ON SOME CLASSES OF OPERATORS DETERMINED BY THE STRUCTURE OF THEIR MEMORY

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Abstract. Two classes of nonlinear operators generalizing the notion of a local operator between ideal function spaces are introduced. The first class, called *atomic*, contains in particular all the linear shifts, while the second one, called *coatomic*, contains all the adjoints to former, and in particular, the conditional expectations. Both classes include local (in particular, Nemytskiĭ) operators and are closed with respect to compositions of operators. Basic properties of operators of introduced classes in the Lebesgue spaces of vector-valued functions are studied. It is shown that both classes inherit from Nemytskiĭ operators the properties of noncompactness in measure and weak degeneracy, while have different relationships of acting, continuity and boundedness, as well as different convergence properties. Representation results for the operators of both classes are provided. The definitions of the introduced classes as well as the proofs of their properties are based on a purely measure theoretic notion of memory of an operator, also introduced in the paper.

1. Introduction

It is well-known that many classical problems for differential equations, both ordinary and partial, usually involve only *local* operators between suitable function spaces. Two most famous examples of local operators, which justify the importance of this notion for applications, are linear differentiation operators between spaces of smooth functions or Sobolev spaces [19], and the nonlinear Nemytskiĭ (also called superposition) operator [1] usually defined between the spaces of summable functions (Lebesgue or Orlicz spaces). Another less known but not less important example of a local operator which cannot be reduced to the above-mentioned ones is given by the stochastic integral defined over some suitable space of summable functions [14]. It is worth emphasizing that many general properties of classical differential equations are provided by the locality of the operators involved.

It is however well-known that there are many applications involving operators which are intuitively "almost local" and in fact have many very similar properties. Such operators are for instance, linear weighted shifts, appearing in hereditary systems and functional-differential equations with deviating argument [3] as well as in many other applications. Since the general definition of a local operator [15] has been formulated, several attempts have been made to obtain its reasonable generalization which would cover many such applications, while not losing most of the important properties of local operators. It seems however, that only the notion of *disjointness preserving* (D-) operators was rather successful in this sense.

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Moreover, even with the latter notion, most significant results were obtained only for the linear operator case [20], while we are inclined to think (and try to justify it in the paper) that no reasonably "nice" properties can be proved for general D-operators in nonlinear case.

In this paper we point out two new extensions of the notion of a local operator in ideal function spaces. The first class of operators, called *atomic*, contains all the (linear) weighted shifts. The second one, in certain sense dual to the first one, called *coatomic*, includes in particular conditional expectations and the adjoints of weighted shifts. Both classes include local operators and are closed with respect to composition of operators. We show that such operators arise naturally in the study of periodic solutions to stochastic differential equations, as well as in functional-differential equations. In the paper we study basic properties of the operators of the identified classes like acting, boundedness, continuity, compactness and convergence, and show that in many cases they just inherit the respective properties of local (and in particular, Nemytskiĭ) operators. However, we show by a number of counterexamples that the introduced objects are essentially new and in fact one may not just blindly extend the properties of local operators to such more general classes. The terminology introduced is justified by the representation theorems we prove for both classes: for instance, it will be clear that atomic operators can be regarded as nonlinear integral operators generated by a random atomic measure.

We find it suitable to define the outlined two new classes of operators on the basis of the notion of the *memory* of an operator also introduced in this paper. Roughly speaking, a memory is a piece of information about the preimages the operator is able to "remember" given a piece of information about images. This notion most naturally arises just from the definition of a local operator [15]. What is more, it will be easy to conclude that most of the common properties of local operators and the new introduced classes comes from nothing else but a measure-theoretic structure of their memory.

2. Memory and comemory of an operator

2.1. **Notation and preliminaries.** Let (Ω, Σ, μ) be a measure space with a finite measure μ, Σ being a σ-algebra of its μ-measurable subsets. A σ-ideal of subsets of measure zero (called further *nullsets* or *μ-nullsets*, if the measure needs to be specified) will be denoted by Σ^0. Identifying in Σ the sets which differ (in the sense of symmetric difference) by a nullset, we come to the σ-algebra $\tilde{\Sigma} := \Sigma/\Sigma^0$ of classes of measurable sets modulo nullsets. It is customary to call $\tilde{\Sigma}$ a *measure algebra of μ* (see, e.g., § 42 of [16]). In general we identify the elements of $\tilde{\Sigma}$ with their representatives from Σ, but whenever there might be a possibility of confusion, we denote for $e \in \Sigma$ the respective equivalence class in $\tilde{\Sigma}$ by either $\tilde{e}$ or $[e]$. We also find it convenient to write $x\,|_e = y\,|_e$, if $x(\omega) = y(\omega)$ a.e. on the set $e \subset \Omega$.

By $L^p(\Omega, \Sigma, \mu; \mathcal{X})$ we denote the classical Lebesgue-Bochner space of (classes of μ-equivalent) functions summable with exponent $p > 0$ over Ω with respect to the measure μ and taking values in some separable normed space $\mathcal{X}$. By $L^0(\Omega, \Sigma, \mu; \mathcal{X})$, where $\mathcal{X}$ is a separable metric space, we denote the space of (classes of) $\mathcal{X}$-valued measurable functions equipped with the usual metric inducing the topology of convergence in measure. Whenever there is no possibility of confusion, the references to $\mathcal{X}$, Ω, Σ and/or μ will be omitted.

In some cases we assume (Ω, Σ, μ) to be a *standard measure space*, where Ω is a Polish space [4], μ is a Borel measure on Ω, Σ is either the σ-algebra of Borel subsets of the space Ω or its μ-completion.

The characteristic function of a subset $e \subset \Omega$ will in the sequel be denoted by $\mathbf{1}_e(\omega)$, omitting the reference to e when $e = \Omega$.

If $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ are two measure spaces, a map F: $\Sigma_1 \to \Sigma_2$ is called a *σ-homomorphism*, if $F(\Omega_1) = \Omega_2$, $F(\Omega_1 \backslash e) = \Omega_2 \backslash F(e)$ whenever $e \in \Sigma_1$ and

$$F\left(\bigsqcup_{i=1}^{\infty} e_i\right) = \bigsqcup_{i=1}^{\infty} F(e_i),$$

for any pairwise disjoint collection of μ_1-measurable sets $\{e_i\}_{i=1}^{\infty}$, where $\sqcup$ stands for the disjoint union. The σ-homomorphisms considered below will silently be assumed to be *nullset preserving*, that is,

$$\mu_2(F(e_1)) = 0 \text{ when } \mu_1(e_1) = 0.$$

Thus F naturally induces a σ-homomorphism $\tilde{F}$: $\tilde{\Sigma}_1 \to \tilde{\Sigma}_2$ by the relationship $\tilde{F}(\tilde{e}_1) := [F(e_1)]$.

2.2. **Memory and comemory.** Let $X_i := L^{p_i}(\Omega_i, \Sigma_i, \mu_i; \mathcal{X}_i)$, $i = 1, 2$, and $0 \leq p_i \leq +\infty$. Consider an operator T: $X_1 \to X_2$. We introduce now the concept of *memory* and the related concept of *comemory*.

Definition 2.1. *We call the* **memory** *of an operator T: $X_1 \to X_2$ on a set $e_2 \in \Sigma_2$ the family of all possible $\tilde{e}_1 \in \tilde{\Sigma}_1$ such that for any $x, y \in X_1$ satisfying $x\,|_{e_1} = y\,|_{e_1}$ it follows that $T(x)\,|_{e_2} = T(y)\,|_{e_2}$. In other words,*

$$Mem_T(\tilde{e}_2) := \left\{\tilde{e}_1 \in \tilde{\Sigma}_1 \,:\, x\,|_{e_1} = y\,|_{e_1} \quad implies\ T(x)\,|_{e_2} = T(y)\,|_{e_2}\right\},$$

Similarly, the **comemory** *of the operator T on a set $e_1 \in \Sigma_1$ is the family*

$$Comem_T(\tilde{e}_1) := \left\{\tilde{e}_2 \in \tilde{\Sigma}_2 \,:\, x\,|_{e_1} = y\,|_{e_1} \quad implies\ T(x)\,|_{e_2} = T(y)\,|_{e_2}\right\}.$$

It is clear that $\tilde{e}_1 \in \mathrm{Mem}_T(\tilde{e}_2)$, if and only if $\tilde{e}_2 \in \mathrm{Comem}_T(\tilde{e}_1)$.

Example 2.1. Define a *shift* operator T_g: $L^0(\Omega_1; \mathcal{X}) \to L^0(\Omega_2; \mathcal{X})$ (sometimes also called inner superposition) by the formula

$$(T_g x)(\omega_2) := x(g(\omega_2)) \tag{1}$$

where g: $\Omega_2 \to \Omega_1$ is a given (Σ_2, Σ_1)-measurable function. For this operator to be well-defined on the classes of measurable functions we require

$$\mu_2(g^{-1}(e_1)) = 0 \text{ for } e_1 \in \Sigma_1,\ \mu_1(e_1) = 0. \tag{2}$$

Then

$$\mathrm{Mem}_{T_g}(\tilde{e}_2) = \{\tilde{e}_1 \in \tilde{\Sigma}_1 \,:\, e_1 \supset g(e_2)\}.$$

Example 2.2. Let $\Omega \subset \mathbb{R}^n$ be a compact set supplied with the n-dimensional Lebesgue measure μ, Σ being the Lebesgue σ-algebra. We define an operator T: $L^1(\Omega) \to L^1(\Omega)$ by the formula

$$(Tx)(\omega) := \int_{\Omega} x(s)\, ds \cdot \mathbf{1}(\omega),$$

i.e. T maps every function into its integral. Then

$$\mathrm{Mem}_T(\tilde{\mathcal{E}}) = \begin{cases} \{\tilde{\Omega}\}, & \mu(\mathcal{E}) \neq 0, \\ \tilde{\Sigma}, & \mu(\mathcal{E}) = 0. \end{cases}$$

2.3. **Local and Nemytskiĭ operators.** Let $X_i = L^0(\Omega, \Sigma, \mu; \mathcal{X}_i)$, $i = 1, 2$.

Definition 2.2. *An operator $T\colon X_1 \to X_2$ is called* **local**, *if*

$$\tilde{e} \in Mem_T(\tilde{e})$$

for all $e \in \Sigma$, that is, if from $x\,|_e = y\,|_e$ for $x, y \in X_1$ follows $T(x)\,|_e = T(y)\,|_e$.

The above general definition reformulates the classical one due to I.V. Shragin [15] using the notion of memory. The following example is also classical and in fact motivated the study of local operators.

