ON THE RELAXATION OF SOME OPTIMAL PARTITION PROBLEMS

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Abstract. We consider optimization problems for which the cost functional depends on a partition of a given domain Ω. It is well known that, unless we assume special monotonicity conditions on the cost functional or geometric constraints on the class of admissible choices, an optimal solution does not exist and a relaxation procedure is then necessary to describe the asymptotic behavior of the minimizing sequences. In this paper we determine the form of the relaxed optimization problem.

1. Introduction

In this paper we consider optimal partition problems of the following type

$$\min\{F(A_1, \ \dots, \ A_k) : A_i \in \mathcal{A}(\Omega), \ A_i \cap A_j = \emptyset \text{ for } i \neq j\}, \tag{1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, k is a fixed positive integer, $\mathcal{A}(\Omega)$ is the class of all admissible domains, i.e. the class of all (quasi) open subsets of Ω and $F : \mathcal{A}(\Omega)^k \to [0, +\infty]$ is the cost functional. A family $\{A_1, \dots, A_k\}$ of pairwise disjoint subsets of Ω will be called *partition* in the following.

Problems of this kind have been considered by Bucur, Buttazzo and Henrot in [1], where the existence of an optimal partition, made of quasi-open subsets, is shown under suitable monotonicity assumptions on the cost functional F. On the other hand, even in the case in which $k = 1$, it is known that, without extra assumptions involving either the monotonicity of the cost functional or some geometric constraints on the class of competing domains (see [1]), the existence of an optimal domain may fail. In this case, if we want to study the asymptotic behavior of the minimizing sequences, a relaxation procedure is needed and we may only speak of a solution in a relaxed sense.

However, this still allows to obtain some necessary conditions of optimality for the relaxed solutions (see [2]).

The notion of "*relaxed Dirichlet problem*" was introduced in [5] in order to describe the asymptotic behavior of the solutions of classical Dirichlet problems in highly perturbed domains. Such a "relaxed" problem can be formally written as

$$\begin{cases} -\Delta u + \mu u = f, \ \text{in } \Omega, \\ u = 0, \ \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, $f \in H^{-1}(\Omega)$ and the "coefficient" μ is a suitable nonnegative Borel measure on Ω, which must vanish on all sets of harmonic capacity zero, but is allowed to take the value $+\infty$ on nonpolar subsets of Ω. We shall denote by $\mathcal{M}_0(\Omega)$ the class of all measures μ allowed in (1.2).

More precisely, problems of type (1.2) arise naturally as asymptotic equations satisfied by the limit u of sequences of "perturbed" solutions u_n, on varying open subsets Ω_n of Ω, of Dirichlet problems of the form

$$\begin{cases} -\Delta u_n = f, \text{ in } \Omega_n, \\ u_n = 0, \text{ on } \partial\Omega_n. \end{cases} \tag{1.3}$$

Here, $\Omega_n = \Omega \setminus S_n$, $\{S_n\}$ is a sequence of compact subsets of Ω and u_n is extended to zero on $\Omega \backslash \Omega_n$.

Such problems correspond, in fact, to the measures $\mu_n = \infty_{S_n}$ of the class $\mathcal{M}_0(\Omega)$, defined by

$$\infty_{S_n}(B) = \begin{cases} 0, \text{ if } cap(B \cap S_n) = 0, \\ +\infty, \text{ if } cap(B \cap S_n) > 0, \end{cases} \tag{1.4}$$

for every Borel subset B of Ω. These measures are sometimes called measures of *Dirichlet* type.

It is well known that the solutions u_n of problems (1.3) may have wild oscillatory behavior as $n \to +\infty$ and, therefore, u_n will converge to some limit u in Ω only in a weak topology, like that of the space $H^1(\Omega)$. Also, the limit u may be no more a solution of a problem of the same type. In fact, the solution u verifies a "relaxed" equation of the type (1.2), with a new "penalization" term μu.

In order to study the dependence of the solution u of (1.2) on the measure μ, one can rely on a notion of convergence for sequences $\{\mu_n\}$ in $\mathcal{M}_0(\Omega)$, called $\gamma-$convergence. We refer to [3] and [4] for more details and applications to the study of perturbation problems in the calculus of variations.

The main properties of the $\gamma-$convergence are the following compactness and density results ([6]):

a) for every sequence $\{\mu_n\}$ in $\mathcal{M}_0(\Omega)$, there exists a subsequence $\{\mu_{n_k}\}$ which $\gamma-$converges to a measure $\mu \in \mathcal{M}_0(\Omega)$;

b) for every $\mu \in \mathcal{M}_0(\Omega)$, there exists a sequence $\{S_n\}$ of compact and regular subsets of Ω such that the sequence of measures $\{\infty_{S_n}\}$ $\gamma-$converges to μ.

In other words, the class $\mathcal{M}_0(\Omega)$ is sequentially compact and the measures of the Dirichlet type are dense in $\mathcal{M}_0(\Omega)$. Therefore, $\mathcal{M}_0(\Omega)$ proves to be a natural variational closure of the class of Dirichlet measures (1.4).

The purpose of this note is to consider the case in which we have k sequences $\{A_n^1\}, \dots, \{A_n^k\}$ of pairwise disjoint subsets of Ω.

