PERCOLATION-TYPE PROBLEMS ON INFINITE RANDOM GRAPHS

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Abstract. We study some percolation problems on the complete graph over $\mathbb{N}$. In particular, we give sharp sufficient conditions for the existence of (finite or infinite) cliques and paths in a random subgraph. No specific assumption on the probability, such as independency, is made. The main tools are a topological version of Ramsey theory, exchangeability theory and elementary ergodic theory.

Contents

1. Introduction

Let $G = (\mathbb{N}, \mathbb{N}^{[2]})$ be the complete oriented graph having vertices in $\mathbb{N}$, with the orientation induced by the usual order of $\mathbb{N}$, and let us randomly choose some of its edges: that is, we associate to the edge $(i, j) \in \mathbb{N}^{[2]}$ (thus $i < j$) a measurable set $X_{ij} \subseteq X$, where $(X\mathcal{A}, \mu)$ is a base probability space. We then ask if the resulting random graph contains an infinite path:

Problem 1. Let $(X, \mathcal{A}, \mu)$ be a probability space. For all $(i, j) \in \mathbb{N}^{[2]}$, let X_{ij} be a measurable subset of X. Is there an infinite increasing sequence $\{n_i\}_{i\in\mathbb{N}}$ such that $\bigcap_{i\in\mathbb{N}} X_{n_i n_{i+1}}$ is non-empty?

More formally, a random subgraph of the oriented graph G is defined by a measurable function $F : X \to 2^{E_G}$[1], where E_G is the set of edges of G and 2^{E_G} its powerset, equipped with the product σ-algebra. We briefly say that F has path percolation, or F contains an infinite path, if the subgraph $F(x)$

[1]this is equivalent to specify a family of measurable sets $\{X_e\}_{e\in E_G}$, with $X_e \subseteq X$

contains an infinite path for some $x \in X$. As in classic percolation theory, we wish to estimate the probability that F has path percolation, that is[2]

$$\mu(\{x \in X : F(x) \text{ contains an infinite path}\})$$

in terms of a parameter λ that bounds from below the probability that an edge e belongs to F, that is $\mu(X_e) \geq \lambda$, where $X_e := \{x \in X : e \in F(x)\}$, for all $e \in E_G$.

It has to be noticed that the analogy with classic bond percolation is only formal, the main difference being that in the usual percolation models (see for instance [GR:99]) the events X_{ij} are supposed *independent*, whereas in the present case the probability distribution is completely general, i.e. we do not impose any restriction on the events X_{ij}.

In Section 3, we show that path percolation occurs with probability strictly greater than $2\lambda - 1$ (see Theorem 3.3 and Corollary 3.4 for a precise statement). Moreover, we show that the estimate $2\lambda - 1$ is optimal; in particular X may fail to contain an infinite path if $\lambda < 1/2$.

In order prove this result, we first observe that a subgraph H of $(\mathbb{N}, \mathbb{N}^{[2]})$ does not contain an infinite path iff it admits a height function with values in ω_1, where ω_1 is the first uncountable ordinal, i.e. there exists a graph map between H and the complete graph over ω_1 with decreasing orientation, that is (α, β) is an edge of the graph if $\alpha, \beta \in \omega_1$ and $\alpha > \beta$.

Therefore, if a random graph F has no infinite paths, introducing the dependence on $x \in X$ and on the vertices of F, it is defined a measurable map from $X \times \mathbb{N}$ to ω_1, which can be also seen as a map $\varphi : X \to \omega_1^{\mathbb{N}}$, where $\omega_1^{\mathbb{N}}$ is equipped with the product σ-algebra generated by the finite subsets of ω_1. It turns out that φ is essentially bounded (see Lemma 3.2), which implies that $\varphi_\#(\mu)$ is a compactly supported Radon measure on $\omega_1^{\mathbb{N}}$, and that $\varphi(X_{ij}) \subseteq A_{ij} := \left\{x \in \omega_1^{\mathbb{N}} : x_i > x_j\right\}$. As a consequence, in the determination of the threshold for existence of infinite paths

$$\lambda_c := \sup\left\{\inf_{i<j\in\mathbb{N}} \mu(X_{ij}) : \ F \text{ random graph without infinite paths}\right\}, \tag{1.1}$$

we can set $X = \omega_1^{\mathbb{N}}$, $X_{ij} = A_{ij}$, and reduce to the variational problem on the convex set $\mathcal{M}_c(\omega_1^{\mathbb{N}})$ of compactly supported probability measures on $\omega_1^{\mathbb{N}}$:

$$\lambda_c = \sup_{m\in\mathcal{M}_c(\omega_1^{\mathbb{N}})} \inf_{i<j\in\mathbb{N}} m\left(A_{ij}\right). \tag{1.2}$$

As a next step, we show that in (1.2) we can equivalently take the supremum in the smaller class of all the compactly supported *exchangeable measures* on $\omega_1^{\mathbb{N}}$ (see Appendix B and references therein for a precise definition). Thanks to this reduction, we can explicitly compute $\lambda_c = 1/2$. We note that the supremum in (1.2) is not attained, which implies that for $\mu(X_{ij}) \geq 1/2$ path percolation occurs with positive probability.

A natural motivation for Problem 1 comes from the following situation, that we state in a very general form.

[2]it is not a priori obvious that this event has a well-defined probability, since it corresponds to the uncountable union of the sets $\bigcap_{k\in\mathbb{N}} X_{(i_k,i_{k+1})}$ over all strictly increasing sequences $i : \mathbb{N} \to \mathbb{N}$. However, it turns out that it belongs to the μ-completion of the σ-algebra generated by the X_{ij}

Suppose we are given a space E and a certain family X of sequences on E (e.g., minimizing sequences of a functional, or orbits of a discrete dynamical system, etc). A typical, general problem ask for existence of a sequence in the family X, that admits a subsequence with a prescribed property. One approach to it is by means of measure theory. The archetypal situation here come from recurrence theorems: one may ask if there exists a subsequence which belongs frequently to a given subset C of the "phase" space E (we refer to such sequences as "C-recurrent orbits"). If we consider the set $X_i := \{x \in X : x_i \in C\}$, then a standard sufficient condition for existence of C-recurrent orbits is $\mu(X_i) \geq \lambda > 0$, for some probability measure μ on X. In fact is easy to check that the set of C-recurrent orbits has measure at least λ by an elementary version of a Borel-Cantelli lemma (see Proposition 5.1). This is indeed the existence argument in the Poincaré Recurrence Theorem for measure preserving transformations. A more subtle question arises when one looks for a subsequence satisfying a given relation between two successive (or possibly more) terms: given a subset R of $E \times E$ we look for a subsequence x_{i_k} such that $(x_{i_k}, x_{i_{k+1}}) \in R$ for all $k \in \mathbb{N}$. As before, we may consider the subset of X, with double indices $i < j$, $X_{ij} := \{x \in X : (x_i, x_j) \in R\}$ and we are then led to Problem 1.

By looking for other properties of the random graph F, we can embed Problem 1 in a wider class of pattern-search problems. Indeed, given a property $\mathcal{P}$ of graphs, if we choose each edge of G with probability greater than λ, so that $\mu(X_e) \geq \lambda$ for all $e \in E_G$, we can ask if the graph $F(x)$ enjoys the property $\mathcal{P}$. Let

$$\begin{aligned} p(\lambda) &:= \inf\{\mu(\{x \in X : F(x) \text{ satisfies } \mathcal{P}\}) : (X, \mathcal{A}, \mu) \text{ probability space}\} \\ \lambda_c &:= \inf\{\lambda \in [0,1] : p(\lambda) > 0\}. \end{aligned}$$

Notice that, if G itself satisfies $\mathcal{P}$, then $p(\lambda) \leq \lambda$, since we can always choose all the edges of G simultaneously with probability λ. In Sections 3 and 4 we show that:

- if $\mathcal{P}$ is the property of having an infinite path, then $p(\lambda) = \min\{2\lambda - 1, 0\}$ and $\lambda_c = 1/2$;
- if $\mathcal{P}$ is the property of having a path of length N, then $p(\lambda) = \min\{(2N\lambda - N + 1)/(N+1), 0\}$ and $\lambda_c = (1 - 1/N)/2$;
- if $\mathcal{P}$ is the property of having chromatic number greater than N, then $p(\lambda) = \min\{N\lambda - N + 1, 0\}$ and $\lambda_c = 1 - 1/N$.

More generally, we can consider analogous percolation problems in an oriented graph G, not necessarily equal to $(\mathbb{N}, \mathbb{N}^{[2]})$. However, it can be shown that, if we replace G with a finitely branching graph (such as a finite dimensional network), then path percolation does not occur without some restriction on the probability, i.e. $p(\lambda) = 0$ for all $\lambda < 1$. On the other hand, if a vertex of G has infinite degree, then F contains an infinite cluster with probability at least $p(\lambda) = \lambda$, so that $\lambda_c = 0$. In a future work, we explicitly determine the path percolation tresholds for a random subgraph of the shift graphs $G = (\mathbb{N}^{[k]}, \mathbb{N}^{[q]})$, with $k < q \in \mathbb{N}$.