Example 2.3. Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be separable metric spaces, $f\colon \Omega \times \mathcal{X}_1 \to \mathcal{X}_2$ be a sup-measurable function (i.e. $f(\cdot, x(\cdot))$ is μ-measurable whenever $x(\cdot)$ is μ-measurable). Then the Nemytskiĭ operator $N_f\colon L^0(\Omega; \mathcal{X}_1) \to L^0(\Omega; \mathcal{X}_2)$ (commonly known also under the name of the *superposition operator* [1]), defined by

$$(N_f x)(\omega) := f(\omega, x(\omega))$$

is local. If $f\colon \Omega \times \mathcal{X}_1 \to \mathcal{X}_2$ is a Carathéodory function (i.e. $f(\omega, \cdot)$ is continuous for μ-almost every $\omega \in \Omega$ and $f(\cdot, x)$ is μ-measurable for all $x \in \mathcal{X}_1$), then the Nemytskiĭ operator N_f becomes continuous in measure (i.e. as an operator in L^0).

By now much research has been done on Nemytskiĭ operators, especially in ideal function spaces. The reader is referred to the recent monographs [1] and [17] for the collection of results on this topic. It is known, however, that in general local operators over spaces of vector-valued summable functions are not reduced to Nemytskiĭ operators, although the former inherit many properties of the latter.

3. Atomic operators

Now we introduce another definition generalizing the notion of a local operator. Here $X_i := L^{p_i}(\Omega_i, \Sigma_i, \mu_i : \mathcal{X}_i)$, $i = 1, 2$ and $0 \leq p_i \leq +\infty$.

Definition 3.1. *An operator $T\colon X_1 \to X_2$ is called* **atomic**, *if there is a σ-homomorphism $F\colon \Sigma_1 \to \Sigma_2$, satisfying*

$$[F(e_1)] \in Comem_T(\tilde{e}_1)$$

for all $e_1 \in \Sigma_1$, that is, if from $x\,|_{e_1} = y\,|_{e_1}$ for $x, y \in X_1$ follows $T(x)\,\big|_{F(e_1)} = T(y)\,\big|_{F(e_1)}$.

Obviously, every local operator is atomic. However, the class of atomic operators is richer, as one can conclude from the following example.

Example 3.1. Every shift operator (see example 2.1) $T_g\colon L^0(\Omega_1; \mathcal{X}) \to L^0(\Omega_2; \mathcal{X})$ is atomic. To show this, it is enough to set $F(e_1) := g^{-1}(e_1)$.

The class of atomic operators is obviously closed under compositions.

Example 3.2. Let Σ_2 be a σ-algebra of Borel subsets (or its μ_2-completion) of a metric space Ω_2, $\mathcal{X}_1$ and $\mathcal{X}_2$ be separable metric spaces, $f\colon \Omega_2 \times \mathcal{X}_1 \to \mathcal{X}_2$ be a Carathéodory function, and $g\colon \Omega_2 \to \Omega_1$ be a (Σ_2, Σ_1)-measurable function satisfying (2). Then the operator T: $L^0(\Omega_1;\mathcal{X}_1) \to L^0(\Omega_2;\mathcal{X}_2)$ defined by the relationship

$$(Tx)(\omega_2) := f(\omega_2, x(g(\omega_2)))$$

is atomic as a composition of a Nemytskiĭ operator N_f, and a shift T_g.

The example below shows however that unlike local operators the class of atomic operators is not closed under finite sums, even if the latter are well-defined.

Example 3.3. Let $\Omega = (0,1)$, Σ be the σ-algebra of Lebesgue measurable subsets, μ being the one-dimensional Lebesgue measure. Consider the shift operator T_g: $L^2(\Omega) \to L^2(\Omega)$ defined by the formula $(T_g x)(\omega) := x(1-\omega)$. We show that $T := Id + T_g$, where Id stands for the identity in L^2, is not atomic even though both summands are. Indeed, for an operator to be atomic, it is necessary that for any disjoint e_1^1 and e_1^2 in Σ_1 there exist such disjoint e_2^1 and e_2^2 in Σ_2 with $e_1^1 \sqcup e_1^2 = \Omega_1$ that $\tilde{e}_2^1 \in \mathrm{Comem}_T(\tilde{e}_1^1)$, $\tilde{e}_2^2 \in \mathrm{Comem}_T(\tilde{e}_1^2)$ and $e_2^1 \sqcup e_2^2 = \Omega_2$. It is enough hence to take $e_1^1 := (0, 1/2]$ and $e_1^2 := (1/2, 1)$ and observe that

$$\mathrm{Comem}_T(\tilde{e}_1^1) = \mathrm{Comem}_T(\tilde{e}_1^2) = \tilde{\emptyset},$$

so there is no possible way to choose sets e_2^1 and e_2^2 in Σ_2 with mentioned properties.

Now present now the key example of this section, which shows how atomic operators arise naturally in a periodic problem for a stochastic differential equation.

Example 3.4. Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space, Σ_0 be a complete σ-subalgebra of Σ. We also suppose given a group of bimeasurable measure preserving bijections $\{\theta_\gamma : \gamma \in \mathbb{R}\}$ on Ω such that $\theta_\gamma(\Sigma_0) \supset \Sigma_0$ for all $\gamma > 0$. An example of such a group is given by the Wiener shifts satisfying the following condition:

$$W(t+\gamma,\omega) - W(s+\gamma,\omega) = W(t,\theta_\gamma(\omega)) - W(s,\theta_\gamma(\omega)) \text{ a.s.}$$

for every real t, s, γ (see e.g. [2]). Here and below $W_t := W(t,\omega)$ is the standard Wiener process and Σ_0 is generated by random values $\{W(s,\omega) : s \le 0\}$. Consider a stochastic differential equation of Itô type in a phase space $\mathcal{B}$ (which is $\mathbb{R}^n$ for ordinary stochastic differential equations and a Banach space otherwise):

$$dx(t) = a(t, x(t))\, dt + b(t, x(t))\, dW_t. \tag{3}$$

Assume that this equation has the unique (up to the natural almost sure equivalence) solution $x_\alpha(t)$ for each initial value $x(0) = \alpha \in L^0(\Omega, \Sigma_0, \mathbb{P}; \mathcal{B})$. This is for instance the case when a and b are Lipschitz-continuous functions. Assume in addition that a and b are τ-periodic in time. Then the evolution (propagation) operator defined by $U(t,\alpha)x := x_\alpha(t)$ corresponding to the equation (3) builds the so-called "cocycle" (see e.g. [2]):

$$T_g \circ U(t+\tau, \alpha) = U(t, T_g \alpha) \tag{4}$$

where T_g is the shift operator generated by the function $g := \theta_{-\tau}$ according to the formula

$$(T_g\xi)(\omega) := \xi(g(\omega)) = \xi(\theta_{-\tau}(\omega)).$$

Suppose now we are looking for solutions of (3) periodic in distribution, i.e. satisfying

$$\begin{aligned} &\mathbb{P}\{x(t_1) \in B_1, x(t_2) \in B_2, ..., x(t_k) \in B_k\} = \\ &\qquad \mathbb{P}\{x(t_1+\tau) \in B_1, x(t_2+\tau) \in B_2, ..., x(t_k+\tau) \in B_k\} \end{aligned} \tag{5}$$

whatever time points $t_i \in \mathbb{R}$, a number $k \in \mathbb{N}$ and Borel sets $B_i \subset \mathcal{B}$, $i = 1, \dots, k$ in the phase space of the equation (3) are. In fact, it is the cocycle property that makes it hopeless to find a periodic solution in the usual sense: $U(t+\tau, \alpha) = U(t, \alpha)$ would typically imply that $T_g \circ U(\tau) \circ T_g^{-1} = U(\tau)$ which is not the case since U is random (it essentially depends on ω). We have then two reasonable notions of stochastically periodic solutions: strong and weak. While the latter is known to be equivalent to (5), the former is stronger. Both definitions are however acceptable for probabilists. For the sake of simplicity we consider only the *strong* periodic solution determined by the condition

$$T_g x(t+\tau) = x(t) \text{ a.s.}$$

The latter condition leads to the fixed point problem

$$\alpha = T(\alpha) \tag{6}$$

with respect the unknown $\alpha \in L^0(\Omega, \Sigma_0, \mathbb{P}; \mathcal{B})$, where $T := T_g \circ U(\tau)$. Using the cocycle property (4) one can show that the solution of (3) starting at α is strongly periodic, if and only if it satisfies (6).

In [13] it has been shown that the operator $U(\tau)$ is local. Therefore, T is atomic as a composition of a shift T_g with the local operator. It is worth emphasizing that for ordinary stochastic differential equations U is usually generated by a Carathéodory function (see e.g. [2]) while even for simple delay equations it is not (see [12]).

Let us discuss now the relationship between *disjointness preserving* operators called further on *D-operators* and atomic operators. For simplicity we restrict ourselves to the case of the operators $T\colon L^0(\Omega; \mathbb{R}) \to L^0(\Omega; \mathbb{R})$. Recall that T is called a D-operator, if it sends functions with disjoint supports into functions with disjoint supports, that is, for every $x, y \in L^0$ one has

$$[\operatorname{supp} x] \cap [\operatorname{supp} y] = \tilde{\emptyset} \Rightarrow [\operatorname{supp} T(x)] \cap [\operatorname{supp} T(y)] = \tilde{\emptyset},$$

where $\operatorname{supp} x := \{\omega \in \Omega : x(\omega) \neq 0\}$ for $x \in L^0$. It is clear that every linear atomic operator is a D-operator. On the other hand, it is easy to prove using either the representation result due to J. von Neumann [18, 10] or the similar theorem of R. Sikorski (theorem 32.3 in [16]) that every linear continuous in measure D-operator $T\colon L^0 \to L^0$ is a weighted shift $(Tx)(\omega) = b(\omega)x(g(\omega))$ for some Σ-measurable function $b\colon \Omega \to \mathbb{R}$ and a (Σ, Σ)-measurable $g\colon \Omega \to \Omega$ satisfying

$$\mu(g^{-1}(e)) = 0 \text{ whenever } \mu(e) = 0,$$

and therefore is a linear atomic operator. An analogous result may be proven also when $\mathbb{R}$ is substituted by an arbitrary separable Banach space $\mathcal{X}$. However, such a perfect correspondence between atomic and D-operators is broken in the nonlinear case. In fact, in the following example we provide a nonlinear D-operator which is nonatomic.