If we consider the associated Dirichlet measures $\mu_n^i = \infty_{\Omega \backslash A_n^i}$, $i = 1, \dots, k$, of the class $\mathcal{M}_0(\Omega)$, from the sequential compactness property of $\mathcal{M}_0(\Omega)$, we know that, up to a subsequence, there exist k measures $\mu_i \in \mathcal{M}_0(\Omega)$, $i = 1, \dots, k$, such that μ_n^i $\gamma-$converge to μ_i, for any $i = 1, \dots, k$.

The natural question arising here is to characterize all $k-$tuples $(\mu_1, \dots \mu_k)$ which are the $\gamma-$limits of $\mu_n^i = \infty_{\Omega \backslash A_n^i}$, with A_n^i pairwise disjoint (we shall call such a $k-$tuple an *admissible* or an *attainable one).* If we denote by A_{μ_i} the set of $\sigma-$finiteness of the measure μ_i, we shall prove that $(\mu_1, \dots \mu_k)$ is admissible if and only if it satisfies the following property:

$$cap(A_{\mu_i} \cap A_{\mu_j}) = 0, \text{ for any } i \neq j, \ i, \ j = 1, \dots, k. \tag{P}$$

In particular, if the property above holds, it is possible to find k sequences of pairwise disjoint domains $\{A_n^i\}$ such that the corresponding Dirichlet measures $\mu_n^i = \infty_{\Omega \backslash A_n^i}$ $\gamma-$converge to μ_i, for any $i = 1, \dots, k$.

The main tool for finding such sequences of disjoint domains $\{A_n^i\}$ will be a "localization" result obtained in [3]: if two $\gamma-$convergent sequences of measures of class $\mathcal{M}_0(\Omega)$ coincide on all finely open subsets of a finely open set, then the same property holds for their $\gamma-$limits.

Then, we shall address the optimal partition problem (1.1) and since is well known that, unless we assume special monotonicity conditions on the cost functional F or geometric constraints on the class of admissible choices, an optimal solution may not exist, we shall consider the relaxed problem associated to (1.1). Using the compactness and density results obtained in Section 3, we shall prove that under suitable hypotheses on the cost functional F, the associated relaxed problem admits a solution. Moreover, we shall prove some necessary optimality conditions for the relaxed solutions.

The paper is organized as follows: in Section 2 we recall some definitions and preliminary results, necessary in the sequel. Section 3 is devoted to the proof of our main results, the relationship between the regular sets of the $\gamma-$limits measures μ_i, $i = 1, \dots, k$ and the possibility of "reconstructing" the disjoint domains A_n^i from the given measures μ_i satisfying the property (P). In the final section of the paper we shall address the optimization problem and we shall give some necessary conditions of optimality for the solution of the relaxed problem associated to (1.1).

2. Notation and Preliminary Results

In this section we fix the notation and recall some definitions and preliminary results necessary in the sequel.

Let Ω be a bounded open subset of $\mathbb{R}^N$, with $N \geq 2$. By $H^1(\Omega)$ we shall denote the usual Sobolev space of all real-valued functions $u \in L^2(\Omega)$ with first-order distribution derivatives $Du \in L^2(\Omega)$, endowed with the scalar product

$$(u, v)_{H^1(\Omega)} = \int_\Omega DuDvdx + \int_\Omega uvdx$$

and with the corresponding norm $\|\cdot\|_{H^1(\Omega)}$.

As usual, by $H_0^1(\Omega)$ we shall denote the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$ and by $H^{-1}(\Omega)$ the dual space of $H_0^1(\Omega)$. Making the convention that each function $u \in H_0^1(\Omega)$ is extended to $\mathbb{R}^N$ by putting $u = 0$ on $\mathbb{R}^N \setminus \Omega$, we have $H_0^1(\Omega) \subseteq H^1(\mathbb{R}^N)$.

The capacity of a subset E of $\mathbb{R}^N$ is defined by

$$cap(E) = \inf_{u \in \mathcal{U}_E} \|u\|^2_{H^1(\mathbb{R}^N)},$$

where $\mathcal{U}_E$ is the set of all functions $u \in H^1(\mathbb{R}^N)$ such that $u \geq 1$ a.e. in a neighborhood of E. If a property $P(x)$ holds for all $x \in E$, except for a set $Z \subseteq E$ with $cap(Z) = 0$, then we say that $P(x)$ holds quasi-everywhere on E (q.e. on E).

A set $A \subseteq \Omega$ is called *quasi-open* (respectively *quasi-closed*, *quasi-compact*) in Ω if, for every $\epsilon > 0$, there exists an open (respectively closed or compact) set $U \subseteq \Omega$ such that $cap(A \Delta U) < \epsilon$, where Δ denotes the symmetric difference of sets.

A function $f : A \to \mathbb{R}$ is said to be *quasi-continuous* in Ω if for every $\epsilon > 0$ there exists a subset $E \subseteq \Omega$ with $cap(\Omega \setminus E) < \epsilon$ such that the restriction of f to E is continuous on E.

Functions in $H^1(\Omega)$ can be defined q.e. in Ω and it is well known (see, for instance, [2]) that every function $u \in H^1(\Omega)$ admits a quasi-continuous representative, which is uniquely defined up to a set of capacity zero. By convention, we shall always identify in the sequel the function u with its quasi-continuous representative.