In Section 5 we let $G = (\mathbb{N}, \mathbb{N}^{[2]})$ and we ask if a random graph F contains an infinite clique, i.e. a copy of G itself. Note that this problem is a random version of the classical Ramsey theorem [R:28] (we refer to [GP:73, PR:05],

and references therein, for various generalization of Ramsey theorem to infinite graphs). We show with an explicit example (see Example 5.2) that in this case $p(\lambda) = 0$ for all $\lambda < 1$, so that the answer is negative unless we impose some restrictions on the probability space.

By Ramsey theorem, we know that if we assign to each element of $\mathbb{N}^{[k]}$ a *colour* taken from a set of n colours, then there is an infinite subset $J \subset \mathbb{N}$ such that all the elements of $J^{[k]}$ have the same colour. As a consequence, the probability is strictly positive if we restrict ourselves to the finite probability spaces with at most n elements. In analogy with Ramsey theorem, in Section 5 we deal with the following natural generalization of the previous problem:

Problem 2. Let $(X, \mathcal{A}, \mu)$ be a probability space. For all $(i_1, \dots, i_k) \in \mathbb{N}^{[k]}$, let $X_{i_1 \dots i_k}$ be a measurable subset of X. Is there an infinite set $J \subset \mathbb{N}$ such that the intersection $\bigcap_{(i_1,\dots,i_k)\in J^{[k]}} X_{i_1\dots i_k}$ is non-empty?

As already observed, if X is a prescribed finite set, then the answer is positive by Ramsey theorem. In fact, if we choose an element $x_{i_1\dots i_k} \in X_{i_1\dots i_k}$, we can intepret $x_{i_1\dots i_k}$ as the *colour* of $(i_1, \dots, i_k) \in \mathbb{N}^{[k]}$. If X is infinite the situation is more complicated, and we show that Problem 2 has a positive answer if the indicator functions of the sets $X_{i_1\dots i_k}$ all belong to a compact subset of $L^1(X, \mu)$ (see Theorem 5.4).

Note: After this paper was completed we learned that Problem 1 had been originally proposed by P. Erdös and A. Hajnal in [EH:64], and a complete answer was later given by D. H. Fremlin and M. Talagrand in the very interesting paper [FT:85], where other related problems are also considered. In particular, Corollary 3.4 is already contained in [FT:85], at least when the probability space (X, μ) is the interval $[0, 1]$ equipped with the Lebesgue measure. As far as we know, the solution of Problem 2 given in Theorem 5.4 is not present in the literature.

We would like to compare our approach and results with those in [FT:85]. Besides the fact that we do not impose any condition on the probability space, as already mentioned, our method allows us solve the following problem:

given a directed graph F, determine the critical treshold λ_c and the probability $p(\lambda)$ that $F(x) \to F$ (that is there exists a graph map between $F(x)$ and F), for some $x \in X$.

In Section 4, we completely solve this problem when F is a finite graph, showing in particular that

$$\lambda_c = c_0(F) := \sup_{\lambda\in\Sigma_F} \sum_{(a,b)\in E_F} \lambda_a \lambda_b \,,$$

where Σ_F is the set of all sequences $\{\lambda_a\}_{a\in V_F}$ with values in $[0, 1]$ and such that $\sum_{a\in V_F} \lambda_a = 1$. As observed above, Problem 1 can be reformulated in this setting by letting F be the complete graph over ω_1.

On the contrary, [FT:85] the following somewhat complementary problem is considered:

given a directed graph F, determine the critical treshold λ_c such that that $F(x)$ contains a copy of F (in particular $F \to F(x)$), for some $x \in [0, 1]$ and for all $\lambda > \lambda_c$.

The authors construct an algorithm which leads to a complete solution of the problem for finite F, and show that

$$\lambda_c = \sup\Big\{c_0(H) : \, H \text{ is finite and does not contain a copy of } F\Big\}.$$

Moreover, they can also solve this problem for some infinite graphs F, thus obtaining a solution of Problem 1. We observe that the notion of capacity we introduce in Section 4 is the same as in [FT:85].

As a final remark, we point out that our method is quite different from the one in [FT:85], since it relies on restating the problem as a variational problem like (1.2) for the probability measures on a suitable Cantor space, and then applying classical reasults of exchangeability theory (see Proposition B.4).

2. Notation

Given a compact metric space Λ, we let $\Lambda^{\mathbb{N}}$ be the space of all sequences taking values in Λ, endowed with the product topology. The space $\mathcal{M}(\Lambda^{\mathbb{N}})$ of Borel measures on $\Lambda^{\mathbb{N}}$ can be identified with $C(\Lambda^{\mathbb{N}})^*$, i.e. the dual of the Banach space of all continuous functions on $\Lambda^{\mathbb{N}}$. By the Banach-Alaoglu theorem the subset $\mathcal{M}_1(\Lambda^{\mathbb{N}}) \subset \mathcal{M}(\Lambda^{\mathbb{N}})$ of probability measures is a compact (metrizable) subspace of $C(\Lambda^{\mathbb{N}})^*$ endowed with the weak* topology. Given $p \in \mathbb{N}$, we identify p with the set $\{0, 1, \dots, p-1\}$, and we denote by $p^{\mathbb{N}}$ the (compact) Cantor space of all sequences taking values in p.

Notice that, when Λ is countable, the space $\mathcal{M}(\Lambda^{\mathbb{N}})$ does not depend on the topology of Λ, and a measure $m \in \mathcal{M}(\Lambda^{\mathbb{N}})$ is uniquely characterized by the values it takes on the cylindrical sets

$$E_{i_1 \dots i_r}(a_1, \dots, a_r) := \Big\{x \in \Lambda^{\mathbb{N}} : \; x_{i_1} = a_1, \dots, x_{i_r} = a_r\Big\}. \tag{2.1}$$

Given a topological space S and $k \in \mathbb{N}$, we let $S^{[k]}$ be the set of all subsets of S of cardinality k, endowed with the product topology. If S is ordered, we can identify $S^{[k]}$ with the set of k-tuples $(i_1, \dots, i_k)$, with $i_1 < \dots < i_k \in S$.

Given a map $\sigma : \mathbb{N} \to \mathbb{N}$, we let $T^\sigma : \Lambda^{\mathbb{N}} \to \Lambda^{\mathbb{N}}$ be defined as $T^\sigma(x)_i = x_{\sigma(i)}$, and we let $T^\sigma_\# : \mathcal{M}(\Lambda^{\mathbb{N}}) \to \mathcal{M}(\Lambda^{\mathbb{N}})$ be the corresponding pushforward map. In particular, when $\sigma(i) = i+1$, $s = T^\sigma$ is the so-called *shift map* on $\Lambda^{\mathbb{N}}$. Given a multi-index $\iota = (i_0, \dots, i_{r-1}) \in \mathbb{N}^{[r]}$, we let $T^\iota \colon \Lambda^{\mathbb{N}} \to \Lambda^r$ be such that $T^\iota(x)_k = x_{i_k}$ for all $k < r$, and we let $T^\iota_\# \colon \mathcal{M}(\Lambda^{\mathbb{N}}) \to \mathcal{M}(\Lambda^r)$ be the corresponding pushforward map. We also let $P_k : \Lambda^{\mathbb{N}} \to \Lambda$ be the projector on the kth coordinate, i.e. $P_k(x) = x_k$ for all $x \in \Lambda^{\mathbb{N}}$. We clearly have $P_{k+1} = P_1 \circ s^k$ for all $k \in \mathbb{N}$.

We say that $f \in L^1(\Lambda^{\mathbb{N}}, m)$ is invariant with respect to $\sigma : \mathbb{N} \to \mathbb{N}$ if $f = f \circ T^\sigma$ m-almost everywhere. We say that a measure $m \in \mathcal{M}(\Lambda^{\mathbb{N}})$ is invariant with respect to σ if $m = T^\sigma_\#(m)$.

We let $\mathfrak{S}_c(\mathbb{N}), \mathrm{Inj}(\mathbb{N}), \mathrm{Incr}(\mathbb{N}) \subset \mathbb{N}^{\mathbb{N}}$ be the families of maps $\sigma : \mathbb{N} \to \mathbb{N}$ which are compactly supported permutations, injective functions and strictly increasing functions, respectively.

We denote by $\overline{\mathbb{N}}$ the Alexandroff compactification of $\mathbb{N}$, equipped with a distance δ. For all $k \in \mathbb{N}$, a corresponding distance on the product space

$\overline{\mathbb{N}}^{[k]}$ can be defined as

$$\delta_k\left((j_1,\dots,j_k),(i_1,\dots,i_k)\right) := \max_{n\in\{1,\dots,k\}} \delta\left(j_n,i_n\right)$$

for all $(j_1,\dots,j_k),(i_1,\dots,i_k)\in\overline{\mathbb{N}}^{[k]}$.

Finally, given $k\in\mathbb{N}$ and $\sigma\in\mathrm{Incr}(\mathbb{N})$, we let $\sigma^*:\overline{\mathbb{N}}^{[k]}\to\overline{\mathbb{N}}^{[k]}$ be defined as $\sigma^*\left(i_1,\dots,i_k\right)=\left(\sigma(i_1),\dots,\sigma(i_k)\right)$, where we set $\sigma(\infty):=\infty$.

