REGULARITY OF STRESSES IN PRANDTL-REUSS PERFECT PLASTICITY

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ABSTRACT. We study the differential properties of solutions of the Prandtl-Reuss model. We prove that the stress tensor has locally square-integrable first derivatives: $\sigma \in L^\infty([0,T]; W^{1,2}_{loc}(\Omega; \mathbb{M}^{n\times n}_{sym}))$. The result is based on discretization of time and uniform estimates of solutions of the incremental problems, which generalize the estimates in the case of Hencky perfect plasticity. Counterexamples to the regularity of displacements and plastic strains in the quasistatic case are presented.

Keywords: quasistatic evolution, rate independent processes, Prandtl-Reuss plasticity, regularity of solutions.
2000 Mathematics Subject Classification: 74C05 (28A33, 49N60, 74G65)

1. INTRODUCTION

A strong formulation of the Prandtl-Reuss model of perfect plasticity is the following: given a domain $\Omega \subset \mathbb{R}^n$,

$$\begin{aligned}&\text{body force } f(t,x) : [0,T]\times\Omega \to \mathbb{R}^n,\\ &\text{boundary displacement } w(t,x) : [0,T]\times\Gamma_0 \to \mathbb{R}^n,\\ &\text{surface force } F(t,x) : [0,T]\times\Gamma_1 \to \mathbb{R}^n,\end{aligned}$$

the problem is to find functions

$$u(t,x),\ e(t,x),\ p(t,x) \quad \text{and } \sigma(t,x)$$

such that for every $t \in [0,T]$, for every $x \in \Omega$ the following hold:

1. kinematic admissibility: $\varepsilon(u(t,x)) = e(t,x) + p(t,x)$ in Ω, $u(t,x) = w(t,x)$ on Γ_0
2. constitutive equation: $\sigma(t,x) = \mathbb{A}^{-1}\, e(t,x)$,
3. equilibrium: $\operatorname{div}_x \sigma(t,x) = -f(t,x)$ in Ω, $\sigma(t,x)\,\nu(x) = g(t,x)$ on Γ_1,
4. stress constraint $\sigma(t,x) \in \mathbb{K}$,
5. associative flow rule: $(\xi - \sigma(t,x)) : \dot{p}(t,x) \le 0$ for every $\xi \in \mathbb{K}$,

where

$$\varepsilon(u) = \frac{\nabla u + \nabla u^T}{2},$$
$$\mathbb{K} = \{\tau \in \mathbb{M}^{n\times n}_{sym} \ :\ |\tau^D| \le \sqrt{2}k_*\}$$

and $\mathbb{A}$ is the compliance tensor (the inverse of the elasticity tensor), which in the isotropic case has the form

$$\mathbb{A}\sigma = \frac{\operatorname{tr}\sigma}{n^2\,K_0}\,\mathbf{1} + \frac{1}{2\mu}\,\sigma^D, \tag{1.1}$$

where nK_0 is the first Lamé constant, and μ is the shear modulus. The problem is supplemented by initial conditions at time $t = 0$.

During the last decades there was an extensive study of this problem in its weak formulation (see e.g. [2, 5, 15]). Due to the linear growth of the functional with respect to $\varepsilon(u)$, arising in this problem, one looks for displacements u in the space $BD(\Omega)$ and for stresses σ in the space $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$. However, one can expect a better regularity of the stress tensor

σ. Namely, as it was shown in [10, 11, 12, 13], in some static situations the stress belongs to the space $W^{1,2}_{loc}(\Omega;\mathbb{R}^n)$.

In this paper we address the issue of a higher regularity of the stress tensor $\sigma(t)$ with respect to spatial variables. The main result (see Theorem 2.1 below) states that for the Prandtl-Reuss model one has

$$\sigma \in L^\infty([0,T]; W^{1,2}_{loc}(\Omega;\mathbb{M}^{n\times n}_{sym})).$$

A similar result was obtained in [1], where the authors used Norton-Hoff approximations and the dual theory of elliptic equations. However, our proof is based on a completely different approach, developed by G. Seregin for proving regularity of stresses in the case of Hencky perfect plasticity (see [4, 10, 14]). Observe, that due to this fact, our assumptions on the data of the problem are different from those of [1].

We believe, that the method proposed in this paper can be used for proving the differentiability of stresses for other models occurring in plasticity.

Shortly, the strategy for proving Theorem 2.1 consists in refining the proof of the existence of a solution to the quasistatic problem, carried out in [2], by generalizing the estimates obtained in [4] for proving the regularity of stresses in the case of Hencky perfect plasticity.

More precisely, we follow the general scheme for proving the existence of weak solutions of the continuous-time energy formulation of rate-independent processes (see e.g. [8] and the references contained therein). Our arguments are similar to the ones used in [14] for the case of plasticity with hardening. Note, that in [5, 15] the existence was proved by visco-plastic approximations, while in order to use the methods of [4] one needs to have some analogue of the static problem. This is why we follow the proof of the existence given in [2], where a quasistatic problem in perfect plasticity was solved by time discretization. In this case the incremental problems one has to solve to get the updated values of solutions, play the role of the static problem, where one can use the machinery of [4].

We perform the standard time-discretization procedure, and for suitable defined approximate solutions $(u_N(t), e_N(t), p_N(t), \sigma_N(t))$, converging to a weak solution of the quasistatic problem, we obtain the estimate

$$\sup_{N\in\mathbb{N}} \sup_{t\in[0,T]} \|\sigma_N(t)\|_{W^{1,2}_{loc}(\Omega;\mathbb{M}^{n\times n}_{sym})} \le C, \tag{1.2}$$

which yields Theorem 2.1. To get (1.2), one looks for solutions of the incremental problems, regarded as saddle points of some minimax problem, similar to the one considered in [4, 10] for the static case of Hencky perfect plasticity. The main difference is the presence of a term which takes into account the preceding history of plastic deformation.

Let us note, that Theorem 2.1 does not give any information about the behavior of the stress tensor near the boundary. As it was observed in [9], the method we use is not suitable for the investigation of regularity up to the boundary, at least in the case of a nonconvex domain Ω. The issue of boundary regularity was discussed also in [3].

To our best knowledge, the only global regularity result for the stress in the case of Hencky perfect plasticity is contained in [6] where under appropriate assumptions it is proved that $\sigma \in W^{1/2-\delta,2}(\Omega)$ for every $\delta > 0$.

