EVOLUTION AND MEMORY EFFECTS IN THE HOMOGENIZATION LIMIT FOR ELECTRICAL CONDUCTION IN BIOLOGICAL TISSUES

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Abstract. We study a problem set in a finely mixed periodic medium, modelling electrical conduction in biological tissues. The unknown electric potential solves standard elliptic equations set in different conductive regions (the intracellular and extracellular spaces), separated by a dielectric surface (the cell membranes), which exhibits both a capacitive and a nonlinear conductive behaviour. Accordingly, dynamical conditions prevail on the membranes, so that the dependence of the solution on the time variable t is not only of parametric character.
As the spatial period of the medium goes to zero, the eletric potential approaches a homogenization limit u_0, solving

$$\operatorname{div}\Big[-\sigma_0\nabla_x u_0 - A^0\nabla_x u_0 - \int_0^t A^1(t-\tau)\nabla_x u_0(x,\tau)\,\mathrm{d}\tau + \mathcal{F}(x,t)\Big] = 0\,,$$

where $\sigma_0 > 0$ and the matrices A^0, A^1 depend on the properties of the tissue, and the vector function $\mathcal{F}$ keeps trace of the initial data of the original problem. In the limit, the current, given as the term in square brackets in the PDE above, is still divergence-free, but it depends on the history of the potential gradient, so that memory effects explicitly appear.
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1. Introduction

We consider a model for the electrical conduction in a medium composed of two different conductive phases, separated by a dielectric interface. The mathematical scheme consists in partial differential equations of elliptic type prescribed in each phase, complemented with suitable boundary conditions at the interface, and at the boundary of the spatial domain. The unknown function is the electric potential.
Since the problem evolves in time, we have a family of elliptic problems parametrized by time; but the dependence of the unknown on time is not merely parametrical. Indeed, in order to take into account the conductive/capacitive behaviour of the interface, the potential jumps across the interface, and the jump satisfies a dynamical

condition (roughly speaking, in the form of a hyperbolic differential equation on the interface itself, see (2.4)).

The physical framework we have described is most obviously applied to electrical conduction in biological tissues [22], where one of the phases is the extracellular space, the other one is the intracellular space, and the interface represents the cell membranes. It is known that cell membranes may exhibit such a nonlinear conductive behaviour [25] taken into account by the function f in (2.4). See Subsection 2.1 for more details. Our model is designed to investigate the response of biological tissues to the injection of electrical currents in the radiofrequency range, that is the Maxwell–Wagner interfacial polarization effect [13], [22]. This effect is relevant in clinical applications like electric tomography and body composition [15], [19]. The applicative interest of the model described here is treated in [3].

In view of the applications we have in mind, we assume that the two phases are finely mixed with a microscopic periodic structure, so that the problem contains a small parameter ε, coinciding with the period of the microstructure. We investigate the homogenization limit when we let $\varepsilon \to 0$. Extensive surveys on this topic are, e.g., in [10], [11], [12], [14], [17], [24], [29], [31], [33], [34]. In view of the applications it is of interest to study the evolution in time of the homogenized potential. It turns out that the partial differential equation obtained in the limit is non standard (see (2.12) below), indeed it is an elliptic equation exhibiting memory effects, i.e., it contains explicitly the history of the unknown. In connection with the applications mentioned above, we remark that the limit equation (2.12) is markedly different from the Laplace equation used as a standard in the bioelectrical impedance literature [15].

The rigorous proof of this limiting behaviour of the approximating problem relies technically on the introduction of a non standard kind of cell functions (containing memory terms), which we identify through the two scales approach (see Section 3).

Homogenization problems leading to the onset of memory terms are treated e.g., in [2], [8], [16], [23], [34], [35] (see also the references therein). However the homogenization process here is characterized by the presence of interfaces carrying a peculiar kind of evolutive differential equations.

Our model can be compared, from the mathematical point of view, to some papers where homogenization theory is applied to linear stationary elliptic problems involving imperfect interfaces, arising in fields like elasticity [28], or heat conduction [30]. Our method differs from the variational approach of [30], and from the use of extension techniques of [28]. Further remarks are given in Subsection 2.2. The main novelty here, with respect to the just mentioned pieces of literature, lies in the fact that those authors were concerned with stationary problems, and therefore no evolutionary behaviour was investigated. However, in the linear case where in (2.4) one has $f(s) = s$, the evolutive problem can be reduced to the stationary setting by means of Laplace's transform, see [6]. We stress the fact that our approach covers also the case where the partial differential equations are nonlinear (see Remark 2.4).

We note that our model is different from the "bidomain model" for the electrical activation of cardiac muscle cells (see [18], [26]), which deals with different length and time scales, therefore resulting in a different scaling in the interface condition.

Indeed, the homogenized bidomain model consists of a degenerate system of parabolic equations, while our model yields in the limit an elliptic problem with memory effects.

1.1. **Content of the paper.** In Section 2 we set the problem, and give our main results, along with the main ideas of the proofs. In Section 3 we apply the two scales method and find the cell functions, formally identifying the limit equation, whose structure is investigated in Section 4.
The limiting behaviour of our model is rigorously determined in Section 5. An estimate of the speed of convergence is found in Section 6. Finally, Section 7 is devoted to technical and auxiliary results.

2. The geometrical setting. Main results

Let Ω be an open connected bounded subset of $\boldsymbol{R}^N$, and let $\Omega = \Omega_1^\varepsilon \cup \Omega_2^\varepsilon \cup \Gamma^\varepsilon$, where Ω_1^ε and Ω_2^ε are two disjoint open subsets of Ω, and $\Gamma^\varepsilon = \partial\Omega_1^\varepsilon \cap \Omega = \partial\Omega_2^\varepsilon \cap \Omega$. Let also $T > 0$ be a given time.
We are interested in the homogenization limit as $\varepsilon \searrow 0$ of the problem for $u_\varepsilon(x,t)$ (here the operators div and ∇ act only with respect to the space variable x)

$$-\operatorname{div}(\sigma_1 \nabla u_\varepsilon) = 0\,, \qquad \text{in } \Omega_1^\varepsilon; \tag{2.1}$$

$$-\operatorname{div}(\sigma_2 \nabla u_\varepsilon) = 0\,, \qquad \text{in } \Omega_2^\varepsilon; \tag{2.2}$$

$$\sigma_1 \nabla u_\varepsilon^{(\mathrm{int})} \cdot \nu = \sigma_2 \nabla u_\varepsilon^{(\mathrm{out})} \cdot \nu\,, \qquad \text{on } \Gamma^\varepsilon; \tag{2.3}$$

$$\frac{\alpha}{\varepsilon}\frac{\partial}{\partial t}[u_\varepsilon] + \frac{1}{\varepsilon} f([u_\varepsilon]) = \sigma_2 \nabla u_\varepsilon^{(\mathrm{out})} \cdot \nu\,, \qquad \text{on } \Gamma^\varepsilon; \tag{2.4}$$

$$[u_\varepsilon](x,0) = S_\varepsilon(x)\,, \qquad \text{on } \Gamma^\varepsilon; \tag{2.5}$$

$$u_\varepsilon(x,t) = 0\,, \qquad \text{on } \partial\Omega. \tag{2.6}$$

The notation in (2.1)–(2.4), (2.6), means that the indicated equations are in force in the relevant spatial domain for $0 < t < T$.
Here σ_1, σ_2 and α are positive constants, and ν is the normal unit vector to Γ^ε pointing into Ω_2^ε. Since u_ε is not in general continuous across Γ^ε we have set

$$u_\varepsilon^{(\mathrm{int})} := \text{ trace of } u_{\varepsilon|\Omega_1^\varepsilon} \text{ on } \Gamma^\varepsilon; \quad u_\varepsilon^{(\mathrm{out})} := \text{ trace of } u_{\varepsilon|\Omega_2^\varepsilon} \text{ on } \Gamma^\varepsilon.$$

Indeed we refer conventionally to Ω_1^ε as to the *interior domain*, and to Ω_2^ε as to the *outer domain*. We also denote

$$[u_\varepsilon] := u_\varepsilon^{(\mathrm{out})} - u_\varepsilon^{(\mathrm{int})}\,.$$

Similar conventions are employed for other quantities; for example (2.3) can be rewritten as

$$[\sigma \nabla u_\varepsilon \cdot \nu] = 0\,, \qquad \text{on } \Gamma^\varepsilon,$$

where

$$\sigma = \sigma_1 \quad \text{in } \Omega_1^\varepsilon, \qquad \sigma = \sigma_2 \quad \text{in } \Omega_2^\varepsilon.$$

The function f fulfils

$$f \in C^2(\boldsymbol{R})\,, \quad f'\,, f'' \in L^\infty(\boldsymbol{R})\,, \qquad f(0) = 0\,. \tag{2.7}$$

The initial data S_ε will be discussed below.

In order to be more specific about the geometry of the domains of interest, let us introduce a periodic open subset E of $\boldsymbol{R}^N$, so that $E + z = E$ for all $z \in \boldsymbol{Z}^N$. For all $\varepsilon > 0$ define $\Omega_1^\varepsilon = \Omega \cap \varepsilon E$, $\Omega_2^\varepsilon = \Omega \setminus \overline{\varepsilon E}$. We assume that Ω, E have regular boundary, say of class C^∞ for the sake of simplicity. We also employ the notation $Y = (0,1)^N$, and $E_1 = E \cap Y$, $E_2 = Y \setminus \overline{E}$, $\Gamma = \partial E \cap \overline{Y}$. As a simplifying assumption, we stipulate that $|\Gamma \cap \partial Y|_{N-1} = 0$.

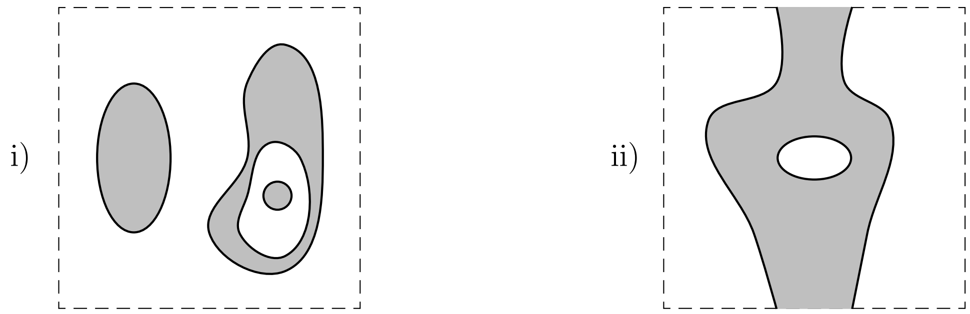

FIGURE 1. Two examples of admissible periodic structures in $\boldsymbol{R}^2$. In both cases Y is the dashed square, and $E \cap Y$ is the shaded region.

Essentially, we will show that, if $\gamma^{-1}\varepsilon \le S_\varepsilon(x) \le \gamma\varepsilon$, where S_ε is the initial jump prescribed in (2.5), for a fixed constant $\gamma > 1$, then u_ε becomes stable as $\varepsilon \to 0$ (i.e., it converges to a nonvanishing bounded function). Therefore, let us stipulate that

$$S_\varepsilon(x) = \varepsilon S_1\big(x, \frac{x}{\varepsilon}\big) + \varepsilon R_\varepsilon(x) \,, \tag{2.8}$$

where $S_1 : \Omega \times \partial E \to \boldsymbol{R}$, and

$$\begin{aligned} &\|S_1\|_{L^\infty(\Omega \times \partial E)} < \infty \,, \qquad \|R_\varepsilon\|_{L^\infty(\Omega)} \to 0 \,, \text{ as } \varepsilon \to 0; \\ &S_1(x,y) \text{ is continuous in } x, \text{ uniformly over } y \in \partial E, \\ &\text{and periodic in } y, \text{ for each } x \in \Omega. \end{aligned} \tag{2.9}$$

In [5], under the assumptions above, we prove existence and uniqueness of a weak solution to (2.1)–(2.6), in the class

$$u_{\varepsilon|\Omega_i^\varepsilon} \in L^2(0,T;H^1(\Omega_i^\varepsilon)) \,, \qquad i = 1\,, 2\,, \tag{2.10}$$

and $u_{\varepsilon|\partial\Omega} = 0$ in the sense of traces. The weak formulation of the problem is

$$\begin{aligned} \int_0^T \int_\Omega \sigma \nabla u_\varepsilon \cdot \nabla \psi \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{\varepsilon} \int_0^T \int_{\Gamma^\varepsilon} f([u_\varepsilon])[\psi] \,\mathrm{d}\sigma\,\mathrm{d}t \\ - \frac{\alpha}{\varepsilon} \int_0^T \int_{\Gamma^\varepsilon} [u_\varepsilon] \frac{\partial}{\partial t}[\psi] \,\mathrm{d}\sigma\,\mathrm{d}t - \frac{\alpha}{\varepsilon} \int_{\Gamma^\varepsilon} [u_\varepsilon](0)[\psi](0) \,\mathrm{d}\sigma = 0 \,, \end{aligned} \tag{2.11}$$

for each $\psi \in L^2(\Omega \times (0,T))$ such that ψ is in the class (2.10), $[\psi] \in H^1(0,T;L^2(\Gamma^\varepsilon))$, and ψ vanishes on $\partial\Omega \times (0,T)$, as well as at $t = T$.

In Section 3 we give a formal asymptotic expansion of the unknown function u_ε in powers of ε (see e.g., [12], [29], [33])

$$u_\varepsilon = u_\varepsilon(x,y,t) = u_0(x,y,t) + \varepsilon u_1(x,y,t) + \varepsilon^2 u_2(x,y,t) + \dots ,$$

where $y \in Y$, $y = x/\varepsilon$ is the microscopic variable. Here u_0, u_1, u_2 are periodic in y, and u_1, u_2 are assumed to have zero integral average over Y.

The coefficients of such an expansion are represented in terms of cell functions, i.e., periodic functions of the microscopic variable. In particular, u_1 is represented in the form

$$u_1(x,y,t) = -\chi^0(y) \cdot \nabla_x u_0(x,t) + \mathcal{T}(S_1(x,\cdot))(y,t) - \int_0^t \nabla_x u_0(x,\tau) \cdot \chi^1(y,t-\tau)\,\mathrm{d}\tau .$$

The definition of the cell function $\chi^0 : Y \to \boldsymbol{R}^N$ is standard. In addition to this function, a new cell function $\chi^1 : Y \times (0,T) \to \boldsymbol{R}^N$ is required, owing to the dynamical boundary condition (2.4). Its definition involves a transform $\mathcal{T}$, which plays an essential role in the following. The transform $\mathcal{T}$ is defined by

$$\mathcal{T}(s)(y,t) = v(y,t) , \qquad y \in Y , \; t > 0 ,$$

where $s : \Gamma \to \boldsymbol{R}$, and v is a periodic null-average function in Y, solving the problem

$$\begin{aligned}
-\sigma\,\Delta_y v &= 0 , && \text{in } E_1,\ E_2; \\
[\sigma \nabla_y v \cdot \nu] &= 0 , && \text{on } \Gamma; \\
\alpha \frac{\partial}{\partial t}[v] + f'(0)[v] &= \sigma_2 \nabla_y v^{(\mathrm{out})} \cdot \nu , && \text{on } \Gamma; \\
[v](y,0) &= s(y) , && \text{on } \Gamma.
\end{aligned}$$

From the point of view of physics, s can be interpreted as an initial potential jump across Γ; $\mathcal{T}$ associates to this initial data, the evolution of the potential itself, in the process determining the discharge of the membrane in the unit cell Y under periodic boundary conditions and in the linear approximation of f.

