FREE-DISCONTINUITY PROBLEMS GENERATED BY SINGULAR PERTURBATION: THE n-DIMENSIONAL CASE

Roberto ALICANDRO and Maria Stella GELLI

SISSA, via Beirut 4, 34013 Trieste, Italy

Abstract

We provide an approximation of some free discontinuity problems by local functionals with a singular perturbation of higher order. More precisely, we study the limiting behaviour of energies of the form

$$F_\varepsilon(u) = \frac{1}{\varepsilon}\int_\Omega f(\varepsilon|\nabla u|^2)\,dx + \varepsilon^3\int_\Omega |Hu|^2\,dx,$$

where Hu denotes the Hessian matrix of u.

e-mail: alicandr@sissa.it gelli@sissa.it

1. Introduction

In many problems of the Calculus of Variations involving competing bulk and surface energies, we encounter functionals of the form

$$\mathcal{E}(u,K)=\int_{\Omega\setminus K} f(\nabla u)\,dx+\int_K \vartheta(u^+-u^-,\nu)\,d\mathcal{H}^{n-1}, \tag{1.1}$$

where $\Omega\subset\mathbf{R}^n$ is a fixed bounded open subset of $\mathbf{R}^n$, K is a (sufficiently smooth) unknown closed hypersurface with normal ν and the unknown function u belongs to a class of (sufficiently smooth) functions defined in $\Omega\setminus K$ with traces $u^\pm$ on both sides of K. As K can be interpreted as the set of discontinuity points of u problems of this kind are usually labelled as "free-discontinuity problems". For examples of energies of the form (1.1) which arise from problems in Computer Vision, Fracture Mechanics, the theory of Liquid Crystals, Minimal Surfaces, etc. we refer to [**25**] and to the Introduction of [**15**].

In order to have a weak formulation of minimum problems involving functionals as in (1.1) to which apply the direct methods of the Calculus of Variations, De Giorgi and Ambrosio introduced the space of *special functions of bounded variation* (see [**25**], [**10**]). A function u belongs to $SBV(\Omega)$ if and only if its distributional derivative Du is a bounded measure that can be split into a bulk and a surface term. This definition can be further specified: if $u\in SBV(\Omega)$ and $S(u)$ stands for the complement of the set of the Lebesgue points for u then a measure-theoretical normal ν_u to $S(u)$ can be defined $\mathcal{H}^{n-1}$-a.e. on $S(u)$, together with the traces $u^\pm$ on both sides of $S(u)$; moreover, the approximate gradient ∇u exists a.e. on Ω, and we have

$$Du=\nabla u\,\mathcal{L}_n+(u^+-u^-)\nu_u\,\mathcal{H}^{n-1}\llcorner S(u)\,. \tag{1.2}$$

In $SBV(\Omega)$ we can thus consider a weak version of the energies (1.1), formally substituting K by $S(u)$. An existence theory for problems involving these kinds of energies has been developed by Ambrosio [**5**]–[**8**]. Various regularity results show that for a wide class of problems the weak solution u in $SBV(\Omega)$ provides a solution to the original free-discontinuity problem, taking $K=\overline{S(u)}$ (see [**26**], [**21**], [**9**], [**13**], [**28**]).

To bypass the difficulties that arise from the presence of energies depending on an unknown discontinuity hypersurface, many variational approximations have been proposed by functionals depending on Sobolev functions (see [**11**], [**12**], [**17**], [**30**], [**19**], [**4**], [**29**], [**1**] and the book [**15**]). In the simplest isotropic case, heuristic arguments suggest the use of functionals of the form

$$\frac{1}{\varepsilon}\int_\Omega f(\varepsilon|\nabla u|^2)\,dx, \tag{1.3}$$

with f satisfying the conditions

$$f'(0)=\alpha \text{ and } \lim_{t\to+\infty} f(t)=\beta, \tag{1.4}$$

with $0<\alpha,\beta<+\infty$ (see the Introduction of [**15**]). Unfortunately, the use of local integral functionals of the form (1.3) as approximate energies is forbidden in the framework of Γ-convergence, as easy convexity arguments show. Hence, the approximations quoted above have to use either more complex functionals with additional variables, or non-local versions of the functionals in (1.3).

The convexity constraint that forbids the use of functionals in (1.3) can be removed by a singular perturbation approach, by considering, for example, energies of the form

$$F_\varepsilon(u)=\frac{1}{\varepsilon}\int_\Omega f(\varepsilon|\nabla u|^2)\,dx+\varepsilon^\gamma\int_\Omega |Hu|^2\,dx, \tag{1.5}$$

where Hu denotes the Hessian matrix of u. In a previous paper in collaboration with A. Braides, we considered functionals of the form (1.5) in the 1-dimensional case $n=1$, showing that if f satisfies conditions (1.4) then the Γ-limit of F_ε as $\varepsilon\to 0^+$ can be expressed on $SBV(\Omega)$ as

$$F(u)=\alpha\int_\Omega |u'|^2\,dx+c\sum_{S(u)}\sqrt{|u^+-u^-|} \tag{1.6}$$

when $\gamma=3$ (the other cases giving trivial limits), with c explicitly computable from β. In this paper we carry on this study to the higher-dimensional case, when the Γ-limit takes the form

$$F(u)=\alpha\int_\Omega |\nabla u|^2\,dx+c\int_{S(u)}\sqrt{|u^+-u^-|}\,d\mathcal{H}^{n-1}. \tag{1.7}$$

The extension of the 1-dimensional case is not obvious, and requires some non-trivial improvements of previous techniques. While a lower bound for the Γ-limit can be obtained by reducing to the 1-dimensional case through standard slicing procedures, the proof of the upper bound requires some extra care to show that the presence of second derivatives of u does not introduce a dependence on the curvature of $S(u)$ in the surface part of the limit energy. To this end, we use a density theorem of functions with polyhedral jump set $S(u)$ in $SBV(\Omega)$ by Cortesani and Toader ([**23**]). For functions of this class the possible effect of Hu is restricted to a neighbourhood of a set of Hausdorff dimension $n-2$, which can be then more easily taken care of.

2. Preliminaries

The natural number n will be fixed. We denote by $\langle \cdot,\cdot \rangle$ the scalar product in $\mathbf{R}^n$; $|\cdot|$ will be the usual euclidean norm. We denote by $\mathbf{M}^{n\times n}$ the space of $n\times n$ real matrices and by $\|\cdot\|$ the norm on $M^{n\times n}$ defined by $\|M\| = \sup_{|\xi|=1} |\langle M\xi,\xi\rangle|$. If $a,b\in\mathbf{R}^n$ the *tensor product* $a\otimes b\in M^{n\times n}$ is the matrix whose entries are a_ib_j with $i,j=1,..n$.

We use standard notation for Sobolev and Lebesgue spaces. We denote by $\mathcal{L}^n$ the Lebesgue measure in $\mathbf{R}^n$ and by $\mathcal{H}^k$ the k-dimensional Hausdorff measure. If Ω is a bounded open subset of $\mathbf{R}^n$, $\mathcal{A}(\Omega)$ and $\mathcal{B}(\Omega)$ are the families of open and Borel subsets of Ω, respectively. If μ is a Borel measure and B is a Borel set, then the measure $\mu \llcorner B$ is defined as $\mu\llcorner B(A) = \mu(A\cap B)$. Let $A',A\in\mathcal{A}(\Omega)$ be such that $A'\subset A$ and $\mathrm{dist}\,(A',\Omega\setminus A)>0$; by a *cut-off function between* A' *and* A we mean a function $\phi\in C^\infty(\overline{\Omega})$ with $0\le\phi\le1$ and such that $\phi=1$ on A' and $\phi=0$ on $\Omega\setminus A$.

Let $u\in L^1(\Omega)$. We say that u is a *function of bounded variation* if its distributional first derivatives D_iu are measures with finite total variation. This space is denoted by $BV(\Omega)$.

If u is a measurable function we denote by $S(u)$ the complement of the Lebesgue set of u, i.e. $x\notin S(u)$ if and only if

$$\lim_{\rho\to0+}\rho^{-n}\int_{B_\rho(x)}|u(y)-z|\,dy=0$$

for some $z\in\mathbf{R}$. If z exists then it is unique and we denote it by $\overline{u}(x)$. For any function $u\in L^1(\Omega)$ the set $S(u)$ is Lebesgue-negligible and $\overline{u}$ is a Borel function equal to u almost everywhere.

If $u\in BV(\Omega)$, then $S(u)$ is $(\mathcal{H}^{n-1},n-1)$-rectifiable, i.e. there is a countable family of C^1 hypersurfaces Γ_i such that $\mathcal{H}^{n-1}(S(u)\setminus\bigcup_{i=0}^\infty\Gamma_i)=0$. Furthermore, there exist Borel functions $\nu_u: S(u)\to S^{n-1}$ and $u^+,u^-: S(u)\to\mathbf{R}$ such that, for $\mathcal{H}^{n-1}$- a.e. $x\in S(u)$,

$$\lim_{\rho\to0+}\rho^{-n}\int_{B^+_\rho(x)\cap\Omega}|u(y)-u^+(x)|\,dy=0,$$

$$\lim_{\rho\to0+}\rho^{-n}\int_{B^-_\rho(x)\cap\Omega}|u(y)-u^-(x)|\,dy=0,$$

where $B^\pm_\rho(x):=\{y\in B_\rho(x):\langle y-x,\pm\nu_u(x)\rangle>0\}$. The triplet $(u^+(x),u^-(x),\nu_u(x))$ is uniquely determined up to a change of sign of $\nu_u(x)$ and a simultaneous interchange between $u^+(x)$ and $u^-(x)$.

