A variational model for phase separation

tesi di dottorato di

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XIV Ciclo
Anno Accademico 2001-02

– ¿Que esperabas? – suspiró Úrsula–. El tiempo pasa.
– Así es – admitió Aureliano–, pero no tanto.

Gabriel García Márquez
Cien años de soledad

Acknowledgements

I wish to thank all the people who have helped me in these years: in first place my thesis advisor Enrico Vitali, who introduced me to the subjects treated in this work. I am extremely grateful to him for all that he taught me, through discussions and examples, during this period, for his advices, for his patience and friendship.

I can not express properly my gratitude to Andrea Braides: it is hard to evaluate, how much his constant interest and his friendship have influenced and encouraged me, and how much his suggestions and ideas, and the many discussions we had, have been essential for me.

My deepest gratitude goes to Gian Pietro Pirola, for his constant help, for his patience and attention in reading this thesis, for precious suggestions and encouragement. It is a great pleasure to thank Paolo Cermelli and Giovanni Leoni for the interest shown in my work, and for their essential suggestions and advices; moreover, I express my gratitude to Lev Truskinovsky, who indicated to me many interesting directions of further investigations.

I would like to thank Maurizio Cornalba for his constant kindness and attention to our problems.

I wish to thank my parents, for more than I can say here, and my friends Gabriele and Paolo, for their so long presence. Finally, I owe special thanks to Marco and Silvia.

CONTENTS

Chapter 1

INTRODUCTION

The object of this work is the mathematical treatment of some theories for multi-phase materials. The model problem is that of the description of the equilibrium configuration for a fluid filling a regular bounded region Ω in space. A configuration of the system can be described by a function on Ω which takes value a on the subset of Ω occupied by one phase, and value b where the material is in the other phase (for instance, $a = 0$ and $b = 1$). We could also think of u as the density of the fluid, taking either of two different values a and b.

A physically reasonable criterion requires that the interface between the two phases at the equilibrium has minimal area. This corresponds to postulate a free energy of the form

$$F(u) = \sigma\mathcal{H}^2(S(u)), \tag{1.0.1}$$

where σ is the *surface tension,* $S(u)$ is the discontinuity set of u and $\mathcal{H}^2$ denotes the 2-dimensional surface measure. The minimization of F is considered among all admissible configurations u.

A different approach assumes that the free energy, per unit volume, is a prescribed function W of the density distribution u. Thus the stable configurations of the fluid minimize the total energy

$$E(u) = \int_\Omega W(u(x))\,dx,$$

under the constraint of prescribed total mass: $\int_\Omega u(x)\,dx = m$. Assume W of "double-well" type, with minima located at $u = a$ and $u = b$. It is not restrictive to assume the same level for both minima at a and b. This can be done by adding an affine function to W; in fact, this only changes the minimum problem by an additive constant, because of the condition $\int_\Omega u(x)\,dx = m$; thus we can assume the same level for both the minima at a and b (see Figure 1.1, where W is obtained from $\overline{W}$ by addition of a suitable affine function).

If $a|\Omega| < m < b|\Omega|$, the minimum problem for $E(u)$ admits infinitely many piecewise constant solutions with values a and b (see, e.g., [19]), without any condition on the interface between the sets $\{u = a\}$ and $\{u = b\}$ corresponding to the phases. A modification of the model therefore is needed to select some sort of "physically preferred" solutions. The Van der Waals and Cahn-Hilliard approach consists in the addition of a penalty on the density gradient, thus considering the total energy

$$G_\varepsilon(u) = \int_\Omega \Big(W(u) + \varepsilon^2|\nabla u|^2\Big)\,dx,$$

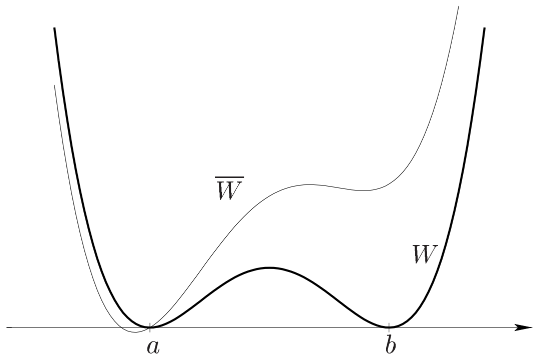

Figure 1.1.

where $\varepsilon > 0$ is a small parameter. A link between the two models can be established via the asymptotic analysis for $\varepsilon \to 0$. More precisely, a suitable rescaling $\lambda_\varepsilon G_\varepsilon$ (which admits the same minimizers) is needed to give rise to a non trivial limit functional. In [35] (see also [37]) Modica proved that, under the constraint of prescribed mass, the Γ-limit of

$$J_\varepsilon(u) = \frac{1}{\varepsilon} G_\varepsilon(u) = \frac{1}{\varepsilon}\int_\Omega W(u)\,dx + \varepsilon \int_\Omega |\nabla u|^2\,dx \tag{1.0.2}$$

is finite only if $u \in \{a, b\}$ a.e. is a function of bounded variation, and in that case is given by (1.0.1), with

$$\sigma = 2\int_a^b \sqrt{W(s)}\,ds = \inf\Big\{ \int_{\mathbb{R}} \big(W(u) + (u')^2\big)\,dt \ : \ \lim_{t\to-\infty} u(t) = a, \quad \lim_{t\to+\infty} u(t) = b \Big\}. \quad (*)$$

Analogous models have been widely studied by several authors; some generalizations of the above convergence result were obtained, e.g., in [14] (where more general energies are considered, and the parameters a and b are allowed to depend on x, e.g., through the temperature at x), in [33], [34], [40] (where the scaled energy has the form $\varepsilon^{-1} f(x, u(x), \varepsilon Du(x))$), and in many other papers.

As to the vectorial setting (see also Chapter 5), where the function u takes values in $\mathbb{R}^d$, the problem was considered and solved by Irene Fonseca and Luc Tartar in [26]; further generalizations are taken into account, e.g., in [12] (anisotropic case), and in [10], [46], for energy functionals with more than two wells.

A variant proposed for the Cahn-Hilliard functional G_ε reads as follows (see e.g. [27],[2], [3], [42])

$$E_\varepsilon(u) = \frac{1}{\varepsilon}\int_\Omega W(u(x))\,dx + \frac{1}{4\varepsilon^{n+1}} \iint_{\Omega\times\Omega} e^{-|x-y|/\varepsilon}\big(u(x) - u(y)\big)^2\,dx\,dy, \tag{1.0.3}$$

n being the dimension of Ω. This corresponds to the replacement of the Dirichlet term $\varepsilon^2 \int_\Omega |\nabla u|^2\,dx$ in G_ε with a nonlocal term where spatial inhomogeneity, weighted with an influence kernel, is penalized.

In such a case (see [4] for interaction potentials more general than the exponential kernel above) the limit as $\varepsilon \to 0$ is again the functional (1.0.1), but with a different surface tension:

$$\begin{aligned}\sigma = \inf\Big\{ \int_{\mathbb{R}} W(u(t))\,dt + \frac{1}{4}\iint_{\mathbb{R}^2} e^{-|s-t|}\big(u(t)-u(s)\big)^2\,dt\,ds \ : \\ \lim_{t\to-\infty} u(t) = a, \quad \lim_{t\to+\infty} u(t) = b \Big\}.\end{aligned} \tag{$**$}$$

We consider the following generalization of J_ε :

$$F_\varepsilon(u,v) = \frac{1}{\varepsilon}\int_\Omega W(u)\,dx + \frac{\alpha}{\varepsilon}\int_\Omega (u-v)^2\,dx + \varepsilon\int_\Omega |\nabla v|^2\,dx,$$

where α is a positive parameter. Let us notice that in the one-dimensional setting these functionals were proposed by Rogers and Truskinovsky in [43] (see also [17] and [42]) as a model for the longitudinal deformation of an elastic bar which could take into account an elastic energy of Ericksen's type together with an internal scalar variable which measures the deviation from one-dimensional deformation.

In Chapter 2 (which essentially forms the content of [45]) we shall study the asymptotic behaviour of functionals F_ε. We shall show that the limit functional $F(u,v)$ is finite if and only if $u = v \in \{a,b\}$ a.e. is a function of bounded variation and again it is given by (1.0.1), with a surface tension of type $(**)$ (here we assume $a=0$ and $b=1$):

$$\begin{aligned}\sigma = \sigma(\alpha) = \sqrt{\alpha}\inf\Big\{ \int_{\mathbb{R}} W(u(t))\,dt + \frac{\alpha^2}{4}\iint_{\mathbb{R}^2} e^{-\alpha|t-s|}\big(u(t)-u(s)\big)^2\,dt\,ds \ : \\ \lim_{t\to-\infty} u(t) = 0, \quad \lim_{t\to+\infty} u(t) = 1 \Big\}.\end{aligned}$$

The surface tension $(*)$ corresponding to the family $J_\varepsilon(u)$ in (1.0.2) will be recovered for $\alpha \to +\infty$. Thus, through the analysis of a "two-variable" model, a precise connection is established between a family of nonlocal models and the classical Cahn-Hilliard theory.

Actually, we shall consider, in place of $\int_\Omega |\nabla v|^2\,dx$ in $F_\varepsilon(u,v)$, a more general anisotropic term $\int_\Omega g^2(\nabla v)\,dx$, with g positively 1-homogeneous. This will give rise, in the limit, to an anisotropic term of the form $\int_{S(u)} \varphi(\nu_u)\,d\mathcal{H}^2$, where ν_u is a normal unit vector field on $S(u)$.

The value σ for the surface tension in $(**)$ can be expressed through a minimum problem involving the non-scaled one dimensional functional F_1 on the real line, i.e.

$$\begin{aligned}\sigma = \sigma(\alpha) = \inf\Big\{ \int_{\mathbb{R}} W(u) + \alpha(u-v)^2 + (v')^2 dt \ : \\ \lim_{t\to-\infty} u(t),\ \lim_{t\to-\infty} v(t) = 0, \quad \lim_{t\to+\infty} u(t),\ \lim_{t\to+\infty} v(t) = 1 \Big\}.\end{aligned} \tag{P}$$

We emphasize the importance to have precise information on the behaviour of minimizer (the so-called *optimal profiles*) for problem (P). Indeed, such a knowledge is relevant both as a tool in the proof of the Γ-convergence results, and for the intrinsic physical meaning: the optimal profiles represent the asymptotic shapes of the minimizers of F_ε close to discontinuity points of their limit.

Chapter 3 is devoted to the study of problem (P). We notice that this approach allows the explicit computation of the constant $\sigma(\alpha)$ appearing in the Γ-limit of the nonlocal functionals E_ε, at least for α greater then a critical value depending on W (see Section 3.3). Furthermore, the same analysis is developed for the analogous problem on a half-line: this will be exploited in Chapter 4, where boundary Dirichlet conditions are taken into account. The asymptotic result here obtained can be useful in modelling the formation of boundary layers due to interactions with the container walls. In the Γ-limit of F_ε with prescribed boundary values, we recover the term $\sigma(\alpha)\mathcal{H}^{n-1}(S(u))$, and a new term measuring the energy of the boundary layer due to the transition from each of the phases to the boundary data, i.e. an integral term of the form $\int_{\partial\Omega} \Phi^\alpha(u(x), h(x))d\mathcal{H}^{n-1}(x)$, where h is the limit of the Dirichlet data.

We shall prove that, also in this case, the result for the functionals J_ε with boundary Dirichlet conditions, obtained by Owen, Rubinstein and Sternberg in [39], can be recovered taking the limit as the parameter α goes to 0.

As for the asymptotic behaviour of minimizers of F_ε, the Γ-convergence result says that a minimizing sequence converges, as $\varepsilon \to 0$, to the characteristic function of a subset E of Ω with finite perimeter in Ω, such that the perimeter $P_\Omega(E)$ is minimal, and such that the angle between the interface ∂E and $\partial\Omega$ (*contact angle*) is a prescribed function of $x \in \partial\Omega$ (see Section 4.1, and [39, Proposition 5.2]). We notice that the *selection criterion* of minimal energy configurations induced by the sequence of two-variables functionals with Dirichlet conditions gives a different constraint on the contact angle, thus the preferred equilibrium configurations have in this case different shapes with respect to those "selected" by J_ε with the same conditions.

In Section 4.5 we again consider the problem of the interactions with the container walls, following the approach of L. Modica (see [36]); we analyze the asymptotic behaviour of a sequence of functionals involving a term of *contact energy* at the boundary instead of Dirichlet conditions.

These problems, concerning a contact angle condition, are connected with the theory of capillary surfaces and other physical models (see, e.g. [25], [36] and the references therein, in particular [18] and [28]).

The close connection between functionals of type F_ε, depending on a phase variable and on an order parameter v, and nonlocal functionals, suggests that through this approach it could be possible to give an interpretation of boundary contact terms also for nonlocal models.

In the last chapter we give a short account on a possible generalization of functionals F_ε to nonlinear elasticity, where the phase variable is replaced by a deformation gradient.

Notation and preliminaries

The Lebesgue measure in $\mathbb{R}^n$ and the $(n-1)$-dimensional Hausdorff measure will be denoted respectively by $\mathcal{L}^n$ and $\mathcal{H}^{n-1}$; we shall also use the notation $|E|$ instead of $\mathcal{L}^n(E)$. $B_r(x)$ will be the open ball with centre x and radius r, and S^{n-1} the boundary of the ball $B_1(0)$.

To shorten the notation, we will sometimes omit the Lebesgue measure in integrals by writing $\int f(u)$ instead of $\int f(u(x))\,dx$ and similar.

Functions of bounded variation and sets of finite perimeter. For the general theory of this topic we refer to [23], [24], [8]; here we recall some definitions and properties we shall use in the sequel.

Given an open subset Ω of $\mathbb{R}^n$, a function $u\colon \Omega \to \mathbb{R}$ is said to be of *bounded variation* ($u \in BV(\Omega)$) if $u \in L^1(\Omega)$ and its distributional derivatives $D_i u$ are Radon measures with finite total variation in Ω.

If $u \in L^1_{loc}(\Omega)$ we say that $z \in \mathbb{R}$ is the *approximate limit* of u in $x \in \Omega$ ($z =$ap-$\lim_{y\to x} u(y)$) if

$$\lim_{\rho\to 0} \rho^{-n} \int_{B_\rho(x)} |u(y) - z|\, dy = 0.$$

The set $S(u)$ of points where this property does not hold is called the *approximate discontinuity set* of u. The set $S(u)$ is a Borel set, and $|S(u)| = 0$. If $u \in BV(\Omega)$, then $S(u)$ is countably $(n-1)$-rectifiable, and there exist Borel functions $\nu_u\colon S(u) \to S^{n-1}$, and $u^+, u^-\colon S(u) \to \mathbb{R}$ such that for $\mathcal{H}^{n-1}$-a.e. $x \in S(u)$

$$\lim_{\rho\to 0} \rho^{-n} \int_{B^+_\rho(x)\cap\Omega} |u(y) - u^+(x)| dy = 0, \qquad \lim_{\rho\to 0} \rho^{-n} \int_{B^-_\rho(x)\cap\Omega} |u(y) - u^-(x)| dy = 0,$$

where $B^+_\rho(x) = \{y \in B_\rho(x) :\ \langle y-x, \nu_u(x)\rangle > 0\}$ and $B^-_\rho(x) = \{y \in B_\rho(x) :\ \langle y-x, \nu_u(x)\rangle < 0\}$.

If $u \in BV(\Omega)$ we denote by ∇u the density of the absolutely continuous part $D^a u$ of the vector measure Du with respect to the Lebesgue measure. Let $D^s u$ be the singular part of Du, and define $D^j u = Du \llcorner S(u)$ and $D^c u = Du \llcorner (\Omega \setminus S(u))$ (the *jump* and *Cantor* part of Du, respectively). We say that u is a *special function of bounded variation* ($u \in SBV(\Omega)$) if $D^c u = 0$; in that case the following decomposition of Du holds:

$$Du = \nabla u \mathcal{L}^n + (u^+ - u^-)\nu_u \mathcal{H}^{n-1} \llcorner S(u).$$

If E is a Borel subset of $\mathbb{R}^n$, the *essential boundary* $\partial^* E$ of E is defined as

$$\partial^* E = \{x \in \mathbb{R}^n \ :\ \limsup_{\rho\to 0} \rho^{-n}|B_\rho(x) \cap E| > 0,\ \limsup_{\rho\to 0} \rho^{-n}|B_\rho(x) \setminus E| > 0\ \}.$$

It turns out that the discontinuity set of the characteristic function χ_E coincides with $\partial^* E$, i.e.

$$S(\chi_E) = \partial^* E. \tag{1.0.4}$$

It can be proved (see [24]) that for any open subset Ω of $\mathbb{R}^n$

$$\int_\Omega |D\chi_E| = \mathcal{H}^{n-1}(\Omega \cap \partial^* E). \tag{1.0.5}$$

In particular, if E is a bounded Borel subset, then $\chi_E \in BV(\Omega)$ if and only if $\mathcal{H}^{n-1}(\Omega \cap \partial^* E) < +\infty$ (in such a case, E is said to have *finite perimeter* in Ω).

In the sequel we shall use extensively results concerning integral representations, decomposition of measures, structure of the space of SBV functions, rearrangement theory, ordinary differential equations and other topics; in particular, in Chapter 4 some notions and theorems of differential geometry will be necessary. We will state the precise results and give the related references where they are useful.

Since it will be often applied, we recall here a result concerning the supremum of a family of measures.

Supremum of a family of measures. Taking into account the regularity of positive Borel measures and the standard properties of the least upper bound of a family of measures, we get the following result (see, e.g., [15, Proposition 1.16]):

Proposition 1.0.1. *Let Ω be an open subset of $\mathbb{R}^n$ and μ a finite, positive set function defined on the family of open subsets of Ω. Let λ be a positive Borel measure on Ω, and $(f_i)_{i\in I}$ a family of positive Borel functions on Ω. Assume that $\mu(A) \geq \int_A f_i \, d\lambda$ for every A and i, and that $\mu(A \cup B) \geq \mu(A) + \mu(B)$ whenever $A, B \subset\subset \Omega$ and $\overline{A} \cap \overline{B} = \emptyset$ (superadditivity). Then $\mu(A) \geq \int_A \big(\sup_i f_i\big) \, d\lambda$ for every A.*

Γ*-convergence.* We recall the notion of Γ-convergence (we refer to [16] and [21] for a complete analysis of the subject, and for the connection with the theory of phase transitions). Let (X, d) be a metric space, $F_\varepsilon \colon X \to \overline{\mathbb{R}}$ $(\varepsilon > 0)$ a family of functionals, and $F \colon X \to \overline{\mathbb{R}}$. We say that $\{F_\varepsilon\}$ Γ-converges to F at $x \in X$ as $\varepsilon \to 0$ if:

i) for every infinitesimal sequence $\{\varepsilon_j\}$ and for every sequence $\{x_j\}$ converging to x in X, we have $F(x) \leq \liminf_{j\to\infty} F_{\varepsilon_j}(x_j)$;

ii) for every infinitesimal sequence $\{\varepsilon_j\}$ there exists a sequence $\{x_j\}$ (*recovery sequence*) converging to x in X such that $F(x) = \lim_{j\to\infty} F_{\varepsilon_j}(x_j)$.

If *i)* and *ii)* hold for every $x \in X$ we say that $\{F_\varepsilon\}$ Γ-converges to F in X, and $F = \Gamma - \lim_{\varepsilon\to 0} F_\varepsilon$.

Remark 1.0.2. a) Condition *ii)* in the previous definition can be replaced by

ii)' for every $\eta > 0$ and for every infinitesimal sequence $\{\varepsilon_j\}$ there exists a sequence $\{x_j\}$ converging to x in X such that $F(x) \geq \limsup_{j\to\infty} F_{\varepsilon_j}(x_j) - \eta$.

b) If $F_\varepsilon = G$ is constant with respect to ε, the Γ-limit exists and coincides with the lower semicontinuous envelope of G with respect to the topology induced by d.

Remark 1.0.3. The Γ-*lower limit* and the Γ-*upper limit* of $\{F_\varepsilon\}$ are defined as follows:

$$\begin{aligned} F'(x) &= \inf\big\{\liminf_{j\to\infty} F_{\varepsilon_j}(x_j) : \ \ \varepsilon_j \to 0, \ x_j \to x\big\} \\ F''(x) &= \inf\big\{\limsup_{j\to\infty} F_{\varepsilon_j}(x_j) : \ \ \varepsilon_j \to 0, \ x_j \to x\big\}. \end{aligned}$$

It is not difficult to see that F' and F'' are both lower semicontinuous and that $\{F_\varepsilon\}$ Γ-converges if and only if $F' = F''$.

We recall the main properties of Γ-convergence:

Proposition 1.0.4. *i)* Lower semicontinuity: *the Γ-limit F is always lower semicontinuous on X.*

ii) Stability under continuous perturbations: *if* $F_\varepsilon \xrightarrow{\Gamma} F$ *and* G *is continuous, then* $F_\varepsilon + G \xrightarrow{\Gamma} F + G$.

iii) Stability of minimizing sequences: *if* $F_\varepsilon \xrightarrow{\Gamma} F$ *and* x_ε *minimizes* F_ε *over* X, *then every cluster point of* $\{x_\varepsilon\}$ *minimizes* F *over* X.

We say that a sequence $\{x_\varepsilon\}$ is a quasi-minimizing sequence for $\{F_\varepsilon\}$ over X if $F_\varepsilon(x_\varepsilon) = \inf\{F_\varepsilon(x) \ : \ x \in X\} + o(1)_{\varepsilon \to 0})$. Thus, the following property holds

iii)' Stability of quasi-minimizing sequences: *if* $F_\varepsilon \xrightarrow{\Gamma} F$ *and* $\{x_\varepsilon\}$ *is a quasi-minimizing sequence for* $\{F_\varepsilon\}$ *over* X, *then every cluster point of* $\{x_\varepsilon\}$ *minimizes* F *over* X.

In view of properties *iii)* and *iii)'*, since we are interested to the asymptotic behaviour of the minimizers or quasi-minimizers of F_ε, a Γ-convergence result for the functionals F_ε should be paired with a *compactness result* for the corresponding minimizing or quasi-minimizing sequences $\{x_\varepsilon\}$.

Indeed, suppose that the following equi-coerciveness property holds for F_ε :

> Given $\{\varepsilon_n\}$ a positive infinitesimal sequence, and $\{x_n\}$ in X such that $F_{\varepsilon_n}(x_n)$ is uniformly bounded, then $\{x_n\}$ is relatively compact in X.

Then, every minimizing or quasi-minimizing sequence admits a subsequence approximating a solution of the problem

$$\min\{F(x) : \ x \in X\}.$$

Chapter 2

ENERGY FUNCTIONALS WITH AN ORDER PARAMETER AND NONLOCAL MODELS

2.1 Setting of the problem and statement of the results

Let $n \geq 1$ be fixed. Let W and g be given functions satisfying the following conditions:

– $W\colon \mathbb{R} \to [0, +\infty)$ belongs to $C^0(\mathbb{R})$ and $W(t) = 0$ if and only if $t \in \{0, 1\}$; moreover W has at least linear growth at $\pm\infty$, i.e.

$$\liminf_{|t| \to +\infty} \frac{W(t)}{|t|} > 0;$$

– $g\colon \mathbb{R}^n \to [0, +\infty)$ is Lipschitz continuous and positively 1-homogeneous, and $g(\xi) > 0$ if $\xi \neq 0$. Moreover, we assume that $g(-\xi) = g(\xi)$ for every ξ.

For every open subset Ω of $\mathbb{R}^n$ and for every $\varepsilon > 0$, we define $F_\varepsilon(\cdot, \cdot\,; \Omega)\colon [L^1(\Omega)]^2 \to [0, +\infty]$ as follows:

$$F_\varepsilon(u, v; \Omega) = \begin{cases} \dfrac{1}{\varepsilon} \displaystyle\int_\Omega W(u) + \frac{\alpha}{\varepsilon} \int_\Omega (u - v)^2 + \varepsilon \int_\Omega g^2(Dv) & \text{if } \ v \in H^1(\Omega) \\ +\infty & \text{otherwise,} \end{cases}$$

where α is a given positive constant. Moreover, for every measurable $\varphi\colon \mathbb{R} \to \mathbb{R}$ and $\psi \in H^1_{\mathrm{loc}}(\mathbb{R})$ let

$$\begin{aligned} \mathcal{F}^\alpha(\varphi, \psi) &= \int_\mathbb{R} W(\varphi) + \alpha \int_\mathbb{R} (\varphi - \psi)^2 + \int_\mathbb{R} (\psi')^2, \\ \mathcal{E}^\alpha(\varphi) &= \int_\mathbb{R} W(\varphi) + \frac{\alpha^2}{4} \iint_{\mathbb{R}^2} e^{-\alpha|x-y|} \big(\varphi(x) - \varphi(y)\big)^2 \, dx dy. \end{aligned} \tag{2.1.1}$$

It will be useful to introduce the following function spaces. Given $a, b \in \mathbb{R}$, and $I \subseteq \mathbb{R}$ a bounded interval, we shall denote by $H^1_{a,b}(I)$ the subset of $H^1(I)$ whose elements have boundary values a, b. Moreover, we set

$$X = \{\varphi\colon \mathbb{R} \to [0, 1] \ \text{ measurable such that } \ \lim_{x \to -\infty} \varphi(x) = 0, \ \lim_{x \to +\infty} \varphi(x) = 1\};$$

$$Y = \{\varphi \in AC(\mathbb{R}) \ : \ \varphi' \in L^2(\mathbb{R})\}.$$

In this work we prove the following theorem about the asymptotic behaviour of $\{F_\varepsilon\}$ as $\varepsilon \to 0$:

Theorem 2.1.1. *For every bounded open subset Ω of $\mathbb{R}^n$ the functionals $F_\varepsilon(\cdot,\cdot\,;\Omega)$ Γ-converge, in $[L^1(\Omega)]^2$ as $\varepsilon \to 0$, to the functional $F(\cdot,\cdot\,;\Omega)\colon [L^1(\Omega)]^2 \to [0,+\infty]$ defined by*

$$F(u,v;\Omega) = \begin{cases} c_W(\alpha) \displaystyle\int_{S(u)} g^{**}(\nu_u) d\mathcal{H}^{n-1} & \textit{if } u,v \in BV(\Omega) \textit{ and } u = v \in \{0,1\} \textit{ a.e.} \\ +\infty & \textit{otherwise,} \end{cases}$$

where g^{**} denotes the convex hull of g, and

$$c_W(\alpha) = \sqrt{\alpha}\inf\{\mathcal{E}^\alpha(\varphi) \ : \ \varphi \in X\}.$$

Furthermore, we shall study the behaviour of the surface tension term $c_W(\alpha)$, recovering, in the limit as $\alpha \to +\infty$, the well-known Modica-Mortola constant ([35], [37]). More precisely we prove (see §5):

Proposition 2.1.2. *a) The following representation holds:*

$$c_W(\alpha) = \inf\{\mathcal{F}^\alpha(\varphi,\psi) : \ \varphi \in X, \quad \psi \in X \cap Y, \quad \varphi,\psi \text{ increasing}\}.$$

Moreover, the infimum in the right-hand side is attained.

b) The map $c_W : \alpha \mapsto c_W(\alpha)$ is strictly increasing, and

$$\lim_{\alpha\to+\infty} c_W(\alpha) = \min\Big\{ \int_{\mathbb{R}} W(\varphi) + \int_{\mathbb{R}} (\varphi')^2 : \ \varphi \in X \cap Y\Big\}.$$

Remark 2.1.3. Let us notice that $\mathcal{F}^\alpha$ is related to $\mathcal{E}^\alpha$ by a minimization over ψ; the term $\sqrt{\alpha}$ comes from the change of variable $t \mapsto \sqrt{\alpha}t$, which allows to reduce the proof of the equality

$$\inf\{\mathcal{F}^\alpha(\varphi,\psi) : \ \varphi \in X, \quad \psi \in X \cap Y\} = \sqrt{\alpha}\ \inf\{\mathcal{E}^\alpha(\varphi) : \ \varphi \in X\}$$

to the case $\alpha = 1$.

Remark 2.1.4. a) One can easily see that it is enough to establish Theorem 2.1.1 when $\alpha = 1$, the general case following with W and g replaced by W/α and $g/\sqrt{\alpha}$, respectively.

b) The proof of Theorem 2.1.1 relies on the study of functionals obtained by suitable restrictions of F_ε to parallel lines ("slicing method"). Thus, we consider the 1-dimensional case first, taking the following family $F_\varepsilon^\beta(\cdot,\cdot\,;I)\colon [L^1(I)]^2 \to [0,+\infty]$ into account:

$$F_\varepsilon^\beta(u,v;I) = \begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_I W(u) + \dfrac{1}{\varepsilon}\int_I (u-v)^2 + \varepsilon\beta^2 \int_I (v')^2 & \text{if } \ v \in H^1(I) \\ +\infty & \text{otherwise,} \end{cases}$$

where $I \subseteq \mathbb{R}$ is an open set and $\beta > 0$. Notice that for every $\sigma > 0$

$$F^{\beta}_{\varepsilon/\sigma} = \sigma F^{\beta/\sigma}_{\varepsilon};$$

in particular

$$F^{\beta}_{\varepsilon/\beta} = \beta F^{1}_{\varepsilon}.$$

Therefore, the proof will come down to the case $\beta = 1$.

Furthermore, we shall prove a compactness result for the functionals F_ε.

Proposition 2.1.5. (Compactness) *Given $\{\varepsilon_n\}$ a positive infinitesimal sequence, and $\{(u_n, v_n)\} \subset [L^1(\Omega)]^2$, such that $F_{\varepsilon_n}(u_n, v_n)$ is uniformly bounded, then $\{(u_n, v_n)\}$ is relatively compact in $[L^1(\Omega)]^2$. Moreover, the cluster points (u, v) are such that $u = v \in BV(\Omega; \{0, 1\})$.*

In Section 2.3.1 we prove that the Γ-convergence result of Theorem 2.1.1 can be refined as follows. Let $u \in BV(\Omega; \{0, 1\})$; then, for every infinitesimal positive sequence $\{\varepsilon_n\}$, we shall construct a recovery sequence $\{(u_n, v_n)\}$ so that $u_n, v_n \to u$ in $L^1(\Omega)$ as $n \to +\infty$, $F_{\varepsilon_n}(u_{\varepsilon_n}, v_{\varepsilon_n}) \to F(u, u)$, and $\int_\Omega v_{\varepsilon_n} = \int_\Omega v$ for every n (see the property *ii*) in the definition of Γ-convergence). This corresponds in the limit to deal with a prescribed mass constraint and can be rephrased as a Γ-limit result: for m such that $0 \le m \le |\Omega|$, the sequence $\{F_\varepsilon + I_m\}$ also Γ-converges to $F + I_m$ in $[L^1]^2$, where $I_m(u, v) = 0$ if $\int_\Omega v = m$, and $I_m(u, v) = +\infty$ otherwise (see Proposition 2.3.4).

Setting

$$U_m = \Big\{(u, v) \in L^1(\Omega) \ : \ \int_\Omega v = m\Big\},$$

we say that a sequence $\{(u_\varepsilon, v_\varepsilon)\}$ in U_m is a minimizing sequence (respectively a quasi-minimizing sequence) if, for every $\varepsilon > 0$, the pair $(u_\varepsilon, v_\varepsilon)$ minimizes F_ε in U_m (respectively, $F_\varepsilon(u_\varepsilon, v_\varepsilon) = \inf\{F_\varepsilon(u, v) \ : \ (u, v) \in U_m\} + o(1)_{\varepsilon \to 0}$).

We define

$$U^0_m = \Big\{(u, v) \in [BV(\Omega)]^2 \ : \ u = v \in \{0, 1\} \ \text{ a.e.}, \ \int_\Omega u = m\Big\}.$$

Thus, the well-known property of Γ-convergence recalled in Proposition 1.0.4, *iii*) and the compactness result of Proposition 2.1.5 allow to deduce the following

Corollary 2.1.6. *Let $\{\varepsilon_n\}$ be a positive infinitesimal sequence, and $\{(u_n, u_n)\}$ be a minimizing or quasi-minimizing sequence for F_{ε_n} in U_m. Then $\{(u_n, v_n)\}$ is relatively compact in $[L^1(\Omega)]^2$, and every cluster point (u, v) minimizes F in U^0_m.*

Therefore, the minimizers (or quasi-minimizers) of F_ε in U_m approximate some particular solutions of the geometric problem

$$\min\Big\{\int_{\partial^* E \cap \Omega} g^{**}(\nu_E)\, d\mathcal{H}^{n-1} \ : \ E \ \text{ has finite perimeter in } \Omega, \ |E| = m\Big\}. \qquad (P_0)$$

In this sense, as noticed in the introduction, the family of perturbed energies $\{\int_\Omega (W(u) + (u-v)^2 + \varepsilon^2|Dv|^2)\}$ constitutes a *selection criterion* for the equilibrium configurations of the system, i.e. for the configurations minimizing the energy $\int_\Omega W(u)$.

We observe that throughout this Chapter we can deal with functions which take values in $[0,1]$. Indeed, we have the following truncation lemma:

Lemma 2.1.7. *For every $u\colon \Omega \to \mathbb{R}$, we set $\tilde{u} = (u \wedge 1) \vee 0$. Then, $F_\varepsilon(u,v) \geq F_\varepsilon(\tilde{u},\tilde{v})$ for every $\varepsilon > 0$, and if $\{(u_\varepsilon, v_\varepsilon)\}$ is such that $F_\varepsilon(u_\varepsilon, v_\varepsilon)$ is uniformly bounded, it follows that $\|u_\varepsilon - \tilde{u}_\varepsilon\|_{L^1}$, $\|v_\varepsilon - \tilde{v}_\varepsilon\|_{L^1}$ are infinitesimal as $\varepsilon \to 0$.*

Proof. The first assertion is obvious.

Consider a sequence $\{(u_\varepsilon, v_\varepsilon)\}$ such that $F_\varepsilon(u_\varepsilon, v_\varepsilon) \leq C$, with C independent of ε. The assumptions on W allow to find, for every fixed $\delta > 0$, c_1, c_2, T strictly positive so that

$$\begin{array}{ll} W(t) \geq c_1 & \text{in } [-T,-\delta] \cup [1+\delta, T] = I_{T,\delta} \\ W(t) \geq c_2|t| & \text{in } |t| \geq T. \end{array}$$

Hence

$$\begin{aligned} \|u_\varepsilon - \tilde{u}_\varepsilon\|_{L^1} &\leq \delta\big|\{u_\varepsilon \in [-\delta, 0] \cup [1, 1+\delta]\}\big| + \big|\{u_\varepsilon \in I_{T,\delta}\}\big| T + \int_{\{|u_\varepsilon| \geq T\}} |u_\varepsilon| \\ &\leq \delta|\Omega| + \Big(\frac{T}{c_1} + \frac{1}{c_2}\Big) \int_\Omega W(u_\varepsilon), \end{aligned}$$

and the arbitrariness of δ, recalling that $\int_\Omega W(u_\varepsilon) \leq C\varepsilon$, gives the convergence of $\|u_\varepsilon - \tilde{u}_\varepsilon\|_{L^1}$.

