^* \in B_0^*,$$

so that $u \in \mathcal{M}_w(0,T;B)$. We want now to prove that

$$\langle w^*, u^{n_k}\rangle \to \langle w^*, u\rangle \quad \text{in measure} \quad \forall\, w^* \in B_0^*.$$

For a fixed $w^* \in B_0^*$ and an arbitrary element $w_m^* \in D$ (to be chosen later on) we observe that

$$\big|\langle w^*, u^{n_k}(t) - u(t)\rangle\big| \le \|w^* - w_m^*\|_{B^*}\big(\|u^{n_k}(t)\|_B + \|u(t)\|_B\big) + |\langle w_m^*, u^{n_k}(t) - u(t)\rangle|.$$

For a given $\varepsilon > 0$ we can find $M > 0$ such that (here $|\cdot|$ denotes the

$$\Big|\big\{t \in (0,T) : \|u^{n_k}(t)\|_B + \|u(t)\|_B > M\big\}\Big| \le \varepsilon/2$$

and $m \in \mathbb{N}$ such that $\|w^* - w_m^*\|_{B^*} \le \varepsilon/M$; finally, by assumption, there exists k_0 such that

$$\Big|\big\{t \in (0,T) : |\langle w_m^*, u^{n_k}(t) - u(t)\rangle| > \varepsilon/2\big\}\Big| < \varepsilon/2 \quad \forall\, k \ge k_0$$

Combining all these estimates we conclude that for $k \ge k_0$

$$\Big|\big\{t \in (0,T) : |\langle w^*, u^{n_k}(t) - u(t)\rangle| \ge \varepsilon\big\}\Big| \le \varepsilon$$

which yields the convergence in measure. □

Proof of Theorem 1.20 It is not restrictive to suppose that S is a linear space. We denote by $\mathscr{F}$ the functional yielding the tightness of $\mathcal{U}$; let $\hat{B}, S_0$ be defined as in the previous Theorem 4.2, $\hat{\mathscr{F}}$ as in (4.9): by Corollary 4.3, we have to show that $\mathcal{U}$ is relatively compact in $\mathcal{M}(0,T;\hat{B})$. But now we can apply Theorem 1.15, taking account of Remark 4.1.

Proof of Theorem 1.21 Let $\mathscr{G} : C \to [0,+\infty]$ be the (weakly) coercive functional yielding the tightness of $\mathscr{V}$, and let $\hat{C}, S_0$ be given by Theorem 4.2; the extended functional $\hat{\mathscr{G}} : \hat{C} \to [0,+\infty]$ is defined as in (4.9).

By Theorem 1.20 and Corollary 4.3, we know that $\mathcal{V}$ is relatively compact in $\mathcal{M}(0,T;\hat{C})$. Then we can proceed as in the proof of Theorem 1.10, by setting

$$\mathscr{H}(t,u,v) := \begin{cases} \mathscr{F}(t,u) + \hat{\mathscr{G}}(v) & \text{if } v \in L(t)u, \\ +\infty & \text{otherwise;} \end{cases}$$

(1.43a), (1.43b), the tightness of $\mathscr{F}$ and $\hat{\mathscr{G}}$ yield that $\mathscr{H}$ is a normal coercive integrand on $(0,T) \times B \times \hat{C}$.

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