3. Problem 1

The following example shows that Problem 1 has in general a negative answer.

Example 3.1. Let $X=p^{\mathbb{N}}$, let $X_{ij}=A_{ij}=\{x\in p^{\mathbb{N}}: x_i>x_j\}$ for $i<j$, and let μ be the Bernoulli probability measure $B_{(1/p,\dots,1/p)}$. Then, the sets A_{ij} have all measure $(1-1/p)/2$ but the intersections of the form $\bigcap_{k=0}^{p} A_{i_k i_{k+1}}$, with $i_0<\ldots<i_p\in\mathbb{N}$, are necessarily empty. It follows that $\lambda_c\geq 1/2$, where λ_c is defined as in (1.1).

In Section 4, we show that Example 3.1 is optimal in the sense that, if $\mu(X_{ij})>(1-1/p)/2$, there exist monotone paths of length at least p, and there exist infinite paths if $\mu(X_{ij})\geq 1/2$.

For all $x\in X$, we consider the ordered graph $F(x)<\mathbb{N}^{[2]}$, whose edges are all the (i,j), with $i<j$, such that $x\in X_{ij}$. Let also $X_i\subseteq X$ be the subset of all $x\in X$ such that $F(x)$ contains an infinite path starting from i, i.e. there exists an increasing sequence $\{j_k\}_{k\in\mathbb{N}}$, with $j_1=i$ and $x\in\bigcap_k X_{j_k j_{k+1}}$.

Recall that a partially ordered set admits a decreasing function into the first uncountable ordinal ω_1 (the height function) if and only if it has no infinite increasing sequences. As a consequence, we can define a map $\varphi: X\times\mathbb{N}\to\omega_1+1$ by setting

$$\varphi(x,i)=\begin{cases}\displaystyle\sup_{j>i:\ x\in X_{ij}}\varphi(x,j)+1 & \text{if } x\notin X_i,\\[2ex] \omega_1 & \text{otherwise.}\end{cases}$$

We identify this map with the map $\varphi: X\to(\omega_1+1)^{\mathbb{N}}$ defined as $\varphi(x)_i=\varphi(x,i)$. We also set $\tilde\varphi: X\to\omega_1+1$ as $\tilde\varphi(x)=\sup_{i\in\mathbb{N}}\varphi(x,i)$. Notice that $\varphi(x,i)<\omega_1$ iff there is no infinite path in $F(x)$ starting from i, and in this case $\varphi(x,i)$ is precisely the height of i in $F(x)$. In particular, if F has no infinite paths, then the function φ takes value in $\omega_1^{\mathbb{N}}$ and, if there are no paths of length p, then it takes values in $p^{\mathbb{N}}$. On the other hand, path percolation occurs if and only if the set $\{x:\ \tilde\varphi(x)=\omega_1\}$ is non-empty. We also observe that the function φ can be equivalently defined by iteration as $\varphi(x,i)=\varphi_{\omega_1}(x,i)$, where

$$\begin{aligned}\varphi_\alpha(x,i) &= \sup_{\beta<\alpha,\ j>i:\, x\in X_{ij}}\varphi_\beta(x,j)+1\\ \varphi_0(x,i) &= 0\,,\end{aligned}\tag{3.1}$$

for all $i\in\mathbb{N}$ and $\alpha\leq\omega_1$.

From definition (3.1) it immediately follows that the sets $\{x:\varphi(x,k)=\alpha\}$ are measurable for all $k\in\mathbb{N}$ and $\alpha<\omega_1$. In Lemma 3.2 we show that the

set $\{x : \tilde{\varphi}(x) = \omega_1\} = \cup_i X_i$ of all x for which $F(x)$ contains an infinite path is also measurable.

We now show that $\tilde{\varphi}$ is always essentially bounded (even if not necessarily bounded everywhere) if F has no infinite paths.

Lemma 3.2. *The set $\{x \in X : \tilde{\varphi}(x) = \omega_1\}$ is measurable. Moreover, if F has no infinite paths, then $\tilde{\varphi} \in L^\infty(X, \mu)$.*

Proof. Let $\alpha_0 < \omega_1$ be such that

$$\mu\left(\{x \in X : \ \varphi(x,k) = \beta\}\right) = 0 \qquad \forall\, k \in \mathbb{N} \text{ and } \alpha_0 \le \beta < \omega_1 \,.$$

This is possible since the sequence of values $\mu\left(\{x : \ \varphi(x,k) \le \beta\}\right)$ is increasing and uniformly bounded by $\mu(X)$. Then, the space X can be decomposed as union of the three disjoint sets

$$\begin{aligned} X_1 &= \{x \in X : \ \tilde{\varphi}(x) < \alpha_0\} \\ X_2 &= \{x \in X : \ \alpha_0 \le \tilde{\varphi}(x) < \omega_1\} \subseteq \bigcup_{k\in\mathbb{N}} \{x \in X : \ \varphi(x,k) = \alpha_0\} \\ X_3 &= X \setminus (X_1 \cup X_2) = \{x \in X : \ \tilde{\varphi}(x) = \omega_1\} \,. \end{aligned}$$

The thesis follows observing that $\mu(X_2) = 0$ by the definition of α_0. □

As a consequence, if F has no infinite paths, then the function φ maps X (up to a set of zero measure) into the Cantor space $\alpha^{\mathbb{N}} \subset \omega_1^{\mathbb{N}}$ for some $\alpha < \omega_1$, so that it induces a Radon measure $m = \varphi_\#(\mu)$ on $\omega_1^{\mathbb{N}}$ concentrated on $\alpha^{\mathbb{N}}$, i.e. $m(\alpha^{\mathbb{N}}) = \mu(X)$. Moreover, $\varphi(X_{ij}) \subseteq A_{ij}$ for all $i < j \in \mathbb{N}$, where $A_{ij} := \{x \in \alpha^{\mathbb{N}} : x_i > x_j\}$ as in Example 3.1, so that $m(A_{ij}) \ge \mu(X_{ij})$ for all $i < j$. We denote by $\mathcal{M}_{\mathrm{c}}(\omega_1^{\mathbb{N}})$ the set of all Radon measures on $\omega_1^{\mathbb{N}}$ with compact support, i.e. with support in $\alpha^{\mathbb{N}}$ for some $\alpha < \omega_1$.

We now state a sufficient condition for path percolation.

Theorem 3.3. *Let $m \in \mathcal{M}_{\mathrm{c}}(\omega_1^{\mathbb{N}})$. Then*

$$\inf_{i<j\in\mathbb{N}} m(A_{ij}) < \frac{m\left(\omega_1^{\mathbb{N}}\right)}{2} \,. \tag{3.2}$$

In particular, path percolation occurs if

$$\lambda := \inf_{i<j\in\mathbb{N}} \mu(X_{ij}) \ge \frac{1}{2} \,. \tag{3.3}$$

Actually the same argument shows that we can replace the "$\inf_{i<j}$" (in both equations) with "$\limsup_{i\to\infty} \liminf_{j\to\infty}$".

Proof. With no loss of generality we can assume that $m \in \mathcal{M}_1(\omega_1^{\mathbb{N}})$, i.e. $m\left(\omega_1^{\mathbb{N}}\right) = 1$. We divide the proof into four steps.

Step 1. Letting $\partial\omega_1$ be the derived set of ω_1, that is the subset of all countable limit ordinals, we can assume that

$$m\left(\{x : \ x_i \in \partial\omega_1\}\right) = 0 \qquad \forall i \in \mathbb{N} \,.$$

Indeed, it is enough to observe that the left-hand side of (3.2) remains unchanged if we replace m with $s_\#(m)$, where $s : \omega_1 \to \omega_1 \setminus \partial\omega_1$ is the shift-map on ω_1, defined as $s(\alpha) = \alpha + 1$ for all $\alpha < \omega_1$.

Step 2. Since the support of m is contained in $\alpha_0^{\mathbb{N}}$, for some compact ordinal $\alpha_0 < \omega_1$, thanks to Proposition B.4 we can assume that m is asymptotically exchangeable, i.e. the sequence $m_k = (s_\#)^k(m)$ converges to an exchangeable measure $m' \in \mathcal{M}_1(\alpha_0^{\mathbb{N}})$ in the weak* topology.
Step 3. We shall prove by induction that for all $\alpha < \omega_1$ there holds

$$\inf_{i<j} m\left(\{x:\ x_j < x_i \le \alpha\}\right) \le m'\left(\{x:\ x_1 < x_0 \le \alpha\}\right). \tag{3.4}$$

Indeed, for $\alpha = 0$ we have $\{x:\ x_j < x_i \le 0\} = \emptyset$, and (3.4) holds.