The fine topology (see for details [7]) is defined as the weakest topology in $\mathbb{R}^N$ making continuous every superharmonic function on $\mathbb{R}^N$.

For every set $E \subseteq \Omega$ we denote by int^*E, cl^*E and $\partial^* E$, respectively, the fine interior, the fine closure and the fine boundary of E in Ω. It is well-known that every finely open (respectively finely closed) subset of Ω is quasi-open (respectively quasi-closed).

Also, a real-valued function u is quasi-continuous on an open set Ω if and only if u is finely continuous q.e. in Ω. So, by the above mentioned convention, every function $u \in H^1(\Omega)$ is finely continuous q.e. in Ω.

If Ω is a finely open subset of $\mathbb{R}^N$, by $H_0^1(\Omega)$ we denote the space of all functions $u \in H^1(\mathbb{R}^N)$ such that $u = 0$ q.e. on $\mathbb{R}^N \setminus \Omega$, with the Hilbert space structure inherited from $H^1(\mathbb{R}^N)$. Note that if Ω is open, the previous definition is equivalent to that one introduced at the beginning of this section.

We recall now the definition of the class of measures $\mathcal{M}_0(\Omega)$, already introduced in [2] and [3]. If $\mathcal{B}(\Omega)$ denotes the $\sigma-$field of all Borel subsets of the open bounded subset Ω of $\mathbb{R}^N$, by a *Borel measure* on Ω we mean a nonnegative countably additive set function $\mu : \mathcal{B}(\Omega) \to [0, +\infty]$ such that $\mu(\emptyset) = 0$.

Definition 2.1. *By $\mathcal{M}_0(\Omega)$ we denote the set of all nonnegative Borel measures μ on Ω such that $\mu(E) = 0$, for every Borel set $E \subseteq \Omega$ with $cap(E) = 0$.*

Let us note that the measures of the class $\mathcal{M}_0(\Omega)$ are not required to be regular nor $\sigma-$finite; for instance, for every set $E \subseteq \Omega$, the Borel measure ∞_E defined by

$$\infty_E(B) = \begin{cases} 0, \text{ if } cap(B \cap E) = 0, \\ +\infty, \text{ if } cap(B \cap E) > 0, \end{cases} \quad \text{for any } B \in \mathcal{B}(\Omega) \tag{2.1}$$

belongs to $\mathcal{M}_0(\Omega)$.

Definition 2.2. *We say that two measures μ, $\nu \in \mathcal{M}_0(\Omega)$ are equivalent if*

$$\int_\Omega u^2 d\mu = \int_\Omega u^2 d\nu, \tag{2.2}$$

for every $u \in H_0^1(\Omega)$.

It is well known (see, for instance, [3]) that two measures of the class $\mathcal{M}_0(\Omega)$ are equivalent if and only if they agree on all finely open subsets of Ω. In fact, we have the following well known result ([3], [4]):

Proposition 2.1. *Let μ and ν be two measures of the class $\mathcal{M}_0(\Omega)$. The following conditions are equivalent:*

(a) $\mu(A) = \nu(A)$, *for every finely open set $A \subseteq \Omega$;*
(b) $\mu(A) = \nu(A)$, *for every quasi-open set $A \subseteq \Omega$;*
(c) μ *and* ν *are equivalent.*

Let us recall now the notion of *regular set* of a measure $\mu \in \mathcal{M}_0(\Omega)$.

Definition 2.3. *For every measure $\mu \in \mathcal{M}_0(\Omega)$, the regular set $A(\mu)$ (or the set of $\sigma-$finiteness) of μ is defined as the union of all finely open subsets A of Ω such that $\mu(A) < +\infty$. The singular set $S(\mu)$ of μ is defined as the complement of $A(\mu)$ in Ω.*

Let us remark here that the regular set $A(\mu)$ of μ is finely open. Also, it is easy to see that if A is a finely open subset of Ω which intersects $S(\mu)$, then $\mu(A) = +\infty$. From Proposition 2.1., we have $A(\mu) = A(\nu)$ and $S(\mu) = S(\nu)$, if μ and ν are equivalent measures of the class $\mathcal{M}_0(\Omega)$.

We recall now the notion of "*relaxed Dirichlet problems*". If $\mu \in \mathcal{M}_0(\Omega)$ and $f \in H^{-1}(\Omega)$, we may consider the elliptic problem formally written as

$$u \in H_0^1(\Omega) \cap L^2_\mu(\Omega), \quad -\Delta u + u\mu = f. \tag{2.3}$$

The precise meaning of problem (2.3), which is called "*relaxed Dirichlet problem*", is given in a weak form (see [2]) as

$$\int_\Omega DuDv dx + \int_\Omega uv d\mu = \langle f, v\rangle_{H^{-1}(\Omega)}, \ \forall v \in H_0^1(\Omega) \cap L^2_\mu(\Omega) \tag{2.4}$$

and it is easy to show that for every $\mu \in \mathcal{M}_0(\Omega)$ the problem (2.3) admits a unique solution $u \in H_0^1(\Omega) \cap L^2_\mu(\Omega)$.