Then we set

$$\alpha \chi^1_h = \mathcal{T}\big(\sigma_2 (\nabla_y \chi_h^{0(\mathrm{out})} - \boldsymbol{e}_h) \cdot \nu\big) .$$

Memory effects appear in the homogenized equation just as a consequence of the transform $\mathcal{T}$.

As usual, the limit equation is then found as a solvability condition for a certain "microscopic" differential problem. Actually, the limit function u_0 solves the equation

$$-\operatorname{div}\left(\sigma_0 \nabla_x u_0 + A^0 \nabla_x u_0 + \int_0^t A^1(t-\tau) \nabla_x u_0(x,\tau)\,\mathrm{d}\tau - \mathcal{F}(x,t)\right) = 0 , \qquad (2.12)$$

where $\sigma_0 = \int_Y \sigma$, and the two matrices A^0, A^1 are defined in (3.31), while $\mathcal{F}$ is a vector function keeping trace of the initial data S_1, defined in (4.18). The evolution in time of A^1 and of $\mathcal{F}$ is ruled by the transform $\mathcal{T}$.
In Section 4 we show that A^0 and A^1 are symmetric, and that $\sigma_0 I + A^0$ is positive definite, so that (2.12) is indeed of elliptic type.
In Section 5 we prove our main result:

Theorem 2.1. *Under the assumptions listed above, as $\varepsilon \to 0$, we have that $u_\varepsilon \to u_0$, weakly in $L^2(\Omega \times (0,T))$, and strongly in $L^1_{\mathrm{loc}}(0,T; L^1(\Omega))$, where the limit $u_0 \in L^2(0,T; H^1_o(\Omega))$ solves* (2.12).

We rely there on the use of suitable testing functions, constructed again via the $\mathcal{T}$ transform, and eventually responsible of the appearance of memory terms in the limit. An essential ingredient in the proof, i.e., the compactness in L^1 of $\{u_\varepsilon\}$, is provided by Lemma 7.7. Then we prove that u_0 has a vanishing trace on $\partial\Omega$, at all time levels, using the energy inequality (2.25) and some facts from BV theory.
We also obtain the following error estimate, which we state for the sake of simplicity under redundant regularity assumptions

Theorem 2.2. *In addition to the assumptions of Theorem 2.1, we require that $R_\varepsilon(x) = \varepsilon S_2(x, x/\varepsilon) + \varepsilon R^1_\varepsilon(x)$, where $\|R^1_\varepsilon\|_{L^\infty(\Omega)} \to 0$ as $\varepsilon \to 0$. Moreover, $S_2(x,y)$ is periodic in y, and of class $C^1(\overline{\Omega}, C^\infty(\Gamma))$, while S_1 is of class $C^{2+\lambda}(\overline{\Omega}, C^\infty(\Gamma))$, for a given $\lambda \in (0,1)$. Finally let $\operatorname{dist}(\partial\Omega, \Gamma^\varepsilon) \geq \gamma_0 \varepsilon$. Then*

$$\|u_\varepsilon - u_0\|_{L^2(\Omega\times(0,T))} \leq \gamma\sqrt{\varepsilon}\,, \tag{2.13}$$

where γ, γ_0 are positive constants not depending on ε.

The proof of this result is contained in Section 6. See also Remark 6.1 for the case when $\partial\Omega \cap \Gamma^\varepsilon \neq \emptyset$.
We have collected in Section 7 some needed technical results. For further information on evolutive problems with boundary conditions involving the time derivative we refer the reader to [9], [20], [21], [32], and to the references therein.

Remark 2.3. By means of minor changes in our approach, we may consider cases with non vanishing sources appearing on the right hand sides of (2.1)–(2.4).
Of special interest in applications is the case of nonvanishing Dirichlet data, where (2.6) is replaced with

$$u_\varepsilon(x,t) = \hat{u}(x,t)\,, \quad \text{on } \partial\Omega, \text{ where } \hat{u} \in L^2(0,T; H^2(\Omega)) \cap H^1(0,T; H^1(\Omega)). \tag{2.14}$$

In this case we look at the homogeneous Dirichlet problem for $v_\varepsilon = u_\varepsilon - \hat{u}$, i.e.,

$$-\operatorname{div}(\sigma \nabla v_\varepsilon) = \sigma\, \Delta\, \hat{u}\,, \qquad \text{in } \Omega^\varepsilon_1,\ \Omega^\varepsilon_2; \tag{2.15}$$

$$[\sigma \nabla v_\varepsilon \cdot \nu] = -[\sigma]\nabla \hat{u} \cdot \nu\,, \qquad \text{on } \Gamma^\varepsilon; \tag{2.16}$$

$$\frac{\alpha}{\varepsilon}\frac{\partial}{\partial t}[v_\varepsilon] + \frac{1}{\varepsilon} f([v_\varepsilon]) = \sigma_2 \nabla v^{(\mathrm{out})}_\varepsilon \cdot \nu + \sigma_2 \nabla \hat{u} \cdot \nu\,, \qquad \text{on } \Gamma^\varepsilon; \tag{2.17}$$

$$[v_\varepsilon](x,0) = S_\varepsilon(x)\,, \qquad \text{on } \Gamma^\varepsilon; \tag{2.18}$$

$$v_\varepsilon(x,t) = 0\,, \qquad \text{on } \partial\Omega. \tag{2.19}$$

Arguing as in Section 5 below, one can show that v_ε converges to a function $v_0 \in L^2(0,T;H^1_o(\Omega))$ such that $u_0 = v_0 + \hat{u}$ solves (2.12).

Remark 2.4. We also consider the case when the elliptic equations in (2.1), (2.2) contain nonlinear terms. More specifically, our approach covers the case when

$$-\sigma\,\Delta\, u_\varepsilon = g(x,t,u_\varepsilon)\,, \qquad \text{in } \Omega^\varepsilon_1,\ \Omega^\varepsilon_2, \tag{2.20}$$

and (2.3)–(2.6) stay the same. Here g is a Caratheodory function such that

$$|g(x,t,s)| \le L|s| + L_0\,, \tag{2.21}$$

$$|g(x,t,s) - g(x,t,s')| \le L|s - s'|\,, \tag{2.22}$$

for all $x \in \Omega$, $t > 0$, s, $s' \in \boldsymbol{R}$, for constants L_0, $L > 0$ independent of x, t, s, s'. As usual in elliptic problems, we require L to satisfy

$$L < \gamma_0\,, \tag{2.23}$$

where γ_0 is a suitable positive constant depending on the parameters in (2.1)–(2.6), and on the constant appearing in Poincaré's inequality (see Lemma 7.1 and [5]). We also need for g some kind of integral continuity in t. We state here a simple assumption suitable for our purposes; the interested reader may easily generalize the proof in Subsection 7.4 to other cases. Assume then

$$|g(x,t,s) - g(x,t',s)| \le \gamma\omega(|t-t'|)|s|\,, \qquad \text{for all } x \in \Omega,\ t,\ t' > 0,\ s \in \boldsymbol{R}, \tag{2.24}$$

where ω is a positive continuous function such that $\omega(0) = 0$.
The proofs and the results are essentially similar to the case when $g \equiv 0$, so we consider only the homogeneous case in the following. Only the compactness estimate (7.19) of Lemma 7.7 requires a different approach, which is discussed in Subsection 7.4. In Section 6 one needs some further assumptions, e.g., $g \in C^2([0,T]\times\overline{\Omega}\times\boldsymbol{R})$, with g_t, g_{x_i} bounded as in (2.21), in order to obtain the smoothness required there. In the case (2.20) the limiting function u_0 satisfies a version of (2.12) where the right hand side is substituted with $g(x,t,u_0)$.

2.1. **Significance of** (2.1)–(2.6) **in electrodynamics.** Our problem actually models conduction of electrical currents in a medium with inclusions, such as for example, a biological tissue. In this connection, Ω^ε_1 represents the intracellular space, and Ω^ε_2 the extracellular space, while Γ^ε represents the cell membranes. Thus, (2.1), (2.2) are the standard equations for the potential u_ε, in the widely accepted quasi-stationary approximation. Continuity condition (2.3) across the cell membranes is also standard. However, the potential in general may have jumps across Γ^ε, because the latter exhibits a capacitive/conductive behaviour. Indeed, α stands here for the membrane capacity per unit of area. The motivation of the scaling $1/\varepsilon$ appearing in (2.4) is discussed in [3]. Here we confine ourselves to point out that the capacity per unit of area of a capacitor is inversely proportional to the gap between the plates. In our setting, such a gap coincides with the physical membrane width, which in turn is a given fraction of the cell diameter, and therefore is of order ε.
The term containing f in (2.4) takes into account the conductive behaviour of the membrane [25]. The nonlinearity of f is interesting from the mathematical and modelling point of view. Indeed, in the general case the two possible scalings in (2.4),

i.e., $f([u_\varepsilon])/\varepsilon$ and $f([u_\varepsilon]/\varepsilon)$ differ substantially. The second one will be dealt with elsewhere.

On multiplying (2.1), (2.2) by u_ε and integrating formally by parts, using also (2.3)–(2.5), we obtain, for $0 < t < T$,

$$\int_0^t \int_\Omega \sigma |\nabla u_\varepsilon|^2 \, \mathrm{d}x \, \mathrm{d}\tau + \frac{\alpha}{2\varepsilon} \int_{\Gamma^\varepsilon} [u_\varepsilon]^2(x,t) \, \mathrm{d}\sigma + \frac{1}{\varepsilon} \int_0^t \int_{\Gamma^\varepsilon} [u_\varepsilon] f([u_\varepsilon]) \, \mathrm{d}\sigma \, \mathrm{d}\tau = \frac{\alpha}{2\varepsilon} \int_{\Gamma^\varepsilon} S_\varepsilon^2(x) \, \mathrm{d}\sigma \,,$$

whence, by Gronwall's inequality, and using the linear growth of f,

$$\int_0^t \int_\Omega \sigma |\nabla u_\varepsilon|^2 \, \mathrm{d}x \, \mathrm{d}t + \frac{1}{\varepsilon} \int_{\Gamma^\varepsilon} [u_\varepsilon]^2(x,t) \, \mathrm{d}\sigma \le \frac{\gamma}{\varepsilon} \int_{\Gamma^\varepsilon} S_\varepsilon^2(x) \, \mathrm{d}\sigma \le \gamma \,, \tag{2.25}$$

where, only for the last inequality, we appealed to (2.8), (2.9); γ is independent of ε. This energy estimate will be instrumental in our mathematical approach. Moreover, on physical grounds, it is natural to require that the energy on the leftmost side of (2.25) stays bounded as $\varepsilon \to 0$, thereby selecting the scaling in (2.8).

We are using in (2.25) the obvious fact that, for a suitable constant $\gamma > 1$,

$$\gamma^{-1}\varepsilon^{-1} \le |\Gamma^\varepsilon|_{N-1} \le \gamma\varepsilon^{-1} \,, \qquad \text{for all } \varepsilon > 0. \tag{2.26}$$

2.2. **Comparison with previous results.** The scaling appearing in (2.4) is essentially the same of [28], where, however, the authors considered a stationary problem in linear elasticity. Modulo the obvious differences in the setting, we may formally obtain the analog of the interface condition corresponding to (2.4), by letting $\partial u_\varepsilon/\partial t = 0$ in (2.4) itself. The corresponding linear evolutive problem was treated in [6].

In [30] the author deals with a stationary problem for heat conduction. As in [28], the problem of [30] is linear. Again, we may formally derive the model of [30] from ours, setting $\partial u_\varepsilon/\partial t = 0$ and $f(s) = \varepsilon\bar{\beta}s$ in (2.4), where $\bar{\beta}$ is independent of ε. Due to this different scaling, the asymptotic behaviour of the problem in [30] is dissimilar from the one investigated here. In particular, our homogenized problem bears memory of the physical parameters appearing in the interface condition (i.e., α and f; see (2.12)), while in [30] this is not the case for $\bar{\beta}$.

Even in the case when neither E nor $\boldsymbol{R}^N \setminus E$ are connected, we identify the Dirichlet boundary data of the limit u_0, a piece of information which, in this case, was not obtained in [28] or [30], for different technical reasons.

Remark 2.5. If E_1 and E_2 have a cubic symmetry, one could prove that the matrices A^0 and A^1 are scalar, i.e., $A^0 = a_0 I$, $A^1(t) = a_1(t) I$. In this case, if moreover $g = g(x,t)$, equation (2.12) may be reduced to a standard Laplace equation. However, the current flux on $\partial\Omega$, which is needed for example in inverse reconstruction problems, is inherently nonlocal in time, see Remark 5.1.

2.3. **Notation.** We denote by γ a generic positive constant (independent from ε), taking in principle different values in different occurrences. These constants depend only on the geometrical properties of E and Ω, and on α, σ_1, σ_2, T, and on the global bounds for f', f''. We also denote by ∇f the pointwise spatial gradient of f, while

Df denotes its variation measure (in the BV sense). In general ∇f and Df differ, as we consider functions with jumps.