As inductive step, let us assume that (3.4) holds for all $\alpha < \beta < \omega_1$, and we distinguish whether β is a limit ordinal or not. In the former case,

$$\bigcap_{\alpha<\beta} \{x:\ \alpha < x_i < \beta\} = \emptyset,$$

so that for all $\varepsilon > 0$ there exists $\alpha < \beta$ such that $m'\left(\{\alpha < x_i < \beta\}\right) < \varepsilon$. Moreover, by assumption $m'\left(\{x_i = \beta\}\right) = 0$ for any $i \in \mathbb{N}$, hence there exists $\alpha \le \alpha_i < \beta$ such that $m\left(\{\alpha_i \le x_i < \beta\}\right) < \varepsilon$. For all $i < j$ we have

$$\begin{aligned}\{x_j < x_i \le \beta\} \quad \subseteq \quad & \{x_j < x_i \le \alpha\} \cup \{x_j \le \alpha < x_i \le \beta\} \\ & \cup \{\alpha < x_j \le \alpha_i\} \cup \{\alpha_i < x_i \le \beta\},\end{aligned}$$

which gives

$$\begin{aligned}m\left(\{x_j < x_i \le \beta\}\right) \quad \le \quad & m\left(\{x_j < x_i \le \alpha\}\right) + m\left(\{x_j \le \alpha < x_i \le \beta\}\right) \\ & + m\left(\{\alpha < x_j \le \alpha_i\}\right) + m\left(\{\alpha_i < x_i \le \beta\}\right).\end{aligned}$$

By induction hypothesis we know that

$$\inf_{i<j} m\left(\{x_j < x_i \le \alpha\}\right) \le m'\left(\{x_1 < x_0 \le \alpha\}\right),$$

and, since m is asymptotically exchangeable, we have

$$m\left(\{x_j \le \alpha < x_i \le \beta\}\right) = m'\left(\{x_1 \le \alpha < x_0 \le \beta\}\right) + o(1),$$

and

$$m\left(\{\alpha < x_j \le \alpha_i\}\right) = m'\left(\{\alpha < x_1 \le \alpha_i\}\right) + o(1),$$

as $(i,j) \to +\infty$, where we used the fact that the sets $\{x_j \le \alpha < x_i \le \beta\}$ and $\{\alpha < x_j \le \alpha_i\}$ are both clopen. Therefore, we get

$$\begin{aligned}\inf_{i<j} m\left(\{x_j < x_i \le \beta\}\right) \quad \le \quad & m'\left(\{x_1 < x_0 \le \alpha\}\right) + m'\left(\{x_1 \le \alpha < x_0 \le \beta\}\right) \\ & + m'\left(\{\alpha < x_1 \le \beta\}\right) + \varepsilon + o(1) \\ \le \quad & m'\left(\{x_1 < x_0 \le \beta\}\right) + 2\varepsilon + o(1),\end{aligned}$$

so that the inequality (3.4) holds true with $\alpha = \beta$, when β is a limit ordinal.

On the other hand, if $\beta = \alpha + 1$, for $(i,j) \to +\infty$ we have

$$\begin{aligned}m\left(\{x_j < x_i \le \beta\}\right) \quad &= \quad m\left(\{x_j < x_i \le \alpha\}\right) + m\left(\{x_j \le \alpha,\ x_i = \beta\}\right) \\ &= \quad m'\left(\{x_1 < x_0 \le \alpha\}\right) + m'\left(\{x_1 \le \alpha,\ x_0 = \beta\}\right) + o(1) \\ &= \quad m'\left(\{x_1 < x_0 \le \beta\}\right) + o(1),\end{aligned}$$

where we used again the induction hypothesis, and the fact that the set $\{x_j \le \alpha,\ x_i = \beta\}$ is clopen.

Inequality (3.4) is then proved for all $\alpha < \omega_1$.

Step 4. We now conclude the proof of the theorem. Since the measure m' is exchangeable, from (3.4) it follows

$$\inf_{i<j} m\left(A_{ij}\right) \leq m'\left(\{x:\ x_1 < x_0\}\right) = \frac{1}{2}\left(1 - m'\left(\{x:\ x_1 = x_0\}\right)\right) \leq \frac{1}{2}\,. \tag{3.5}$$

Moreover, from (B.5) and the fact that $\Lambda = \alpha_0$ is countable, it follows that $m'\left(\{x:\ x_1 = x_0\}\right) = 0$ iff $m' = 0$, so that the strict inequality holds in (3.5). □

Combining Example 3.1 with Theorem 3.3 we obtain a complete solution to Problem 1 in terms of $\mu(X_{ij})$: assume that $\mu(X_{ij}) \geq \lambda$ for all $i < j$, then path percolation occurs if $\lambda \geq 1/2$, on the contrary if $\lambda < 1/2$ there are random subgraphs F of $(\mathbb{N}, \mathbb{N}^{[2]})$ with no infinite paths.

The next result provides a sharp lower bound on the probability of path percolation.

Corollary 3.4. *Assume that the sets X_{ij} are such that $\mu(X_{ij}) \geq \lambda \geq 1/2$ for all $(i,j) \in \mathbb{N}^{[2]}$. Let P_λ be the set of all $x \in X$ such that $F(x)$ contains an infinite path. Then $\mu(P_\lambda) > 2\lambda - 1$.*

Proof. Let $\tilde{\varphi} : X \to \omega_1 + 1$ as above, so that $\tilde{\varphi}(x) = \omega_1$ iff $x \in P_\lambda$, and let $m = \left(\tilde{\varphi}|_{X\setminus P_\lambda}\right)_{\#}(\mu) \in \mathcal{M}_{\mathrm{c}}(\omega_1^{\mathbb{N}})$. By Theorem 3.3 we then have

$$\lambda - \mu(P_\lambda) \leq \inf_{i<j\in\mathbb{N}} \mu\left(X_{ij} \cap (X \setminus P_\lambda)\right) = \inf_{i<j\in\mathbb{N}} m(A_{ij}) < \frac{1-\mu(P_\lambda)}{2}\,,$$

which gives $\mu(P_\lambda) > 2\lambda - 1$. □

4. Extensions and related problems

4.1. **A notion of capacity for directed graphs.** A *directed graph* F is a couple of sets (V_F, E_F), which are respectively the set of vertices and the set of edges of F, such that E_F is a subset of $V_F \times V_F$. We denote by $\mathcal{G}$ the class of all directed graphs $F = (V_F, E_F)$. Notice that it is possible that both the edges (a,b) and (b,a) belong to F. Given $F \in \mathcal{G}$, we let the *clique number* $\mathrm{cl}(F)$ of F be the maximum $n \in \overline{\mathbb{N}}$ such that F has a complete subgraph of cardinality n.

For all $F \in \mathcal{G}$, we define the *capacity* of F as

$$c_0(F) := \sup_{\lambda\in\Sigma_F} \sum_{(a,b)\in E_F} \lambda_a\lambda_b \quad \in [0,1]\,, \tag{4.1}$$

where Σ_F is the symplex of all sequences $\{\lambda_a\}_{a\in V_F}$ such that $\lambda_a \geq 0$ and $\sum_{a\in V_F}\lambda_a = 1$. Notice that the capacity is an invariant for directed graphs, up to isomorphism, and it is equal to 1 if F contains an arcloop.

Given two graphs $F, G \in \mathcal{G}$ we write $G < F$ to indicate that G is a subgraph of F, i.e. $V_G = V_F$ and $E_G \subseteq E_F$. More generally, we say that G maps into F, and we write $G \to F$, if there is a map $\varphi : V_G \to V_F$ such that $(\varphi(a), \varphi(b)) \in E_F$ for all $(a,b) \in E_G$. Notice that

$$c_0(G) \leq c_0(F) \qquad \text{for all} \quad F, G \in \mathcal{G} \quad \text{such that} \quad G \to F\,. \tag{4.2}$$

The following result shows that the capacity reduces to the clique number, for suitable finite graphs.

Proposition 4.1. *Let $F \in \mathcal{G}$ be a finite graph. If F is oriented, that is*

$$(a,b) \in E_F \quad \Rightarrow \quad (b,a) \notin E_F \quad \forall a,b \in V_F\,,$$

then

$$c_0(F) = \frac{1}{2}\left(1 - \frac{1}{\mathrm{cl}(F)}\right). \tag{4.3}$$

If F is symmetric with no arcloops, that is

$$(a,a) \notin E_F \quad \text{and} \quad (a,b) \in E_F \quad \Rightarrow \quad (b,a) \in E_F \quad \forall a,b \in V_F\,,$$

then

$$c_0(F) = 1 - \frac{1}{\mathrm{cl}(F)}. \tag{4.4}$$

Proof. Let F be a finite oriented graph, and let $\lambda \in \Sigma_F$ be a maximizing distribution, meaning that $c_0(F) = \sum_{(a,b)\in E_F} \lambda_a \lambda_b$, and let S_λ be the subgraph of F spanned by the support of λ, that is $V_{S_\lambda} = \{a \in V_F : \lambda_a > 0\}$. From Lagrange's multiplier Theorem it follows that, for all $a \in V_{S_\lambda}$, we have

$$\sum_{b\in V_F:\,(a,b)\in E_F \text{ or } (b,a)\in E_F} \lambda_b = 2\,c_0(F). \tag{4.5}$$

If $a, a' \in V_{S_\lambda}$, we can consider the distribution $\lambda' \in \Sigma_F$ such that $\lambda'_a = 0$, $\lambda'_{a'} = \lambda_a + \lambda_{a'}$, and $\lambda'_b = \lambda_b$ for all $b \in V_F \setminus \{a, a'\}$. From (4.5) it then follows that λ' is also a maximizing distribution whenever a and a' are independent vertices, that is neither (a, a') nor (a', a) belong to E_F.