Following [2], we may then introduce the *resolvent operator* R_μ defined by $R_\mu(f) = u$, where $f \in H^{-1}(\Omega)$ and $u \in H_0^1(\Omega) \cap L^2_\mu(\Omega)$ is the unique solution of (2.3). It can be proved that R_μ is a continuous, symmetric linear operator and that there exists a constant $C(\Omega)$ independent of μ such that

$$\|R_\mu(f)\|_{H^1(\Omega)} \leq C \|f\|_{H^{-1}(\Omega)}, \tag{2.5}$$

for every $f \in H^{-1}(\Omega)$.

Definition 2.4. *A sequence $\{\mu_n\}$ of measures of the class $\mathcal{M}_0(\Omega)$ $\gamma-$converges to a measure $\mu \in \mathcal{M}_0(\Omega)$ if and only if $R_{\mu_n}(f) \to R_\mu(f)$, strongly in $L^2(\Omega)$, for every $f \in L^2(\Omega)$. In this case, we shall write $\mu_n \xrightarrow{\gamma} \mu$.*

From (2.5) we see immediately that $\{\mu_n\}$ $\gamma-$converges to μ if and only if $R_{\mu_n}(f)$ is weakly convergent in $H_0^1(\Omega)$, for any $f \in H^{-1}(\Omega)$.

The following compactness and density results were proved in [6]:

Theorem 2.2. *For every sequence $\{\mu_n\}$ of measures of class $\mathcal{M}_0(\Omega)$, there exists a subsequence $\{\mu_{n_k}\}$ which $\gamma-$converges to a measure $\mu \in \mathcal{M}_0(\Omega)$.*

Theorem 2.3. *For every measure $\mu \in \mathcal{M}_0(\Omega)$, there exists a sequence $\{S_n\}$ of compact and smooth subsets of Ω such that the sequence $\{\infty_{S_n}\}$ of measures of the class $\mathcal{M}_0(\Omega)$ γ-converges to μ.*

Let $\mu \in \mathcal{M}_0(\Omega)$ and let $\{S_n\}$ be the sequence of compact subsets of Ω given by Theorem 2.3.. For any $f \in H^{-1}(\Omega)$, let $A_n = \Omega \setminus S_n$ and let $u_n \in H_0^1(A_n)$ be the solution of the Dirichlet problem

$$u_n \in H_0^1(A_n), \quad -\Delta u_n = f \text{ in } H^{-1}(A_n). \tag{2.6}$$

If $\mu_n = \infty_{S_n}$, the solution u_n of (2.6) can be written as $u_n = R_{\mu_n}(f)$ and from the $\gamma-$convergence of μ_n to μ, we get immediately the weak convergence in $H_0^1(\Omega)$ of the sequence of solutions $\{u_n\}$ of (2.6) to the solution u of the "*relaxed Dirichlet problem*" (2.3).

So, the solution u of (2.3) can be approximated by the solutions of the classical Dirichlet problems (2.6) and this justifies the name "*relaxed Dirichlet problems*" given to such kind of problems.

We conclude this section by recalling the following fine localization theorem, which was proved in [3].

Theorem 2.4. *Let $\{\mu_n\}$ and $\{\nu_n\}$ be two sequences of measures of class $\mathcal{M}_0(\Omega)$ which $\gamma-$converge to μ and, respectively, ν and let E be a quasi-open subset of Ω. If μ_n and ν_n coincide on all quasi-open subsets of E, for every $n \in \mathbb{N}$, then their $\gamma-$limits μ and ν coincide on all quasi-open subsets of E.*

3. The Relaxation Result

Consider k sequences of compact subsets $\{S_n^i\}$ of Ω, $i = 1, \dots, k$, such that the corresponding domains $A_n^i = \Omega \setminus S_n^i$ are pairwise disjoint. We define the associated measures μ_n^i, $i = 1, \dots, k$, of the class $\mathcal{M}_0(\Omega)$, by putting $\mu_n^i = \infty_{S_n^i}$.

From the sequential compactness property of the $\gamma-$convergence (given by Theorem 2.2.), we get, by passing possibly to some subsequences, the existence of k measures μ_i, $i = 1, \dots, k$, of class $\mathcal{M}_0(\Omega)$, such that $\mu_n^i \xrightarrow{\gamma} \mu_i$, for any $i = 1, \dots, k$.

We shall prove that the above $\gamma-$limit measures μ_i, $i = 1, \dots, k$, are related by the following property:

$$cap(A_{\mu_i} \cap A_{\mu_j}) = 0, \text{ for any } i \neq j,\ i,\ j = 1, \dots,\ k. \tag{3.1}$$

If μ_i satisfy (3.1), the k-tuple $(\mu_1, \dots,\ \mu_k)$ will be called an *admissible* or an *attainable* one.

So, the limit measures are not "independent". For example, it will be not possible to get an attainable k-tuple formed by k measures of the Lebesgue type.

Conversely, we shall prove that if $\mu_1, \dots,\ \mu_k$ satisfy (3.1), then there exist k sequences of pairwise disjoint domains $\{A_n^i\}$ such that the corresponding Dirichlet measures $\infty_{\Omega \setminus A_n^i}$ $\gamma-$converge to μ_i, for any $i = 1, \dots,\ k$.