We say that a set E is of *a set of finite perimeter* in Ω if $\chi_E \in BV(\Omega)$. We will set $\partial^* E = S(\chi_E)$, the *reduced boundary* of u in Ω, and $\nu_E = \nu_{\chi_E}$.

We recall that if $u \in BV(\Omega)$, then for a.e. $t \in \mathbf{R}$ the set $\{u > t\}$ is of finite perimeter in Ω, and the *co-area formula* holds:

$$|Du|(\Omega) = \int_{-\infty}^{+\infty} \mathcal{H}^{n-1}(\partial^*\{u > t\} \cap \Omega)\, dt. \tag{2.1}$$

We recall also the *Fleming and Rishel co-area formula*: let u be a Lipschitz function; then for every $v \in BV(\Omega)$ we have that

$$\int_\Omega v|\nabla u|\, dx = \int_{-\infty}^{+\infty} dt \int_{\partial^*\{u>t\}} \overline{v}\, d\mathcal{H}^{n-1}. \tag{2.2}$$

For $u \in BV(\Omega)$ the Lebesgue decomposition $Du = D^a u + D^s u$ holds, where $D^a u$ is the absolutely continuous part of Du with respect to the Lebesgue measure and $D^s u \perp \mathcal{L}^n$. The density of $D^a u$ with respect to the Lebesgue measure is denoted by ∇u.

We say that $u \in BV(\Omega)$ is a *special function of bounded variation* if the singular part of Du is concentrated on $S(u)$, i.e. $|D^s u|(\Omega \setminus S(u)) = 0$. In this case $|D^s u| << \mathcal{H}^{n-1} \llcorner S(u)$. This space is denoted by $SBV(\Omega)$.

For each $p \geq 1$ the space of all functions $u \in SBV(\Omega)$ such that

$$\nabla u \in L^p(\Omega) \quad \text{and} \quad \mathcal{H}^{n-1}(S(u)) < +\infty$$

is denoted by $SBV^p(\Omega)$. The subspace of $u \in SBV^\infty(\Omega)$ such that $\nabla u = 0$ a.e. in Ω is denoted by $SBV_0(\Omega)$.

Let $\mathcal{W}(\Omega)$ be the space of all functions $w \in SBV(\Omega)$ satisfying the following properties:

(i) $\mathcal{H}^{n-1}\left(\overline{S(w)} \setminus S(w)\right) = 0$;

(ii) $\overline{S}(w)$ is the intersection of Ω with the union of a finite number of $(n-1)$-dimensional simplexes;

(iii) $w \in W^{k,\infty}\left(\Omega \setminus \overline{S(w)}\right)$ for every $k \in \mathbf{N}$.

The following density result of $\mathcal{W}(\Omega)$ in $SBV^p(\Omega)$ is due to Cortesani and Toader [**23**] (see also [**16**]).

Theorem 2.1. *Let Ω be an open subset of $\mathbf{R}^n$ with Lipschitz boundary. Let $u \in SBV^p(\Omega) \cap L^\infty(\Omega)$. Then there exists a sequence (w_h) in $\mathcal{W}(\Omega)$ such that*

$$w_h \to u \quad \text{strongly in } L^1(\Omega), \tag{2.3}$$

$$\nabla w_h \to \nabla u \quad \text{strongly in } L^p(\Omega), \tag{2.4}$$

$$\limsup_{j\to\infty} \|w_h\|_\infty \le \|u\|_\infty \tag{2.5}$$

and

$$\limsup_{h\to+\infty} \int_{S_{w_h}} \phi(w_h^+, w_h^-, \nu_{w_h})\, d\mathcal{H}^{n-1} \le \int_{S_u} \phi(u^+, u^-, \nu_u)\, d\mathcal{H}^{n-1} \tag{2.6}$$

for every upper semicontinuous function $\phi : \mathbf{R} \times \mathbf{R} \times S^{n-1} \to [0, +\infty)$ such that

$$\phi(a, b, \nu) = \phi(b, a, -\nu),$$

for every $a, b \in \mathbf{R}$ and $\nu \in S^{n-1}$.

In the sequel we will consider also the larger spaces $GSBV(\Omega)$ and $GSBV^p(\Omega)$, which are the spaces of all functions $u \in L^1(\Omega)$ whose truncations $u_k = (-k) \vee (u \wedge k)$ belong to $SBV(\Omega)$ and $SBV^p(\Omega)$, respectively, for any $k > 0$.

Let $\xi \in S^{n-1}$ and let $\Pi_\xi := \{y \in \mathbf{R}^n : \langle y, \xi\rangle = 0\}$ be the linear hyperplane orthogonal to ξ. If $y \in \Pi_\xi$ and $E \subset \mathbf{R}^n$ we set

$$E_{\xi y} := \{t \in \mathbf{R} : y + t\xi \in E\}. \tag{2.7}$$

Moreover, if $u : E \to \mathbf{R}$ we define the function $u_{\xi y} : E_{\xi y} \to \mathbf{R}$ by

$$u_{\xi y}(t) = u(y + t\xi). \tag{2.8}$$

The following theorem describes the properties of 1-dimensional sections of $GSBV(\Omega)$-functions (see e.g. [**10**], [**15**]).

Theorem 2.2. (a) *Let $u \in GSBV(\Omega)$. Then, for all $\xi \in S^{n-1}$ the function $u_{\xi y}$ belongs to $GSBV(\Omega_{\xi y})$ for $\mathcal{H}^{n-1}$-a.a. $y \in \pi_\xi$. Moreover for such y we have*

$$u'_{\xi y}(t) = \langle \nabla u(y + t\xi), \xi\rangle \quad \text{for a.a. } t \in \Omega_{\xi y},$$

$$S(u_{\xi y}) = \{t \in \mathbf{R} : y + t\xi \in S(u)\},$$

$$u_{\xi y}(t\pm) = u^{\pm}(y+t\xi) \quad or \quad u_{\xi y}(t\pm) = u^{\mp}(y+t\xi),$$

according to the case $\langle \nu_u, \xi\rangle > 0$ *or* $\langle \nu_u, \xi\rangle < 0$ *(the case* $\langle \nu_u, \xi\rangle = 0$ *being negligible) and for all Borel functions* g

$$\int_{\pi_\xi} \sum_{t\in S(u_{\xi y})} g(t)\, d\mathcal{H}^{n-1}(y) = \int_{S(u)} g(x) |\langle \nu_u, \xi\rangle|\, d\mathcal{H}^{n-1}.$$

(b) *Conversely, if* $u \in L^1(\Omega)$ *and for all* $\xi \in \{e_1, ..., e_n\}$ *and for a.a.* $y \in \pi_\xi$ $u_{\xi y} \in SBV(\Omega_{\xi y})$ *and*

$$\int_{\pi_\xi} |Du_{\xi y}|\, d\mathcal{H}^{n-1}(y) < +\infty,$$

then $u \in SBV(\Omega)$.

We recall the notion of Γ-convergence (see [**27**]). Let (X, d) be a metric space. A family $(F_\varepsilon)_{\varepsilon>0}$ of functionals $F_\varepsilon : X \to [0,+\infty]$ is said to Γ-*converge* to a functional $F : X \to [0, +\infty]$ at $u \in X$, and we write $F(u) = \Gamma\text{-}\lim_{\varepsilon\to 0^+} F_\varepsilon(u)$, if for every sequence (ε_j) of positive numbers decreasing to 0 the following two conditions hold:

(i) (*lower semicontinuity inequality*) for all sequences (u_j) converging to u in X we have $F(u) \le \liminf_j F_{\varepsilon_j}(u_j)$;

(ii) (*existence of a recovery sequence*) there exists a sequence (u_j) converging to u in X such that $F(u) \ge \limsup_j F_{\varepsilon_j}(u_j)$.

We say that F_ε Γ-converges to F if $F(u) = \Gamma\text{-}\lim_{\varepsilon\to 0^+} F_\varepsilon(u)$ at all points $u \in X$ and that F is the Γ-*limit* of F_ε. If we define the *lower and upper* Γ-*limits* by

$$F''(u) = \Gamma\text{-}\limsup_{\varepsilon\to 0+} F_\varepsilon(u) = \inf\{\limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) : \ u_\varepsilon \to u\},$$

$$F'(u) = \Gamma\text{-}\liminf_{\varepsilon\to 0+} F_\varepsilon(u) = \inf\{\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) : \ u_\varepsilon \to u\},$$

respectively, then the conditions (i) and (ii) are equivalent to $F'(u) = F''(u) = F(u)$. Note that the functions F' and F'' are lower semicontinuous.

The reason for the introduction of this notion is explained by the following fundamental theorem.