As a first consequence, S_λ is a clique whenever λ has minimal support. Indeed, let K be a maximal clique contained in S_λ, and assume by contradiction that there exists $a \in V_{S_\lambda} \setminus V_K$. Letting $a' \in V_K$ be a vertex of F independent of a (such a' exists since K is a maximal clique), and letting $\lambda' \in \Sigma_F$ as above, we have $c_0(F) = \sum_{(a,b)\in E_F} \lambda'_a \lambda'_b$, contradicting the minimality of V_{S_λ}.

Once we know that S_λ is a clique, again from (4.5) we get that λ is a uniform ditribution, that is $\lambda_a = \lambda_b$, for all $a, b \in V_{S_\lambda}$. It follows

$$c_0(F) = \frac{1}{2}\left(1 - \frac{1}{|S_\lambda|}\right) \le \frac{1}{2}\left(1 - \frac{1}{\mathrm{cl}(F)}\right),$$

which in turn implies (4.4), the opposite inequality being realized by a uniform distribution on a maximal clique.

The case of a symmetric graph follows immediately from the oriented case. □

Notice that a finite graph F is oriented if and only if $c_0(F) < 1/2$. Notice also that, if F is a finite directed graph (not necessarily oriented) the proof of Proposition 4.1 shows that there exists a maximizing $\lambda \in \Sigma_F$ whose support is a clique (not necessarily of maximal order).

Let us denote by F^G the Cantor space of all functions $u : V_G \to V_F$, endowed with the product topology induced by the discrete topology on V_F.

Given two oriented graphs F, G, we can define the *relative capacity* of F with respect to G as

(4.6)
$$c(F,G) := \sup_{m\in\mathcal{M}_1(F^G)} \inf_{(a,b)\in E_G} m\left(\{u \in F^G : (u(a),u(b)) \in E_F\}\right) \in [0,1].$$

The relative capacity is in general quite difficult to compute, but it reduces to the previous notion of capacity when V_F is finite and $G = (\mathbb{N}, \mathbb{N}^{[2]})$.

Proposition 4.2. *For all $F \in \mathcal{G}$ such that $|V_F| < \infty$, it holds*

$$c\left(F, (\mathbb{N}, \mathbb{N}^{[2]})\right) = c_0(F)\,. \tag{4.7}$$

Proof. Reasoning as in the proof of Theorem 3.3, from Proposition B.4 it follows that we can equivalently take the supremum in (4.6) among the measures $m \in \mathcal{M}_1(V_F^{\mathbb{N}})$ which are exchangeable. Moreover, recalling (B.6), every exchangeable measure is a convex integral combination of Bernoulli measures B_λ, with $\lambda \in \Sigma_F$. It follows that it is sufficient to compute the supremum on the Bernoulli measures, so that (4.6) reduces to (4.1). □

Given $F, G \in \mathcal{G}$, let us now consider a random subgraph of G, that is we associate to each $(a,b) \in E_G$ a measurable set $X_{ab} \subset X$, with $\mu(X_{ab}) \geq \lambda$ for some $\lambda \in [0,1]$. In the same spirit of Problem 1, we then ask which is the probability that the random graph *does not* map into F.

As above, for all $x \in X$ we let $F(x) < G$ be such that

$$E_{F(x)} = \{(a,b) \in E_G : x \in X_{ab}\}$$

and we let

$$P_\lambda := \{x \in X : F(x) \not\to F\}\,.$$

Proposition 4.3. *Let $F, G \in \mathcal{G}$, with $G < (\mathbb{N}, \mathbb{N}^{[2]})$. If $c(F,G) < 1$, there holds*

$$\mu\left(P_\lambda\right) \geq p(\lambda) = \frac{\lambda - c(F,G)}{1 - c(F,G)}\,. \tag{4.8}$$

Proof. We proceed as in the first part of Section 3. Letting $\widetilde{X} := X \setminus P_\lambda$, for all $x \in \widetilde{X}$ we have $F(x) \to F$, where the map is realized by a function from $V_{F(x)} = \mathbb{N}$ to V_F, which in turn defines a map $\varphi : \widetilde{X} \to F^G$. Let now

$$m := \frac{1}{\mu\left(\widetilde{X}\right)} \varphi_\#(\mu) \in \mathcal{M}_1\left(F^G\right)\,.$$

Notice that

$$\varphi\left(X_{ab} \cap \widetilde{X}\right) \subseteq \left\{u \in F^G : (u(a),u(b)) \in E_F\right\}$$

for all $(a,b) \in E_G$, so that

$$\frac{\mu\left(X_{ab} \cap \widetilde{X}\right)}{\mu\left(\widetilde{X}\right)} \leq m\left(\left\{u \in F^G : (u(a),u(b)) \in E_F\right\}\right) \leq c(F,G)\,. \tag{4.9}$$

The thesis now follows from (4.9) and the inequality

$$\frac{\mu\left(X_{ab}\cap\widetilde{X}\right)}{\mu\left(\widetilde{X}\right)}\geq\frac{\lambda-\mu\left(P_\lambda\right)}{1-\mu\left(P_\lambda\right)}\,.$$

□

4.2. **Finite monotone paths and chromatic number.** For all $p\in\mathbb{N}$, we shall consider the graphs $(p,p^{[2]})$ and Q_p, where

$$Q_p=\left(V_{Q_p},E_{Q_p}\right)\qquad\text{with}\qquad V_{Q_p}=p,\quad E_{Q_p}=\{(i,j)\in p\times p:\,i\neq j\}\,.$$

A direct computation as in the proof of Proposition 4.1 gives

$$c_0\left((p,p^{[2]})\right)=\frac{1}{2}\left(1-\frac{1}{p}\right),\qquad c_0(Q_p)=1-\frac{1}{p}\,. \tag{4.10}$$

Notice that $G\to Q_p$ iff $\chi(G)\leq p$, where $\chi(G)$ is the chromatic number of G [B:79], and $G\to(p,p^{[2]})$ iff G does not contain a path of length p. Indeed, the first assertion is equivalent to the definition of chromatic number, whereas the second follows by associating to each vertex $v\in V_G$ the number $(p-1)-d(v)\in p$, where $d(v)$ is the maximal length of a path in G starting from v.

By Propositions 4.2 and 4.3 with $G=(\mathbb{N},\mathbb{N}^{[2]})$, for all $\lambda\in[0,1]$ we have

$$\mu\left(P_\lambda\right)\geq\frac{\lambda-c_0(F)}{1-c_0(F)}\,. \tag{4.11}$$

When $F=(p,p^{[2]})$, then $x\in P_\lambda$ iff $F(x)\not\to(p,p^{[2]})$, i.e. $F(x)$ contains a path of length p, and from (4.10) and (4.11) it follows

$$\mu\left(P_\lambda\right)\geq\frac{2p\lambda-p+1}{p+1}\,.$$

Example 3.1 shows that such estimate is optimal, so that

$$p(\lambda):=\inf\{\mu(P_\lambda):\,(X,\mu)\text{ probability space}\}=\frac{2p\lambda-p+1}{p+1}\,.$$

In particular, if $\lambda>\lambda_c=(1-1/p)/2$, then the random subgraph $F(x)$ contains a path of length p with probability at least $p(\lambda)>0$.

Remark 4.4. Notice that for all $p\in\mathbb{N}$ and $G\in\mathcal{G}$ the following equivalent statements hold:

$$G\text{ contains a path of length }p\quad\Leftrightarrow\quad C_p\to G\quad\Leftrightarrow\quad G\not\to(p,p^{[2]})\,,$$

where $C_p<(p,p^{[2]})$ is such that $(i,j)\in E_{C_p}$ iff $j=i+1$. In particular, one may consider $(p,p^{[2]})$ as *dual* of the graph C_p with respect to graph mapping, so that it naturally arises the question of which graphs, other than C_p, admit such dual representation.

When $F=Q_p$, then $x\in P_\lambda$ iff $\chi(F(x))>p$, and we have

$$\mu\left(P_\lambda\right)\geq p\lambda-p+1=p(\lambda)\,.$$

Example 5.2 shows that also this estimate is optimal. As a consequence, if $\lambda>\lambda_c=1-1/p$, then the random subgraph $F(x)$ has chromatic number strictly greater than p with probability at least $p(\lambda)>0$.

5. Problem 2

We recall the following standard Borel-Cantelli type result.

Proposition 5.1. *Let $k = 1$ and let $X_i \subseteq X$ be such that $\mu(X_i) \geq \lambda$, for all $i \in \mathbb{N}$ and for some $\lambda > 0$. Then, Problem 2 has a positive answer, i.e. there is an infinite set $J \subset \mathbb{N}$ such that*

$$\bigcap_{i \in J} X_i \neq \emptyset.$$

Proof. The set $Y := \bigcap_n \bigcup_{i>n} X_i$ is a decreasing intersection of sets of (finite) measure greater than $\lambda > 0$, hence $\mu(Y) \geq \lambda$ and, in particular, Y is non-empty. Now it suffices to note that any element x of Y belongs to infinitely many X_i's. □

Proposition 5.1 has the following interpretation in terms of percolation: if we choose each element of $\mathbb{N}$ with probability greater or equal to λ, we obtain an infinite random subset with probability grater or equal to $p(\lambda) = \lambda$ (we recall that $p(\lambda)$ is always less than or equal to λ).