For simplicity, in what follows we shall restrict ourselves to the case $k = 2$, but all the results we shall obtain hold for the general case, in which k is an arbitrary fixed positive integer.

So, for the case in which $k = 2$, let us denote the two sequences of compact subsets of Ω by $\{S_n\}$ and, respectively, $\{T_n\}$ and the corresponding disjoint domains by $A_n = \Omega \setminus S_n$ and, respectively, $B_n = \Omega \setminus T_n$.

Let us introduce the associated measures of class $\mathcal{M}_0(\Omega)$, by putting

$$\mu_n = \infty_{S_n} \text{ and } \nu_n = \infty_{T_n}. \tag{3.2}$$

Proposition 3.1. *Let Ω be an open bounded subset of $\mathbb{R}^N$ and let $\{S_n\}$ and $\{T_n\}$ be two sequences of compact subsets of Ω such that the corresponding domains $A_n = \Omega \setminus S_n$ and $B_n = \Omega \setminus T_n$ are disjoint. Then, there exist two measures μ and ν of the class $\mathcal{M}_0(\Omega)$ such that, up to a subsequence, the sequences of measures $\{\mu_n\}$ and $\{\nu_n\}$ defined by (3.2) γ-converge to μ and, respectively, ν. Moreover, the limit measures μ and ν satisfy the property (3.1).*

Proof. Let $\{S_n\}$ and $\{T_n\}$ be two sequences of compact subsets of Ω such that the domains $A_n = \Omega \setminus S_n$ and $B_n = \Omega \setminus T_n$ are disjoint and let the corresponding measures μ_n and ν_n be defined by (3.2).

For any domain $A \subseteq \Omega$, let us denote by w_A the solution of the auxiliary problem

$$w_A \in H_0^1(A), \quad -\Delta w_A = 1, \text{ on } A. \tag{3.3}$$

Replacing A with A_n, respectively B_n and using the Poincaré inequality, we see immediately that the sequences of solutions $\{w_{A_n}\}$ and $\{w_{B_n}\}$ are bounded in $H_0^1(\Omega)$. So, there exist w_μ and w_ν in $H_0^1(\Omega)$ such that $w_{A_n} \rightharpoonup w_\mu$ and $w_{B_n} \rightharpoonup w_\nu$ weakly in $H_0^1(\Omega)$. It is easy to see that $A_\mu = \{w_\mu > 0\}$ and $A_\nu = \{w_\nu > 0\}$.

On the other hand, since the domains A_n and B_n are disjoint, we get

$$w_{A_n} \cdot w_{B_n} = 0. \tag{3.4}$$

The weak convergence of w_{A_n} to w_μ in $H_0^1(\Omega)$ implies that $w_{A_n} \to w_\mu$ strongly in $L^2(\Omega)$. In a similar manner, we obtain that $w_{B_n} \to w_\nu$ strongly in $L^2(\Omega)$.

Therefore

$$w_\mu \cdot w_\nu = 0 \text{ a.e. in } \Omega. \tag{3.5}$$

But since $w_\mu \cdot w_\nu$ is finely continuous q.e. in Ω and every nonempty finely open set has positive Lebesgue measure, it follows that

$$w_\mu \cdot w_\nu = 0 \text{ q.e. in } \Omega \tag{3.6}$$

and this proves that the regular sets A_μ and A_ν are disjoint (up to a set of capacity zero), so the γ-limit measures μ and ν satisfy the property (3.1). □

Let us consider now the converse problem of the "reconstruction" of the disjoint domains A_n and B_n from the given measures μ and ν satisfying the property (3.1).

As already mentioned, we shall prove that given two measures μ and ν satisfying (3.1), it is possible to find two sequences of disjoint domains $\{A_n\}$ and $\{B_n\}$ such that the corresponding Dirichlet measures $\mu_n = \infty_{\Omega\setminus A_n}$ and $\nu_n = \infty_{\Omega\setminus B_n}$ $\gamma-$converge to μ and, respectively, ν.

The main tool for finding such sequences of disjoint domains is the "localization" result given by Theorem 2.4..

Proposition 3.2. *Let μ and ν be two given measures of class $\mathcal{M}_0(\Omega)$ satisfying the property (3.1). Then, there exist two sequences of disjoint domains $\{A_n\}$ and $\{B_n\}$ such that the corresponding Dirichlet measures $\mu_n = \infty_{\Omega\setminus A_n}$ and $\nu_n = \infty_{\Omega\setminus B_n}$ $\gamma-$converge to μ and, respectively, ν.*

Proof. Let μ and ν be two given measures of class $\mathcal{M}_0(\Omega)$ satisfying (3.1). From the density result given by Theorem 2.3., there exist two sequences of compact subsets of Ω, $\{S_n\}$ and, respectively, $\{T_n\}$, such that the measures $\mu_n = \infty_{S_n}$ and $\nu_n = \infty_{T_n}$ are $\gamma-$convergent to μ and, respectively, ν. So

$$\mu_n \xrightarrow{\gamma} \mu \text{ and } \nu_n \xrightarrow{\gamma} \nu. \tag{3.7}$$

Let $\widetilde{A}_n = \Omega \setminus S_n$ and $\widetilde{B}_n = \Omega \setminus T_n$. If A_μ and A_ν are the regular sets of μ and, respectively, ν, consider the Dirichlet measures

$$\overline{\mu}_n = \infty_{\Omega\setminus(\widetilde{A}_n\cap A_\mu)} \text{ and } \overline{\nu}_n = \infty_{\Omega\setminus(\widetilde{B}_n\cap A_\nu)}. \tag{3.8}$$

Of course, the domains $A_n = \widetilde{A}_n \cap A_\mu$ and $B_n = \widetilde{B}_n \cap A_\nu$ will be disjoint (up to a set of capacity zero).