3. The formal homogenization asymptotics

In this section we aim at identifying the form of the homogenized equation, via the two-scale method (see [12], [29], [34]). Introduce the microscopic variables $y \in Y$, $y = x/\varepsilon$, assuming

$$u_\varepsilon = u_\varepsilon(x, y, t) = u_0(x, y, t) + \varepsilon u_1(x, y, t) + \varepsilon^2 u_2(x, y, t) + \dots . \tag{3.1}$$

Note that u_0, u_1, u_2 are periodic in y, and u_1, u_2 are assumed to have zero integral average over Y. Recalling that

$$\operatorname{div} = \frac{1}{\varepsilon} \operatorname{div}_y + \operatorname{div}_x , \qquad \nabla = \frac{1}{\varepsilon} \nabla_y + \nabla_x , \tag{3.2}$$

we compute

$$\Delta\, u_\varepsilon = \frac{1}{\varepsilon^2} A_0 u_0 + \frac{1}{\varepsilon}(A_0 u_1 + A_1 u_0) + (A_0 u_2 + A_1 u_1 + A_2 u_0) + \dots , \tag{3.3}$$

Here

$$A_0 = \Delta_y , \quad A_1 = \operatorname{div}_y \nabla_x + \operatorname{div}_x \nabla_y , \quad A_2 = \Delta_x . \tag{3.4}$$

Let us recall explicitly that

$$\nabla u_\varepsilon = \frac{1}{\varepsilon} \nabla_y u_0 + \big(\nabla_x u_0 + \nabla_y u_1\big) + \varepsilon\big(\nabla_y u_2 + \nabla_x u_1\big) + \dots , \tag{3.5}$$

and stipulate, in addition to (2.8),

$$S_\varepsilon = S_\varepsilon(x, y) = \varepsilon S_1(x, y) + \varepsilon^2 S_2(x, y) + \dots . \tag{3.6}$$

Finally we expand

$$f([u_\varepsilon]) = f([u_0]) + \varepsilon f'([u_0])[u_1] + \varepsilon^2 \big\{ f'([u_0])[u_2] + \frac{1}{2} f''([u_0])[u_1]^2 \big\} + \dots .$$

3.1. **The term of order ε^{-2}.** Equating the first term on the right hand side of (3.3) to zero, and applying (3.1), (3.5) to (2.1)–(2.5) we find

$$-\sigma\, \Delta_y\, u_0 = 0 , \qquad \text{in } E_1,\ E_2; \tag{3.7}$$

$$\sigma_1 \nabla_y u_0^{(\mathrm{int})} \cdot \nu = \sigma_2 \nabla_y u_0^{(\mathrm{out})} \cdot \nu , \qquad \text{on } \mathit{\Gamma}; \tag{3.8}$$

$$\alpha \frac{\partial}{\partial t}[u_0] + f([u_0]) = \sigma_2 \nabla_y u_0^{(\mathrm{out})} \cdot \nu , \qquad \text{on } \mathit{\Gamma}; \tag{3.9}$$

$$[u_0](x, y, 0) = 0 , \qquad \text{on } \mathit{\Gamma}. \tag{3.10}$$

In (3.10) we have also exploited the expansion (3.6). It follows (see [5], and recall that $f(0) = 0$) that

$$u_0 = u_0(x, t) . \tag{3.11}$$

3.2. **The term of order ε^{-1}.** Proceeding as above, but taking into consideration the second term on the right hand side of (3.3) we obtain

$$-\sigma\,\Delta_y\,u_1 = \sigma A_1 u_0 = 0\,, \qquad \text{in } E_1,\, E_2; \tag{3.12}$$

$$[\sigma\nabla_y u_1\cdot\nu] = -[\sigma\nabla_x u_0\cdot\nu]\,, \qquad \text{on } \mathit{\Gamma}; \tag{3.13}$$

$$\alpha\frac{\partial}{\partial t}[u_1] + f'(0)[u_1] = \sigma_2\nabla_y u_1^{(\mathrm{out})}\cdot\nu + \sigma_2\nabla_x u_0\cdot\nu\,, \qquad \text{on } \mathit{\Gamma}; \tag{3.14}$$

$$[u_1](x,y,0) = S_1(x,y)\,, \qquad \text{on } \mathit{\Gamma}. \tag{3.15}$$

In (3.12) and in (3.14) we have made use of (3.11), and of its consequence $[u_0]=0$.

3.3. **The $\mathcal{T}$ transform. Cell functions.** Let $s:\mathit{\Gamma}\to\boldsymbol{R}$ be a jump function. Consider the problem

$$-\sigma\,\Delta_y\,v = 0\,, \qquad \text{in } E_1,\, E_2; \tag{3.16}$$

$$[\sigma\nabla_y v\cdot\nu] = 0\,, \qquad \text{on } \mathit{\Gamma}; \tag{3.17}$$

$$\alpha\frac{\partial}{\partial t}[v] + f'(0)[v] = \sigma_2\nabla_y v^{(\mathrm{out})}\cdot\nu\,, \qquad \text{on } \mathit{\Gamma}; \tag{3.18}$$

$$[v](y,0) = s(y)\,, \qquad \text{on } \mathit{\Gamma}, \tag{3.19}$$

where v is a periodic function in Y, such that $\int_Y v = 0$. Define the transform $\mathcal{T}$ by

$$\mathcal{T}(s)(y,t) = v(y,t)\,, \qquad y\in Y\,, t>0\,,$$

and extend the definition of $\mathcal{T}$ to vector (jump) functions, by letting it act componentwise on its argument.
Introduce also the functions $\chi^0 : Y\to\boldsymbol{R}^N$, $\chi^1 : Y\times(0,T)\to\boldsymbol{R}^N$ as follows. The components χ_h^0, $h=1,\dots,N$, satisfy

$$-\sigma\,\Delta_y\,\chi_h^0 = 0\,, \qquad \text{in } E_1,\, E_2; \tag{3.20}$$

$$[\sigma(\nabla_y\chi_h^0 - \boldsymbol{e}_h)\cdot\nu] = 0\,, \qquad \text{on } \mathit{\Gamma}; \tag{3.21}$$

$$[\chi_h^0] = 0\,, \qquad \text{on } \mathit{\Gamma}. \tag{3.22}$$

We also require χ_h^0 to be a periodic function with vanishing integral average over Y. Moreover χ_h^1 is defined from

$$\alpha\chi_h^1 = \mathcal{T}\big(\sigma_2(\nabla_y\chi_h^{0\,(\mathrm{out})} - \boldsymbol{e}_h)\cdot\nu\big)\,. \tag{3.23}$$

Let us stipulate that u_1 may be written in the form

$$u_1(x,y,t) = -\chi^0(y)\cdot\nabla_x u_0(x,t) + \mathcal{T}(S_1(x,\cdot))(y,t) - \int_0^t \nabla_x u_0(x,\tau)\cdot\chi^1(y,t-\tau)\,\mathrm{d}\tau\,. \tag{3.24}$$

3.4. **Reconciling** (3.24) **with** (3.12)–(3.15)**.** Equations (3.12) are equivalent to (3.20), when we remember that the terms χ^1_h and $\mathcal{T}(S_1(x,\cdot))$ in (3.24) fulfil (3.16). Next, let us impose (3.13) in (3.24). We get, on recalling (3.17)

$$[\sigma\nabla_y u_1\cdot\nu] = -[\sigma\nabla_y\chi^0_h(y)\cdot\nu]u_{0x_h}(x,t) = -[\sigma\nabla_x u_0\cdot\nu] = -[\sigma u_{0x_h}(x,t)\nu_h]\,.$$

In order to satisfy this requirement, we prescribe (3.21). Note that (3.15) is obviously satisfied, owing to the definition of $\mathcal{T}$.
Finally we get to (3.14), which we combine with (3.24) obtaining

$$\begin{aligned}\alpha\frac{\partial}{\partial t}[u_1] + f'(0)[u_1] = -\alpha[\chi^0(y)\cdot\frac{\partial}{\partial t}\nabla_x u_0(x,t)] + \alpha\frac{\partial}{\partial t}[\mathcal{T}(S_1(x,\cdot))](y,t)\\ -\alpha\nabla_x u_0(x,t)\cdot[\chi^1](y,0) - \alpha\int_0^t \nabla_x u_0(x,\tau)\cdot\frac{\partial}{\partial t}[\chi^1](y,t-\tau)\,\mathrm{d}\tau\\ - f'(0)[\chi^0(y)\cdot\nabla_x u_0(x,t)] + f'(0)[\mathcal{T}(S_1(x,\cdot))](y,t)\\ - f'(0)\int_0^t \nabla_x u_0(x,\tau)\cdot[\chi^1](y,t-\tau)\,\mathrm{d}\tau\,.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\sigma_2\nabla_y u_1^{(\mathrm{out})}\cdot\nu + \sigma_2\nabla_x u_0\cdot\nu = -\sigma_2\nabla_y\chi_h^{0\,(\mathrm{out})}(y)\cdot\nu u_{0x_h}(x,t)\\ &+\sigma_2\nabla_y\mathcal{T}(S_1(x,\cdot))^{(\mathrm{out})}\cdot\nu - \int_0^t u_{0x_h}(x,\tau)\sigma_2\nabla_y\chi^1_h(y,t-\tau)\cdot\nu\,\mathrm{d}\tau + \sigma_2 u_{0x_h}(x,t)\nu_h\,.\end{aligned}$$

Hence, (3.22)–(3.23) follow, on equating the quantities above.

3.5. **The term of order** ε^0**.** Let us first calculate

$$A_1u_1 = 2\frac{\partial^2 u_1}{\partial x_j\partial y_j}\,,$$

where we employ the summation convention. Therefore, the complete problem involving the third term on the right hand side of (3.3) is

$$-\sigma\,\Delta_y\,u_2 = \sigma\,\Delta_x\,u_0 + 2\sigma\frac{\partial^2 u_1}{\partial x_j\partial y_j}\,,\qquad E_1,E_2\quad(3.25)$$

$$[\sigma\nabla_y u_2\cdot\nu] = -[\sigma\nabla_x u_1\cdot\nu]\,,\qquad \text{on }\Gamma;\quad(3.26)$$

$$\alpha\frac{\partial}{\partial t}[u_2] + f'(0)[u_2] + \frac{f''(0)}{2}[u_1]^2 = \sigma_2\nabla_x u_1^{(\mathrm{out})}\cdot\nu + \sigma_2\nabla_y u_2^{(\mathrm{out})}\cdot\nu\,,\qquad \text{on }\Gamma;\quad(3.27)$$

$$[u_2](x,y,0) = S_2(x,y)\,,\qquad \text{on }\Gamma.\quad(3.28)$$

3.6. **Formal derivation of the homogenized equation.** Integrating by parts equation (3.25) both in E_1 and in E_2, and adding the two contributions, we get

$$\left[\int\limits_{E_1} + \int\limits_{E_2}\right] \left\{ \sigma\, \Delta_x\, u_0(x,t) + 2\sigma \frac{\partial^2 u_1}{\partial x_j \partial y_j} \right\} \mathrm{d}y$$

$$= \int\limits_{\Gamma} \left\{ \sigma_2 \nabla_y u_2^{(\mathrm{out})} \cdot \nu - \sigma_1 \nabla_y u_2^{(\mathrm{int})} \cdot \nu \right\} \mathrm{d}\sigma = \int\limits_{\Gamma} [\sigma \nabla_y u_2 \cdot \nu]\, \mathrm{d}\sigma = - \int\limits_{\Gamma} [\sigma \nabla_x u_1 \cdot \nu]\, \mathrm{d}\sigma\,.$$

Thus

$$\sigma_0\, \Delta_x\, u_0 = 2 \int\limits_{\Gamma} [\sigma \nabla_x u_1 \cdot \nu]\, \mathrm{d}\sigma - \int\limits_{\Gamma} [\sigma \nabla_x u_1 \cdot \nu]\, \mathrm{d}\sigma = \int\limits_{\Gamma} [\sigma \nabla_x u_1 \cdot \nu]\, \mathrm{d}\sigma\,,$$

where we denote

$$\sigma_0 = \sigma_1 |E_1| + \sigma_2 |E_2|\,. \tag{3.29}$$

We use next the expansion (3.24); namely, we recall that, in it, only last two terms on the right hand side have a non zero jump across Γ. Thus we infer from the equality above

$$\sigma_0\, \Delta_x\, u_0 = \int\limits_{\Gamma} [\sigma \nabla_x u_1 \cdot \nu]\, \mathrm{d}\sigma = - \int\limits_{\Gamma} [\sigma] \chi_h^0(y) \nu_j\, \mathrm{d}\sigma u_{0 x_h x_j}(x,t)$$

$$+ \frac{\partial}{\partial x_j} \int\limits_{\Gamma} [\sigma \mathcal{T}(S_1(x,\cdot))](y,t) \nu_j\, \mathrm{d}\sigma - \int\limits_0^t u_{0 x_h x_j}(x,\tau) \int\limits_{\Gamma} [\sigma \chi_h^1](y, t-\tau) \nu_j\, \mathrm{d}\sigma\ \mathrm{d}\tau\,.$$

We finally write the PDE for u_0 in $\Omega \times (0,T)$ as

$$- \operatorname{div} \Bigg(\sigma_0 \nabla_x u_0 + A^0 \nabla_x u_0 + \int\limits_0^t A^1(t-\tau) \nabla_x u_0(x,\tau)\, \mathrm{d}\tau - \int\limits_{\Gamma} [\sigma \mathcal{T}(S_1(x,\cdot))](y,t) \nu\, \mathrm{d}\sigma \Bigg) = 0\,. \tag{3.30}$$

The two matrices A^i are defined by

$$(A^0)_{jh} = \int\limits_{\Gamma} [\sigma] \chi_h^0(y) \nu_j\, \mathrm{d}\sigma\,, \qquad (A^1(t))_{jh} = \int\limits_{\Gamma} [\sigma \chi_h^1](y,t) \nu_j\, \mathrm{d}\sigma\,. \tag{3.31}$$

The matrices A^0 and A^1 are symmetric, and $\sigma_0 I + A^0$ is positive definite (see Section 4).

Remark 3.1. In the case when $f(s) = s$, $g = 0$, one can give the following representation of u_2. For any given pair of jump functions s_1, $s_2 : \Gamma \to \boldsymbol{R}$ define $\widetilde{\mathcal{T}}_j(s_1, s_2) = v$,

$j = 1, \dots, N$, where v is a periodic function with zero average in Y, solving

$$-\sigma\, \Delta_y\, v = \frac{\sigma}{\sigma_0} \int_\Gamma [\sigma \mathcal{T}(s_1)] \nu_j \,\mathrm{d}\sigma + 2\sigma \frac{\partial}{\partial y_j} \mathcal{T}(s_1)\,, \qquad \text{in } E_1,\, E_2; \tag{3.32}$$

$$[\sigma \nabla_y v \cdot \nu] = -[\sigma \mathcal{T}(s_1) \nu_j]\,, \qquad \text{on } \Gamma; \tag{3.33}$$

$$\alpha \frac{\partial}{\partial t}[v] = \sigma_2 \nabla_y v^{(\mathrm{out})} \cdot \nu + \sigma_2 \mathcal{T}(s_1)^{(\mathrm{out})} \nu_j \qquad \text{on } \Gamma; \tag{3.34}$$

$$[v](y, 0) = s_2(y)\,, \qquad \text{on } \Gamma. \tag{3.35}$$

Note that $\mathcal{T}(s) = \widetilde{\mathcal{T}}_j(0, s)$ for any choice of j. Moreover define the functions $\tilde{\chi}^0_{hj} : Y \to \boldsymbol{R}$, $\tilde{\chi}^1_{hj} : \Gamma \to \boldsymbol{R}$, for h, $j = 1, \dots, N$, by means of

$$-\sigma\, \Delta_y\, \tilde{\chi}^0_{hj} = -\frac{\sigma}{\sigma_0} A^0_{hj} - 2\sigma \frac{\partial}{\partial y_j} \chi^0_h\,, \qquad \text{in } E_1,\, E_2; \tag{3.36}$$

$$[\sigma(\nabla_y \tilde{\chi}^0_{hj} - \chi^0_h \boldsymbol{e}_j) \cdot \nu] = 0\,, \qquad \text{on } \Gamma; \tag{3.37}$$

$$[\tilde{\chi}^0_{hj}] = 0\,, \qquad \text{on } \Gamma; \tag{3.38}$$

and of

$$\alpha \tilde{\chi}^1_{hj} = \sigma_2 (\nabla_y \tilde{\chi}^{0\,(\mathrm{out})}_{hj} - \chi^0_h \boldsymbol{e}_j) \cdot \nu\,, \qquad \text{on } \Gamma. \tag{3.39}$$

We require $\tilde{\chi}^0_{hj}$ to be a periodic function with vanishing integral average over Y. Then one can check that the problem for u_2, (3.25)–(3.28) is equivalent to the representation

$$u_2(x, y, t) = \tilde{\chi}^0_{hj}(y) \frac{\partial^2 u_0}{\partial x_h \partial x_j}(x, t) + \widetilde{\mathcal{T}}_j\Big(\frac{\partial S_1}{\partial x_j}(x, \cdot), S_2(x, \cdot)\Big)(y, t) + \int_0^t \frac{\partial^2 u_0}{\partial x_h \partial x_j}(x, \tau) \bar{\mathcal{T}}_{jh}(y, t - \tau) \,\mathrm{d}\tau\,, \tag{3.40}$$

where $\bar{\mathcal{T}}_{jh}(y, t) = \widetilde{\mathcal{T}}_j(-\chi^1_h, \tilde{\chi}^1_{hj})(y, t)$.