Example 3.5. Let $\Omega := \{0, 1\}$, $\Sigma := 2^\Omega$ (the family of all subsets of Ω), μ being any measure on Ω assigning a strictly positive value to each point, $\mathcal{X} := \Omega$. Consider a function $h\colon \Omega \times \mathcal{X} \to \Omega$ such that $h(\omega, 0) = \omega$, but $h(\omega, 1) \neq \omega$ for all $\omega \in \Omega$.

Define the *nonlinear shift* operator $H\colon L^0(\Omega; \mathcal{X}) \to L^0(\Omega; \mathcal{X})$ by the relationship

$$H(x)(\omega) = x\left(h(\omega, x)\right). \tag{7}$$

Clearly, H is continuous in measure. To show that it is a D-operator, assume $\operatorname{supp} x = e$, $\operatorname{supp} y \subset \Omega\backslash e$ and denote $h_z(\cdot) := h(\cdot, z(\cdot))$. Then

$$\begin{aligned}\operatorname{supp} H(x) &= \{\omega \in \Omega : h(\omega, x(\omega)) \in e\} = [(\Omega\backslash e) \cap h_0^{-1}(e)] \sqcup [e \cap h_x^{-1}(e)] \\ &= e \cap h_x^{-1}(e),\end{aligned}$$

since $h_0^{-1}(e) = e$. Similarly,

$$\operatorname{supp} H(y) \subset \{\omega \in \Omega : h(\omega, y(\omega)) \in \Omega\backslash e\} = (\Omega\backslash e) \cap h_y^{-1}(\Omega\backslash e).$$

Thus, $\operatorname{supp} H(x) \cap \operatorname{supp} H(y) = \emptyset$ and H is a D-operator. On the other hand, taking $x = \mathbf{1}_e$ and $y \equiv 0$ we get $\{\omega \in \Omega : H(x)(\omega) = H(y)(\omega)\} = e$. If H were atomic, the respective σ-homomorphism F would have satisfied the relationship $F(e) \subset e$, so that necessarily $F(e) = e$, $h(\omega, x) = \omega$ and hence $H(x) = x$ which of course is not the case. Therefore, H is not atomic.

This example explains why we could not directly use the definition of D-operators in the nonlinear situation, in spite of the fact that every atomic operator is evidently a D-operator. Otherwise, we would be forced to cover in our theory also nonlinear shifts which usually have rather bad properties [7] and which therefore we wanted to avoid. On the other hand, our definition of atomic operators is equivalent to the definition of D-operators in the linear case, but it fits better into the nonlinear framework as one can conclude from the results below.

3.1. **Representation.** We will prove that all the atomic operators acting between "nice" spaces of measurable functions, as the compositions of local operators and shifts. To begin with, we need the following auxiliary technical statement.

Lemma 3.1. *Every σ-homomorphism $F\colon \Sigma_1 \to \Sigma_2$ determines a unique linear continuous (in measure) operator $T_F\colon L^0(\Omega_1;\mathcal{X}) \to L^0(\Omega_2;\mathcal{X})$, satisfying*

$$T_F(\mathbf{1}_{e_1} z) = \mathbf{1}_{F(e_1)} z \tag{8}$$

for all $e_1 \in \Sigma_1$ and $z \in \mathcal{X}$, where $\mathcal{X}$ is a separable Banach space. For this operator, called further generalized shift, one has $Im T_F = L^0(\Omega_2, F\Sigma_1; \mathcal{X})$.

PROOF:

Step 1. Define for all $z \in \mathcal{X}$ and for all $e_1 \in \Sigma_1$ the value of the operator T_F by means of (8), extend it by linearity to the set of all simple (i.e. finite-valued) functions in $L^0(\Omega_1, \Sigma_1; \mathcal{X})$, and then extend T_F to the whole space $L^0(\Omega_1, \Sigma_1; \mathcal{X})$ by continuity (see [5, pp.453-454]). The uniqueness of the desired operator is immediate from this construction.

Step 2. We are to show now that $\operatorname{Im} T_F = L^0(\Omega_2, F\Sigma_1; \mathcal{X})$. In fact, the inclusion

$$\operatorname{Im} T_F \subset L^0(\Omega_2, F\Sigma_1; \mathcal{X}).$$

is obvious. To prove the reverse inclusion, take an arbitrary $y \in L^0(\Omega_2, F\Sigma_1; \mathcal{X})$, and represent it as a limit of a sequence of simple functions $\{y_\nu\} \subset L^0(\Omega_2, F\Sigma_1; \mathcal{X})$, i.e. $y_\nu := \sum_{i=1}^{N_\nu} \mathbf{1}_{F(e_1^{i,\nu})} z_{i,\nu} \to y$ in measure μ_2 on Ω_2, where $z_{i,\nu} \in \mathcal{X}$, $e_1^{i,\nu} \in \Sigma_1$ and $N_\nu \in \mathbb{N}$. Then $y_\nu = T_F x_\nu$, where $x_\nu = \sum_{i=1}^{N_\nu} \mathbf{1}_{e_1^{i,\nu}} z_{i,\nu}$ are simple functions from $L^0(\Omega_1, \Sigma_1; \mathcal{X})$. Set

$$E_1 := \left\{\omega_1 \in \Omega_1 : \frac{d\mu_F}{d\mu_1} > 0\right\}.$$

Letting now $x'_\nu = \sum_{i=1}^{N_\nu} \mathbf{1}_{e_1^{i,\nu} \cap E_1} z_{i,\nu}$, one observes immediately that $T_F x_\nu = T_F x'_\nu$, but since $\{T_F x'_\nu\}$ is a Cauchy sequence in $L^0(\Omega_2, F\Sigma_1; \mathcal{X})$, then so is $\{x'_\nu\}$ in

$L^0(\Omega_1, \Sigma_1; \mathcal{X})$, because μ_F is equivalent to μ_1 over E_1. Therefore there is an $x \in L^0(\Omega_1, \Sigma_1; \mathcal{X})$ such that $x'_\nu \to x$ in measure μ_1 on Ω_1, and hence $T_F x = y$, thus proving $\operatorname{Im} T_F \supset L^0(\Omega_2, F\Sigma_1; \mathcal{X})$ and showing the statement. $\square$

Now we may proclaim the principal result of this paragraph.

Theorem 3.1. *Let $\mathcal{X}_i$ be separable Banach spaces. Then for every atomic operator $T\colon L^0(\Omega_1; \mathcal{X}_1) \to L^0(\Omega_2; \mathcal{X}_2)$ there is a local operator $N\colon L^0(\Omega_2; \mathcal{X}_1) \to L^0(\Omega_2; \mathcal{X}_2)$ such that*

$$T = N \circ T_F. \tag{9}$$

The operator T is continuous (in measure), if and only if so is the restriction of the operator N to the subspace $L^0(\Omega_2, F\Sigma_1; \mathcal{X}_1) \subset L^0(\Omega_2, \Sigma_2; \mathcal{X}_1)$, where $F\colon \Sigma_1 \to \Sigma_2$ is the σ-homomorphism corresponding to T. Moreover, if $(\Omega_1, \Sigma_1, \mu_1)$ is a standard measure space, then there is a (Σ_1, Σ_2)-measurable function $g\colon \Omega_2 \to \Omega_1$ satisfying (2) such that $T = N \circ T_g$.

PROOF:

Step 1. Define on $\operatorname{Im} T_F$ an operator $N\colon \operatorname{Im} T_F \to L^0(\Omega_2, \Sigma_2, \mu_2; \mathcal{X}_2)$ by setting $Ny := Tx$, if $y = T_F x$. First we show that the above definition of N is correct. Assume that $y = T_F x$ and $y = T_F x'$, while $x \neq x'$. Let $\bar{x} := x - x'$ and $e'_1 := \{\omega_1 \in \Omega_1 : \bar{x}(\omega_1) = 0\}$, $e_1 := \Omega_1 \setminus e'_1$. We will prove that $\mu_F(e_1) := \mu_2(F(e_1)) = 0$, hence $\tilde{F}(\tilde{e}'_1) = \tilde{\Omega}_2$, and $Tx = Tx'$ thus showing the correctness of the definition of N. In fact, if $\mu_F(e_1) > 0$, then $\mu_F(e_1 \cap E_1) > 0$, where $E_1 \subset \Omega_1$ is defined as in the step 2 of the proof of the lemma 3.1. Consider a sequence of simple functions $\{x_\nu\} \subset L^0(\Omega_1, \Sigma_1, \mu_1; \mathcal{X}_1)$ with $x_\nu\big|_{e'_1} = 0$ and $x_\nu \to \bar{x}$ in measure μ_1 on Ω_1, while $T_F x_\nu \to T_F \bar{x} = 0$ in measure μ_2 on Ω_2. Since $T_F(\mathbf{1}_{E_1} x_\nu) = T_F x_\nu$ and μ_F is equivalent to μ_1 over E_1, one has that $\mathbf{1}_{E_1} x_\nu \to 0$ in measure μ_1 on Ω_1, and hence $\mathbf{1}_{E_1 \cap e_1} x = 0$, contradicting the definition of e_1.

Step 2. We show that for all $e_2 \in \Sigma_2$ from

$$y_1\big|_{e_2} = y_2\big|_{e_2}, \tag{$*$}$$

where $\{y_1, y_2\} \subset L^0(\Omega_2, F\Sigma_1, \mu_2; \mathcal{X}_1)$, it follows that

$$(Ny_1)\big|_{e_2} = (Ny_2)\big|_{e_2}. \tag{$**$}$$

For $e_2 \in F\Sigma_1$ this is immediate, while if $e_2 \notin F\Sigma_1$, then $(*)$ implies the existence of $E_2 \in F\Sigma_1$, $e_2 \subset E_2$ such that $y_1\big|_{E_2} = y_2\big|_{E_2}$.

Step 3. We are now to extend N over the whole $L^0(\Omega_2, \Sigma_2, \mu_2; \mathcal{X}_1)$ as a local operator, i.e. so that $(*)$ imply $(**)$ for all $\{y_1, y_2\} \subset L^0(\Omega_2, \Sigma_2, \mu_2; \mathcal{X}_1)$. Assume

$$z \in L^0(\Omega_2, \Sigma_2, \mu_2; \mathcal{X}_1) \setminus L^0(\Omega_2, F\Sigma_1, \mu_2; \mathcal{X}_1).$$

Then for every $e_2 \in \Sigma_2$ there are two possibilities:

(i) either there is a $v \in L^0(\Omega_2, F\Sigma_1; \mathcal{X}_1)$ which coincides with z over e_2,

(ii) or there is no such $v \in L^0(\Omega_2, F\Sigma_1; \mathcal{X}_1)$.