We shall prove that these new measures are also $\gamma-$convergent to μ and, respectively, ν and, since their domains are disjoint, this will conclude the proof.

From the compactness result given by Theorem 2.2., we know that there exist two measures $\widetilde{\mu}$ and, respectively, $\widetilde{\nu}$, of the class $\mathcal{M}_0(\Omega)$, such that, up to a subsequence, $\widetilde{\mu}_n \xrightarrow{\gamma} \widetilde{\mu}$ and $\widetilde{\nu}_n \xrightarrow{\gamma} \widetilde{\nu}$. We shall prove that $\mu = \widetilde{\mu}$ and $\nu = \widetilde{\nu}$ and this will end, in fact, the proof.

Let us show that $\mu = \widetilde{\mu}$. By construction, $\mu_n \le \widetilde{\mu}_n$ and so, $\mu \le \widetilde{\mu}$.

Using the localization result given by Theorem 2.4., it suffices to prove that μ_n and $\widetilde{\mu}_n$ coincide on all finely open subsets of A_μ. This will imply that μ and $\widetilde{\mu}$ coincide on A_μ and since outside of A_μ they are obviously equal (being both equal to $+\infty$), this gives $\mu = \widetilde{\mu}$. In a similar manner we obtain $\nu = \widetilde{\nu}$.

So, it remains only to prove that μ_n and $\widetilde{\mu}_n$ coincide on all finely open subsets of A_μ. Let E be an arbitrary finely open subset of A_μ.

We can write E in the form $E = (E \cap \widetilde{A}_n) \cup (E \setminus \widetilde{A}_n)$.

If $cap(E\setminus\widetilde{A}_n)=0$, then $\mu_n(E)=\mu_n(E\cap\widetilde{A}_n)\le\mu_n(\widetilde{A}_n)=0$. On the other hand, $\widetilde{\mu}_n(E)\le\widetilde{\mu}_n(E\cap\widetilde{A}_n)\le\widetilde{\mu}_n(A_\mu\cap\widetilde{A}_n)=0$ and so, in this case, we get that μ_n and $\widetilde{\mu}_n$ coincide on all finely open subsets of A_μ.

If $cap(E\setminus\widetilde{A}_n)>0$, then $\mu_n(E)\ge\mu_n(E\setminus\widetilde{A}_n)=+\infty$ and $\widetilde{\mu}_n(E)\ge\mu_n(E)=+\infty$. Then, in this case too, μ_n and $\widetilde{\mu}_n$ coincide on all finely open subsets of A_μ.

This means that in any case μ_n and $\widetilde{\mu}_n$ coincide on all finely open subsets of A_μ and the proposition is proved. □

4. Some Necessary Conditions of Optimality

Let us consider two given functions, $f=(f_1,...,f_k)\in L^2(\Omega)^k$ and $j:\Omega\times\mathbb{R}^k\to\mathbb{R}$, satisfying the following conditions:

(i) $j(\cdot,\,s)$ is $\mathcal{L}^N-$measurable in Ω, for every $s\in\mathbb{R}^k$;

(ii) $j(x,\,\cdot)$ is continuous in $\mathbb{R}^k$, for a.e. $x\in\Omega$;

(iii) there exist $a_0\in L^1(\Omega)$ and $c_0\in\mathbb{R}$ such that, for a.e. $x\in\Omega$ and for every $s\in\mathbb{R}^k$,

$$|j(x,\;s)|\le a_0(x)+c_0\,|s|^2\,.$$

For every $(u_1,\dots,\,u_k)\in L^2(\Omega)^k$, let us define

$$J(u_1,\dots,\;u_k)=\int_\Omega j(x,\;u_1(x),\;\;\dots,\;u_k(x))dx. \tag{4.1}$$

The aim of this section is to study the optimization problem

$$\min\{J(u_{A_1},\dots,\;u_{A_k})\,:\,A_i\in\mathcal{A}(\Omega),\;A_i\cap A_j=\emptyset,\;\text{for }i\ne j\}, \tag{4.2}$$

where $\mathcal{A}(\Omega)$ is the class of all admissible domains, i.e. the class of all quasi-open subsets of Ω and u_{A_i}, $i=1,\dots,\,k$, are the solutions of the Dirichlet problems

$$u_{A_i}\in H_0^1(A_i),\;\;-\Delta u_{A_i}=f_i\text{ in }H^{-1}(A_i). \tag{4.3}$$

It is well known that, in general, problem (4.2) has no solution (see, for instance, Example 4.1. below) and in order to study the minimizing sequences, we introduce the relaxed optimization problem

$$\min\{J(u_{\mu_1},\dots,\;u_{\mu_k})\text{: }\mu_i\in\mathcal{M}_0(\Omega),\;\mu_i\text{ satisfy (3.1)}\}, \tag{4.4}$$

where u_{μ_i}, $i=1,\dots,k$, are the solutions, in the sense of (2.4), of the relaxed problems

$$u_{\mu_i}\in H_0^1(\Omega)\cap L^2_{\mu_i}(\Omega),\;\;-\Delta u_{\mu_i}+u_{\mu_i}\mu_i=f_i. \tag{4.5}$$

The connection between problems (4.2) and (4.4) is given, like in [2], by the following theorem.