As to v_ε, clearly

$$\begin{aligned} \|v_\varepsilon - \tilde{v}_\varepsilon\|_{L^1} &\leq \|v_\varepsilon - u_\varepsilon\|_{L^1} + \|u_\varepsilon - \tilde{u}_\varepsilon\|_{L^1} + \|\tilde{u}_\varepsilon - \tilde{v}_\varepsilon\|_{L^1} \\ &\leq 2\|v_\varepsilon - u_\varepsilon\|_{L^1} + \|u_\varepsilon - \tilde{u}_\varepsilon\|_{L^1}; \end{aligned}$$

recalling that $\int_\Omega (u_\varepsilon - v_\varepsilon)^2 \leq C\varepsilon$, the convergence follows. ∎

2.2 The one dimensional case

As noticed in Remark 2.1.4, the asymptotic behaviour of $\{F_\varepsilon^\beta\}$, stated in Corollary 2.2.2 below, can be immediately deduced from the case $\beta = 1$; hence, for every open subset I of $\mathbb{R}$ we shall consider the functionals $F_\varepsilon(\cdot,\cdot;I)\colon [L^1(I)]^2 \to [0,+\infty]$ defined by

$$F_\varepsilon(u,v;I) = \begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_I W(u) + \frac{1}{\varepsilon}\int_I (u-v)^2 + \varepsilon \int_I (v')^2 & \text{if } v \in H^1(I), \\ +\infty & \text{otherwise.} \end{cases} \tag{2.2.1}$$

Theorem 2.2.1. *For every $I \subseteq \mathbb{R}$ open subset, $\{F_\varepsilon(\cdot,\cdot;I)\}$ Γ-converges, in $[L^1(I)]^2$ as $\varepsilon \to 0$, to the functional $F\colon [L^1(I)]^2 \to [0,+\infty]$ defined by*

$$F(u,v;I) = \begin{cases} c_W \# S(u) & \text{if } u,v \in BV(I),\ u=v \text{ and } u \in \{0,1\} \text{ a.e.} \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$\begin{aligned} c_W = c_W(1) &= \inf\{\mathcal{E}^1(\varphi): \quad \varphi \in X\} \\ &= \inf\Big\{\int_{\mathbb{R}} W(\varphi) + \frac{1}{4}\int\int_{\mathbb{R}^2} e^{-|x-y|}\big(\varphi(x)-\varphi(y)\big)^2\,dxdy : \quad \varphi \in X\Big\}. \end{aligned}$$

Corollary 2.2.2. *Let F^β_ε be as in Remark 2.1.4 (b). Then for every $I \subseteq \mathbb{R}$ open subset $\{F^\beta_\varepsilon(\cdot,\cdot;I)\}$ Γ-converges, in $[L^1(I)]^2$ as $\varepsilon \to 0$, to $\beta F(\cdot,\cdot;I)$, where F is defined in Theorem 2.2.1.*

In the proof of Theorem 2.2.1 it will be enough to consider the case $I = (-1,1)$. The extension to an arbitrary interval (a,b) is immediate since $F_\varepsilon(u,v,(a,b)) = F_\sigma(\varphi,\psi,(-1,1))$ with $\sigma = \frac{2\varepsilon}{b-a}$, $u(x) = \varphi(\frac{2x-a-b}{b-a})$ and $v(x) = \psi(\frac{2x-a-b}{b-a})$. Moreover, it is not difficult to see that the Γ-lim inf inequality for a single bounded interval implies the same inequality for a disjoint countable union of open intervals (bounded or unbounded), hence for an arbitrary open set (recall property (1.0.5) for characteristic functions). Finally, the recovery sequence exibited in the proof of the Γ-lim sup inequality is obtained by a local argument which naturally extends to the case of a general open set.

The Γ-lim inf and the Γ-lim sup inequalities will be proved in Proposition 2.2.3 and 2.2.7, respectively, when $I = (-1,1)$ (recall Remark 1.0.2). We shall refer to $F_\varepsilon(u,v;(-1,1))$ simply as $F_\varepsilon(u,v)$.

2.2.1 The Γ-lower limit

This section is devoted to prove the following:

Proposition 2.2.3. *Let $\{\varepsilon_j\}$ be a positive infinitesimal sequence. Then*

$$\liminf_{j\to\infty} F_{\varepsilon_j}(u_j,v_j) \geq F(u,v) \tag{2.2.2}$$

whenever $\{u_j\},\{v_j\}$ are sequences converging in $L^1(-1,1)$ to u and v, respectively.

In order to prove Proposition 2.2.3, let u,v and $\{u_j\},\{v_j\}$ be as in the statement. We can suppose that the left-hand side in (2.2.2) is finite and is a limit; therefore we can also assume that $\sup_j F_{\varepsilon_j}(u_j,v_j) = k < +\infty$, and that $\{u_j\},\{v_j\}$ also converge a.e. Since $F_{\varepsilon_j}(u_j,v_j) \leq k$ for every j, an application of the Fatou Lemma gives:

$$\int_{-1}^1 (W(u)+(u-v)^2)dx \leq \liminf_{j\to\infty}\int_{-1}^1 (W(u_j)+(u_j-v_j)^2)dx \leq \lim_{j\to\infty}(\varepsilon_j k) = 0.$$

Hence $u \in \{0,1\}$ a.e. and $u = v$ a.e. We can assume $u_j, v_j \in [0,1]$; indeed, if $\hat{u}_j = 0 \vee u_j \wedge 1$ and $\hat{v}_j = 0 \vee v_j \wedge 1$ then $\{\hat{u}_j\}$ and $\{\hat{v}_j\}$ converge to u and v in L^1, and $F_{\varepsilon_j}(\hat{u}_j, \hat{v}_j) \leq F_{\varepsilon_j}(u_j, v_j)$.

If $S(u) = \emptyset$ there is nothing to prove. Let $N \leq \#S(u)$, and $\{t_1, \ldots, t_N\} \subseteq S(u)$. Then we can find intervals $I_i = (a_i, b_i)$, $i = 1, \ldots, N$, such that $-1 < a_1 < t_1 < b_1 \leq \cdots \leq a_N < t_N < b_N < 1$, and, as $j \to \infty$,

$$u_j(a_i), v_j(a_i) \to u(a_i) \in \{0,1\}, \qquad u_j(b_i), v_j(b_i) \to u(b_i) \in \{0,1\},$$

with $u(a_i) \neq u(b_i)$. We have:

$$F_{\varepsilon_j}(u_j, v_j) \geq \sum_{i=1}^{N} F_{\varepsilon_j}(u_j, v_j, I_i). \tag{2.2.3}$$

Now, consider the i-th term of the sum; without loss of generality we can suppose $u(a_i) = 0$, $u(b_i) = 1$; indeed, the following argument is easily adapted to the case $u(a_i) = 1$, $u(b_i) = 0$, and gives exactly the same result. For every j define

$$\tilde{u}_j^i(x) = \begin{cases} 0 & \text{if } x \in (-1, a_i - \varepsilon_j) \\ u_j(a_i) + \dfrac{u_j(a_i)}{\varepsilon_j}(x - a_i) & \text{if } x \in [a_i - \varepsilon_j, a_i) \\ u_j(x) & \text{if } x \in [a_i, b_i] \\ u_j(b_i) + \dfrac{1 - u_j(b_i)}{\varepsilon_j}(x - b_i) & \text{if } x \in (b_i, b_i + \varepsilon_j] \\ 1 & \text{if } x \in (b_i + \varepsilon_j, 1) \end{cases}$$

and analogously $\tilde{v}_j^i$ (clearly, we can suppose $\varepsilon_j < \min\{a_i + 1, 1 - b_i\}$ for every j). It is easy to verify that

$$F_{\varepsilon_j}(u_j, v_j, I_i) = F_{\varepsilon_j}(\tilde{u}_j^i, \tilde{v}_j^i) + o(1)_{j \to \infty}. \tag{2.2.4}$$

Since $\tilde{u}_j^i$ and $\tilde{v}_j^i$ take the boundary values 0 and 1 in a neighborhood of -1 and 1, respectively, we have

$$F_{\varepsilon_j}(\tilde{u}_j^i, \tilde{v}_j^i) \geq \inf\{F_{\varepsilon_j}(\tilde{u}_j^i, \psi) : \ \psi \in H^1_{0,1}(-1,1)\}. \tag{2.2.5}$$

We want to show that

$$\inf\{F_{\varepsilon_j}(\tilde{u}_j^i, \psi) : \ \psi \in H^1_{0,1}(-1,1)\} = E_{\varepsilon_j}(\tilde{u}_j^i, (-1,1)) + o(1)_{j \to \infty}, \tag{2.2.6}$$

where, for every open $I \subseteq \mathbb{R}$ and $\varphi \colon I \to \mathbb{R}$ measurable, we define

$$E_\varepsilon(\varphi, I) = \frac{1}{\varepsilon} \int_I W(\varphi) + \frac{1}{4\varepsilon^2} \iint_{I \times I} e^{-\frac{|x-y|}{\varepsilon}} (\varphi(x) - \varphi(y))^2 \, dxdy.$$

This is the key point of the proof, since it leads to a family of functionals whose asymptotic behaviour is known (a similar procedure has been used, e.g., in [43]). Indeed, the one dimensional version of a result due to G. Alberti and G. Bellettini reads as follows (see [2] Th. 1.4 and §1.9, with $J(t) = e^{-|t|}$):

Theorem 2.2.4. *For every $I \subseteq \mathbb{R}$ open interval, $\{E_\varepsilon(\,\cdot\,; I)\}$ Γ-converges, in $L^1(I)$ as $\varepsilon \to 0$, to $\widetilde{F}\colon L^1(I) \to [0, +\infty]$ defined by*

$$\widetilde{F}(u) = \begin{cases} c_W \# S(u) & \text{if } u \in BV(I), \text{ and } u \in \{0,1\} \text{ a.e.} \\ +\infty & \text{otherwise.} \end{cases}$$

where c_W is as in Theorem 2.2.1.

In order to obtain (2.2.6), we start by proving the following lemma. Set

$$K_\varepsilon(\varphi, \psi) = \frac{1}{\varepsilon} \int_{-1}^{1} (\varphi - \psi)^2 + \varepsilon \int_{-1}^{1} (\psi')^2,$$

whenever $\varphi \in L^2(-1,1)$ and $\psi \in H^1(-1,1)$.

Lemma 2.2.5. *Let $a, b \in \mathbb{R}$ and $0 < \delta < 1$. Let $\mathcal{A}$ be a subset of $L^\infty(-1,1)$ such that $\varphi = a$ in $(-1, -1+\delta)$ and $\varphi = b$ in $(1-\delta, 1)$, whenever $\varphi \in \mathcal{A}$. Assume that $\mathcal{A}$ is bounded in $L^\infty(-1,1)$. Then for every $\varepsilon > 0$ and $\varphi \in \mathcal{A}$:*

$$\begin{aligned} &\inf\{K_\varepsilon(\varphi,\psi) : \ \psi \in H^1_{a,b}(-1,1)\} = \\ \frac{1}{4\varepsilon^2} &\int_{-1}^{1}\int_{-1}^{1} e^{-\frac{|x-y|}{\varepsilon}} (\varphi(x) - \varphi(y))^2 dxdy + \sigma(\varepsilon, \varphi), \end{aligned} \tag{2.2.7}$$

where $\lim\limits_{\varepsilon\to 0} \sigma(\varepsilon, \varphi) = 0$ uniformly with respect to $\varphi \in \mathcal{A}$.

Proof. In the sequel we denote by $\sigma(\varepsilon)$ any function $\sigma(\varepsilon, \varphi)$ such that $\lim_{\varepsilon\to 0} \sigma(\varepsilon, \varphi) = 0$ uniformly with respect to $\varphi \in \mathcal{A}$. It is easy to see that the infimum in (2.2.7) is attained when ψ is the solution of the following Euler equation:

$$\begin{cases} \psi(x) - \varepsilon^2 \psi''(x) = \varphi(x), & x \in [-1, 1] \\ \psi(-1) = a, \quad \psi(1) = b. \end{cases} \tag{2.2.8}$$

The solution ψ of (2.2.8) is the sum of ψ_o, solution of the homogeneous problem with the given boundary conditions, and ψ_g, solution of the Euler equation with Dirichlet homogeneous conditions; we obtain (see, e.g., [41, pp. 312-315] for the calculus of the Green function):

$$\begin{aligned} \psi_o(x) &= \frac{1}{\sinh(2/\varepsilon)} \big(b \sinh \frac{x+1}{\varepsilon} - a \sinh \frac{x-1}{\varepsilon}\big) \\ \psi_g(x) &= \frac{1}{2\varepsilon \sinh(2/\varepsilon)} \int_{-1}^{1} \big(\cosh \frac{2 - |x-y|}{\varepsilon} - \cosh \frac{x+y}{\varepsilon}\big) \varphi(y) dy. \end{aligned}$$

Integrating by parts and taking (2.2.8) into account, we have:

$$K_\varepsilon(\varphi, \psi) = \frac{1}{\varepsilon} \int_{-1}^{1} \varphi^2 - \frac{1}{\varepsilon} \int_{-1}^{1} (2\varphi\psi - (\psi)^2) - \varepsilon \int_{-1}^{1} \psi\psi'' + \varepsilon[\psi\psi']_{-1}^{1} =$$

$$\frac{1}{\varepsilon}\int_{-1}^{1}\varphi^2-\frac{1}{\varepsilon}\int_{-1}^{1}\varphi\psi_o-\frac{1}{\varepsilon}\int_{-1}^{1}\varphi\psi_g+\varepsilon[\psi\psi']_{-1}^{1}=:$$
$$\mathcal{I}_1^\varepsilon(\varphi)+\mathcal{I}_2^\varepsilon(\varphi)+\mathcal{I}_3^\varepsilon(\varphi)+\mathcal{I}_4^\varepsilon(\varphi).$$

A simple computation yields

$$-\frac{1}{\varepsilon}\int_{-1}^{-1+\delta}\varphi\psi_o-\frac{1}{\varepsilon}\int_{1-\delta}^{1}\varphi\psi_o=-\frac{a^2+b^2}{\tanh(2/\varepsilon)}+\sigma(\varepsilon),$$

and

$$\Big|\frac{1}{\varepsilon}\int_{-1+\delta}^{1-\delta}\varphi\psi_o\Big|\le\|\varphi\|_{L^1(-1,1)}(|a|+|b|)\frac{\sinh[(2-\delta)/\varepsilon]}{\varepsilon\sinh(2/\varepsilon)}.$$

Hence

$$\mathcal{I}_2^\varepsilon(\varphi)=-(a^2+b^2)+\sigma(\varepsilon).$$

Now we consider $\mathcal{I}_3^\varepsilon(\varphi)$.
Since $\cosh(s-t)=e^{-t}\sinh s+e^{-s}\cosh t$ and $\cosh(s+t)=\cosh s\cosh t+\sinh s\sinh t$, it turns out that:

$$\mathcal{I}_3^\varepsilon(\varphi)=\frac{1}{2\varepsilon^2\sinh(2/\varepsilon)}\Big[\Big(\int_{-1}^{1}\varphi(x)\cosh\frac{x}{\varepsilon}dx\Big)^2+\Big(\int_{-1}^{1}\varphi(x)\sinh\frac{x}{\varepsilon}dx\Big)^2\Big]-$$
$$\frac{1}{2\varepsilon^2}\int_{-1}^{1}\int_{-1}^{1}\varphi(x)\varphi(y)e^{-\frac{|x-y|}{\varepsilon}}dxdy-\frac{1}{2\varepsilon^2\sinh(2/\varepsilon)}\int_{-1}^{1}\int_{-1}^{1}\varphi(x)\varphi(y)e^{-\frac{2}{\varepsilon}}\cosh\frac{|x-y|}{\varepsilon}dxdy.$$

It is immediate to see that the last term is infinitesimal with ε uniformly with respect to $\varphi\in\mathcal{A}$. Consider now the first one; splitting the integral as above, we have:

$$\int_{-1}^{-1+\delta}\varphi(x)\cosh\frac{x}{\varepsilon}dx=a\varepsilon\big(\sinh\frac{1}{\varepsilon}-\sinh\frac{1-\delta}{\varepsilon}\big),$$
$$\Big|\int_{-1+\delta}^{1-\delta}\varphi(x)\cosh\frac{x}{\varepsilon}dx\Big|\le\|\varphi\|_{L^1(-1,1)}\cosh\frac{1-\delta}{\varepsilon},$$
$$\int_{1-\delta}^{1}\varphi(x)\cosh\frac{x}{\varepsilon}dx=b\varepsilon\big(\sinh\frac{1}{\varepsilon}-\sinh\frac{1-\delta}{\varepsilon}\big).$$

Therefore

$$\frac{1}{2\varepsilon^2\sinh(2/\varepsilon)}\Big(\int_{-1}^{1}\varphi(x)\cosh\frac{x}{\varepsilon}dx\Big)^2=\frac{1}{4}(a^2+b^2)+\sigma(\varepsilon).$$

In the same way we compute the analogous quadratic term. Thus we conclude that

$$\mathcal{I}_3^\varepsilon(\varphi)=-\frac{1}{2\varepsilon^2}\int_{-1}^{1}\int_{-1}^{1}e^{-\frac{|x-y|}{\varepsilon}}\varphi(x)\varphi(y)\,dxdy+\frac{1}{2}(a^2+b^2)+\sigma(\varepsilon).$$

Finally, we want to show that

$$I_4^\varepsilon(\varphi)=\sigma(\varepsilon). \tag{2.2.9}$$

A straightforward computation gives

$$[\psi_o\psi_o']_{-1}^{1} = \frac{a^2+b^2}{\varepsilon\tanh(2/\varepsilon)} - \frac{2ab}{\sinh(2/\varepsilon)}$$
$$[\psi_o\psi_g']_{-1}^{1} = -\frac{1}{\varepsilon^2\sinh(2/\varepsilon)}\Big(a\int_{-1}^{1}\varphi(y)\sinh\frac{1-y}{\varepsilon}dy + b\int_{-1}^{1}\varphi(y)\sinh\frac{1+y}{\varepsilon}dy\Big).$$

Splitting the integral as above we obtain

$$[\psi_o\psi_g']_{-1}^{1} = -\frac{a^2+b^2}{\varepsilon\tanh(2/\varepsilon)} + \sigma(\varepsilon).$$

Therefore (2.2.9) follows.

The results just proved allow us to say that:

$$K_\varepsilon(\varphi,\psi) = \frac{1}{\varepsilon}\int_{-1}^{1}\varphi^2 - \frac{1}{2\varepsilon^2}\int_{-1}^{1}\int_{-1}^{1} e^{-\frac{|x-y|}{\varepsilon}}\varphi(x)\varphi(y)dxdy - \frac{1}{2}(a^2+b^2) + \sigma(\varepsilon). \qquad (2.2.10)$$

Notice now that, since $\int_{-1}^{1} e^{-\frac{|x-y|}{\varepsilon}}dy = 2\varepsilon(1-e^{-\frac{1}{\varepsilon}}\cosh\frac{x}{\varepsilon})$,

$$\frac{1}{4\varepsilon^2}\int_{-1}^{1}\int_{-1}^{1} e^{-\frac{|x-y|}{\varepsilon}}(\varphi(x)-\varphi(y))^2dxdy = \frac{1}{\varepsilon}\int_{-1}^{1}\varphi^2 - \frac{e^{-\frac{1}{\varepsilon}}}{\varepsilon}\int_{-1}^{1}\varphi^2(x)\cosh\frac{x}{\varepsilon}dx$$
$$-\frac{1}{2\varepsilon^2}\int_{-1}^{1}\int_{-1}^{1} e^{-\frac{|x-y|}{\varepsilon}}\varphi(x)\varphi(y)dxdy.$$

Recalling (2.2.10), it is now sufficient to show that $\dfrac{e^{-\frac{1}{\varepsilon}}}{\varepsilon}\displaystyle\int_{-1}^{1}\varphi^2(x)\cosh\frac{x}{\varepsilon}dx - \frac{1}{2}(a^2+b^2) = \sigma(\varepsilon)$. This can be easily done splitting the integral in the same way as above. ■

Let us go back to the proof of (2.2.6), and apply the lemma just proved to the case $\mathcal{A} = \{\tilde{u}_j^i : j \in \mathbb{N}\}$. In view of the properties of $\{\tilde{u}_j^i\}$, the assumptions on $\mathcal{A}$ are satisfied for a suitable δ (depending on i), with $a = 0$ and $b = 1$. Since $F_{\varepsilon_j}(\tilde{u}_j^i,\psi) = \dfrac{1}{\varepsilon}\displaystyle\int_{-1}^{1} W(\tilde{u}_j^i) + K_{\varepsilon_j}(\tilde{u}_j^i,\psi)$, we obtain (2.2.6).

Collecting (2.2.3), (2.2.4), (2.2.5) and (2.2.6), it turns out that

$$F_{\varepsilon_j}(u_j,v_j) \geq \sum_{i=1}^{N} E_{\varepsilon_j}(\tilde{u}_j^i,(-1,1)) + o(1)_{j\to\infty}.$$

Notice now that $\tilde{u}_j^i \to \tilde{u}^i$ in $L^1(-1,1)$, where $\tilde{u}^i$ is defined by:

$$\tilde{u}^i(x) = \begin{cases} 0 & \text{if } x\in(-1,a_i) \\ u(x) & \text{if } x\in(a_i,b_i) \\ 1 & \text{if } x\in(b_i,1). \end{cases}$$

Then Theorem 2.2.4 (with $I = (-1,1)$) yields:

$$\liminf_{j\to\infty} F_{\varepsilon_j}(u_j, v_j) \geq \sum_{i=1}^{N} \liminf_{j\to\infty} E_{\varepsilon_j}(\tilde{u}_j^i, (-1,1)) \geq \sum_{i=1}^{N} \widetilde{F}(\tilde{u}^i) \geq c_W N.$$

The left-hand side is finite by assumption, and $N \leq \#S(u)$ is arbitrary: it follows that $\#S(u) < +\infty$, therefore $u \in BV(-1,1)$ (recall (1.0.5)). Furthermore

$$\liminf_{j\to\infty} F_{\varepsilon_j}(u_j, v_j) \geq c_W \#S(u).$$

This implies the required inequality (2.2.2).

2.2.2 A compactness result

Using the connection with the nonlocal functionals $\{E_\varepsilon\}$, we prove a one-dimensional version of the compactness result stated in Proposition 2.1.5.

Proposition 2.2.6. *Let $\{\varepsilon_n\}$ be an infinitesimal positive sequence, and $\{(u_n, v_n)\}$ a sequence in $[L^1(-1,1)]^2$, such that $F_{\varepsilon_n}(u_n, v_n)$ is uniformly bounded. Then, $\{(u_n, v_n)\}$ is relatively compact in $[L^1(-1,1)]^2$, and every cluster point (u,v) is such that $u, v \in BV(-1,1)$, and $u = v \in \{0,1\}$ a.e. in $(-1,1)$.*

Proof. Thanks to Lemma 2.1.7, we have that the sequence $\{(\tilde{u}_n, \tilde{v}_n)\}$ defined by $\tilde{u}_n = (u_n \wedge 1) \vee 0$ and $\tilde{v}_n = (v_n \wedge 1) \vee 0$ satisfies

$$F_{\varepsilon_n}(u_n, v_n) \geq F_{\varepsilon_n}(\tilde{u}_n, \tilde{v}_n), \quad \|u_n - \tilde{u}_n\|_{L^1},\ \|v_n - \tilde{v}_n\|_{L^1} \to 0 \ \text{ as } \ n \to +\infty.$$

Step 1. Let us suppose that $\mathcal{A} = \{\tilde{u}_n\}$ satisfies the hypotheses of Lemma 2.2.5 with $a, b \in \{0,1\}$, and $\tilde{v}_n(-1) = a$, $\tilde{v}_n(1) = b$ for every n.

Then

$$\inf\{K_{\varepsilon_n}(\tilde{u}_n, \psi) : \ \psi \in H^1_{a,b}(-1,1)\} =$$
$$\frac{1}{4\varepsilon^2} \int_{-1}^{1}\int_{-1}^{1} e^{-\frac{|x-y|}{\varepsilon}} (\tilde{u}_n(x) - \tilde{u}_n(y))^2 dx dy + o(1)_{n\to+\infty}.$$

Hence,

$$E_{\varepsilon_n}(\tilde{u}_n) \quad \text{uniformly bounded,}$$

and the compactness theorem for the functionals E_ε proven by G. Alberti and G. Bellettini (see [2, Theorem 3.1]) gives the existence of a subsequence of $\{\tilde{u}_n\}$, not relabelled, converging to u in $L^1(-1,1)$. Since $\int_{-1}^{1} (\tilde{u}_n - \tilde{v}_n)^2 \leq C\varepsilon_n$, it follows that also $\{\tilde{v}_n\}$ has a subsequence converging in $L^1(-1,1)$ to u. Since $\|\tilde{u}_n - u_n\|_{L^1}$ and $\|\tilde{v}_n - v_n\|_{L^1}$ are infinitesimal, we obtain the relative compactness for the sequence $\{(u_n, v_n)\}$, and the result follows recalling Proposition 2.2.3.

Step 2. Fixed $\delta \in (0,1)$, we define, for every n such that $\varepsilon_n \in (0,\delta)$:

$$\overline{u}_n(t) = \begin{cases} 0 & \text{in} \quad (-1,-1+\delta] \\ \tilde{u}_n(t) & \text{in} \quad (-1+\delta, 1-\delta) \\ 1 & \text{in} \quad [1-\delta, 1) \end{cases}$$

$$\overline{v}_n(t) = \begin{cases} 0 & \text{in} \quad (-1,-1+\delta-\varepsilon_n] \\ \frac{v_n(-1+\delta)}{\varepsilon_n}(t+1-\delta+\varepsilon_n) & \text{in} \quad (-1+\delta-\varepsilon_n, -1+\delta] \\ \tilde{v}_n(t) & \text{in} \quad (-1+\delta, 1-\delta) \\ \frac{1-v_n(1-\delta)}{\varepsilon_n}(t-1+\delta-\varepsilon_n)+1 & \text{in} \quad [1-\delta, 1-\delta+\varepsilon_n) \\ 1 & \text{in} \quad [1-\delta+\varepsilon_n, 1). \end{cases}$$

It follows that

$$F_{\varepsilon_n}(\overline{u}_n, \overline{v}_n) \le F_{\varepsilon_n}(\tilde{u}_n, \tilde{v}_n) + c,$$

where c is a positive constant independent of n and δ.

The sequence $\{(\overline{u}_n, \overline{v}_n)\}$ satisfies the hypotheses required for the Step 1, then, possibly passing to a subsequence, $\overline{u}_n$ and $\overline{v}_n$ converge in $L^1(-1,1)$ to $u^\delta \in BV(-1,1)$ such that $u^\delta \in \{0,1\}$) a.e. Clearly, this implies that for every $\delta \in (0,1)$ the sequence $\{(u_n, v_n)\}$ is relatively compact in $[L^1(-1+\delta, 1-\delta)]^2$.

A diagonalization argument concludes the proof. ■

Clearly, the assert of Proposition 2.1.5 also holds for the functionals $F_\varepsilon^\beta(\cdot,\cdot;I)$, for every $\beta > 0$ and I bounded open subset of $\mathbb{R}$.

2.2.3 The Γ-upper limit

In this subsection we prove the following:

Proposition 2.2.7. *Let $\eta > 0$ and $\{\varepsilon_j\}$ be a positive infinitesimal sequence. For every u, v in $L^1(-1,1)$ there exist sequences $\{u_j^\eta\}, \{v_j^\eta\}$ converging, in $L^1(-1,1)$ as $j \to \infty$, to u and v respectively, and such that:*

$$\limsup_{j\to\infty} F_{\varepsilon_j}(u_j^\eta, v_j^\eta) \le F(u,v) + \eta. \tag{2.2.11}$$

Proof. Notice that if $F(u,v) = +\infty$ or $S(u) = \emptyset$ the inequality (2.2.11) is obvious. Then, since the approximation is invariant under translations and reflections with respect to $u = 1/2$, and modifies u only in a small neighbourhood of $S(u)$, we can deal only with the case $\#S(u) = 1$, and it is enough to prove (2.2.11) for $u = v = \chi_{(0,1)}$, the characteristic function of $(0,1)$. Then $F(u,v) = c_W = c_W(1)$. In view of the representation of $c_W(\alpha)$ which will be given in Proposition 2.4.1, for every $\eta > 0$ we can find $\varphi_\eta \in X$ and $\psi_\eta \in X \cap Y$ such that $\varphi_\eta = \psi_\eta = \chi_{(0,+\infty)}$ on $\mathbb{R} \setminus (-T_\eta, T_\eta)$ for some $T_\eta > 0$, and

$$\mathcal{F}^1(\varphi_\eta, \psi_\eta) \le c_W + \eta.$$

Define now $u_\varepsilon^\eta(x) = \varphi_\eta(\frac{x}{\varepsilon})$ and $v_\varepsilon^\eta(x) = \psi_\eta(\frac{x}{\varepsilon})$ for every $\varepsilon > 0$. Clearly, $\{u_\varepsilon^\eta\}$ and $\{v_\varepsilon^\eta\}$ converge to $u = \chi_{(0,1)}$ in $L^1(-1,1)$. Moreover:

$$F_\varepsilon(u_\varepsilon^\eta, v_\varepsilon^\eta) = F_\varepsilon(u_\varepsilon^\eta, v_\varepsilon^\eta; \mathbb{R}) = \mathcal{F}^1(\varphi_\eta, \psi_\eta).$$

Then, we have

$$F(u,v)+\eta \geq F_\varepsilon(u_\varepsilon^\eta, v_\varepsilon^\eta)$$

for every $\varepsilon < 1/T_\eta$. ■

2.3 The n-dimensional case

In this section we prove Theorem 2.1.1 in the general case Ω open subset of $\mathbb{R}^n$. We assume $\alpha = 1$ (recall Remark 2.1.4 (a)).

For every $u, v \in L^1(\Omega)$ and A open subset of Ω let

$$F'(u,v;A) = \inf\bigl\{\liminf_{j\to\infty} F_{\varepsilon_j}(u_j, v_j; A) : \quad \varepsilon_j \to 0,\ u_j \to u,\ v_j \to v \text{ in } L^1(A)\bigr\}$$

(Γ-lower limit of $\{F_\varepsilon(\cdot,\cdot;A)\}$).

Proposition 2.3.1. *For every $u, v \in L^1(\Omega)$*

$$F'(u,v;\Omega) \geq F(u,v;\Omega).$$

Proof. The proof follows the steps outlined in [15], where the *slicing method* is applied to get the lower semicontinuity inequality for Γ-limits.

Let $E \subseteq \mathbb{R}^n$, and $f\colon E \to \mathbb{R}$; for every $\xi \in S^{n-1}$ let E^ξ be the orthogonal projection of E on $\xi^\perp$ (the $(n-1)$-dimensional vector space orthogonal to ξ); for every $y \in E^\xi$ let $E^{\xi,y} = \{t \in \mathbb{R} :\ y + t\xi \in E\}$, and let $f^{\xi,y}\colon E^{\xi,y} \to \mathbb{R}$ be defined by $f^{\xi,y}(t) = f(y+t\xi)$.

We recall the following basic facts (see, e.g., [8, § 3.11]).

(i) Let $u \in BV(\Omega)$ and let $D^k u$ stands for any of D^a, D^j or D^c (the absolutely continuous, jump or Cantor part of the derivative). Then, for every $\xi \in S^{n-1}$ and for $\mathcal{H}^{n-1}$-a.e. $y \in \Omega^\xi$ we have $u^{\xi,y} \in BV(\Omega^{\xi,y})$; moreover, denoting by $\langle D^k u, \xi\rangle$ the component of $D^k u$ along ξ, the following representation holds:

$$\int_B \langle D^k u, \xi\rangle = \int_{B^\xi} D^k u^{\xi,y}(B^{\xi,y})\, d\mathcal{H}^{n-1}(y).$$

(ii) Let $u \in L^1(\Omega)$; assume that $u^{\xi,y} \in BV(\Omega^{\xi,y})$ for every ξ in a basis of $\mathbb{R}^n$ and for a.e. $y \in \Omega^\xi$, and that

$$\int_{\Omega^\xi} |Du^{\xi,y}|(\Omega^{\xi,y})\, d\mathcal{H}^{n-1}(y) < +\infty.$$

Then $u \in BV(\Omega)$.

Let g_* be the polar function of g, i.e.

$$g_*(\xi) = \sup\Bigl\{\frac{\xi\cdot\eta}{g(\eta)} :\quad \eta \in S^{n-1}\Bigr\}, \qquad \xi \in \mathbb{R}^n.$$

It is not difficult to see that $(g_*)_*$ is the convex hull of g. For every I open subset of $\mathbb{R}$ and $\xi \in S^{n-1}$ denote by ϕ_ε^ξ the functional F_ε^β in Remark 2.1.4 (b), with $\beta = 1/g_*(\xi)$, i.e.

$$\phi_\varepsilon^\xi(\varphi, \psi; I) := \begin{cases} \frac{1}{\varepsilon}\int_I W(\varphi) + \frac{1}{\varepsilon}\int_I (\varphi-\psi)^2 + \frac{\varepsilon}{g_*^2(\xi)}\int_I (\psi)'^2 & \text{if } \psi \in H^1(I) \\ +\infty & \text{otherwise in } \left[L^1(I)\right]^2. \end{cases}$$

By Corollary 2.2.2 the Γ-limit as $\varepsilon \to 0$ is

$$\phi^\xi(\varphi, \psi; I) = \begin{cases} \frac{c_W}{g_*(\xi)}\#S(\varphi) & \varphi, \psi \in BV(I),\ \varphi = \psi \text{ and } \varphi \in \{0,1\} \text{ a.e.} \\ +\infty & \text{otherwise} \end{cases}$$

where c_W is as in Theorem 2.2.1. Let

$$F_\varepsilon^\xi(u, v; A) = \int_{A^\xi} \phi_\varepsilon^\xi(u^{\xi,y}, v^{\xi,y}; A^{\xi,y})d\mathcal{H}^{n-1}(y) \tag{2.3.1}$$

for every open set A of Ω and for every $u, v \in L^1(A)$. An application of Fubini's Theorem yields that if $u, v \in L^1(A)$ and $v \in H^1(A)$ then

$$F_\varepsilon^\xi(u, v; A) = \frac{1}{\varepsilon}\int_A W(u) + \frac{1}{\varepsilon}\int_A (u-v)^2 + \frac{\varepsilon}{g_*^2(\xi)}\int_A |\langle Dv, \xi\rangle|^2$$

Clearly, $F_\varepsilon^\xi \le F_\varepsilon$ (recall that $g(-\eta) = g(\eta)$ for every η). Therefore, by Fatou's lemma

$$F'(u, v; A) \ge F^\xi(u, v; A),$$

where

$$F^\xi(u, v; A) = \int_{A^\xi} \phi^\xi(u^{\xi,y}, v^{\xi,y}; A^{\xi,y})d\mathcal{H}^{n-1}(y).$$

In view of the explicit form of ϕ^ξ and by property (ii) above, if $F'(u, v; A)$ is finite, then $u, v \in BV(A)$, $u = v$ and $u \in \{0,1\}$ a.e.; furthermore, by (i):

$$F^\xi(u, v; A) = c_W \int_{S(u)\cap A} \frac{|\langle \xi, \nu_u\rangle|}{g_*(\xi)} d\mathcal{H}^{n-1}.$$

Then,

$$F'(u, v; A) \ge c_W \int_{S(u)\cap A} \frac{|\langle \xi, \nu_u\rangle|}{g_*(\xi)} d\mathcal{H}^{n-1}$$

for every open subset A, $\xi \in S^{n-1}$ and $u, v \in L^1(A)$. Since the functionals F_ε are local, the set function $\mu(A) = F'(u, v; A)$ is superadditive on disjoint open sets; thus, by Proposition 1.0.1,

$$F'(u, v; A) \ge c_W \int_{S(u)\cap A} \sup_{\xi \in S^{n-1}} \frac{|\langle \xi, \nu_u\rangle|}{g_*(\xi)} d\mathcal{H}^{n-1} = c_W \int_{S(u)\cap A} \sup_{\xi \in S^{n-1}} \frac{\langle \xi, \nu_u\rangle}{g_*(\xi)} d\mathcal{H}^{n-1}.$$

The conclusion follows since, as noticed above, the integrand function of the last term is the convex hull of g. ■

The slicing method gives some compactness criteria for the n-dimensional case. Following [5], we recall the simplest result which connects the relative compactness of a subset $\mathcal{U}$ of $L^1(\Omega)$ with the relative compactness of the set

$$\mathcal{U}^{\xi,y} = \{u^{\xi,y} \ : \ u \in \mathcal{U}\}$$

in $L^1(\Omega^{\xi,y})$. If there exists n linearly independent unit vectors ξ such that

$$\mathcal{U}^{\xi,y} \quad \text{is relatively compact for} \quad \mathcal{H}^{n-1}\text{- a.a.} \quad y \in \Omega^{\xi},$$

then $\mathcal{U}$ is relatively compact in $L^1(\Omega)$.

As in [1], we observe that this criterion can not apply to our case. A more general result is obtained in [5] replacing $\mathcal{U}$ with a perturbation. Precisely, for every $\delta > 0$, we say that a family $\mathcal{U}'$ is δ-dense in $\mathcal{U}$ if $\mathcal{U}$ lies in a δ-neighbourhood of $\mathcal{U}'$ in $L^1(\Omega)$. Then:

Lemma 2.3.2. (Compactness criterion, [5, Theorem 6.6]) *Let $\mathcal{U}$ be a family of functions $u\colon \Omega \to [0,1]$ and assume that there exist n linearly independent unit vectors ξ which satisfy the following property:*

> *for every $\delta > 0$ there exists a family $\mathcal{U}^{\delta}$ δ-dense in $\mathcal{U}$ such that $(\mathcal{U}^{\delta})^{\xi,y}$ is relatively compact in $L^1(\Omega^{\xi,y})$ for $\mathcal{H}^{n-1}$-a.a. $y \in \Omega^{\xi,y}$.*

Then, $\mathcal{U}$ is relatively compact in $L^1(\Omega)$.