The paper is organized as follows: in Section 2 we introduce the definitions and state the main result. In Section 3 we present a weak formulation of the quasistatic problem, outline the proof of existence of the quasistatic evolution and obtain some time-continuity estimates for the approximate solutions. In Section 4 an abstract scheme of relaxation of convex functionals in non reflexive spaces is described. A minimax formulation of the incremental problems is given in Section 5. In Section 6 we formulate the regularized problems, which are used for obtaining the differentiability of stresses, and show the convergence properties of their solutions. Section 7 contains the estimates of the $W^{1,2}_{loc}$ norms of the solutions to the regularized problems, which imply that for every approximate solution we actually have

$$\sup_{t\in[0,T]} \|\sigma_N\|_{W^{1,2}_{loc}(\Omega;\mathbb{M}^{n\times n}_{sym})} \le C(N),$$

however, without any uniformity with respect to N. The uniform estimates (1.2) and the proof of Theorem 2.1 are contained in Section 8. In Section 9 we consider the examples which show that there is no analogue of regularity theorem, as in [4, 10, 11], for the displacement u and the plastic strain p.

2. Preliminary definitions and the main result

We use the following notations:

$\mathbb{R}^n$ denotes the n-dimensional Euclidian space,

$\mathbb{M}^{n\times n}_{sym}$ denotes the space of all $n\times n$ symmetric matrices, equipped with a Hilbert-Schmidt scalar product $\sigma:\xi = \sigma_{ij}\xi_{ij}$,

$\mathbf{1}$ stands for the identity matrix, and we consider the orthogonal decomposition $\mathbb{M}^{n\times n}_{sym} = \mathbb{M}^{n\times n}_D \oplus \mathbb{R}\mathbf{1}$ of the space $\mathbb{M}^{n\times n}_{sym}$ into the subspace of trace-free matrices $\mathbb{M}^{n\times n}_D$ and of the multiples of identity $\mathbb{R}\mathbf{1}$,

$\mathbf{1}^D$ represents an orthogonal projection onto the subspace $\mathbb{M}^{n\times n}_D$,

$a\odot b$ stands for the symmetrized tensor product of two vectors $a,b\in\mathbb{R}^n$, given by the formula $(a\odot b)_{ij} = \frac{1}{2}(a_ib_j + a_jb_i)$,

$L^p(\Omega;\mathbb{R}^m)$ is the Lebesgue space of all functions from Ω into $\mathbb{R}^m$, having the finite norm $(\int_\Omega |f|^p\,dx)^{1/p}$,

$W^{l,p}(\Omega;\mathbb{R}^m)$ is the Sobolev space of all functions from Ω into $\mathbb{R}^m$ with the norm

$$\|f\|_{l,p,\Omega} := \Big(\int_\Omega \sum_{\alpha=0}^{l} |\nabla^\alpha f|^r\Big)^{1/r},$$

$M_b(\Omega;\mathbb{R}^m)$ is the space of all bounded Radon measures on Ω with values in $\mathbb{R}^m$,

$BD(\Omega)$ is the space of all functions in $L^1(\Omega;\mathbb{R}^n)$ such that $\varepsilon(u)\in M_b(\Omega;\mathbb{M}^{n\times n}_{sym})$,

$\mathcal{L}^n$ stands for the Lebesgue measure on $\mathbb{R}^n$,

$\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorf meausure.

In the sequel we will make use of the spaces

$$D^{2,1}(\Omega) = \Big\{v\in L^1(\Omega;\mathbb{R}^n) : \|v\|_{2,1} = \|\operatorname{div} v\|_{L^2(\Omega)} + \|v\|_{L_1(\Omega)} + \|\varepsilon^D(v)\|_{L^1(\Omega)} < +\infty\Big\},$$

$$D^{2,1}_0(\Omega) = \Big\{v\in D^{2,1}(\Omega) : v = 0 \quad \text{on } \Gamma_0\Big\},$$

which are well-known spaces of weakly differentiable vector-valued functions. For their properties we refer to [4, Appendix A.2]. Let us introduce the notation

$$\Sigma = \Big\{\sigma\in L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) \ : \ \operatorname{div}\sigma\in L^n(\Omega;\mathbb{R}^n),\ \sigma^D\in L^\infty(\Omega;\mathbb{M}^{n\times n}_D)\Big\},$$

$$\mathcal{K} = \Big\{\sigma\in L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) \ : \ \sigma(x)\in\mathbb{K} \text{ for a.e. } x\in\Omega\Big\}.$$

2.1. **The main result.** We impose the following assumptions on the data of the problem

$$\begin{aligned} &f\in AC([0,T];L^n(\Omega;\mathbb{R}^n))\cap L^\infty([0,T];C^1_{loc}(\Omega;\mathbb{R}^n))\\ &F\in AC([0,T];L^\infty(\Gamma_1))\\ &w\in AC([0,T];W^{1,2}(\Omega;\mathbb{R}^n)).\end{aligned} \tag{2.1}$$

We also assume the so-called uniform safe-load condition:

$$\text{(2.2)}\qquad \begin{aligned}&\text{there exists a function } \varrho \in AC([0,T]; L^2(\Omega;\mathbb{M}^{n\times n}_{sym})), \text{ such that}\\ &\operatorname{div}_x \varrho(t) = -f(t) \text{ in } \Omega \text{ and } [\varrho\nu] = F(t) \text{ on } \Gamma_1 \text{ for every } t\in[0,T],\\ &|\varrho^D(t,x)| \le (1-\lambda)\sqrt{2}k_* \text{ for some } 0<\lambda<1, \text{ a.e. } x\in\Omega, \text{ for every } t\in[0,T],\\ &\text{and } \varrho^D \in AC([0,T]; L^\infty(\Omega;\mathbb{M}^{n\times n}_D)).\end{aligned}$$

Suppose that $\partial\Omega \in C^2$ is partitioned into two disjoint open sets Γ_0, Γ_1 and their common interface $\gamma = \partial\Gamma_0 = \partial\Gamma_1$:

$$\partial\Omega = \Gamma_0 \cup \gamma \cup \Gamma_1.$$

Further, assume that

$$\text{(2.3)}\qquad \begin{aligned}&\text{for each } x\in\gamma, \text{ there exists a } C^2 \text{ diffeomorphism defined in a}\\ &\text{neighbourhood of } x \text{ which maps } \partial\Omega \text{ to an } (n-1)-\text{dimensional}\\ &\text{hyperplane, and } \gamma \text{ to an } (n-2)\text{-dimensional plane.}\end{aligned}$$

The main result of this paper is the following theorem.

Theorem 2.1. *Suppose that $n = 2,3$, $\partial\Omega \in C^2$, $\mathbb{A}$ has the form* (1.1) *and the assumptions* (2.1)-(2.3) *are satisfied. Then for the solution (u,e,p) of the quasistatic problem, see Definition 3.6, we have*

$$\sigma \in L^\infty([0,T]; W^{1,2}_{loc}(\Omega;\mathbb{M}^{n\times n}_{sym})),$$

with $\sigma(t,x) = \mathbb{A}^{-1}e(t,x)$.