Theorem 4.1. *Let $f\in L^2(\Omega)^k$ and $j:\Omega\times\mathbb{R}^k\to\mathbb{R}$ satisfy conditions (i)-(iii). Then, the relaxed problem (4.4) admits a solution and*

$$\min\{J(u_{\mu_1},\dots,\;u_{\mu_k})\,:\,\mu_i\in\mathcal{M}_0(\Omega),\;\mu_i\textit{ satisfy (3.1)}\}=$$

$$=\inf\{J(u_{A_1},\dots,\;u_{A_k})\,:\,A_i\in\mathcal{A}(\Omega),\;A_i\cap A_j=\emptyset,\;\textit{for }i\ne j\}.$$

Proof. Let us notice that the functional J is continuous in the strong topology of $L^2(\Omega)^k$. So, the function $(\mu_1, \ldots, \mu_k) \mapsto J(u_{\mu_1}, \ldots, u_{\mu_k})$ is continuous on $\mathcal{M}_0(\Omega)^k$ with respect to the $\gamma-$convergence and the conclusion of the theorem follows immediately from the compactness and density properties stated in Propositions 3.1. and 3.2.. □

In what follows, our aim is to obtain, as in [2], some necessary conditions of optimality for the solutions of the relaxed optimization problem (4.4).

The methods used to prove these conditions are quite similar to those used in [2] for the case of only one measure μ.

Let us suppose that, in addition, the function $j : \Omega \times \mathbb{R}^k \to \mathbb{R}$ satisfies:

(iv) $j(x, \cdot)$ is of class C^1 on $\mathbb{R}^k$;

(v) $j_{s_i}(\cdot, s)$ are $\mathcal{L}^N-$measurable on Ω, for every $s \in \mathbb{R}^k$;

(vi) there exist $a_1 \in L^2(\Omega)$ and $c_1 \in \mathbb{R}$ such that, for a.e. $x \in \Omega$ and for every $s \in \mathbb{R}^k$,

$$\sum_{i=1}^{k} |j_{s_i}(x, s)| \leq a_1(x) + c_1 |s| .$$

It follows immediately that the map J defined by (4.1) is differentiable on $L^2(\Omega)^k$ and its differential J' can be written as

$$\langle J'(u), v\rangle = \sum_{i=1}^{k} \int_\Omega j_{s_i}(x, u) v_i dx,$$

for any $u, v \in L^2(\Omega)^k$. Here, $\langle \cdot, \cdot\rangle$ denotes the duality pair between $L^2(\Omega)^k$ and its dual.

Following [2, Theorem 6.1.], we introduce the boundary measure ν_A associated with every finely open subset A of $\mathbb{R}^N$ by writing equation (3.3) in the weak form

$$w_A \in H_0^1(A), \quad \int_A Dw_A Dv dx = \int_A v dx, \ \forall v \in H_0^1(A).$$

Then, there exists a unique nonnegative Radon measure ν_A belonging to $H^{-1}(\mathbb{R}^N)$ such that

$$-\Delta w_A + \nu_A = 1_{cl^* A} \text{ in } H^{-1}(\mathbb{R}^N).$$

Moreover, we have $\nu_A(\partial^* A) = \mathcal{L}^N(cl^* A)$ and $\nu_A(\mathbb{R}^N \setminus \partial^* A) = 0$, i.e. ν_A is carried by $\partial^* A$.

Similarly to what was done in [2, Theorem 7.1.], we can prove that for every $\mu \in \mathcal{M}_0(\Omega)$, there exists a unique function $\alpha \in L^2(\Omega, \nu_{A_\mu})$ such that

$$-\Delta u_\mu + \mu u_\mu + \alpha \nu_A = f 1_{cl^* A_\mu} \text{ in } H^{-1}(\Omega),$$

where A_μ is the regular set of μ and u_μ is the solution of the relaxed problem (2.3).

Remark 4.1. We notice that when μ is a measure of the form $\infty_{\Omega\setminus A} + q\mathcal{L}^N\lfloor_A$, with $q \in L^\infty(A)$ and A an open set of class C^2, then u_μ satisfies the problem

$$u_\mu \in H_0^1(A), \ -\Delta u_\mu + q u_\mu = f \text{ in } H^{-1}(A)$$

and we get

$$\nu_A = -\frac{\partial w_A}{\partial n} \mathcal{H}^{N-1} \lfloor_{\Omega\cap\partial A}$$

and

$$\alpha = \frac{\partial u}{\partial n}\left[\frac{\partial w_A}{\partial n}\right]^{-1} \ \mathcal{H}^{N-1}\text{-a.e. on } \Omega \cap \partial A.$$

Let u_{μ_i} be the solutions of the state equation (4.5). We introduce now the solutions v_{μ_i} of the corresponding adjoint state equations

$$v_{\mu_i} \in H_0^1(\Omega) \cap L^2_{\mu_i}(\Omega), \quad -\Delta v_{\mu_i} + v_{\mu_i}\mu_i = j_{s_i}(x,\ u_{\mu_1}, ..., u_{\mu_k}). \tag{4.6}$$

Notice that, since $\{\mu_i\}$ satisfy property (3.1), u_{μ_i} vanish q.e. on $\Omega \setminus \bigcup_{j\neq i} A_{\mu_j}$.