Proof of Proposition 2.1.5. It is clear from Lemma 2.1.7 that we can restrict ourselves to functions which take values in $[0,1]$.

Let $\{(u_\varepsilon, v_\varepsilon)\}$ be a sequence in $[L^1(\Omega;[0,1])]^2$, such that $F_\varepsilon(u_\varepsilon, v_\varepsilon) \leq C$ for every ε, where C is a constant independent of ε. We have to prove that $\{(u_\varepsilon, v_\varepsilon)\}$ satisfies the hypotheses of Lemma 2.3.2.

Let $\xi \in S^{n-1}$. As in [1, §3.11], for $\delta > 0$ fixed and for every $\varepsilon > 0$ we define $u_\varepsilon^\delta, v_\varepsilon^\delta : \Omega \to [0,1]$ so that

$$(u_\varepsilon^\delta)^{\xi,y} = \begin{cases} u_\varepsilon^{\xi,y} & \text{for } \ y \in \Omega^{\xi} \quad \text{s.t.} \quad \phi_\varepsilon^{\xi}(u_\varepsilon^{\xi,y}, v_\varepsilon^{\xi,y}; \Omega^{\xi,y}) \leq C/\delta \\ 0 & \text{otherwise in } \ \Omega^{\xi}, \end{cases} \tag{2.3.2}$$

and the same for $(v_\varepsilon^\delta)^{\xi,y}$. Recalling that $C \geq F_\varepsilon(u_\varepsilon, v_\varepsilon) \geq F_\varepsilon^\xi(u_\varepsilon, v_\varepsilon)$, where F_ε^ξ is the functional defined in (2.3.1), we obtain that $(u_\varepsilon^\delta)^{\xi,y} = u_\varepsilon^{\xi,y}$ and $(v_\varepsilon^\delta)^{\xi,y} = v_\varepsilon^{\xi,y}$ for y in Ω^ξ except for a set R_δ such that $|R(\delta)| \leq \delta$. It follows that

$$||u_\varepsilon - u_\varepsilon^\delta||_{L^1}, \ ||v_\varepsilon - v_\varepsilon^\delta||_{L^1} \leq c\delta,$$

where c depends only on Ω. Thus, $\{u_\varepsilon^\delta\}$ and $\{v_\varepsilon^\delta\}$ are δ-dense in $\{u_\varepsilon\}$ and $\{v_\varepsilon\}$ respectively.

Recalling that $\phi_\varepsilon^\xi(0,0;\Omega^{\xi,y})) = 0$, we have that

$$\phi_\varepsilon^\xi((u_\varepsilon^\delta)^{\xi,y}, (v_\varepsilon^\delta)^{\xi,y}; \Omega^{\xi,y}) \leq C/\delta;$$

Then, by the one-dimensional version of Theorem 2.1.5 stated in Proposition 2.2.6, the sequences $\{(u_\varepsilon^\delta)^{\xi,y}\}$ and $\{(v_\varepsilon^\delta)^{\xi,y}\}$ are relatively compact in $L^1(\Omega^{\xi,y})$ for $\mathcal{H}^{n-1}$-a.a. $y \in \Omega^\xi$.

Then, the thesis follows from Lemma 2.3.2, choosing n independent $\xi \in S^{n-1}$. ∎

We have now to estimate the Γ-upper limit of $\{F_\varepsilon(\cdot,\cdot;\Omega)\}$, i.e.

$$F''(u,v;\Omega)=\inf\bigl\{\limsup_{j\to\infty}F_{\varepsilon_j}(u_j,v_j;\Omega):\ \ \varepsilon_j\to 0,\ u_j\to u,\ v_j\to v \text{ in } L^1(\Omega)\bigr\}.$$

Let us consider the functional

$$\mathcal{G}(u)=\int_{S(u)}g(\nu_u)d\mathcal{H}^{n-1},$$

defined for $u\in BV(\Omega;\{0,1\})$. By general results on functionals defined on partitions (see [6] and [7], in particular § 3) the lower semicontinuous envelope of $\mathcal{G}$ can be written in the form

$$\overline{\mathcal{G}}(u)=\int_{S(u)}j(\nu_u)d\mathcal{H}^{n-1},$$

where j is the greatest BV-elliptic function less than g; hence ([7, Example 2.8]) j is convex, so that

$$\overline{\mathcal{G}}(u)\le\int_{S(u)}g^{**}(\nu_u)d\mathcal{H}^{n-1}, \tag{2.3.3}$$

with g^{**} denoting the convex hull of g.

Proposition 2.3.3. *For every $u,v\in L^1(\Omega)$*

$$F''(u,v;\Omega)\le F(u,v;\Omega).$$

Proof. Let $\mathcal{D}$ be the family of the characteristic functions of the sets $E=\Omega\cap A$, with A of class C^∞ in a neighbourhood of Ω. If $F(u,v;\Omega)$ is finite, it can be proved that sequences $\{u_k\}$ and $\{v_k\}$ exist in $\mathcal{D}$ such that $F(u_k,v_k;\Omega)\to F(u,v;\Omega)$ (see [15], Proposition 4.7 and Remark 4.8). By the lower semicontinuity of F'' it is enough to prove the proposition when $u=v\in\mathcal{D}$.

Let $u=v=\chi_E\in\mathcal{D}$. Then $E=\Omega\cap A$, with A set of class C^∞ in a neighbourhood of Ω. Let Ω' be fixed, with $\Omega\subset\subset\Omega'$. There exists $\eta_0>0$ such that for every $0<\eta<\eta_0$, the projection

$$\pi\colon E_\eta=\{x\in\Omega:\ d(x,\partial A)<\eta\}\to\partial A\cap\Omega'$$

is well-defined. Let $d(x)=d(x,\Omega\setminus A)-d(x,A)$; it turns out that for η_0 small enough d is smooth in E_η and $\nu(x)=\nabla d(x)$ is a normal unit vector to ∂A in $\pi(x)$ if $x\in E_\eta$.

Let $0<\eta<\eta_0$ be fixed and $M=\max\{1/g(\xi):\ \xi\in S^{n-1}\}$. As in the proof of Proposition 2.2.7, for any fixed $\eta>0$ we can find $\varphi_\eta\in X$, $\psi_\eta\in X\cap Y$ such that $\varphi_\eta=\psi_\eta=\chi_{(0,+\infty)}$ on $\mathbb{R}\setminus(-MT,MT)$, for some $T=T_\eta>0$, and

$$\mathcal{F}^1(\varphi_\eta,\psi_\eta)\le c_W+\eta.$$

For every $\varepsilon>0$, with $T\varepsilon<\eta$, define $u_\varepsilon\colon\Omega\to\mathbb{R}$ as follows:

$$u_\varepsilon(x)=\begin{cases}\varphi_\eta\bigl(\frac{d(x)}{\varepsilon g(\nu(x))}\bigr) & \text{if } |d(x)|\le T\varepsilon\\ 0 & \text{otherwise in } \Omega\setminus E\\ 1 & \text{otherwise in } E.\end{cases}$$

In the same way we define v_ε on Ω.

Now we want to estimate $F_\varepsilon(u_\varepsilon, v_\varepsilon; \Omega)$. Since $d, g, 1/g, \nu$ are Lipschitz on $\{|d| \le T\varepsilon\}$, we can find a positive constant L such that

$$\varepsilon g^2(Dv_\varepsilon(x)) = \frac{1}{\varepsilon}\Big|(\psi_\eta)'\Big(\frac{d(x)}{\varepsilon g(\nu(x))}\Big)\Big|^2 g^2\left(\frac{\nabla d(x)}{g(\nu(x))} + d(x)\nabla\big(\frac{1}{g(\nu(x))}\big)\right)$$

$$\le \frac{1}{\varepsilon}\Big|(\psi_\eta)'\Big(\frac{d(x)}{\varepsilon g(\nu(x))}\Big)\Big|^2\left[g^2\left(\frac{\nu(x)}{g(\nu(x))}\right) + \varepsilon L\right] \le \Big|(\psi_\eta)'\Big(\frac{d(x)}{\varepsilon g(\nu(x))}\Big)\Big|^2\Big(\frac{1}{\varepsilon} + L\Big).$$

In the following computation we shall drop the subscript η. Taking the coarea formula into account (see, e.g., [8, § 2.12]), we get:

$$F_\varepsilon(u_\varepsilon, v_\varepsilon; \Omega) \le$$

$$\le \big(\frac{1}{\varepsilon} + L\big)\int_{\Omega\cap\{|d|\le T\varepsilon\}}\Big[W\Big(\varphi\big(\frac{d(x)}{\varepsilon g(\nu(x))}\big)\Big) + \Big(\varphi\big(\frac{d(x)}{\varepsilon g(\nu(x))}\big) - \psi\big(\frac{d(x)}{\varepsilon g(\nu(x))}\big)\Big)^2 + \Big|\psi'\big(\frac{d(x)}{\varepsilon g(\nu(x))}\big)\Big|^2\Big]\,dx$$

$$\le \big(\frac{1}{\varepsilon} + L\big)\int_{-T\varepsilon}^{T\varepsilon}\int_{\{d(x)=t\}}\Big[W\Big(\varphi\big(\frac{t}{\varepsilon g(\nu(x))}\big)\Big) + \Big(\varphi\big(\frac{t}{\varepsilon g(\nu(x))}\big) - \psi\big(\frac{t}{\varepsilon g(\nu(x))}\big)\Big)^2 + \Big|\psi'\big(\frac{t}{\varepsilon g(\nu(x))}\big)\Big|^2\Big]\,d\mathcal{H}^{n-1}(x)\,dt$$

$$\le \big(\frac{1}{\varepsilon} + L\big)\int_{-T\varepsilon}^{T\varepsilon}\int_{\partial E\cap\Omega'}\Big[W\Big(\varphi\big(\frac{t}{\varepsilon g(\nu(y))}\big)\Big) + \Big(\varphi\big(\frac{t}{\varepsilon g(\nu(y))}\big) - \psi\big(\frac{t}{\varepsilon g(\nu(y))}\big)\Big)^2 + \Big|\psi'\big(\frac{t}{\varepsilon g(\nu(y))}\big)\Big|^2\Big]\,d\mathcal{H}^{n-1}(y)\,dt + o(1)_{\varepsilon\to 0}$$

$$\le \big(1 + \varepsilon L\big)\int_{-\infty}^{+\infty}\Big[W\big(\varphi(t)\big) + \big(\varphi(t) - \psi(t)\big)^2 + |\psi'(t)|^2\Big]\,dt\int_{\partial E\cap\Omega'} g\big((\nu(y)\big)\,d\mathcal{H}^{n-1}(y) + o(1)_{\varepsilon\to 0}$$

$$\le \big(1 + \varepsilon L\big)(c_W + \eta)\int_{\partial E\cap\Omega'} g\big(\nu(y)\big)\,d\mathcal{H}^{n-1}(y) + o(1)_{\varepsilon\to 0}$$

Since Ω' and η are arbitrary, we deduce that

$$F''(u, v; \Omega) \le c_W\ \mathcal{G}(u).$$

The same inequality holds for the relaxed functionals; by the lower semicontinuity of F'' and by (2.3.3), we conclude that

$$F''(u, v; \Omega) \le c_W\int_{S(u)} g^{**}(\nu_u)\,d\mathcal{H}^{n-1},$$

and Proposition 2.3.3 is completely proved. ■

2.3.1 Mass constraint

Let us consider, for $0 < m < |\Omega|$, the set

$$U_m^0 = \Big\{(u,v) \in [BV(\Omega)]^2 \ : \ u = v \in \{0,1\} \ \text{ a.e.}, \ \int_\Omega v = m\Big\}.$$

For $(u,u) \in U_m^0$, let $\{(u_\varepsilon, v_\varepsilon)\}$ be the recovery sequence converging to (u,u) constructed in the proof of Proposition 2.3.3.

Now, by modifying $\{(u_\varepsilon, v_\varepsilon)\}$, we shall construct a sequence $\{(\tilde u_\varepsilon, \tilde v_\varepsilon)\}$ such that $\tilde u_\varepsilon, \tilde v_\varepsilon$ converge to u in $L^1(\Omega)$ and

$$\limsup_{\varepsilon\to 0} F_\varepsilon(\tilde u_\varepsilon, \tilde v_\varepsilon) = \limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, v_\varepsilon),$$

satisfying the additional property

$$\int_\Omega v_\varepsilon = \int_\Omega u = m \quad \text{for every } \ \varepsilon > 0.$$

The proof follows from the following steps.

1. Notice that

$$\Big|\int_\Omega v_\varepsilon - m\Big| \le \int_\Omega |v_\varepsilon - v| \le \int_{I_\varepsilon} |v_\varepsilon - v| \le 2|I_\varepsilon| = O(\varepsilon)_{\varepsilon\to 0},$$

where $I_\varepsilon = \{|d(x)| \le T_\eta \varepsilon\}$, and d stands for the signed distance from ∂E (see the proof of Proposition 2.3.3). Then,

$$\lambda_\varepsilon = m - \int_\Omega v_\varepsilon = O(\varepsilon)_{\varepsilon\to 0}.$$

2. Set

$$\tilde v_\varepsilon = v_\varepsilon + \frac{\lambda_\varepsilon}{|\Omega|}, \qquad \tilde u_\varepsilon = u_\varepsilon;$$

we prove that

$$\limsup_{\varepsilon\to 0} F_\varepsilon(\tilde u_\varepsilon, \tilde v_\varepsilon) = \limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, v_\varepsilon).$$

Clearly, it is sufficient to deal with the term $\frac{1}{\varepsilon}\int_\Omega(\tilde u_\varepsilon - \tilde v_\varepsilon)^2$.

Since $\lambda_\varepsilon = O(\varepsilon)_{\varepsilon\to 0}$ and $\lim_{\varepsilon\to 0}\int_\Omega |u_\varepsilon - v_\varepsilon| = 0$, then

$$\frac{1}{\varepsilon}\int_\Omega(\tilde u_\varepsilon - \tilde v_\varepsilon)^2 \le \frac{1}{\varepsilon}\int_\Omega (u_\varepsilon - v_\varepsilon)^2 + \frac{\lambda_\varepsilon^2}{\varepsilon|\Omega|} + \frac{2}{\varepsilon}\int_\Omega \frac{|\lambda_\varepsilon|\,|u_\varepsilon - v_\varepsilon|}{|\Omega|} = \frac{1}{\varepsilon}\int_\Omega (u_\varepsilon - v_\varepsilon)^2 + o(1)_{\varepsilon\to 0}.$$

Setting, in $[L^1(\Omega)]^2$:

$$I_m(u,v) = \begin{cases} 0 & \text{if } \int_\Omega v = m, \\ +\infty & \text{otherwise,} \end{cases}$$

the previous construction proves the following

Proposition 2.3.4. **(Γ-convergence with mass constraint)** *The sequence of functionals* $\{F_\varepsilon + I_m\}$ Γ*-converges in* $[L^1(\Omega)]^2$ *to* $F + I_m$.

Remark 2.3.5. The compactness result for the sequence $\{F_\varepsilon + I_m\}$ follows *a fortiori* from the corresponding result for $\{F_\varepsilon\}$.

2.4 Some properties of $c_W(\alpha)$

Proposition 2.4.1. *Let* $c_W(\alpha)$ *be as in Theorem 2.1.1. Then*

$$\begin{aligned} c_W(\alpha) &= \sqrt{\alpha}\inf\{\mathcal{E}^\alpha(\varphi) :\ \varphi \in \mathcal{U}\} \\ &= \inf\{\mathcal{F}^\alpha(\varphi,\psi) :\ \varphi \in \mathcal{U}, \quad \psi \in \mathcal{U} \cap Y\}, \end{aligned}$$

where $\mathcal{U}$ *is any of the following spaces:*

> X : *the set of measurable* $\varphi\colon \mathbb{R} \to [0,1]$ *such that* $\lim_{x\to-\infty} \varphi(x) = 0$ *and* $\lim_{x\to-\infty} \varphi(x) = 1$;
>
> X^* : *the set of measurable* $\varphi\colon \mathbb{R} \to [0,1]$ *with* $\varphi = \chi_{(0,+\infty)}$ *on* $\mathbb{R} \setminus (-R,R)$ *for some* $R > 0$;
>
> Z^* : *the subset of the increasing elements of* X^*,

and, as usual, Y *is the set of* $\varphi \in AC(\mathbb{R})$ *such that* $\varphi' \in L^2(\mathbb{R})$.

Proof. It is enough to deal with the case $\alpha = 1$ (see Remarks 2.1.3 and 2.1.4).
Let

$$\Phi_{\mathcal{U}} = \inf\{\mathcal{F}^1(\varphi,\psi) :\ \varphi \in \mathcal{U}, \quad \psi \in \mathcal{U} \cap Y\}, \quad I_{\mathcal{U}} = \inf\{\mathcal{E}^1(\varphi) :\ \varphi \in \mathcal{U}\}.$$

The proof consists in showing that $\Phi_X = \Phi_{X^*} = \Phi_{Z^*}$, $I_X = I_{X^*} = I_{Z^*}$ and $\Phi_{X^*} = I_{X^*}$. For the first set of equalities, we refer to Chapter 3 (Sec. 3.1), where the optimal profile problem for the functional $\mathcal{F}^1$ is studied.

Step 1. $\inf\{\mathcal{E}^1(\varphi) :\ \varphi \in X\} = \inf\{\mathcal{E}^1(\varphi) :\ \varphi \in X^*\}$.

We have only to prove that the right-hand side does not exceed the left-hand one. Let $\sigma > 1$ and $\varphi_\sigma \in X$ such that $\mathcal{E}^1(\varphi_\sigma) \le \inf\{\mathcal{E}^1(\varphi) :\ \varphi \in X\}$; let $\overline{\varphi}_\sigma = \frac{1}{2} + \sigma(\varphi_\sigma - \frac{1}{2})$ and $\widetilde{\varphi}_\sigma = (\overline{\varphi}_\sigma \vee 0) \wedge 1$. Then $\widetilde{\varphi}_\sigma \in X^*$, and

$$\mathcal{E}^1(\widetilde{\varphi}_\sigma) \le \mathcal{E}^1(\overline{\varphi}_\sigma) = \sigma^2 \mathcal{E}^1(\varphi_\sigma) \le \sigma^3 \inf\{\mathcal{E}^1(\varphi) :\ \ \varphi \in X\}.$$

Step 2. Let $\varphi \in X^*$. We claim that

$$\inf\{\mathcal{F}^1(\varphi,\psi) :\ \ \psi \in X^* \cap Y\} = \mathcal{E}^1(\varphi). \tag{2.4.1}$$

For every $R > 0$ let $X^*(R) = \{\varphi \in X : \varphi = \chi_{(0,+\infty)}$ on $\mathbb{R} \setminus (-R, R)\}$; then $X^* = \bigcup_{R>0} X^*(R)$.

By assumption $\varphi \in X^*(R)$ for some $R > 0$. If $T > 2R$, let $\varphi_T(x) = \varphi(Tx)$. Apply Lemma 2.2.5 with $\mathcal{A} = \{\varphi_T : T > 2R\}$. Then for every $\varepsilon > 0$

$$\inf\Big\{\frac{1}{\varepsilon}\int_{\mathbb{R}}(\varphi_T - \psi)^2 + \varepsilon\int_{\mathbb{R}}(\psi')^2 : \quad \psi \in X \cap Y\Big\} =$$

$$\frac{1}{4\varepsilon^2}\int_{-1}^{1}\int_{-1}^{1} e^{-|x-y|/\varepsilon}(\varphi(x) - \varphi(y))^2\, dxdy + \sigma(\varepsilon),$$

with $\lim_{\varepsilon\to 0}\sigma(\varepsilon) = 0$ uniformly with respect to $T > 2R$. Let $\varepsilon = 1/T$ and $\widetilde{\psi}(t) = \psi(t/T)$ whenever $\psi \in H^1_{0,1}(-1,1)$; then, by a linear change of variable, we get

$$\inf\Big\{\int_{-T}^{T}(\varphi - \widetilde{\psi})^2 + \varepsilon\int_{-T}^{T}(\widetilde{\psi}')^2 : \quad \psi \in H^1_{0,1}(-1,1)\Big\} =$$

$$\frac{1}{4}\int_{-T}^{T}\int_{-T}^{T} e^{-|t-s|}(\varphi(t) - \varphi(s))^2\, dtds + o(1)_{T\to+\infty},$$

i.e.

$$\inf\Big\{\int_{\mathbb{R}}(\varphi - \psi)^2 + \varepsilon\int_{\mathbb{R}}(\psi')^2 : \quad \psi \in X^*(T) \cap Y\Big\} =$$

$$\frac{1}{4}\int_{\mathbb{R}}\int_{\mathbb{R}}^{-|t-s|}(\varphi(t) - \varphi(s))^2\, dtds + o(1)_{T\to+\infty}.$$

It is now enough to notice that

$$\inf\{\mathcal{F}^1(\varphi,\psi) : \quad \psi \in X^* \cap Y\} =$$

$$\int_{\mathbb{R}} W(\varphi) + \lim_{T\to+\infty}\inf\Big\{\int_{\mathbb{R}}(\varphi - \psi)^2 + \varepsilon\int_{\mathbb{R}}(\psi')^2 : \quad \psi \in X^*(T) \cap Y\Big\}.$$

<u>Step 3.</u> Consider the infimum of both sides of (2.4.1) when φ varies in X^*; then

$$\Phi_{X^*} = \inf\{\mathcal{E}^1(\varphi) : \varphi \in X^*\}.$$

If we restrict φ in (2.4.1) to vary in Z^*, then

$$\inf\{\mathcal{E}^1(\varphi) : \varphi \in Z^*\} = \inf\{\mathcal{F}^1(\varphi,\psi) : \varphi \in Z^*,\ \psi \in X^* \cap Y\} \le \Phi_{Z^*} = \Phi_{X^*}.$$

We conclude that

$$\Phi_{X^*} = \inf\{\mathcal{E}^1(\varphi) : \varphi \in Z^*\} = \inf\{\mathcal{E}^1(\varphi) : \varphi \in X^*\}.$$

■

Proof of Proposition 2.1.2. (a) The equality

$$c_W(\alpha) = \inf\{\mathcal{F}^\alpha(\varphi,\psi) : \varphi \in X, \quad \psi \in X \cap Y, \quad \varphi,\psi \text{ increasing}\}$$

easily follows from Proposition 2.4.1; for the existence result, we refer to Proposition 3.1.3, where it is proven in the context of the optimal profile problem for the functional $\mathcal{F}$.

(b) Let

$$C_W = \min\Big\{ \int_{\mathbb{R}} W(\varphi) + \int_{\mathbb{R}} (\varphi')^2 \ : \ \varphi \in X \cap Y \Big\}.$$

We claim that for every α

$$0 < c_W(\alpha) < C_W.$$

By (a) we have $c_W(\alpha) > 0$ for every α. To prove that $c_W(\alpha) < C_W$ we argue by contradiction, assuming that $c_W(\alpha) \geq C_W$. Let $\varphi \in X \cap Y$ such that $\int_{\mathbb{R}} W(\varphi) + \int_{\mathbb{R}} (\varphi')^2 = C_W$. Given $\zeta \in \mathcal{C}_c^\infty(\mathbb{R})$ and $t \in \mathbb{R}$, define $\Phi(t) = \mathcal{F}^\alpha(\varphi, \varphi + t\zeta)$. Then $\Phi(t) \geq c_W(\alpha) \geq C_W = \Phi(0)$ for every t. Since $\Phi \in C^1(\mathbb{R})$, it follows that

$$\Phi'(0) = 2\int_{\mathbb{R}} \varphi'\zeta' = 0 \qquad \forall \zeta \ \in C_c^\infty(\mathbb{R}).$$

Therefore $\varphi'' = 0$, and this gives a contradiction.

It is immediate to check that the map $\alpha \mapsto c_W(\alpha)$ is increasing. We claim that it is strictly increasing. For $\sigma > 0$ let (φ,ψ) be a solution of the minimum problem dealt with in part (a) for $c_W(\alpha+\sigma)$. Then

$$\begin{aligned} c_W(\alpha+\sigma) &= \int_{\mathbb{R}} W(\varphi) + \alpha \int_{\mathbb{R}} (\varphi-\psi)^2 + \int_{\mathbb{R}} (\psi')^2 + \sigma \int_{\mathbb{R}} (\varphi-\psi)^2 \\ &\geq c_W(\alpha) + \sigma \int_{\mathbb{R}} (\varphi-\psi)^2; \end{aligned}$$

if $c_W(\alpha) = c_W(\alpha+\sigma)$, then $c_W(\alpha+\sigma) = C_W$, and this gives a contradiction.

Let $(\varphi_\alpha, \psi_\alpha)$ be a solution of the minimum problem dealt with in part (a) for $c_W(\alpha)$. The translation invariance of the functional allows to choose φ_α such that $\varphi_\alpha \leq 1/2$ in $(-\infty, 0)$ and $\varphi_\alpha \geq 1/2$ in $(0,+\infty)$. Since $c_W(\alpha) \leq C_W$, weak compactness arguments allow to deduce that, up to a subsequence:

$$\begin{aligned} \psi_\alpha \rightharpoonup \varphi \ \text{ in } \ H^1_{loc}(\mathbb{R}) \quad \text{as } \alpha \to +\infty \\ \varphi_\alpha \to \varphi \ \text{ in } \ L^2_{loc}(\mathbb{R}) \quad \text{as } \alpha \to +\infty. \end{aligned}$$

Moreover, φ_α can be assumed to converge pointwise a.e. The Fatou Lemma and the weak lower semicontinuity of the norm yield that

$$\liminf_{\alpha\to+\infty} \Big(\int_{\mathbb{R}} W(\varphi_\alpha) + \alpha \int_{\mathbb{R}} (\varphi_\alpha - \psi_\alpha)^2 + \int_{\mathbb{R}} (\psi'_\alpha)^2 \Big) \geq \int_{\mathbb{R}} W(\varphi) + \int_{\mathbb{R}} (\varphi')^2.$$

As in (a) we prove that $\varphi \in X \cap Y$. This implies, since $c_W(\alpha) \leq C_W$,

$$\lim_{\alpha\to+\infty} c_W(\alpha) = C_W,$$

thus concluding the proof. ∎

Remark 2.4.2. It is easy to check directly that

$$\lim_{\alpha\to 0^+} c_W(\alpha) = 0.$$

Remark 2.4.3. If $W \in C^3(\mathbb{R})$, then the infimum in

$$\inf\{\mathcal{E}^\alpha(\varphi) : \quad \varphi \in X\}$$

is attained when φ is a travelling wave solution for the one-dimensional parabolic problem:

$$w_t = J * w - w - f(w),$$

where f is the derivative of W (see [3, p. 549]). The regularity result in [13, Theorem 3.1(c)] allows us to construct a minimizing sequence in $X \cap Y$ converging in $L^1_{loc}(\mathbb{R})$ to the minimum point; hence:

$$\min\{\mathcal{E}^\alpha(\varphi) \mid \varphi \in X\} = \inf\{\mathcal{E}^\alpha(\varphi) \mid \varphi \in X \cap Y\}.$$

Chapter 3

THE OPTIMAL PROFILE PROBLEM

In this chapter we consider in more detail the optimal profile problem which arises in the study of the Γ-limit of the functionals F_ε introduced in the previous chapter. Sections 3.1 through 3.3 are devoted to the optimal profile problem on the whole line, showing some monotonicity and uniqueness results for the minimizers and making some remarks on the explicit computation of the minimum.

In Section 3.4 we analyze the corresponding minimum problem on the half-line, connected with the asymptotic behaviour of the functionals F_ε with Dirichlet boundary conditions which will be dealt with in the next chapter.

In the last section the asymptotic behaviour of the minimum values with respect to the parameter α is studied, in connection with other classical problems.

3.1 The optimal profile on the real line: monotonicity results

The function $W\colon \mathbb{R} \to \mathbb{R}$, as in the previous Chapter, satisfies the following conditions:

$W\colon \mathbb{R} \to [0,+\infty)$ belongs to $C^0(\mathbb{R})$ and $W(t) = 0$ if and only if $t \in \{0,1\}$; moreover W has at least linear growth at $\pm\infty$, i.e.:

$$\liminf_{|t|\to+\infty} \frac{W(t)}{|t|} > 0.$$

Further hypotheses will be specified when necessary.

We recall some definition given in Chapter 2:

$$X = \{\varphi\colon \mathbb{R} \to [0,1] \ \text{ measurable such that } \lim_{t\to-\infty} \varphi(t) = 0, \lim_{t\to+\infty} \varphi(t) = 1\};$$

$$Y = \{\varphi \in AC(\mathbb{R}) \ : \ \varphi' \in L^2(\mathbb{R})\}.$$

Consider now the problem

$$(P) \qquad \begin{array}{l} \text{minimize} \quad \mathcal{F}(\varphi,\psi) = \displaystyle\int_{-\infty}^{+\infty} W(\varphi) + (\varphi-\psi)^2 + (\psi')^2 \\ \text{in the set } \ \{\varphi \in X,\ \psi \in X \cap Y\}. \end{array}$$

The functional $\mathcal{F}$ is nothing but the functional $\mathcal{F}^\alpha$ of the previous Chapter with $\alpha = 1$; in Sec. 3.5 we shall study the dependence on α.

Remark 3.1.1. Notice that we deal only with functions which take values in $[0,1]$; in fact, this is clearly not restrictive for the minimum problem (P).

We set

$$\Phi = \inf\{\mathcal{F}(\varphi,\psi) \ : \ \varphi \in X, \ \psi \in X \cap Y\}, \tag{3.1.1}$$

noticing that, as stated in Proposition 2.4.1,

$$\Phi = c_W = \inf\Big\{ \int_{\mathbb{R}} W(\varphi) + \frac{1}{4} \iint_{\mathbb{R}^2} e^{-|x-y|} \big(\varphi(x) - \varphi(y)\big)^2 \, dxdy \ : \ \varphi \in X \Big\},$$

the constant which appears in the Γ-limit of the sequence F_ε (see Theorem 2.1.1). In this context, the number Φ stands for a *surface tension*, and represents the minimal cost in terms of the energy $\mathcal{F}$, i.e. the non-scaled energy F_1, for the transition from 0 to 1 on the real line. The minimum problem (P) is then called the *optimal profile problem*, and a solution (φ, ψ) is an optimal profile for transition (with respect to the non-scaled energy).

We emphasize here that the proof of Theorem 2.1.1 is essentially based on the analysis of problem (P). Indeed, also Proposition 2.2.3 (the lower bound for the Γ-lower limit in one dimension) can be deduced exploiting the optimal profile problem (see, below, Remark 3.1.2). Thus, the great flexibility of this kind of approach (pointed out in [1] revisiting the classical Modica-Mortola convergence result) is further confirmed; moreover, in the next chapter, we shall follow the same general line to deal with the case of boundary conditions for the functionals F_ε.

The proof of Proposition 2.2.3 given in Chapter 2 allows to establish a connection with a family of nonlocal models studied in the literature.

Remark 3.1.2. It is possible to give a proof of Proposition 2.2.3 based on the optimal profile problem for the functional $\mathcal{F}$, obtaining directly

$$c_W = \inf\{\mathcal{F}(\varphi,\psi) \ : \ \varphi \in X, \ \psi \in X \cap Y\}.$$

Indeed, following the proof given in §2.2.1, we obtain the equality (2.2.4):

$$F_{\varepsilon_j}(u_j, v_j, I_i) = F_{\varepsilon_j}(\tilde{u}^i_j, \tilde{v}^i_j) + o(1)_{j\to\infty}.$$

Thus, a rescaling argument and the translation invariance of $\mathcal{F}$ allow to deduce

$$F_{\varepsilon_j}(u_j, v_j, I_i) \geq \Phi + o(1)_{j\to\infty};$$

recalling (2.2.3),

$$F_{\varepsilon_j}(u_j, v_j) \geq \sum_{i=1}^{N} F_{\varepsilon_j}(u_j, v_j, I_i) \geq N\Phi + o(1)_{j\to\infty},$$

for every $N \leq \#S(u)$. This concludes the proof of the lower bound.

Proposition 3.1.3. *The following equality holds:*

$$\Phi = \inf\{\mathcal{F}(\varphi,\psi) \ : \ \varphi \in X, \quad \psi \in X \cap Y, \quad \varphi,\psi \quad \textit{non decreasing}\}, \tag{3.1.2}$$

and the infimum is attained.

For the proof we introduce, as in the proof of Proposition 2.4.1

$$\Phi_{\mathcal{U}} = \inf\{\mathcal{F}(\varphi,\psi) : \ \varphi \in \mathcal{U}, \quad \psi \in \mathcal{U} \cap Y\},$$

where $\mathcal{U}$ is any of the following spaces:

X^* : the set of measurable $\varphi\colon \mathbb{R} \to [0,1]$ with $\varphi = \chi_{(0,+\infty)}$ on $\mathbb{R} \setminus (-R,R)$ for some $R > 0$;

Z^* : the subset of the increasing elements of X^*.

Lemma 3.1.4.

$$\Phi = \Phi_{X^*} = \Phi_{Z^*}.$$

Proof. Clearly, $\Phi \le \Phi_{X^*} \le \Phi_{Z^*}$.

Step 1. $\Phi \ge \Phi_{X^*}$.

Let $\sigma > 0$ and $\varphi_\sigma \in X$, $\psi_\sigma \in X \cap Y$ such that $(\varphi_\sigma, \psi_\sigma) \le \Phi + \sigma$. There exists $R > 0$ with the following property

$$|\varphi_\sigma - \chi_{(0,+\infty)}|, \ |\psi_\sigma - \chi_{(0,+\infty)}| < \sigma \qquad \text{on } \ \mathbb{R} \setminus (-R,R).$$

Define $\widetilde{\psi}_\sigma\colon \mathbb{R} \to \mathbb{R}$ as the continuous function which is affine in both the intervals $J = (-R-1,-R)$, $J' = (R,R+1)$ and such that

$$\widetilde{\psi}_\sigma = \psi_\sigma \quad \text{on} \ \ (-R,R), \qquad \widetilde{\psi}_\sigma = \chi_{(0,+\infty)} \quad \text{on} \ \ \mathbb{R} \setminus (-R-1,R+1).$$

Moreover, let $\widetilde{\varphi}_\sigma\colon \mathbb{R} \to \mathbb{R}$ be defined by: $\widetilde{\varphi}_\sigma = \varphi_\sigma$ on $\mathbb{R} \setminus (-R,R)$, $\widetilde{\varphi}_\sigma = \widetilde{\psi}_\sigma$ on $J \cup J'$, and $\varphi_\sigma = \chi_{(0,+\infty)}$ on $\mathbb{R} \setminus (-R-1,R+1)$. Then $\widetilde{\varphi}_\sigma \in X^*$, $\widetilde{\psi}_\sigma \in X^* \cap Y$ and, as one can easily check,

$$\mathcal{F}(\widetilde{\varphi}_\sigma, \widetilde{\psi}_\sigma) \le \mathcal{F}(\varphi_\sigma, \psi_\sigma) + o(1)_{\sigma \to 0}.$$

Step 2. $\Phi_{X^*} \ge \Phi_{Z^*}$.

For every $\varphi \in X^*$ and $\psi \in X^* \cap Y$ we construct $\overline{\varphi} \in Z^*$ and $\overline{\psi} \in Z^* \cap Y$, such that

$$\mathcal{F}(\overline{\varphi}, \overline{\psi}) \le \mathcal{F}(\varphi,\psi). \tag{3.1.3}$$

Let $R > 0$ such that $\varphi = \psi = \chi_{(0,+\infty)}$ on $\mathbb{R} \setminus (-R,R)$. Following [3], we define the right rearrangement of a subset A of $[-R,R]$ as the set $A^* = [R - m(A), R]$. Then, for a measurable function $u\colon [-R,R] \to [0,1]$ we define the increasing rearrangement of u as

$$u^*(x) = \sup\{c \in \mathbb{R} : \ x \in I_c^*\}, \quad x \in [-R,R], \tag{3.1.4}$$

where $I_c = \{x \in [-R, R] : u(x) \geq c\}$. Let

$$\overline{\varphi}(x) = \begin{cases} 0 & \text{if } x < -R \\ \varphi^*(x) & \text{if } -R \leq x \leq R \\ 1 & \text{if } x > R \end{cases}$$

and let $\overline{\psi}$ be analogously defined. Then $\overline{\varphi} \in Z^*$ and $\overline{\psi} \in Z^* \cap Y$; moreover, properties [32, (C) - p. 22, (M9) - p. 23, (G1a) - p. 35] allow to deduce the inequality (3.1.3). ■

Proof of the Proposition 3.1.3. The equality (3.1.2) follows immediately from Lemma 3.1.4 ($\Phi = \Phi_{Z^*}$).