In the case (i) we set $(Nz)\big|_{e_2} := (Nv)\big|_{e_2}$, while in the case (ii) $(Nz)\big|_{e_2} := 0$. This extension is well-defined by Step 2 and is obviously local.

Step 4. Suppose that T is continuous. By lemma 3.1 the generalized shift T_F is always continuous. Therefore the restriction of N to $\operatorname{Im} T_F = L^0(\Omega_2, F\Sigma_1; \mathcal{X}_1)$ must also be continuous.

Step 5. To prove the last claim, we observe that if $(\Omega_1, \Sigma_1, \mu_1)$ is a standard measure space, then by the theorem 32.3 of [16] there is a map $g\colon \Omega_2 \to \Omega_1$

satisfying $g^{-1}(e_1) = F(e_1)$ for all Borel sets $e_1 \subset \Omega_1$. If Σ_1 contains, besides Borel sets, also some non-Borel μ_1-nullsets, then from the above relationship easily follows

$$[g^{-1}(e_1)] = \tilde{F}(\tilde{e}_1).$$

This implies that g is (Σ_1, Σ_2)-measurable and satisfies (2), while $T_g x = T_F x$ for every simple $x \in L^0(\Omega_1; \mathcal{X}_1)$, and hence, by continuity, everywhere over this space. $\square$

Corollary 3.1. *Let $(\Omega_1, \Sigma_1, \mu_1)$ be a standard measure space, $\mathcal{X}_i$ be separable metric spaces and let $T\colon L^0(\Omega_1; \mathcal{X}_1) \to L^0(\Omega_2; \mathcal{X}_2)$ be an atomic operator. Under the continuum hypothesis there is a measurable function $g\colon \Omega_2 \to \Omega_1$ satisfying (2) and a function $f\colon \Omega_2 \times \mathcal{X}_1 \to \mathcal{X}_2$ such that $T = N_f \circ T_g$, i.e. for all $x \in L^0(\Omega_1; \mathcal{X}_1)$ one has*

$$(Tx)(\omega_2) = f(\omega_2, x(g(\omega_2))) \qquad \mu_2 - a.e. \text{ on } \Omega_2.$$

PROOF: It is enough to apply the theorem 3.1 using the representation of every local operator as a Nemytskiĭ operator provided in [11] (see also [1]). $\square$

This corollary justifies our terminology: we may think now of an atomic operator as a kind of an integral operator with respect to the *random atomic measure* concentrated in the point $g(\omega_2)$. A similar terminology is widely used in the linear case (see e.g. [20]).

We note that even if $T\colon L^0(\Omega_1; \mathcal{X}_1) \to L^0(\Omega_2; \mathcal{X}_2)$ is continuous, the operator N defined in the theorem 3.1 needs not to be continuous (only its restriction to a certain subspace is). Therefore, even in the situation of the corollary 3.1 the function $f\colon \Omega_2 \times \mathcal{X}_1 \to \mathcal{X}_2$, generating together with $g\colon \Omega_2 \to \Omega_1$ the operator T, needs not be a Carathéodory function. An example from the theory of stochastic processes, which we are going to discuss now, shows that there are indeed atomic operators T not representable by a composition of a Nemytskiĭ operator generated by a Carathéodory function and a shift operator.

Example 3.6. Consider as in the example 3.4 a probability space $(\Omega, \Sigma, \mathbb{P})$, the standard Wiener process W_t, the Wiener shift $g := \theta_{-1}\colon \Omega \to \Omega$ inducing the isomorphism of the σ-subalgebrae Σ_0 and $\Sigma_1 := g^{-1}(\Sigma_0)$. Letting $\mathcal{X} := L^2[0,1]$, define the operator $T\colon L^0(\Omega, \Sigma_1, \mathbb{P}; \mathcal{X}) \to L^0(\Omega, \Sigma_1, \mathbb{P}; \mathcal{X})$ as the stochastic integration with respect to the Wiener process

$$(Tx)(\omega) := \int_0^{(\cdot)} x(s, g(\omega)) dW_s(\omega).$$

Note that we shifted the Σ_1-measurable integrand $x(t, \omega)$ with the help of g. In this way the stochastic process $x(s, g(\omega))$ becomes Σ_0-measurable, so that the stochastic integral is well-defined. The operator T is atomic since it is a composition of the stochastic integral (which is local) and the shift T_g. However, the stochastic integral cannot be represented by a Nemytskiĭ operator generated by a Carathéodory function. Otherwise, the stochastic integral could have been, by the Riesz representation theorem, reduced to the ordinary Lebesgue-Stieltjes integral, which is impossible (see e.g. [13] or [14]).

3.2. **Acting, continuity and boundedness.** We suppose throughout this paragraph $(\Omega_2, \Sigma_2, \mu_2)$ to support a nonatomic measure and let both $\mathcal{X}_1$ and $\mathcal{X}_2$ be separable Banach spaces.

Theorem 3.2. *If a continuous in measure atomic operator $T\colon\ L^0(\Omega_1;\mathcal{X}_1)\to L^0(\Omega_2;\mathcal{X}_2)$, continuously maps $L^p(\Omega_1;\mathcal{X}_1)$ into $L^q(\Omega_2;\mathcal{X}_2)$, where $1\le p,q<\infty$, then T sends bounded sets of L^p into bounded sets of L^q.*

PROOF:

Step 1. According to the representation theorem 3.1, the continuity in measure of the operator T is equivalent to that of the restriction of the local operator N in the representation formula (9) to the space $L^0(\Omega_2, F\Sigma_1, \mu_2;\mathcal{X}_1)$, where $F\colon \Sigma_1\to\Sigma_2$ is the respective σ-homomorphism making T atomic. We make then the following auxiliary construction. Suppose without loss of generality $T(0)=0$. Let $\mu_F(e_1):=\mu_2(F(e_1))$, observe that $\mu_F << \mu_1$ and consider the set

$$E_1 := \left\{\omega_1\in\Omega_1\ :\ \frac{d\mu_F}{d\mu_1}>0\right\}.$$

To show the statement, it is enough to prove, that if $T\colon\ L^p(\Omega_1;\mathcal{X}_1)\to L^q(\Omega_2;\mathcal{X}_2)$ is continuous, then its restriction to $L^p(E_1;\mathcal{X}_1)$ is bounded. In fact, this follows from $Tx = N(T_Fx) = N(T_F(\mathbf{1}_{E_1}x))$.

Step 2. Consider the operator P defined formally by $(Px)(\omega_2) := |(Nx)(\omega_2)|^q_{\mathcal{X}_2}$. Note that since

$$\int_{\Omega_2}|(T_Fx)(\omega_2)|^p_{\mathcal{X}_1}\,d\mu_2(\omega_2) = \int_{\Omega_1}\frac{d\mu_F}{d\mu_1}|x(\omega_1)|^p_{\mathcal{X}_1}\,d\mu_1(\omega_1)$$

and $T_F(x\cdot y) = (T_Fx)\cdot(T_Fy)$, then

$$\int_{E_1}|x(\omega_1)|^p_{\mathcal{X}_1}\,d\mu_1(\omega_1) = \int_{F(E_1)} T_F\left(\frac{d\mu_F}{d\mu_1}\right)^{-1}|(T_Fx)(\omega_2)|^p_{\mathcal{X}_1}\,d\mu_2(\omega_2),$$

and hence T_F is an isometry between $L^p(E_1,\mu_1;\mathcal{X}_1)$ and $L^p(F(E_1),\mu_2';\mathcal{X}_1)$, where

$$\mu_2' = T_F\left(\frac{d\mu_F}{d\mu_1}\right)^{-1}\mu_2.$$

Therefore, if $T\colon\ L^p(E_1,\Sigma_1,\mu_1;\mathcal{X}_1)\to L^q(\Omega_2,\Sigma_2,\mu_2;\mathcal{X}_2)$ is continuous, then

$$P\ :\ L^p(F(E_1),F(\Sigma_1\cap E_1),\mu_2';\mathcal{X}_1)\to L^1(F(E_1),\Sigma_2,\mu_2;\mathbb{R})$$

is also continuous. Hence, so is the operator

$$Q\ :\ L^p(F(E_1),F(\Sigma_1\cap E_1),\mu_2';\mathcal{X}_1)\to L^1(F(E_1),\Sigma_2,\mu_2';\mathbb{R})$$

defined by

$$(Qx)(\omega_2) := \left(T_F\frac{d\mu_F}{d\mu_1}\right)(\omega_2)(Px)(\omega_2) = \left(T_F\frac{d\mu_F}{d\mu_1}\right)(\omega_2)\,|(Nx)(\omega_2)|^q_{\mathcal{X}_2}.$$

Moreover, both P and Q are clearly the restrictions of some local operators.

Step 3. Let now $\mathcal{E}$ stand for the conditional expectation with respect to the σ-subalgebra $F(\Sigma_1\cap E_1)$ (see e.g. chapter XI of [6] or the example 4.3 below). Since $\mathcal{E}(\mathbf{1}_ex) = \mathbf{1}_e\mathcal{E}(x)$ for all $x\in L^1(\Omega_2,\Sigma_2,\mu_2;\mathbb{R})$ and for arbitrary $e\in F(\Sigma_1\cap E_1)$, the composition

$$\mathcal{E}\circ Q\ :\ L^p(F(E_1),F(\Sigma_1\cap E_1),\mu_2';\mathcal{X}_1)\to L^q(F(E_1),F(\Sigma_1\cap E_1),\mu_2';\mathbb{R})$$

is a continuous local operator. Now both images and preimages of $\mathcal{E}\circ Q$ "live" in the same measure space and we may apply the second representation result from [13] which provides a Carathéodory function $\varphi\colon\ \Omega_2\times\mathcal{X}_1\to\mathbb{R}$, for which

$$(\mathcal{E}\circ Q)(x)(\omega_2) = \varphi(\omega_2, x(\omega_2))\qquad \mu_2' - \text{a.e. on } F(E_1).$$

Choose an arbitrary bounded $B \subset L^p(E_1, \Sigma_1, \mu_1; \mathcal{X}_1)$. Then the set $T_F B$ is bounded in $L^p(F(E_1), F(\Sigma_1 \cap E_1), \mu_2'; \mathcal{X}_1)$. Since for Nemytskiĭ operators generated by Carathéodory functions continuity and boundedness in Lebesgue spaces are equivalent (theorems 3.1 and 3.2 of [1]), then also

$$(\mathcal{E} \circ Q)(T_F B) \subset L^1(F(E_1), F(\Sigma_1 \cap E_1), \mu_2'; \mathbb{R})$$

is bounded. But

$$\begin{array}{rcll} \|(\mathcal{E} \circ Q)(x)\|_1 & = & \int_{F(E_1)} (\mathcal{E} \circ Q)(x)(\omega_2)\, d\mu_2'(\omega_2) & = \\ & & \int_{F(E_1)} Q(x)(\omega_2)\, d\mu_2'(\omega_2) & = \quad \|Q(x)\|_1 \end{array}$$

by the property of conditional expectation, where $\|\cdot\|_1$ stands for the norm in $L^1(F(E_1), \Sigma_1 \cap E_1, \mu_2'; \mathbb{R})$. Thus $Q(T_F B)$ is bounded in $L^1(F(E_1), \Sigma_1 \cap E_1, \mu_2'; \mathbb{R})$, wherefrom follows also the boundedness of $P(T_F B)$ in $L^1(F(E_1), \Sigma_1 \cap E_1, \mu_2; \mathbb{R})$, which shows the statement. $\square$

Still unlike the Nemytskiĭ operators, for atomic operators between Lebesgue spaces no continuity is guaranteed by boundedness, to say nothing about acting, as the following example shows.