3. Weak formulation of the quasistatic problem

There are several equivalent ways to state the original problem in a weak form. In this section we present a formulation, expressed in terms of energy balance and energy dissipation, presented in [2]. Then we state the existence and regularity results for this quasistatic problem and briefly discuss the method of the proof, which consists in time-discretization procedure. Finally, in the end of the section, we obtain a discrete version of the absolute continuity with respect to time, which holds also at the level of incremental problems.

3.1. **Weak formulation: quasistatic evolution.** The variational formulation of rate-independent processes, for which we refer to [8], expresses the evolution in terms of energy balance and dissipation. In the rest of this section we follow the exposition of [2]. First, we recall two definitions, which are needed to deal with boundary conditions in a relaxed form and to have the duality between the plastic part of the strain and functions from the set Σ, defined above. We note that the latter definition generalizes the well-known stress-strain duality, studied in [7].

Definition 3.1. A triple $(u,e,p) \in BD(\Omega) \times L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) \times M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D)$ is said to be admissible for a given boundary data $w \in W^{1,2}(\Omega;\mathbb{R}^n)$, if

(1) $\varepsilon(u) = e + p$ in Ω,
(2) $p = (w-u)\odot\nu\,\mathcal{H}^{n-1}$ on Γ_0.

The set of all admissible triples for a given w is denoted by $A(w)$.

Remark 3.2. We point out that the first part of this definition is responsible for the additive decomposition, while the second condition reflects the weak form of the boundary conditions, which are typical in the variational theory of functionals with linear growth.

Definition 3.3. For $w \in W^{1,2}(\Omega;\mathbb{M}^{n\times n}_{sym})$, an admissible triple $(u,e,p) \in A(w)$ and $\sigma\in\Sigma$ we define a measure $[\sigma^D : p] \in M_b(\Omega\cup\Gamma_0)$ by

$$\int_{\Omega\cup\Gamma_0} \varphi\, d[\sigma^D : p] = \int_\Omega \varphi\, d[\sigma^D : \varepsilon^D(u)] - \int_\Omega \varphi\sigma^D : e^D\,dx + \int_{\Gamma_0} \varphi(w-u)\cdot[\sigma\nu]^\perp\, d\mathcal{H}^{n-1},$$

for every $\varphi \in C(\Omega \cup \Gamma_0)$. Thus, the following duality is well-defined:

$$\langle \sigma^D : p \rangle_{\Sigma,\Pi} = [\sigma^D : p](\Omega \cup \Gamma_0).$$

Remark 3.4. Here $[\sigma^D : \varepsilon^D(u)]$ is the measure, defined in [7]. As in the case of stress-strain duality, here the difficulty is due to the fact, that σ^D is an L^∞ function, while p is just a bounded Radon measure.

One can show, that for the duality defined in this way, the usual integration by parts formula holds:

Proposition 3.5. *Let $\sigma \in \Sigma$, $f \in L^n(\Omega;\mathbb{R}^n)$, $F \in L^\infty(\Gamma_1;\mathbb{R}^n)$ and let $(u,e,p) \in A(w)$ with $w \in H^1(\Omega;\mathbb{R}^n)$. Assume that $\operatorname{div}\sigma = -f$ a.e. in Ω and $[\sigma\nu] = F$ on Γ_1. Then*

$$\langle \sigma^D, p \rangle_{\Sigma;\Pi} + \int_\Omega \sigma : (e - \varepsilon(w))\, dx = \int_\Omega f \cdot (u-w)\, dx + \int_{\Gamma_1} g \cdot (u-w)\, d\mathcal{H}^{n-1}. \tag{3.1}$$

Now let us define the functionals which appear in the energy formulation of the problem. We start by defining the quadratic form $\mathcal{Q} : L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) \to \mathbb{R}$, corresponding to the stored elastic energy, by

$$\mathcal{Q}(e) = \frac{1}{2}\int_\Omega \mathbb{A}^{-1} e : e\, dx.$$

Denoting by $H : \mathbb{M}^{n\times n}_D \to \mathbb{R}$ the support function to the sections of $\mathbb{K}$, which in the case of Prandtl-Reuss perfect plasticity has a very simple form, we introduce in the usual way the convex functional of measures $\mathcal{H} : M_b(\Omega \cup \Gamma_0;\mathbb{M}^{n\times n}_D) \to \mathbb{R}$. Then the dissipation associated with $\mathcal{H}$ in any time interval $[s,t] \subset [0,T]$ is given by

$$\mathcal{D}_\mathcal{H}(p;s,t) = \sup\left\{ \sum_{j=1}^{M} \mathcal{H}(p(t_j) - p(t_{j-1})) \; : \; s = t_0 \le \dots \le t_M,\ M \in \mathbb{N} \right\}.$$

Finally, we define the total load $M : [0,T] \to BD(\Omega)'$ by

$$M[t]u = \int_\Omega f(t)\cdot u\, dx + \int_{\Gamma_1} F(t) \cdot u\, d\mathcal{H}^{n-1}. \tag{3.2}$$

Now we are in a position to give a variational formulation of the quasistatic problem.

Definition 3.6. A quasistatic evolution is a function

$$(u,e,p) : [0,T] \to BD(\Omega) \times L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) \times M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D),$$

which satisfies the following conditions

(qs1) (global stability): For every $t \in [0,T]$ the triple $(u,e,p)(t) \in A(w(t))$ and

$$\mathcal{Q}(e(t)) - M[t]u(t) \le \mathcal{Q}(\eta) + \mathcal{H}(q - p(t)) - M[t]v$$

for every $(v,\eta,q) \in A(w(t))$,

(qs2) (energy balance): $p : [0,T] \to M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D)$ has bounded variation and for every $t \in [0,T]$

$$\mathcal{Q}(e(t)) + \mathcal{D}_\mathcal{H}(p;0,t) - M[t]u(t) =$$
$$= \mathcal{Q}(e(0)) - M[0]u(0) + \int_0^t \Big[\langle \sigma(s), \varepsilon(\dot w(s))\rangle_{L^2;L^2} - M[s]\dot w(s) - \dot M[s]u(s) \Big]\, ds$$

where $\sigma(t) = \mathbb{A}^{-1} e(t)$.

3.2. **Existence result and time-discretization.** The following theorem establishes the existence of a solution to the quasistatic problem in perfect plasticity.