4. The structure of the limit equation.

Owing to the results of Subsection 7.2, the functions χ^0 and χ^1 are of class C^∞ separately in $\overline{E_1}$, and in $\overline{E_2}$. As a by product, the two matrices A^1 and A^0 are well defined.

Proposition 4.1. *$\sigma_0 I + A^0$ is symmetric and positive definite.*

Proof. From the definition of χ^0 we get

$$\int_Y \sigma \nabla \chi^0_h \cdot \nabla \chi^0_j \,\mathrm{d}y = -\int_\Gamma [\sigma \nabla \chi^0_j \cdot \nu] \chi^0_h \,\mathrm{d}\sigma = -\int_\Gamma [\sigma] \nu_j \chi^0_h \,\mathrm{d}\sigma = -(A^0)_{jh}\,, \tag{4.1}$$

implying that A^0 is symmetric (and negative semidefinite). Moreover

$$-\int_Y \sigma \nabla \chi^0_j \cdot \boldsymbol{e}_h \,\mathrm{d}y = -\int_Y \sigma \frac{\partial \chi^0_j}{\partial y_h} \,\mathrm{d}y = \int_\Gamma [\sigma] \chi^0_j \nu_h \,\mathrm{d}\sigma = (A^0)_{hj}\,. \tag{4.2}$$

Thus, taking into account (4.1), (4.2), and the definition of σ_0,

$$\int_Y \sigma(\nabla\chi_j^0 - \boldsymbol{e}_j)\cdot(\nabla\chi_h^0 - \boldsymbol{e}_h)\,\mathrm{d}y = -(A^0)_{jh} + 2(A^0)_{jh} + \sigma_0\delta_{jh} = (\sigma_0 I + A^0)_{jh}\,. \quad (4.3)$$

The strict positivity of $\sigma_0 I + A^0$ can be proven in the following standard fashion, making use of (4.3) and of elementary linearity properties. Denoting $\sigma_m = \min(\sigma_1, \sigma_2)$, we have for all $\xi \in \boldsymbol{R}^N$

$$\textstyle\sum_{j,h}(\sigma_0 I + A^0)_{jh}\xi_j\xi_h \geq \sigma_m \displaystyle\int_Y |\nabla \textstyle\sum_j (\chi_j^0\xi_j - y_j\xi_j)|^2\,\mathrm{d}y \geq$$

$$\sigma_m \textstyle\sum_k \displaystyle\Big(\textstyle\sum_j \displaystyle\int_Y \frac{\partial\chi_j^0}{\partial y_k}\xi_j\,\mathrm{d}y - \int_Y \xi_k\,\mathrm{d}y\Big)^2 = \sigma_m \textstyle\sum_k \displaystyle\Big(\textstyle\sum_j \displaystyle\int_{\partial Y} \chi_j^0\nu_k\,\mathrm{d}y\,\xi_j - \xi_k\Big)^2 = \sigma_m|\xi|^2\,.$$

□

The ideas employed in the proof of next result will turn out to be essential also in proving Theorem 2.1.

Lemma 4.2. *Let s_1, $s_2 : \Gamma \to \boldsymbol{R}$ be traces on Γ of two functions of class $H^1(Y)$, periodic in y. Then, for all $t > 0$:*

$$\int_\Gamma [\mathcal{T}(s_1)](t)\,[\mathcal{T}(s_2)](0)\,\mathrm{d}\sigma = \int_\Gamma [\mathcal{T}(s_1)](0)\,[\mathcal{T}(s_2)](t)\,\mathrm{d}\sigma\,. \quad (4.4)$$

Note that the transforms $\mathcal{T}(s_1)$, $\mathcal{T}(s_2)$ are well defined, according to the results in [5].

Proof. Let us define $v_h = \mathcal{T}(s_h)$, and, for a fixed $t > 0$,

$$\hat{v}_h(y,\tau) = v_h(y, t-\tau)\,, \qquad 0 < \tau < t\,. \quad (4.5)$$

Thus $\hat{v}_h$ solves for $0 < \tau < t$

$$-\sigma\,\Delta_y\,\hat{v}_h = 0\,, \qquad \text{in } E_1,\ E_2; \quad (4.6)$$

$$[\sigma\nabla_y\hat{v}_h\cdot\nu] = 0\,, \qquad \text{on } \Gamma; \quad (4.7)$$

$$\alpha\frac{\partial}{\partial\tau}[\hat{v}_h] - f'(0)[\hat{v}_h] = -\sigma_2\nabla_y\hat{v}_h^{(\mathrm{out})}\cdot\nu\,, \qquad \text{on } \Gamma; \quad (4.8)$$

$$[\hat{v}_h](y,t) = s_h(y)\,, \qquad \text{on } \Gamma. \quad (4.9)$$

Use v_j as a testing function in (4.6)–(4.8), and obtain

$$\int_Y \sigma\nabla\hat{v}_h\cdot\nabla v_j\,\mathrm{d}y = -\int_\Gamma [v_j]\sigma_2\nabla\hat{v}_h^{(\mathrm{out})}\cdot\nu\,\mathrm{d}\sigma = \alpha\int_\Gamma [v_j]\frac{\partial}{\partial\tau}[\hat{v}_h]\,\mathrm{d}\sigma - f'(0)\int_\Gamma [v_j][\hat{v}_h]\,\mathrm{d}\sigma\,.$$

Note that the formal calculations appearing here can be easily made rigorous invoking the regularity obtained in [5]. Integrating in time over $(0,t)$, we calculate

$$\int_0^t \int_Y \sigma \nabla \hat{v}_h \cdot \nabla v_j \, \mathrm{d}y \, \mathrm{d}\tau + f'(0) \int_0^t \int_\Gamma [v_j][\hat{v}_h] \, \mathrm{d}\sigma \, \mathrm{d}\tau = \alpha \int_0^t \int_\Gamma [v_j] \frac{\partial}{\partial \tau} [\hat{v}_h] \, \mathrm{d}\sigma \, \mathrm{d}\tau$$

$$= \alpha \int_\Gamma [v_j](t)[\hat{v}_h](t) \, \mathrm{d}\sigma - \alpha \int_\Gamma [v_j](0)[\hat{v}_h](0) \, \mathrm{d}\sigma - \alpha \int_0^t \int_\Gamma [\hat{v}_h] \frac{\partial}{\partial \tau} [v_j] \, \mathrm{d}\sigma \, \mathrm{d}\tau \, . \quad (4.10)$$

Next using $\hat{v}_h$ as a testing function in the problem (3.16)–(3.18) solved by v_j we find

$$\int_0^t \int_Y \sigma \nabla v_j \cdot \nabla \hat{v}_h \, \mathrm{d}y \, \mathrm{d}\tau = - \int_0^t \int_\Gamma [\hat{v}_h] \sigma_2 \nabla v_j^{(\mathrm{out})} \cdot \nu \, \mathrm{d}\sigma \, \mathrm{d}\tau = -\alpha \int_0^t \int_\Gamma [\hat{v}_h] \frac{\partial}{\partial \tau} [v_j] \, \mathrm{d}\sigma \, \mathrm{d}\tau$$

$$- f'(0) \int_0^t \int_\Gamma [v_j][\hat{v}_h] \, \mathrm{d}\sigma \, \mathrm{d}\tau \, . \quad (4.11)$$

Subtracting (4.11) from (4.10) we find

$$\alpha \int_\Gamma [v_j](t)[\hat{v}_h](t) \, \mathrm{d}\sigma = \alpha \int_\Gamma [v_j](0)[\hat{v}_h](0) \, \mathrm{d}\sigma \, , \quad (4.12)$$

thereby proving (4.4). □

Corollary 4.3. *A^1 is symmetric.*

Proof. Let us define now $v_h = \chi_h^1$, and, for a fixed $t > 0$, $\hat{v}_h$ as in (4.5). We start from (4.12) written for this choice of v_h.
Let us compute explicitly, recalling the definitions of $\hat{v}_h$, χ_h^1,

$$\alpha \int_\Gamma [v_j](t)[\hat{v}_h](t) \, \mathrm{d}\sigma = \alpha \int_\Gamma [v_j](t)[v_h](0) \, \mathrm{d}\sigma$$

$$= \alpha \int_\Gamma [v_j](t)[\chi_h^1](0) \, \mathrm{d}\sigma = \int_\Gamma [v_j](t) \sigma_2 \nabla \chi_h^{0\,(\mathrm{out})} \cdot \nu \, \mathrm{d}\sigma - \int_\Gamma [v_j](t) \sigma_2 \nu_h \, \mathrm{d}\sigma \, . \quad (4.13)$$

In order to obtain a different expression for the penultimate integral, we multiply (3.20) against $v_j(t)$ and integrate by parts, so that we get

$$\int_Y \sigma \nabla \chi_h^0 \cdot \nabla v_j(t) \, \mathrm{d}\sigma = \int_\Gamma \{ \sigma_1 \nabla \chi_h^{0\,(\mathrm{int})} \cdot \nu v_j^{(\mathrm{int})}(t) - \sigma_2 \nabla \chi_h^{0\,(\mathrm{out})} \cdot \nu v_j^{(\mathrm{out})}(t) \} \, \mathrm{d}\sigma$$

$$= - \int_\Gamma [v_j](t) \sigma_2 \nabla \chi_h^{0\,(\mathrm{out})} \cdot \nu \, \mathrm{d}\sigma - \int_\Gamma v_j^{(\mathrm{int})}(t) [\sigma] \nu_h \, \mathrm{d}\sigma \, , \quad (4.14)$$

where we have made use of (3.21). However, on using χ_h^0 as a testing function in the problem (3.16)–(3.19) defining v_j, one readily sees that the leftmost side of (4.14) vanishes. Thus, combining (4.13) and (4.14), we prove

$$\alpha \int_\Gamma [v_j](t)[\hat{v}_h](t)\,\mathrm{d}\sigma = -\int_\Gamma v_j^{(\mathrm{int})}(t)[\sigma]\nu_h\,\mathrm{d}\sigma - \int_\Gamma [v_j](t)\sigma_2\nu_h\,\mathrm{d}\sigma = -\int_\Gamma [\sigma v_j](t)\nu_h\,\mathrm{d}\sigma\,. \tag{4.15}$$

Reasoning as above

$$\alpha \int_\Gamma [v_j](0)[\hat{v}_h](0)\,\mathrm{d}\sigma = \alpha \int_\Gamma [v_j](0)[v_h](t)\,\mathrm{d}\sigma = -\int_\Gamma [\sigma v_h](t)\nu_j\,\mathrm{d}\sigma\,. \tag{4.16}$$

Combining (4.12) with (4.15), (4.16) we finally infer

$$(A^1(t))_{jh} = \int_\Gamma [\sigma v_h](t)\nu_j\,\mathrm{d}\sigma = \int_\Gamma [\sigma v_j](t)\nu_h\,\mathrm{d}\sigma = (A^1(t))_{hj}\,. \tag{4.17}$$

□

Next result reconciles the seemingly different forms of $\mathcal{F}$ as found in (3.30) and in the rigorous limiting procedure carried out in Section 5.

Corollary 4.4. *For all $h = 1, \dots, N$,*

$$\mathcal{F}_h(x,t) := -\alpha \int_\Gamma S_1(x,y)[\chi_h^1](y,t)\,\mathrm{d}\sigma = \int_\Gamma [\sigma\mathcal{T}(S_1(x,\cdot))](y,t)\nu_h\,\mathrm{d}\sigma\,. \tag{4.18}$$

Of course we assume here that $S_1(x,\cdot)$ is regular enough for $\mathcal{T}(S_1(x,\cdot))$ to be defined, e.g., that $S_1(x,\cdot)$ has the regularity mentioned in Lemma 4.2.

Proof. This is a by-product of Lemma 4.2, and of the proof of Corollary 4.3. Indeed, note that the calculations leading to (4.15) actually use only the special form of $[v_h](0) = [\chi_h^1](0)$. Therefore, (4.12) and (4.15) are still in force if we let v_h as in Corollary 4.3, while formally replacing there v_j with $\mathcal{T}(S_1(x,\cdot))$. This yields (4.18). □

5. The homogenization limit

Introduce for $i = 1, \dots, N$, the functions

$$w_i^\varepsilon(x,t) = x_i - \varepsilon\chi_i^0\Big(\frac{x}{\varepsilon}\Big) - \varepsilon \int_t^T \chi_i^1\Big(\frac{x}{\varepsilon}, \tau - t\Big)\,\mathrm{d}\tau\,, \tag{5.1}$$

so that explicit calculations reveal

$$-\sigma\,\Delta\, w_i^\varepsilon = 0\,, \qquad \text{in } \Omega_1^\varepsilon,\ \Omega_2^\varepsilon; \tag{5.2}$$

$$[\sigma\nabla w_i^\varepsilon\cdot\nu] = 0\,, \qquad \text{on } \Gamma^\varepsilon; \tag{5.3}$$

$$\frac{\alpha}{\varepsilon}\frac{\partial}{\partial t}[w_i^\varepsilon] - \frac{f'(0)}{\varepsilon}[w_i^\varepsilon] = -\sigma_2\nabla w_i^{\varepsilon(\mathrm{out})}\cdot\nu\,, \qquad \text{on } \Gamma^\varepsilon. \tag{5.4}$$

Let $\varphi \in C_o^\infty(\Omega)$, and select $w_i^\varepsilon \varphi$ as a testing function in the weak formulation (2.11). We obtain

$$\int_0^T\!\!\int_\Omega \sigma \nabla u_\varepsilon \cdot \nabla w_i^\varepsilon \, \varphi \,\mathrm{d}x\,\mathrm{d}t + \int_0^T\!\!\int_\Omega \sigma \nabla u_\varepsilon \cdot \nabla \varphi \, w_i^\varepsilon \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{\varepsilon}\int_0^T\!\!\int_{\Gamma^\varepsilon} f([u_\varepsilon])[w_i^\varepsilon]\varphi \,\mathrm{d}\sigma\,\mathrm{d}t$$
$$- \frac{\alpha}{\varepsilon}\int_0^T\!\!\int_{\Gamma^\varepsilon} [u_\varepsilon]\frac{\partial}{\partial t}[w_i^\varepsilon]\varphi \,\mathrm{d}\sigma\,\mathrm{d}t - \frac{\alpha}{\varepsilon}\int_{\Gamma^\varepsilon} [u_\varepsilon](0)[w_i^\varepsilon](0)\varphi \,\mathrm{d}\sigma = 0\,, \quad (5.5)$$

once we use the obvious relation $[w_i^\varepsilon](x,T) = 0$. Next select $u_\varepsilon \varphi$ as a testing function in the weak formulation of (5.2)–(5.4); in this second step, no integration by parts in t is needed on Γ^ε. We get

$$\int_0^T\!\!\int_\Omega \sigma \nabla w_i^\varepsilon \cdot \nabla u_\varepsilon \, \varphi \,\mathrm{d}x\,\mathrm{d}t + \int_0^T\!\!\int_\Omega \sigma \nabla w_i^\varepsilon \cdot \nabla \varphi \, u_\varepsilon \,\mathrm{d}x\,\mathrm{d}t$$
$$- \frac{\alpha}{\varepsilon}\int_0^T\!\!\int_{\Gamma^\varepsilon} \frac{\partial}{\partial t}[w_i^\varepsilon][u_\varepsilon]\varphi \,\mathrm{d}\sigma\,\mathrm{d}t + \frac{f'(0)}{\varepsilon}\int_0^T\!\!\int_{\Gamma^\varepsilon} [w_i^\varepsilon][u_\varepsilon]\varphi \,\mathrm{d}\sigma\,\mathrm{d}t = 0\,. \quad (5.6)$$