Example 3.7. Let $\Omega_1 := (0,1)$ and $\Omega_2 := (0,1) \times (0,1)$, μ_1 and μ_2 be one- and two-dimensional Lebesgue measures in Ω_1 and Ω_2 respectively, while Σ_1 and Σ_2 stand for the respective Lebesgue σ-algebrae. We will construct an atomic operator T: $L^p(\Omega_1; \mathbb{R}) \to L^p(\Omega_2; \mathbb{R})$, which is bounded and continuous in measure, but is discontinuous as an operator between the mentioned Lebesgue spaces.

For this purpose choose a sequence $\{\alpha_\nu\}$ in the unit ball $B \subset L^p(0,1)$ converging to zero *in measure*, but not in the norm of $L^p(0,1)$. Without loss of generality we may assume that the sequence consists of different members. Set $\gamma(1/\nu) := \alpha_\nu$ for $\nu \in \mathbb{N}$ and extend γ to be a measure continuous function γ: $\mathbb{R} \to B$. The latter extension can be obtained by setting first

$$\gamma((1-\tau)/\nu + \tau/(\nu+1)) := (1-\tau)\alpha_\nu + \tau\alpha_{\nu+1}$$

for $0 \le \tau \le 1$ which gives a measure continuous function γ: $(0,1) \to B$ and then extending the latter to the whole $\mathbb{R}$ by means of a projection υ: $\mathbb{R} \to (0,1)$. By construction, the function γ is discontinuous with respect to the norm topology of $L^p(0,1)$, but it is bounded in the space $L^p(0,1)$.

Note that $L^p(\Omega_2; \mathbb{R})$ may be naturally identified with $L^p(\Omega_1; L^p(0,1))$. In fact, if $\omega_2 \in \Omega_2$ is represented as $\omega_2 = (t,s)$, where $t, s \in \Omega_1$, then every element $z \in L^p(\Omega_2; \mathbb{R})$ may be viewed as a function z: $\Omega_1 \to L^p(0,1)$ which maps $t \in \Omega_1 \mapsto z(t,\cdot) \in L^p(0,1)$ and belongs to $L^p(\Omega_1; L^p(0,1))$. We set now

$$T(x)(\omega_2) \equiv T(x)(t,s) := \gamma(x(t))(s),$$

where $x \in L^p(\Omega_1; \mathbb{R})$. Clearly, T is continuous in measure since so is γ. Moreover, T sends bounded sets of $L^p(\Omega_1; \mathbb{R})$ into bounded sets of $L^p(\Omega_2; \mathbb{R})$, for

$$\|Tx\|^p = \int_0^1 dt \int_0^1 |(Tx)(t,s)|^p \, ds = \int_0^1 \|\gamma(x(t))\|^p_{L^p(0,1)} \, dt \le 1.$$

This operator is, however, discontinuous as an operator between the mentioned Lebesgue spaces, because the sequence $\{T(1/\nu)\}$ does not converge in $L^p(0,1)$. At last, it remains to note that T is atomic with respect to the natural imbedding F:

$\Sigma_1 \to \Sigma_2$ defined by $F(e_1) := e_1 \times (0,1)$. Indeed, if $x(t) = y(t)$ for a.e. $t \in e_1 \in \Sigma_1$, then

$$T(x)(\omega_2) = \gamma(x(t))(s) = \gamma(y(t))(s) = T(y)(\omega_2) \qquad \mu_2 - \text{a.e. on } F(e_1).$$

3.3. **Noncompactness.** The *atomic* operators acting in the spaces of measurable functions possess, much like the *local* operators, an important property of noncompactness.

Theorem 3.3. *Assume $0 \le p, q \le \infty$ and $\mathcal{X}_i$ to be separable Banach spaces. Then a nonconstant atomic operator $T\colon L^p(\Omega_1; \mathcal{X}_1) \to L^q(\Omega_2; \mathcal{X}_2)$ is not compact, provided μ_1 is a nonatomic measure.*

PROOF: Since T is nonconstant, there exists an $x \in L^p(\Omega_1; \mathcal{X}_1)$ and $E_2 \in \Sigma_2$, $\mu_2(E_2) > 0$, such that $(T(x))(\omega_2) \ne (T(0))(\omega_2)$ for all $\omega_2 \in E_2$. Without loss of generality we may suppose $T(0) = 0$ (a function $T(0)$, which always can be added, does not make any influence on the compactness properties of the operator). Define then $\hat{\mu}_F(e_1) := \mu_2(F(e_1) \cap E_2)$. Clearly, $\hat{\mu}_F$ is a measure on Σ_1, while $\hat{\mu}_F << \mu_1$ and hence is nonatomic. Note that $\hat{\mu}_F(\Omega_1) = \mu_2(E_2)$.

We construct partitions of Ω_1 by the following inductive process. First find such $\Omega_1^0 \in \Sigma_1$, $\Omega_1^1 \in \Sigma_1$, that $\Omega_1 := \Omega_1^0 \sqcup \Omega_1^1$ with

$$\hat{\mu}_F(\Omega_1^0) = \hat{\mu}_F(\Omega_1^1) = \mu_2(E_2)/2.$$

On the k-th step, having constructed a partition of Ω_1 into sets $\Omega_1^{i_1 \ldots i_k}$, $i_j \in \{0,1\}$, such that

$$\Omega_1 = \bigsqcup_{\substack{i_j \in \{0,1\} \\ j=1,\ldots,k}} \Omega_1^{i_1 \ldots i_k}, \qquad \hat{\mu}_F(\Omega_1^{i_1 \ldots i_k}) = \frac{1}{2^k}\mu_2(E_2),$$

we divide each of the sets $\Omega_1^{i_1 \ldots i_k}$ into two, so that

$$\Omega_1^{i_1 \ldots i_k} = \Omega_1^{i_1 \ldots i_k 0} \sqcup \Omega_1^{i_1 \ldots i_k 1}, \qquad \hat{\mu}_F(\Omega_1^{i_1 \ldots i_k 0}) = \hat{\mu}_F(\Omega_1^{i_1 \ldots i_k 1}) = \frac{1}{2^{k+1}}\mu_2(E_2).$$

Now we set

$$E_1(k) := \bigcup_{\substack{i_j \in \{0,1\} \\ j=1,\ldots,k}} \Omega_1^{i_1 \ldots i_k 0},$$

observing that $\hat{\mu}_F(E_1(k)) = \mu_2(E_2)/2$ for all $k \in \mathbb{N}$. The sequence of functions $x_k := \mathbf{1}_{E_1(k)} x$ is bounded in $L^p(\Omega_1; \mathcal{X}_1)$, but the sequence of images $T(x_k)$ is not precompact in $L^q(\Omega_2; \mathcal{X}_2)$, because

$$|(Tx_j)(\omega_2) - (Tx_k)(\omega_2)|_{\mathcal{X}_2} = \mathbf{1}_{F(E_1(j)) \Delta F(E_1(k))} \, |T(x)|_{\mathcal{X}_2},$$

while for $j < k$ the latter is equal to $|T(x)|_{\mathcal{X}_2} > 0$ on the set

$$E_2(j,k) := F(E_1(j) \Delta E_1(k)) = F\left(\bigcup_{i_j \ne i_k} \Omega_1^{i_1 \ldots i_j \ldots i_k} \right)$$

of the measure $\mu_2(E_2(j,k)) \ge \hat{\mu}_F(E_1(j) \Delta E_1(k)) = \mu_2(E_2)/2 > 0$. $\square$

The proof of the theorem 3.3 extends the well-known proof of noncompactness of the Nemytskiĭ operator (see theorem 1.8 of [1]), its main idea going back to M.A. Krasnosel'skiĭ. We remark also that the same argument gives noncompactness of the atomic operators in more general ideal Banach spaces.

3.4. **Weak continuity.** Our next result extends the well-known theorem on the degeneracy of the Nemytskiĭ operator in the weak topologies of Lebesgue spaces (theorem 2.10 of [1]) to the case of atomic operators.

Theorem 3.4. *Let $1 \leq p, q < \infty$, μ_1 be a nonatomic measure and $\mathcal{X}_i$ be separable Banach spaces. Then an atomic operator T: $L^p(\Omega_1; \mathcal{X}_1) \to L^q(\Omega_2; \mathcal{X}_2)$ is weakly continuous, if and only if T is affine, i.e. $T(\cdot) - T(0)$ is a linear bounded operator.*

PROOF: The only nontrivial part of the statement is the "only if" part. To prove this part, it clearly suffices to show that

$$T(\lambda x + (1-\lambda)y) = \lambda T(x) + (1-\lambda)T(y) \tag{10}$$

for all $\lambda \in [0,1]$ and $x, y \in L^p(\Omega_1; \mathcal{X}_1)$. Given such a λ, we may, since μ_1 is nonatomic, choose a sequence of sets $\{e_1^\nu\} \subset \Sigma_1$ such that $\mathbf{1}_{e_1^\nu} \rightharpoonup \lambda$ $*$-weakly in $L^\infty(\Omega_1; \mathbb{R})$. Then also $\mathbf{1}_{F(e_1^\nu)} = T_F \mathbf{1}_{e_1^\nu} \rightharpoonup \lambda$ $*$-weakly in $L^\infty(\Omega_2; \mathbb{R})$, where F: $\Sigma_1 \to \Sigma_2$ is the σ-homomorphism making T atomic, and hence

$$\mathbf{1}_{e_1^\nu} x + \mathbf{1}_{\Omega_1 \setminus e_1^\nu} y \rightharpoonup \lambda x + (1-\lambda) y \text{ $*$-weakly in } L^\infty(\Omega_1; \mathcal{X}_1). \tag{11}$$

On the other hand, supposing without loss of generality that $T(0) = 0$, one has $T(\mathbf{1}_{e_1} x) = \mathbf{1}_{F(e_1)} T(x)$ for every $x \in L^p(\Omega_1; \mathcal{X}_1)$ and $e_1 \in \Sigma_1$. Therefore,

$$T(\mathbf{1}_{e_1^\nu} x + \mathbf{1}_{\Omega_1 \setminus e_1^\nu} y) \rightharpoonup \lambda T(x) + (1-\lambda) T(y) \text{ $*$-weakly in } L^\infty(\Omega_2; \mathcal{X}_2). \tag{12}$$

Combining now (11) and (12) with weak continuity of T, one obtains (10) thus concluding the proof. □

3.5. **Convergence.** In [9] it has been proven (though in a bit more general terminology) that the strong (pointwise) limit of a sequence of shift operators in Lebesgue spaces is still a shift operator. In [8] this assertion was generalized for weighted shifts. This corresponds well to the theorem below which asserts the atomicity of the strong (pointwise) limits of general atomic operators.