Theorem 3.7. *Let* $(u_0, e_0, p_0) \in A(w(0))$ *satisfy the stability condition*

$$Q(e_0) - M[0]u_0 \leq Q(\eta) + \mathcal{H}(q - p_0) - M[0]v,$$

for every $(v, \eta, q) \in A(w(0))$. *Then there exists a quasistatic evolution*

$$(u(t), e(t), p(t)),$$

such that

$$u(0) = u_0,\ e(0) = e_0,\ p(0) = p_0.$$

Moreover, the elastic part of the symmetrized gradient $t \mapsto e(t)$ *is unique and a quasistatic evolution* (u, e, p) *as a function from* $[0, T]$ *to* $BD(\Omega) \times L^2(\Omega; \mathbb{M}^{n\times n}_{sym}) \times M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$ *is absolutely continuous in time.*

In [2] this theorem is proved by a discretization of time. We divide the interval $[0, T]$ into N equal parts of length T/N by points $(t^m_N)_{m=0,\dots,N}$. For $m = 0, \dots, N$ we set

$$w^m_N = w(t^m_N),\ f^m_N = f(t^m_N),\ F^m_N = F(t^m_N),\ M^m_N = M[t^m_N], \quad \text{and } \varrho^m_N = \varrho(t^m_N). \tag{3.3}$$

For every N we define u^m_N, e^m_N and p^m_N by induction. We set

$$(u^0_N, e^0_N, p^0_N) = (u_0, e_0, p_0) \in A(w(0)),$$

while for every $m = 1, \dots, N$ we define (u^m_N, e^m_N, p^m_N) as a solution to the incremental problem

$$\min_{(u,e,p)\in A(w^m_N)} \Big\{ \mathcal{Q}(e) + \mathcal{H}(p - p^{m-1}_N) - M^m_N(u) \Big\}. \tag{3.4}$$

Remark 3.8. We note, that (u, e, p) is a solution to (3.4) if and only if one of the following conditions holds:

(1) $-\mathcal{H}(q) \leq \langle \sigma | \eta \rangle_{L^2;L^2} - \langle f^m_N | v \rangle_{L^n;L^{n'}} \leq \mathcal{H}(-q)$ for every $(v, \eta, q) \in A(0)$.
(2) $\sigma \in \Sigma \cap \mathcal{K}$ with $\operatorname{div} \sigma = -f^m_N$ and $[\sigma\nu] = F^m_N$.

For $m = 0, \dots, N$ we set $\sigma^m_N = \mathbb{A}^{-1} e^m_N$ and for every $t \in [0, T]$ we define piecewise constant interpolations

$$u_N(t) = u^m_N, \quad e_N(t) = e^m_N, \quad p_N(t) = p^m_N, \quad \sigma_N(t) = \sigma^m_N,$$

$$w_N(t) = w^m_N, \quad f_N(t) = f^m_N, \quad F_N(t) = F^m_N, \quad M_N(t) = M^m_N, \quad \varrho_N(t) = \varrho^m_N,$$

where m is the largest integer such that $t^m_N \leq t$. By definition $(u_N(t), e_N(t), p_N(t)) \in A(w_N(t))$.

In the proof of the existence, it was shown that for approximate solutions one has the estimate

$$\sup_{t\in[0,T]} \|e_N(t)\|_{L^2} + \operatorname{Var}(p_N; 0, T) + \sup_{t\in[0,T]} \|u_N\|_{BD} \leq C, \tag{3.5}$$

which is uniform with respect to N, and it was established, that these functions converge pointwise (with respect to t) to a solution of the quasistatic evolution problem.

3.3. **Continuity estimates of solutions of the incremental problems.** In [2] it was established that the quasistatic evolution is absolutely continuous in time. However, as we will deal precisely with the solutions of the time-discretized problems, we would need the continuity estimates of solutions at the level of incremental problems.

Below the following notations will be often used: given a function $h : [0,T] \to X$,

$$\delta h_N^m := h(t_N^m) - h(t_N^{m-1}). \tag{3.6}$$

We also consider the increment of the data of the problem, defined by

$$D_N^m := \|\delta \varrho_N^m\|_{L^2} + \|\delta \varrho_N^{mD}\|_{L^\infty} + \|\delta w_N^m\|_{W^{1,2}} + \|\delta f_N^m\|_{L^n} + \|\delta F_N^m\|_{L^\infty}. \tag{3.7}$$

We note, that by (2.1), we may assume the data of the problem to be Lipschitz with respect to time. Indeed, the absolutely continuous functions can be made Lipschitz just by time reparametrization.

Theorem 3.9. *For solutions of the incremental problems* (u_N^m, e_N^m, p_N^m) *the following inequality holds:*

$$\|\delta e_N^m\|_{L^2} + \|\delta p_N^m\|_{M_b} + \|\varepsilon(\delta u_N^m)\|_{M_b} + \|\delta u_N^m\|_{L^1} \le C\, D_N^m, \tag{3.8}$$

where δh_N^m *in understood as in* (3.6) *and* D_N^m *denotes the increment of the data of the problem, defined by* (3.7)*.*

Proof: As the triple

$$(u_N^{m-1} + w_N^m - w_N^{m-1}, e_N^{m-1} + \varepsilon(w_N^m) - \varepsilon(w_N^{m-1}), p_N^{m-1}) \in A(w_N^m),$$

the minimality condition (3.4) and the integration by parts formula (3.1) imply

$$\begin{aligned} &\mathcal{Q}(e_N^m) - \int_\Omega \varrho_N^m : e_N^m \, dx + \mathcal{H}(p_N^m - p_N^{m-1}) - \langle \varrho_N^m, p_N^m - p_N^{m-1}\rangle_{\Sigma;\Pi} \le \\ &\le \mathcal{Q}(e_N^{m-1} + \varepsilon(w_N^m) - \varepsilon(w_N^{m-1})) - \int_\Omega \varrho_N^m : (e_N^{m-1} + \varepsilon(w_N^m) - \varepsilon(w_N^{m-1}))\, dx \end{aligned}$$

Developing the quadratic form in the right-hand side we arrive at:

$$\begin{aligned} &\frac{1}{2}\int_\Omega \sigma_N^m : e_N^m \, dx - \frac{1}{2}\int_\Omega \sigma_N^{m-1} : e_N^{m-1}\, dx + \mathcal{H}(p_N^m - p_N^{m-1}) \le \\ &\le \mathcal{Q}(\varepsilon(w_N^m) - \varepsilon(w_N^{m-1})) + \int_\Omega \sigma_N^{m-1} : (\varepsilon(w_N^m) - \varepsilon(w_N^{m-1}))\, dx + \\ &+ \langle \varrho_N^m, p_N^m - p_N^{m-1}\rangle_{\Sigma;\Pi} - \int_\Omega \varrho_N^m : (e_N^{m-1} + \varepsilon(w_N^m) - \varepsilon(w_N^{m-1}))\, dx + \int_\Omega \varrho_N^m : e_N^m \, dx. \end{aligned} \tag{3.9}$$

Now consider the functions

$$v = u_N^m - u_N^{m-1} - (w_N^m - w_N^{m-1}),\ \eta = e_N^m - e_N^{m-1} - (\varepsilon(w_N^m) - \varepsilon(w_N^{m-1})),\quad q = p_N^m - p_N^{m-1}.$$