Let us prove that the minimum is attained. Let (φ_n, ψ_n) be a minimizing sequence for $\mathcal{F}$ on

$$\widetilde{Z} = \{(\varphi, \psi) : \ \varphi \in X, \ \psi \in X \cap Y, \ \varphi, \psi \text{ increasing}\}.$$

Thanks to the translation invariance of $\mathcal{F}$ it is not restrictive to suppose that $\varphi_n \leq 1/2$ in $(-\infty, 0)$ and $\varphi_n \geq 1/2$ in $(0, +\infty)$. Since the total variation of the measure $D\varphi_n$ is 1 for every n, it follows that $\{\varphi_n\}$ is bounded in $BV_{loc}(\mathbb{R})$; moreover ψ_n' is bounded in $L^2(\mathbb{R})$; by compactness there exist $u \in BV_{loc}(\mathbb{R})$ and $v \in Y$ such that, possibly passing to a subsequence,

$$\varphi_n \to u \ \text{ in } \ L^1_{loc}(\mathbb{R}) \quad \text{and} \quad \psi_n \rightharpoonup v \ \text{ in } \ H^1_{loc}(\mathbb{R}).$$

We can also suppose that the convergences are pointwise a.e. By the Fatou Lemma and the weak lower semicontinuity of the norm

$$\liminf_{n\to\infty} \mathcal{F}(\varphi_n, \psi_n) \geq \mathcal{F}(u, v).$$

If we prove that $(u, v) \in \widetilde{Z}$, this inequality implies that

$$\mathcal{F}(u, v) = \min\{\mathcal{F}(\varphi, \psi) \ : \ (\varphi, \psi) \in \widetilde{Z}\}.$$

It is clear that u and v are increasing; since $\int_{\mathbb{R}} (u - v)^2 < +\infty$, we get $\lim_{x\to+\infty} u(x) = \lim_{x\to+\infty} v(x)$, and the same for $x \to -\infty$. Then, it is sufficient to see that $\lim_{x\to+\infty} u(x) = 1$ and $\lim_{x\to-\infty} u(x) = 0$. This is an immediate consequence of the fact that $\int_{\mathbb{R}} W(u) < +\infty$.

Then, the minimum is attained in $\widetilde{Z}$. The thesis follows recalling (3.1.2). ■

Now, we prove some further properties of minimizers of problem (P).

Proposition 3.1.5. *If (φ, ψ) minimizes $\mathcal{F}$ in $X \times (X \cap Y)$, then $\psi \in C^1(\mathbb{R})$.*

Proof. For a minimum (φ, ψ) we can write the following Euler equation:

$$-\psi'' + \psi = \varphi. \tag{3.1.5}$$

Then, $\psi' \in H^1(\mathbb{R})$, and $\psi \in C^1(\mathbb{R})$. ■

The monotonicity is a necessary condition for a minimizer of $\mathcal{F}$ in $X \times (X \cap Y)$. More precisely, the following result holds:

Proposition 3.1.6. *If (φ, ψ) minimizes $\mathcal{F}$ in $X \times (X \cap Y)$, then φ and ψ are non decreasing.*

Proof. Let $(\varphi, \psi) \in X \times (X \cap Y)$ be such that $\mathcal{F}(\varphi, \psi) = \Phi$.

We prove that the restriction of ψ to $\{t \in \mathbb{R} : 0 < \psi(t) < 1\}$ is injective. The monotonicity of ψ easily follows.

Suppose that there exist T_1, T_2 such that $T_1 < T_2$ and $0 < \psi(T_1) = \psi(T_2) < 1$. If we set

$$\overline{\psi}(t) = \begin{cases} \psi(t) & \text{if } t \le T_1 \\ \psi(t + T_2 - T_1) & \text{if } t > T_1 \end{cases} \tag{3.1.6}$$

$$\overline{\varphi}(t) = \begin{cases} \varphi(t) & \text{if } t \le T_1 \\ \varphi(t + T_2 - T_1) & \text{if } t > T_1, \end{cases} \tag{3.1.7}$$

the pair $(\overline{\varphi}, \overline{\psi})$ belongs to $X \times (X \cap Y)$, and clearly $\mathcal{F}(\overline{\varphi}, \overline{\psi}) \le \mathcal{F}(\varphi, \psi)$. Since (φ, ψ) is a minimum point, $\mathcal{F}(\overline{\varphi}, \overline{\psi}) = \mathcal{F}(\varphi, \psi)$, and this implies $\psi(t) \equiv 0$ or $\psi(t) \equiv 1$ in $[T_1, T_2]$.

Then ψ is non decreasing, and strictly increasing where $\psi \notin \{0, 1\}$.

Let I be an arbitrary bounded interval, and consider in this interval the monotone n.d. rearrangments φ_r and ψ_r of φ and ψ respectively (see (3.1.4)). Since ψ is non decreasing, it coincides with its rearrangment. Then

$$\mathcal{F}(\widetilde{\varphi}, \psi) \le \mathcal{F}(\varphi, \psi),$$

where $\widetilde{\varphi} = \varphi$ in $\mathbb{R} \setminus I$ and $\widetilde{\varphi} = \varphi_r$ in I.

Since (φ, ψ) is a minimum point, it follows that $\mathcal{F}(\widetilde{\varphi}, \psi) = \mathcal{F}(\varphi, \psi)$, and then:

$$\begin{aligned} -\psi'' + \psi &= \varphi \qquad \text{in } \mathcal{D}'(\mathbb{R}) \\ -\psi'' + \psi &= \widetilde{\varphi} \qquad \text{in } \mathcal{D}'(\mathbb{R}). \end{aligned}$$

We conclude $\varphi = \widetilde{\varphi}$, and then φ non decreasing in I, therefore in $\mathbb{R}$. ∎

The following property is a further specification of the previous monotonicity results, and it will be useful in the costruction of the recovery sequence for the Γ-lim sup inequality in the next chapter, and to prove some uniqueness result for the minimum points of $\mathcal{F}$.

Proposition 3.1.7. *If (φ, ψ) minimizes $\mathcal{F}$ in $X \times (X \cap Y)$, then:*

$$0 < \psi(t) < 1 \Longrightarrow \psi'(t) > 0.$$

Proof. Let us suppose by contradiction that $\psi(t) \notin \{0, 1\}$, and $\psi'(t) = 0$.

As first step, we prove that

$$0 < \lim_{s \to t^-} \varphi(s) \le \lim_{s \to t^+} \varphi(s) < 1. \tag{3.1.8}$$

If $\lim_{s \to t^-} \varphi(s) = 0$, then $\varphi = 0$ in $(-\infty, t)$, and from (3.1.5):

$$\frac{1}{2}(\psi'(t))^2 = \frac{1}{2}\psi^2(t);$$

this gives a contradiction, hence $\lim_{s \to t^-} \varphi(s) > 0$.

If $\lim_{s\to t^+}\varphi(s)=1$, then $\varphi=1$ in $(t,+\infty)$, and from (3.1.5):

$$-\frac{1}{2}(\psi'(t))^2=\frac{1}{2}-\frac{1}{2}\psi^2(t)-1+\psi(t);$$

since $\psi'(t)=0$, it follows that $\psi(t)=1$, a contradiction; hence, $\lim_{s\to t^+}\varphi(s)<1$.

From (3.1.8) and the continuity of W we deduce that there exist $a>0$ and $\tau_1>0$ such that:

$$\forall\ s\in(t-\tau_1,t)\qquad W(\varphi(s))>a.$$

Fixed $\delta>0$ arbitrary, there exists τ_2 such that $0<\tau_2\le\tau_1$ and:

$$\forall\ s\in(t-\tau_2,t)\qquad 0\le\psi'(s)\le\delta.$$

Set

$$\tilde{\varphi}(s)=\begin{cases}\varphi(s) & \text{if } s\le t-\frac{\tau_2}{2}\\ \varphi(s+\frac{\tau_2}{2}) & \text{if } s>t-\frac{\tau_2}{2}\end{cases}$$

$$\tilde{\psi}(s)=\begin{cases}\psi(s) & \text{if } s\le t-\tau_2\\ \psi(t-\tau_2)+\frac{\psi(t)-\psi(t-\tau_2)}{\tau_2/2}(s-t+\tau_2) & \text{if } t-\tau_2<s<t-\frac{\tau_2}{2}\\ \psi(s+\frac{\tau_2}{2}) & \text{if } s\ge t-\frac{\tau_2}{2}.\end{cases}$$

Recalling that $\psi(t)-\psi(t-\tau_2)=\psi'(\xi)\tau_2$ for some $\xi\in(t-\tau_2,t)$, and hence $\psi(t)-\psi(t-\tau_2)\le\delta\tau_2$, an easy computation gives:

$$\mathcal{F}(\varphi,\psi)-\mathcal{F}(\tilde{\varphi},\tilde{\psi})\ge\frac{\tau_2}{2}(a-\delta^2\tau_2^2-4\delta^2).$$

Clearly, we can choose $\tau_2\le 1$, and the arbitrariness of δ allows to deduce:

$$\mathcal{F}(\varphi,\psi)-\mathcal{F}(\tilde{\varphi},\tilde{\psi})>0,$$

which contradicts the minimality. ■

Proposition 3.1.8. *Let W satisfy one of the following hypotheses*

(a) $W\in C^1$ and convex in a neighbourhood of 0 and 1;

(b) $W\in C^2$.

If (φ,ψ) minimizes $\mathcal{F}$ in $X\times(X\cap Y)$, then:

$$0<\varphi<1\qquad\textit{and}\qquad 0<\psi<1.$$

Proof. We observe that, if $W\in C^1(\mathbb{R})$, a minimum point (φ,ψ) satisfies also the following Euler equation:

$$\frac{W'(\varphi)+2\varphi}{2}=\psi\quad\text{a.e. in }\mathbb{R}. \tag{3.1.9}$$

Now, we suppose by contradiction that $\psi(t)=0$ for some t. Since $\psi\ge 0$ and ψ non decreasing, if we pose $T=\max\{t\in\mathbb{R}\ :\ \psi(t)=0\}$, it follows that $\psi'(T)=0$.

We claim that φ is continuous in T. Indeed, clearly $\varphi(t) = 0$ in $(-\infty, T)$. Let us suppose by contradiction that $\varphi^+(T) > 0$. Then, recalling that $\psi'(0) = 0$, there exists $\delta > 0$ such that

$$\psi(t) \le \varphi^+(T) \quad \text{and} \quad \psi(t) \le \frac{\psi(T+\delta)}{\delta}(t-T) \quad \text{if} \quad (T, T+\delta).$$

If we set

$$\tilde{\psi}(t) = \begin{cases} \psi(t) & \text{if } t \in (-\infty, T) \\ \dfrac{\psi(T+\delta)}{\delta}(t-T) & \text{if } t \in [T, T+\delta] \\ \psi(t) & \text{if } t \in (T+\delta, +\infty); \end{cases}$$

since $\varphi \ge \varphi^+(T)$, then $|\varphi - \tilde{\psi}| \le |\varphi - \psi|$; clearly, $\int_T^{T+\delta} (\tilde{\psi}')^2$. Thus, $\mathcal{F}(\varphi, \tilde{\psi}) < \mathcal{F}(\varphi, \tilde{\psi})$, a contradiction.

Either condition (a) or condition (b) on W imply that the function $\frac{W'(s)+2s}{2}$ is invertible in a neighbourhood of 0, and the local inverse, denoted by h, is Lipschitz. Thanks to the previous claim, ψ solves the following Cauchy problem:

$$\begin{cases} \psi'' + (h(\psi) - \psi) = 0 \\ \psi(T) = 0 \\ \psi'(T) = 0 \end{cases}$$

The function h is Lipschitz in a neighbourhood of 0; hence, the Cauchy problem has a unique local solution. Since the constant 0 is a solution, we find a contradiction, and $\psi > 0$. The proof of the fact that $\psi < 1$ is completely similar. Clearly, this implies the strict monotonicity of ψ, and the same result holds for φ (see Proposition 3.1.6). ■

We observe that if $W \notin C^1(\mathbb{R})$ the strict monotonicity can fail. If we consider, e.g., $W(t) = \min\{|t|, |t-1|\}$, a simple computation shows that the pair (φ, ψ), where

$$\varphi(t) = \chi_{(0,+\infty)}(t)$$

$$\psi(t) = \begin{cases} \dfrac{1}{2}e^t & \text{if } t \le 0 \\ 1 - \dfrac{1}{2}e^{-t} & \text{if } t > 0, \end{cases}$$

is a minimum point, and clearly φ is not strictly increasing.

As a consequence of Propositions 3.1.7 and 3.1.8, we deduce that if $W \in C^1$ and convex in a neighbourhood of 0 and 1, or if $W \in C^2$, then $\psi'(t) > 0$ for every $t \in \mathbb{R}$.

3.2 Uniqueness results

In this section we assume the further hypothesis

$$W \in C^2(\mathbb{R}).$$

As we noticed in the proof of Proposition 3.1.6, the C^1-regularity is sufficient to deduce that a necessary condition for a minimizer $(\varphi, \psi) \in X \times (X \cap Y)$ is to satisfy the following Euler system (see (3.1.5) and (3.1.9)):

$$\begin{cases} \dfrac{W'(\varphi)+2\varphi}{2} = \psi & \text{a.e. in } \mathbb{R} \\ -\psi''+\psi=\varphi & \text{in } \mathcal{D}'(\mathbb{R}) \end{cases} \tag{3.2.1}$$

Let us notice that some of the results that we prove in this section continue to hold if W is only of class C^1. The hypothesis W of class C^2 allows a simplification of the proofs.

Let us define $g\colon [0,1] \to \mathbb{R}$ as:

$$g(s) = \frac{W'(s)+2s}{2}, \tag{3.2.2}$$

and denote by $\gamma(g)$ the set $\{(x,y) \in \mathbb{R}\times[0,1] \ : \ g(y)=x\}$.

3.2.1 Case A: g invertible

It is clear that if g is invertible, $\gamma(g)$ is the graph of the inverse function g^{-1}. Since $W \in C^1(\mathbb{R})$, g is invertible if and only if it is strictly increasing (note that $g(0)=0$ and $g(1)=1$). An example of W such that the corresponding function g is invertible is given by the classical double-well potential $W(s)=ks^2(1-s)^2$, for $k \le 2$. Clearly, a sufficient condition is $W''+2>0$ in $[0,1]$.

If g is invertible, we deduce from (3.2.1):

$$\psi'' + (g^{-1}(\psi)-\psi) = 0 \qquad \text{in } \mathcal{D}'(\mathbb{R}). \tag{3.2.3}$$

Then the continuity of g^{-1} allows to conclude that $\varphi \in C^0(\mathbb{R})$ and $\psi \in C^2(\mathbb{R})$.

The following uniqueness result holds.

Proposition 3.2.1. *The minimum problem (P) has a unique solution up to translation.*

Proof. The proof follows from Proposition 3.1.7.

Let $(g^{-1}(\psi_1),\psi_1)$ and $(g^{-1}(\psi_2),\psi_2)$ be minimizers of $\mathcal{F}$ in $X\times(X\cap Y)$. Propositions 3.1.7 and 3.1.8 ensure that the inverse functions ψ_1^{-1} and ψ_2^{-1} belong to $C^1(0,1)$, and we can write $(\psi_i^{-1})'(s) = \big(\psi_i'(\psi_i^{-1}(s))\big)^{-1}$ for $i=1,2$. Then, consider the equation (3.2.3). Since $\psi_i' \in H^1(\mathbb{R})$, we can multiply by ψ_i' and integrate by parts in $(-\infty,t)$. Recalling that $\lim_{t\to-\infty}\psi_i(s)=0$, $\lim_{t\to-\infty}\psi_i'(s)=0$, and $g^{-1}(0)=0$, it follows that:

$$(\psi_i'(t))^2 = \psi_i^2(t) - 2\int_0^{\psi_i(t)} g^{-1}(s)ds. \tag{3.2.4}$$

Hence

$$\big(\psi_1'(\psi_1^{-1}(s))\big)^2 = s^2 - 2\int_0^s g^{-1}(\sigma)d\sigma = \big(\psi_2'(\psi_2^{-1}(s))\big)^2 \qquad \text{for every} \quad \sigma \in (0,1).$$

This implies $(\psi_1^{-1}-\psi_2^{-1})'=0$, and then there exists a constant c such that $\psi_1^{-1}-\psi_2^{-1}=c$. Clearly, this gives:

$$\psi_1(t) = \psi_2(t-c)$$

and concludes the proof. ∎

3.2.2 Case B: g non invertible

Let us consider $\psi \in X \cap Y$ such that, for some $\varphi \in X$, the pair (φ, ψ) is a minimum point for $\mathcal{F}$.

It is clear from the second equation in (3.2.1) that ψ determines a unique φ such that (φ, ψ) minimizes $\mathcal{F}$ in $X \times (X \cap Y)$.

Since ψ is injective, then we set

$$h = \varphi \circ \psi^{-1}.$$

Proposition 3.1.6 implies that the function h is strictly increasing, and it is clear from the first equation in (3.2.1) that $graph(h) \subset \gamma(g)$, i.e. h is a right inverse of g, where g is the function defined in (3.2.2).

The limit conditions for φ and ψ ensure that

$$\lim_{s\to 0^+} h(s) = 0, \qquad \lim_{s\to 1^-} h(s) = 1.$$

Moreover, since 0 is a minimum point for W, the function g is strictly increasing in a right neighbourhood U of 0, then h coincides with the inverse of $g|_U$ in $[0, s_1]$ for some $s_1 > 0$. The same argument shows that in $[s_2, 1]$, for some $s_2 < 1$, h is the inverse of the restriction of g on a neighbourhood of 1.

Writing the second equation in (3.2.1) as

$$\psi'' + h(\psi) - \psi = 0,$$

we can multiply by ψ'; an integration by parts and the limit conditions give the integral condition:

$$\int_0^1 h(s)ds = \frac{1}{2}. \tag{3.2.5}$$

Finally, we observe that, since clearly h can not be continuous, it follows that φ can not be continuous, and then $\psi \notin C^2(\mathbb{R})$.

Setting

- $Y_{min} = \{\psi \in X \cap Y \ : \ \exists\, \varphi \in X \ \text{ s.t. } \ \mathcal{F}(\varphi, \psi) = \Phi\}$,
- H: the set of the functions $h\colon (0,1) \to (0,1)$ strictly increasing, such that $g \circ h = id$, $\lim_{s\to 0} h(s) = 0$, $\lim_{s\to 1} h(s) = 1$, and $\int_0^1 h(s)ds = \frac{1}{2}$,

we can summarize previous results in the following:

Proposition 3.2.2. *For every $\psi \in Y_{min}$ there exists a (unique) function $h \in H$ such that*

$$\mathcal{F}(h(\psi), \psi) = \Phi.$$

We can easily prove the following result:

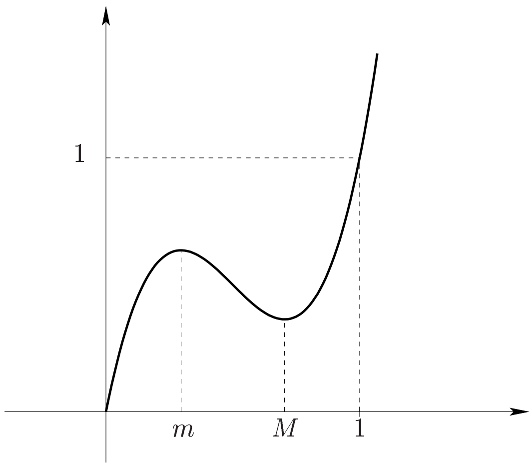

Figure 3.1.

Lemma 3.2.3. *If W is such that the following condition holds*

$$\exists\ h \in H \text{ such that } \quad \forall\ \psi \in Y_{min} \quad \mathcal{F}(h(\psi), \psi) = \Phi, \tag{$\star$}$$

then the minimum problem (P) has a unique solution (up to translations).

Proof. We use the same argument that in the proof of the Proposition 3.2.1, simply replacing g^{-1} by h. Clearly, the key point is the fact that h does not depend on the particular minimizer. ∎

Remark 3.2.4. If g is invertible, $(\star)$ is clearly satisfied by g^{-1}.

Now, we analyze a particular case where we can prove that the condition $(\star)$ holds. Let us suppose that there exist m, M such that $0 < m < M < 1$, and

$$g \quad \text{strictly increasing in} \quad [0, m] \cup [M, 1], \quad \text{strictly decreasing in} \quad [m, M], \tag{$*$}$$

as shown in Figure 3.1.

This hypothesis is satisfied, e.g., if W is the classical double-well potential $W(s) = ks^2(s-1)^2$, with $k > 2$.

Let us notice that the assumption $(*)$ is very close to the assumption on the function W which allows to prove a uniqueness result for the minimizers of the nonlocal functional $\mathcal{F}_{NL}$, defined by

$$\mathcal{F}_{NL}(\varphi) = \int_{\mathbb{R}} W(\varphi) + \frac{1}{4} \int\int_{\mathbb{R}^2} J(x-y)(\varphi(x) - \varphi(y))^2 dxdy$$

(see Chapter 2). In fact, if J is of class C^1, and

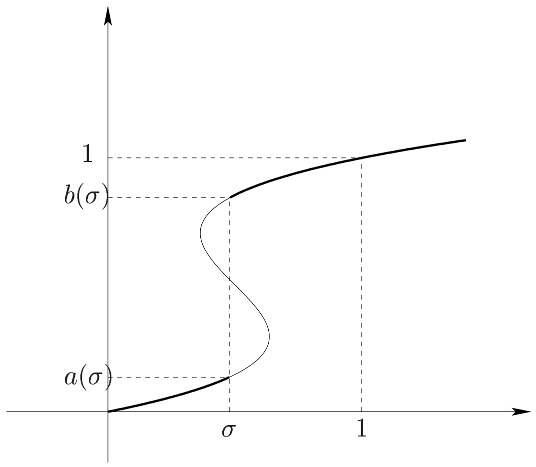

Figure 3.2.

W is of class C^2, and the interval $(-1, 1)$ splits into three sub-intervals where W is either convex or concave,

then the solution in X of the Euler equation associated with $\mathcal{F}_{NL}$, i.e.

$$J * u - u = W'(u) \quad \text{a. e. in } \mathbb{R},$$

is unique up to translations (see [3, Sect. 4, (a)] and [13, Theorem 4.1]).

Let us consider, for every $\sigma \in (\max\{0, m\}, \min\{1, M\})$, the (only) function $h_\sigma : (0, 1) \to (0, 1)$ such that $graph(h_\sigma) \subset \gamma(g)$, and h_σ is continuous in $(0, \sigma) \cup (\sigma, 1)$ (see Figure 3.2).

If $\psi \in Y_{min}$, and h is the corresponding function in H such that $(h(\psi), \psi)$ is a minimum point, then the monotonicity of h, joint with the assumption $(*)$, implies that there exists exactly one point $\sigma(\psi)$ in $[\max\{0, m\}, \min\{1, M\}]$, such that $h = h_{\sigma(\psi)}$.

For every σ, h_σ coincides with the inverse of the restriction of g to $(0, a(\sigma)) \cup (b(\sigma), 0)$, where $a(\sigma)$ and $b(\sigma)$ are respectively the left and right limits of h_σ in σ.

The function $I : [\max\{0, m\}, \min\{1, M\}] \to \mathbb{R}$ defined by:

$$I(\sigma) = \int_0^1 h_\sigma(s) ds$$

is strictly decreasing and continuous; since it is straightforward to see that $I(\max\{0, m\}) > \frac{1}{2}$ and $I(\min\{1, M\}) < \frac{1}{2}$, we deduce that there exists exactly one value σ^* in the interval $(\max\{0, m\}, \min\{1, M\})$ such that (3.2.5) is satisfied by h_{σ^*}. Hence, $\sigma(\psi) = \sigma^*$ for every $\psi \in Y_{min}$ (moreover, $H = \{h_{\sigma^*}\}$), and the condition $(\star)$ is satisfied with $h = h_{\sigma^*}$.

An application of Lemma 3.2.3 gives the following:

Proposition 3.2.5. *Let W be such that the corresponding function g satisfies the hypothesis $(*)$. Then the minimum point of $\mathcal{F}$ is unique up to translations.*

Moreover, an easy computation gives, for $\sigma \in (\max\{0,m\},\min\{1,M\})$:

$$\begin{aligned} 2\int_0^1 h_\sigma(s)ds &= \int_0^{a(\sigma)} \tau(W''(\tau)+2)d\tau + \int_{b(\sigma)}^1 \tau(W''(\tau)+2)d\tau \\ &= W(b(\sigma)) + (b(\sigma)-\sigma)^2 - W(a(\sigma)) - (a(\sigma)-\sigma)^2 + 1. \end{aligned}$$

If we set

$$R_\sigma(t) = W(h_\sigma(t)) + (h_\sigma(t)-t)^2,$$

it follows that R_σ is continuous if and only if

$$W(a(\sigma)) + (a(\sigma)-\sigma)^2 = W(b(\sigma)) + (b(\sigma)-\sigma)^2. \tag{3.2.6}$$

Equation (3.2.6) is satisfied if and only if h_σ satisfies the integral condition (3.2.5), i.e. if and only if $\sigma = \sigma^*$. In particular, the function $R(t) = W(h(t)) + (h(t)-t)^2 = R_{\sigma^*}(t)$ belongs to $C^0([0,1])$. We shall use this result in the next section, to compute the explicit value of Φ.

3.3 Explicit calculus of the minimum

In this section we assume again the hypothesis W of class C^2.

We deal firstly with the simplest case

$$g \quad \text{invertible.}$$

The results of §3.2.1 allows to say that

$$\Phi = \min\Big\{ \int_{\mathbb{R}} R(\psi) + (\psi')^2 \ : \ \psi \in X \cap Y \Big\},$$

where the function R is defined by $R(s) = W(g^{-1}(s)) + (g^{-1}(s)-s)^2$. Since R is clearly positive and continuous, and vanishes only in $\{0,1\}$, it follows that

$$\min\Big\{ \int_{\mathbb{R}} R(\psi) + (\psi')^2 \ : \ \psi \in X \cap Y \Big\} = 2\int_0^1 \sqrt{R(s)}ds.$$

Thus, recalling the definition of g,

$$\Phi = \int_0^1 \sqrt{W(s) + \frac{(W'(s))^2}{4}}\Big(W''(s)+2\Big)ds. \tag{3.3.1}$$

Then, we consider the case of a function g satisfying the condition $(*)$ (see §3.2.2). Since $H = \{h_{\sigma^*}\}$, it follows

$$\Phi = \min\Big\{ \int_{\mathbb{R}} R(\psi) + (\psi')^2 \ : \ \psi \in X \cap Y \Big\},$$

where in this case $R(s) = W(h_{\sigma^*}(s)) + (h_{\sigma^*}(s) - s)^2$. The continuity of R follows from the condition $\int_0^1 h_{\sigma^*} = 1/2$ (see (3.2.6)). As in the previous case, this gives

$$\Phi = 2\int_0^1 \sqrt{R(s)}ds.$$

Here, the value of Φ is determined if we can explicitly compute the values $a(\sigma^*)$ and $b(\sigma^*)$, respectively the left and right limits of h^*_σ in σ^*. Indeed,

$$\Phi = \int_0^{a(\sigma^*)} \sqrt{W(s) + \frac{(W'(s))^2}{4}\Big(W''(s)+2\Big)}ds + \int_{b(\sigma^*)}^1 \sqrt{W(s) + \frac{(W'(s))^2}{4}\Big(W''(s)+2\Big)}ds.$$

These examples suggest a more general result. We introduce

- H: the set of functions $h\colon [0,1] \to [0,1]$ strictly increasing, such that $\lim_{s\to 0} h(s) = 0$, $\lim_{s\to 1} h(s) = 1$, $g \circ h = id$, and $\int_0^1 h(s)ds = \frac{1}{2}$
- for $h \in H$: $R(s,h) = W(h) + (h-s)^2$.

Proposition 3.3.1. *The following equality holds:*

$$\Phi = 2\inf\Big\{ \int_0^1 \sqrt{R(s,h(s))}ds \;:\; h \in H\Big\}, \tag{3.3.2}$$

and the minimum in the right hand side is attained. Moreover, if h is a minimum point, then the function $s \mapsto R(s,h(s))$ is continuous.

Proof. Recalling Proposition 3.2.2, it is easy to check that

$$\Phi = \min\Big\{\int_{\mathbb{R}} R(\psi,h(\psi)) + (\psi')^2 \;:\; h\in H,\ \psi \in X\cap C^1(\mathbb{R}),\ \psi \text{ increasing}\Big\}. \tag{3.3.3}$$

Now, for h in H, consider

$$\phi(h) = \inf\Big\{\int_{\mathbb{R}} R(\psi,h(\psi)) + (\psi')^2 \;:\; \psi \in X\cap C^1(\mathbb{R}),\ \psi \text{ increasing}\Big\}.$$

Clearly, for every $h \in H$ we can write

$$\phi(h) \geq 2\int_0^1 \sqrt{R(s,h(s))}ds.$$

We show that the inequality is in fact an equality, and moreover that, if h is such that $\Phi = \phi(h)$, the function $s \mapsto R(s,h(s))$ is continuous.

It is clear that, if we *a priori* know that the map $s \mapsto R(s,h(s))$ is continuous, then, as in the previous particular cases, we deduce the equality, since $R(s,h(s))$ is positive and vanishes only in $\{0,1\}$.

Since h is monotone, it is straightforward to construct a sequence $\{R_n\}$ of continuous functions vanishing only in $\{0,1\}$, equibounded and such that

$$R_n(s) \geq R(s,h(s)) \text{ in } [0,1], \text{ and } R_n(s) \to R(s,h(s)) \text{ a.e. in } [0,1].$$

Let ψ_n be a solution of $\psi' = \sqrt{R_n(\psi)}$, satisfying e.g. $\psi_n(0) = 1/2$. Then, $\psi_n \in X \cap C^1(\mathbb{R})$, and

$$\int_{\mathbb{R}} R_n(\psi_n) + (\psi_n')^2 = 2\int_0^1 \sqrt{R_n(s)}ds.$$

Recalling that $R_n \geq R(\cdot,h(\cdot))$, and $\lim_{n\to+\infty}\int_0^1 \sqrt{R_n(s)}ds = 2\int_0^1 \sqrt{R_h(s)}ds$, the inequality

$$\phi(h) \leq 2\int_0^1 \sqrt{R(s,h(s))}ds$$

follows since $\psi_n \in X \cap C^1$ for every n.

Therefore,

$$\phi(h) = 2\int_0^1 \sqrt{R(s,h(s))}ds. \tag{3.3.4}$$

Now, assume that h is such that $\Phi = \phi(h)$. It is clear that there exists at least one $h \in H$ satisfying this condition. Then, the minimum in $\phi(h)$ is attained, i.e. there exists $\psi \in X \cap C^1(\mathbb{R})$ such that

$$\phi(h) = \int_{\mathbb{R}} R(\psi, h(\psi)) + (\psi')^2.$$

Recalling (3.3.4), we obtain

$$\psi' = \sqrt{R(\psi,h(\psi))},$$

therefore, since ψ is is an homeomorphism, $s \mapsto R(s,h(s))$ is continuous. ■

We notice that the minimization problem in (3.3.2) can be solved, for a wide class of W, as a minimization problem in $\mathbb{R}^n$ with constraints. Indeed, for W such that the corresponding g is piecewise monotone, i.e. $(0,1)$ splits in $N+1$ intervals of monotonicity, a function $h \in H$ can have at most N jump points. Thus, for every $n \leq N$, we consider $h \in H$ with n jump points $s_1,\ldots,s_n$, and setting, for every $i \in \{1,\ldots,n\}$, $a_i = h^-(s_i)$, $b_i = h^+(s_i)$ we deal with the problem of minimize

$$\sum_{i=1}^{n+1}\int_{b_{i-1}}^{a_i} \sqrt{W(s) + \frac{(W'(s))^2}{4}\Big(W''(s)+2\Big)},$$

with $b_0 = 0$, $a_{n+1} = 1$ and $a_i < b_i < a_{i+1}$ for each i, and further constraints on a_i, b_i due to the continuity of $R(\cdot,h(\cdot))$ and to the properties satisfied by $h \in H$.

Let us notice that, recalling the equality

$$\Phi = \inf\Big\{\int_{\mathbb{R}} W(\varphi) + \frac{1}{4}\iint_{\mathbb{R}^2} e^{-|x-y|}\big(\varphi(x)-\varphi(y)\big)^2\,dxdy \;:\; \varphi \in X\Big\},$$

the results of this section allow the computation of the explicit value of the constant which appears in the Γ-limit of some families of nonlocal functionals.

3.4 The optimal profile on the half-line

In this section we consider, for $i \in \{0, 1\}$, the problem

$$\Phi(i, \lambda) = \inf\{\mathcal{F}^\circ(\varphi, \psi) \ : \ \varphi \in X_i, \ \psi \in X_i \cap Y, \ \psi(0) = \lambda\} \tag{3.4.1}$$

where $\lambda \in \mathbb{R}$, and

$$X_i = \{u \colon (-\infty, 0) \to \mathbb{R} \ \text{ measurable s.t. } \ \exists \lim_{t \to -\infty} u(t) = i\};$$

$$\mathcal{F}^\circ(\varphi, \psi) = \int_{-\infty}^{0} W(\varphi) + (\varphi - \psi)^2 + (\psi')^2.$$

With a slight abuse of notation, in this context Y stands for the set of the functions $u \in AC(-\infty, 0)$ such that $u' \in L^2(-\infty, 0)$.

In this Section we deal with the case $i = 0$, noticing that every result shown for $\Phi(0, \lambda)$ has an obvious counterpart for $\Phi(1, \lambda)$.

The results concerning the problem of the optimal profile for the functional $\mathcal{F}^\circ$ are clearly in a close connection with the ones obtained in the previous sections.

Here we notice essentially the differences between the minimum problem on the real line and its counterpart on the half-line, especially emphasizing the results which will be useful in the next chapter. In fact, the optimal profile problem for the functional $\mathcal{F}^\circ$ is the key to the proof of the Γ-convergence theorem for the functionals with boundary Dirichlet conditions (Theorem 4.1.1).

In view of these applications, we give some approximation results, especially in the cases where the optimal profile problem does not have solution ($\lambda \geq 1$).

It is easy to check, thanks to a rearrangement argument, following the steps of the proof of (3.1.2), that

$$\Phi(0, \lambda) = \begin{cases} \inf\{\widetilde{\mathcal{F}}(\varphi, \psi) : \ \varphi \in X_0 \cap Z^+, \ \psi \in X_0 \cap Y \cap Z^+, \ \psi(0) = \lambda\} & \text{if } \lambda > 0 \\ \\ \inf\{\widetilde{\mathcal{F}}(\varphi, \psi) : \ \varphi \in X_0 \cap Z^-, \ \psi \in X_0 \cap Y \cap Z^-, \ \psi(0) = \lambda\} & \text{if } \lambda < 0, \end{cases} \tag{3.4.2}$$

where

$$Z^+ = \{u \colon (-\infty, 0) \to \mathbb{R} \ \text{ non decreasing}\}$$

$$Z^- = \{u \colon (-\infty, 0) \to \mathbb{R} \ \text{ non increasing}\}.$$

We give a sketch of the proof in the case $\lambda < 0$, the other being completely similar.

1. $\Phi(0, \lambda) = \inf\{\mathcal{F}^\circ(\varphi, \psi) \ : \ \varphi \in X_0^*, \ \psi \in X_0^* \cap Y, \ \psi(0) = \lambda\}$, where

   X_0^* is the subset of the elements φ in X_0 such that $\varphi = 0$ in $(-\infty, -R)$ for some $R > 0$.