Subtract (5.6) from (5.5) and find, taking (2.8) into account,

$$\int_0^T\!\!\int_\Omega \sigma \nabla u_\varepsilon \cdot \nabla \varphi \, w_i^\varepsilon \,\mathrm{d}x\,\mathrm{d}t = K_{1\varepsilon} + K_{2\varepsilon} + K_{3\varepsilon}\,, \quad (5.7)$$

where we have defined

$$K_{1\varepsilon} = \int_0^T\!\!\int_\Omega \sigma \nabla w_i^\varepsilon \cdot \nabla \varphi \, u_\varepsilon \,\mathrm{d}x\,\mathrm{d}t\,,$$

$$K_{2\varepsilon} = -\alpha\varepsilon \int_{\Gamma^\varepsilon} \big(S_1\big(x, \frac{x}{\varepsilon}\big) + R_\varepsilon(x)\big)\varphi(x) \int_0^T [\chi_i^1]\big(\frac{x}{\varepsilon}, \tau\big)\,\mathrm{d}\tau\,\mathrm{d}\sigma\,,$$

$$K_{3\varepsilon} = \frac{1}{\varepsilon}\int_0^T\!\!\int_{\Gamma^\varepsilon} \big\{f'(0)[u_\varepsilon] - f([u_\varepsilon])\big\}[w_i^\varepsilon]\varphi \,\mathrm{d}\sigma\,\mathrm{d}t\,.$$

We rely here on the energy inequality (2.25). Although it was derived formally in the Introduction, its proof can be made rigorous, starting from the weak formulation (2.11), and using a Steklov averaging procedure. Estimate (2.25), together with lemma 7.1 and 7.7 imply that, extracting subsequences if needed, we may assume

$$-\sigma \nabla u_\varepsilon \to \xi\,, \quad u_\varepsilon \to u_0\,, \qquad \text{weakly in } L^2(\Omega \times (0,T)), \quad (5.8)$$

$$u_\varepsilon \to u_0\,, \qquad \text{strongly in } L^1_{\mathrm{loc}}(0,T; L^1(\Omega)), \quad (5.9)$$

for some $\xi \in L^2(\Omega\times(0,T))^N$, $u_0 \in L^2(\Omega\times(0,T))$. On the other hand, clearly $w_i^\varepsilon \to x_i$ strongly in $L^2(\Omega \times (0,T))$, as $\varepsilon \to 0$. Let us investigate the limiting behaviour of $\sigma\nabla w_i^\varepsilon$. Due to the periodicity of the functions χ^i, and to (4.2), one immediately gets

$$\sigma\nabla\Big(x_i - \varepsilon\chi_i^0\big(\frac{x}{\varepsilon}\big)\Big) \to (\sigma_0 I + A^0)\boldsymbol{e}_i\,, \quad \text{weakly in } L^2(\Omega).$$

By the same token, in the same weak sense,

$$-\sigma\nabla\Big(\varepsilon\int_t^T \chi_i^1\big(\frac{x}{\varepsilon},\tau-t\big)\,\mathrm{d}\tau\Big) \to -\int_t^T\int_Y \sigma\nabla_y\chi_i^1(y,\tau-t)\,\mathrm{d}y\,\mathrm{d}\tau = \int_t^T A^1(\tau-t)\boldsymbol{e}_i\,\mathrm{d}\tau\,,$$

where last equality follows from the definition (3.31) of A^1 and from a trivial integration by parts. Thus, invoking Lemma 7.9 and Remark 7.10 below,

$$K_{1\varepsilon} \to \int_0^T\int_\Omega (\sigma_0 I + A^0)\boldsymbol{e}_i\cdot\nabla\varphi u_0\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega\int_t^T A^1(\tau-t)\boldsymbol{e}_i\,\mathrm{d}\tau\cdot\nabla\varphi u_0\,\mathrm{d}x\,\mathrm{d}t =: K_{10}\,.$$

Elementary manipulations show that

$$K_{10} = \int_0^T\int_\Omega \Big\{u_0(x,t)(\sigma_0 I + A^0)\boldsymbol{e}_i + \int_0^t u_0(x,\tau)A^1(t-\tau)\boldsymbol{e}_i\,\mathrm{d}\tau\Big\}\cdot\nabla\varphi(x)\,\mathrm{d}x\,\mathrm{d}t\,.$$

Next we turn to the task of evaluating the limiting behaviour of $K_{2\varepsilon}$. Clearly the term involving R_ε vanishes in the limit. Then we appeal to the stipulated regularity of S_1, and apply, with minor changes, the ideas of [36] Lemma 3; we infer

$$K_{2\varepsilon} \to -\alpha\int_\Omega \varphi(x)\int_\Gamma S_1(x,y)\int_0^T [\chi_i^1](y,\tau)\,\mathrm{d}\tau\,\mathrm{d}\sigma\,\mathrm{d}x = \int_0^T\int_\Omega \varphi(x)\mathcal{F}_i(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau\,,$$

where $\mathcal{F}$ has been defined in the first equality of (4.18).

One can easily check that $K_{3\varepsilon} \to 0$; actually

$$|K_{3\varepsilon}| \le \left|\frac{1}{\varepsilon}\int_0^T\int_{\Gamma^\varepsilon} \frac{f''(\mu_\varepsilon)}{2}[u_\varepsilon]^2[w_i^\varepsilon]\varphi\,\mathrm{d}\sigma\,\mathrm{d}t\right| \le \gamma\int_0^T\int_{\Gamma^\varepsilon}[u_\varepsilon]^2\,\mathrm{d}\sigma\,\mathrm{d}t \le \gamma\varepsilon\,, \qquad \mu_\varepsilon = \mu_\varepsilon(x,t)\,,$$

where we have used the properties of f, the definition of w_i^ε, and the energy inequality (2.25), as well as the regularity of χ^0, χ^1 obtained in Subsection 7.2.
Collecting the results above, let $\varepsilon \to 0$ in (5.7) to arrive at

$$\begin{aligned}-\int_0^T\int_\Omega \xi\cdot\nabla\varphi\,x_i\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\int_\Omega \varphi(x)\mathcal{F}_i(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau \\ &+ \int_0^T\int_\Omega \Big\{u_0(x,t)(\sigma_0 I + A^0)\boldsymbol{e}_i + \int_0^t u_0(x,\tau)A^1(t-\tau)\boldsymbol{e}_i\,\mathrm{d}\tau\Big\}\cdot\nabla\varphi(x)\,\mathrm{d}x\,\mathrm{d}t\,. \quad (5.10)\end{aligned}$$

As usual, next we take φx_i as a testing function in (2.11). This test essentially does not detect the boundary Γ^ε, due to (2.3); on letting $\varepsilon \to 0$

$$\int_0^T\!\!\int_\Omega \xi \cdot \nabla\varphi \, x_i \,\mathrm{d}x\,\mathrm{d}t + \int_0^T\!\!\int_\Omega \xi \cdot \boldsymbol{e}_i \,\varphi \,\mathrm{d}x\,\mathrm{d}t = 0\,. \tag{5.11}$$

We substitute (5.11) in (5.10), and differentiate in T the resulting equality; in fact the choice of T is essentially arbitrary in this setting. We obtain (reverting to t as the time variable)

$$\int_\Omega \Big\{ u_0(x,t)(\sigma_0 I + A^0) + \int_0^t u_0(x,\tau) A^1(t-\tau)\,\mathrm{d}\tau \Big\} \nabla\varphi(x)\,\mathrm{d}x \\ = \int_\Omega \xi(x,t)\varphi(x)\,\mathrm{d}x - \int_\Omega \varphi(x)\mathcal{F}(x,t)\,\mathrm{d}x\,,$$

a.e. t. It follows that $u_0 \in L^2(0,T;H^1(\Omega))$ (see Lemma 7.3 below), and that

$$\xi(x,t) = -(\sigma_0 I + A^0)\nabla u_0(x,t) - \int_0^t A^1(t-\tau)\nabla u_0(x,\tau)\,\mathrm{d}\tau + \mathcal{F}(x,t)\,, \tag{5.12}$$

a.e. (x,t). Clearly $\operatorname{div}\xi = 0$ in the sense of distributions (see e.g., (5.11) above). This shows that (2.12) is in force.

Remark 5.1. The equality (5.12), which is the constitutive relationship of the homogenized material, expresses the limiting current ξ as a function of the history of the gradient of the potential, ∇u_0.

5.1. u_0 **vanishes on** $\partial\Omega$**.** The trace of u_0 on $\partial\Omega$ exists for a.e. $t \in (0,T)$, because of the already proven regularity of u_0. It is left to show that this trace is zero.
We understand here u_0 and each u_ε to be defined on $\boldsymbol{R}^N \times (0,T)$, by extending them as zero outside Ω. Also define,

$$U_\varepsilon(x) = \int_0^T u_\varepsilon(x,t)\,\mathrm{d}t\,, \qquad U_0(x) = \int_0^T u_0(x,t)\,\mathrm{d}t\,.$$

Since we already know that the trace on $\partial\Omega$ of each u_ε, and therefore of U_ε, is zero, we infer that for each bounded open set $G \subset \boldsymbol{R}^N$, the variation $|DU_\varepsilon|(G)$ is given by

$$|DU_\varepsilon|(G) = \int_G \left| \int_0^T \nabla u_\varepsilon \,\mathrm{d}t \right| \mathrm{d}x + \int_{\Gamma^\varepsilon \cap G} \left| \int_0^T [u_\varepsilon]\,\mathrm{d}t \right| \mathrm{d}\sigma \\ \leq \gamma\big(|G|^{1/2} + (\varepsilon |\Gamma^\varepsilon \cap G|_{N-1})^{1/2} \big)\,, \tag{5.13}$$

where we have made use of Hölder's inequality and of (2.25). As a first consequence of this estimate, we may invoke classical compactness and semicontinuity results to show that (extracting subsequences if needed)

$$U_\varepsilon \to U_0\,, \qquad \text{in } L^1(\boldsymbol{R}^N), \qquad |DU_0|(G) \le \liminf_{\varepsilon\to 0} |DU_\varepsilon|(G)\,, \tag{5.14}$$

for every set $G \subset \boldsymbol{R}^N$ as above. On the other hand, according to [7] Theorem 3.77,

$$|DU_0|(\partial\Omega) = \int_{\partial\Omega} |U_0^+ - U_0^-|\,\mathrm{d}\sigma = \int_{\partial\Omega} |U_0^+|\,\mathrm{d}\sigma\,, \tag{5.15}$$

where the symbol U_0^+ (respectively, U_0^-) denotes the trace on $\partial\Omega$ of $U_{0|\Omega}$ (respectively, of $U_{0|\boldsymbol{R}^N\setminus\overline{\Omega}} \equiv 0$).
Define for $0 < h < 1$ the open set

$$V_h = \{x \in \boldsymbol{R}^N \mid \operatorname{dist}(x, \partial\Omega) < h\}\,.$$

Combining (5.13)–(5.15), we obtain, as $\partial\Omega \subset V_h$ for all h,

$$\int_{\partial\Omega} |U_0^+|\,\mathrm{d}\sigma \le |DU_0(V_h)| \le \gamma \liminf_{\varepsilon\to 0} \big(|V_h|^{1/2} + (\varepsilon|\Gamma^\varepsilon \cap V_h|_{N-1})^{1/2}\big) \le \gamma h^{1/2}\,.$$

Indeed, it is readily seen that $|V_h| \le \gamma h$, and that $|\Gamma^\varepsilon \cap V_h|_{N-1} \le \gamma h/\varepsilon$ for all sufficiently small h. Therefore, letting $h \to 0$ above we obtain that $U_0^+ = 0$ a.e. on $\partial\Omega$. However, U_0^+ and the trace u_0^+ of u_0 are related by

$$U_0^+(x) = \int_0^T u_0^+(x,t)\,\mathrm{d}t\,, \qquad x \in \partial\Omega\,.$$

Clearly, T stands here for any positive time, so that, differentiating the last equality in time, we obtain $u_0^+(x,t) = 0$ a.e. on $\partial\Omega \times (0,T)$.

Remark 5.2. Due to the uniqueness results of [4], actually the whole sequence $\{u_\varepsilon\}$ converges to u_0.