The following example shows that Problem 2 has in general a negative answer for $k > 1$.

Example 5.2. Let $p \in \mathbb{N}$ and consider the Cantor space $X = p^{\mathbb{N}}$, equipped with the Bernulli measure $B_{(1/p,\dots,1/p)}$, and let $X_{ij} := \{x \in X : x_i \neq x_j\}$. Then each X_{ij} has measure $\lambda = 1 - 1/p$, and for all $x \in X$ the graph $F(x) := \{(i,j) \in \mathbb{N}^{[2]} : x \in X_{ij}\}$ does not contains cliques (i.e. complete subgraphs) of cardinality $(p+1)$.

In view of Example 5.2, we need to impose further restrictions on the sets $X_{i_1 \dots i_k}$, in order to get a positive answer to Problem 2. In the following, we shall always assume that

$$\mu(X_{i_1 \dots i_k}) \geq \lambda \qquad \forall (i_1, \dots, i_k) \in \mathbb{N}^{[k]}, \tag{5.1}$$

for some $\lambda > 0$.

Notice that, if each set $X_{i_1 \dots i_k}$ has the form $X_{i_1} \cap \dots \cap X_{i_k}$ and satisfies (5.1), then Problem 2 has a positive answer by Proposition 5.1. Moreover, by Ramsey theorem, Problem 2 has a positive answer if there is a finite set $S \subset X$ such that each $X_{i_1,\dots,i_k}$ has a non-empty intersection with S. In particular, this is the case if X is a countable set and (5.1) holds.

Proposition 5.3. *Let X be a compact metric space and assume that each set $X_{i_1 \dots i_k}$ contains a ball $B_{i_1,\dots,i_k}$ of radius $r > 0$. Then Problem 2 has a positive answer.*

Proof. Applying Lemma A.1 to the centers of the balls $B_{i_1,\dots,i_k}$ it follows that for all $0 < r' < r$ there exists an infinite set J and a ball B of radius r' such that

$$B \subset \bigcap_{(j_1,\dots,j_k) \in J^{[k]}} X_{j_1 \dots j_k}.$$

□

We now give a sufficient condition for a positive answer to Problem 2.

Theorem 5.4. *Assume that the sets $X_{i_1\dots i_k}$ satisfy* (5.1)*, and the indicator functions of $X_{i_1\dots i_k}$ belong to a compact subset $\mathcal{K}$ of $L^1(X,\mu)$. Then, for any $\varepsilon > 0$ there exists an infinite set $J \subset \mathbb{N}$ such that*

$$\mu\left(\bigcap_{(i_1,\dots,i_k)\in J^{[k]}} X_{i_1\dots i_k}\right) \geq \lambda - \varepsilon.$$

Proof. Consider first the case $k = 1$. By compactness of $\mathcal{K}$, for all $\varepsilon > 0$ there exist an increasing sequence $\{i_n\}$ and a set $X_\infty \subset X$, with $\mu(X_\infty) \geq \lambda$, such that

$$\mu\left(X_\infty \Delta X_{i_n}\right) \leq \frac{\varepsilon}{2^n} \qquad \forall n \in \mathbb{N}\,.$$

As a consequence, letting $J := \{i_n : n \in \mathbb{N}\}$ we have

$$\mu\left(\bigcap_{n\in\mathbb{N}} X_{i_n}\right) \geq \mu\left(X_\infty \cap \bigcap_{n\in\mathbb{N}} X_{i_n}\right) \geq \mu\left(X_\infty\right) - \sum_{n\in\mathbb{N}} \mu\left(X_\infty \Delta X_{i_n}\right) \geq \lambda - \varepsilon.$$

For $k > 1$, we apply Lemma A.1 with

$$\begin{aligned} M &= \mathcal{K} \subset L^1(X) \\ f(i_1,\dots,i_k) &= \chi_{X_{i_1\dots i_k}} \in L^1(X). \end{aligned}$$

In particular, recalling Remark A.2, for all $\varepsilon > 0$ there exist $J = \sigma(\mathbb{N})$, $X_\infty \subset X$, and $X_{i_1\dots i_m} \subset X$, for all $(i_1,\dots,i_m) \in J^{[m]}$ with $1 \leq m < k$, such that $\mu(X_\infty) \geq \lambda$ and for all $(i_1,\dots,i_k) \in J^{[k]}$ it holds

$$\begin{aligned} \mu\left(X_\infty \Delta X_{i_1}\right) &\leq \frac{\varepsilon}{2^{\sigma^{-1}(i_1)}} \\ \mu\left(X_{i_1\dots i_m} \Delta X_{i_1\dots i_{m+1}}\right) &\leq \frac{\varepsilon}{2^{\sigma^{-1}(i_{m+1})}}\,. \end{aligned}$$

Reasoning as above, it then follows

$$\begin{aligned} &\mu\left(X_\infty \Delta \bigcap_{(i_1,\dots,i_k)\in J^{[k]}} X_{i_1\dots i_k}\right) \leq \\ &\sum_{i_1\in\mathbb{N}} \mu\left(X_\infty \Delta X_{i_1}\right) + \sum_{i_1<i_2} \mu\left(X_{i_1} \Delta X_{i_1 i_2}\right) + \\ &\cdots + \sum_{i_1<\dots<i_k} \mu\left(X_{i_1\dots i_{k-1}} \Delta X_{i_1\dots i_k}\right) \leq C(k)\varepsilon\,, \end{aligned}$$

where $C(k) > 0$ is a constant depending only on k. Therefore

$$\begin{aligned} \mu\left(\bigcap_{(i_1,\dots,i_k)\in J^{[k]}} X_{i_1\dots i_k}\right) &\geq \mu\left(X_\infty \cap \bigcap_{(i_1,\dots,i_k)\in J^{[k]}} X_{i_1\dots i_k}\right) \\ &\geq \mu\left(X_\infty\right) - \mu\left(X_\infty \Delta \bigcap_{(i_1,\dots,i_k)\in J^{[k]}} X_{i_1\dots i_k}\right) \\ &\geq \lambda - C(k)\varepsilon. \end{aligned}$$

□

Notice that from Theorem 5.4 it follows that Problem 2 has a positive answer if there exist an infinite $J \subseteq \mathbb{N}$ and sets $\widetilde{X}_{i_1\dots i_k} \subseteq X_{i_1\dots i_k}$ with $(i_1,\dots,i_k) \in J^{[k]}$, such that $\mu\left(\widetilde{X}_{i_1\dots i_k}\right) \geq \lambda$ for some $\lambda > 0$, and the indicator functions of $\widetilde{X}_{i_1\dots i_k}$ belong to a compact subset of $L^1(X)$.

Remark 5.5. We recall that, when X is a compact subset of $\mathbb{R}^n$ and the perimeters of the sets $X_{i_1\dots i_k}$ are uniformly bounded, then the family $\chi_{X_{i_1\dots i_k}}$ has compact closure in $L^1(X)$ (see for instance [AFP:00, Thm. 3.23]). In particular, if the sets $X_{i_1\dots i_k}$ have equibounded Cheeger constant, i.e. if there exists $C > 0$ such that

$$\min_{E\subset X_{i_1\dots i_k}} \frac{\mathrm{Per}(E)}{|E|} \leq C \qquad \forall (i_1,\dots,i_k) \in \mathbb{N}^{[k]},$$

then Problem 2 has a positive answer.

Appendix A. A topological Ramsey theorem

We prove the following topological lemma, which is a generalization of the well-known Ramsey theorem [R:28] (see also [C:74]).

Lemma A.1. *Let M be a compact metric space, let $k \in \mathbb{N}$, and let $f : \mathbb{N}^{[k]} \to M$. Then, for any distance δ on $\overline{\mathbb{N}}$ there exists $\sigma \in \mathrm{Incr}(\mathbb{N})$ such that $f \circ \sigma^* : \mathbb{N}^{[k]} \to M$ is 1-Lipschitz. As a consequence, it can be extended to a 1-Lipschitz function on the whole of $\overline{\mathbb{N}}^{[k]}$.*

Proof. We proceed by induction on k. When $k = 1$, by compactness of M there exist $x \in M$ and a subsequence $f \circ \sigma$ of f converging to x with the property

$$\begin{aligned} d_M\left(f(\sigma(n)), x\right) &\qquad \text{is decreasing in } n \\ d_M\left(f(\sigma(n)), x\right) &\leq \frac{1}{2}\inf_{m>n} \delta(n,m) \quad \forall n \in \mathbb{N}\,. \end{aligned}$$

For all $n \leq m$, it then follows

$$d_M\left(f(\sigma(n)), f(\sigma(m))\right) \leq 2 d_M\left(f(\sigma(n)), x\right) \leq \delta(n,m)\,.$$

Assuming that the thesis is true for some $k \in \mathbb{N}$, we now prove that it is true also for $k+1$. By inductive assumption, for all $j \in \mathbb{N}$ there exists $\sigma_j \in \mathrm{Incr}(\mathbb{N})$ such that $f(j, \sigma_j^*(\cdot))$ is 1-Lipschitz on $\mathbb{N}^{[k]}$. (This makes sense if σ_j has values bigger than j, and it is easy to see that we can choose it in this way.)