Theorem 2.3. *Let $F = \Gamma\text{-}\lim_{\varepsilon\to 0+} F_\varepsilon$, and let a compact set $K \subset X$ exist such that $\inf_X F_\varepsilon = \inf_K F_\varepsilon$ for all ε. Then*

$$\exists \min_X F = \lim_{\varepsilon\to 0+} \inf_X F_\varepsilon. \tag{2.9}$$

Moreover, if (u_j) is a converging sequence such that $\lim_j F_j(u_j) = \lim_j \inf_X F_j$ then its limit is a minimum point for F.

We refer to [**24**] for an exposition of the main properties of Γ-convergence (see also [**18**]).

Finally, we include the following lemma on the supremum of a family of measures which will be useful in the sequel (for a proof see e.g. [**15**]).

Lemma 2.4. *Let $\mu : \mathcal{A}(\Omega) \to [0, +\infty)$ be an open-set superadditive function, let λ be a positive measure on Ω, let ψ_i be positive Borel functions such that $\mu(A) \geq \int_A \psi_i \, d\lambda$ for all $A \in \mathcal{A}(\Omega)$ and let $\psi(x) = \sup_i \psi_i(x)$. Then $\mu(A) \geq \int_A \psi \, d\lambda$ for all $A \in \mathcal{A}(\Omega)$.*

3. The main result

The main result of the paper is the following

Theorem 3.1. *Let $f : [0, +\infty) \to [0, +\infty)$ be a lower semicontinuous increasing function, such that $\alpha, \beta \in \mathbf{R}$ exist with*

$$\alpha = \lim_{t\to 0+} \frac{f(t)}{t}, \qquad \beta = \lim_{t\to +\infty} f(t). \tag{3.1}$$

Let Ω be a bounded open subset of $\mathbf{R}^n$ with Lipschitz boundary and let $F_\varepsilon : L^2(\Omega) \to [0, +\infty]$ be defined by

$$F_\varepsilon(u) = \begin{cases} \dfrac{1}{\varepsilon} \displaystyle\int_\Omega f(\varepsilon |\nabla u|^2)\, dx + \varepsilon^3 \int_\Omega \|Hu\|^2 \, dx & \text{if } u \in H^2(\Omega) \\ +\infty & \text{otherwise.} \end{cases} \tag{3.2}$$

Then for any $u \in GSBV^2(\Omega) \cap L^2(\Omega)$ there exists the Γ-limit $\Gamma\text{-}\lim_{\varepsilon\to 0+} F_\varepsilon(u) = F(u)$ with respect to the $L^2(\Omega)$-topology, where

$$F(u) = \begin{cases} \alpha \displaystyle\int_I |\nabla u|^2 \, dx + m(\beta) \int_{S(u)} \sqrt{|u^+ - u^-|}\, d\mathcal{H}^{n-1} & \text{if } u \in GSBV(\Omega) \cap L^2(\Omega) \\ +\infty & \text{otherwise} \end{cases} \tag{3.3}$$

and $m(\beta) = \beta^{\frac{3}{4}} \left(2\sqrt{3/2} + \sqrt{2/3} \right)$.

Remark 3.2. In the 1-dimensional case functional (3.2) can be written on $L^1(I)$, where I is a bounded open set of $\mathbf{R}$, as

$$\mathcal{F}_\varepsilon(u) = \begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_I f(\varepsilon|u'|^2)\,dx + \varepsilon^3\int_I |u''|^2\,dx & \text{if } u \in H^2(I) \\ +\infty & \text{otherwise.} \end{cases}$$

In this case Theorem 3.1 of [**3**] shows that $\mathcal{F}_\varepsilon$ Γ-converges to the functional $\mathcal{F} : L^1(I) \to [0,+\infty]$ given by

$$\mathcal{F}(u) = \begin{cases} \alpha\displaystyle\int_I |u'|^2\,dx + m(\beta)\sum_{S(u)}\sqrt{|u^+ - u^-|} & \text{if } u \in SBV(I) \\ +\infty & \text{otherwise,} \end{cases}$$

with respect to the $L^1(I)$-topology, where f and $m(\beta)$ are given by the previous Theorem. Note that $\mathcal{F}_\varepsilon, \mathcal{F} \equiv +\infty$ on $L^1(I) \setminus L^2(I)$ and $(\mathcal{F}_\varepsilon)$ Γ-converges to $\mathcal{F}$ also with respect to the $L^2(I)$-topology.

The proof of Theorem 3.1 will be a consequence of the propositions in the rest of the chapter, which deal with lower and upper Γ-limits separately.

Remark 3.3. Note that we do not recover the Γ-limit of F_ε on the whole $L^2(\Omega)$. The result would be complete if we showed that for all $u \in GSBV(\Omega) \cap L^2(\Omega)$ there exists a sequence $u_j \in GSBV^2(\Omega) \cap L^2(\Omega)$ such that $u_j \to u$ and $F(u_j) \to F(u)$. The corresponding result for the Mumford-Shah functional relies on regularity results which are not extensible to our case. However the *lower semicontinuity inequality* holds for any $u \in L^2(\Omega)$, as stated in Proposition 3.4.

We define also a "localized version" of our functionals by setting

$$F_\varepsilon(u,A) = \begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_A f(\varepsilon|\nabla u|^2)\,dt + \varepsilon^3\int_A \|Hu\|^2\,dt & \text{if } u \in H^2(\Omega) \\ +\infty & \text{otherwise,} \end{cases} \tag{3.4}$$

and

$$F(u,A) = \begin{cases} \alpha\displaystyle\int_A |\nabla u|^2\,dx + m(\beta)\int_{A\cap S(u)}\sqrt{|u^+ - u^-|}\;\;d\mathcal{H}^{n-1} & \\ & \text{if } u \in GSBV(A) \cap L^2(A) \\ +\infty & \text{otherwise,} \end{cases} \tag{3.5}$$

for all $u \in L^2(\Omega)$ and $A \subseteq \Omega$ open set.

Proposition 3.4. *We have*

$$F(u,A) \le \Gamma\text{-}\liminf_{\varepsilon\to 0+} F_\varepsilon(u,A)\,,$$

for all $u \in L^2(\Omega)$ *and for all* $A \in \mathcal{A}(\Omega)$.

Proof. We recover now the thesis from the 1-dimensional case (see Remark 3.2), by using Theorem 2.2.

Fix $\xi \in S^{n-1}$, $A \in \mathcal{A}(\Omega)$, $v \in H^2(\Omega)$ and let $A_{\xi y}$, $v_{\xi y}$ be defined as in (2.7) and (2.8). Since

$$v'_{\xi y}(t) = \langle \nabla v(y+t\xi), \xi\rangle \quad \text{and} \quad v''_{\xi y}(t) = \langle (Hv(y+t\xi))\,\xi, \xi\rangle,$$

we have, by Fubini's Theorem,

$$\begin{aligned}
F_\varepsilon(v,A) &= \int_{\Pi_\xi}\int_{A_{\xi y}} \Big(\frac{1}{\varepsilon} f(\varepsilon|\nabla v(y+t\xi)|^2) + \varepsilon^3 \|Hv(y+t\xi)\|^2\Big)\,dt\,d\mathcal{H}^{n-1}(y)\\
&\ge \int_{\Pi_\xi}\int_{A_{\xi y}} \Big(\frac{1}{\varepsilon} f(\varepsilon|\langle\nabla v(y+t\xi),\xi\rangle|^2) + \varepsilon^3 |\langle (Hv(y+t\xi))\,\xi,\xi\rangle|^2\Big)\,dt\,d\mathcal{H}^{n-1}(y)\\
&= \int_{\Pi_\xi} \mathcal{F}_\varepsilon(v_{\xi y}, A_{\xi y})\,d\mathcal{H}^{n-1}(y),
\end{aligned}$$

where

$$\mathcal{F}_\varepsilon(g,I) = \begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_I f(\varepsilon|g'|^2)\,dt + \varepsilon^3 \int_I |g''|^2\,dt & \text{if } g \in H^2(I)\\ +\infty & \text{otherwise,}\end{cases}$$

for all $g \in L^2_{\mathrm{loc}}(\mathbf{R})$ and $I \subseteq \mathbf{R}$ bounded open set.

Let $u \in L^2(\Omega)$ be fixed and let $\varepsilon_j \to 0$, $u_{\varepsilon_j} \to u$ in $L^2(\Omega)$ be such that $\sup_j F_{\varepsilon_j}(u_{\varepsilon_j}, A) < +\infty$. Then $(u_{\varepsilon_j})_{\xi y} \in H^2(A_{\xi y})$ for all $\xi \in S^{n-1}$ and for a.e. $y \in \Pi_\xi$, and, by Fubini's Theorem and Fatou's Lemma, $(u_{\varepsilon_j})_{\xi y} \to u_{\xi y}$ in $L^2(A_{\xi y})$ for a.e. $y \in \Pi_\xi$. Hence, by [**3**] Proposition 3.5.,

$$\liminf_j \mathcal{F}_{\varepsilon_j}((u_{\varepsilon_j})_{\xi y}, A_{\xi y}) \ge \alpha \int_{A_{\xi y}} |u'_{\xi y}|^2\,dt + m(\beta) \sum_{S_{u_{\xi y}}\cap A_{\xi y}} \sqrt{|u^+_{\xi y} - u^-_{\xi y}|}\,,$$

By Fatou's Lemma, we have

$$\begin{aligned}
\liminf_j F_{\varepsilon_j}(u_{\varepsilon_j}, A) &\ge \int_{\Pi_\xi} \liminf_j \mathcal{F}_{\varepsilon_j}((u_{\varepsilon_j})_{\xi y}, A_{\xi y})\,d\mathcal{H}^{n-1}(y)\\
&\ge \int_{\Pi_\xi} \Big(\alpha \int_{A_{\xi y}} |u'_{\xi y}|^2\,dt + m(\beta) \sum_{S_{u_{\xi y}}\cap A_{\xi y}} \sqrt{|u^+_{\xi y} - u^-_{\xi y}|}\,\Big)\,d\mathcal{H}^{n-1}(y).
\end{aligned} \tag{3.6}$$

Let $T > 0$ and set $u_T = (-T) \vee (u \wedge T)$. By (3.6) we deduce that $(u_T)_{\xi y} \in SBV(A_{\xi y})$ and

$$\int_{\Pi_\xi} |D(u_T)_{\xi y}|(A_{\xi y})\, d\mathcal{H}^{n-1}(y) < +\infty$$

for every $\xi \in S^{n-1}$ and for a.e. $y \in \Pi_\xi$. Then by Theorem 2.2 $u_T \in SBV(A)$, i.e. $u \in GSBV(A)$.