6. Error estimate

Here u_0 is the limit function of Theorem 2.1, while u_1 and u_2 are defined as in subsections 3.2, 3.5. In what follows we understand that

$$u_1 = u_1(x, \frac{x}{\varepsilon}, t)\,, \qquad u_2 = u_2(x, \frac{x}{\varepsilon}, t)\,.$$

We use extensively in this section the notation in (3.2)–(3.4).
Define the rest function

$$r_\varepsilon = (u_\varepsilon - u_0 - \varepsilon u_1)\varepsilon^{-1}\,, \qquad x \in \Omega\,, t > 0\,.$$

6.1. **The conditions satisfied by the rest function.** We calculate, separately in Ω_1^ε and in Ω_2^ε,

$$-\operatorname{div}\sigma\nabla r_\varepsilon = \frac{1}{\varepsilon}\Big\{-\operatorname{div}\sigma\nabla u_\varepsilon + \operatorname{div}\sigma\nabla u_0 + \varepsilon\operatorname{div}\sigma\nabla u_1\Big\}$$

(by means of (2.1), (2.2))

$$= \frac{1}{\varepsilon}\Big\{\sigma\,\Delta_x\, u_0 + \varepsilon\sigma\,\Delta_x\, u_1 + 2\sigma u_{1x_h y_h} + \frac{1}{\varepsilon}\sigma\,\Delta_y\, u_1\Big\}$$

(by means of (3.12), (3.25))

$$= -\frac{1}{\varepsilon}\sigma\,\Delta_y\, u_2 + \sigma\,\Delta_x\, u_1 =: E_\varepsilon\,.$$

Define

$$Q_\varepsilon = \frac{1}{\varepsilon}f([u_\varepsilon]) - f'(0)[u_1] = \frac{1}{\varepsilon}\big\{f([u_\varepsilon]) - \varepsilon f'(0)[u_1]\big\}\,.$$

Then (recalling (2.4), (3.14)), straightforward calculations yield that on Γ^ε

$$\begin{aligned}\alpha\frac{\partial}{\partial t}[r_\varepsilon] &= \Big\{\sigma_2\nabla u_\varepsilon^{(\text{out})} - \sigma_2\nabla u_0^{(\text{out})} - \sigma_2\nabla_y u_1^{(\text{out})}\Big\}\cdot\nu - Q_\varepsilon \\ &= \varepsilon\sigma_2\Big\{\frac{1}{\varepsilon}\nabla u_\varepsilon^{(\text{out})} - \frac{1}{\varepsilon}\nabla u_0^{(\text{out})} - \nabla u_1^{(\text{out})}\Big\}\cdot\nu + \varepsilon\sigma_2\nabla_x u_1^{(\text{out})}\cdot\nu - Q_\varepsilon \\ &= \varepsilon\sigma_2\nabla r_\varepsilon^{(\text{out})}\cdot\nu + \varepsilon\sigma_2\nabla_x u_1^{(\text{out})}\cdot\nu - Q_\varepsilon\,.\end{aligned}$$

Finally

$$[\sigma\nabla r_\varepsilon\cdot\nu] = \frac{1}{\varepsilon}[\sigma(\nabla u_\varepsilon - \nabla u_0 - \varepsilon\nabla u_1)\cdot\nu]$$

(by means of (2.3), (3.13))

$$= -\frac{1}{\varepsilon}[\sigma\nabla u_0\cdot\nu] - \frac{1}{\varepsilon}[\sigma\nabla_y u_1\cdot\nu] - [\sigma\nabla_x u_1\cdot\nu] = -[\sigma\nabla_x u_1\cdot\nu]\,.$$

6.2. **Estimating the L^2 norm of the rest.** Introduce the corrected rest function

$$\tilde r_\varepsilon = r_\varepsilon + u_1\phi_\varepsilon\,,$$

where ϕ_ε is a cut off function equal to 1 in a neighbourhood of $\partial\Omega$, and such that

$$\phi_\varepsilon(x) = 0 \quad\text{if}\quad \operatorname{dist}(x,\partial\Omega)\ge\gamma_0\varepsilon\,.$$

Here γ_0 is the constant appearing in in the statement of Theorem 2.2. Hence $\phi_\varepsilon\equiv 0$ on Γ^ε. We may assume $0\le\phi_\varepsilon\le 1$, $|\nabla\phi_\varepsilon|\le\gamma/\varepsilon$. The function $\tilde r_\varepsilon$ satisfies

$$-\sigma\,\Delta\,\tilde r_\varepsilon = E_\varepsilon - \sigma\,\Delta(u_1\phi_\varepsilon)\,, \qquad \text{in } \Omega_1^\varepsilon,\ \Omega_2^\varepsilon; \tag{6.1}$$

$$[\sigma\nabla\tilde r_\varepsilon\cdot\nu] = -[\sigma\nabla_x u_1\cdot\nu]\,, \qquad \text{on } \Gamma^\varepsilon; \tag{6.2}$$

$$\alpha\frac{\partial}{\partial t}[\tilde r_\varepsilon] = \varepsilon\sigma_2\nabla\tilde r_\varepsilon^{(\text{out})}\cdot\nu + \varepsilon\sigma_2\nabla_x u_1^{(\text{out})}\cdot\nu - Q_\varepsilon\,, \qquad \text{on } \Gamma^\varepsilon; \tag{6.3}$$

$$\tilde r_\varepsilon = 0\,, \qquad \text{on } \partial\Omega. \tag{6.4}$$

Note that the correction $u_1\phi_\varepsilon$ has been introduced precisely in order to guarantee (6.4).

Multiply (6.1) by $\tilde{r}_\varepsilon$ and integrate by parts, obtaining, by virtue of (6.4),

$$\int_0^t\!\!\int_\Omega \sigma|\nabla \tilde{r}_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}\tau + \int_0^t\!\!\int_{\Gamma^\varepsilon} [\sigma\nabla\tilde{r}_\varepsilon \cdot \nu]\tilde{r}_\varepsilon^{(\mathrm{int})} \,\mathrm{d}\sigma\,\mathrm{d}\tau + \int_0^t\!\!\int_{\Gamma^\varepsilon} [\tilde{r}_\varepsilon]\sigma_2\nabla\tilde{r}_\varepsilon^{(\mathrm{out})} \cdot \nu \,\mathrm{d}\sigma\,\mathrm{d}\tau$$
$$= \int_0^t\!\!\int_\Omega \big\{E_\varepsilon - \sigma\,\Delta(u_1\phi_\varepsilon)\big\}\tilde{r}_\varepsilon \,\mathrm{d}x\,\mathrm{d}\tau\,. \quad (6.5)$$

Next compute

$$\int_0^t\!\!\int_\Omega E_\varepsilon\tilde{r}_\varepsilon \,\mathrm{d}x\,\mathrm{d}\tau = \int_0^t\!\!\int_\Omega \sigma\big\{-\frac{1}{\varepsilon}\,\Delta_y\,u_2 + \Delta_x\,u_1\big\}\tilde{r}_\varepsilon \,\mathrm{d}x\,\mathrm{d}\tau$$
$$= \int_0^t\!\!\int_\Omega \sigma\big\{-\frac{1}{\varepsilon}\,\Delta_y\,u_2 - \mathrm{div}_x(\nabla_y u_2)\big\}\tilde{r}_\varepsilon \,\mathrm{d}x\,\mathrm{d}\tau + \int_0^t\!\!\int_\Omega \sigma\big\{\,\mathrm{div}_x(\nabla_y u_2) + \Delta_x\,u_1\big\}\tilde{r}_\varepsilon \,\mathrm{d}x\,\mathrm{d}\tau$$
$$= -\int_0^t\!\!\int_\Omega \mathrm{div}(\sigma\nabla_y u_2)\tilde{r}_\varepsilon \,\mathrm{d}x\,\mathrm{d}\tau + \int_0^t\!\!\int_\Omega \big\{\sigma\,\mathrm{div}_x(\nabla_y u_2) + \sigma\,\Delta_x\,u_1\big\}\tilde{r}_\varepsilon \,\mathrm{d}x\,\mathrm{d}\tau$$
$$= \int_0^t\!\!\int_{\Gamma^\varepsilon} [\sigma\nabla_y u_2 \cdot \nu\tilde{r}_\varepsilon] \,\mathrm{d}\sigma\,\mathrm{d}\tau + \int_0^t\!\!\int_\Omega \sigma\nabla_y u_2 \cdot \nabla\tilde{r}_\varepsilon \,\mathrm{d}x\,\mathrm{d}\tau$$
$$+ \int_0^t\!\!\int_\Omega \big\{\sigma\,\mathrm{div}_x(\nabla_y u_2) + \sigma\,\Delta_x\,u_1\big\}\tilde{r}_\varepsilon \,\mathrm{d}x\,\mathrm{d}\tau \quad (6.6)$$

Note that the last integral in (6.6) can be majorized by

$$\int_0^t\!\!\int_\Omega \big\{\sigma\,\mathrm{div}_x(\nabla_y u_2) + \sigma\,\Delta_x\,u_1\big\}\tilde{r}_\varepsilon \,\mathrm{d}x\,\mathrm{d}\tau \le \gamma(\delta) + \delta\int_0^t\!\!\int_\Omega \tilde{r}_\varepsilon^2 \,\mathrm{d}x\,\mathrm{d}\tau\,,$$

where $\delta > 0$ will be chosen in the following. We exploit here the estimate (see Subsection 7.2, and [4])

$$\int_0^T\!\!\int_\Omega (u_{2x_iy_i}^2 + u_{1x_ix_i}^2) \,\mathrm{d}x\,\mathrm{d}t \le \gamma\,. \quad (6.7)$$

Similarly, for $\delta' = \min(\sigma_1, \sigma_2)/2$,

$$-\int_0^t\!\!\int_\Omega \sigma\,\Delta(u_1\phi_\varepsilon)\tilde{r}_\varepsilon\,\mathrm{d}x\,\mathrm{d}\tau = \int_0^t\!\!\int_\Omega \sigma\nabla(u_1\phi_\varepsilon)\cdot\nabla\tilde{r}_\varepsilon\,\mathrm{d}x\,\mathrm{d}\tau \le \delta'\int_0^t\!\!\int_\Omega |\nabla\tilde{r}_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}\tau$$

$$+\frac{\gamma(\delta')}{\varepsilon^2}|\{x\in\Omega \mid \mathrm{dist}(x,\partial\Omega)\le\gamma_0\varepsilon\}| \le \delta'\int_0^t\!\!\int_\Omega |\nabla\tilde{r}_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}\tau + \frac{\gamma(\delta')}{\varepsilon}\,. \quad (6.8)$$

We have used here (see again Subsection 7.2 and [4])

$$\sup_{x\in\Omega\,,\,y\in Y\,,\,0<t<T}\big\{|u_1| + |\nabla_x u_1| + |\nabla_y u_1|\big\}(x,y,t) < \infty\,. \quad (6.9)$$

Let us also rewrite the integral appearing in (6.6)

$$\int_0^t\!\!\int_{\Gamma^\varepsilon} [\sigma\nabla_y u_2\cdot\nu\tilde{r}_\varepsilon]\,\mathrm{d}\sigma\,\mathrm{d}\tau = \int_0^t\!\!\int_{\Gamma^\varepsilon} [\sigma\nabla_y u_2\cdot\nu]\tilde{r}_\varepsilon^{(\mathrm{int})}\,\mathrm{d}\sigma\,\mathrm{d}\tau + \int_0^t\!\!\int_{\Gamma^\varepsilon} \sigma_2\nabla_y u_2^{(\mathrm{out})}\cdot\nu[\tilde{r}_\varepsilon]\,\mathrm{d}\sigma\,\mathrm{d}\tau$$

$$= -\int_0^t\!\!\int_{\Gamma^\varepsilon} [\sigma\nabla_x u_1\cdot\nu]\tilde{r}_\varepsilon^{(\mathrm{int})}\,\mathrm{d}\sigma\,\mathrm{d}\tau + \int_0^t\!\!\int_{\Gamma^\varepsilon} \sigma_2\nabla_y u_2^{(\mathrm{out})}\cdot\nu[\tilde{r}_\varepsilon]\,\mathrm{d}\sigma\,\mathrm{d}\tau\,, \quad (6.10)$$

where we have made use of (3.26) too.
Combining the previous estimates, absorbing the gradient term in (6.8) into the left hand side of (6.5), and also recalling (6.2),

$$\frac{1}{2}\int_0^t\!\!\int_\Omega \sigma|\nabla\tilde{r}_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}\tau + \int_0^t\!\!\int_{\Gamma^\varepsilon} [\tilde{r}_\varepsilon]\sigma_2\nabla\tilde{r}_\varepsilon^{(\mathrm{out})}\cdot\nu\,\mathrm{d}\sigma\,\mathrm{d}\tau - \int_0^t\!\!\int_{\Gamma^\varepsilon} [\sigma\nabla_x u_1\cdot\nu]\tilde{r}_\varepsilon^{(\mathrm{int})}\,\mathrm{d}\sigma\,\mathrm{d}\tau$$

$$\le \frac{\gamma(\delta)}{\varepsilon} + \delta\int_0^t\!\!\int_\Omega \tilde{r}_\varepsilon^2\,\mathrm{d}x\,\mathrm{d}\tau - \int_0^t\!\!\int_{\Gamma^\varepsilon} [\sigma\nabla_x u_1\cdot\nu]\tilde{r}_\varepsilon^{(\mathrm{int})}\,\mathrm{d}\sigma\,\mathrm{d}\tau + \int_0^t\!\!\int_{\Gamma^\varepsilon} \sigma_2\nabla_y u_2^{(\mathrm{out})}\cdot\nu[\tilde{r}_\varepsilon]\,\mathrm{d}\sigma\,\mathrm{d}\tau$$

$$+\int_0^t\!\!\int_\Omega \sigma\nabla_y u_2\cdot\nabla\tilde{r}_\varepsilon\,\mathrm{d}x\,\mathrm{d}\tau\,. \quad (6.11)$$

Here we drop the equal terms appearing on both sides, and then consider that, owing to (6.3),

$$\int_0^t\int_{\Gamma^\varepsilon} [\tilde{r}_\varepsilon]\sigma_2\nabla\tilde{r}_\varepsilon^{(\mathrm{out})}\cdot\nu\,\mathrm{d}\sigma\,\mathrm{d}\tau = \frac{\alpha}{\varepsilon}\int_0^t\int_{\Gamma^\varepsilon} \frac{\partial}{\partial t}[\tilde{r}_\varepsilon]\,[\tilde{r}_\varepsilon]\,\mathrm{d}\sigma\,\mathrm{d}\tau$$

$$+\frac{1}{\varepsilon}\int_0^t\int_{\Gamma^\varepsilon} [\tilde{r}_\varepsilon]Q_\varepsilon\,\mathrm{d}\sigma\,\mathrm{d}\tau - \int_0^t\int_{\Gamma^\varepsilon} [\tilde{r}_\varepsilon]\sigma_2\nabla_x u_1^{(\mathrm{out})}\cdot\nu\,\mathrm{d}\sigma\,\mathrm{d}\tau\,.$$

Also recall that, taking into account (2.5), (3.15), and the expansion in ε we assume for S_ε, we have at time $t=0$

$$\int_{\Gamma^\varepsilon} [\tilde{r}_\varepsilon]^2(0)\,\mathrm{d}\sigma \le \gamma\varepsilon\,.$$

Hence, we obtain from (6.11) that

$$\begin{aligned}\int_0^t\int_\Omega |\nabla\tilde{r}_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}\tau + \frac{1}{\varepsilon}\int_{\Gamma^\varepsilon}[\tilde{r}_\varepsilon]^2(t)\,\mathrm{d}\sigma &\le \frac{\gamma(\delta)}{\varepsilon} + \gamma\delta\int_0^t\int_\Omega \tilde{r}_\varepsilon^2\,\mathrm{d}x\,\mathrm{d}\tau \\ &+ \gamma\int_0^t\int_{\Gamma^\varepsilon}[\tilde{r}_\varepsilon]\sigma_2\Big\{\nabla_y u_2^{(\mathrm{out})} + \nabla_x u_1^{(\mathrm{out})}\Big\}\cdot\nu\,\mathrm{d}\sigma\,\mathrm{d}\tau + \gamma\int_0^t\int_\Omega \sigma\nabla_y u_2\cdot\nabla\tilde{r}_\varepsilon\,\mathrm{d}x\,\mathrm{d}\tau \\ &+ \frac{\gamma}{\varepsilon}\int_0^t\int_{\Gamma^\varepsilon}|f'(0)-f'(\mu_\varepsilon)|\,|[u_1][\tilde{r}_\varepsilon]|\,\mathrm{d}\sigma\,\mathrm{d}\tau + \frac{\gamma}{\varepsilon}\int_0^t\int_{\Gamma^\varepsilon}|f'(\mu_\varepsilon)|[\tilde{r}_\varepsilon]^2\,\mathrm{d}\sigma\,\mathrm{d}\tau\,, \end{aligned} \tag{6.12}$$

where $|\mu_\varepsilon(x,t)| \le |[u_\varepsilon]|$. By means of a simple application of Cauchy-Schwarz inequality, we may partially absorb the third and fourth terms on the right hand side of (6.12) into the left hand side. Moreover, note that

$$\begin{aligned}\frac{1}{\varepsilon}\int_0^t\int_{\Gamma^\varepsilon}|f'(0)-f'(\mu_\varepsilon)|\,|[u_1][\tilde{r}_\varepsilon]|\,\mathrm{d}\sigma\,\mathrm{d}\tau &\le \frac{\gamma}{\varepsilon}\int_0^t\int_{\Gamma^\varepsilon}|[u_\varepsilon][u_1][\tilde{r}_\varepsilon]|\,\mathrm{d}\sigma\,\mathrm{d}\tau \\ &\le \frac{\gamma}{\varepsilon}\int_0^t\int_{\Gamma^\varepsilon}[\tilde{r}_\varepsilon]^2\,\mathrm{d}\sigma\,\mathrm{d}\tau + \frac{\gamma}{\varepsilon}\int_0^t\int_{\Gamma^\varepsilon}[u_\varepsilon]^2\,\mathrm{d}\sigma\,\mathrm{d}\tau\,,\end{aligned}$$

where the last term can be bounded uniformly by means of (2.25). We have used here the boundedness of f''. Thus (6.12) yields

$$\int_0^t\int_\Omega |\nabla\tilde{r}_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}\tau + \frac{1}{\varepsilon}\int_{\Gamma^\varepsilon}[\tilde{r}_\varepsilon]^2(t)\,\mathrm{d}\sigma \le \frac{\gamma(\delta)}{\varepsilon} + \gamma\delta\int_0^t\int_\Omega \tilde{r}_\varepsilon^2\,\mathrm{d}x\,\mathrm{d}\tau + \frac{\gamma}{\varepsilon}\int_0^t\int_{\Gamma^\varepsilon}[\tilde{r}_\varepsilon]^2\,\mathrm{d}\sigma\,\mathrm{d}\tau\,.$$