By applying the above mentioned Theorem 7.1. of [2], for the relaxed equations (4.5) we get the existence of unique functions $\alpha_i \in L^2(\Omega,\ \nu_{A_{\mu_i}})$ such that

$$-\Delta u_{\mu_i} + \mu_i u_{\mu_i} + \alpha_i \nu_{A_{\mu_i}} = f_i 1_{cl^* A_{\mu_i}} \text{ in } H^{-1}(\Omega). \tag{4.7}$$

Similarly, for the adjoint equation (4.6), we get the existence of unique functions $\beta_i \in L^2(\Omega,\ \nu_{A_{\mu_i}})$ such that

$$-\Delta v_{\mu_i} + \mu_i v_{\mu_i} + \beta_i \nu_{A_{\mu_i}} = j_{s_i}(x,\ u_{\mu_1}, ..., u_{\mu_k}) 1_{cl^* A_{\mu_i}} \text{ in } H^{-1}(\Omega). \tag{4.8}$$

We can now perform the first variation arguments used in [2] and follow exactly the same steps in the proof of Theorem 9.3. of [2]. The method is based on perturbing the optimal measures μ_i by approximating measures μ_i^ϵ; the only difference with [2] consists in taking care of fulfilling for every ϵ the admissibility condition (3.1). We then obtain the following necessary conditions of optimality.

Theorem 4.2. *Consider $f \in L^2(\Omega)^k$ and $j : \Omega \times \mathbb{R}^k \to \mathbb{R}$, satisfying the conditions (i)-(vi). Let $(\mu_1, ..., \mu_k)$ be a solution of the relaxed optimization problem (4.4) and let u_{μ_i} and v_{μ_i} be the solutions of the problems (4.5) and (4.6). Then, for every $i = 1, ..., k$, we have*

(a) $u_{\mu_i} v_{\mu_i} \leq 0$ *q.e. on* Ω,
(b) $u_{\mu_i} v_{\mu_i} = 0$ μ_i*–a.e. on* Ω,
(c) $f_i(\cdot) j_{s_i}(\cdot, 0, ..., 0) \geq 0$ *a.e. on* $int^* S_{\mu_i}$,
(d) $\alpha_i \beta_i \geq 0$ $\nu_{A_{\mu_i}}$*–a.e. on* $\Omega \cap [\partial^* A_{\mu_i} \setminus (\bigcup_{j\neq i} \partial^* A_{\mu_j})]$,

where α_i and β_i are defined by (4.7) and (4.8).

Remark 4.2. Assume that the original optimization problem (4.2) admits a solution $(u_{A_1}, ..., u_{A_k})$, with A_i of class C^2 (which happens, for instance, in the cases considered in [1]). Then, the measures $\mu_i = \infty_{\Omega\setminus A_i}$ are solutions of the relaxed problem (4.4). In this case, conditions (a)-(d) of Theorem 4.2. take the form

(a′) $u_{A_i} v_{A_i} \leq 0$ *q.e. on* Ω,
(c′) $f_i(\cdot) j_{s_i}(\cdot, 0, ..., 0) \geq 0$ *a.e. on* S_{μ_i},
(d′) $\frac{\partial u_{A_i}}{\partial n}\frac{\partial v_{A_i}}{\partial n} = 0$ $\mathcal{H}^{N-1}$*–a.e. on* $\Omega \cap [\partial A_i \setminus (\bigcup_{j\neq i} \partial A_j)]$,

while condition (b) is trivially satisfied in this case.

We conclude this section by exhibiting a simple example where the original problem (4.2) has no solution.

Example 4.1. Consider a function $f \in L^2(\Omega)$ such that $f(x) > 0$ for a.e. $x \in \Omega$ and let $\{\Omega_1,\ \Omega_2\}$ be a given partition of Ω. Take $j : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$j(x,\ s_1,\ s_2) = (2s_1 - w_1(x))^2 + (2s_2 - w_2(x))^2,$$

where w_i are the solutions of the following auxiliary problems

$$w_i \in H_0^1(\Omega_i), \ -\Delta w_i = f \text{ in } \Omega_i, \ i = 1, \ 2.$$

It is not difficult to see that the relaxed problem (4.4) attains its minimum value 0 at the measures $\mu = \infty_{\Omega \setminus \Omega_1} + \frac{f}{w_1}\mathcal{L}^N \lfloor_{\Omega_1}$ and $\nu = \infty_{\Omega \setminus \Omega_2} + \frac{f}{w_2}\mathcal{L}^N \lfloor_{\Omega_2}$. On the other hand, it is clear that there are no domains A_1 and A_2 such that $j(x, u_{A_1}, u_{A_2}) = 0$, for a.e. $x \in \Omega$ and this implies that the original problem (4.2) has no solution.

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