Theorem 3.5. *Assume that $0 < p < \infty$, $0 \leq q \leq \infty$, Σ_1 is countably generated and $\mathcal{X}_i$, $i = 1, 2$, are separable Banach spaces. Let a sequence of atomic operators T_ν: $L^p(\Omega_1; \mathcal{X}_1) \to L^q(\Omega_2; \mathcal{X}_2)$, converge strongly (pointwise) to an operator T, which maps L^p-convergent sequences into measure convergent ones. Then T is atomic.*

PROOF:

Step 1. Before starting the proof we make some necessary arrangements. First, observe that since $L^q(\Omega_2; \mathcal{X}_2) \hookrightarrow L^0(\Omega_2; \mathcal{X}_2)$, we may assume without loss of generality that $q = 0$. Second, we may let all the operators satisfy $T_\nu(0) = 0$, hence also $T(0) = 0$. Finally, note that due to our assumption on Σ_1, the space $L^p(\Omega_1, \Sigma_1, \mu_1; \mathcal{X}_1)$ is separable, so that we can choose its countable dense subset $\{z_i\}$. Now we set

$$A_i := \{\omega_2 \in \Omega_2 \,:\, T(z_i)(\omega_2) \neq 0\}, \qquad E_2 := \bigcup_{i=1}^{\infty} A_i, \ B_1 := A_1, \ B_i := A_i \setminus \bigcup_{j=1}^{i-1} B_j,$$

Without loss of generality we may suppose that $\mu_2(B_i) > 0$ for all $i \in \mathbb{N}$ (otherwise one just renumbers the sequence throwing out the nullsets). Clearly, $T(x)(\omega_2) = 0$ for μ_2-a.e. $\omega_2 \in \Omega_2 \setminus E_2$ and for every $x \in L^p(\Omega_1; \mathcal{X}_1)$.

Step 2. Assume that F_ν: $\Sigma_1 \to \Sigma_2$ are the σ-homomorphisms making the respective T_ν atomic. We construct the "limit" σ-homomorphism F to be used in the proof. For this purpose define the operators $\Gamma_{i,\nu}$: $L^p(\Omega_1;\mathcal{X}_1) \to L^0(\Omega_2;\bar{\mathbb{R}})$, where $\bar{\mathbb{R}} := \mathbb{R}\cup\{+\infty\}$, by the formula

$$\Gamma_{i,\nu}(x)(\omega_2) := \frac{|T_\nu(x)(\omega_2)|_{\mathcal{X}_2}}{|T_\nu(z_i)(\omega_2)|_{\mathcal{X}_2}}\mathbf{1}_{B_i}(\omega_2).$$

Observe that the denominators may vanish for some ω_2 so that images of the operators $\Gamma_{i,\nu}$ admit infinite values. Therefore, for technical reasons, we assume that $0/0 = 0$ and $a/0 = +\infty$, if $a \neq 0$. Clearly, $\Gamma_{i,\nu}$ are atomic operators and converge strongly (pointwise) as $\nu \to \infty$ to the operator Γ_i: $L^p(\Omega_1;\mathcal{X}_1) \to L^0(\Omega_2;\bar{\mathbb{R}})$ given by the formula

$$\Gamma_i(x)(\omega_2) := \frac{|T(x)(\omega_2)|_{\mathcal{X}_2}}{|T(z_i)(\omega_2)|_{\mathcal{X}_2}}\mathbf{1}_{B_i}(\omega_2)$$

For an arbitrary $e_1 \in \Sigma_1$ we take $x_i := \mathbf{1}_{e_1}z_i$ and observe that

$$\Gamma_{i,\nu}(x_i) = \mathbf{1}_{F_\nu(e_1)\cap A_i^\nu\cap B_i},$$

where

$$A_i^\nu := \{\omega_2 \in \Omega_2 \,:\, T_\nu(z_i)(\omega_2) \neq 0\}.$$

Since, up to a subsequence, $\mathbf{1}_{A_i^\nu} \to \mathbf{1}_{A_i}$ μ_2-a.e. on Ω_2 and $\Gamma_{i,\nu}(x_i) \to \Gamma_i(x_i)$ in measure μ_2 on Ω_2, as $\nu \to \infty$, then also $\{\mathbf{1}_{F_\nu(e_1)\cap B_i}\}$ converges in measure μ_2 on Ω_2. The limit is again a characteristic function of some set which we denote by $\Psi_i(e_1)$. Note that $\Psi_i(e_1) \subset B_i$ by construction. Moreover, each Ψ_i: $\Sigma_1 \to \Sigma_2 \cap B_i$ is a σ-homomorphism, since so do all the maps $e_1 \mapsto F_\nu(e_1)\cap B_i$ do. To show that Ψ_i preserves nullsets, it is enough to apply the Vitaly-Hahn-Saks theorem (see section IX.10 of [6]) to the sequence of measures $\{\mu_1^\nu\}$ over Σ_1 defined by $\mu_1^\nu(e_1) := \mu_2(F_\nu(e_1)\cap B_i)$. In fact, according to this theorem the measure μ_1^∞ defined by $\mu_1^\infty(e_1) := \lim_\nu \mu_1^\nu(e_1) = \mu_2(\Psi_i(e_1))$ is absolutely continuous with respect to μ_1, since so so are all μ_1^ν.

Now, noting that each of B_i is disjoint with $\Omega_2\setminus E_2$, we define a map F: $\Sigma_1 \to \Sigma_2$ by

$$F(e_1) := \left(\bigsqcup_{i=1}^{\infty}\Psi_i(e_1)\right)\bigsqcup(F_1(e_1)\setminus E_2).$$

Clearly, F is a σ-homomorphism from Σ_1 to Σ_2. It is also obvious that for every $e_1 \in \Sigma_1$ we have $\mathbf{1}_{F_\nu(e_1)\cap E_2} \to \mathbf{1}_{F(e_1)\cap E_2}$ in measure as $\nu\to\infty$.

Step 3. Observe now that if $x|_{e_1} = y|_{e_1}$, then

$$\mathbf{1}_{E_2}T(\mathbf{1}_{e_1}x) = \lim_{\nu\to\infty}\mathbf{1}_{E_2}T_\nu(\mathbf{1}_{e_1}x) = \lim_{\nu\to\infty}\mathbf{1}_{F_\nu(e_1)\cap E_2}T_\nu(x) = \mathbf{1}_{F(e_1)\cap E_2}T(x)$$

and, analogously, $\mathbf{1}_{E_2}T(\mathbf{1}_{e_1}y) = \mathbf{1}_{F(e_1)\cap E_2}T(y)$, so that $T(x)|_{e_1\cap E_2} = T(y)|_{e_1\cap E_2}$ μ_2-a.e. on Ω_2. On the other hand,

$$T(x)|_{\Omega_2\setminus E_2} = T(y)|_{\Omega_2\setminus E_2} = 0,$$

and hence T is atomic. $\square$

4. Coatomic operators

Another concept which seems to be interesting to study, is given by the following definition. Again, here $X_i := L^{p_i}(\Omega_i,\Sigma_i,\mu_i;\mathcal{X}_i)$, $i = 1,2$ and $0 \leq p_i \leq +\infty$.

Definition 4.1. *An operator $T\colon X_1 \to X_2$ is called* **coatomic**, *if there is a σ-homomorphism $\Phi\colon \Sigma_2 \to \Sigma_1$, satisfying*

$$[\Phi(e_2)] \in Mem_T(\tilde{e}_2)$$

for all $e_2 \in \Sigma_2$, that is, if from $x\,\big|_{\Phi(e_2)} = y\,\big|_{\Phi(e_2)}$ for $x, y \in X_1$ follows $T(x)\,|_{e_2} = T(y)\,|_{e_2}$.

It is not difficult to observe that a notion of a *coatomic* operator is in certain sense dual to the notion of an *atomic* operator. However, both classes contain local operators. Certainly, the class of coatomic operators is wider than that of local operators.

Example 4.1. Assume that $g\colon \Omega_2 \to \Omega_1$ is a bijection satisfying (2) and the inverse function $g^{-1}\colon \Omega_1 \to \Omega_2$ has the same property

$$\mu_1(g(e_2)) = 0 \text{ for } e_2 \in \Sigma_2,\ \mu_2(e_2) = 0. \tag{13}$$

Then the respective shift operator $T_g\colon L^0(\Omega_1) \to L^0(\Omega_2)$ is coatomic, for one can take $\Phi(e) := g(e)$.

In relation to the latter example we observe that, unlike the class of *atomic* operators, the class of *coatomic* operators contains only rather particular shifts, namely, only those described in the example 4.1.