We now invoke Lemma 7.1 to get, for a suitable choice of δ,

$$\int_0^t\int_\Omega |\nabla\tilde{r}_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}\tau + \frac{1}{\varepsilon}\int_{\Gamma^\varepsilon}[\tilde{r}_\varepsilon]^2(t)\,\mathrm{d}\sigma \le \frac{\gamma}{\varepsilon} + \frac{\gamma}{\varepsilon}\int_0^t\int_{\Gamma^\varepsilon}[\tilde{r}_\varepsilon]^2\,\mathrm{d}\sigma\,\mathrm{d}\tau\,. \tag{6.13}$$

Finally, after an application of Gronwall's inequality, we get

$$\int_0^t\int_\Omega |\nabla\tilde{r}_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}\tau + \frac{1}{\varepsilon}\int_{\Gamma^\varepsilon}[\tilde{r}_\varepsilon]^2(t)\,\mathrm{d}\sigma \le \frac{\gamma}{\varepsilon}\,. \tag{6.14}$$

On recalling the definition of $\tilde{r}_\varepsilon$, and invoking again Lemma 7.1, we obtain

$$\int_0^t \int_\Omega (u_\varepsilon - u_0 - \varepsilon u_1 (1-\phi_\varepsilon))^2 \,\mathrm{d}x\,\mathrm{d}\tau \le \gamma\varepsilon \,, \tag{6.15}$$

whence finally the estimate (2.13), on making use of

$$\int_0^t \int_\Omega (\varepsilon u_1(1-\phi_\varepsilon))^2 \,\mathrm{d}x\,\mathrm{d}\tau \le \gamma\varepsilon^2 \,.$$

Remark 6.1. The proof in the case when $\partial\Omega \cap \Gamma^\varepsilon \neq \emptyset$ differs from the one we present here only for minor changes, required by the slightly different form taken by the interface conditions (6.2) and (6.3). However the additional terms, appearing in formulae (6.8) and following can be dealt with by means of the same techniques employed above.

7. Auxiliary results

7.1. **Poincaré's inequality.** We give first a result central to our approach; for example, it is required to prove a uniform L^2 estimate for the sequence $\{u_\varepsilon\}$.

Lemma 7.1. *Let $v : \Omega \to \boldsymbol{R}$ be given by*

$$v_{|\Omega_1^\varepsilon} = v_{1|\Omega_1^\varepsilon} \,, \quad v_{|\Omega_2^\varepsilon} = v_{2|\Omega_2^\varepsilon} \,, \qquad v_1 \,,\, v_2 \in H^1_o(\Omega) \,.$$

Then

$$\int_\Omega v^2 \,\mathrm{d}x \le C \Big\{ \int_\Omega |\nabla v|^2 \,\mathrm{d}x + \varepsilon^{-1} \int_{\Gamma^\varepsilon} [v]^2 \,\mathrm{d}\sigma \Big\} \,. \tag{7.1}$$

Here C depends only on Ω and E.

Proof. As v^2 is of class $W^{1,1}$ both in Ω_1^ε and in Ω_2^ε, $v^2 \in BV(\Omega)$, and the usual contradiction argument, exploiting $v^2 = 0$ on $\partial\Omega$ in the sense of traces, shows that

$$\int_\Omega v^2 \,\mathrm{d}x \le \gamma |Dv^2(\Omega)| \le \gamma \int_\Omega |v||\nabla v| \,\mathrm{d}x + \gamma \int_{\Gamma^\varepsilon} |[v^2]| \,\mathrm{d}\sigma \,, \qquad \gamma = \gamma(\Omega) \,. \tag{7.2}$$

Indeed the singular part of the variation of v (and therefore of v^2) is concentrated on Γ^ε. We estimate above last integral by

$$\int_{\Gamma^\varepsilon} |[v]| (|v^{(\mathrm{int})}| + |v^{(\mathrm{out})}|) \,\mathrm{d}\sigma \le (\delta\varepsilon)^{-1} \int_{\Gamma^\varepsilon} [v]^2 \,\mathrm{d}\sigma + \delta\varepsilon \int_{\Gamma^\varepsilon} (|v^{(\mathrm{int})}|^2 + |v^{(\mathrm{out})}|^2) \,\mathrm{d}\sigma \,, \tag{7.3}$$

for a $\delta \in (0,1)$ to be chosen presently. Exploiting the periodicity of E, and standard trace inequalities, we check that for each cell $Q_i = \varepsilon(Y + z_i)$, $z_i \in \boldsymbol{Z}^N$,

$$\int_{\Gamma^\varepsilon \cap Q_i} (|v^{(\mathrm{int})}|^2 + |v^{(\mathrm{out})}|^2) \,\mathrm{d}\sigma \le \gamma\varepsilon^{-1} \int_{\Omega\cap Q_i} \big(v^2 + \varepsilon^2 |\nabla v|^2 \big) \,\mathrm{d}x \,, \tag{7.4}$$

where $\gamma = \gamma(E)$ does not depend on Q_i. Next we add (7.4) over all the cells covering Ω, and use the resulting inequality in (7.3). A further application of Cauchy-Schwartz inequality to (7.2) yields

$$\int_\Omega v^2 \,\mathrm{d}x \le \gamma\delta^{-1} \int_\Omega |\nabla v|^2 \,\mathrm{d}x + \gamma(\delta\varepsilon)^{-1} \int_{\Gamma^\varepsilon} [v]^2 \,\mathrm{d}\sigma + \gamma\delta \int_\Omega v^2 \,\mathrm{d}x \,,$$

whence (7.1) on selecting a small enough δ. □

Remark 7.2. The factor ε^{-1} in (7.1) is necessary in general, as one can show easily by counterexample. However, if E is connected, this is not the case: actually, one can prove an estimate similar to (7.1), but with the factor ε^{-1} formally replaced by ε (in this spirit, see Lemma 6 of [28]).

7.2. **Regularity results.**

Lemma 7.3. *Let M_0, M_1 be two $N\times N$ real matrices. We assume that M_0 is constant and non singular, while the entries of M_1 are functions of t of class $L^2(0,T)$. Let $\eta \in L^2(\Omega \times (0,T))^N$, $u \in L^2(\Omega \times (0,T))$, and assume that a.e. $t > 0$, for all $\varphi \in C^1_o(\Omega)$,*

$$\int_\Omega \Big\{ M_0 u(x,t) + \int_0^t M_1(t-\tau) u(x,\tau) \,\mathrm{d}\tau \Big\} \nabla\varphi(x) \,\mathrm{d}x = \int_\Omega \eta(x,t)\varphi(x) \,\mathrm{d}x \,. \qquad (7.5)$$

Then $u \in L^2(0,T;H^1(\Omega))$, and

$$-\eta(x,t) = M_0 \nabla u(x,t) + \int_0^t M_1(t-\tau)\nabla u(x,\tau) \,\mathrm{d}\tau \,. \qquad (7.6)$$

Proof. First, we may clearly assume $M_0 = I$, by multiplying (7.5) by M_0^{-1}.
Next, introduce a smooth symmetric mollifying kernel $k_h(x)$ such that $\operatorname{supp} k_h \subset \{|x| \le h\}$, and approaching Dirac's delta as $h \to 0$. Fix $h > 0$, and select φ as above such that $h < \operatorname{dist}(\operatorname{supp}\varphi, \partial\Omega)$. Denoting by $*$ the standard convolution operator in $\boldsymbol{R}^N$, choose $\varphi * k_h$ as a testing function in (7.5). An application of Fubini's theorem shows that $u_h = u * k_h$ still satisfies (7.5), with η replaced by $\eta_h = \eta * k_h$. Then, integrating by parts, and using the arbitrariness of φ we obtain

$$-\eta_h(x,t) = \nabla u_h(x,t) + \int_0^t M_1(t-\tau)\nabla u_h(x,\tau) \,\mathrm{d}\tau \,,$$

a.e. $x \in \Omega_h = \{x \in \Omega \mid \operatorname{dist}(x,\partial\Omega) > h\}$. Whence, for a.e. $t > 0$,

$$\int_{\Omega_h} |\nabla u_h(x,t)|^2 \,\mathrm{d}x \le 2 \int_\Omega |\eta(x,t)|^2 \,\mathrm{d}x + \gamma \int_0^t \int_{\Omega_h} |\nabla u_h(x,\tau)|^2 \,\mathrm{d}x \,\mathrm{d}\tau \,,$$

where $\gamma = \gamma(M_1, T, \Omega)$, which in turn yields, by virtue of Gronwall's lemma (this is the observation that makes the result possible),

$$\int_0^T \int_{\Omega_h} |\nabla u_h(x,t)|^2 \, \mathrm{d}x \, \mathrm{d}t \le \gamma(\eta, M_1, T, \Omega) \,, \tag{7.7}$$

for a γ independent of h. This shows that u is as regular as claimed. Equality (7.6) follows again from integrating (7.5) by parts. □

Our next aim is to prove the smoothness of the cell functions used in previous sections. In view of the existence and uniqueness results proved in [5], it is enough to find the required regularity results in the form of a priori estimates. We denote by $C^\infty_{\mathrm{per}}(A)$ the space of functions of class C^∞ which are Y-periodic in A. Similarly we define $H^m_{\mathrm{per}}(E_i)$ the space of Y-periodic functions in E_i, of class H^m. Finally $H^{m-\frac{1}{2}}_{\mathrm{per}}(\Gamma)$ is the space of traces on Γ of functions in $H^m_{\mathrm{per}}(E_i)$.

Lemma 7.4. *Let $P_{|E_i} \in C^\infty_{\mathrm{per}}(\overline{E}_i)$, $i = 1, 2$, and Q, $s \in C^\infty_{\mathrm{per}}(\Gamma)$; we also assume that*

$$\int_Y P \, \mathrm{d}y = \int_\Gamma Q \, \mathrm{d}\sigma \,. \tag{7.8}$$

Then the solution v to

$$-\sigma \, \Delta \, v = P \,, \qquad \text{in } E_1, \, E_2; \tag{7.9}$$

$$[\sigma \nabla v \cdot \nu] = Q \,, \qquad \text{on } \Gamma; \tag{7.10}$$

$$[v] = s \,, \qquad \text{on } \Gamma; \tag{7.11}$$

satisfies $v_{|E_i} \in C^\infty_{\mathrm{per}}(\overline{E}_i)$, $i = 1, 2$.

Proof. First we reduce to the case of a standard diffraction problem, where $Q \equiv s \equiv 0$ on Γ, by subtracting from $v_{|E_2}$ a suitable function $w \in C^\infty_{\mathrm{per}}(\overline{E_2})$ such that on Γ, $w = s$, $\sigma_2 \nabla w \cdot \nu = Q$. Then the result follows by local rectification of Γ, and iterated use of Theorem 16.2 of [27]. □

Corollary 7.5. *Assume that $P \in C^\infty([0,T]; C^\infty_{\mathrm{per}}(\overline{E}_i))$, $i = 1, 2$, and that Q, $h \in C^\infty([0,T]; C^\infty_{\mathrm{per}}(\Gamma))$, $s \in C^\infty_{\mathrm{per}}(\Gamma)$, and that (7.8) holds at every time level. For all integers $k \ge 0$ set $v^k(x,t) = \partial^k v / \partial t^k(x,t)$, where v is the solution to*

$$-\sigma \, \Delta \, v = P(t) \,, \qquad \text{in } E_1, \, E_2; \tag{7.12}$$

$$[\sigma \nabla v \cdot \nu] = Q(t) \,, \qquad \text{on } \Gamma; \tag{7.13}$$

$$\alpha \frac{\partial}{\partial t}[v] + f'(0)[v] = \sigma_2 \nabla v^{(\mathrm{out})} \cdot \nu + h(t) \,, \qquad \text{on } \Gamma; \tag{7.14}$$

$$[v](y, 0) = s \,, \qquad \text{on } \Gamma \,. \tag{7.15}$$

Then for all $k \ge 0$, $v^k(\cdot, 0)_{|E_i} \in C^\infty(\overline{E}_i)$ $i = 1, 2$.