Moreover, by (3.6) and Theorem 2.2(a), we get

$$\Gamma\text{-}\liminf_{\varepsilon \to 0+} F_\varepsilon(u, A) \geq \alpha \int_A |\langle \nabla u, \xi \rangle|^2\, dx + m(\beta) \int_{S(u) \cap A} \sqrt{|u^+ - u^-|}|\langle \nu_u, \xi \rangle|\, d\mathcal{H}^{n-1}(y). \tag{3.7}$$

For a fixed $u \in GSBV(\Omega)$ consider the superadditive increasing function γ defined on $\mathcal{A}(\Omega)$ by

$$\gamma(A) := \Gamma\text{-}\liminf_{\varepsilon \to 0+} F_\varepsilon(u, A)$$

and the Radon measure

$$\lambda := \mathcal{L}_n \llcorner \Omega + \sqrt{|u^+ - u^-|}\mathcal{H}^{n-1} \llcorner S(u).$$

Fixed a sequence $(\xi_i)_{i \in \mathbf{N}}$, dense in S^{n-1}, we have, by (3.7),

$$\gamma(A) \geq \int_A \psi_i(x)\, d\lambda \quad \forall i \in \mathbf{N},$$

where

$$\psi_i = \begin{cases} |\langle \nabla u, \xi_i \rangle|^2 & \text{if } x \in A \setminus S(u) \\ \\ |\langle \nu_u, \xi_i \rangle| & \text{if } x \in S(u). \end{cases}$$

Hence, applying Lemma 2.4, we get

$$\begin{aligned} F(u, A) \geq \Gamma\text{-}\liminf_{\varepsilon \to 0+} F_\varepsilon(u, A) &\geq \int_A \sup_i \psi_i(x)\, d\lambda \\ &= \alpha \int_A |\nabla u|^2\, dx + m(\beta) \int_{S(u) \cap A} \sqrt{|u^+ - u^-|}\, d\mathcal{H}^{n-1}(y), \end{aligned}$$

as desired. $\square$

The rest of the section is devoted to the proof of the upper inequality for the Γ-limit.

Proposition 3.5. *Let Ω' be a bounded open set such that $\Omega \subset\subset \Omega'$ and let E be such that $E = E' \cap \Omega$, where E' is a set of finite perimeter in Ω' such that $\partial E' \cap \Omega'$ is a smooth $(n-1)$-manifold. Then*

$$\Gamma\text{-}\limsup_{\varepsilon \to 0+} F_\varepsilon(z\chi_E, A) \le m(\beta)\sqrt{|z|}\mathcal{H}^{n-1}(\partial E \cap A)$$

for all $z \in \mathbf{R}$, for all $A \in \mathcal{A}(\Omega)$.

Proof. Let

$$d(x) := d(x, \mathbf{R}^n \setminus E') - d(x, E'), \qquad x \in \mathbf{R}^n.$$

Under our hypotheses there exists $\delta > 0$ such that $d \in \mathcal{C}^\infty(\overline{B_\delta \cap \Omega})$, where

$$B_\delta := \{x \in \mathbf{R}^n : |d(x)| < \delta\}.$$

Moreover $|\nabla d| = 1$ on B_δ.

Consider the recovery sequence $v_\varepsilon : \mathbf{R} \to \mathbf{R}$ defined by

$$v_\varepsilon(t) = \begin{cases} z & \text{if } t \ge \varepsilon\eta \\ v\left(\frac{t}{\varepsilon}\right) + \frac{z}{2} & \text{if } t \in (-\varepsilon\eta, \varepsilon\eta) \\ 0 & \text{if } t \le -\varepsilon\eta, \end{cases}$$

with (η, v) solution of the minimum problem

$$\min\Big\{2c\eta + \int_{-\eta}^{\eta} |v''|^2\,dt :\ \eta > 0,\ v \in H^2(-\eta,\eta),\ v(\pm\eta) = \pm\frac{z}{2},\ v'(\pm\eta) = 0\Big\}.$$

Note that $\lim_{\varepsilon\to 0+} \mathcal{F}_\varepsilon(v_\varepsilon, (-\delta,\delta)) = m(\beta)\sqrt{|z|}$ (see Proposition 3.6 in [**3**]).
Define

$$\widetilde{v}_\varepsilon(x) = v_\varepsilon(d(x)), \qquad x \in \Omega. \tag{3.8}$$

We have $\widetilde{v}_\varepsilon \in H^2(\Omega)$ and $\widetilde{v}_\varepsilon \to z\chi_E$ in $L^2(\Omega)$. Indeed

$$\int_\Omega |\widetilde{v}_\varepsilon(x) - z\chi_E|^2\,dx \le 2|z|^2\mathcal{L}^n(B_{\eta\varepsilon}) \to 0.$$

Using the co-area formula (2.2), we get

$$\begin{aligned} F_\varepsilon(\widetilde{v}_\varepsilon, A) = \frac{1}{\varepsilon}\int_{A\cap B_{\eta\varepsilon}} & f(\varepsilon|v_\varepsilon'(d)\nabla d|^2)\,dx \\ &+ \varepsilon^3 \int_{A\cap B_{\eta\varepsilon}} \|v_\varepsilon''(d)(\nabla d \otimes \nabla d) + v_\varepsilon'(d)Hd\|^2\,dx \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\varepsilon}\int_{-\eta\varepsilon}^{\eta\varepsilon} dt \int_{\{x\in A: d(x)=t\}} f(\varepsilon|v_\varepsilon'(t)|^2)\, d\mathcal{H}^{n-1} \\
&\quad + \varepsilon^3 \int_{-\eta\varepsilon}^{\eta\varepsilon} dt \int_{\{x\in A: d(x)=t\}} (|v_\varepsilon''(t)|^2 + |v_\varepsilon'(t)|^2\|Hd\|_\infty^2)\, d\mathcal{H}^{n-1} \\
&= \int_{-\eta\varepsilon}^{\eta\varepsilon} \Big(\frac{1}{\varepsilon} f(\varepsilon|v_\varepsilon'(t)|^2) + \varepsilon^3|v_\varepsilon''(t)|^2\Big)\mathcal{H}^{n-1}(\{x\in A: d(x)=t\})\, dt \\
&\quad + \varepsilon\int_{-\eta\varepsilon}^{\eta\varepsilon} |v'(\frac{t}{\varepsilon})|^2\|Hd\|_\infty^2\, \mathcal{H}^{n-1}(\{x\in A: d(x)=t\}) dt \\
&\leq \sup_{t\in(-\eta\varepsilon,\eta\varepsilon)} \mathcal{H}^{n-1}(\{x\in A: d(x)=t\})\mathcal{F}_\varepsilon(v_\varepsilon, (-\eta\varepsilon,\eta\varepsilon)) + o(\varepsilon); \qquad (3.9)
\end{aligned}
$$

hence, since

$$\lim_{t\to 0+} \mathcal{H}^{n-1}(\{x\in A: d(x)=t\}) = \mathcal{H}^{n-1}(\partial E\cap A),$$

we have

$$\limsup_{\varepsilon\to 0} F_\varepsilon(\widetilde{v}_\varepsilon, A) \leq m(\beta)\sqrt{|z|}\,\mathcal{H}^{n-1}(\partial E\cap A)$$

for all $A\in\mathcal{A}(\Omega)$ and for all $z\in\mathbf{R}$. □

Proposition 3.6. *Let $u=\sum_{i=1}^k z_i\chi_{E_i}$ with E_i closed polyhedra and $\overset{o}{E}_i\cap\overset{o}{E}_j=\emptyset$ if $i\neq j$. Then*

$$\Gamma\text{-}\limsup_{\varepsilon\to 0+} F_\varepsilon(u,A)\leq F(u,A)$$

for all $A\in\mathcal{A}(\Omega)$.