Proposition 4.1. *Let $(\Omega_2, \Sigma_2, \mu_2)$ be a standard measure space and a (Σ_1, Σ_2)-measurable function $g\colon \Omega_2 \to \Omega_1$ satisfy (2). Then the shift operator $T_g\colon L^0(\Omega_1) \to L^0(\Omega_2)$ is coatomic, if and only if g is μ_2-equivalent to a bijection and satisfies (13).*

PROOF: It is only necessity that needs to be proved. Assume that T_g is coatomic with respect to a σ-homomorphism $\Phi\colon \Sigma_2 \to \Sigma_1$. Taking $x = \mathbf{1}_{\Phi(e)}$ for an arbitrary $e \in \Sigma_2$ we observe that $T_g(x) = \mathbf{1}_{g^{-1}(\Phi(e))}$. The definition of coatomic operators yields then $g^{-1}(\Phi(e)) \subset e$. On the other hand, taking $\Omega_2 \setminus e$ instead of e implies

$$\Omega_2 \setminus g^{-1}(\Phi(e)) = g^{-1}(\Phi(\Omega_2)) \setminus g^{-1}(\Phi(e)) = g^{-1}(\Phi(\Omega_2 \setminus e)) \subset \Omega_2 \setminus e,$$

so that $g^{-1}(\Phi(e)) = e$ (up to a μ_2-null set), and the property (13) holds. Since $(\Omega_2, \Sigma_2, \mu_2)$ is a standard measure space, then by theorem 32.3 of [16] there is a function $h\colon \Omega_1 \to \Omega_2$ satisfying $\Phi(e_2) = h^{-1}(e_2)$ for all Borel sets $e_2 \subset \Omega_2$. Define $\varphi := h \circ g\colon \Omega_2 \to \Omega_2$. Noting that $[\varphi^{-1}(e_2)] = [e_2]$ for every Borel set $e_2 \in \Sigma_2$, we see that the shift $T_\varphi\colon L^0(\Omega_2) \to L^0(\Omega_2)$ is an identity operator in $L^0(\Omega_2)$, since it coincides with the identity on the set of simple (i.e. finite-valued) functions. Therefore, $\varphi = T_\varphi(\mathrm{id}) = \mathrm{id}$ μ_2-a.e. on Ω_2, which shows the statement. $\square$

Thus we see that the class of *coatomic* operators does not coincide with that of *atomic* operators. Moreover, it is not contained in the latter one, as we will show using the following statement.

Proposition 4.2. *Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be reflexive Banach spaces. A linear bounded operator $T\colon L^p(\Omega_1; \mathcal{X}_1) \to L^q(\Omega_2; \mathcal{X}_2)$, $1 \le p, q < +\infty$, is coatomic (resp. atomic), if and only if its adjoint $T'\colon L^{q'}(\Omega_2; \mathcal{X}_2') \to L^{p'}(\Omega_1; \mathcal{X}_1')$ is atomic (resp. coatomic).*

PROOF: Let T be *atomic*, while $F\colon \Sigma_1 \to \Sigma_2$ be the respective σ-homomorphism. Consider the identity

$$\int_{\Omega_2} \langle (Tx)(\omega_2), y'(\omega_2) \rangle_2 \, d\mu_2(\omega_2) = \int_{\Omega_1} \langle x(\omega_1), (T'y')(\omega_1) \rangle_1 \, d\mu_1(\omega_1),$$

where $x \in L^p(\Omega_1; \mathcal{X}_1)$, $y' \in L^{q'}(\Omega_2; \mathcal{X}_2')$ and $\langle \cdot, \cdot \rangle_i$ is the natural duality between the spaces $\mathcal{X}_i$ and $\mathcal{X}_i'$. We prove that T' is coatomic showing that $y'|_{F(e_1)} = 0$ implies $(T'y')|_{e_1} = 0$. Supposing the contrary, we choose according to the corollary A.2.1 of [17] an $x \in L^p$ such that

$$\langle x(\omega_1), (T'y')(\omega_1) \rangle_1 \geq \frac{1}{2} |(T'y')(\omega_1)|_{\mathcal{X}_1'}, \text{ if } \omega_1 \in e_1 \text{ and } x|_{\Omega_1 \setminus e_1} = 0,$$

thus having $(Tx)|_{\Omega_2 \setminus F(e_1)} = 0$, and obtain the contradiction

$$0 = \frac{1}{2} \int_{e_1} |T'y'(\omega_1)|_{\mathcal{X}_1'} \, d\mu_1(\omega_1) > 0.$$

If T is *coatomic*, Φ: $\Sigma_2 \to \Sigma_1$ being the respective σ-homomorphism, then we are to prove that $y'|_{e_2} = 0$ implies $(T'y')|_{\Phi(e_2)} = 0$. Supposing the contrary, we choose x such that

$$\langle x(\omega_1), (T'y')(\omega_1) \rangle_1 \geq \frac{1}{2} |(T'y')(\omega_1)|_{\mathcal{X}_1'}, \text{ if } \omega_1 \in \Phi(e_2) \text{ and } x|_{\Phi(\Omega_2 \setminus e_2)} = 0,$$

thus having $(Tx)|_{\Omega_2 \setminus e_2} = 0$, and obtain the contradiction

$$0 = \frac{1}{2} \int_{\Phi(e_2)} |T'y'(\omega_1)|_{\mathcal{X}_1'} \, d\mu_1(\omega_1) > 0.$$

The "only if" part of the statement is proved in a completely symmetric way by interchanging T and T'. □

From the latter proposition, which is of independent interest, we may conclude that, roughly speaking, while the class of *atomic* operators contains all the shift operators between Lebesgue spaces, the class of *coatomic* operators contains all their adjoints. Now we are able to show that there are *coatomic* operators, which are not *atomic*.

Example 4.2. Let $\Omega = (0, 1)$, Σ be the σ-algebra of Lebesgue-measurable subsets of $(0, 1)$, μ be the Lebesgue measure. Consider the function

$$g(\omega) := \sin^2(1/\omega).$$

The corresponding shift operator T_g: $L^\infty(\Omega) \to L^\infty(\Omega)$ is order continuous and, therefore, continuous in $*$-weak topology $\sigma(L^\infty, L^1)$. Then there is a dual operator (i.e. adjoint with respect to the duality) T: $L^1(\Omega) \to L^1(\Omega)$ such that $T' = T_g$. As it has been proven, that T_g is atomic (being a shift), then by proposition 4.2 we have that T is coatomic. However, it cannot be atomic, because otherwise by the same proposition T_g would be coatomic.

Much like the class of *atomic* operators, the class of *coatomic* operators is also obviously closed with respect to compositions, but not closed with respect to finite sums. In fact, the operator T: $L^2(\Omega) \to L^2(\Omega)$ defined in the example 3.3 is self-adjoint. Since it is not atomic, it is by proposition 4.2 neither coatomic, although both of the summands are.

We give now an interesting example of a coatomic operator arising in probability theory.

Example 4.3. Let $\Sigma_1 \subset \Sigma_2$ be two different σ-algebrae on the same set Ω, the probability $\mathbb{P}$ being defined on Σ_2. For every $1 \leq p \leq \infty$ the natural imbedding $\imath$: $L^p(\Omega, \Sigma_1, \mathbb{P}) \to L^p(\Omega, \Sigma_2, \mathbb{P})$ is an atomic operator, the respective σ-homomorphism F: $\Sigma_1 \to \Sigma_2$ being induced by the imbedding of Σ_1 into Σ_2. Its adjoint, which by proposition 4.2 is coatomic, is the operator of *conditional expectation* $\mathcal{E}(\cdot\,|\Sigma_1)$.

4.1. **Representation.** The following assertion gives the relation between coatomic operators, local operators and shifts.

Theorem 4.1. *For every coatomic operator $T\colon L^0(\Omega_1;\mathcal{X}_1) \to L^0(\Omega_2;\mathcal{X}_2)$ the operator*

$$N := T \circ T_\Phi \,:\, L^0(\Omega_2;\mathcal{X}_1) \to L^0(\Omega_2;\mathcal{X}_2),$$

where $\Phi\colon \Sigma_2 \to \Sigma_1$ is the respective σ-homomorphism, is local. The continuity of T (in measure), implies that of the operator N. Moreover, if $(\Omega_1, \Sigma_1, \mu_1)$ is a standard measure space, then there is a (Σ_2, Σ_1)-measurable function $g\colon \Omega_1 \to \Omega_2$ satisfying (2), such that $N = T \circ T_g$.

PROOF: The generalized shift $T_\Phi\colon L^0(\Omega_2;\mathcal{X}_1) \to L^0(\Omega_1;\mathcal{X}_1)$ is defined and is automatically continuous by lemma 3.1. It is clear now that $N = T \circ T_\Phi\colon L^0(\Omega_1;\mathcal{X}_1) \to L^0(\Omega_2;\mathcal{X}_2)$ is local. The continuity of N is evidently implied by that of T. If $(\Omega_1, \Sigma_1, \mu_1)$ is a standard measure space, then as in the proof of the theorem 3.1, we construct a (Σ_2, Σ_1)-measurable function $g\colon \Omega_1 \to \Omega_2$ satisfying

$$\tilde{\Phi}(\tilde{e}_2) = [g^{-1}(e_2)] \text{ for all } e_2 \in \Sigma_2,$$

and hence also (2), and observe that the latter relationship implies $T_\Phi = T_g$. □

Corollary 4.1. *Let $(\Omega_1, \Sigma_1, \mu_1)$ be a standard measure space, $\mathcal{X}_i$ be separable metric spaces. Under the continuum hypothesis one can find for every atomic operator $T\colon L^0(\Omega_1;\mathcal{X}_1) \to L^0(\Omega_2;\mathcal{X}_2)$ a measurable function $g\colon \Omega_2 \to \Omega_1$ satisfying (2) and a function $f\colon \Omega_1 \times \mathcal{X}_1 \to \mathcal{X}_2$ such that $T \circ T_g = N_f$, i.e. for every $x \in L^0(\Omega_2;\mathcal{X}_1)$ one has*

$$(Tx)(g(\omega_2)) = f(\omega_2, x(\omega_2)) \qquad \mu_2 - a.e. \text{ on } \Omega_2.$$

Moreover, if T is continuous in measure, then (independently of the validity of continuum hypothesis) the function f may be assumed a Carathéodory function.

4.2. **Acting, boundedness and continuity.** Unlike atomic, but similar to Nemytskiĭ operators, the coatomic operators possess better continuity properties in Lebesgue spaces. In this paragraph we assume that μ_1 and μ_2 are nonatomic, $\mathcal{X}_1$ and $\mathcal{X}_2$ are separable Banach spaces.