Since $(v, \eta, q) \in A(0)$ and $(u_N^{m-1}, e_N^{m-1}, p_N^{m-1})$ is a solution of the corresponding minimum problem at the previous step, we obtain, by means of Remark 3.8 and the integration by parts formula (3.1)

$$\begin{aligned} &-\int_\Omega \sigma_N^{m-1} : (e_N^m - e_N^{m-1})\, dx + \int_\Omega \varrho_N^{m-1} : (e_N^m - e_N^{m-1})\, dx + \\ &\langle \varrho_N^{m-1\,D}, p_N^m - p_N^{m-1}\rangle_{\Sigma;\Pi} + \int_\Omega (\sigma_N^{m-1} - \varrho_N^{m-1}) : (\varepsilon(w_N^m) - \varepsilon(w_N^{m-1})) \le \\ &\le \mathcal{H}(p_N^m - p_N^{m-1}). \end{aligned} \tag{3.10}$$

By combining (3.9) and (3.10) we get the following

$$
\begin{gathered}
\mathcal{Q}(e_N^m - e_N^{m-1}) = \frac{1}{2}\int_\Omega \sigma_N^m : e_N^m \, dx - \frac{1}{2}\int_\Omega \sigma_N^{m-1} : e_N^{m-1} \, dx - \\
- \int_\Omega \sigma_N^{m-1} : (e_N^m - e_N^{m-1}) \, dx \leq \mathcal{Q}(\varepsilon(w_N^m) - \varepsilon(w_N^{m-1})) + \int_\Omega \sigma_N^{m-1} : (\varepsilon(w_N^m) - \\
\varepsilon(w_N^{m-1})) \, dx + \langle \varrho_N^m, p_N^m - p_N^{m-1} \rangle_{\Sigma;\Pi} - \\
- \int_\Omega \varrho_N^m : (e_N^{m-1} + \varepsilon(w_N^m) - \varepsilon(w_N^{m-1})) \, dx + \int_\Omega \varrho_N^m : e_N^m \, dx - \\
- \int_\Omega \varrho_N^{m-1} : (e_N^m - e_N^{m-1}) \, dx - \\
- \langle \varrho_M^{m-1\,D}, p_N^m - p_N^{m-1} \rangle_{\Sigma;\Pi} - \int_\Omega (\sigma_N^{m-1} - \varrho_N^{m-1}) : (\varepsilon(w_N^m) - \varepsilon(w_N^{m-1})).
\end{gathered} \tag{3.11}
$$

Let us apply the integration by parts formula (3.1) to compute $\langle \varrho_N^m, p_N^m - p_N^{m-1} \rangle_{\Sigma;\Pi}$:

$$
\begin{gathered}
\langle \varrho_N^m, p_N^m - p_N^{m-1} \rangle_{\Sigma;\Pi} = - \int_\Omega \varrho_N^m : (e_N^m - \varepsilon(w_N^m) - e_N^{m-1} + \varepsilon(w_N^{m-1})) \, dx + \\
+ \int_\Omega f_N^m \cdot (u_N^m - w_N^m - u_N^{m-1} + w_N^{m-1}) \, dx + \\
+ \int_{\Gamma_1} F_N^m \cdot (u_N^m - w_N^m - u_N^{m-1} + w_N^{m-1}) \, d\mathcal{H}^{n-1},
\end{gathered} \tag{3.12}
$$

with the analogous expression for $\langle \varrho_N^{m-1}, p_N^m - p_N^{m-1} \rangle_{\Sigma;\Pi}$.

Putting the identity (3.12) into the inequality (3.11) we end up with the estimate

$$
\begin{gathered}
\mathcal{Q}(e_N^m - e_N^{m-1}) \leq \mathcal{Q}(\varepsilon(w_N^m) - \varepsilon(w_N^{m-1})) + \\
+ \int_\Omega (f_N^m - f_N^{m-1}) \cdot (u_N^m - u_N^{m-1} - (w_N^m - w_N^{m-1})) \, dx + \\
+ \int_{\Gamma_1} (F_N^m - F_N^{m-1}) \cdot (u_N^m - u_N^{m-1} - (w_N^m - w_N^{m-1})) \, d\mathcal{H}^{n-1} \leq \\
\leq C \, \|\varepsilon(w_N^m) - \varepsilon(w_N^{m-1})\|_{L^2}^2 + \\
+ \Big(\|f_N^m - f_N^{m-1}\|_{L^n} + \|F_N^m - F_N^{m-1}\|_{L^\infty} \Big) \|u_N^m - w_N^m - (u_N^{m-1} - w_N^{m-1})\|_{BD}.
\end{gathered} \tag{3.13}
$$

Now let us estimate $\|p_N^m - p_N^{m-1}\|_1$ in terms of the data of the problem. First of all, the safe load condition yields

$$
\alpha \|p_N^m - p_N^{m-1}\|_1 \leq \mathcal{H}(p_N^m - p_N^{m-1}) - \langle \varrho_N^{m\,D}, p_N^m - p_N^{m-1} \rangle.
$$

Now, the relation (3.9) and the boundedness of $\|\varrho_N^m\|_{L^2}$, $\|\varrho_N^{m\,D}\|_{L^\infty}$, $\|e_N^m\|_{L^2}$ and $\|p_N^m\|_1$ imply

$$
\|p_N^m - p_N^{m-1}\|_1 \leq C(\|e_N^m - e_N^{m-1}\|_{L^2} + D_N^m) \tag{3.14}
$$

Taking into account the inequality

$$
\|u_N^m - w_N^m - (u_N^{m-1} - w_N^{m-1})\|_{BD} \leq C(\|e_N^m - e_N^{m-1}\|_{L^2} + \|p_N^m - p_N^{m-1}\|_1 + \|\varepsilon(w_N^m) - \varepsilon(w_N^{m-1})\|_{L^2}),
$$

proved in [2, relations (3.24) and (3.25) in Theorem 3.8], the estimate

$$
\|p_N^m - p_N^{m-1}\|_{M_b} + \|e_N^m - e_N^{m-1}\|_{L^2} \leq C D_N^m \tag{3.15}
$$

follows now from (3.13), (3.14) and the application of the Cauchy inequality.

To prove

$$
\|\varepsilon(u_N^m) - \varepsilon(u_N^{m-1})\|_{M_b} \leq C D_N^m, \tag{3.16}
$$

we recall the additive decomposition $\varepsilon(u) = e + p$ and make use of (3.15).