Proof. Up to further subdivide the family (E_i), it is not restrictive to suppose that each E_i is convex. Set

$$E_{k+1} := \Omega\setminus\bigcup_{i=1}^k E_i,$$

$$\mathcal{I} := \{(i,j): i,j\in\{1,....,k+1\}, i<j, \mathcal{H}^{n-1}(E_i\cap E_j)\neq 0\}.$$

For $r>0$, let

$$B^r := \{x\in\Omega: d(x,\bigcup_{i=1}^k\partial E_i)<r\}$$

and, for $(i,j)\in\mathcal{I}$, let

$$B^r_{ij} := \{x\in\Omega: d(x,E_i\cap E_j)<r\}.$$

Then, for a fixed $\delta>0$, we can find $g_{ij}\in\mathcal{C}_0^\infty(\mathbf{R}^n)$ such that

$$g_{ij}=1 \quad\text{on } B^\delta_{ij}, \quad g_{ij}=0 \quad\text{on } \Omega\setminus B^{2\delta}_{ij},$$

$$\|\nabla g_{ij}\|_\infty \le \frac{c}{\delta}, \quad \|Hg_{ij}\|_\infty \le \frac{c}{\delta^2},$$

for any $(i,j) \in \mathcal{I}$, with c independent of δ, i and j. Define

$$h_{ij} := g_{ij}\Big(\sum_{(l,m)\in\mathcal{I}} g_{lm} + \prod_{(l,m)\in\mathcal{I}} (1-g_{lm})\Big)^{-1},$$

$$h_0 := \prod_{(l,m)\in\mathcal{I}} (1-g_{lm})\Big(\sum_{(l,m)\in\mathcal{I}} g_{lm} + \prod_{(l,m)\in\mathcal{I}} (1-g_{lm})\Big)^{-1}.$$

Note that

$$\sum_{(i,j)\in\mathcal{I}} h_{ij} + h_0 = 1,$$

$$h_{ij} = 1 \quad \text{on } B_{ij}^{\delta} \setminus \bigcup_{(l,m)\neq(i,j)} B_{lm}^{2\delta}, \quad h_0 = 1 \quad \text{on } \Omega \setminus B^{2\delta}.$$

Moreover, an easy computation shows that

$$\|\nabla h_{ij}\|_\infty \le \frac{c}{\delta}, \quad \|\nabla h_0\|_\infty \le \frac{c}{\delta},$$

$$\|Hh_{ij}\|_\infty \le \frac{c}{\delta^2}, \quad \|Hh_0\|_\infty \le \frac{c}{\delta^2}.$$

Let $v_{ij}^{\varepsilon} \in H^2(\Omega)$ be the recovery sequence, constructed as in Proposition 3.5, related to

$$u_{ij} = \begin{cases} z_i - z_j & \text{on } S_{ij}^{+} \\ 0 & \text{on } S_{ij}^{-}, \end{cases}$$

where S_{ij} is the hyperplane containing $E_i \cap E_j$, that is uniquely determined since E_i, E_j are convex and belong to $\mathcal{I}$, and

$$S_{ij}^{\pm} := \{x \in \Omega : x = y \pm t\nu_{ij},\ y \in S_{ij},\ t \in \mathbf{R}^+\},$$

where ν_{ij} is the internal normal to E_i in $E_i \cap E_j$. We can now choose $\delta = \eta\varepsilon$ with η as in Proposition 3.5 and set

$$v_\varepsilon := \sum_{(i,j)\in\mathcal{I}} h_{ij}(v_{ij}^{\varepsilon} + z_j) + h_0 u.$$

We have that $v_\varepsilon \in H^2(\Omega)$ and $v_\varepsilon \to u$ in $L^2(\Omega)$. Indeed

$$\int_\Omega |v_\varepsilon - u|^2\, dx \le \|u\|_\infty^2 \mathcal{L}^n(B^{2\eta\varepsilon}) = O(\varepsilon).$$

Moreover,

$$
\begin{aligned}
F_\varepsilon(v_\varepsilon, A) \le & \sum_{(i,j)\in\mathcal{I}} F_\varepsilon\Big(g_{ij}v^\varepsilon_{ij} + (1-g_{ij})u, \Big(B^{2\eta\varepsilon}_{ij} \setminus \bigcup_{(l,m)\neq(i,j)} B^{2\eta\varepsilon}_{lm}\Big) \cap A\Big) \\
& + c \sum_{(i,j)\neq(l,m)} \int_{B^{2\eta\varepsilon}_{ij}\cap B^{2\eta\varepsilon}_{lm}\cap A} \Bigg(\frac{1}{\varepsilon} + \varepsilon^3 \sup_{(i,j)\in\mathcal{I}} \big\{\|Hh_{ij}\|^2_\infty + \|Hv^\varepsilon_{ij}\|^2_\infty \\
& \qquad + \|Hh_0\|^2_\infty + \|\nabla h_{ij}\|^2_\infty \|\nabla v^\varepsilon_{ij}\|^2_\infty\big\}\Bigg)\, dx \\
\le & \sum_{(i,j)\in\mathcal{I}} F_\varepsilon(v^\varepsilon_{ij}, B^{\eta\varepsilon}_{ij} \cap A) \\
& + c \sum_{(l,m)\neq(i,j)} \left(\frac{1}{\varepsilon} + \varepsilon^3\left(\frac{1}{\eta^4\varepsilon^4} + \frac{1}{\varepsilon^2}\frac{1}{\eta^2\varepsilon^2} + \frac{1}{\varepsilon^4}\right)\right)\mathcal{L}^n(B^{2\eta\varepsilon}_{ij} \cap B^{2\eta\varepsilon}_{lm}) \\
= & \, O(\varepsilon) + \sum_{(i,j)\in\mathcal{I}} m(\beta)\sqrt{|z_i - z_j|}\,\mathcal{H}^{n-1}(E_i \cap E_j \cap A).
\end{aligned}
$$

Note that $\mathcal{L}^n(B^{2\eta\varepsilon}_{ij} \cap B^{2\eta\varepsilon}_{lm}) = O(\varepsilon^2)$ if $(i,j) \neq (l,m)$, due to the fact that $\mathcal{H}^{n-2}(E_i \cap E_j \cap E_l \cap E_m) < +\infty$. $\square$

Remark 3.7. By using a density argument and the previous Proposition, it can be easily seen that if $u \in L^\infty(\Omega)$ and $u = \sum_{i=1}^{+\infty} z_i \chi_{E_i}$, with E_i closed polyhedra, $\overset{\circ}{E}_i \cap \overset{\circ}{E}_j = \emptyset$ and $\mathcal{H}^{n-1}(S(u)) < +\infty$, then

$$F''(u, A) \le \sum_{i<j} m(\beta)\sqrt{|z_i - z_j|}\mathcal{H}^{n-1}(E_i \cap E_j \cap A)$$

for all $A \in \mathcal{A}(\Omega)$.

Proposition 3.8. *Let $A', A, B \in \mathcal{A}(\Omega)$ be such that $A' \subset A$ and $\mathrm{dist}(A', \Omega \setminus A) > 0$ and let ϕ be a cut-off function between A' and A. Then for all $u, v \in L^2(\Omega)$ and for all $u_\varepsilon,\ v_\varepsilon \in H^2(\Omega)$ such that $u_\varepsilon \to u,\ v_\varepsilon \to v$ in $L^2(\Omega)$ and*

$$F''(u, A) = \lim_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon, A),\ \ F''(v, B) = \lim_{\varepsilon\to 0^+} F_\varepsilon(v_\varepsilon, B),$$

the following estimate holds

$$
\begin{aligned}
F''(\phi u + (1-\phi)v, A' \cup B) \le & F''(u, A) + F''(v, B) + c\|\nabla\phi\|^2_\infty \|u - v\|^2_{L^2((A\setminus\overline{A'})\cap B)} \\
& + c \limsup_{\varepsilon\to 0} \left(F_\varepsilon(u_\varepsilon, (A \setminus \overline{A'}) \cap B) + F_\varepsilon(v_\varepsilon, (A \setminus \overline{A'}) \cap B)\right).
\end{aligned}
$$

Proof. Set

$$S = (A \setminus \overline{A'}) \cap B.$$

Then, using the convexity of $|\cdot|^2$, $\|\cdot\|^2$, and the monotonicity of f, we have

$$\begin{aligned}
&F_\varepsilon(\phi u_\varepsilon + (1-\phi)v_\varepsilon, A' \cup B)\\
&= F_\varepsilon(u_\varepsilon, A') + F_\varepsilon(v_\varepsilon, B \setminus A)\\
&\quad + \frac{1}{\varepsilon}\int_S f(\varepsilon|(u_\varepsilon - v_\varepsilon)\nabla\phi + \phi\nabla u_\varepsilon + (1-\phi)\nabla v_\varepsilon|^2)\,dx\\
&\quad + \varepsilon^3\int_S \|2\nabla\phi\otimes\nabla(u_\varepsilon - v_\varepsilon) + (u_\varepsilon - v_\varepsilon)(H\phi) + \phi(Hu_\varepsilon - Hv_\varepsilon)\|^2\,dx\\
&\le F_\varepsilon(u_\varepsilon, A) + F_\varepsilon(v_\varepsilon, B)\\
&\quad + \frac{1}{\varepsilon}\int_S f(2\varepsilon|(u_\varepsilon - v_\varepsilon)\nabla\phi|^2 + 2\varepsilon\phi|\nabla u_\varepsilon|^2 + 2\varepsilon(1-\phi)|\nabla v_\varepsilon|^2)\,dx\\
&\quad + c\varepsilon^3\int_S (|\nabla\phi|^2|\nabla u_\varepsilon - \nabla v_\varepsilon|^2 + |u_\varepsilon - v_\varepsilon|^2\|H\phi\|^2 + \|Hu_\varepsilon - Hv_\varepsilon\|^2)\,dx\\
&\le F_\varepsilon(u_\varepsilon, A) + F_\varepsilon(v_\varepsilon, B)\\
&\quad + \frac{1}{\varepsilon}\int_S \big(f(4\varepsilon|(u_\varepsilon - v_\varepsilon)\nabla\phi|^2) + f(4\varepsilon|\nabla u_\varepsilon|^2) + f(4\varepsilon|\nabla v_\varepsilon|^2)\big)\,dx\\
&\quad + c\varepsilon^3\int_S (|\nabla\phi|^2|\nabla u_\varepsilon - \nabla v_\varepsilon|^2 + |u_\varepsilon - v_\varepsilon|^2\|H\phi\|^2 + \|Hu_\varepsilon\|^2 + \|Hv_\varepsilon\|^2)\,dx.
\end{aligned}$$