By a recursive construction we can also assume that $J_{j+1} \subseteq J_j$, where we set $J_j := \sigma_j(\mathbb{N})$. Let $x_j \in M$ be the limit of $f(j, \sigma_j^*(\iota))$ for $\min(\iota) \to \infty$. By compactness of M there is $x_\infty \in M$ and an increasing function $\sigma\colon \mathbb{N} \to \mathbb{N}$ such that $x_{\sigma(n)}$ converges to x_∞. Moreover, we can choose σ such that

$\sigma(n+1) \in J_{\sigma(n)}$ and the following holds:

$$\begin{aligned} d_M\left(x_{\sigma(n)}, x_\infty\right) & \quad \text{is decreasing in } n \\ d_M\left(x_{\sigma(n)}, x_\infty\right) & \leq \frac{1}{4} \inf_{m>n} \delta(n,m) \\ \sup_{\iota \in \mathbb{N}^{[k]},\, \min(\iota)>n} d_M\left(f\left(\sigma^*(n,\iota)\right), x_{\sigma(n)}\right) & \quad \text{is decreasing in } n \\ \sup_{\iota \in \mathbb{N}^{[k]},\, \min(\iota)>n} d_M\left(f\left(\sigma^*(n,\iota)\right), x_{\sigma(n)}\right) & \leq \frac{1}{4} \inf_{m>n} \delta(n,m)\,. \end{aligned}$$

Note that the last two properties follows from the fact that $f(\sigma^*(n,\iota))$ can be made arbitrarily close to its limit $x_{\sigma(n)}$ provided σ maps the integers $> n$ into sufficiently large elements of $J_{\sigma(n)}$.

For all $(n,\iota), (m,\kappa) \in \mathbb{N}^{[k+1]}$, with $n \leq m$, it then follows

$$d_M\left(f\left(\sigma^*(n,\iota)\right), f\left(\sigma^*(m,\kappa)\right)\right) \leq \delta_k(\iota,\kappa) \quad \text{if} \quad n = m\,,$$

by inductive assumption, while for $n < m$ we have

$$\begin{aligned} d_M\left(f\left(\sigma^*(n,\iota)\right), f\left(\sigma^*(m,\kappa)\right)\right) & \leq \sup_{\iota \in \mathbb{N}^{[k]}} d_M\left(f\left(\sigma^*(n,\iota)\right), x_{\sigma(n)}\right) \\ & \quad + d_M\left(x_{\sigma(n)}, x_\infty\right) + d_M\left(x_{\sigma(m)}, x_\infty\right) \\ & \quad + \sup_{\kappa \in \mathbb{N}^{[k]}} d_M\left(f\left(\sigma^*(m,\kappa)\right), x_{\sigma(m)}\right) \\ & \leq \delta(n,m) \leq \delta_{k+1}\left((n,\iota),(m,\kappa)\right)\,, \end{aligned}$$

that is, $f \circ \sigma^*$ is 1-Lipschitz on $\mathbb{N}^{[k+1]}$. □

Lemma A.1 is a sort of asymptotic Ramsey theorem with colours in a compact metric space, and reduces to the classical Ramsey theorem when the space M is finite.

Remark A.2. Notice also that Lemma A.1 implies that there exists an infinite set $J = \sigma(\mathbb{N}) \subset \mathbb{N}$ such that, for all $0 \leq m < k$ and $(i_1, \ldots, i_m) \in J^{[m]}$, there are limit points $x_{i_1 \ldots i_m} \in M$ with the property

$$x_{i_1 \ldots i_m} = \lim_{\substack{(i_{m+1},\ldots,i_k) \to \infty \\ (i_1 \ldots i_k) \in J^{[k]}}} x_{i_1 \ldots i_k},$$

where we set $x_{i_1 \ldots i_k} := f\left(i_1, \ldots, i_k\right)$. Moreover, by choosing the distance $\delta(n,m) = \varepsilon |2^{-n} - 2^{-m}|$, we may also require

$$d_M\left(x_{i_1 \ldots i_m}, x_{i_1 \ldots i_k}\right) \leq \frac{\varepsilon}{2^{\sigma^{-1}(i_{m+1})}} \qquad \forall (i_1, \ldots, i_k) \in J^{[k]}.$$

Appendix B. A few facts on exchangeable measures

Let Λ be a compact metric space and let $\Lambda^{\mathbb{N}}$ be the space of all sequences $u : \mathbb{N} \to \Lambda$ endowed with the product topology. Given $m \in \mathcal{M}(\Lambda^{\mathbb{N}})$ and $f \in L^p(\Lambda^{\mathbb{N}})$, with $p \in [1, +\infty]$, we let

$$\tilde{f} = E\left(f | \mathcal{A}_s\right) \in L^p(\Lambda^{\mathbb{N}})$$

be the conditional probability of f with respect to the σ-algebra $\mathcal{A}_s$ of the shift-invariant Borel subsets of $\Lambda^{\mathbb{N}}$. In particular, $\tilde{f}$ is shif-invariant, and by Birkhoff's theorem (see f.e. [P:82]) we have

$$\tilde{f} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f \circ s^k \,,$$

where the limit holds almost everywhere and in the strong topology of $L^1(\Lambda^{\mathbb{N}})$.

We now recall a classical notion of *exchangeable measure* due to De Finetti [DF:74], showing some equivalent conditions.

Proposition B.1. *Given $m \in \mathcal{M}_1(\Lambda^{\mathbb{N}})$, the following conditions are equivalent:*

a) *m is $\mathfrak{S}_c(\mathbb{N})$-invariant;*
b) *m is $\mathrm{Inj}(\mathbb{N})$-invariant;*
c) *m is $\mathrm{Incr}(\mathbb{N})$-invariant.*

If m satisfies one of these equivalent conditions we say that m is *exchangeable*. Notice that an exchangeable measure is always shift-invariant, while there are shift-invariant measures which are not exchangeable.

Proof. Since $\mathfrak{S}_c(\mathbb{N}) \subset \mathrm{Inj}(\mathbb{N})$ and $\mathrm{Incr}(\mathbb{N}) \subset \mathrm{Inj}(\mathbb{N})$, the implications b) $\Rightarrow$ a) and b) $\Rightarrow$ c) are obvious.

The implication a) $\Rightarrow$ b) is also obvious since it is trivially true on the cylindrical sets (2.1), which generate the whole Borel σ-algebra of $\Lambda^{\mathbb{N}}$.

Let us prove that c) $\Rightarrow$ b). We first show that, if c) holds, then for all $f \in L^\infty(\Lambda^{\mathbb{N}})$ it holds

$$\tilde{f} = \lim_{n\to\infty} f \circ s^n \,, \tag{B.1}$$

where the limit is taken in the weak* topology of $L^\infty(\Lambda^{\mathbb{N}})$. Indeed, since the sequence $f \circ s^n$ is bounded in $L^\infty(\Lambda^{\mathbb{N}})$, it is enough to prove the convergence of

$$\int_{\Lambda^{\mathbb{N}}} \big(f \circ s^n\big)\, g\, dm \tag{B.2}$$

for all g in a dense subset D of $L^1(\Lambda^{\mathbb{N}})$. Letting

$$\begin{aligned} D \;=\; & \big\{ g \in L^\infty(\Lambda^{\mathbb{N}}) : \, g(x) = g_1(x_1) \cdots g_r(x_r) \\ & \text{for some } r \in \mathbb{N} \text{ and } g_1, \dots, g_r \in L^\infty(\Lambda) \big\}, \end{aligned}$$

the convergence of (B.2) follows at once from the fact that m is $\mathrm{Incr}(\mathbb{N})$-invariant and $g \in D$, which implies that the quantity in (B.2) is constant for all $n > r$. To conclude the proof, it remains to show that

$$\int_{\Lambda^{\mathbb{N}}} g\, dm = \int_{\Lambda^{\mathbb{N}}} g \circ T^\sigma\, dm \,, \tag{B.3}$$

for all $g \in D$ and $\sigma \in \mathrm{Inj}(\mathbb{N})$. Notice that, by assumption, the right-hand side of (B.3) does not depend on σ as long as $\sigma \in \mathrm{Incr}(\mathbb{N})$, in particular

$$\int_{\Lambda^{\mathbb{N}}} g\, dm = \int_{\Lambda^{\mathbb{N}}} (g_1 \circ P_{i_1}) \cdots (g_r \circ P_{i_r})\, dm \,,$$

for all $(i_1, \dots, i_r) \in \mathbb{N}^{[r]}$. Recalling (B.1) and passing to the limit as $i_r \to +\infty, \dots, i_1 \to +\infty$, we then obtain

$$\int_{\Lambda^{\mathbb{N}}} g \, dm = \int_{\Lambda^{\mathbb{N}}} \widetilde{g_1 \circ P_1} \cdots \widetilde{g_r \circ P_1} \, dm \, .$$