Finally to show the validity of (3.8), it remains to estimate $\|u_N^m - u_N^{m-1}\|_{L^1}$. By the Poincare inequality for BD it suffices to estimate $\|u_N^m - u_N^{m-1}\|_{L^1(\Gamma_0)}$:

$$
\|u_N^m - u_N^{m-1}\|_{L^1(\Gamma_0)} \leq \sqrt{2} \|p_N^m - p_N^{m-1}\|_1 + C \|w_N^m - w_N^{m-1}\|_{W^{1,2}},
$$

so the result follows from (3.15), (3.16) and the latter inequality. □

4. Relaxation of convex variational problems in non-reflexive spaces

For the reader's convenience, here we state the general construction of the relaxed convex variational problems in non-reflexive spaces, which is well-suited for studying the problems in plasticity theory. For the details, we refer to [4, Chapter 1]. We remark that, by abuse of notations, in this section the symbol u_0 stands for the boundary data of a saddle-point problem, which corresponds to w_N^m, the boundary data of the incremental problems, and has nothing to do with the initial data u_0 of the quasistatic problem.

Let V, U and P be Banach spaces, $V \subset U$, and let V_0 be a subspace of V. Let $A : V \to P$ denote a linear bounded operator, and suppose that $G : P \to \overline{\mathbb{R}}$ and $\widehat{M} : U \to \overline{\mathbb{R}}$ are convex, proper, lower semicontinuous functionals. We denote by P^* and U^* the dual spaces to P and U, and by $\langle \cdot, \cdot \rangle_{P,P^*}$ and $\langle \cdot, \cdot \rangle_{U,U^*}$ the duality relations between the corresponding spaces.

By G^* we denote the conjugate functional to G, i.e. $G^*(p^*) = \sup\{\langle p^*, p\rangle_{P,P^*} - G(p) \ : \ p \in P\}$, for $p^* \in P^*$. Let us consider the variational problem

$$\text{find } u \in u_0 + V_0 \text{ such that } I(u) = \inf\{I(v) \ : \ v \in u_0 + V_0\}, \tag{4.1}$$

where $u_0 \in V$ is fixed, and

$$I(v) = G(Av) + \widehat{M}(v).$$

Let us intoduce the Largangian ℓ by letting

$$\ell(v, q^*) = \langle q^*, Av\rangle_{P^*,P} - G^*(q^*) + \widehat{M}(v). \tag{4.2}$$

The dual problem thus takes the form

$$\text{find } p^* \in P^* \text{ such that } R(p^*) = \sup\{R(q^*) \ : \ q^* \in P^*\}, \tag{4.3}$$

where $R(q^*) = \inf\{\ell(v, q^*) \ : \ v \in u_0 + V_0\}$. The following theorem (see [4, Chapter 1]) states that the problem (4.3) has a solution.

Theorem 4.1. *Suppose that the following two conditions hold*

$$\widehat{C} := \inf\{I(v) \ : \ v \in u_0 + V_0\} \in \mathbb{R}; \tag{4.4}$$

$$\begin{cases} \textit{there exists } u_1 \in u_0 + V_0 \textit{ such that } G(Au_1) < +\infty, \ \widehat{M}(u_1) < +\infty \\ \textit{and the function } p \mapsto G(Au_1 + p) \textit{ is continuous at zero.} \end{cases} \tag{4.5}$$

Then the problem (4.3) has at least one solution and the identity

$$\widehat{C} = \sup\{R(q^*) \ : \ q^* \in P^*\} \tag{4.6}$$

is valid.

Together with problems (4.1) and (4.3) let us consider the following minimax problem

$$\begin{cases} \text{find a pair } (u, p^*) \in (u_0 + V_0) \times P^* \text{ such that} \\ \ell(u, q^*) \le \ell(u, p^*) \le \ell(v, p^*), \text{ for all } v \in u_0 + V_0, \ q^* \in P^*. \end{cases} \tag{4.7}$$

Since $G : P \to \mathbb{R}$ is a proper, convex, l.s.c. functional, then $G = G^{**}$, and therefore

$$I(v) = \sup\{\ell(v, q^*) \ : \ q^* \in P^*\}. \tag{4.8}$$

Thus under conditions (4.4) and (4.5) we have the identity

$$\inf_{v \in u_0 + V_0} \sup_{q^* \in P^*} \ell(v, q^*) = \widehat{C} = \sup_{q^* \in P^*} \inf_{v \in u_0 + V_0} \ell(v, q^*) \tag{4.9}$$

and the general duality theory of provides the following statement:

$$\begin{cases} \text{a pair } (u,p^*) \in (u_0+V_0)\times P^* \\ \text{is a saddle point of the minimax problem (4.7) if and only if} \\ u\in u_0+V_0 \text{ is a minimizer of problem (4.1) and} \\ p^*\in P^* \text{ is a maximizer of problem (4.3).} \end{cases} \tag{4.10}$$

So by Theorem 4.1 and (4.10), the solvability of problem (4.1) is equivalent to the solvability of the minimax problem (4.7).

Let us assume the following additional properties:

$$\begin{cases} \text{the embedding } V\hookrightarrow U \text{ is continuous;} \\ V_0 \text{ is dense in } U; \\ U \ \text{ is a reflexive space;} \end{cases} \tag{4.11}$$

$$\begin{cases} \text{there exists } u_2\in u_0+V_0, \text{ such that } u_2\in \operatorname{int}\operatorname{dom}\widehat{M}, \\ \operatorname{dom}\widehat{M} = \{u\in U \ : \ \widehat{M}(u)<+\infty\}. \end{cases} \tag{4.12}$$

$$I(v)\to+\infty \text{ if } \|v\|_V\to+\infty \text{ and } v\in u_0+V_0. \tag{4.13}$$

If the space V is nonreflexive, in general, problems (4.1) and (4.7) have no solutions. Thus, we need to relax our problem, and the desired relaxation should satisfy the following two requirements:

(1) conservation of the greatest lower bound for problem (4.1),
(2) conservation of the dual problem.

Remark 4.2. The first requirement needs no explanations: speaking about relaxation, we should not change the infimum of the problem. While the second point is due to the fact, that in many physical applications the solution of the dual problem is unique and has a clear geometrical or mechanical interpretation, so there is no necessity to change the dual problem. In the case of perfect plasticity the stress tensor is responsible for the distribution of elastic and plastic zones.