By our hypotheses on f, it can be easily proved that for all $\alpha > 0$

$$f(\alpha t) \le c_\alpha f(t) \quad \forall t > 0.$$

Thus we have

$$\begin{aligned}
&F_\varepsilon(\phi u_\varepsilon + (1-\phi)v_\varepsilon, A' \cup B)\\
&\le F_\varepsilon(u_\varepsilon, A) + F_\varepsilon(v_\varepsilon, B)\\
&\quad + c\Big(F_\varepsilon(u_\varepsilon, S) + F_\varepsilon(v_\varepsilon, S)\Big) + c\frac{1}{\varepsilon}\int_S f(\varepsilon|\nabla\phi|^2|u_\varepsilon - v_\varepsilon|^2)\,dx\\
&\quad + c\varepsilon^3\int_S \Big(|\nabla\phi|^2|\nabla u_\varepsilon - \nabla v_\varepsilon|^2 + |u_\varepsilon - v_\varepsilon|^2\|H\phi\|^2\Big)\,dx\\
&\le F_\varepsilon(u_\varepsilon, A) + F_\varepsilon(v_\varepsilon, B)\\
&\quad + c\Big(F_\varepsilon(u_\varepsilon, S) + F_\varepsilon(v_\varepsilon, S)\Big) + c\int_S |\nabla\phi|^2|u_\varepsilon - v_\varepsilon|^2\,dx\\
&\quad + c\varepsilon^3\int_S \Big(|\nabla\phi|^2|\nabla u_\varepsilon - \nabla v_\varepsilon|^2 + |u_\varepsilon - v_\varepsilon|^2\|H\phi\|^2\Big)\,dx.
\end{aligned} \tag{3.10}$$

Note that, by Gagliardo-Nirenberg inequality (see [**20**]), the equiboundness of the family

$$\left(\|u_\varepsilon - v_\varepsilon\|_{L^2(S)} + \varepsilon^3 \|Hu_\varepsilon - Hv_\varepsilon\|^2_{L^2(S)}\right)_\varepsilon$$

implies that

$$\lim_{\varepsilon\to 0^+} \varepsilon^3 \int_S |\nabla u_\varepsilon - \nabla v_\varepsilon|^2\, dx = 0$$

The thesis follows passing to the limit as $\varepsilon \to 0^+$ in (3.10). $\square$

Corollary 3.9. *Let A', A, $B \in \mathcal{A}(\Omega)$ be as in Proposition 3.8. Then for any u, $v \in L^2(\Omega)$, $N \in \mathbf{N}$ there exists a cut-off function ϕ_N between A' and A, such that*

$$\begin{aligned} &F''(\phi_N u + (1-\phi_N)v, A'\cup B) \\ &\le (1+\frac{c}{N})\,(F''(u,A) + F''(v,B)) + c\left|\frac{N}{\mathrm{dist}\,(A',\Omega\setminus A)}\right|^2 \|u-v\|^2_{L^2((A\setminus\overline{A'})\cap B)}. \end{aligned} \tag{3.11}$$

Proof. We will use a standard argument by De Giorgi. Let $N \in \mathbf{N}$ be fixed and set $d = \mathrm{dist}\,(A', \Omega\setminus A)$. Then for $k \in \{0, ..., N\}$ consider the set

$$A_k^N = \left\{x \in A : \mathrm{dist}\,(x, A') < k\frac{d}{N}\right\},$$

For any $k \in \{0, ..., N-1\}$ we can find a cut-off function ϕ_k^N between A_k^N and A_{k+1}^N such that

$$\|\nabla \phi_k^N\|_\infty \le 2\frac{N}{d}. \tag{3.12}$$

Set

$$S_k^N = \left(A_{k+1}^N \setminus \overline{A}_k^N\right) \cap B.$$

Then, by (3.10) and taking into account (3.12), we get

$$\begin{aligned} &F_\varepsilon(\phi_k^N u_\varepsilon + (1-\phi_k^N)v_\varepsilon, A'\cup B) \\ &\le F_\varepsilon(u_\varepsilon, A) + F_\varepsilon(v_\varepsilon, B) + c\Big(F_\varepsilon(u_\varepsilon, S_k^N) + F_\varepsilon(v_\varepsilon, S_k^N)\Big) + c\frac{N^2}{d^2}\int_{S_k^N} |u_\varepsilon - v_\varepsilon|^2\, dx \\ &\quad + c\varepsilon^3 \int_{S_k^N} \Big(|\nabla\phi_k^N|^2 |\nabla u_\varepsilon - \nabla v_\varepsilon|^2 + |u_\varepsilon - v_\varepsilon|^2 \|H\phi_k^N\|^2\Big)\, dx. \end{aligned}$$

Thus, there exists $k_N \in \{0,\dots,N-1\}$ such that

$$
\begin{aligned}
&F_\varepsilon(\phi^N_{k_N} u_\varepsilon + (1-\phi^N_{k_N})v_\varepsilon, A'\cup B)\\
&\le \frac{1}{N}\sum_{k=0}^{N-1} F_\varepsilon(\phi^N_k u_\varepsilon + (1-\phi^N_k)v_\varepsilon, A'\cup B)\\
&\le \Big(1+\frac{c}{N}\Big)\left(F_\varepsilon(u_\varepsilon, A) + F_\varepsilon(v_\varepsilon, B)\right)\\
&\quad + c\frac{N^2}{d^2}\int_S |u_\varepsilon - v_\varepsilon|^2\,dx + c(N)\varepsilon^3\int_S \left(|\nabla u_\varepsilon - \nabla v_\varepsilon|^2 + |u_\varepsilon - v_\varepsilon|^2\right)\,dx.
\end{aligned}
$$

The thesis follows taking $\phi_N = \phi^N_{k_N}$ and letting ε tend to 0^+. □

Remark 3.10 Note that for any $A \in \mathcal{A}(\Omega)$ and $u \in L^2(\Omega)$ such that $u|_A \in H^1(A)$ we have

$$
F''(u,A) \le \alpha \int_A |\nabla u|^2\,dx. \tag{3.13}
$$

Indeed, if $u|_A \in H^2(A)$ it suffices to choose $u_\varepsilon = u$ as the recovery sequence, while in the general case (3.13) follows by the density of $H^2(A)$ in $H^1(A)$ and the lower semicontinuity of F''.

Proposition 3.11. *Let $u \in SBV^2(\Omega)$. Then*

$$
F''(u,\Omega) \le \alpha\int_\Omega |\nabla u|^2\,dx + m(\beta)\int_\Omega \sqrt{|u^+ - u^-|}\,d\mathcal{H}^{n-1}.
$$

Proof. Step 1. Suppose that $u \in \mathcal{W}(\Omega)$. In particular, for any $h \in \mathbf{N}$ the sets

$$
B_h = \left\{ x \in \Omega \;:\; d(x, S(u)) < \frac{1}{h} \right\}
$$

satisfy

$$
\mathcal{L}^n(B_h) \le \frac{c}{h}.
$$

Then, fixed a sequence (ρ_h) decreasing to 0, we can find $(v_h) \subset SBV_0(\Omega)$ satisfying the hypotheses of Proposition 3.6 such that

$$
\|u - v_h\|_\infty \le \rho_h, \quad \mathcal{H}^{n-1}((S(v_h)\cap B_h)\setminus S(u)) \le \frac{1}{\rho_h}\int_{B_h} |\nabla u|\,dx + O(1). \tag{3.14}
$$

Indeed by the co-area formula (2.1) we have

$$|Du|(B_h\setminus S(u)) = \int_{B_h} |\nabla u|\,dx = \sum_{j\in I_h}\int_{j\rho_h}^{(j+1)\rho_h} \mathcal{H}^{n-1}(\partial^*\{u>t\}\cap(B_h\setminus S(u)))\,dt,$$

where $I_h = \{j\in\mathbf{Z} : |j| \le \frac{\|u\|_\infty}{\rho_h}+2\}$. Hence, by the Mean Value Theorem, for every $j\in\mathbf{Z}$ we can find $t_h^j\in(j\rho_h,(j+1)\rho_h)$ such that

$$\rho_h\mathcal{H}^{n-1}(\partial^*\{u>t_h^j\}\cap(B_h\setminus S(u))) \le \int_{B_h}|\nabla u|\,dx. \tag{3.15}$$

Let P_h^j be a polyhedron such that

$$B_h\cap\{u>(j+1)\rho_h\}\subseteq P_h^j\subseteq B_h\cap\{u>j\rho_h\} \tag{3.16}$$

and

$$\mathcal{H}^{n-1}(\partial P_h^j\cap B_h)\le\mathcal{H}^{n-1}(\partial^*\{u>t_h^j\}\cap B_h)+\rho_h 2^{-|j|}. \tag{3.17}$$

Then we can define $v_h\in SBV_0(B_h)$ by setting

$$v_h(x) = (j+1)\rho_h \quad \text{on } P_h^j\setminus P_h^{(j+1)}.$$

Taking into account (3.15)–(3.17), it is not difficult to verify that v_h satisfies (3.14).