2. We observe that $\psi \in X_0^* \cap Y$ is clearly bounded, and that it is sufficient to consider $\varphi \in X_0^*$ bounded since $\lim_{|t| \to +\infty} W(t) = +\infty$. Then, for such a pair (φ, ψ) we construct the decreasing rearrangment (φ_r, ψ_r) in $(-R, 0)$. As in the proof of Proposition 3.1.4, $\mathcal{F}^\circ(\varphi_r, \psi_r) \leq \mathcal{F}^\circ(\varphi, \psi)$.

3. Clearly, $\psi_r(0) \leq \lambda < 0$. Since $\psi_r \in H^1(-\infty, 0)$, then it is continuous, there exists $T \geq 0$ such that $\psi_r(T) = \lambda$. The pair (φ_T, ψ_T) defined by

$$\varphi_T = \varphi_r(t+T), \quad \psi_T = \psi_r(t+T)$$

belongs to $(X_0 \cap Z^-) \times ((X_0 \cap Y) \cap Z^-)$, $\psi_T(0) = \lambda$, and

$$\mathcal{F}^\circ(\varphi_T, \psi_T) \leq \mathcal{F}^\circ(\varphi, \psi).$$

This concludes the proof of (3.4.2).

3.4.1 The case $\lambda < 0$

We start by proving the following

Proposition 3.4.1. *If $\lambda < 0$, the infimum in (3.4.1) is attained.*

Proof. As in the proof of the Proposition 3.1.3, the proof is based on the fact that we can work with monotone functions only (see (3.4.2)).

Let $\{(\varphi_n, \psi_n)\}$ a minimizing sequence in $(X_0 \cap Z^-) \times (X_0 \cap Y \cap Z^-)$. We can clearly choose $\varphi_n, \psi_n \leq 0$, and since $\lim\limits_{s \to +\infty} W(s) = +\infty$, and $\psi_n(0) = \lambda$, we can assume also φ_n uniformly bounded in $L^\infty(-\infty, 0)$. Then:

$$|D\varphi_n| \leq c, \qquad |\varphi_n| \leq c,$$

where c is a positive constant which does not depend on n. By the compactness theorem in BV, up to the extraction of a subsequence:

$$\exists\ \varphi \in BV_{loc} \ \text{ s.t. } \ \varphi_n \longrightarrow \varphi \ \text{ in } \ L^1_{loc}. \tag{3.4.3}$$

Clearly, φ is non increasing, and $0 \geq \varphi \geq -c$. The Fatou Lemma ensures that

$$\int_{-\infty}^0 W(\varphi) \leq \liminf \int_{-\infty}^0 W(\varphi_n),$$

and since the right-hand side is uniformly bounded, this implies $\lim\limits_{t \to -\infty} \varphi(t) = 0$; hence, $\varphi \in X_0 \cap Z^+$.

Since ψ_n' is uniformly bounded in $L^2(\mathbb{R})$, and $0 \geq \psi_n \geq \lambda$, a weak compactness argument gives

$$\exists\ \psi \in Y \ \text{ s.t. } \ \psi_n \longrightarrow \psi \ \text{ in } \ H^1(-T, 0) \quad \forall\ T > 0.$$

The function ψ is non increasing, and, since $\lim\limits_{t \to -\infty} \varphi(t) = 0$, the same property holds for ψ; then, $\psi \in X_0 \cap Y \cap Z^-$. Moreover, the weak convergence in $H^1(-T, 0)$ implies the convergence of the traces, i.e.:

$$\psi_n(0) \longrightarrow \psi(0).$$

Hence, $\psi(0) = \lambda$. The weak lower semincontinuity of the norm and the Fatou Lemma give:

$$\widetilde{\mathcal{F}}(\varphi, \psi) \leq \lim_{n \to \infty} \widetilde{\mathcal{F}}(\varphi_n, \psi_n) = \Phi(0, \lambda).$$

∎

As in the Section 3.1, we can prove that a minimum point (φ, ψ) is such that $\psi \in C^1(-\infty, 0)$, and $\varphi, \psi \in Z^-$, and in particular $\psi' < 0$ when $\psi \neq 0$.

In appropriate hypotheses on W, very similar to the case of the functional $\mathcal{F}$, it is possible to prove a uniqueness result. Clearly, we are interested in the invertibility or the piecewise monotonicity of the function $\dfrac{W'(s) + 2s}{2}$ in the domain $[\lambda, 0]$ instead of $[0, 1]$.

3.4.2 The case $\lambda > 0$

I It is obvious that if $\lambda \in (0, 1)$ the existence, regularity and uniqueness results follows from the corresponding results for the functional $\mathcal{F}$ considered in the previous section. It is sufficient to consider a pair (φ, ψ) minimizing $\mathcal{F}$ in $X \times (X \cap Y)$ such that $\psi(0) = \lambda$, and take the restriction to $(-\infty, 0)$. This proves the existence and regularity results.

Moreover, if the minimum problem for $\mathcal{F}$ has a unique solution (up to translation), the uniqueness is preserved.

II If $\lambda \geq 1$, in the case $W \in C^1(\mathbb{R})$ and convex in a neighbourhood of 0 and 1, or $W \in C^2(\mathbb{R})$, the problem does not have solution. The existence of (φ, ψ) such that $\mathcal{F}^\circ(\varphi, \psi) = \Phi(0, \lambda)$ implies the existence of a minimum for $\mathcal{F}$ which does not satisfy the strict monotonicity condition. In fact, we can choose $\overline{\varphi}(t) = \max\{\varphi(t), 1\}$ if $t \leq 0$, $\overline{\varphi}(t) = 1$ if $t > 0$, and in the same way we define $\overline{\psi}$. Clearly, $(\overline{\varphi}, \overline{\psi})$ minimizes $\mathcal{F}$ and gives a contradiction.

3.4.3 Approximation results

We want to show that, even if the problem does not have solution, it is possible to approximate the value $\Phi(0, \lambda)$ with pairs of functions (φ, ψ) preserving some regularity properties useful in the sequel, in particular the strict positiveness of the derivative of ψ.

Fix $\eta > 0$, and $\lambda > 1$.

As shown in the proof of Proposition 3.1.4, we can find $T_\eta > 0$, and a pair $(\varphi_{1,\eta}, \psi_{1,\eta})$ in $X \times (X \cap Y)$ such that $\varphi_{1,\eta} = \psi_{1,\eta} = \chi_{(0,+\infty)}$ in $\mathbb{R} \setminus (-T_\eta, T_\eta)$, satisfying:

$$\mathcal{F}(\varphi_{1,\eta}, \psi_{1,\eta}) \leq \Phi + \eta.$$

Moreover, following the proof of (3.4.2), we can find $R_\eta > 0$ and $(\varphi_{2,\eta}, \psi_{2,\eta})$ in $X_1 \times (X_1 \cap Y)$, such that $\varphi_{2,\eta} = \psi_{2,\eta} = 1$ in $(-\infty, -R_\eta)$, $\psi_{2,\eta}(0) = \lambda$, satisfying:

$$\mathcal{F}^\circ(\varphi_{2,\eta}, \psi_{2,\eta}) \leq \Phi(1, \lambda) + \eta.$$

Let us notice that $\psi_{1,\eta}$ and $\psi_{2,\eta}$ are obtained, by a truncation with an affine function, modifying the minimizers ψ_1 and ψ_2 for Φ and $\Phi(1, \lambda)$ respectively, which are of class C^1 and such that $\psi_i' > 0$ when $\psi_i \neq 0, 1$.

Setting

$$\varphi(t) = \begin{cases} \varphi_{1,\eta}(t + T_\eta + R_\eta) & \text{if } t \leq -R_\eta \\ \varphi_{2,\eta}(t) & \text{if } t > -R_\eta, \end{cases}$$

$$\psi(t) = \begin{cases} \psi_{1,\eta}(t + T_\eta + R_\eta) & \text{if } t \leq -R_\eta \\ \psi_{2,\eta}(t) & \text{if } t > -R_\eta, \end{cases}$$

the monotonicity properties are preserved, ψ is piecewise C^1, with the derivative ψ' satisfying:

$$\psi' \geq c(\eta) > 0 \qquad \text{in} \;\; (-R_\eta - 2T_\eta, 0).$$

For every pair $(u,v) \in X_0 \times (X_0 \cap Y)$ with $v(0) = \lambda$ there exists T such that $v(T) = 1$; clearly, this implies $\mathcal{F}^\circ(u,v) \geq \Phi + \Phi(1,\lambda)$. Thus

$$\Phi(0,\lambda) \geq \Phi + \Phi(1,\lambda).$$

It follows that

$$\mathcal{F}^\circ(\varphi,\psi) \leq \Phi(0,\lambda) + 2\eta.$$

As a consequence, we can prove the following *additivity property*:

$$\Phi(0,\lambda) = \Phi + \Phi(1,\lambda) \qquad \text{for} \quad \lambda > 1.$$

Indeed,

$$\Phi(0,\lambda) \leq \mathcal{F}^\circ(\varphi,\psi) \leq \Phi + \Phi(1,\lambda) + 2\eta,$$

and the arbitrariness of η gives the inequality

$$\Phi(0,\lambda) \leq \Phi + \Phi(1,\lambda).$$

In the same way

$$\Phi(1,\lambda) = \Phi + \Phi(0,\lambda) \qquad \text{for} \quad \lambda < 0.$$

These constructions will be useful in the proof of the Γ-lim sup inequality for the sequence of functionals considered in the next chapter (see, e.g., Lemma 4.4.2).

3.5 Dependence on the parameter α and asymptotic behaviour

In the Proposition 2.1.2(b) we studied the asymptotic behaviour of

$$\begin{aligned} c_W(\alpha) &= \sqrt{\alpha}\inf\Big\{ \int_{\mathbb{R}} W(\varphi) + \frac{\alpha^2}{4}\int\int_{\mathbb{R}^2} e^{-\alpha|t-s|}(\varphi(t)-\varphi(s))^2 \;:\; \varphi \in X\Big\} \\ &= \inf\Big\{ \int_{\mathbb{R}} W(\varphi) + \alpha(\varphi-\psi)^2 + (\psi')^2 \;:\; \varphi \in X,\; \psi \in X \cap Y\Big\} \end{aligned}$$

as the parameter α goes to $+\infty$, obtaining $\lim\limits_{\alpha\to+\infty} c_W(\alpha) = C_W$, where

$$C_W = \min\Big\{ \int_{\mathbb{R}} W(\varphi) + \int_{\mathbb{R}} (\varphi')^2 \;:\; \varphi \in X \cap Y\Big\}.$$

Recalling Remark 2.1.4, it is clear that the study of the optimal profile for the functional

$$\mathcal{F}^\alpha(\varphi,\psi) = \int_{\mathbb{R}} W(\varphi) + \alpha(\varphi-\psi)^2 + (\psi')^2$$

is equivalent to the study of the problem concerning the functional $\mathcal{F}$ with W replaced by W/α, and the same holds, in the case of the half line, for $\mathcal{F}^{\circ}$. Therefore, the corresponding existence and regularity results follow. More precisely

$$\begin{aligned}\Phi^{\alpha} &:= \min\{\mathcal{F}^{\alpha}(\varphi,\psi) \ : \ \varphi \in X, \ \psi \in X \cap Y\} \\ &= \sqrt{\alpha}\ \min\Big\{\int_{\mathbb{R}} \frac{W(\varphi)}{\alpha} + (\varphi-\psi)^2 + (\psi')^2 \ : \ \varphi \in X, \ \psi \in X\cap Y\Big\}.\end{aligned}$$

If $W \in C^2$, let us set

$$\alpha_{crit} = \frac{|\min\{W''(s) \ : \ s \in [0,1]\}|}{2}.$$

Then, for $\alpha > \alpha_{crit}$ the corresponding function $g_{\alpha}(s) = \frac{W'(s)}{2\alpha} + s$ is invertible ($\frac{W''(s)}{2\alpha} + 1 > 0$), and the results of §3.2.1 hold for the minimizers of the functional $\mathcal{F}^{\alpha}$.

In particular,

$$c_W(\alpha) = \sqrt{\alpha} \int_0^1 \sqrt{\frac{W(s)}{\alpha} + \frac{(W'(s))^2}{4\alpha^2}} \Big(\frac{W''(s)}{\alpha} + 2\Big) ds \tag{3.5.1}$$

(see Proposition 2.4.1 and (3.3.1)), and this provides a direct proof of the convergence result

$$\lim_{\alpha\to+\infty} c_W(\alpha) = 2\int_0^1 \sqrt{W(s)}ds = C_W.$$

Let us notice that this different proof of Proposition 2.1.2, (b) has been obtained in the hypothesis W of class C^2, while the proof given in the Chapter 2 holds if W is only continuous. This is clearly the natural assumption, since it is sufficient in order to prove the Γ-convergence theorem.

Note that formula (3.5.1) for the representation of $c_W(\alpha)$ is a consequence of properties concerning the Euler equations associated to $\mathcal{F}$. Therefore, for this kind of results, W can not be less regular than C^1.

Finally, let us consider, for $\alpha > 0$:

$$\Phi^{\alpha}(i,\lambda) = \inf\Big\{\int_{-\infty}^{0} W(\varphi) + \alpha(\varphi-\psi)^2 + (\psi')^2 \ : \ \varphi \in X_i, \ \psi \in X_i \cap Y, \ \psi(0) = \lambda\Big\}.$$

Setting

$$I(i,\lambda) = \inf\Big\{\int_{-\infty}^{0} W(\varphi) + (\varphi')^2 \ : \ \varphi \in X_i \cap Y, \ \varphi(0) = \lambda\Big\},$$

it is easy to prove that:

$$\lim_{\alpha\to+\infty} \Phi^{\alpha}(i,\lambda) = I(i,\lambda). \tag{3.5.2}$$

We outline only a sketch of the proof, since it closely follows the proof of Proposition 2.1.2, (b) given in the Chapter 2, noticing however that here the inf values are not attained for some pairs i,λ (see §3.4.1).

1. We prove that the function $\alpha \mapsto \Phi^\alpha(i,\lambda)$ is monotone not decreasing, and $\Phi^\alpha(i,\lambda) \leq I(i,\lambda)$.

2. Fixed $\eta > 0$, let $(\varphi_\alpha, \psi_\alpha) \in X_i \times (X_i \cap Y)$, with $\psi_\alpha(0) = \lambda$, such that

$$\mathcal{F}^\circ(\varphi_\alpha, \psi_\alpha) \leq \Phi^\alpha(i,\lambda) + \eta,$$

we show that φ_α and ψ_α converge to a $\varphi \in X_i \cap Y$, with $\varphi(0) = \lambda$, and

$$\liminf_{\alpha \to +\infty} \mathcal{F}^\circ(\varphi_\alpha, \psi_\alpha) \geq \int_{-\infty}^0 W(\varphi) + (\varphi')^2.$$

This implies $\lim_{\alpha \to +\infty} \Phi^\alpha(i,\lambda) \geq I(i,\lambda)$, and concludes the proof of (3.5.2).

In order to connect the Γ-convergence result of Theorem 4.1.1, for a sequence of functionals with boundary conditions, with Theorem 2.1 in [39] (see Section 4.1), we observe that, as in the case of the functional of Modica and Mortola:

$$I(i,\lambda) = 2\Big|\int_i^\lambda \sqrt{W(s)}ds\Big|. \tag{3.5.3}$$

Let us suppose $i = 0$. Recalling that, if $a, b > 0$, it follows $a^2 + b^2 \geq 2ab$, then, for every $\varphi \in X_0$ monotone (non decreasing if $\lambda > 0$, and non increasing if $\lambda < 0$):

$$\int_{-\infty}^0 W(\varphi) + (\varphi')^2 \geq 2\int_{-\infty}^0 \sqrt{W(\varphi)}|\varphi'| = 2\Big|\int_0^\lambda \sqrt{W(s)}ds\Big|.$$

Thus, $I(0,\lambda) \geq 2|\int_0^\lambda \sqrt{W(s)}ds|$. Clearly, the same inequality holds with $i = 1$.

To prove the other inequality, let us consider the Cauchy problems:

$$(1) \quad \begin{cases} \varphi' = \sqrt{W(\varphi)} \\ \varphi(0) = \lambda \end{cases} \quad \text{and} \quad (2) \quad \begin{cases} \varphi' = -\sqrt{W(\varphi)} \\ \varphi(0) = \lambda \end{cases} \tag{3.5.4}$$

Since $\sqrt{W}$ is continuous, the problems admit a global solution on $(-\infty, 0)$, respectively increasing the first, and decreasing the second.

The constant functions 0 and 1 are solution of both the equations, then:

if $0 \leq \lambda < 1$, the first problem admits a solution in $X_0 \cap Y$;

if $\lambda < 0$, the second problem admits a solution in $X_0 \cap Y$.

The equality holds in $a^2 + b^2 \geq 2ab$ when $a = b$; thus, if φ solves the first problem in (3.5.4) with $0 \leq \lambda < 1$, or the second with $\lambda < 0$:

$$\int_{-\infty}^0 W(\varphi) + (\varphi')^2 = 2\Big|\int_0^\lambda \sqrt{W(s)}ds\Big|,$$

and $\varphi \in Y_0$. Hence, for $\lambda < 1$, we obtain $I(0,\lambda) = 2|\int_0^\lambda \sqrt{W(s)}ds|$.

As to the case $i = 1$, in the same way it follows that, if $\lambda > 0$, then $I(1,\lambda) = 2|\int_1^\lambda \sqrt{W(s)}ds|$.

Now, we deal with the remaining cases. If $i = 0$ and $\lambda \geq 1$, let us observe that an approximation argument gives:

$$I(0,\lambda) \leq C_W + I(1,\lambda).$$

(the construction is closely similar to the one in §3.4.3).

Recalling that $C_W = 2\int_0^1 \sqrt{W(s)}ds$, therefore:

$$I(0,\lambda) \leq 2\int_0^1 \sqrt{W(s)}ds + 2\int_1^\lambda \sqrt{W(s)}ds = 2\Big|\int_0^\lambda \sqrt{W(s)}ds\Big|.$$

The same argument allows to treat the case $i = 1$ and $\lambda \leq 0$, and this concludes the proof of (3.5.3).

Chapter 4

DIRICHLET BOUNDARY CONDITIONS

In the paper [39], the authors consider, in connection with phase separation models, the asymptotic behaviour of minimizers of the penalized energy functionals

$$G_\varepsilon(u) = \int_\Omega W(u) + \varepsilon^2 |Du|^2$$

defined for $u \in H^1(\Omega)$ satisfying the Dirichlet condition

$$u = h_\varepsilon \quad \text{on} \quad \partial\Omega,$$

where h_ε are prescribed functions converging to h in $L^1(\partial\Omega)$, and W is a double-well potential.

In this chapter we analyze the problem with Dirichlet conditions on v for the functionals $F_\varepsilon(u, v)$ introduced in Chapter 2. The constraint is imposed to the function v since v naturally lies in $H^1(\Omega)$, while the functional does not impose regularity constraint on the phase variable u.

As for the functionals G_ε, the problem of determining the structure of minimizers of the functionals F_ε (or $\varepsilon F_\varepsilon$) as $\varepsilon \to 0$, studied in Chapter 2 via Γ-convergence, corresponds to consider Neumann homogeneous conditions. We recall that the minimizers of F_ε converge, as $\varepsilon \to 0$, to the charachteristic function of a subset E of Ω with finite perimeter in Ω, such that the perimeter $P_\Omega(E)$ is minimal.

Following [39], we notice that the structure of minimizers, in the case of Dirichlet boundary condition, can be more complicated. We prove that, also in the case of the two-variable functionals F_ε, the minimizers (or quasi-minimizers) converge to the solution of a variational problem involving the minimization of a functional which consists of two terms:

- a term measuring the energy associated with the interior interfaces between the phases, as in the case of Neumann conditions;

- a term measuring the energy of the boundary layer due to the transition from each of the phases to the prescribed Dirichlet data.

Then, the minimizers or quasi-minimizers converge, as $\varepsilon \to 0$, to the characteristic function of a subset E of Ω with finite perimeter in Ω, such that the perimeter $P_\Omega(E)$ is minimal, and such that the angle between the interface ∂E and $\partial\Omega$ (*contact angle*) is a prescribed function of $x \in \partial\Omega$ (see Section 4.1, and [39, Proposition 5.2]). Problems of this kind are connected with the theory of capillary surfaces and other physical models (see, e.g. [25], and also Section 4.5, [36] and the references therein).

We notice that the *selection criterion* of the equilibrium configurations induced by the sequence of two-variable functionals with Dirichlet conditions gives a different constraint on the contact angle, as we shall see.

In Section 4.5 we shall consider, following the approach of L. Modica (see [36]), a sequence of functionals involving a term of *contact energy* at the boundary instead of Dirichlet conditions.

4.1 Setting of the problem and main results

We assume, as in Chapter 2:

> $W\colon \mathbb{R} \to [0, +\infty)$ belongs to $C^0(\mathbb{R})$ and $W(t) = 0$ if and only if $t \in \{0, 1\}$; moreover W has at least linear growth at $\pm\infty$, i.e.
>
> $$\liminf_{|t|\to+\infty} \frac{W(t)}{|t|} > 0.$$

Moreover, we require that:

> $\Omega \subset \mathbb{R}^n$ is bounded, open, connected, and the boundary $\partial\Omega$ is of class C^2.

If $\{h_\varepsilon\}$ is a sequence in $H^{\frac{1}{2}}(\partial\Omega)$, we can consider, for every $\varepsilon > 0$, the following functional defined on $[L^1(\Omega)]^2$:

$$F^\circ_\varepsilon(u, v) = F_\varepsilon(u, v) + I_{h_\varepsilon}(v) \tag{4.1.1}$$

where, for $g \in H^{\frac{1}{2}}(\partial\Omega)$, I_g is defined on $L^1(\Omega)$ by:

$$I_g(v) = \begin{cases} 0 & \text{if } v \in H^1(\Omega) \quad \text{and} \quad v|_{\partial\Omega} = g \\ +\infty & \text{otherwise,} \end{cases}$$

and $F_\varepsilon\colon [L^1(\Omega)]^2 \to [0, +\infty]$ is the functional

$$F_\varepsilon(u, v) = \begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_\Omega W(u) + \frac{\alpha}{\varepsilon}\int_\Omega (u - v)^2 + \varepsilon\int_\Omega |Dv|^2 & \text{if } v \in H^1(\Omega) \\ +\infty & \text{otherwise,} \end{cases}$$

α being a real positive parameter.

As to the boundary values h_ε, we shall require pointwise convergence together with some regularity and boundedness conditions (in $(h2) - (h4)$ below, c denotes a constant independent on ε).

Hypotheses on the Dirichlet data h_ε**.**

(h1) $h_\varepsilon\colon \partial\Omega \to \mathbb{R}$, $h_\varepsilon \in C^0(\partial\Omega) \cap H^{\frac{1}{2}}(\partial\Omega)$;

(h2) $\|h_\varepsilon\|_{L^\infty} \leq c$;

(h3) $\|h_\varepsilon\|_{H^{\frac{1}{2}}} \leq c$;

(h4) the distributional derivative $\dfrac{\partial h_\varepsilon}{\partial \sigma}$, where σ is a surface parameter in $\partial\Omega$, belongs to $L^\infty(\partial\Omega)$, and $\left|\dfrac{\partial h_\varepsilon}{\partial \sigma}\right| \leq c\varepsilon^{-r}$ for some $r < 1$.

(h5) $\exists\, h\colon \partial\Omega \to \mathbb{R}$, $h \in L^\infty(\partial\Omega) \cap H^{\frac{1}{2}}(\partial\Omega)$, such that

$$h_\varepsilon(x) \to h(x)$$

pointwise for $\mathcal{H}^{n-1}-$ a.a. $x \in \partial\Omega$.

Before giving the statement of the theorem, we recall the definition of the function $\Phi^\alpha(i, \lambda)$ (see Chapter 3, in particular Section 3.5).

Fixed $\alpha > 0$, for $i \in \{0, 1\}$, and $\lambda \in \mathbb{R}$, we set:

$$\Phi^\alpha(i, \lambda) = \inf\Big\{\int_{-\infty}^0 W(\varphi) + \alpha(\varphi - \psi)^2 + (\psi')^2 \ : \ \varphi \in X_i,\ \psi \in X_i \cap Y,\ \psi(0) = \lambda\Big\}$$

where

$$X_i = \{u\colon (-\infty, 0) \to \mathbb{R} \ \text{ measurable s.t. } \lim_{t \to -\infty} u(t) = i\}.$$

Moreover, we recall that

$$c_W(\alpha) = \inf\Big\{\int_{\mathbb{R}} W(\varphi) + \alpha(\varphi - \psi)^2 + (\psi')^2 \ : \ \varphi \in X,\ \psi \in X \cap Y\Big\},$$

where

$$X = \{u\colon \mathbb{R} \to [0, 1] \ \text{ measurable s.t. } \lim_{t \to -\infty} u(t) = 0, \lim_{t \to +\infty} u(t) = 1\}.$$

In the previous definitions, Y denotes the space of the functions $u \in AC(-\infty, 0)$ (resp. $AC(\mathbb{R})$) such that $u' \in L^2(-\infty, 0)$ (resp. $L^2(\mathbb{R})$).

Now we can state the following

Theorem 4.1.1. *Let the hypotheses* $(h1) - (h5)$ *hold. The sequence* $\{F_\varepsilon^\circ\}$ Γ*-converges, as* $\varepsilon \to 0$, *to* $F^\circ\colon L^1(\Omega) \times L^1(\Omega) \to [0, +\infty]$ *defined by:*

$$F^\circ(u, v) = \begin{cases} c_W(\alpha)\mathcal{H}^{n-1}(Su) + \displaystyle\int_{\partial\Omega} \Phi^\alpha(\tilde{u}(x), h(x))\, d\mathcal{H}^{n-1}(x) & \text{if } \ u \in SBV(\Omega), \\ & u = v \in \{0, 1\} \ \text{ a.e.} \\ +\infty & \text{otherwise,} \end{cases} \tag{4.1.2}$$

where $\tilde{u}$ *stands for the trace of* u *on the boundary.*

Remark 4.1.2. Let us notice that, as we precise in the sequel, to prove the Γ-$\liminf$ inequality we shall use only the pointwise a.e. convergence and the uniform boundedness of h_ε, and as to the set Ω it will be sufficient to assume that the boundary is of class C^1.

As in Theorem 2.1.1, it is sufficient to prove the result for $\alpha = 1$ (see Remark 2.1.4). To shorten the notation, we denote $c_W(1)$ and Φ^1 by c_W and Φ respectively.

Thanks to the results obtained in Section 3.5, we have:

$$\lim_{\alpha\to+\infty}\int_{\partial\Omega}\Phi^\alpha(\tilde{u}(x),h(x))d\mathcal{H}^{n-1}(x) = 2\int_{\partial\Omega}\Big|\int_{\tilde{u}(x)}^{h(x)}\sqrt{W(s)}ds\Big|d\mathcal{H}^{n-1}(x),$$

and, recalling Proposition 2.1.2, (b), we can say, even in the case with boundary Dirichlet conditions, that the "two-variables" functional *approximates* the classical Modica-Mortola one.

Indeed, following [39], we define for $\varepsilon > 0$ the functional $J_\varepsilon^\circ : L^1(\Omega) \to [0,+\infty]$ as:

$$J_\varepsilon^\circ(u) = \begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_\Omega W(u) + \varepsilon\int_\Omega |Du|^2 & \text{if } u\in H^1(\Omega) \text{ and } u|_{\partial\Omega} = h_\varepsilon \\ +\infty & \text{otherwise.}\end{cases} \tag{4.1.3}$$

The Γ-convergence results for the sequence of functionals $\{J_\varepsilon^\circ\}$, assuming W of class C^2 and $\{h_\varepsilon\}$ satisfying some regularity and convergence properties slightly stronger than $(h1)-(h5)$, reads as follows:

Theorem 4.1.3. ([39, Theorem 2.1]) *The sequence $\{J_\varepsilon^\circ\}$ Γ-converges in $L^1(\Omega)$ to the functional*

$$J^\circ(u) = \begin{cases} \Psi(0,1)\mathcal{H}^{n-1}(Su\cap\Omega) + \displaystyle\int_{\partial\Omega}\Psi(\tilde{u}(x),h(x))\,d\mathcal{H}^{n-1}(x) & \text{if } u\in BV(\Omega;\{0,1\}) \\ +\infty & \text{otherwise,}\end{cases}$$

where $\Psi(s,t) = |2\int_s^t\sqrt{W(\tau)}\,d\tau|$, and h is the pointwise limit of $\{h_\varepsilon\}$.

Remark 4.1.4. (Compactness) The compactness result for the functionals $\{F_\varepsilon^\circ\}$ follows *a fortiori* from Proposition 2.1.5 (compactness for $\{F_\varepsilon\}$), by the inequality $F_\varepsilon \le F_\varepsilon^\circ$.

Thus, as for the sequence $\{F_\varepsilon\}$:

Corollary 4.1.5. *Let $\{\varepsilon_n\}$ be a positive infinitesimal sequence, and $\{(u_{\varepsilon_n},u_{\varepsilon_n})\}$ be a minimizing or quasi-minimizing sequence for $F_{\varepsilon_n}^\circ$. Then $\{(u_{\varepsilon_n},v_{\varepsilon_n})\}$ is relatively compact in $[L^1(\Omega)]^2$, and every cluster point (u,v) minimizes F°.*

The contact angle

Let us consider the minimum problem associated to the Γ-limit functional $F^*(u) = F^\circ(u,u)$ (with $\alpha = 1$), i.e.

$$\min\Big\{c_W\mathcal{H}^{n-1}(Su) + \int_{\partial\Omega}\Phi(\tilde{u}(x),h(x))d\mathcal{H}^{n-1}(x) \ : \ u\in BV(\Omega;\{0,1\})\Big\}.$$

The problem has a solution. Indeed, the compactness in $L^1(\Omega)$ is given by the compactness theorem in BV, and the lower semicontinuity with respect to the topology of $L^1(\Omega)$ is a consequence of the fact that F° is a Γ-limit (see Proposition 1.0.4, i)).

We notice that a non-constant minimizer (local or global) u_0 of F^* in $BV(\Omega; \{0,1\})$ is such that $S(u)$ has zero mean curvature; this is a well-known result concerning the theory of minimal surfaces (see, e.g. [29]).

Moreover, as in [39, Section 5], dealing with the case $\Omega \subset \mathbb{R}^2$, it is reasonable to suppose that the second term in the Γ-limit stands for a prescribed *contact angle* between the interface $S(u_0)$ and the boundary $\partial\Omega$ (see also Section 4.5).

An adaptation of the proof of [39, Proposition 5.2], which relies on the arguments used in [25, Sec. 1.5 and 1.6] to derive the contact angle condition for the capillary problem, should give the following further property for a minimizer u_0 of F^* in $BV(\Omega; \{0,1\})$:

> if h is continuous in x, then the contact angle $\vartheta(x)$ between $S(u_0)$ and $\partial\Omega$ at x in $\partial\Omega$ is given by
> $$\cos\vartheta(x) = \frac{\Phi(0, h(x)) - \Phi(1, h(x))}{c_W}. \tag{4.1.4}$$

We refer to Section 4.5 for further discussion on this problem.

Let us notice that, since

$$\begin{aligned} \Phi(0,h) &= c_W + \Phi(1,h) \quad \text{if} \;\; h \geq 1 \\ \Phi(1,h) &= c_W + \Phi(0,h) \quad \text{if} \;\; h \leq 0 \end{aligned}$$

(see §3.4.3), then

$$\cos\vartheta(x) = \begin{cases} 1 & \text{if} \;\; h(x) \geq 1 \\ -1 & \text{if} \;\; h(x) \leq 0 \end{cases} \tag{4.1.5}$$

This is consistent with the nature of the equlibrium configurations.

The contact angle prescribed for a minimum of the functional F^*, which appears as limit of two-variables functionals, is different from the one determined by the minimization of the limit G° of the sequence G°_ε, thus it could give an alternative selection criterion for the equilibrium configurations of the two-phase system.

4.2 The one dimensional case

The proof of the Γ-lim inf inequality in the general n-dimensional case relies again on the *slicing method*, which considers functionals obtained by suitable restrictions of F°_ε to parallel lines. Thus, this Section is devoted to show the 1-dimensional version of Theorem 4.1.1.

Let us prove some further results concerning the function Φ. We first show a continuity property which will be useful in the sequel.

Lemma 4.2.1. *For $i \in \{0,1\}$, the function $\Phi(i,\cdot)\colon \mathbb{R} \to \mathbb{R}$ is continuous.*

Proof. Fixed $\lambda_0 \in \mathbb{R}$ and $\eta > 0$ arbitrary, let $(\varphi_0, \psi_0) \in X_i \times X_i \cap Y$ such that $\psi_0(0) = \lambda_0$, and

$$\mathcal{F}^\circ(\varphi_0, \psi_0) \leq \Phi(i, \lambda_0) + \eta.$$

For $\lambda \in \mathbb{R}$ we define, for every $\sigma > 0$:

$$\varphi_\sigma(t) = \begin{cases} \varphi_0(t+\sigma) & \text{if } \ t \le -\sigma \\ 0 & \text{if } \ -\sigma < t \le 0 \end{cases}$$

$$\psi_\sigma(t) = \begin{cases} \psi_0(t+\sigma) & \text{if } \ t \le -\sigma \\ \lambda + \dfrac{\lambda - \lambda_0}{\sigma} t & \text{if } \ -\sigma < t \le 0. \end{cases}$$

Clearly, $(\varphi_\sigma, \psi_\sigma) \in X_i \times (X_i \cap Y)$ and $\psi_\sigma(0) = \lambda$. Moreover,

$$\begin{aligned} \mathcal{F}^\circ(\varphi_\sigma, \psi_\sigma) &= \mathcal{F}^\circ(\varphi_0, \psi_0) + \int_{-\sigma}^{0} \psi_\sigma^2 + \int_{-\sigma}^{0} (\psi_\sigma')^2 \\ &\le \Phi(i, \lambda_0) + \eta + \sigma \max\{\lambda^2, \lambda_0^2\} + \frac{|\lambda - \lambda_0|^2}{\sigma}. \end{aligned}$$

Then, if we choose $\sigma = |\lambda - \lambda_0|$, it follows that:

$$\Phi(i, \lambda) \le \Phi(i, \lambda_0) + |\lambda - \lambda_0|(\max\{\lambda^2, \lambda_0^2\} + 1) + \eta,$$

and the arbitrariness of λ_0, λ and η gives the result. ∎

Now we recall some further definitions given in Chapter 3. For $T > 0$, set:

$$X_i^*(T) = X_i \cap \{u \ : \ u(t) = i \ \text{ for } \ t \le -T\}, \qquad \text{and} \qquad X_i^* = \bigcup_{T>0} X_i^*(T).$$

Remark 4.2.2. Recalling the proof of (3.1.2):

$$\Phi(i, \lambda) = \inf\{\mathcal{F}^\circ(\varphi, \psi) \ : \ \varphi \in X_i^*, \ \psi \in X_i^* \cap Y, \ \psi(0) = \lambda\}.$$

0

Lemma 4.2.3. *If $\underline{\lambda} < \overline{\lambda}$, and $i \in \{0, 1\}$, then for every $\eta > 0$ there exists a $T_\eta > 0$ such that*

$$\forall \lambda \in [\underline{\lambda}, \overline{\lambda}] \quad \exists\, \varphi_\lambda \in X_i^*(T_\eta), \ \psi_\lambda \in X_i^*(T_\eta) \cap Y \ \text{ with } \ \psi_\lambda(0) = \lambda :$$
$$\mathcal{F}^\circ(\varphi_\lambda, \psi_\lambda) \le \Phi(i, \lambda) + \eta.$$

Proof. Fix $\lambda \in [\underline{\lambda}, \overline{\lambda}]$. Assume $\lambda > i$.