Reasoning in the same way for the function $g \circ T^\sigma$, we finally get

$$\int_{\Lambda^{\mathbb{N}}} g \circ T^\sigma \, dm = \int_{\Lambda^{\mathbb{N}}} \widetilde{g_1 \circ P_1} \cdots \widetilde{g_r \circ P_1} \, dm = \int_{\Lambda^{\mathbb{N}}} g \, dm \, .$$

□

Remark B.2. If Λ is countable, a measure m is exchangeable iff for all $r \in \mathbb{N}$ there exists a symmetric function $f : \Lambda^r \to \mathbb{R}$ such that for all $(i_1, \dots, i_r) \in \mathbb{N}^{[r]}$ and $(a_1, \dots, a_r) \in \Lambda^r$ it holds

$$m \left(E_{i_1 \dots i_r}(a_1, \dots, a_r) \right) = f \left(a_1, \dots, a_r \right) . \tag{B.4}$$

In other words, an exchangeable measure on $\Lambda^{\mathbb{N}}$, with Λ countable, is such that the measure of the cylindrical set $E_{i_1 \dots i_r}(a_1, \dots, a_r)$ only depends on $a_1, \dots, a_r$, and does not depend on the sequence of indices $i_1, \dots, i_r$.

Lemma B.3. *Let $m \in \mathcal{M}_1(\Lambda^{\mathbb{N}})$ be exchangeable, then for all $f \in L^1(\Lambda^{\mathbb{N}})$ the following conditions are equivalent:*

a) *f is $\mathfrak{S}_c(\mathbb{N})$-invariant;*
b) *f is $\mathrm{Inj}(\mathbb{N})$-invariant;*
c) *f is shift-invariant.*

Proof. Since $\mathfrak{S}_c(\mathbb{N}) \subset \mathrm{Inj}(\mathbb{N})$ and $s \in \mathrm{Inj}(\mathbb{N})$, the implications b) $\Rightarrow$ a) and b) $\Rightarrow$ c) are obvious.

In order to prove that a) $\Rightarrow$ b), we let $\mathcal{F} = \{\sigma \in \mathrm{Inj}(\mathbb{N}) : f = f \circ T^\sigma\}$, which is a closed subset of $\mathrm{Inj}(\mathbb{N})$ containing $\mathfrak{S}_c(\mathbb{N})$. Then, it is enough to observe that $\mathfrak{S}_c(\mathbb{N})$ is a dense subset of $\mathrm{Inj}(\mathbb{N}) \subset \mathbb{N}^{\mathbb{N}}$, with respect to the product topology of $\mathbb{N}^{\mathbb{N}}$, so that $\mathcal{F} = \overline{\mathfrak{S}_c(\mathbb{N})} = \mathrm{Inj}(\mathbb{N})$.

Let us prove that c) $\Rightarrow$ a). Let $\sigma \in \mathfrak{S}_c(\mathbb{N})$ and let n be such that $\sigma(i) = i$ for all $i \geq n$. It follows that $s^k \circ T^\sigma = s^k$, for all $k \geq n$. As a consequence, for m-almost every $x \in \Lambda^{\mathbb{N}}$ it holds

$$f \circ T^\sigma(x) = f \circ s^n \circ T^\sigma(x) = f \circ s^n(x) = f(x),$$

where the first equality holds since the measure m is $\mathfrak{S}_c(\mathbb{N})$-invariant. □

Notice that from Lemma B.3 it follows that $\tilde{f}$ is $\mathrm{Inj}(\mathbb{N})$-invariant for all $f \in L^1(\Lambda^{\mathbb{N}})$. In particular, for an exchangeable measure, the σ-algebra of the shift-invariant sets coincides with the (a priori smaller) σ-algebra of the $\mathrm{Inj}(\mathbb{N})$-invariant sets.

Thanks to a theorem of De Finetti, suitably extended in [HS:55], there is an integral representation *à la Choquet* for the exchangeable measures on $\Lambda^{\mathbb{N}}$. More precisely, in [HS:55] it is shown that the extremal points of the (compact) convex set of all exchangeable measures are given by the product measures $\sigma^{\mathbb{N}}$, with $\sigma \in \mathcal{M}_1(\Lambda)$. As a consequence, Choquet theorem [C:69] provides an integral representation for any exchangeable measure m on $\Lambda^{\mathbb{N}}$, i.e. there is a probability measure $\mu \in \mathcal{M}_1(\Lambda)$ such that

$$m = \int_{\mathcal{M}_1(\Lambda)} \sigma^{\mathbb{N}} \, d\mu(\sigma) \, . \tag{B.5}$$

When Λ is finite, i.e. $\Lambda = p = \{0, \dots, p-1\}$ for some $p \in \mathbb{N}$, we can identify $\mathcal{M}_1(\Lambda)$ with the symplex Σ_p of all $\lambda \in [0,1]^p$ such that $\sum_{i=0}^{p-1} \lambda_i = 1$. Given $\lambda \in \Sigma_p$, we denote by B_λ the (product) Bernoulli measure on $p^{\mathbb{N}}$ such that all the events $E_i(a)$ are independent and $B_\lambda(E_i(a)) = B_\lambda(E_j(a)) = \lambda_a$, for all $i, j \in \mathbb{N}$ and $a \in p$. In this case, (B.5) becomes

$$m = \int_{\Sigma_p} B_\lambda \, d\mu(\lambda) \,, \tag{B.6}$$

where μ is a probability measure on Σ_p.

We say that a measure $m \in \mathcal{M}(\Lambda^{\mathbb{N}})$ is *asymptotically exchangeable* if the sequence $m_k := (s_\#)^k (m)$ weakly* converges to an exchangeable measure [K:78].

We now prove that any probability measure on $\Lambda^{\mathbb{N}}$ is asymptotically exchangeable on a suitable subsequence of indeces (we refer to [C:74, FS:76, K:78, K:05] for similar results).

Proposition B.4. *Given $m \in \mathcal{M}_1(\Lambda^{\mathbb{N}})$ there is an increasing function $\sigma\colon \mathbb{N} \to \mathbb{N}$ such that $T^\sigma_\#(m)$ is asymptotically exchangeable.*

Proof. Thanks to Lemma A.1, applied with $M = \mathcal{M}_1(S^r)$, for all $r \in \mathbb{N}$ there is an infinite set $J_r \subset \mathbb{N}$ such that $T^\iota_\#(m)$ is convergent in $\mathcal{M}_1(S^r)$, for $\iota = (i_0, \dots, i_{r-1}) \in [J_r]^r$ and $i_0 \to \infty$. By a diagonal argument, we can choose the same set $J \subset \mathbb{N}$ for all $r \in \mathbb{N}$. Letting $\sigma \in \mathrm{Incr}(\mathbb{N})$ be such that $\sigma(\mathbb{N}) = J$, we claim that $T^\sigma_\#(m)$ is asymptotically exchangeable.

Let us first show that $m_k := s^k_\# \circ T^\sigma_\#(m)$ is convergent in $\mathcal{M}_1(S^{\mathbb{N}})$. Indeed, since the sequence m_k is precompact in $\mathcal{M}_1(S^{\mathbb{N}})$, it is enough to show that $T^\iota_\#(m_k)$ is convergent in $\mathcal{M}_1(S^r)$ for all $\iota \in \mathbb{N}^{[r]}$ and $r \in \mathbb{N}$, and the latter follows as above from Lemma A.1 and the choice of J.

It remains to prove that the limit m' of m_k is exchangeable. By Proposition B.1, it is enough to show that $T^\theta_\#(m') = m'$, for every $\theta \in \mathrm{Incr}(\mathbb{N})$. Again by the choice of J, the sequence of measures $T^\theta_\# \circ T^\iota_\#(m_k)$ has the same limit of $T^\iota_\#(m_k)$ for all $\iota \in \mathbb{N}^{[r]}$ and $r \in \mathbb{N}$, which in turn implies $T^\theta_\#(m') = m'$. □

Remark B.5. Notice that, if m is asymptotically exchangeable, then for all $(a_1, \dots, a_r) \in S^r$ the limit exchangeable measure m' satisfies

$$\begin{aligned} m\left(\{x_{i_1+k} = a_1, \dots, x_{i_r+k} = a_r\}\right) &\le m'\left(\{x_{i_1} = a_1, \dots, x_{i_r} = a_r\}\right) \\ &+ o(1) \quad \text{as } k \to +\infty \,, \end{aligned} \tag{B.7}$$

and the equality holds if the points $\{a_j\}$ are open in S for all $1 \le j \le r$ (which is always the case if S is finite). More generally, the equality holds in (B.7) for all clopen (i.e. both open and closed) $C \subseteq S^r$, that is

$$m'\left(\{(x_1, \dots, x_r) \in C\}\right) = \lim_{k \to +\infty} m\left(\{(x_{i_1+k}, \dots, x_{i_r+k}) \in C\}\right).$$

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