In order to extend the domain of definition of the functional G, we should construct a suitable extension of the operator A. We begin by introducing an auxiliary operator A^* with a domain $D(A^*)$ defined as

$$\begin{cases} D(A^*)=\{p^*\in P^* \ : \text{ there exists } u^*\in U^*, \text{ such that} \\ \langle p^*,Au\rangle_{P^*;P}=\langle u^*,u\rangle_{U^*;U} \text{ for all } u\in V_0\}. \end{cases} \tag{4.14}$$

The density condition (4.11) implies that for each $p^*\in D(A^*)$ there exists only one element $u^*\in U^*$ satisfying the identity $\langle p^*,Au\rangle_{P^*;P}=\langle u^*,u\rangle_{U^*;U}$ on V_0. Thus we can define the linear operator $A^*:D(A^*)\to U^*$ through the relation

$$\langle p^*,Au\rangle_{P^*;P}=\langle A^*p^*,u\rangle_{U^*;U} \quad \text{for every } p^*\in D(A^*),\ u\in V_0.$$

If u_0 is a fixed element from V, then we have the identity

$$\langle p^*,Au\rangle_{P^*;P}=\mathcal{E}(u_0,p^*)+\langle A^*p^*,u\rangle_{U^*;U}, \quad \text{for all } u\in u_0+V_0,\ p^*\in D(A^*), \tag{4.15}$$

where $u-u_0\in V_0$ and

$$\mathcal{E}(u_0,p^*)=\langle p^*,Au_0\rangle_{P^*;P}-\langle A^*p^*,u_0\rangle_{U^*;U}.$$

We enlarge the set u_0+V_0 by letting

$$V_+=\Big\{u\in U \ : \sup_{p^*\in D(A^*),\ \|p^*\|_{P^*}\le 1} |\mathcal{E}(u_0,p^*)+\langle A^*p^*,u\rangle_{U^*;U}|<+\infty\Big\}, \tag{4.16}$$

and introduce a relaxation Φ of the functional I by means of the Lagrangian L:

$$\begin{cases} L(v, q^*) = \mathcal{E}(u_0, q^*) + \langle A^* q^*, v\rangle_{U^*;U} - G^*(q^*) + \widehat{M}(v) \\ q^* \in D(A^*),\ v \in V_+; \\ \Phi(v) = \sup\limits_{q^* \in D(A^*)} L(v, q^*), \quad \Phi : V_+ \to \mathbb{R}. \end{cases} \tag{4.17}$$

Let us collect some consequences of these definitions.

Lemma 4.3. *The following relations hold:*

$$u_0 + V_0 \subset V_+, \tag{4.18}$$

$$\Phi(v) \le I(v), \quad \textit{for all } v \in u_0 + V_0. \tag{4.19}$$

Moreover, under certain hypotheses the equality holds in (4.19).

Lemma 4.4. *Suppose that for any $p \in \operatorname{dom} G^*$ there exists a sequence $p^*_m \in D(A^*)$ such that*

$$\begin{cases} p^*_m \overset{*}{\rightharpoonup} p^* \quad \textit{in } P^*, \\ G^*(p^*_m) \to G^*(p^*). \end{cases} \tag{4.20}$$

Then the identity

$$\Phi(v) = I(v) \quad \textit{for all } v \in u_0 + V_0 \tag{4.21}$$

is valid.

The following Lemma clarifies the meaning of the relaxation considered:

Lemma 4.5. *Consider a sequence $u_m \in u_0 + V_0$, bounded in the norm of the space V and converging to u weakly in U. Then*

$$u \in V_+,$$

$$\liminf_{m \to +\infty} I(u_m) \ge \Phi(u).$$

Now we consider the minimax problem

$$\begin{cases} \text{find a pair } (u^*, p) \in V_+ \times D(A^*) \text{ such that} \\ L(u, q^*) \le L(u, p^*) \le L(v, p^*), \quad \text{for all } v \in V_+,\ q^* \in D(A^*). \end{cases} \tag{4.22}$$

This minimax problem generates two variational problems being in duality:

$$\begin{cases} \text{find } u \in V_+ \text{ such that} \\ \Phi(u) = \inf\{\Phi(v) \ :\ v \in V_+\}, \end{cases} \tag{4.23}$$

where $\Phi(v) = \sup\{L(v, q^*) \ :\ q^* \in D(A^*)\}$, and

$$\begin{cases} \text{find } p^* \in D(A^*) \text{ such that} \\ \tilde{R}(p^*) = \sup\{\tilde{R}(q^*) \ :\ q^* \in D(A^*)\}), \end{cases} \tag{4.24}$$

with $\tilde{R}(q^*) = \inf\{L(v, q^*) \ :\ v \in V_+\}$.

Remark 4.6. Lemma 4.5 shows that there is a hope to apply the Direct methods: the coercivity implies the boundedness of a minimizing sequence of the problem (4.1) in U and the potential minimizer of (4.23) will be a weak cluster point of this sequence, which belongs to the set V_+ and such that the $\liminf$ inequality is satisfied.

Indeed, this remark leads us to the following conclusion:

Theorem 4.7. *Suppose that conditions (4.4), (4.5) and (4.11)-(4.13) hold. Then:*

(1) *Problems* (4.23) *and* (4.24) *are solvable. Moreover, if* $u \in V_+$ *is a solution to problem* (4.23) *and* $p^* \in D(A^*)$ *is a solution to problem* (4.24), *then the identity*

$$\Phi(u) = \widehat{C} = \tilde{R}(p^*) \tag{4.25}$$

holds true.

(2) *Problems* (4.3) *and* (4.24) *are equivalent, i.e. they have the same set of solutions.*

(3) *A pair* $(u, p^*) \in V_+ \times D(A^*)$ *is a saddle point of the minimax problem* (4.22) *if and only if* $u \in V_+$ *is a minimizer of problem* (4.23) *and* $p^* \in D(A^*)$ *is a maximizer of problem* (4.24).

(4) *Any minimizing sequence of problem* (4.1) *contains a subsequence converging to some solution of problem* (4.23) *weakly in* U.

5. Minimax formulation of the incremental problem

Recall that, during the proof of existence of a weak solution to the quasistatic evolution problem of perfect plasticity, the time-discretization procedure leads one to solving the following incremental problem at every step (see (3.4)):

$$\min_{(u,e,p)\in A(w_N^m)} \{\mathcal{Q}(e) + \mathcal{H}(p - p_N^{m-1}) - M_N^m(u)\}, \tag{5.1}$$

with p_N^{m-1} be a solution of the corresponding incremental problem, obtained at the previous step.

In the rest of this section, to simplify the notations, we will omit writing the indices m and N when dealing with some functionals and spaces. So, in what follows the functionals G, $\widehat{M}$, M, ℓ, L, I, R, Φ and the space V_+ should be understood as G_N^m, $\widehat{M}_N^m$, M_N^m, ℓ_N^m, L_N^m, I_N^m, R_N^m, Φ_N^m and $(V_+)_N^m$, written, however, without an explicit dependence on t_N^m.

We state the minimax formulation of the incremental problem and briefly sketch the ideas, leading to the notion of a weak solution. Note, that it is a generalization of the functional formulation of the classical boundary value problem, describing the equilibrium of a perfect elastoplastic body (see [4, 10, 11, 12, 13, 14]).