Theorem 4.2. *If a coatomic operator $T\colon L^p(\Omega_1;\mathcal{X}_1) \to L^q(\Omega_2;\mathcal{X}_2)$, where $1 \leq p, q < \infty$, maps L^p-convergent sequences into measure convergent ones, then it is continuous as an operator between these two spaces.*

PROOF:

Step 1. Assuming without loss of generality that $T(0) = 0$, we suppose that continuity we need fails. Then there is a converging sequence $\{x_\nu\} \subset L^p(\Omega_1;\mathcal{X}_1)$ such that the sequence $\{T(x_\nu)\}$ does not converge in $L^q(\Omega_2;\mathcal{X}_2)$. Denote

$$\gamma(\delta) := \sup\{\|\mathbf{1}_{e_1} x_\nu\|_p\,,\ e_1 \in \Sigma_1,\ \mu_1(e_1) < \delta,\ \nu \in \mathbb{N}\}$$

where $\|\cdot\|_p$ stands for the norm in the space $L^p(\Omega_1;\mathcal{X}_1)$. Since $\{x_\nu\}$ is norm convergent, hence equiintegrable, we have that $\gamma(\delta) \to 0$ as $\delta \to 0$. Moreover, since $\{T(x_\nu)\} \subset L^q(\Omega_2;\mathcal{X}_2)$ is not convergent, we can choose an appropriate subsequence of $\{x_\nu\}$, denoted further by the same index, and a respective family $\{E_\nu\} \subset \Sigma_2$, so that for some $\varepsilon > 0$ one has

$$\|\mathbf{1}_{E_\nu} T(x_\nu)\|_q > \varepsilon, \text{ while } \gamma(\mu_1(\Phi(E_\nu))) < 2^{-\nu} \text{ and } \mu_2(E_\nu) < 2^{-\nu}.$$

Step 2. Define by induction a sequence $\{G_k\} \subset \Sigma_2$ such that

(i) G_k are pairwise disjoint;
(ii) $\mu_2(G_k) \to 0$ as $k \to \infty$;
(iii) $\|\mathbf{1}_{G_k} T(x_{\nu_k})\|_q > \varepsilon/2$ for some subsequence $\{x_{\nu_k}\}$;
(iv) $\left(\bigcup_{j=1}^{k} G_j\right) \bigcap \left(\bigcup_{j=\nu_k}^{\infty} E_j\right) = \emptyset$ for some $\nu_k \in \mathbb{N}$.

If $k = 1$, choose $G_1 := E_1 \setminus \bigcup_{i=\nu_1}^{\infty} E_\nu$, where $\nu_1 \in \mathbb{N}$ is large enough to provide

$$\left\| \mathbf{1}_{\cup_{j=\nu_1}^{\infty} E_j} T(x_1) \right\|_q < \varepsilon/2.$$

The choice of such $\nu_1 \in \mathbb{N}$ is possible, since by construction $\mu_2(\bigcup_{j=\nu}^{\infty} E_j) \to 0$. If there already exists a family $G_1, \dots, G_k$, satisfying (i)–(iv), then we again can find a number ν_{k+1} such that

$$\left\| \mathbf{1}_{\cup_{j=\nu_{k+1}}^{\infty} E_j} T(x_{k+1}) \right\|_q < \frac{\varepsilon}{2},$$

and then we set

$$G_{k+1} := E_{\nu_k} \setminus \bigcup_{\nu=\nu_{k+1}}^{\infty} E_\nu.$$

Step 3. Since $\gamma(\mu_1(\Phi(E_\nu))) < 2^{-\nu}$ we have also that $\sum_{k=1}^{\infty} \gamma(\mu_1 \Phi(G_k)) < \infty$. Therefore, a measurable function $y := \sum_{k=1}^{\infty} \mathbf{1}_{\Phi(G_k)} x_{\nu_k}$ belongs to the space $L^p(\Omega_1; \mathcal{X}_1)$, because

$$\|y\|_p \le \sum_{k=1}^{\infty} \left\| \mathbf{1}_{\Phi(G_k)} x_{\nu_k} \right\|_p \le \sum_{k=1}^{\infty} \gamma(\mu_1(\Phi(G_k))) < \infty.$$

On the other hand, since T is coatomic, one has

$$\|T(y)\|_q^q = \left\| \sum_{k=1}^{\infty} \mathbf{1}_{G_k} T(x_{\nu_k}) \right\|_q^q = \sum_{k=1}^{\infty} \|\mathbf{1}_{G_k} T(x_{\nu_k})\|_q^q = \infty,$$

because each summand in the right-hand side is greater than $\varepsilon/2$. This contradiction completes the proof. □

4.3. **Noncompactness.** The *coatomic* operators acting in ideal function spaces, possess similarly to local and atomic operators, the property of noncompactness.

Theorem 4.3. *Assume $0 \le p, q \le \infty$ and $\mathcal{X}_i$ to be separable Banach spaces. A nonconstant coatomic operator $T\colon L^p(\Omega_1; \mathcal{X}_1) \to L^q(\Omega_2; \mathcal{X}_2)$, $1 \le p, q \le +\infty$, is not compact, if μ_2 is a nonatomic measure.*

PROOF: Again, as in the proof of the theorem 3.3, we choose $x \in L^p(\Omega_1; \mathcal{X}_1)$ and $E_2 \in \Sigma_2$, $\mu_2(E_2) > 0$, such that $(T(x))(\omega_2) \ne (T(0))(\omega_2)$ for all $\omega_2 \in E_2$, and assume without loss of generality that $T(0) = 0$. Now in a completely analogous way with the proof of the theorem 3.3 construct a sequence $\{E_2(k)\}_{k=1}^{\infty}$ of measurable subsets of E_2 with the properties

$$\mu_2(E_2(k)) = \mu_2(E_2)/2 \text{ and } \mu_2(E_2(k) \Delta E_2(j)) = \mu_2(E_2)/2$$

for all $k \in \mathbb{N}$ and for all $j \in \mathbb{N}$, $j \ne k$. The sequence of functions $x_k := \mathbf{1}_{\Phi(E_2(k))} x$ is bounded in $L^p(\Omega_1; \mathcal{X}_1)$, but the sequence of the images $T(x_k)$ is not precompact in $L^q(\Omega_2; \mathcal{X}_2)$, because

$$|(Tx_j)(\omega_2) - (Tx_k)(\omega_2)|_{\mathcal{X}_2} = \mathbf{1}_{E_2(j) \Delta E_2(k)} |T(x)|_{\mathcal{X}_2},$$

while for $j < k$ the latter is equal to $|T(x)|_{\mathcal{X}_2} > 0$ on the set $E_2(j)\Delta E_2(k)$ of the measure $\mu_2(E_2)/2$.□

4.4. **Weak continuity.** Like both local and atomic operators, coatomic operators are as a rule degenerate in weak topology of L^p.

Theorem 4.4. *Let $1 \leq p, q < \infty$, μ_2 be a nonatomic measure and $\mathcal{X}_i$ be separable Banach spaces. Then a coatomic operator T: $L^p(\Omega_1; \mathcal{X}_1) \to L^q(\Omega_2; \mathcal{X}_2)$ is weakly continuous, if and only if T is affine, i.e. $T(\cdot) - T(0)$ is a linear bounded operator.*

The PROOF is completely analogous to that of the theorem 3.4.

4.5. **Convergence.** Convergence properties of coatomic operators are much worse than those of atomic operators. We first consider linear coatomic operators.

Example 4.4. Let $\Omega = (0, 1)$ and the functions g^ν: $\Omega \to \mathbb{R}$, $\nu \in \mathbb{N}$, be defined as

$$g^\nu(\omega) = \nu\omega - j, \quad j/\nu \leq \omega < (j + 1)/\nu, \qquad j = 0, \ldots , \nu - 1.$$

In [9] it has been noted that the operators T_{g^ν}: $L^2 \to L^2$, converge to an operator T: $L^2 \to L^2$ given by the formula

$$(Tx)(\omega) := \int_\Omega x(\omega')\, d\omega' \cdot 1(\omega)$$

in such a way, that their duals T'_{g^ν}: $L^2 \to L^2$ converge strongly (pointwise) to the dual T': $L^2 \to L^2$ of the operator T. In fact, to prove this convergence, it is enough to verify that for every weakly convergent sequence $\{x_\nu\} \subset L^2$, $x_\nu \rightharpoonup x$, one has $T_{g^\nu} x_\nu \rightharpoonup Tx$ (see [9]). We see now that all T'_{g^ν} are coatomic as duals to the shift operators, while their strong limit $T' = T$ is obviously not coatomic.

Nevertheless, the following proposition concerning convergence of a sequence of linear coatomic operators can be obtained.

Proposition 4.3. *A uniform limit T of a sequence of linear bounded coatomic operators T_ν: $L^p(\Omega_1; \mathcal{X}_1) \to L^q(\Omega_2; \mathcal{X}_2)$, where $1 \leq p, q < \infty$, is coatomic.*

PROOF: The assumptions imply that the sequence of duals $\{T'_\nu\}$ converges strongly to a bounded linear operator T'. By the proposition 4.2, the operators T'_ν are atomic and therefore, by the theorem 3.5, so is T'. Again applying the proposition 4.2, we complete the proof. □

Unfortunately, nothing can be said about convergence of nonlinear coatomic operators, as the example below shows.

Example 4.5. Take $\Omega_1 := \mathbb{N}$, $\Omega_2 := \{0\} \cup \mathbb{N}$, $\Sigma_i := 2^{\Omega_i}$, $\mu_1(\{i\}) := 2^{-i}$ $(i \in \Omega_1)$, $\mu_2(\{i\}) := 2^{-i}$ $(i \in \Omega_2)$. Consider some continuous function φ: $\mathbb{R} \to [0, 1]$ and define the sequence of operators T_ν: $L^p(\Omega_1; \mathbb{R}) \to L^q(\Omega_2; \mathbb{R})$ by the relationships

$$\bar{T}_\nu(\{\xi_i\}) := 2^{-\nu-1}\{2^{\nu+1}\bar{\xi}_\nu, \xi_{\nu+1}, \xi_{\nu+2}, \ldots\} \text{ and } T_\nu(\{\xi_i\}) := \bar{T}_\nu(\{\varphi(\xi_i)\}).$$

Here $\{\xi_i\}$ is an arbitrary element from $L^p(\Omega_1, \Sigma_1, \mu_1; \mathcal{X}_1)$ which consists of sequences indexed by the natural numbers, while $T_\nu(\{\xi_i\})$ is a sequence indexed by the nonnegative integers,

$$\bar{\xi}_\nu := \frac{\sum_{i=1}^\nu 2^{-i}\xi_i}{1 - 2^{-\nu-1}}.$$

Clearly, the sequence of the operators T_ν converges uniformly operator

$$T(\{\xi_i\}) := \{\bar{\xi}, 0, 0, ...\},$$

where $\bar{\xi} = \sum_{i=1}^{\infty} 2^{-i}\varphi(\xi_i)$, which is obviously not coatomic. On the other hand, T_ν is coatomic with respect to the σ-homomorphism defined by

$$\Phi_\nu(\{i\}) := \begin{cases} \{i+\nu\}, & i \geq 1, \\ \{1,2,\ldots,\nu\}, & i = 0. \end{cases}$$

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