Recalling Remark 4.2.2, for $\eta > 0$ we can find $\overline{T}_\eta > 0$, $\varphi \in X_i^*(\overline{T}_\eta)$ and $\psi \in X_i^*(\overline{T}_\eta) \cap Y$ with $\psi(0) = \overline{\lambda}$ such that:

$$\mathcal{F}^\circ(\varphi, \psi) \le \Phi(i, \overline{\lambda}) + \frac{\eta}{2}.$$

Let $\tau(\lambda) = \max\{t \in (-\infty, 0) \ : \ \psi(t) \le \lambda\}$. We shall show that

$$\mathcal{F}^\circ(\varphi_\lambda, \psi_\lambda) \le \Phi(i, \lambda) + \eta, \tag{4.2.1}$$

where

$$\begin{aligned} \varphi_\lambda(t) &= \varphi(t + \tau(\lambda)) \\ \psi_\lambda(t) &= \psi(t + \tau(\lambda)). \end{aligned}$$

By construction

$$\mathcal{F}^\circ(\varphi_\lambda,\psi_\lambda)=\int_{-\infty}^{\tau(\lambda)} W(\varphi)+(\varphi-\psi)^2+(\psi')^2.$$

We can find $\tilde\varphi\in X_i$, $\tilde\psi\in X_i\cap Y$, with $\tilde\psi(0)=\lambda$, such that $\mathcal{F}^\circ(\tilde\varphi,\tilde\psi)\le\Phi(i,\lambda)+\eta/2$, and define:

$$\hat\varphi(t)=\begin{cases}\tilde\varphi(t-\tau(\lambda)) & \text{if } t<\tau(\lambda)\\ \varphi(t) & \text{otherwise}\end{cases}$$

$$\hat\psi(t)=\begin{cases}\tilde\psi(t-\tau(\lambda)) & \text{if } t<\tau(\lambda)\\ \psi(t) & \text{otherwise.}\end{cases}$$

If $\mathcal{F}^\circ(\varphi_\lambda,\psi_\lambda)>\Phi(i,\lambda)+\eta$, then

$$\begin{aligned}\mathcal{F}^\circ(\varphi,\psi)\ &>\Phi(i,\lambda)+\eta+\int_{\tau(\lambda)}^0 W(\varphi)+(\varphi-\psi)^2+(\psi')^2\\ &\ge\mathcal{F}^\circ(\hat\varphi,\hat\psi)+\frac{\eta}{2};\end{aligned}$$

This implies

$$\Phi(i,\overline{\lambda})>\mathcal{F}^\circ(\hat\varphi,\hat\psi),$$

and gives a contradiction. Then, (4.2.1) follows.

We observe that $\varphi_\lambda,\psi_\lambda=i$ in $(\overline{T}_\eta-\tau(\lambda),0)$, and then (since $\tau(\lambda)<0$) in $(\overline{T}_\eta,0)$ for every $\lambda\in[\underline{\lambda},\overline{\lambda}]$ with $\lambda>i$.

If $\lambda\in[\underline{\lambda},\overline{\lambda}]$ and $\lambda<i$, the same argument gives a $\underline{T}_\eta>0$ satisfying the corresponding properties, and the proof of the Lemma is complete with the choice $T_\eta=\max\{\underline{T}_\eta,\overline{T}_\eta\}$.

■

Let $I=(a,b)$ be an open and bounded interval. We can prove a Γ-convergence result for the functionals $F^\circ_\varepsilon\colon[L^1(I)]^2\to[0,+\infty]$ defined by:

$$F^\circ_\varepsilon(u,v;I)=\begin{cases}\dfrac{1}{\varepsilon}\displaystyle\int_I W(u)+\frac{1}{\varepsilon}\int_I(u-v)^2+\varepsilon\int_I(v')^2 & \text{if } v\in H^1(I)\\ & \text{and } v(a)=h_\varepsilon(a),\ v(b)=h_\varepsilon(b)\\ +\infty & \text{otherwise,}\end{cases}\tag{4.2.2}$$

where $(h_\varepsilon(a)),(h_\varepsilon(b))$ are given converging sequences; we denote their limits by $h(a)$ and $h(b)$ respectively.

Theorem 4.2.4. *The sequence of functionals $\{F^\circ_\varepsilon\}$, as $\varepsilon\to0$, Γ-converges in $[L^1(I)]^2$ to $F^\circ\colon[L^1(I)]^2\to[0,+\infty]$ defined by:*

$$F^\circ(u,v)=\begin{cases}c_W\#Su+c^-+c^+ & \text{if } u\in BV(I),u=v\in\{0,1\}\ \text{a.e.}\\ +\infty & \text{otherwise,}\end{cases}$$

where $c^-=\Phi(\tilde u(a),h(a))$ and $c^+=\Phi(\tilde u(b),h(b))$.

As for Theorem 2.2.1, it is sufficient to prove the result for $I = (-1,1)$ (see Section 2.2). We denote $F^\circ_\varepsilon(\cdot,\cdot;I)$ by $F^\circ_\varepsilon(\cdot,\cdot)$.

Proof. **Γ-liminf inequality.**
Let $u, v \in L^1(-1,1)$; then, for every $\{u_\varepsilon\}$, $\{v_\varepsilon\}$ such that $u_\varepsilon \to u$, $v_\varepsilon \to v$ in $L^1(-1,1)$, we show that:

$$\liminf_{\varepsilon\to 0} F^\circ_\varepsilon(u_\varepsilon, v_\varepsilon) \geq F^\circ(u,v). \tag{4.2.3}$$

We can suppose that $u_\varepsilon \in L^2(-1,1)$ and $v_\varepsilon \in H^1(-1,1)$ with $v_\varepsilon(\pm 1) = h(\pm 1)$; moreover, that $\liminf_{\varepsilon\to 0} F^\circ_\varepsilon(u_\varepsilon, v_\varepsilon)$ is a limit and is finite. We can assume, up to subsequences, that the convergences are pointwise a.e..

Since $F^\circ_\varepsilon \geq F_\varepsilon$, where F_ε is the one-dimensional functional defined in (2.2.1), the Proposition 2.2.3 allows to deduce:

$$u = v \in \{0,1\}, \quad \text{and} \quad u \in BV(-1,1).$$

Hence, $S(u) = S(u) \cap (-a,a)$ for some $a \in (0,1)$. Since, again by Proposition 2.2.3, $\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, v_\varepsilon; (-a,a)) \geq c_W \# S(u)$, we obtain:

$$\liminf_{\varepsilon\to 0} F^\circ_\varepsilon(u_\varepsilon, v_\varepsilon) \geq c_W \# S(u) + \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, v_\varepsilon; (-1,-a)) + \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, v_\varepsilon; (a,1)).$$

Now, let us consider the subset $(a,1)$. Up to a slight modification, it is possible to assume $u_\varepsilon(a) = v_\varepsilon(a) = \tilde{u}(1)$. Then, setting $\overline{u}_\varepsilon(x) = u_\varepsilon(x+1)$, and $\overline{v}_\varepsilon(x) = v_\varepsilon(x+1)$:

$$\begin{aligned} F_\varepsilon(u_\varepsilon, v_\varepsilon; (a,1)) &= F_\varepsilon(\overline{u}_\varepsilon, \overline{v}_\varepsilon; (a-1,0)) \\ &= \int_{\frac{a-1}{\varepsilon}}^{0} \big(W(\overline{u}_\varepsilon(\varepsilon y)) + (\overline{u}_\varepsilon(\varepsilon y) - \overline{v}_\varepsilon(\varepsilon y))^2 + (\overline{v}_\varepsilon(\varepsilon y))'^2\big)\,dy \\ &= F_1(\varphi_\varepsilon, \psi_\varepsilon; (\frac{a-1}{\varepsilon}, 0)), \end{aligned}$$

where $\varphi_\varepsilon(y) = \overline{u}_\varepsilon(\varepsilon y)$, and $\psi_\varepsilon(y) = \overline{v}_\varepsilon(\varepsilon y)$. Hence

$$F^\circ_\varepsilon(u_\varepsilon, v_\varepsilon; (a,1)) \geq \Phi(\tilde{u}(1), h_\varepsilon(1)).$$

Taking the lower limit as ε goes to 0, the continuity property of Lemma 4.2.1 yields:

$$\liminf_{\varepsilon\to 0} F^\circ_\varepsilon(u_\varepsilon, v_\varepsilon; (a,1)) \geq \Phi(\tilde{u}(1), h(1)) = c^+.$$

For the term $F_\varepsilon(u_\varepsilon, v_\varepsilon; (-1,-a))$ we can argue in a completely similar way, and (4.2.3) follows.

Γ-limsup inequality.
Let $\eta > 0$ and $\{\varepsilon_j\}$ be a positive infinitesimal sequence. We prove that for every u, v in $L^1(-1,1)$ there exist sequences $\{u_j\}, \{v_j\}$ converging, in $L^1(-1,1)$ as $j \to \infty$, to u and v respectively, and such that:

$$\limsup_{j\to\infty} F^\circ_{\varepsilon_j}(u_j, v_j) \leq F^\circ(u,v) + \eta. \tag{4.2.4}$$

It is enough to prove the Γ-lim sup inequality (4.2.4) for $u = v \in \{0,1\}$, $u \in BV(-1,1)$. Let $a \in (0,1)$ be such that $S(u) = S(u) \cap (-a,a)$. By Theorem 2.2.1 there exist $\{\widehat{u}_j\}, \{\widehat{v}_j\}$ converging to u in $L^1(-1,1)$ such that

$$\limsup_{j\to+\infty} F_{\varepsilon_j}(\widehat{u}_j, \widehat{v}_j; (-a,a)) \leq F(u,v;(-a,a)).$$

On the other hand, since the sequence $\{h_{\varepsilon_j}(1)\}$ is bounded, Lemma 4.2.3 gives $T_\eta > 0$ and sequences $\{\varphi_j^+\}, \{\psi_j^+\}$ in $X^*_{\tilde{u}(1)}(T_\eta)$ and $X^*_{\tilde{u}(1)}(T_\eta) \cap Y$ respectively, with $\psi_j^+(0) = h_{\varepsilon_j}(1)$ and

$$\mathcal{F}^\circ(\varphi_j^+, \psi_j^+) \leq \Phi(\tilde{u}(1), h_{\varepsilon_j}(1)) + \frac{\eta}{2}.$$

In the same way we determine $\{\varphi_j^-\}, \{\psi_j^-\}$ in $X^*_{\tilde{u}(-1)}(T_\eta)$ and $X^*_{\tilde{u}(-1)}(T_\eta) \cap Y$ (we can assume the same T_η as before) with $\psi_j^-(0) = h_{\varepsilon_j}(-1)$ and

$$\mathcal{F}^\circ(\varphi_j^-, \psi_j^-) \leq \Phi(\tilde{u}(-1), h_{\varepsilon_j}(-1)) + \frac{\eta}{2}.$$

For $\varepsilon_j \leq \frac{1-a}{T_\eta}$ we set:

$$u_j(x) = \begin{cases} \varphi_j^-(-\frac{x+1}{\varepsilon_j}) & \text{if } \; -1 < x \leq -a \\ \widehat{u}_j(x) & \text{if } \; -a < x < a \\ \varphi_j^+(\frac{x-1}{\varepsilon_j}) & \text{if } \; a \leq x < 1, \end{cases}$$

and in the same way we define v_j. We notice that $v_j \in H^1(-1,1)$, and $v_j(\pm 1) = h_{\varepsilon_j}(\pm 1)$; it is easy to see that the sequences $\{u_j\}$ and $\{v_j\}$ converge in L^1 to u. Moreover:

$$F^\circ_{\varepsilon_j}(u_j, v_j) \leq c_W \# S(u) + \Phi(\tilde{u}(-1), h_{\varepsilon_j}(-1)) + \Phi(\tilde{u}(1), h_{\varepsilon_j}(1)) + \eta.$$

An application of Lemma 4.2.1 gives (4.2.4). ■

Let us notice that, in the application of the slicing method, the one-dimensional sets involved need not be intervals. Therefore, we state the following easy extension of Theorem 4.2.4.

Proposition 4.2.5. *Let $J \subset \mathbb{R}$ be open and bounded. Let $\tau = \{t_1, \dots, t_N\} \subset \partial J$ such that every t_i belongs to one and only one of the closures of the connected component of J. If $\{\underline{h}_\varepsilon\} = \{(h^1_\varepsilon, \dots, h^N_\varepsilon)\}$ is a sequence in $\mathbb{R}^N$ converging to $\underline{h} = (h^1, \dots, h^N)$, we define the sequence of functionals*

$$F^\circ_\varepsilon(u,v;J) = F_\varepsilon(u,v;J) + I(v),$$

where

$$I(v) = \begin{cases} 0 & \text{if } \; v \in H^1(J) \;\; \text{and} \;\; v(t_i) = h^i_\varepsilon \;\; \forall i \\ +\infty & \text{otherwise.} \end{cases}$$

Then, $\{F^\circ_\varepsilon\}$ Γ-converges in $[L^1(J)]^2$ to the functional F° defined by:

$$F^\circ(u,v) = \begin{cases} c_W \# S(u) + \displaystyle\sum_{i=1}^N \Phi(\tilde{u}(t_i), h_i) & \text{if } \; u = v \in BV(J;\{0,1\}) \\ +\infty & \text{otherwise.} \end{cases}$$

We remark that, if the hypothesis of "disjoint closures" is not satisfied, the Γ-limit has a different representation. Consider, e.g., $J = (1,2) \cup (2,3)$, and $\tau = \{2\}$. Clearly, in the limit, the contribution of the point 2 is double, and the Γ-limit is:

$$F^\circ(u,v) = \begin{cases} c_W \#S(u) + 2\Phi(\tilde{u}(2), h) & \text{if } u = v \in BV(J;\{0,1\}) \\ +\infty & \text{otherwise} \end{cases}$$

If the Dirichlet data are defined in an internal point, clearly we have the same behaviour.

4.3 The Γ-lower inequality

Let $\Omega \subset \mathbb{R}^n$ be open, bounded, connected and of class C^1, and let $\{h_\varepsilon\}$ be a sequence in $H^{\frac{1}{2}}(\partial\Omega)$, pointwise converging $\mathcal{H}^{n-1}$-a.e. in $\partial\Omega$ to a function $h \in L^\infty(\partial\Omega)$.

For every $u, v \in L^1(\Omega)$ let

$$(F^\circ)'(u,v) = \inf\{\liminf_{j\to\infty} F^\circ_{\varepsilon_j}(u_j, v_j) : \quad \varepsilon_j \to 0,\ u_j \to u,\ v_j \to v \text{ in } L^1(\Omega)\}$$

(Γ-lower limit of $\{F_\varepsilon\}$).

Proposition 4.3.1. *For every $u, v \in L^1(\Omega)$*

$$(F^\circ)'(u,v) \geq F^\circ(u,v).$$

The proof follows again the steps outlined in [15]; the application of the *slicing method* in this case needs some further regularity properties of $\partial\Omega$, since the boundary data are involved.

4.3.1 Some preliminary remarks on regularity

For $\xi \in S^{n-1}$ we consider the projection $\pi_\xi : \partial\Omega \to \xi^\perp$ defined by:

$$\pi_\xi(x) = x - \langle x, \xi \rangle \xi.$$

We choose in $\mathbb{R}^n$ a basis $\{\xi, e^1_\xi, \ldots, e^{n-1}_\xi\}$, where $\{e^1_\xi, \ldots, e^{n-1}_\xi\}$ is an orthonormal basis of $\xi^\perp$; in this way $\xi^\perp$ is naturally identified with $\mathbb{R}^{n-1}$.

Since $\partial\Omega$ is a C^1-submanifold of $\mathbb{R}^n$, and π_ξ is a C^1-map, for any Borel $A \subseteq \partial\Omega$ we can write the co-area formula as:

$$\int_A J^*(\pi_\xi) d\mathcal{H}^{n-1} = \int_{\xi^\perp} \mathcal{H}^0(A \cap \pi_\xi^{-1}(y)) d\mathcal{H}^{n-1}(y), \tag{4.3.1}$$

where

$$J^*(\pi_\xi)(x) = \sqrt{\det(d\pi_\xi)_x \circ ((d\pi_\xi)_x)^*},$$

and $(d\pi_\xi)_x : T_x(\partial\Omega) \to \mathbb{R}^{n-1}$ denotes the differential map (see, e.g., [44, p. 55]). Clearly, $J^*(\pi_\xi) = |\langle \xi, \nu \rangle|$.

Denoting by C the set of the x in $\partial\Omega$ such that $\operatorname{rank}(d\pi_\xi)_x < n-1$, i.e. $J^*(\pi_\xi)(x) = 0$, we have

$$C \cap (\pi_\xi)^{-1}(y) = \emptyset \qquad \mathcal{H}^{n-1} - \text{a.a. } y \in \xi^\perp. \tag{4.3.2}$$

Notice that, if $x \in \partial\Omega$ and $y \in \xi^\perp$, then $x \in C \cap (\pi_\xi)^{-1}(y)$ if and only if $r(\xi, y) \subset T^a_x(\partial\Omega)$, where $T^a_x(\partial\Omega)$ is the affine tangent space applied in x, and $r(\xi, y)$ is the line:

$$r(\xi, y) = \{t\xi + y \ : \ t \in \mathbb{R}\}.$$

This result is essentially the C^1 version of the Sard Theorem (see, e.g., [44, p. 56]). For further details, we refer to [44].

Moreover, we have that

$$\int_{\partial\Omega} J^*(\pi_\xi) d\mathcal{H}^{n-1} < +\infty,$$

and from (4.3.1) we deduce that $\partial\Omega \cap (\pi_\xi)^{-1}(y)$ is a finite set for $\mathcal{H}^{n-1}$-a.a. $y \in \mathbb{R}^{n-1}$.

This proves the following:

Lemma 4.3.2. *For $\xi \in S^{n-1}$:*

1. *for $\mathcal{H}^{n-1}$-a.a. $y \in \xi^\perp$ the set $\{t \ : \ t\xi + y \in \partial\Omega\}$ is finite;*
2. *for $\mathcal{H}^{n-1}$-a.a. $y \in \xi^\perp$: $r(\xi, y) \not\subset T^a_x(\partial\Omega)$ for every x.*

Hence, we deduce that for every $\xi \in S^{n-1}$ and for $\mathcal{H}^{n-1}$-a.a. $y \in \xi^\perp$ the set $\Omega^{\xi,y}$ is a finite union of intervals with disjoint closure, and $\partial(\Omega^{\xi,y}) = (\partial\Omega)^{\xi,y}$.

4.3.2 Proof of Proposition 4.3.1

In order to apply the *slicing method* to obtain a lower bound for the Γ-limit, let us consider, for every $\varepsilon > 0$, $\xi \in S^{n-1}$ and $y \in \xi^\perp$, the function $h^{\xi,y}_\varepsilon : (\partial\Omega)^{\xi,y} \to \mathbb{R}$ defined by $h^{\xi,y}_\varepsilon(t) = h_\varepsilon(t\xi + y)$. The convergence hypothesis for the sequence $\{h_\varepsilon\}$ implies that, for every $\xi \in S^{n-1}$ and for $\mathcal{H}^{n-1}-$a.a. $y \in \xi^\perp$:

$$h^{\xi,y}_\varepsilon(t) \to h^{\xi,y}(t) = h(t\xi + y) \quad \text{as} \ \ \varepsilon \to 0 \qquad \forall\, t \in (\partial\Omega)^{\xi,y}.$$

For $J \subset \mathbb{R}$ open and bounded, we set:

$$\phi^{\xi,y}_\varepsilon(\varphi, \psi; J) = \begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_J W(\varphi) + \dfrac{1}{\varepsilon}\int_J (\varphi - \psi)^2 + \varepsilon \int_J (\psi')^2 & \text{if} \ \ \psi \in H^1(J), \\ & \psi = h^{\xi,y}_\varepsilon \text{ in } \partial J \cap (\partial\Omega)^{\xi,y} \\ +\infty & \text{otherwise} \end{cases}$$

If J is a finite union of intervals with disjoint closures, then it satisfies the hypothesis of Proposition 4.2.5, with $\tau = \partial J \cap (\partial\Omega)^{\xi,y}$; from Proposition 4.2.5 we deduce that $\phi^{\xi,y}_\varepsilon$ Γ-converges to:

$$\phi^{\xi,y}(\varphi, \psi; J) = \begin{cases} c_W \# S(\varphi) + \displaystyle\sum_{t \in \partial J \cap (\partial\Omega)^{\xi,y}} \Phi(\widetilde{\varphi}(t), h^{\xi,y}(t)) & \text{if} \ \ \varphi, \psi \in BV(J), \\ & \varphi = \psi \in \{0,1\} \text{a.e.} \\ +\infty & \text{otherwise} \end{cases}$$

Now define

$$(F_\varepsilon^\circ)^\xi(u,v) = \int_{\xi^\perp} \phi_\varepsilon^{\xi,y}(u^{\xi,y}, v^{\xi,y}; \Omega^{\xi,y}) d\mathcal{H}^{n-1}(y) \tag{4.3.3}$$

for every $(u,v) \in [L^1(\Omega)]^2$.

For every $u \in L^1(\Omega)$ and $v \in H^1(\Omega)$, with $v = h_\varepsilon$ on $\partial\Omega$, we can write $(F_\varepsilon^\circ)^\xi(u,v)$ as an integral over Ω by applying Fubini's Theorem and taking into account Lemma 4.3.2. We get

$$(F_\varepsilon^\circ)^\xi(u,v) = \frac{1}{\varepsilon}\int_\Omega W(u) + \frac{1}{\varepsilon}\int_\Omega (u-v)^2 + \varepsilon \int_\Omega |\langle Dv, \xi\rangle|^2.$$

Therefore $(F_\varepsilon^\circ)^\xi \le F_\varepsilon^\circ$; it follows, by Fatou's Lemma, that for every $\xi \in S^{n-1}$ and for every $(u,v) \in [L^1(\Omega)]^2$

$$(F^\circ)' \ge (F^\circ)^\xi(u,v),$$

where

$$(F^\circ)^\xi(u,v) = \int_{\xi^\perp} \phi^{\xi,y}(u^{\xi,y}, v^{\xi,y}; \Omega^{\xi,y}) d\mathcal{H}^{n-1}(y).$$

Thus, if $(F^\circ)'(u,v)$ is finite, then $u = v \in \{0,1\}$, $u^{\xi,y} \in SBV(\Omega^{\xi,y})$ for a.a. $y \in \xi^\perp$, and

$$\int_{\xi^\perp} \#S(u^{\xi,y}) d\mathcal{H}^{n-1}(y) < +\infty,$$

i.e.

$$\int_{\xi^\perp} |D(u^{\xi,y})|(\Omega^{\xi,y}) d\mathcal{H}^{n-1}(y) < +\infty;$$

recalling remark (ii) in the proof of Proposition 2.3.1, we deduce that if $(F^\circ)'$ is finite, then $u, v \in SBV(\Omega)$ and $u = v \in \{0,1\}$ a.e.. For such pairs (u,v) we have

$$\begin{aligned}(F^\circ)^\xi(u,v) &= \int_{\xi^\perp} c_W \#S(u^{\xi,y}) d\mathcal{H}^{n-1}(y) \\ &+ \int_{\xi^\perp} \sum_{t \in (\partial\Omega)^{\xi,y}} \Phi(\tilde{u}^{\xi,y}(t), h(t\xi + y)) d\mathcal{H}^{n-1}(y).\end{aligned} \tag{4.3.4}$$

Then, applying the co-area formula we can deduce:

$$(F^\circ)^\xi(u,v) = c_W \int_{S(u)} |\langle \xi, \nu_u(x)\rangle| d\mathcal{H}^{n-1}(x) + \int_{\partial\Omega} \Phi(\tilde{u}(x), h(x))\ |\langle \xi, \nu(x)\rangle| d\mathcal{H}^{n-1}(x).$$

Therefore, the Γ-lower limit

$$(F^\circ)'(u,v) = \Gamma - \liminf_{\varepsilon\to 0} F_\varepsilon^\circ(u,v)$$

is finite only if $u, v \in SBV(\Omega)$ and $u = v \in \{0,1\}$ a.e.. Moreover, $F^\circ(u,v) \ge c\mathcal{H}^{n-1}(S(u))$ (thus, finite only if $\mathcal{H}^{n-1}(S(u))$ is finite).

Then, we consider in $\mathcal{A}(\mathbb{R}^n) = \{A: \text{ open subset of } \Omega\}$ the set function

$$\mu(A) = \sup_i \int_A g_i d\lambda,$$

where $g_i : \mathbb{R}^n \to \mathbb{R}$ is defined by:

$$g_i(x) = \begin{cases} c_W|\langle \xi_i, \nu_u(x)\rangle| & \text{if } x \in S(u) \\ \Phi(\tilde{u}(x), h(x))|\langle \xi_i, \nu(x)\rangle| & \text{if } x \in \partial\Omega \\ 0 & \text{otherwise,} \end{cases}$$

and $\lambda = \mathcal{H}^{n-1} \llcorner (S(u) \cup \partial\Omega)$.

The function μ is superadditive on disjoint open sets of $\mathbb{R}^n$ when $u = v \in SBV(\Omega, \{0,1\})$, and we can conclude, applying the Proposition 1.0.1:

$$\mu(A) \geq \int_A \sup_i g_i d\lambda.$$

Since (ξ_i) is dense in S^{n-1} :

$$g(x) = \sup_i g_i(x) = \begin{cases} c_W & \text{if } x \in S(u) \\ \Phi(\tilde{u}(x), h(x)) & \text{if } x \in \partial\Omega \\ 0 & \text{otherwise.} \end{cases}$$

In particular, choosing $A = \mathbb{R}^n$:

$$\sup_i \Big(\int_{S(u)} c_W|\langle \xi_i, \nu_u\rangle| d\mathcal{H}^{n-1} + \int_{\partial\Omega} \Phi(\tilde{u}(x), h(x))|\langle \xi_i, \nu_u\rangle| d\mathcal{H}^{n-1} \Big) = \mu(\mathbb{R}^n) \geq$$
$$\int_{\mathbb{R}^n} g(x) d\lambda = c_W \mathcal{H}^{n-1}(S(u)) + \int_{\partial\Omega} \Phi(\tilde{u}(x), h(x)) d\mathcal{H}^{n-1}(x)$$

Then:

$$(F^\circ)'(u,v) \geq c_W \mathcal{H}^{n-1}(S(u)) + \int_{\partial\Omega} \Phi(\tilde{u}(x), h(x)) d\mathcal{H}^{n-1}(x). \tag{4.3.5}$$

This concludes the proof of the Γ-lim inf inequality.

4.4 The Γ-upper inequality

In this section we conclude the proof of the Γ-convergence result by showing the inequality for the Γ-upper limit of $\{F_\varepsilon^\circ\}$, i.e.

$$(F^\circ)''(u,v) = \inf\big\{\limsup_{j\to\infty} F^\circ_{\varepsilon_j}(u_j, v_j; \Omega) : \quad \varepsilon_j \to 0,\ u_j \to u,\ v_j \to v \text{ in } L^1(\Omega)\big\}.$$

Proposition 4.4.1. *Let $\Omega \subset \mathbb{R}^n$ bounded, open, connected, with boundary of class C^2. Under conditions $(h1) - (h5)$,*

$$(F^\circ)''(u,v) \leq F^\circ(u,v) \tag{4.4.1}$$

for every $u, v \in L^1(\Omega)$.

We recall that the Γ-lim inf inequality, i.e.

$$(F^\circ)'(u,v) \geq F^\circ(u,v),$$

holds in weaker hypotheses on both Ω and the boundary data.

In view of Remark 1.0.2, a), we shall prove Proposition 4.4.1 by exibiting, for every $\eta > 0$, a *recovery sequence* $\{(u_\varepsilon, v_\varepsilon)\}$, converging in $[L^1(\Omega)]^2$ to (u, v), such that $v_\varepsilon \in H^1(\Omega)$, tr $v_\varepsilon = h_\varepsilon$ on $\partial\Omega$, and

$$\limsup_{\varepsilon\to 0} F_\varepsilon^\circ(u_\varepsilon, v_\varepsilon) = \limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, v_\varepsilon) \le F^\circ(u, v) + \eta. \tag{4.4.2}$$

4.4.1 Construction of $\varphi_i(h,\cdot), \psi_i(h,\cdot)$

This subsection is devoted to a preliminary construction which will be useful in the sequel of the proof, in particular for the definition of the recovery sequence in the boundary layer near $\partial\Omega$ (see §4.4.4).

Let us fix

$$M > \max\{1,\ \sup_\varepsilon M_\varepsilon\},$$

$$m < \min\{0,\ \inf_\varepsilon m_\varepsilon\},$$

where $M_\varepsilon = \max\limits_{x\in\partial\Omega} h_\varepsilon(x)$ and $m_\varepsilon = \min\limits_{x\in\partial\Omega} h_\varepsilon(x)$. The hypotheses $(h1)$ and $(h2)$ ensure that M and m are finite.

We recall that, for $i \in \{0, 1\}$, $\lambda \in \mathbb{R}$:

$$\Phi(i, \lambda) = \inf\{\mathcal{F}^\circ(\varphi, \psi)\ :\ \varphi \in X_i,\ \psi \in X_i \cap Y,\ \psi(0) = \lambda\}.$$

The results obtained in Chapter 3, in particular the constructions in §3.4.3, give a prove of the following:

Lemma 4.4.2. *Let $i \in \{0, 1\}$, and $\lambda \in \{m, M\}$. For every $\eta > 0$, we can find $R_\eta^{i,\lambda} > 0$ and*

1. *$\varphi_{i,\lambda}^\eta \in X_i^*(R_\eta^{i,\lambda})$ monotone,*
2. *$\psi_{i,\lambda}^\eta \in X_i^*(R_\eta^{i,\lambda}) \cap Y$ with the properties:*
   (a) *$\psi_{i,\lambda}^\eta$ piecewise C^1 in $[-R_\eta^{i,\lambda}, 0]$,*
   (b) *if $\lambda > i$: $(\psi_{i,\lambda}^\eta)' \ge c(\eta) > 0$ in $(-R_\eta^{i,\lambda}, 0)$,*
   if $\lambda < i$: $(\psi_{i,\lambda}^\eta)' \le -c(\eta) < 0$ in $(-R_\eta^{i,\lambda}, 0)$,
   (c) *$\psi_{i,\lambda}^\eta(0) = \lambda$;*

such that

$$\mathcal{F}^\circ(\varphi_{i,\lambda}^\eta, \psi_{i,\lambda}^\eta) \le \Phi(i, \lambda) + \frac{\eta}{2}.$$

Remark 4.4.3. The function $\psi_{i,\lambda}^\eta$ restricted to $[-R_\eta^{i,\lambda}, 0]$ is invertible, and we denote the inverse function by $(\psi_{i,\lambda}^\eta)^{-1}$. It follows that $(\psi_{i,\lambda}^\eta)^{-1}$ is piecewise C^1 in its domain, and $\big((\psi_{i,\lambda}^\eta)^{-1}\big)' \ge 1/c(\eta) > 0$.

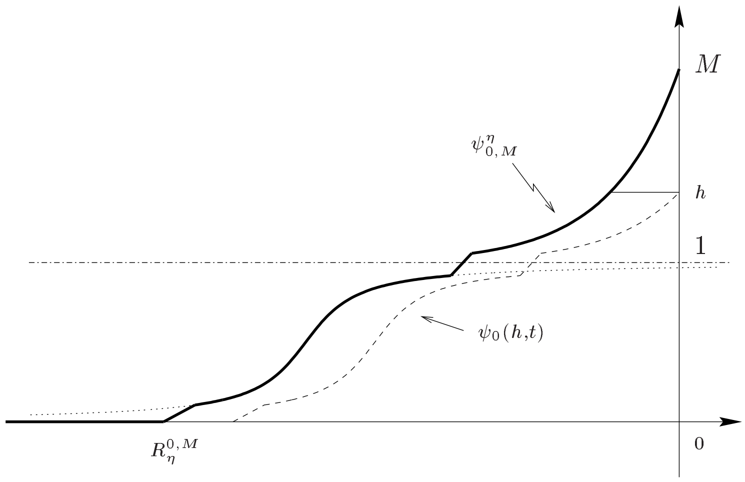

Figure 4.1.

From now on, unless confusion may occur, we shall drop the dependence on η. For $i \in \{0,1\}$ and $h \in [m,M]$, we set:

$$\varphi_i(h,t) = \begin{cases} \varphi^{\eta}_{i,M}((\psi^{\eta}_{i,M})^{-1}(h)+t) & \text{if } i < h \leq M \\ i & \text{if } h = i \\ \varphi^{\eta}_{i,m}((\psi^{\eta}_{i,m})^{-1}(h)+t) & \text{if } m \leq h < i, \end{cases}$$

$$\psi_i(h,t) = \begin{cases} \psi^{\eta}_{i,M}((\psi^{\eta}_{i,M})^{-1}(h)+t) & \text{if } i < h \leq M \\ i & \text{if } h = i \\ \psi^{\eta}_{i,m}((\psi^{\eta}_{i,m})^{-1}(h)+t) & \text{if } m \leq h < i, \end{cases}$$

Recalling the proof of Lemma 4.2.3, it follows that:

$$\mathcal{F}^{\circ}(\varphi_i(h,\cdot),\psi_i(h,\cdot)) \leq \Phi(i,h) + \eta. \tag{4.4.3}$$

Moreover, for every $h \in [m,M]$, $i \in \{0,1\}$:

$$\varphi_i(h,\cdot) = \psi_i(h,\cdot) = i \ \text{ in } \ (-\infty, -R_\eta + 1), \tag{4.4.4}$$

with R_η sufficiently large, say

$$R_\eta = \max\{R^{\eta}_{1,M}, R^{\eta}_{1,m}, R^{\eta}_{0,M}, R^{\eta}_{0,m}\} + 1. \tag{4.4.5}$$

In the sequel we will use the fact that the function $\psi_i(h,\cdot)$ is identically constant in a right neighbourhood of $-R_\eta$ to glue together the partial constructions and obtain the recovery sequence.

We conclude this subsection noting the following property, easy to prove.

Proposition 4.4.4. *The function ψ_i is Lipschitz in $[m,M] \times [-R_\eta, 0]$.*

4.4.2 Regularity and density results

We introduce the following notation: given a subset S of $\mathbb{R}^n$ and $\varrho > 0$ define

$$S_\varrho = \{x \in \mathbb{R}^n \, : \, d(x, S) < \varrho\}.$$

If G is a bounded open subset of $\mathbb{R}^n$ with C^2 boundary, then there exists $\varrho > 0$ such that the function $\Lambda \colon \partial G \times (-\varrho, \varrho) \to \mathbb{R}$ defined by:

$$\Lambda(x, t) = x + t\nu_G(x),$$

is injective, where $\nu_G \colon \partial G \to S^{n-1}$ denotes a unit normal field to ∂G. Then we can define the projection $\pi_G \colon (\partial G)_\varrho \to \partial G$ which maps x to the unique minimizer in ∂G of the function $y \mapsto d(x, y)$.

Moreover (see [9, Theorem 3.1]), by the implicit function theorem, we can choose ϱ so that Λ is a local diffeomorphism; as a consequence, π_G is C^1.

For convenience we simply set

$$\pi = \pi_\Omega \quad \text{and} \quad \nu = \nu_\Omega,$$

where ν_Ω is the inner normal field.

In order to prove the Γ-$\limsup$ inequality, it is convenient, as in Chapter 2, to use a *density argument.* We recall the steps of this procedure (see, e.g., [15, p.97]):

1. Find a subset $\mathcal{T}$ of the domain of F° such that for each (u, v) in the domain of F° there exists a sequence $\{u_j, v_j\}$ in $\mathcal{T}$ satisfying $u_j \to u$, $v_j \to v$ in $L^1(\Omega)$, and $F^\circ(u, v) = \lim_{j\to\infty} F^\circ(u_j, v_j)$;
2. Prove that $(F^\circ)''(u, v) \le F^\circ(u, v)$ for each $(u, v) \in \mathcal{T}$.

The lower semicontinuity of $(F^\circ)''$ entails:

$$(F^\circ)''(u, v) \le \liminf_j (F^\circ)''(u_j, v_j) \le \lim_j F^\circ(u_j, v_j) = F^\circ(u, v)$$

for every (u, v) in the domain of F°.

It is clear that $F^\circ(u, v)$ is finite if and only if $u = v = \chi_E$, where E is a subset of $\mathbb{R}^n$ having finite perimeter in Ω.

In order to define the set $\mathcal{T}$, we need the following definition (see[31, §3.2, p. 74]):

Definition 4.4.5. *Let M, N be C^1-submanifolds of $\mathbb{R}^n$. We say that M is transverse with respect to N (or that M and N are mutually transverse) if:*

$$\forall \, x \in M \cap N \quad T_x M \oplus T_x N = \mathbb{R}^n.$$

Now, we define $\mathcal{T}$ as the set of the pairs (χ_E, χ_E) where E is the intersection with Ω of an open subset of $\mathbb{R}^n$ of class C^∞, whose boundary is transverse with respect to $\partial\Omega$, i.e.:

$$\mathcal{T} = \{(\chi_E, \chi_E) \, : \, E \in \mathcal{S}\},$$

where

$$\mathcal{S} = \{E = \Omega \cap A \, : \, A \subset \mathbb{R}^n \ \text{ open, such that } \ \partial A \ \text{ of class } \ C^\infty \ \text{ transverse to } \ \partial\Omega\}.$$

The required density of $\mathcal{T}$ in the domain of F° follows from Lemma 4.4.6 and Lemma 4.4.7.