Proof. The statement follows directly from Lemma 7.4 in the case $k = 0$. Then we observe that each $v^k(\cdot, 0)$ solves an elliptic problem similar to the one solved by $v(\cdot, 0)$, where the partial differential equations and the interface condition are found by differentiating in time (7.12) and (7.13), while the initial condition is given by (7.14), in terms of v^{k-1}. We conclude the proof reasoning by induction: at each step k, the data in those conditions are smooth, as a consequence of the regularity of v^{k-1}. $\square$

Theorem 7.6. *Under the same assumptions of Corollary 7.5, the solution v to (7.12)–(7.15) satisfies $v \in C^\infty([0,T]; C^\infty_{\mathrm{per}}(\overline{E}_i))$, $i = 1, 2$.*

Proof. First of all, we note that if we set $\beta = f'(0)$, and $\tilde{v}(x,t) = v(x,t)e^{\frac{\beta}{\alpha}t}$, then $\tilde{v}$ satisfies (7.12)–(7.15) where P, Q and h are replaced by $P(x,t)e^{\frac{\beta}{\alpha}t}$, $Q(x,t)e^{\frac{\beta}{\alpha}t}$, $h(x,t)e^{\frac{\beta}{\alpha}t}$, respectively, and (7.14) is rewritten with $f'(0) = 0$. Hence, without loss of generality, we will assume that $f'(0) = 0$ in (7.14).
Now, we differentiate k times with respect to t the problem satisfied by v, obtaining a problem for $v^k = \partial^k v/\partial t^k$. Routine calculations performed on this formulations yield for each $k \in \boldsymbol{N}$

$$\int\limits_Y |\nabla v^k(y,t)|^2 \,\mathrm{d}y + \int\limits_\Gamma [v^k]^2(y,t)\,\mathrm{d}\sigma \le \gamma(k)\,, \qquad 0 < t < T\,. \tag{7.16}$$

The constant γ here depends also on the values of the functions $v^k(\cdot, 0)$, which however are bounded because of Corollary 7.5. In fact, on multiplying (7.12) written for v^k by v^k itself and integrating formally by parts on Y, using also (7.13)–(7.15) and the periodicity of the involved functions, it follows

$$\begin{aligned}
\int\limits_Y |\nabla v^k(y,t)|^2 \,\mathrm{d}y &\le \gamma \left(1 + \int\limits_\Gamma |[v^{k+1}][v^k]|(y,t)\,\mathrm{d}\sigma \right) \\
&\le \gamma \left(1 + \int\limits_\Gamma [v^{k+1}]^2(y,t)\,\mathrm{d}\sigma + \int\limits_\Gamma [v^k]^2(y,t)\,\mathrm{d}\sigma \right) \le \gamma(v^{k+1}(0), v^k(0))\,,
\end{aligned}$$

where we used also (2.25) applied to v^k and v^{k+1} in the periodic cell, with non zero sources.
Estimate (7.16), together with the Poincaré's inequality of [30], implies that $v^{k\,(\mathrm{int})}$, $v^{k\,(\mathrm{out})}$ have traces belonging to $L^\infty(0,T; H^{1/2}_{\mathrm{per}}(\Gamma))$.
As a consequence of (7.16), v^k can be seen, separately in E_1 and in E_2, as the local solution to smooth elliptic problems of Neumann type, with boundary data on Γ of class $L^\infty(0,T; H^{1/2}_{\mathrm{per}}(\Gamma))$. These boundary conditions are in fact obtained on differentiating k times, with respect to t, (7.13) and (7.14). At this stage, we use the fact that $[v^{k+1}]$ belongs to $H^{1/2}_{\mathrm{per}}(\Gamma)$, uniformly in time, as implied by (7.16). Thus the classical results of [1] yield (after local rectification of Γ) that v^k is of class $L^\infty(0,T; H^2(E_i))$, $i = 1, 2$. This regularity for v^{k+1}, together with the condition obtained differentiating (7.14) k times with respect to t, implies that the Neumann data for v^k are in $H^{3/2}_{\mathrm{per}}(\Gamma)$, uniformly in t. Therefore $v^k \in L^\infty(0,T; H^3_{\mathrm{per}}(E_i))$, $i = 1$, 2. A standard bootstrap argument gives that, for every $k \in \boldsymbol{N}$ and every $m \ge 1$, $v^k \in L^\infty(0,T; H^m_{\mathrm{per}}(E_i))$, $i = 1$, 2; i.e., for every $m \ge 1$, $v \in H^m(E_i \times (0,T))$, $i = 1$, 2, which implies the required regularity for v. $\square$

7.3. **Compactness results.** We first note that $[u_\varepsilon] \in H^1_{\mathrm{loc}}(0,T;L^2(\Gamma^\varepsilon))$ can be proven by means of Steklov averages techniques. More exactly, it holds

$$\frac{1}{\varepsilon}\int_{t_0}^{T}\int_{\Gamma^\varepsilon}|[u_\varepsilon]_t|^2 \,\mathrm{d}\sigma\,\mathrm{d}t \le \gamma(t_0)\,, \qquad \text{for all } 0 < t_0 < T. \tag{7.17}$$

This will be used in the following result.

Lemma 7.7. *Let u_ε be the solution to* (2.1)–(2.6)*. For all $\varepsilon > 0$, extend u_ε to 0 in $\boldsymbol{R}^N \setminus \Omega$. Then*

$$\int_0^T\int_\Omega |u_\varepsilon(x+h,t) - u_\varepsilon(x,t)|\,\mathrm{d}x\,\mathrm{d}t \le \gamma_1|h|\,, \qquad \text{for all } h \in B_1(0); \tag{7.18}$$

$$\int_{t_0}^{T-\tau}\int_\Omega |u_\varepsilon(x,t+\tau) - u_\varepsilon(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t \le \gamma_2\tau^2\,, \qquad \text{for all } 0 < \tau < T - t_0, \tag{7.19}$$

for all $0 < t_0 < T$. Here γ_1 and γ_2 do not depend on ε, h or τ; γ_2 depends on t_0 as $t_0 \to 0$, and on T as well.

Proof. We use the fact that $u_\varepsilon(t) \in BV(G)$ for each open set G containing $\overline{\Omega}$ (a.e. t). Then

$$\begin{aligned}\int_0^T\int_\Omega |u_\varepsilon(x+h,t) - u_\varepsilon(x,t)|\,\mathrm{d}x\,\mathrm{d}t &\le |h|\int_0^T |Du_\varepsilon(t)|(G)\,\mathrm{d}t = |h|\int_0^T |Du_\varepsilon(t)|(\Omega)\,\mathrm{d}t \\ &\le \gamma|h|\Big\{\int_0^T\int_\Omega |\nabla u_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t + \varepsilon^{-1}\int_0^T\int_{\Gamma^\varepsilon}[u_\varepsilon]^2\,\mathrm{d}\sigma\,\mathrm{d}t\Big\}^{1/2},\end{aligned}$$

yielding (7.18), when we take into account the energy estimate (2.25).
Next we turn to the proof of (7.19). Define

$$z_\varepsilon(x,t) = u_\varepsilon(x,t+\tau) - u_\varepsilon(x,t)\,. \tag{7.20}$$

Clearly z_ε solves a problem similar to (2.1)–(2.5), so that we can prove the analog of (2.25), for $t_1 < t < T - \tau$

$$\int_{t_1}^{t}\int_\Omega \sigma|\nabla z_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}\theta + \frac{\alpha}{2\varepsilon}\int_{\Gamma^\varepsilon}[z_\varepsilon]^2(t)\,\mathrm{d}\sigma \;\le\; \frac{\gamma}{\varepsilon}\int_{\Gamma^\varepsilon}[z_\varepsilon]^2(t_1)\,\mathrm{d}\sigma + \frac{\gamma}{\varepsilon}\int_{t_1}^{t}\int_{\Gamma^\varepsilon}[z_\varepsilon]^2\,\mathrm{d}\sigma\,\mathrm{d}\theta\,. \tag{7.21}$$

By an application of Gronwall's lemma, we may essentially drop the last term in the formula above, also by redefining the constant γ appearing in the first term on the right hand side. Using the definition of z_ε, we arrive at

$$\int_{t_1}^{t}\int_{\Omega}\sigma|\nabla z_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}\theta+\frac{\alpha}{2\varepsilon}\int_{\Gamma^\varepsilon}[z_\varepsilon]^2(t)\,\mathrm{d}\sigma\le\tau\frac{\gamma}{\varepsilon}\int_{t_1}^{t_1+\tau}\int_{\Gamma^\varepsilon}|[u_\varepsilon]_\theta|^2\,\mathrm{d}\sigma\,\mathrm{d}\theta\,. \tag{7.22}$$

On integrating the inequality above in t_1 over $(t_0/2,t)$ we find after straightforward calculations, for all $0<t_0<t\le T-\tau$,

$$t_0\int_{t_0}^{t}\int_{\Omega}\sigma|\nabla z_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}\theta+t_0\frac{\alpha}{2\varepsilon}\int_{\Gamma^\varepsilon}[z_\varepsilon]^2(t)\,\mathrm{d}\sigma\le\tau^2\frac{\gamma}{\varepsilon}\int_{t_0/2}^{T}\int_{\Gamma^\varepsilon}|[u_\varepsilon]_\theta|^2\,\mathrm{d}\sigma\,\mathrm{d}\theta\le\gamma(t_0)\tau^2\,,$$

where we invoked (7.17). The bound in (7.19) follows now from the estimate above and from Lemma 7.1. □

Remark 7.8. a) When dealing with the non homogeneous problem (2.15)–(2.18), the moduli of continuity on the right hand sides of (7.18) and of (7.19) depend on $\hat{u}$ too.
b) Estimate (7.19) actually implies that $u_{\varepsilon t}$, $u_{0t}\in L^2_{\mathrm{loc}}(0,T;L^2(\Omega))$.

Lemma 7.9. *Let $G\subset\boldsymbol{R}^N$ be a bounded open set. Assume that $v_n\to v$ weakly in $L^2(G)$, and $u_n\to u$ strongly in $L^1_{\mathrm{loc}}(G)$. Assume moreover that the norms of u_n and u in $L^2(G)$ are uniformly bounded, and that for all $\eta>0$ there exist a $k>0$, and a compact set $I\subset G$ such that*

$$\int_{\{|v_n|>k\}}v_n^2\,\mathrm{d}x<\eta\,;\qquad\int_{G\setminus I}v_n^2\,\mathrm{d}x<\eta\,. \tag{7.23}$$

Then

$$\int_G u_nv_n\,\mathrm{d}x\to\int_G uv\,\mathrm{d}x\,. \tag{7.24}$$

Proof. Since

$$\int_G u_nv_n-\int_G uv=\int_G u(v_n-v)+\int_G v_n(u_n-u)\,,$$

we only need show that the last integral becomes vanishingly small as $n\to\infty$. Fix $\eta>0$ and k, I as in (7.23). Write, denoting $V_k=\{|v_n|>k\}$

$$\int_G v_n(u_n-u)=\int_{V_k\cup(G\setminus I)}v_n(u_n-u)+\int_{(G\setminus V_k)\cap I}v_n(u_n-u)=:J_{1n}+J_{2n}\,.$$

Next we bound

$$|J_{1n}|^2\le2\Big\{\int_{V_k}v_n^2+\int_{G\setminus I}v_n^2\Big\}\int_G(u_n^2+u^2)\le\text{constant}\cdot\eta\,.$$

Finally,

$$|J_{2n}|\le k\int_I|u_n-u|\to0\,,\qquad\text{as }n\to\infty.$$

□

Remark 7.10. In Section 5 we apply Lemma 7.9 with $v_n = \partial w_i^\varepsilon / \partial x_j$, $u_n = u_\varepsilon \partial\varphi/\partial x_j$. The equiintegrability in (7.23) follows by periodicity.

7.4. **The case of $g \not\equiv 0$.** In this case all the proofs stay the same, with minor changes. The only result requiring a different approach is the compactness estimate (7.19).

Estimate (7.17) is not implied by a Steklov averages technique, which instead yields

$$\frac{\alpha}{\varepsilon}\int_{t_0}^{t}\int_{\Gamma^\varepsilon}[z_\varepsilon]^2(x,\theta)\,\mathrm{d}\sigma\,\mathrm{d}\theta \le \gamma(t_0)\tau^2 + \tau\int_{t_0}^{t}\int_{\Omega}(g)_\tau z_\varepsilon(x,t)\,\mathrm{d}x\,\mathrm{d}\theta\,, \qquad (7.25)$$

where z_ε is as in (7.20), and $0 < t_0 < t < T-\tau$. Moreover we set

$$(g)_\tau(x,t) = \frac{1}{\tau}\int_{t}^{t+\tau} g(x,\theta,u_\varepsilon(x,\theta))\,\mathrm{d}\theta\,.$$

Note that, owing to the uniform boundedness of $(g)_\tau$ and of z_ε in $L^2(\Omega\times(0,T))$, (7.25) immediately yields

$$\frac{\alpha}{\varepsilon}\int_{t_0}^{t}\int_{\Gamma^\varepsilon}[z_\varepsilon]^2(x,\theta)\,\mathrm{d}\sigma\,\mathrm{d}\theta \le \gamma(t_0)\tau\,. \qquad (7.26)$$

Now inequality (7.21) is replaced with

$$\begin{aligned}\int_{t_1}^{t}\int_{\Omega}\sigma|\nabla z_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}\theta + \frac{\alpha}{2\varepsilon}\int_{\Gamma^\varepsilon}[z_\varepsilon]^2(t)\,\mathrm{d}\sigma \le \frac{\gamma}{\varepsilon}\int_{\Gamma^\varepsilon}[z_\varepsilon]^2(t_1)\,\mathrm{d}\sigma \\ + \frac{\gamma}{\varepsilon}\int_{t_1}^{t}\int_{\Gamma^\varepsilon}[z_\varepsilon]^2\,\mathrm{d}\sigma\,\mathrm{d}\theta + L\int_{t_1}^{t}\int_{\Omega}z_\varepsilon^2\,\mathrm{d}x\,\mathrm{d}\theta + \gamma\omega(\tau)\,, \quad (7.27)\end{aligned}$$

where $0 < t_1 < t < T-\tau$. Indeed, owing to (2.22), (2.24), one has

$$\begin{aligned}&|g(x,t+\tau,u_\varepsilon(x,t+\tau)) - g(x,t,u_\varepsilon(x,t))|\,|z_\varepsilon(x,t)| \\ &\qquad\le Lz_\varepsilon(x,t)^2 + \gamma\omega(\tau)|u_\varepsilon(x,t)||z_\varepsilon(x,t)|\,.\end{aligned}$$

Recalling that L is small, according to (2.23), we may invoke Poincaré's inequality (7.1) to partially absorb the third integral on the right hand side of (7.27) into the left hand side. We obtain in this fashion

$$\int_{t_1}^{t} \int_{\Omega} \sigma |\nabla z_\varepsilon|^2 \,\mathrm{d}x \,\mathrm{d}\theta + \frac{\alpha}{2\varepsilon} \int_{\Gamma^\varepsilon} [z_\varepsilon]^2(t) \,\mathrm{d}\sigma \le \frac{\gamma}{\varepsilon} \int_{\Gamma^\varepsilon} [z_\varepsilon]^2(t_1) \,\mathrm{d}\sigma + \frac{\gamma}{\varepsilon} \int_{t_1}^{t} \int_{\Gamma^\varepsilon} [z_\varepsilon]^2 \,\mathrm{d}\sigma \,\mathrm{d}\theta + \gamma\omega(\tau)\,.$$

On integrating this inequality in t_1 over $(t_0/2, t)$, and applying (7.26) we get for $t \ge t_0$

$$t_0 \int_{t_0}^{t} \int_{\Omega} \sigma |\nabla z_\varepsilon|^2 \,\mathrm{d}x \,\mathrm{d}\theta + t_0 \frac{\alpha}{2\varepsilon} \int_{\Gamma^\varepsilon} [z_\varepsilon]^2(t) \,\mathrm{d}\sigma \le \gamma\omega(\tau) + \frac{\gamma}{\varepsilon} \int_{t_0/2}^{t} \int_{\Gamma^\varepsilon} [z_\varepsilon]^2(t_1) \,\mathrm{d}\sigma \le \gamma(t_0)\omega(\tau)$$

(we may clearly assume $\omega(\tau) \ge \tau$), whence, on applying a last time Poincaré's inequality, we infer a compactness estimate similar to (7.19), but with modulus of continuity given in terms of ω, rather than as a linear function. However, the compactness obtained in this way is sufficient for our purposes in this paper, see Section 5. If $\omega(\tau) = \tau$, a refinement of the proof above gives exactly (7.19) for u_ε.

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