By Corollary 3.9 we can find for any $N\in\mathbf{N}$ a cut-off function ϕ_N^h between B_{2h} and B_h, such that (3.11) holds with $A=B_h$, $A'=B_{2h}$, $B=\Omega\setminus\overline{B}_{3h}$, i.e.

$$\begin{aligned} F''(\phi_N^h v_h+(1-\phi_N^h)u,\Omega) \le \Big(1+\frac{1}{N}\Big)\Big(F''(u,\Omega\setminus\overline{S(u)})+F''(v_h,B_h)\Big)&\\ +cN^2h^2\rho_h^2\mathcal{L}^n(B_h\setminus B_{2h}).& \end{aligned} \tag{3.18}$$

By Proposition 3.6,

$$\begin{aligned} F''(v_h,B_h) &\le m(\beta)\int_{B_h\cap S(v_h)}\sqrt{|v_h^+-v_h^-|}\,d\mathcal{H}^{n-1}\\ &\le 2\|u\|_\infty m(\beta)\mathcal{H}^{n-1}(S(v_h)\cap B_h)\setminus S(u))+O(1)\\ &\quad + m(\beta)\int_{S(u)}\sqrt{|u^+-u^-|}\,d\mathcal{H}^{n-1}. \end{aligned} \tag{3.19}$$

Choose

$$\rho_h = h^{-\frac{1}{2}}\Big(\int_{B_h}|\nabla u|^2\,dx\Big)^{\frac{1}{4}}.$$

Then, by (3.14) and (3.19), $\limsup_h F''(v_h,B_h)\le m(\beta)\int_{S(u)}\sqrt{|u^+-u^-|}\,d\mathcal{H}^{n-1}$. Letting $h\to+\infty$ in (3.18), by the lower semicontinuity of F'', we get the thesis by the arbitrariness of N.

Step 2. Suppose now that $u\in SBV^2(\Omega)\cap L^\infty(\Omega)$. By Theorem 2.1 there exists $(w_h)\subseteq\mathcal{W}(\Omega)$ such that (2.3)–(2.6) hold with $\phi(a,b,\nu)=\sqrt{|a-b|}$. Then we get

$$
\begin{aligned}
F''(u,\Omega)&\le\liminf_{h\to+\infty}F''(w_h,\Omega)\\
&\le\lim_{h\to+\infty}\left(\alpha\int_\Omega|\nabla w_h|^2\,dx+m(\beta)\int_{S(v_h)}\sqrt{|w_h^+-w_h^-|}\,d\mathcal{H}^{n-1}\right)=F(u,\Omega).
\end{aligned}
$$

Step 3. We recover the general case by a truncation argument. Let $u\in GSBV^2(\Omega)$ and let $u_h=(-h)\vee u\wedge h$. By the Monotone Convergence Theorem we have

$$\lim_{h\to+\infty}F(u_h,\Omega)=F(u,\Omega).$$

Since $u_h\to u$ in $L^2(\Omega)$, by Step 2 we get the thesis. □

4. Some generalizations

The choice of the norm $\|\cdot\|$ that occurs in the definition of F_ε in the singular perturbation term is justified by the technical fact that such norm seems the most natural to recover the lower semicontinuity inequality in the n-dimensional case from the 1-dimensional one, by applying a slicing tecnique. From a numerical point of view, it is easier to deal with other norms, like the euclidean one. In this section we replace $\|\cdot\|$ by a generic norm and give a sufficient condition in order to have a generalization of the previous results.

Let $\varphi:M^{n\times n}\to\mathbf{R}^+$ be a norm and let $F_\varepsilon^\varphi,F^\varphi:L^2(\Omega)\to\mathbf{R}^+$ be the functionals defined by

$$
F_\varepsilon^\varphi(u)=\begin{cases}\dfrac{1}{\varepsilon}\displaystyle\int_\Omega f(\varepsilon|\nabla u|^2)\,dx+\varepsilon^3\int_\Omega\varphi^2(Hu)\,dx & \text{if } u\in H^2(\Omega)\\[2mm] +\infty & \text{otherwise,}\end{cases}
$$

$$
F^\varphi(u)=\begin{cases}\alpha\displaystyle\int_\Omega|\nabla u|^2\,dx+m(\beta)\int_{S(u)}\sqrt{\varphi(\nu_u\otimes\nu_u)}\;\sqrt{|u^+-u^-|}\,d\mathcal{H}^{n-1} & \\ & \text{if } u\in GSBV(\Omega)\cap L^2(\Omega)\\ +\infty & \text{otherwise,}\end{cases}
$$

where f and $m(\beta)$ are as in Proposition 3.1. A localized version of the functionals F_ε^φ, F^φ is obtained by extending in a natural way the definitions (3.4) and (3.5).

Proposition 4.1. *Let the norm φ satisfy the following property: for any $M \in M^{n\times n}$*

$$\varphi(M) \geq |\langle M\xi, \xi\rangle| \varphi(\xi \otimes \xi) \qquad \text{for all} \quad \xi \in \mathbf{R}^n\,, \ |\xi| = 1. \tag{4.1}$$

Then, for any $u \in GSBV^2(\Omega) \cap L^2(\Omega)$ there exists the Γ-limit $\Gamma\text{-}\lim_{\varepsilon\to 0+} F^\varphi_\varepsilon(u) = F^\varphi(u)$ with respect to the $L^2(\Omega)$-topology.

Remark 4.2. It can be easily verified that condition (4.1) is satisfied, in particular, by the euclidean norm. In this case the limit functional F^φ coincides with the functional F defined in the previous section.

Proposition 4.3. *Under the hypotheses of Proposition 4.1 we have*

$$F^\varphi(u,A) \leq \Gamma\text{-}\liminf_{\varepsilon\to 0+} F^\varphi_\varepsilon(u,A)$$

for all $u \in L^2(\Omega)$ and for all $A \in \mathcal{A}(\Omega)$.

Proof. Proceeding as in the proof of Proposition 3.4, taking into account (4.1), we get

$$\begin{aligned}\Gamma\text{-}\liminf_{\varepsilon\to 0+} F^\varphi_\varepsilon(u,A) &\geq \alpha \int_A |\langle \nabla u, \xi\rangle|^2\,dx \\ &\quad + m(\beta)\int_{S(u)\cap A} \sqrt{\varphi(\xi\otimes\xi)}\sqrt{|u^+ - u^-|}|\langle \nu_u, \xi\rangle|\,d\mathcal{H}^{n-1}(y).\end{aligned}$$

Since, again by (4.1), $\sup_{|\xi|=1} \sqrt{\varphi(\xi\otimes\xi)}|\langle\nu_u,\xi\rangle| = \sqrt{\varphi(\nu_u\otimes\nu_u)}$, we get the thesis using Lemma 2.4. ☐

We state now the analogue of Proposition 3.5.

Proposition 4.4. *Let $E \subseteq \Omega$ satisfy the hypotheses of Proposition 3.5. Then*

$$\Gamma\text{-}\liminf_{\varepsilon\to 0+} F_\varepsilon(z\chi_E, A) \leq m(\beta)\sqrt{|z|}\int_{A\cap\partial E} \sqrt{\varphi(\nu_E\otimes\nu_E)}\,d\mathcal{H}^{n-1}$$

for all $A \in \mathcal{A}(\Omega)$.