Lemma 4.4.6. *Let $\Omega \subset \mathbb{R}^n$ open, with compact and Lipschitz boundary. Given $E \subset \Omega$ with finite perimeter in Ω, there exists a sequence $\{E'_j\}$ of subsets of $\mathbb{R}^n$ with boundaries of class C^∞ in a neighbourhood of $\overline{\Omega}$, such that the sequence $\{E_j\}$ defined by $E_j = E'_j \cap \Omega$ approximates E in $L^1(\Omega)$, i.e. $|E \Delta E_j| \to 0$ as $j \to +\infty$. Moreover:*

1. $|D\chi_{E_j}|(\Omega) \to |D\chi_E|(\Omega)$;
2. $\tilde{\chi}_{E_j} \to \tilde{\chi}_E$ *in* $L^1(\partial\Omega; \mathcal{H}^{n-1} \llcorner \partial\Omega)$,

where $\tilde{\chi}_{E_j}, \tilde{\chi}_E$ stand for the traces of χ_{E_j} and χ_E respectively.

Proof. Except for the assertion 2, the proof is the same of the Proposition 4.7 in [15], To prove 2, recall that

$$\chi_{E_j} \to \chi_E \qquad \text{strictly in } \; BV(\Omega)$$

(see [8, Definition 3.14, p. 125]). Since, for Ω open subset of $\mathbb{R}^n$ with bounded Lipschitz boundary, the trace operator is continuous between $BV(\Omega)$, endowed with the topology induced by strict convergence, and $L^1(\partial\Omega; \mathcal{H}^{n-1} \llcorner \partial\Omega)$ ([8, Theorem 3.88, p. 181]), the assertion 2 follows. ■

Lemma 4.4.6 proves the density in the domain of F° of the set

$$\tilde{\mathcal{T}} = \{(\chi_E, \chi_E) \; : \; E \subset \Omega \quad \text{s.t.} \quad E = \Omega \cap A \quad \text{for some} \quad A \quad \text{of class} \quad C^\infty\}.$$

The required density of the set $\mathcal{T}$ in the domain of F° follows from the application of the transversality lemma below to the smooth sets A which define the element of $\tilde{\mathcal{T}}$.

Lemma 4.4.7. *Let Ω be a bounded subset of $\mathbb{R}^n$ with boundary of class C^2. If $A \subset \mathbb{R}^n$ is bounded and of class C^∞, for every $\eta > 0$ there exists $f \colon \partial A \to (0, +\infty)$ of class C^∞ such that the set*

$$A_f = A \cup \{x + t\nu_A(x) \; : \; t \in [0, f(x)), \; x \in \partial A\},$$

where ν_A stands for the outer unit normal field, satisfies the following properties:

1. ∂A_f *is transverse with respect to* $\partial\Omega$;
2. $|(A \Delta A_f) \cap \Omega| \leq \eta$;
3. $\big| |D\chi_{A_f}|(\Omega) - |D\chi_A|(\Omega) \big| \leq \eta$;
4. $\|\tilde{\chi}_{A_f} - \tilde{\chi}_A\|_{L^1(\partial\Omega)} \leq \eta$,

where $\tilde{\chi}_{A_f}$ and $\tilde{\chi}_A$ denote the trace of χ_{A_f} and χ_A on $\partial\Omega$ respectively.

Proof. Given A a bounded subset of $\mathbb{R}^n$ with C^∞ boundary, its boundary can be represented by the injection $i_A \colon \partial A \to \mathbb{R}^n$ of class C^∞. As observed at the beginning of this subsection, the regularity hypothesis allows us to find $\delta > 0$ such that the projection $\pi_A \colon (\partial A)_\delta \to \partial A$ is well-defined and smooth.

We set $\mathcal{G} = \{g \colon \partial A \to \mathbb{R}^n$ of class $C^\infty\}$, endowed with a C^1-type topology, which relies on the uniform convergence of the functions and their differentials on compact sets.

More precisely, this is the *compact-open* C^1-topology, and, since ∂A is compact, it coincides with the *strong* C^1-topology (for the precise definitions and further details, we refer to [31, p. 34–35]).

We construct a neighbourhood $\mathcal{U}$ of i_A in $\mathcal{G}$ such that $g(\partial A) \subset (\partial A)_\delta$ for every $g \in \mathcal{U}$, and moreover for every $g \in \mathcal{U}$ we can find $f \colon \partial A \to (-\delta, \delta)$ of class C^∞ such that $g(\partial A) = \{x + f(x)\nu_A(x) \ : \ x \in \partial A\}$. Indeed, since ∂A is compact, the subset of $\mathcal{G}$ such that the differential of $\pi_A \circ g$ has rank $n-1$ for every point of ∂A is open in $\mathcal{G}$ and contains i_A. This defines the set $\mathcal{U}$. It follows that $\pi_A \circ g$, for $g \in \mathcal{U}$, is a topological covering map. Moreover, it has degree one: indeed, the signed distance labels the branches, and the monodromy is trivial. Since, for $g \in \mathcal{G}$, the composition $\pi_A \circ g$ is globally invertible, the implicit function theorem allows to find the required f.

Now, for an open neighbourhood $\mathcal{W} \subset \mathcal{U}$ of i_A, we consider the subset $\widetilde{\mathcal{W}}$ of the elements g in $\mathcal{W}$ such that the image of the corresponding f is in $(0, \delta)$. Let us notice that for such f we can define

$$A_f = A \cup \{x + t\nu_A(x) \ : \ x \in \partial A, \ t \in [0, f(x))\},$$

and clearly $A \subset A_f$.

The set $\widetilde{\mathcal{W}}$ is an open subset of $\mathcal{G}$, then [31, Theorem 2.1, p. 74] ensures the existence of a $\tilde{g} \in \widetilde{\mathcal{W}}$ such that $\tilde{g}(\partial A)$ is a submanifold transverse with respect to $\partial\Omega$; we denote by $\tilde{f}$ the corresponding function.

Fixed $\eta > 0$, the neighbourhood $\mathcal{W}$ can be chosen such that, for every g in $\mathcal{W}$ (then in $\widetilde{\mathcal{W}}$), the corresponding f satisfies $\|f\|_{C^1(\partial A)} \le \eta$. Then, the set $A_{\tilde{f}}$, which is by construction transverse with respect to $\partial\Omega$, satisfies the assertions 2, 3, 4. ∎

4.4.3 Preliminar construction

Now, it is sufficient to prove (4.4.1) for $(u, v) \in \mathcal{T}$: let A be an open subset of $\mathbb{R}^n$ with C^∞ boundary transverse with respect to $\partial\Omega$ and such that $u = v = \chi_E$, with $E = A \cap \Omega$.

Let $\varrho > 0$ and $\delta > 0$ be such that the projections

$$\pi \colon (\partial\Omega)_\varrho \to \partial\Omega, \quad \pi_A \colon (\partial A)_\delta \to \partial A$$

are well-defined and smooth.

Let $\eta > 0$ be fixed, $R_\eta > 0$ as in (4.4.5); let T_η be a positive value which will be chosen in §4.4.5: εT_η will be the width of the transition layer between the two phases.

Then, for $\varepsilon \le \min\{\frac{\varrho}{R_\eta}, \frac{\delta}{T_\eta}\}$, we set:

$$\begin{aligned} I_\varepsilon &= (\partial A)_{\varepsilon T_\eta} \cap \Omega = \{x \in \Omega \ : \ d(x, \partial E) < \varepsilon T_\eta\}, \\ S_\varepsilon &= (\partial\Omega)_{\varepsilon R_\eta} \cap \Omega = \{x \in \Omega \ : \ d(x, \partial\Omega) < \varepsilon R_\eta\} \end{aligned}$$

(see Figure 4.2).

Moreover, we define the set:

$$Q_\varepsilon = \{\pi(x) + t\nu(\pi(x)) \ : \ t \in (0, \varepsilon R_\eta), \ x \in I_\varepsilon \cap S_\varepsilon\}$$

(see Figure 4.3), where we recall that ν and π are respectively the inner normal field to $\partial\Omega$

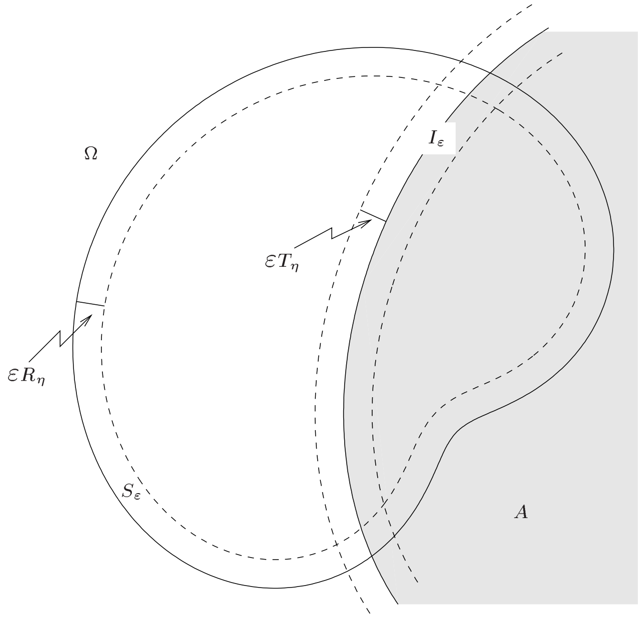

Figure 4.2.

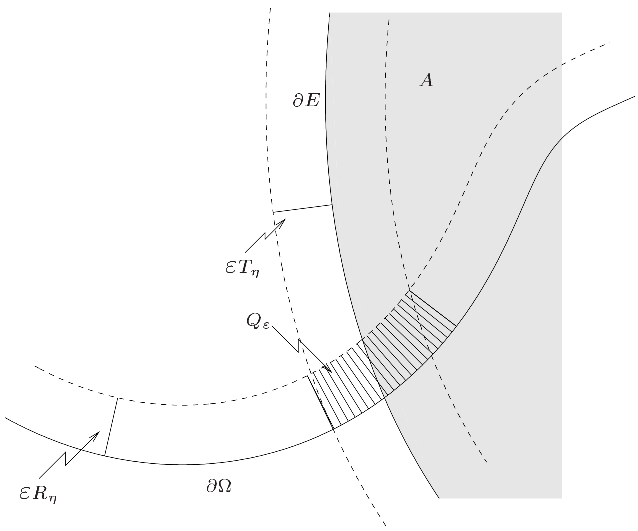

Figure 4.3.

and the projection on $\partial\Omega$ (see §4.4.2). Obviously, Q_ε includes $I_\varepsilon \cap S_\varepsilon$.

Now we give the definition of the Minkowsky content of a closed subset of $\mathbb{R}^n$ (see, e.g., [8, Paragraph 2.13, p. 108-112]). The upper and lower k dimensional Minkowsky content of S are respectively:

$$\mathcal{M}^{*k}(S) = \limsup_{\sigma\to 0} \frac{|S_\sigma|}{\omega_{n-k}\sigma^{n-k}} \qquad \text{and} \qquad \mathcal{M}^k_*(S) = \liminf_{\sigma\to 0} \frac{|S_\sigma|}{\omega_{n-k}\sigma^{n-k}},$$

where ω_{n-k} is the Lebesgue measure of the $n-k$ unit ball.

If the two values coincide, the common value is denoted by $\mathcal{M}^k(S)$. In particular, if S is a compact $n-2$ submanifold without boundary and of class C^2, from [8, Theorem 2.104, p.110] we deduce that:

$$\mathcal{M}^{n-2}(S) = \mathcal{H}^{n-2}(S).$$

This implies

$$|S_\sigma| = O(\sigma^2)_{\sigma\to 0}.$$

The transversality and the assumption Ω of class C^2 allow to deduce that there exists a constant $\gamma > 0$ such that, for ε sufficiently small,

$$Q_\varepsilon \subset (\partial A \cap \partial\Omega)_{\gamma\varepsilon} \cap \Omega = \{x \in \Omega \; : \; d(x, \partial A \cap \partial\Omega) < \gamma\varepsilon\}.$$

To prove this, let $S = \partial A \cap \partial\Omega$, and $\tilde{Q}_\varepsilon = I_\varepsilon \cap S_\varepsilon$; we first prove that there exist $\varepsilon_0 > 0$ and $\gamma > 0$ such that

$$\tilde{Q}_\varepsilon \subseteq S_{\gamma\varepsilon} \tag{4.4.6}$$

for every $0 < \varepsilon < \varepsilon_0$.

Let us argue by contradiction: we can find an infinitesimal sequence $\{\varepsilon_k\}$ of positive numbers and a sequence $\{z_k\}$, with $z_k \in \tilde{Q}_{\varepsilon_k}$ for every $k \in \mathbb{N}$, such that

$$\varepsilon_k = o(\sigma_k)$$

where $\sigma_k = d(z_k, S) > 0$. By compactness we can assume that $\{z_k\}$ converges; clearly the limit z is in S.

Let ν and ν_A be C^1 unit normal fields to $\partial\Omega$ and ∂A, respectively. Each point z_k can be written in both of the following forms:

$$z_k = x_k + t_k\nu(x_k), \qquad z_k = y_k + s_k\nu_A(y_k), \tag{4.4.7}$$

where x_k and y_k are on $\partial\Omega$ and ∂A, respectively, and

$$|t_k| \le \varepsilon_k R_\eta, \qquad |s_k| \le \varepsilon_k T_\eta.$$

Let $w_k \in S$ be such that

$$|z_k - w_k| = \sigma_k.$$

The triangular inequality yields

$$\lim_{k\to+\infty} \frac{\sigma_k}{|x_k - w_k|} = 1 = \lim_{k\to+\infty} \frac{\sigma_k}{|y_k - w_k|}. \tag{4.4.8}$$

Clearly, we can assume that

$$\lim_{k\to+\infty}\frac{x_k-w_k}{|x_k-w_k|}=\xi,\qquad \lim_{k\to+\infty}\frac{y_k-w_k}{|y_k-w_k|}=\zeta \tag{4.4.9}$$

for suitable $\xi,\zeta\in S^{n-1}$. It is easy to see (e.g. in local coordinates) that

$$\xi\in T_z\partial\Omega,\qquad \zeta\in T_z\partial A.$$

From (4.4.7), (4.4.8) and (4.4.9) it follows that

$$\lim_{k\to+\infty}\frac{z_k-w_k}{|z_k-w_k|}=\xi=\zeta.$$

Thus, the tranversality of ∂A and $\partial\Omega$ imply

$$\xi,\zeta\in T_zS. \tag{4.4.10}$$

On the other hand, the minimality condition satified by w_k requires that $(z_k-w_k)/|z_k-w_k|$ is normal to S in w_k; hence also its limit is normal to S in z: this contradicts (4.4.10).

From (4.4.6) we conclude the proof noticing that each point of Q_ε has distance at most εR_η from $\tilde{Q}_\varepsilon$.

Remark 4.4.8. The property does not hold without the transversality hypothesis on A. As an example, if $\Omega=\{(x,y)\in\mathbb{R}^2:\ y>0\}$ and E defined by $E=\{(x,y)\in\mathbb{R}^2: y>x^2\}$, it is clear that $d\big((\sqrt{R_\eta\varepsilon},R_\eta\varepsilon),(0,0)\big)>\sqrt{R_\eta\varepsilon}$. Then, for every $\gamma>0$ there exists ε sufficiently small that $\sqrt{R_\eta\varepsilon}>\gamma\varepsilon$, and $Q_\varepsilon\not\subset M_{\gamma\varepsilon}(\partial E\cap\partial\Omega)$.

We denote $(\partial E\cap\partial\Omega)_{\gamma\varepsilon}\cap\Omega$ by M_ε. Since $\partial E\cap\partial\Omega$ is a $n-2$ submanifold without boundary and of class C^2, it is clear that:

$$|M_\varepsilon|=O(\varepsilon^2)_{\varepsilon\to0}. \tag{4.4.11}$$

4.4.4 Connection with the boundary data

From now on, to shorten the notation, we set:

$$\overline{h}_\varepsilon(x)=h_\varepsilon(\pi(x))\quad\text{and}\quad d_\varepsilon(x)=-\frac{d(x,\partial\Omega)}{\varepsilon}.$$

Moreover, if:

$$S^1_\varepsilon=(S_\varepsilon\setminus Q_\varepsilon)\cap E\quad\text{and}\quad S^0_\varepsilon=(S_\varepsilon\setminus Q_\varepsilon)\setminus E,$$

we define, for $x\in S_\varepsilon\setminus Q_\varepsilon$:

$$\hat{u}_\varepsilon(x)=\begin{cases}\varphi_1\big(\overline{h}_\varepsilon(x),d_\varepsilon(x)\big) & \text{if } x\in S^1_\varepsilon\\[2mm] \varphi_0\big(\overline{h}_\varepsilon(x),d_\varepsilon(x)\big) & \text{if } x\in S^0_\varepsilon\end{cases} \tag{4.4.12}$$

$$\hat{v}_\varepsilon(x)=\begin{cases}\psi_1\big(\overline{h}_\varepsilon(x),d_\varepsilon(x)\big) & \text{if } x\in S^1_\varepsilon\\[2mm] \psi_0\big(\overline{h}_\varepsilon(x),d_\varepsilon(x)\big) & \text{if } x\in S^0_\varepsilon\end{cases} \tag{4.4.13}$$

where φ_i and ψ_i $(i = 0, 1)$ are given in §4.4.1. Since π is Lipschitz and $|\nabla d(x, \partial\Omega)| = 1$ a.e., recalling Proposition 4.4.4 and the hypothesis $(h4)$ on h_ε we easily obtain:

$$|D\hat{v}_\varepsilon(x)|^2 \leq \begin{cases} \dfrac{|\partial_2\psi_1(\overline{h}_\varepsilon(x), d_\varepsilon(x))|^2}{\varepsilon^2} + \dfrac{c}{\varepsilon^{2r}} & \text{if } x \in S^1_\varepsilon, \\[2ex] \dfrac{|\partial_2\psi_0(\overline{h}_\varepsilon(x), d_\varepsilon(x))|^2}{\varepsilon^2} + \dfrac{c}{\varepsilon^{2r}} & \text{if } x \in S^0_\varepsilon, \end{cases} \tag{4.4.14}$$

where $\partial_2\psi_i$ stands for the derivative of ψ_i with respect to the second variable, and c denotes a posive constant which does not depend on ε[1].

Since $\overline{S^1_\varepsilon} \cap \overline{S^0_\varepsilon} = \emptyset$, and $|\partial_2\psi_i|$ is bounded by Proposition 4.4.4, it follows that $D\hat{v}_\varepsilon \in L^\infty(S^1_\varepsilon \cup S^0_\varepsilon)$, and

$$|D\hat{v}_\varepsilon(x)| \leq \frac{c}{\varepsilon} \qquad \text{a.a. } x \in S^1_\varepsilon \cup S^0_\varepsilon.$$

Moreover, we observe that the sequence $\hat{v}_\varepsilon$ is uniformly bounded in L^∞.

Recalling the co-area formula, and the fact that $|\nabla d(x, \partial\Omega)| = 1$ a.e., we obtain:

$$\begin{aligned} &\frac{1}{\varepsilon}\int_{S^1_\varepsilon} W(\hat{u}_\varepsilon)dx + \frac{1}{\varepsilon}\int_{S^1_\varepsilon}(\hat{u}_\varepsilon - \hat{v}_\varepsilon)^2 dx + \varepsilon\int_{S^1_\varepsilon}|D\hat{v}_\varepsilon|^2 dx \leq \\ &\qquad \int_{-R_\eta}^0 \int_{\{d=s\varepsilon\}\cap S^1_\varepsilon} W\big(\varphi_1(\overline{h}_\varepsilon(x), s)\big) d\mathcal{H}^{n-1}(x)ds + \\ &\qquad \int_{-R_\eta}^0 \int_{\{d=s\varepsilon\}\cap S^1_\varepsilon} \big(\varphi_1(\overline{h}_\varepsilon(x), s) - \psi_1(\overline{h}_\varepsilon(x), s)\big)^2 d\mathcal{H}^{n-1}(x)ds + \\ &\qquad \int_{-R_\eta}^0 \int_{\{d=s\varepsilon\}\cap S^1_\varepsilon} |\partial_2\psi_1(\overline{h}_\varepsilon(x), s)|^2 d\mathcal{H}^{n-1}(x)ds + c\varepsilon^{1-r}, \end{aligned}$$

where d stands for the distance $d(x, \partial\Omega)$.

An application of the change of variables formula gives:

$$\begin{aligned} &\frac{1}{\varepsilon}\int_{S^1_\varepsilon} W(\hat{u}_\varepsilon)dx + \frac{1}{\varepsilon}\int_{S^1_\varepsilon}(\hat{u}_\varepsilon - \hat{v}_\varepsilon)^2 dx + \varepsilon\int_{S^1_\varepsilon}|D\hat{v}_\varepsilon|^2 dx \leq \\ &\qquad \int_{-R_\eta}^0 \int_{\{d=0\}\cap S^1_\varepsilon} W\big(\varphi_1(\overline{h}_\varepsilon(x), s)\big) d\mathcal{H}^{n-1}(x)ds + \\ &\qquad \int_{-R_\eta}^0 \int_{\{d=0\}\cap S^1_\varepsilon} \big(\varphi_1(\overline{h}_\varepsilon(x), s) - \psi_1(\overline{h}_\varepsilon(x), s)\big)^2 d\mathcal{H}^{n-1}(x)ds + \\ &\qquad \int_{-R_\eta}^0 \int_{\{d=0\}\cap S^1_\varepsilon} |\partial_2\psi_1(\overline{h}_\varepsilon(x), s)|^2 d\mathcal{H}^{n-1}(x)ds + o(1)_{\varepsilon\to 0} = \\ &\qquad \int_{\partial\Omega\cap\partial S^1_\varepsilon} \mathcal{F}^\circ(\varphi_1(h_\varepsilon(x), \cdot), \psi_1(h_\varepsilon(x), \cdot)) d\mathcal{H}^{n-1}(x) + o(1)_{\varepsilon\to 0}. \end{aligned}$$

[1] In §4.4.4 through §4.4.7 we shall denote by $c, c_1, \ldots$ positive constants which do not depend on ε and η.

By (4.4.3), we obtain:

$$F_\varepsilon(\hat{u}_\varepsilon, \hat{v}_\varepsilon; S^1_\varepsilon) \leq \int_{\partial\Omega\cap\partial S^1_\varepsilon} \Phi(1, h_\varepsilon(x))d\mathcal{H}^{n-1}(x) + c\eta + o(1)_{\varepsilon\to 0}.$$

The same arguments give:

$$F_\varepsilon(\hat{u}_\varepsilon, \hat{v}_\varepsilon; S^0_\varepsilon) \leq \int_{\partial\Omega\cap\partial S^0_\varepsilon} \Phi(0, h_\varepsilon(x))d\mathcal{H}^{n-1}(x) + c\eta + o(1)_{\varepsilon\to 0}.$$

Since $\Phi(i,\cdot)$ is continuous, and $h_\varepsilon(x)$ converges to $h(x)$ $\mathcal{H}^{n-1}-$ a.e. in $\partial\Omega$, the Lebesgue Theorem gives:

$$\limsup_{\varepsilon\to 0} F_\varepsilon(\hat{u}_\varepsilon, \hat{v}_\varepsilon; S_\varepsilon \setminus Q_\varepsilon) \leq \int_{\partial\Omega} \Phi(\tilde{u}(x), h(x))d\mathcal{H}^{n-1}(x) + c\eta. \tag{4.4.15}$$

4.4.5 Connection between the two phases

Let us go back to Proposition 2.3.3; the proof yielded, for every $\eta > 0$, a family $\{(u_\varepsilon, v_\varepsilon)\}$ which we rename as $\{(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon)\}$, satisfying the following properties:

$\tilde{u}_\varepsilon, \tilde{v}_\varepsilon \to u$ in $L^1(\Omega)$;

$0 \leq \tilde{u}_\varepsilon, \tilde{v}_\varepsilon \leq 1$, $D\tilde{v}_\varepsilon \in L^\infty(\Omega)$, and $|D\tilde{v}_\varepsilon| \leq c/\varepsilon$ a.e. in Ω;

$\tilde{u}_\varepsilon = \tilde{v}_\varepsilon = 1$ in $\{x \in E : d(x, \partial A) > \varepsilon T_\eta\}$,
$\tilde{u}_\varepsilon = \tilde{v}_\varepsilon = 0$ in $\{x \in \Omega \setminus E : d(x, \partial A) > \varepsilon T_\eta\}$,

and

$$\limsup_{\varepsilon\to 0} F_\varepsilon(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon; \Omega_\varepsilon) \leq c_W \mathcal{H}^{n-1}(S(u) \cap \Omega) + \eta \tag{4.4.16}$$

(the last inequality can be obtained immediately from the proof of Proposition 2.3.3 taken the arbitrariness of Ω' and η into account). Notice that, choosing T_η suitably we can assume $\tilde{u}_\varepsilon$ and $\tilde{v}_\varepsilon$ taking the values 0 and 1 also in a neighbourhood of $\partial I_\varepsilon \cap \Omega$.

4.4.6 Construction of the recovery sequence

To connect the previous constructions, we define:

$$\overline{u}_\varepsilon(x) = \begin{cases} \tilde{u}_\varepsilon(x) & \text{if } x \in \Omega_\varepsilon \\ \hat{u}_\varepsilon(x) & \text{if } x \in S_\varepsilon \setminus Q_\varepsilon \end{cases}$$

$$\overline{v}_\varepsilon(x) = \begin{cases} \tilde{v}_\varepsilon(x) & \text{if } x \in \Omega_\varepsilon \\ \hat{v}_\varepsilon(x) & \text{if } x \in S_\varepsilon \setminus Q_\varepsilon \end{cases}$$

It is easy to see that the sequence $\overline{v}_\varepsilon$ is uniformly bounded in L^∞; moreover, we recall that $|D\hat{v}_\varepsilon| \leq c/\varepsilon$ a.e. in $S^1_\varepsilon \cup S^0_\varepsilon$, and $|D\tilde{v}_\varepsilon| \leq c/\varepsilon$ a.e. in Ω_ε. Since, by construction, $\overline{v}_\varepsilon = 1$ in a neighbourhood of $\{x \in S^1_\varepsilon : d(x, \partial\Omega) = \varepsilon R_\eta\}$ and $\overline{v}_\varepsilon = 0$ in a neighbourhood of $\{x \in S^0_\varepsilon : d(x, \partial\Omega) = \varepsilon R_\eta\}$ we can deduce that $D\overline{v}_\varepsilon \in L^\infty(\Omega \setminus Q_\varepsilon)$, and moreover :

$$|D\overline{v}_\varepsilon(x)| \leq \frac{c}{\varepsilon} \qquad \text{a.a. } x \in \Omega \setminus Q_\varepsilon. \tag{4.4.17}$$

Now, in order to extend the definition of the functions to the whole Ω, let us consider, for every ε, the following problem:

$$\begin{cases} \Delta H_\varepsilon = 0 & \text{in } \ \Omega \\ \text{tr } H_\varepsilon = h_\varepsilon & \text{in } \ \partial\Omega, \end{cases} \tag{4.4.18}$$

which has, thanks to the hypothesis $(h1)$ on h_ε, a unique solution $H_\varepsilon \in C^0(\overline{\Omega}) \cap H^1(\Omega)$. The maximum principle and the hypothesis $(h2)$ give:

$$\|H_\varepsilon\|_{L^\infty} \le \|h_\varepsilon\|_{L^\infty} \le c \tag{4.4.19}$$

for some constant c not depending on ε. The solution of (4.4.18) minimizes the functional $\int_\Omega |DH|^2$ in the class of the functions in $H^1(\Omega)$ with fixed trace, i.e.:

$$\int_\Omega |DH_\varepsilon|^2 dx = \min\Big\{ \int_\Omega |DH|^2 dx \ : \ H \in H^1(\Omega), \ \ \text{tr } H = h_\varepsilon \Big\}.$$

Since there exists a continuous operator $\tau \colon H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega)$ such that, for every $g \in H^{\frac{1}{2}}(\partial\Omega)$, tr $\tau(g) = g$, it follows, recalling the uniform boundedness of $\{h_\varepsilon\}$ in $H^{\frac{1}{2}}(\partial\Omega)$, that

$$\int_\Omega |DH_\varepsilon|^2 \quad \text{is uniformly bounded.} \tag{4.4.20}$$

We can finally give the definition of the recovery sequence. We set:

$$u_\varepsilon(x) = \begin{cases} \overline{u}_\varepsilon(x) & \text{if } \ x \in \Omega \setminus M_{2\varepsilon} \\ 0 & \text{otherwise in } \ \Omega, \end{cases}$$

$$v_\varepsilon(x) = \begin{cases} \overline{v}_\varepsilon(x) & \text{if } \ x \in \Omega \setminus M_{2\varepsilon} \\[2ex] \Big(1 - \dfrac{d(x, M_\varepsilon)}{\gamma\varepsilon}\Big) H_\varepsilon(x) + \dfrac{d(x, M_\varepsilon)}{\gamma\varepsilon} \overline{v}_\varepsilon(x) & \text{if } \ x \in M_{2\varepsilon} \setminus M_\varepsilon \\[2ex] H_\varepsilon(x) & \text{if } \ x \in M_\varepsilon \end{cases}$$

Thanks to the estimates (4.4.19) and (4.4.20), we easily obtain:

$$\frac{1}{\varepsilon} \int_{M_\varepsilon} H_\varepsilon^2 dx + \varepsilon \int_{M_\varepsilon} |DH_\varepsilon|^2 dx \le \frac{c|M_\varepsilon|}{\varepsilon} + c\varepsilon.$$

Since $|M_\varepsilon| = O(\varepsilon^2)_{\varepsilon \to 0}$, it follows that:

$$\lim_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon, v_\varepsilon; M_\varepsilon) = 0. \tag{4.4.21}$$

Recalling (4.4.19) and the uniform boundedness in L^∞ of $\overline{v}_\varepsilon$, it is easy to see that:

$$\frac{1}{\varepsilon} \int_{M_{2\varepsilon} \setminus M_\varepsilon} v_\varepsilon^2(x) \le \frac{c|M_{2\varepsilon} \setminus M_\varepsilon|}{\varepsilon}.$$

Moreover, we have:

$$|Dv_\varepsilon| \le |DH_\varepsilon| + \frac{1}{\gamma\varepsilon}|H_\varepsilon||\nabla d(x, M_\varepsilon)| + \frac{1}{\gamma\varepsilon}|\overline{v}_\varepsilon||\nabla d(x, M_\varepsilon)| + |D\overline{v}_\varepsilon|.$$

Since $|\nabla d(x, M_\varepsilon)| = 1$ a.e., we can use again the uniform boundedness in L^∞ of $\hat{v}_\varepsilon$ and H_ε, and the estimate $|D\overline{v}_\varepsilon| \le c/\varepsilon$, to obtain:

$$|Dv_\varepsilon| \le |DH_\varepsilon| + \frac{c}{\varepsilon}.$$

This inequality immediately implies:

$$\varepsilon \int_{M_{2\varepsilon}\setminus M_\varepsilon} |Dv_\varepsilon|^2 \le c\varepsilon \int_{M_{2\varepsilon}\setminus M_\varepsilon} |DH_\varepsilon|^2 + \frac{c|M_{2\varepsilon} \setminus M_\varepsilon|}{\varepsilon}.$$

An application of (4.4.20) allows to deduce that:

$$\varepsilon \int_{M_{2\varepsilon}\setminus M_\varepsilon} |Dv_\varepsilon|^2 \le c\varepsilon + \frac{c|M_{2\varepsilon} \setminus M_\varepsilon|}{\varepsilon},$$

and since $|M_{2\varepsilon} \setminus M_\varepsilon| = O(\varepsilon^2)_{\varepsilon\to 0}$ we conclude that

$$\lim_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, v_\varepsilon; M_{2\varepsilon} \setminus M_\varepsilon) = 0. \tag{4.4.22}$$

4.4.7 Conclusion of the proof

Proposition 4.4.9. *The sequences $\{u_\varepsilon\}, \{v_\varepsilon\}$ satisfy the following properties:*

1. $u_\varepsilon, v_\varepsilon \to u$ *in* $L^1(\Omega)$;
2. $v_\varepsilon \in H^1(\Omega)$;
3. *tr* $v_\varepsilon = h_\varepsilon$.

Proof. Assertion 1 is obvious by construction; clearly, the restrictions of v_ε to $\Omega \setminus M_\varepsilon$, $M_{2\varepsilon} \setminus M_\varepsilon$ and M_ε are of class H^1. Moreover, the traces on the common boundaries coincide. This proves 2. Since tr $H_\varepsilon = h_\varepsilon$ on $\partial\Omega$, and tr $\overline{v}_\varepsilon = h_\varepsilon$ on $\partial\Omega \setminus M_\varepsilon$, property 3 easily follows. ■

Splitting the domain Ω as the previous construction suggests, estimates (4.4.15), (4.4.16), (4.4.21) and (4.4.22) easily give the result:

$$\limsup_{\varepsilon\to 0} F^\circ_\varepsilon(u_\varepsilon, v_\varepsilon) \le F^\circ(u, v) + c\eta,$$

and this concludes the proof of Proposition 4.4.1.

4.5 The boundary contact energy

In the paper [36], a sequence of functionals related with phase transitions is considered, taking into account, instead of the boundary Dirichlet conditions, a term which represents the contact energy between the fluid and the container walls.

The functionals considered in [36] are

$$G_\varepsilon(u) = \begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_\Omega W(u) + \dfrac{1}{\varepsilon}\int_\Omega |Du|^2 + \int_{\partial\Omega} \sigma(\tilde{u})d\mathcal{H}^{n-1} & \text{if } u \in H^1(\Omega) \text{ and } u \geq 0 \\ +\infty & \text{otherwise,} \end{cases}$$

where $\tilde{u}$ stands for the trace of u, and

1. W is a double–well potential, with the minimum points in a and b, $0 < a < b$;
2. σ continuous and positive.

These assumptions are coherent with the fact that the function u stands for a density distribution, hence is naturally positive, and with the equilibrium configurations of the Gibbs free-energy per unit volume W.

The result proven by L. Modica in [36, Theorem 2.1] can be read as a Γ-convergence theorem for the sequence of functionals $\{G_\varepsilon + I_m\}$, where $I_m(u) = 0$ if $\int_\Omega u = m$, and $I_m(u) = +\infty$ otherwise in $L^1(\Omega)$. Precisely:

Theorem 4.5.1. **([36])** *Fixed $m \in (a|\Omega|, b|\Omega|)$, the sequence $\{G_\varepsilon + I_m\}$ Γ-converges in $L^1(\Omega)$ to the functional $G + I_m$, where G is defined by*

$$G(u) = \begin{cases} C_W(a,b)\mathcal{H}^{n-1}(Su \cap \Omega) + \displaystyle\int_{\partial\Omega} \hat{\sigma}(\tilde{u})d\mathcal{H}^{n-1} & \text{if } u \in BV(\Omega; \{a,b\}) \\ +\infty & \text{otherwise,} \end{cases}$$

and

$$C_W(a,b) = 2\int_a^b \sqrt{W(s)}ds,$$
$$\hat{\sigma}(t) = \inf\Big\{\sigma(s) + 2\Big|\int_s^t \sqrt{W(\tau)}d\tau\Big| \ : \ s \geq 0\Big\}.$$

The aim of this Section is to study the asymptotic behaviour of a sequence of two-variables functionals with a boundary contact energy term, i.e.:

$$\widetilde{F}^+_\varepsilon(u,v) = \begin{cases} F_\varepsilon(u,v) + \displaystyle\int_{\partial\Omega} \sigma(\tilde{v})d\mathcal{H}^{n-1} & \text{if } v \in H^1(\Omega) \text{ and } u,v \geq 0 \\ +\infty & \text{otherwise,} \end{cases} \tag{4.5.1}$$

where we recall that $F_\varepsilon(u,v) = \dfrac{1}{\varepsilon}\displaystyle\int_\Omega W(u) + \dfrac{1}{\varepsilon}\int_\Omega (u-v)^2 + \varepsilon\int_\Omega |Dv|^2$ if $v \in H^1(\Omega)$, and $\tilde{v}$ stands for the trace of v.

The rôle of modified contact energy played in [36] by the function $\hat{\sigma}$ is assumed, in our case, by:

$$\overline{\sigma}(i) = \inf\{\sigma(h) + \Phi(i,h) \ : \ h \geq 0\}, \tag{4.5.2}$$

for $i = a$ and $i = b$; for the definition (and the properties) of $\Phi(i,h)$, see (3.4.1) and the Section 3.4, clearly replacing 0 by a and 1 by b. It is easy to check that the minimum in (4.5.2) is attained.

According with the other models considered here, we prove a first result concerning a sequence of functionals without the constraint $u, v \geq 0$, i.e.

$$\widetilde{F}_\varepsilon(u,v) = \begin{cases} F_\varepsilon(u,v) + \displaystyle\int_{\partial\Omega} \sigma(\tilde{v}) d\mathcal{H}^{n-1} & \text{if } \ v \in H^1(\Omega) \\ +\infty & \text{otherwise} \end{cases} \tag{4.5.3}$$

Thanks to the relations between these functionals and the sequence $\{F^\circ_\varepsilon\}$, we can apply the results of the previous sections to prove the following Γ-convergence result, where for the sake of simplicity we assume $a = 0$ and $b = 1$:

Theorem 4.5.2. *The sequence of functionals $\widetilde{F}_\varepsilon$ Γ-converges in $[L^1(\Omega)]^2$ to the functional $\widetilde{F}\colon [L^1(\Omega)]^2 \to [0, +\infty]$ defined by:*

$$\widetilde{F}(u,v) = \begin{cases} c_W \mathcal{H}^{n-1}(S(u) \cap \Omega) + \displaystyle\int_{\partial\Omega} \tilde{\sigma}(\tilde{u}) d\mathcal{H}^{n-1} & \text{if } \ u, v \in BV(\Omega) \\ & \text{and } \ u = v \in \{0,1\} \ \text{ a.e.} \\ +\infty & \text{otherwise,} \end{cases}$$

where, for $i \in \{0,1\}$:

$$\tilde{\sigma}(i) = \inf\{\sigma(h) + \Phi(i,h) \ : \ h \in \mathbb{R}\}.$$

Let us note that, if $\widetilde{F}$ is finite, we can write, for $u = v = \chi_E$:

$$\widetilde{F}(u,v) = \widetilde{F}^*(E) = c_W P_\Omega(E) + \tilde{\sigma}(1)\mathcal{H}^{n-1}(\partial^* E \cap \partial\Omega) + \tilde{\sigma}(0)\mathcal{H}^{n-1}(\partial\Omega \setminus \partial^* E),$$

where $P_\Omega(E)$ and $\partial^* E$ stand respectively for the perimeter of E in Ω and the reduced boundary of E. Thus, even in our case, as in [36], the asymptotic behaviour of the functionals is connected with the minimization of a geometric area-like quantity of the form:

$$P_\Omega(E) + \gamma \mathcal{H}^{n-1}(\partial^* E \cap \partial\Omega), \tag{4.5.4}$$

where $|\gamma| \leq 1$, in correspondence with the geometrical meaning of γ, which should represent the cosine of the contact angle between one phase and the container walls (for this interpretation of the meaning of the parameter γ, we refer to [28] and [36]). We shall consider again this problem at the end of the section.

Outline of the Proof of Theorem 4.5.2. **One-dimensional result.**
Let $\Omega = (-1, 1)$. Note that, as usual, this case is sufficient to conclude for an arbitrary finite union of bounded intervals with disjoint closure.

1. Γ-lim inf inequality.
   Since $\widetilde{F}_\varepsilon \geq F_\varepsilon$, it is not restrictive to assume $u = v \in BV((-1,1);\{0,1\})$.

   Let $\{u_\varepsilon\} \subset L^2(-1,1)$, $\{v_\varepsilon\} \subset H^1(-1,1)$, both converging in $L^1(-1,1)$ to u. Since $u \in BV(-1,1)$, we can suppose that $a > 0$ is such that $u = \tilde{u}(1)$ and $u = \tilde{u}(-1)$ in $(a,1)$ and $(-1,-a)$ respectively. Following exactly the steps of the proof of (4.2.3), and recalling the definition of $\tilde{\sigma}$, we immediately obtain:

$$\begin{aligned} \widetilde{F}_\varepsilon(u_\varepsilon, v_\varepsilon) &\geq F_\varepsilon(u_\varepsilon, v_\varepsilon; (-a,a)) + \Phi(\tilde{u}(-1), v_\varepsilon(-1)) \\ &\quad + \sigma(v_\varepsilon(-1)) + \Phi(\tilde{u}(1), v_\varepsilon(1)) + \sigma(v_\varepsilon(1)) \\ &\geq F_\varepsilon(u_\varepsilon, v_\varepsilon; (-a,a)) + \tilde{\sigma}(\tilde{u}(-1)) + \tilde{\sigma}(\tilde{u}(1)). \end{aligned}$$

   Since $\liminf F_\varepsilon(u_\varepsilon, v_\varepsilon; (-a,a)) \geq c_W \# S(u)$, the lower inequality for the Γ-lim inf follows.

2. Γ-lim sup inequality.
   Clearly, we can assume $u = v \in BV((-1,1);\{0,1\})$. Setting $h(\pm 1) =$arg $\min\{\sigma(s) + \Phi(u(\pm 1), s) \ : \ s \in \mathbb{R}\}$, we choose $\{(u_\varepsilon, v_\varepsilon)\}$ as the recovery sequence for the sequence F_ε° with $h_\varepsilon(\pm 1) = h(\pm 1)$.

   From (4.2.4), it follows that

$$\begin{aligned} \limsup_{\varepsilon\to 0} \widetilde{F}_\varepsilon(u_\varepsilon, v_\varepsilon) &= \limsup_{\varepsilon\to 0} F_\varepsilon^\circ(u_\varepsilon, v_\varepsilon) + \sigma(h(1)) + \sigma(h(-1)) + \eta \\ &\leq c_W \# S(u) + \Phi(\tilde{u}(1), h(1)) + \sigma(h(1)) \\ &\quad + \Phi(\tilde{u}(-1)) + \sigma(h(-1)) + \eta \\ &\leq \widetilde{F}(u,v) + \eta. \end{aligned}$$

The lower inequality.
Note that, if the sequence $\{(u_\varepsilon, v_\varepsilon)\}$ converging to (u,v) satisfies the further property that the traces of v_ε converge pointwise $\mathcal{H}^{n-1}$-a.e. to a bounded function h, the lower inequality

$$\liminf \widetilde{F}_\varepsilon(u_\varepsilon, v_\varepsilon) \geq \widetilde{F}(u,v)$$

follows from the corresponding inequality for the sequence F_ε° (with the choice $h_\varepsilon =$tr v_ε) and from the continuity of σ.

The proof of the general case can be obtained again applying the slicing method. We omit the details, since it is closely similar to the proof of the Proposition 4.3.1.

The upper inequality.
As in the proof the Proposition 4.4.1, it is sufficient to prove the inequality

$$\widetilde{F}''(u,v) \leq \widetilde{F}(u,v)$$

for couples (u,v) such that $u = v = \chi_E$, $E \in \mathcal{S}$, recalling that

$$\mathcal{S} = \{E = \Omega \cap A \ : \ A \subset \mathbb{R}^n \ \text{ with boundary of class } \ C^\infty \ \text{ transverse w.r.t. } \ \partial\Omega\}.$$

We set $h_0 = \mathrm{argmin}\{\sigma(s) + \Phi(0,s) \ : \ s \in \mathbb{R}\}$, and $h_1 = \mathrm{argmin}\{\sigma(s) + \Phi(1,s) \ : \ s \in \mathbb{R}\}$, and define:

$$h(x) = \begin{cases} h_0 & \text{if} \quad \tilde{u}(x) = 0 \\ h_1 & \text{if} \quad \tilde{u}(x) = 1 \end{cases}$$

Fixed $\eta > 0$, there exist:

a neighbourhood I_η of $\partial A \cap \partial\Omega$ in $\partial\Omega$, with $\mathcal{H}^{n-1}(I_\eta) \leq \eta$;

$h_\eta \colon \partial\Omega \to \mathbb{R}$, of class $C^1(\partial\Omega)$, such that $\min\{h_0, h_1\} \leq h_\eta \leq \max\{h_0, h_1\}$, and

$$h_\eta = h \quad \text{in} \quad \partial\Omega \setminus I_\eta.$$

Thus, for each pair (u, v), let $\{(u_\varepsilon, v_\varepsilon)\}$ be the recovery sequence converging to (u, v) for the sequence $\{F_\varepsilon^\circ\}$, with the Dirichlet data $h_\varepsilon = h_\eta$ (see Proposition 4.4.1). It follows that

$$\begin{aligned} \limsup\nolimits_{\varepsilon\to 0} \widetilde{F}_\varepsilon(u_\varepsilon, v_\varepsilon) \ &\leq c_W \mathcal{H}^{n-1}(S(u) \cap \Omega) \\ &\quad + \int_{\partial\Omega} \Phi(\tilde{u}(x), h_\eta(x)) + \sigma(h_\eta(x)) d\mathcal{H}^{n-1}(x) + \eta. \end{aligned}$$

The choice of h_η and the continuity of Φ and σ imply that:

$$\int_{I_\eta} \Phi(\tilde{u}(x), h_\eta(x)) d\mathcal{H}^{n-1}(x) + \int_{I_\eta} \sigma(h_\eta(x)) d\mathcal{H}^{n-1}(x) \leq c\eta,$$

where c is a positive constant not depending on η. Therefore,

$$\begin{aligned} &\int_{\partial\Omega} \Phi(\tilde{u}(x), h_\eta(x)) d\mathcal{H}^{n-1}(x) + \int_{\partial\Omega} \sigma(h_\eta(x)) d\mathcal{H}^{n-1}(x) \\ &\leq \int_{\partial\Omega} \Phi(\tilde{u}(x), h(x)) d\mathcal{H}^{n-1}(x) + \int_{\partial\Omega} \sigma(h(x)) d\mathcal{H}^{n-1}(x) + c\eta, \end{aligned}$$

and the definition of h allows to conclude:

$$\limsup_{\varepsilon\to 0} \widetilde{F}_\varepsilon(u_\varepsilon, v_\varepsilon) \leq c_W \mathcal{H}^{n-1}(S(u) \cap \Omega) + \int_{\partial\Omega} \tilde{\sigma}(\tilde{u}(x)) d\mathcal{H}^{n-1}(x) + (c+1)\eta.$$

■

As in §2.3.1, we can refine the construction of the recovery sequence. For $m \in [0, |\Omega|]$, let $\{(u_\varepsilon, v_\varepsilon)\}$ as in the proof of the upper inequality in Theorem 4.5.2. It is easy to check, as in §2.3.1, that

$$\lambda_\varepsilon = m - \int_\Omega v_\varepsilon = O(\varepsilon)_{\varepsilon\to 0}. \tag{4.5.5}$$

Then, setting

$$\tilde{v}_\varepsilon = v_\varepsilon + \frac{\lambda_\varepsilon}{|\Omega|}, \qquad \tilde{u}_\varepsilon = u_\varepsilon,$$

clearly $\tilde{u}_\varepsilon, \tilde{v}_\varepsilon \to u$ in $L^1(\Omega)$, and $\int_\Omega \tilde{v}_\varepsilon = m$ for every ε. Moreover, (4.5.5) implies the convergences $\frac{1}{\varepsilon}\int_\Omega(\tilde{v}_\varepsilon - v_\varepsilon)^2 \to 0$ and $\frac{2|\lambda_\varepsilon|}{\varepsilon|\Omega|}\int_\Omega |u_\varepsilon - v_\varepsilon| \to 0$, as $\varepsilon \to 0$, recalling that $u_\varepsilon, v_\varepsilon \to u$ in $L^1(\Omega)$. Then

$$\frac{1}{\varepsilon}\int_\Omega (\tilde{u}_\varepsilon - \tilde{v}_\varepsilon)^2 = \frac{1}{\varepsilon}\int_\Omega (u_\varepsilon - v_\varepsilon)^2 + o(1)_{\varepsilon\to 0}.$$

Thus, the continuity of σ gives

$$\limsup_{\varepsilon\to 0} \widetilde{F}_\varepsilon(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon) = \limsup_{\varepsilon\to 0} \widetilde{F}_\varepsilon(u_\varepsilon, v_\varepsilon).$$

Setting

$$I_m(u,v) = \begin{cases} 0 & \text{if } \int_\Omega v = m \\ +\infty & \text{otherwise,} \end{cases}$$

we have the following

Corollary 4.5.3. *The sequence* $\{\widetilde{F}_\varepsilon + I_m\}$ Γ*-converges in* $[L^1(\Omega)]^2$ *to the functional* $\widetilde{F} + I_m$.

Remark 4.5.4. **(Compactness)** The compactness result for the sequences $\{\widetilde{F}_\varepsilon\}$ and $\{\widetilde{F}_\varepsilon + I_m\}$ is an immediate consequence of the corresponding result for $\{F_\varepsilon\}$ (Proposition 2.1.5).

Now, in order to give a precise comparison with the result of L. Modica (see Theorem 4.5.1), we state a Γ-convergence result for the sequence $\widetilde{F}_\varepsilon^+$. Notice that in this statement we do not reduce to the case $a = 0$, $b = 1$ for the minimum points of the double-well potential W, emphasizing the dependence on a, b. We set

$$c_W(a,b) = \inf\{\mathcal{F}(\varphi,\psi) \ : \ \varphi \in X_{a,b},\ \psi \in X_{a,b} \cap Y\},$$

where

$$X_{a,b} = \{u\colon \mathbb{R} \to [a,b] \ \text{ measurable such that } \lim_{x\to-\infty} u(x) = a,\ \lim_{x\to+\infty} u(x) = b\},$$

and Y as usual.

It is easy to prove the following

Corollary 4.5.5. *The sequence of functionals* $\widetilde{F}_\varepsilon^+$ Γ*-converges, in* $[L^1(\Omega)]^2$*, to* $\widetilde{F}^+\colon \mathcal{U}^+ \to [0,+\infty]$ *defined by:*

$$\widetilde{F}^+(u,v) = \begin{cases} c_W(a,b)\mathcal{H}^{n-1}(S(u)\cap\Omega) + \displaystyle\int_{\partial\Omega} \overline{\sigma}(\tilde{u})\,d\mathcal{H}^{n-1} & \text{if } u,v \in BV(\Omega) \\ & \text{and } \ u = v \in \{a,b\} \ \text{ a.e.} \\ +\infty & \text{otherwise,} \end{cases}$$

where $\overline{\sigma}$ *is the function defined in* (4.5.2).

Proof. To prove this result, we observe that the lower inequality for the Γ-lim inf with u and v positive, and $\overline{\sigma}$ replacing $\tilde{\sigma}$, can be easily obtained as in Theorem 4.5.2, noticing that the traces are positive.

Since $\overline{\sigma} \geq \tilde{\sigma}$, the upper inequality for the Γ-lim sup follows immediately from the corresponding inequality shown in Theorem 4.5.2 if we can choose the recovery sequence $\{u_\varepsilon, v_\varepsilon\}$ such that $u_\varepsilon, v_\varepsilon \geq 0$. This is straightforward: indeed, let $\overline{h}_0$ and $\overline{h}_1$ be minimizers in $[0,+\infty)$ of $\sigma(s) + \Phi(0,s)$ and $\sigma(s) + \Phi(1,s)$ respectively. Then, we define h_η as in the proof of the upper inequality in Theorem 4.5.2, with $\overline{h}_0, \overline{h}_1$ replacing h_0, h_1; our construction gives in this case positive functions u_ε, v_ε, since h_η, a and b are positive. ■

Clearly, also in Corollary 4.5.5 the Γ-convergence result holds with a volume constraint on the second variable, as in Proposition 2.3.4 and in Corollary 4.5.3. Indeed, it is straightforward to prove that, for $m \in [a|\Omega|, b|\Omega|]$, the sequence $\{\widetilde{F}^+_\varepsilon + I_m\}$ Γ-converges to $\widetilde{F}^+ + I_m$.

For $m \in [a|\Omega|, b|\Omega|]$ we set, as in Section 2.1,

$$U_m = \{(u,v) \in L^1(\Omega) \ : \ \textstyle\int_\Omega v = m\},$$
$$U_m^+ = \{(u,v) \in L^1(\Omega) \ : \ u,v \geq 0, \ \textstyle\int_\Omega v = m\};$$
$$U_m^0 = \{(u,v) \in [BV(\Omega)]^2 \ : \ u = v \in \{a,b\} \ \text{ a.e.}, \ \textstyle\int_\Omega u = m\}.$$

The previous results and the stability of minimizing sequences recalled in Proposition 1.0.4, *iii*), allow to the state the following

Corollary 4.5.6. *Let $\{\varepsilon_n\}$ be a positive infinitesimal sequence, and $\{(u_{\varepsilon_n}, u_{\varepsilon_n})\}$ be a minimizing or quasi-minimizing sequence for $\widetilde{F}_{\varepsilon_n}$ (respectively $\widetilde{F}^+_{\varepsilon_n}$) in U_m. Then $\{(u_{\varepsilon_n}, v_{\varepsilon_n})\}$ is relatively compact in $[L^1(\Omega)]^2$, and every cluster point (u,v) minimizes $\widetilde{F}$ (respectively $\widetilde{F}^+$) in U_m^0.*

Then we come back to the problem (4.5.4), setting, for $\gamma \in \mathbb{R}, \tilde{m} \in [0, |\Omega|]$:

$$(P_0) \qquad \min\{P_\Omega(E) + \gamma\, \mathcal{H}^{n-1}(\partial^* E \cap \partial\Omega) \ : \ E \subset \Omega, \ |E| = \tilde{m}\}.$$

This problem is known as the *liquid-drop* problem, and for Ω bounded and $|\gamma| \leq 1$ it always admits at least a solution (see [30], [28], and [36]). Notice that a proof of this existence result can be given via Γ-convergence.

Let us notice that a minimizer (u_0, v_0) of $\widetilde{F}$ (respectively $\widetilde{F}^+$) in U_m^0 determines the set

$$E_0 = \{x \in \Omega \ : \ u_0(x) = b\}$$

which is a solution of the problem (P_0) with

$$\gamma = \frac{\tilde{\sigma}(b) - \tilde{\sigma}(a)}{c_W(a,b)} \quad (\text{resp. } \ \gamma = \frac{\overline{\sigma}(b) - \overline{\sigma}(a)}{c_W(a,b)}), \qquad \tilde{m} = \frac{m - a|\Omega|}{b-a};$$

a minimizer u_0 of G such that $\int_\Omega u_0 = m$ (see Theorem 4.5.1) determines in the same way a solution E_0 of the problem (P_0) with

$$\gamma = \frac{\hat{\sigma}(b) - \hat{\sigma}(a)}{C_W(a,b)}, \qquad \tilde{m} = \frac{m - a|\Omega|}{b-a}.$$

Thus, a sequence $\{(u_\varepsilon, v_\varepsilon)\}$ of minimizers or quasi-minimizers approximates solutions of problem (P_0). As for the problem associated to F° (see Section 4.1), we notice that this kind of approximation leads to a different selection of the configurations u which minimize the energy $\int_\Omega W(u)$, i.e. the equilibrium configurations of the two-phase physical sistem (see the introduction).

For a solution E_0 of the problem (P_0) with $|\gamma| \leq 1$, the parameter γ stands for the cosine of the contact angle ϑ between ∂E_0 and $\partial\Omega$ (see [28]). Notice that also in our case the condition $|\gamma| \leq 1$ is clearly satisfied. If we come back to the problem analyzed in Section 4.1, concerning the Γ-limit of functionals F_ε°, this result gives some evidence the conjecture that

$$\cos\vartheta(x) = \frac{\Phi(a, h(x)) - \Phi(b, h(x))}{c_W(a, b)}$$

(see (4.1.4)), where $\vartheta(x)$ is the contact angle at $x \in \partial\Omega$, and h is the limit of the Dirichlet data.

As to the wetting phenomena considered in [36], referring to the work of J. W. Cahn [18], we observe that also in our case, in appropriate hypotheses on the energies W and σ, the critical point wetting phenomenon appears, i.e. $\cos\vartheta = 1$. This corresponds to a situation where the phase a ("liquid") does not wet the container, and a layer of the phase b, perfectly wetting, appears between the phase a and the boundary. As noticed in [36], the limit contact energy $\overline{\sigma}(a) - \overline{\sigma}(b)$ can be interpreted as coming from an infinitely thin boundary layer of the phase b.

Let us notice that, recalling (4.1.5), in the case of F° the critical phenomenon appears in the points $x \in \partial\Omega$ such that $h(x) \geq b$ (or $h(x) \leq a$ for the opposite case).

Chapter 5

SOME REMARKS ON THE VECTORIAL CASE

As in the Van der Waals and Cahn-Hilliard theory for fluids, variational models of Modica-Mortola type naturally arise also in the study of nonlinear elasticity. Indeed, for elastic solids which undergo phase transition, Ericksen [22] proposed a non-convex double-well density energy function. More precisely, if u denotes the deformation of a body $\Omega \subseteq \mathbb{R}^3$, the stored energy can be written in the form

$$\int_\Omega W(\nabla u)\,dx,$$

where $W \geq 0$ and the set $\{W = 0\}$ consists of two "wells" which, on account of the usual frame-indifference condition, take the form $SO(3)A \cup SO(3)B$; the matrices A and B are required to satisfy the compatibility condition that $A - B$ has rank 1 (this guarantees the existence of classical non-affine solutions). For more details on this kind of energies, and related problems, we refer e.g. to [11] and [38].

The minimization problem for such a model energy naturally gives rise to strong nonuniqueness; to avoid the drawback the problem is singularly perturbed by a higher order term, obtaining the following modified energy functional:

$$G_\varepsilon(u) = \frac{1}{\varepsilon}\int_\Omega W(\nabla u)\,dx + \varepsilon\int_\Omega |\nabla^2 u|^2\,dx,$$

where $\varepsilon > 0$ is a small parameter. Despite the extensive study by several authors, the general problem of the convergence of $\{G_\varepsilon\}$ when $\varepsilon \to 0$ is still open. Recently, Sergio Conti, Irene Fonseca and Giovanni Leoni, in [20], proved a convergence result for a family of these functionals; apart from some technical simmetry assumptions on W, the key reduction is that the wells are here required to consist of exactly two rank one connected matrices A and B (thus removing the frame indifference constraint). Assuming $A - B = a \otimes \nu$ for some $a \in \mathbb{R}^3$ and some unit vector ν, the interfaces of minimizers are planar and orthogonal to ν (see [11], and [20, Theorem 3.3]); even with such a simplification, the constraint for the admissible fields to be gradients makes the construction of the recovery sequence for the Γ-lim sup very delicate (the problem is in general non one-dimensional).

It can be proved ([20, Theorem 1.3 and Theorem 1.4]) that there exists a constant $K^* > 0$ such that, if $u \in W^{1,1}(\Omega;\mathbb{R}^3)$ and $\nabla u \in BV(\Omega;\{A,B\})$, then the Γ-limit at u in the L^1 topology is

$$K^* P_\Omega(E),$$

where E satisfies $\nabla u = (1-\chi_E)A + \chi_E B$ a.e. (and $P_\Omega(E)$ stands for the perimeter of E in Ω).

It is important to notice that the result is obtained also in hypotheses on W which do not force K^* to assume a one-dimensional character (see [20, Theorem 1.4]).

5.1 A two-variable model in nonlinear elasticity

The energy functionals F_ε introduced in Chapter 2 have a natural counterpart in the framework just outlined. We sketch below some problems on the matter, whose analysis still have to be completely developed.

Let Ω be an open subset of $\mathbb{R}^n$, and $N \geq 1$. For every $\varepsilon > 0$ define $F_\varepsilon\colon \left[L^1(\Omega;\mathbb{R}^N)\right]^2 \to [0,+\infty]$ as follows:

$$F_\varepsilon(u,v) = \begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_\Omega W(\nabla u)\,dx + \dfrac{1}{\varepsilon}\int_\Omega |\nabla u - M|^2\,dx + \varepsilon\int_\Omega |\nabla M|^2\,dx & \\ & \text{if } u \in W^{1,1}_{loc}(\Omega;\mathbb{R}^N) \text{ and } M \in V, \\ +\infty & \text{otherwise,} \end{cases}$$

where ($M^{N\times n}$ is the space of real $N\times n$ matrices):

– $W\colon M^{N\times n} \to [0,+\infty)$ is a continuous function satisfying

$$\{\xi \in M^{N\times n} : W(\xi) = 0\} = \{A, B\} \qquad \text{with } \mathrm{rk}(B-A) = 1;$$

we assume, without loss of generality, that

$$A = -B = a \otimes e_n$$

for a suitable $a \in \mathbb{R}^N$.

– V is any of the following spaces

a) $H^1(\Omega; M^{N\times n})$;

b) $V_{\nu_0} = \{v \otimes \nu_0 : \ v \in H^1(\Omega;\mathbb{R}^N)\}$ for a suitable $\nu_0 \in S^{n-1}$.

Moreover, we assume that

$$(A) \qquad \liminf_{|\xi|\to+\infty} \frac{W(\xi)}{|\xi|} > 0.$$

It is possible to prove, for the sequence $\{F_\varepsilon\}$, a compactness result. We start by recalling a first weak compactness lemma.

Lemma 5.1.1. *Let (A) hold. Let $\{u_h\}$ be a sequence in $W^{1,1}(\Omega;\mathbb{R}^N)$ such that*

$$\frac{1}{\varepsilon_h}\int_\Omega W(\nabla u_h)\,dx \le C$$

for a suitable $C>0$ independent of h. Then the sequence

$$u_h - \frac{1}{|\Omega|}\int_\Omega u_h\,dx, \qquad h\in\mathbb{N}$$

is weakly compact in $W^{1,1}(\Omega;\mathbb{R}^N)$.

Proof. By (A) the sequence $\{\nabla u_h\}$ is bounded in $L^1(\Omega)$. Therefore, by the Poincaré-Friedrichs inequality,

$$w_h = u_h - \frac{1}{|\Omega|}\int u_h,$$

is bounded in $W^{1,1}(\Omega;\mathbb{R}^N)$. Let us check the equi-integrability of $\{\nabla u_h\}$, in the form

$$\lim_{k\to+\infty}\int_{\{|\nabla u_h|\ge k\}} |\nabla u_h|\,dx = 0 \qquad \text{uniformly in } h.$$

By (A) there exist $c_1, c_2 > 0$ such that

$$|t| \le c_1 W(t) + c_2 \qquad \text{for every } t\in\mathbb{R}.$$

Then, for every $h,k\in\mathbb{N}$

$$\begin{aligned}\int_{\{|\nabla u_h|\ge k\}} |\nabla u_h|\,dx &\le c_1\int_{\{|\nabla u_h|\ge k\}} W(\nabla u_h)\,dx + c_2|\{|\nabla u_h|\ge k\}|\\ &\le c_1\int_\Omega W(\nabla u_h)\,dx + \frac{c_2}{k}\int_\Omega |\nabla u_h|\,dx\end{aligned}$$

Fix $\sigma>0$ and let h_σ, k_σ satisfy

$$c_1\int_\Omega W(\nabla u_h)\,dx \le \sigma/2 \qquad \text{for every } h > h_\sigma$$

and (taking into account the boundedness of $\{\nabla u_h\}$ in $L^1(\Omega)$),

$$\frac{c_2}{k}\int_\Omega |\nabla u_h|\,dx \le \sigma/2 \qquad \text{for every } k>k_\sigma \text{ and for every } h.$$

Clearly, we can assume that k_σ also satisfies the following:

$$\int_{\{|\nabla u_h|\ge k_\sigma\}} |\nabla u_h|\,dx \le \sigma \qquad \text{for } 1\le h\le h_\sigma.$$

We conclude that

$$\int_{\{|\nabla u_h|\ge k\}} |\nabla u_h|\,dx \le \sigma$$

for every $k \geq k_\sigma$ and for every $h \in \mathbb{N}$.

The lemma now follows by the Dunford-Pettis Theorem. ■

In order to prove a compactness result for the sequence $\{F_\varepsilon\}$, we introduce the simplest generalization of the family of functionals studied in Chapter 2 to vector valued phase models.

For every open subset Ω of $\mathbb{R}^n$ and for every $\varepsilon > 0$, we denote by $f_\varepsilon(\cdot,\cdot;\Omega)$ the following functionals, defined on $[L^1(\Omega;\mathbb{R}^d)]^2$, with $d \geq 1$:

$$f_\varepsilon(u,v;\Omega) = \begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_\Omega W(u) + \dfrac{1}{\varepsilon}\int_\Omega |u-v|^2 + \varepsilon\int_\Omega |Dv|^2 & \text{if } v \in H^1(\Omega;\mathbb{R}^d) \\ +\infty & \text{otherwise,} \end{cases}$$

where $W\colon \mathbb{R}^d \to [0,+\infty)$ is a continuous function with at least linear growth (see assumption (A)), and is such that the set $\{\xi \in \mathbb{R}^d : W(\xi) = 0\}$ consists of two vectors, denoted by η and ζ.

A compactness result for this sequence of functionals can be easily obtained by adjusting the proof of Proposition 2.1.5, and reads as follows:

Lemma 5.1.2. *Given $\{\varepsilon_h\}$ a positive infinitesimal sequence, and $\{(u_h,v_h)\} \subset [L^1(\Omega;\mathbb{R}^d)]^2$, such that $f_{\varepsilon_h}(u_h,v_h)$ is uniformly bounded, then the sequence $\{(u_h,v_h)\}$ is relatively compact in $[L^1(\Omega);\mathbb{R}^d]^2$. Moreover, the cluster points (u,v) are such that $u = v \in BV(\Omega;\{\eta,\zeta\})$.*

Outline of the proof. Step 1: scalar case ($n = 1$). The proof is exactly the same given for Proposition 2.2.6, and relies on the corresponding result concerning nonlocal functionals (see [2, §1.12, p. 268]).

Step 2: general case. The result can be obtained from the previous step via *slicing*, as in the proof of Proposition 2.1.5, by applying the compactness criterion of Lemma 2.3.2 for vector-valued functions.

Finally, if (u,v) is a cluster point, it is immediate to see that $u = v$ $\mathcal{H}^{n-1}$-a.e. in Ω, and v is in $BV(\Omega;\mathbb{R}^d)$ as L^1 limit of a sequence of $H^1(\Omega;\mathbb{R}^d)$. An application of the Fatou Lemma gives that $v \in \{\eta,\zeta\}$ $\mathcal{H}^{n-1}$-a.e. in Ω. ■

Therefore, we can prove the following compactness result for the sequence $\{F_\varepsilon\}$.

Proposition 5.1.3. (Compactness) *Given $\{\varepsilon_h\}$ a positive infinitesimal sequence, and $\big\{(u_h,M_h)\big\}$ a sequence in $W^{1,1}(\Omega;\mathbb{R}^N) \times L^1(\Omega;M^{N\times n})$ such that $F_{\varepsilon_h}(u_h,M_h)$ is uniformly bounded, then there exist a subsequence $\big\{(u_{h_k},M_{h_k})\big\}$ and $u \in W^{1,1}(\Omega;\mathbb{R}^N)$, with $\nabla u \in BV(\Omega;\{A,B\})$, such that*

$$\begin{aligned} & u_{h_k} - \frac{1}{|\Omega|}\int_\Omega u_{h_k}\,dx \longrightarrow u && \text{in} \quad W^{1,1}(\Omega;\mathbb{R}^N) \\ & M_{h_k} \longrightarrow \nabla u && \text{in} \quad L^1(\Omega;M^{N\times n}). \end{aligned}$$

Proof. Given $\{\varepsilon_h\}$ and $\{(u_h, M_h)\}$ as in the hypothesis, Lemma 5.1.1 ensures that there exist a subsequence $\{u_{h_k}\}$ and a function $u \in W^{1,1}(\Omega; \mathbb{R}^N)$ such that

$$u_{h_k} - \frac{1}{|\Omega|}\int_\Omega u_{h_k}\, dx \rightharpoonup u \quad \text{in} \quad W^{1,1}(\Omega; \mathbb{R}^N).$$

Moreover, the compactness result of Lemma 5.1.2, applied to the sequence $f_{\varepsilon_{h_k}}(\nabla u_{h_k}, M_{h_k})$, implies in particular the existence of $M \in BV(\Omega; \{A, B\})$ such that, possibly passing to a subsequence (not relabelled)

$$M_{h_k} \to M \quad \text{in} \quad L^1(\Omega; M^{N\times n}).$$

Clearly $\nabla u_{h_k} - M_{h_k} \to 0$ in $L^2(\Omega; M^{N\times n})$, thus $\nabla u_h \to M$ in $L^1(\Omega; M^{N\times n})$. From the weak convergence of $\left\{u_{h_k} - \frac{1}{|\Omega|}\int_\Omega u_{h_k}\, dx\right\}$, it turns out that $\nabla u = M$, and the proof is complete. ■

Remark 5.1.4. In the case $V = V_{\nu_0}$, if the Γ-upper limit is finite in at least one point (u, M), then $\nu_0 = \pm e_n$. Indeed, if $\{(u_\varepsilon, M_\varepsilon)\}$ converges to (u, M) and $F_\varepsilon(u_\varepsilon, M_\varepsilon) < +\infty$, then $M_\varepsilon = v_\varepsilon \otimes \nu_0$ with $v_\varepsilon \in H^1(\Omega; \mathbb{R}^n)$, and, by the previous lemma, ∇u has the form $v \otimes \nu_0$, too. On the other hand, by assumption $A = -B = a \otimes e_n$; then $\nabla u = \pm a \otimes e_n$ a.e.; it follows that $v = \pm a$ a.e. and $\nu_0 = \pm e_n$.

As to the asymptotic behaviour of $\{F_\varepsilon\}$ as $\varepsilon \to 0$, it can be dealt with following the line of the proof of the corresponding results in [20] for the functionals G_ε. However, a general simplification should occur since only first order term are involved.

Here, we only would like to give the idea of a first simple convergence result for the sequence $\{F_\varepsilon\}$, under special assumptions. Precisely, if we assume

$$(B) \qquad W(\xi) \geq W(0, \xi_n) \quad \text{where} \quad \xi = (\xi', \xi_n) \in M^{N\times(n-1)} \times \mathbb{R}^N$$

(see (H_3) in [20]), we can again follow a slicing procedure to reduce the analysis to the functionals f_ε defined above, whose asymptotic behaviour is easy to study. Notice that (B) implies a one-dimensional character for the asymptotic problem.

In particular, if we consider a domain Ω of the form $\omega \times I$, with ω an open bounded simply connected domain in $\mathbb{R}^{n-1}$, and I an open and bounded interval, it is easy to prove that there exists a constant $K > 0$ such that the Γ-limit of the sequence $\{F_\varepsilon\}$ in the L^1 topology is

$$F(u, M; \Omega) = \begin{cases} K\mathcal{H}^{n-1}(S(\nabla u) \cap \Omega) & \text{if } u \in W^{1,1}(\Omega; \mathbb{R}^N),\ \nabla u = M \in BV(\Omega; \{A, B\}) \\ +\infty & \text{otherwise.} \end{cases}$$

The constant K is in this case the natural generalization to the vectorial setting of the constant denoted in previous chapters by c_W or Φ (see, e.g., (3.1.1)), i.e.

$$K = \inf\left\{\int_{-L}^{L} W(0, \varphi) + |\varphi - \psi|^2 + |\psi'|^2 \ :\ L > 0,\ \varphi \in X,\ \psi \in X \cap Y\right\},$$

where (recall that $A = -B = a \otimes e_n$) :

$$X = \{\varphi \in AC([-L, L]; \mathbb{R}^N) \ : \ \varphi(-L) = -a, \ \varphi(L) = a\},$$
$$Y = \{\varphi \in AC([-L, L]; \mathbb{R}^N) \ : \ \varphi' \in L^2((-L, L); \mathbb{R}^N)\}.$$

It could be interesting to understand if, considering more general domains (see, e.g., [20, §5.3 and §5.4]), or less restrictive assumptions on W (in particular, assumptions which do not entail a one-dimensional character for the limit problem, as in [20, §6]), the analysis of this kind of "two-variable" models would give rise to a partial simplification of some technical difficulties.

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