Proof. Let $\widetilde{v_\varepsilon}$ be as in the proof of Proposition 3.5. Arguing as in (3.9), we get

$$\begin{aligned} F_\varepsilon(\widetilde{v}_\varepsilon, A) &= \frac{1}{\varepsilon}\int_{A\cap B_{\eta\varepsilon}} f(\varepsilon|v'_\varepsilon(d)\nabla d|^2\,dx \\ &\quad + \varepsilon^3 \int_{A\cap B_{\eta\varepsilon}} \varphi^2(v''_\varepsilon(d)(\nabla d\otimes\nabla d) + v'_\varepsilon(d)Hd)\,dx \\ &\leq \frac{1}{\varepsilon}\int_{-\eta\varepsilon}^{\eta\varepsilon} dt \int_{\{x\in A: d(x)=t\}} f(\varepsilon|v'_\varepsilon(t)|^2)\,d\mathcal{H}^{n-1} \\ &\quad + \varepsilon^3\int_{-\eta\varepsilon}^{\eta\varepsilon} dt\int_{\{x\in A: d(x)=t\}} |v''_\varepsilon(t)|^2\varphi^2(\nabla d\otimes\nabla d)\,d\mathcal{H}^{n-1} \ + \ o(\varepsilon)\ . \end{aligned} \tag{4.2}$$

By the uniform continuity of the functions $f_1(x) = \varphi^2(\nabla d(x) \otimes \nabla d(x))$, $f_2(x) = \sqrt{\varphi(\nabla d(x) \otimes \nabla d(x))}$, for any fixed $\gamma > 0$, we can find $\delta > 0$ such that

$$|x_1 - x_2| < \delta \;\Rightarrow\; |f_i(x_1) - f_i(x_2)| \leq \gamma \quad i = 1, 2\,.$$

Let $(M_k)_{k=1,\ldots,N}$ be a finite family of measurable, pairwise disjoint sets such that $\Omega \subseteq \bigcup_{k=1}^N M_k$ and $\operatorname{diam}(M_k) < \delta$, and let $x_k \in M_k$ be fixed. Then, by (4.2), we obtain

$$\begin{aligned} F_\varepsilon(\widetilde{v}_\varepsilon, A) \leq \sum_{k=1}^N \int_{-\eta\varepsilon}^{\eta\varepsilon} \Big(\frac{1}{\varepsilon} f(\varepsilon |v_\varepsilon'(t)|^2) + \varepsilon^3 |v_\varepsilon''(t)|^2 \varphi^2(\nu_E(Px_k) \otimes \nu_E(Px_k)\Big)\, dt \\ \times \sup_{t \in (-\eta\varepsilon, \eta\varepsilon)} \mathcal{H}^{n-1}(\{x \in M_k \cap A : d(x) = t\}) + \gamma c + o(\varepsilon)\,, \end{aligned}$$

where by Px_k we denote the projection of x_k on ∂E. Hence, letting $\varepsilon \to +\infty$ we get

$$\begin{aligned} \limsup_{\varepsilon \to 0+} F_\varepsilon(\widetilde{v}_\varepsilon, A) &\leq m(\beta) \sum_{k=1}^N \sqrt{|z|} \int_{M_k \cap \partial E \cap A} \sqrt{\varphi(\nu_E(Px_k) \otimes (Px_k))}\, d\mathcal{H}^{n-1} + c\gamma \\ &\leq c\gamma + m(\beta)\sqrt{|z|} \int_{A \cap \partial E} \sqrt{\varphi(\nu_E \otimes \nu_E)}\, d\mathcal{H}^{n-1}\,. \end{aligned}$$

The thesis follows by the arbitrariness of γ. □

Proposition 4.5. *Under the hypotheses of Proposition* 4.1 *we have*

$$\Gamma\text{-}\limsup_{\varepsilon \to 0^+} F_\varepsilon^\varphi(u, \Omega) \leq \alpha \int_\Omega |\nabla u|^2\, dx + m(\beta) \int_\Omega \sqrt{\varphi(\nu_u \otimes \nu_u)} \sqrt{|u^+ - u^-|}\, d\mathcal{H}^{n-1}.$$

for $u \in GSBV^2(\Omega) \cap L^2(\Omega)$.

Proof. First we note that the analogues of Propositions 3.6, 3.8 and Corollary 3.9 hold, since the argument used in the proofs is not affected by the particular choice of the norm in the definition of F_ε^φ. Hence the thesis follows proceeding as in the proof of Proposition 3.11, noticing that, in this case, we apply Theorem 2.1 taking in (2.6) $\phi(a, b, \nu) = \sqrt{\varphi(\nu \otimes \nu)} \sqrt{|a - b|}$. □

Acknowledgements

The authors gratefully acknowledge useful remarks by Andrea Braides. This work is part of the CNR research project "Equazioni differenziali e calcolo delle variazioni".

References

[1] E. Acerbi and A. Braides, Approximation of free-discontinuity problems by elliptic functionals via Γ-convergence, Preprint 1998.

[2] G. Alberti and C. Mantegazza, A note on the theory of SBV functions, *Boll. Un. Mat. It.* **11** (1997), 375-382.

[3] R. Alicandro, A. Braides and M. S. Gelli, Free-discontinuity problems generated by singular perturbation, *The Royal Society of Edinburgh*, to appear.

[4] R. Alicandro, A. Braides and J. Shah, Approximation of non-convex functionals in GBV, *Interfaces and Free Boundaries - modelling, analysis and computation*, to appear.

[5] L. Ambrosio, A compactness theorem for a new class of functions of bounded variation, *Boll. Un. Mat. Ital.* **3-B** (1989), 857-881.

[6] L. Ambrosio, Existence theory for a new class of variational problems, *Arch. Rational Mech. Anal.* **111** (1990), 291-322.

[7] L. Ambrosio, On the lower semicontinuity of quasi-convex integrals defined in $SBV(\Omega;\mathbf{R}^k)$, *Nonlinear Anal.* **23** (1994), 405-425.

[8] L. Ambrosio, A new proof of the SBV compactness theorem, *Calc. Var.* **3** (1995), 127-137.

[9] L. Ambrosio, N. Fusco and D. Pallara, Partial regularity of free-discontinuity sets II, *Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser.* **24** (1997), 39-62.

[10] L. Ambrosio, N. Fusco and D. Pallara, *Special Functions of Bounded Variation and Free Discontinuity Problems*, Oxford University Press, Oxford, to appear.

[11] L. Ambrosio and V. M. Tortorelli, Approximation of functionals depending on jumps by elliptic functionals via Γ-convergence, *Comm. Pure Appl. Math.* **43** (1990), 999-1036.

[12] L. Ambrosio and V. M. Tortorelli, On the approximation of free-discontinuity problems, *Boll. Un. Mat. Ital.* **6-B** (1992), 105-123.

[13] A. Bonnet, On the regularity of edges in image segmentation, *Ann. Inst. H. Poincaré Anal. Nonlin.* **13** (1996), 485-528.

[14] G. Bouchitté, C. Dubs and P. Seppecher, Transitions de phases avec un potentiel dégénéré a l'infini, *C. R. Acad. Sci. Paris* **323** (1996), 1103-1108.

[15] A. Braides, *Approximation of Free-discontinuity Problems*, Lecture Notes in Mathematics, Springer Verlag, Berlin, 1998.

[16] A. Braides and V. Chiadò Piat, Integral representation results for functionals defined on $SBV(\Omega;\mathbf{R}^m)$, *J. Math. Pures Appl.* **75** (1996), 595-626.

[17] A. Braides and G. Dal Maso, Nonlocal approximation of the Mumford-Shah functional, *Calc. Var.* **5** (1997), 293-322.

[18] A. Braides and A. Defranceschi, *Homogenization of Multiple Integrals*, Oxford University Press, Oxford, 1998.

[19] A. Braides and A. Garroni, On the nonlocal approximation of free-discontinuity problems, *Comm. Partial. Diff. Equations*, to appear.

[20] H. Brezis *Analyse fonctionelle*, Masson, Paris, 1983.

[21] M. Carriero and A. Leaci, S^k-valued maps minimizing the L^p-norm of the gradient with free-discontinuities, *Ann. Scuola Norm. Sup. Pisa ser. IV* **18** (1991), 321-352.

[22] G. Cortesani, Sequences of non-local functionals which approximate free discontinuity problems, *Arch. Rational Mech. Anal.*, to appear.

[23] G. Cortesani and R. Toader, A density result in SBV with respect to non isotropic energies, *Nonlinear Analysis*, to appear

[24] G. Dal Maso, *An Introduction to Γ-convergence*, Birkhäuser, Boston, 1993.

[25] E. De Giorgi and L. Ambrosio, Un nuovo funzionale del calcolo delle variazioni, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* **82** (1988), 199-210.

[26] E. De Giorgi, M. Carriero, and A. Leaci, Existence theorem for a minimum problem with free discontinuity set, *Arch. Rational Mech. Anal.* **108** (1989), 195-218.

[27] E. De Giorgi and T. Franzoni, Su un tipo di convergenza variazionale, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Mat.* **58** (1975), 842-850.

[28] F. Dibos, Uniform rectifiability of image segmentations obtained by a variational method, *J. Math. Pures Appl.*, to appear.

[29] M. Gobbino and M.G. Mora, Finite difference approximation of free discontinuity problems, Preprint 1998.

[30] M. Gobbino, Finite difference approximation of the Mumford-Shah functional, *Comm. Pure Appl. Math.* **51** (1998), 197-228.

[31] A.D. Ioffe, On the lower semicontinuity of integral functionals, *SIAM J. Cont. Optim.* **15** (1977), 521-538 and 991-1000

[32] L. Modica and S. Mortola, Un esempio di Γ-convergenza, *Boll. Un. Mat. It.* **14-B** (1977), 285-299.

[33] J. M. Morel and S. Solimini, *Variational Models in Image Segmentation*, Birkhäuser, Boston, 1995.

[34] D. Mumford and J. Shah, Optimal approximation by piecewise smooth functions and associated variational problems, *Comm. Pure Appl. Math.* **17** (1989), 577-685.

[35] L. Truskinovsky and G. Zanzotto, Ericksen's bar revisited: energy wiggles, *J. Mech. Phys. Solids* **44** (1996), 1372-1408.