First (subsection 5.1) we introduce the functional spaces and define the functionals of the minimax problem. Then (subsection 5.2) we define the Lagrangian and state the primal and dual problems. In subsection 5.3 we check the conditions (4.4), (4.5) and (4.11)-(4.13), that allow us to apply the abstract theory from Section 4. The relaxed problem and the properties of its solutions are presented in the same subsection. In subsection 5.4 we show, that every saddle point of the relaxed minimax problem generates a solution to the incremental problem (5.1).

5.1. **Functional formulation.**

In order to handle this problem using the abstract relaxation scheme described in Section 4 we set

$$V = D^{2,1}(\Omega),\ V_0 = D_0^{2,1}(\Omega),\ U = L^{n/(n-1)}(\Omega),$$

$$\begin{cases} P = \{p = \{\tau, a\} : \|p\|_P^2 = \|\tau^D\|_{L^1(\Omega)}^2 + \frac{1}{n}\|\mathrm{tr}\,\tau\|_{L^2(\Omega)}^2 + \\ \|a\|_{L^1(\Gamma_1)} < +\infty\} \subset L^1(\Omega; \mathbb{M}_{sym}^{n\times n}) \times L^1(\Gamma_1; \mathbb{R}^n). \end{cases} \tag{5.2}$$

Then

$$\begin{cases} P^* = \{p^* = \{\sigma, b\} : \sigma^D \in L^\infty(\Omega; \mathbb{M}_{sym}^{n\times n}),\ \mathrm{tr}\,\sigma \in L^2(\Omega), \\ b \in L^\infty(\Gamma_1; \mathbb{R}^n)\} \end{cases} \tag{5.3}$$

Next, let us introduce the functionals $G : P \to \overline{\mathbb{R}}$ and $\widehat{M} : U \to \overline{\mathbb{R}}$

$$
\begin{aligned}
G(p) &= \int_\Omega g(\tau + e_N^{m-1}) + \int_{\Gamma_1} F_N^m \cdot a \, d\mathcal{H}^{n-1}, \quad p = \{\tau, a\} \in P, \\
\widehat{M}(v) &= -\int_\Omega f_N^m \cdot v \, dx, \quad v \in U.
\end{aligned} \tag{5.4}
$$

Then it is easy to see, that for $p^* = \{\sigma, b\} \in P^*$ its Legendre transform G^* takes the form

$$
G^*(p^*) = \begin{cases} \int_\Omega \Big(g^*(\sigma) - \sigma : e_N^{m-1} \Big) \, dx, & \text{if } b \equiv F_N^m \\ +\infty, & \text{otherwise.} \end{cases} \tag{5.5}
$$

Here

$$
g^*(\sigma) = \frac{1}{2n^2 K_0} \mathrm{tr}^{\,2} \tau + g_0^*(|\sigma^D|) = \sup \Big\{ \sigma : \varkappa - g(\varkappa) \ : \ \varkappa \in \mathbb{M}^{n\times n}_{sym} \Big\}
$$

is the Legendre transform of

$$
g : \mathbb{M}^{n\times n}_{sym} \to \overline{\mathbb{R}}, \quad g(\varkappa) = \frac{1}{2} K_0 \mathrm{tr}^{\,2} \varkappa + g_0(|\varkappa^D|), \quad \varkappa \in \mathbb{M}^{n\times n}_{sym}.
$$

Here $g_0 : \mathbb{R} \to \overline{\mathbb{R}}$, and g_0^* is its Legendre transform. In the case of Hencky and Prandtl-Reuss models of plasticity this function has the form:

$$
g_0(t) = \begin{cases} \mu t^2, & |t| \le t_0 = \frac{k_*}{\sqrt{2}\mu}; \\ k_*(\sqrt{2}|t| - \frac{k_*}{2\mu}), & |t| > t_0. \end{cases}
$$

5.2. **Lagrangian and a saddle-point problem.** The linear operator $A : V \to P$ is introduced as follows:

$$
Av = \{\varepsilon(v), \ -v_{|\Gamma_1}\}, \quad v \in V,
$$

and in view of the estimate

$$
\|Av\|_P = \left(\frac{1}{n} \|\mathrm{div}\, v\|^2_{L^2(\Omega)} + \|\varepsilon^D(v)\|^2_{L^1(\Omega)} + \|v\|^2_{L^1(\Gamma_1)} \right)^{1/2} \le c(\Omega, n) \|v\|_{2,1},
$$

one concludes that A is continuous.

Following the ideas, outlined in Section 4 (see (4.7)), the minimax problem is

$$
\begin{cases} \text{find a pair } (u, \sigma) \in (\delta w_N^m + V_0) \times K, \text{ such that} \\ \ell(u, \tau) \le \ell(u, \sigma) \le \ell(v, \sigma), \quad \text{for all } v \in w_N^m + V_0, \ \tau \in \mathcal{K}, \end{cases} \tag{5.6}
$$

where the Lagrangian, according to (4.2), is given by

$$
\ell(v, \tau) = \int_\Omega \Big(\varepsilon(v) : \tau + \tau : e_N^{m-1} \Big) \, dx - \int_\Omega g^*(\tau) \, dx + \widehat{M}(v),
$$

and δw_N^m is defined, according to (3.6). The functional I takes the form

$$
I(v) = G(Av) + \widehat{M}(v) = \int_\Omega g(\varepsilon(v) + e_N^{m-1}) - \int_{\Gamma_1} F_N^m \cdot v \, d\mathcal{H}^{n-1} - \int_\Omega f_N^m \cdot v \, dx.
$$

Recall that the functions f_N^m, F_N^m and δw_N^m satisfy the following conditions:

$$
f_N^m \in L^n(\Omega; \mathbb{R}^n), \quad F_N^m \in L^\infty(\Gamma_1; \mathbb{R}^n), \quad \delta w_N^m \in W^{1,2}(\Omega; \mathbb{R}^n). \tag{5.7}
$$

The minimax problem (5.6) generates two dual variational problems:

$$
\begin{cases} \text{find } u \in \delta w_N^m + V_0 \text{ such that} \\ I(u) = \inf\{ I(v) \ : \ v \in w_N^m + V_0 \}, \end{cases} \tag{5.8}
$$

and

$$
\begin{cases} \text{find } \sigma_N^m \in Q_{f_N^m} \cap \mathcal{K} \text{ such that} \\ R(\sigma) = \sup\{ R(\tau) \ : \ \tau \in Q_{f_N^m} \cap \mathcal{K}, \} \end{cases} \tag{5.9}
$$

where

$$R(\sigma)=\begin{cases}\ell(\delta w_N^m,\tau), & \tau\in Q_{f_N^m}\cap\mathcal{K}\\ -\infty, & \tau\notin Q_{f_N^m}\cap\mathcal{K}\end{cases}\quad \text{for } \tau\in\mathcal{K},$$

with $Q_{f_N^m}$ being defined as

$$Q_{f_N^m}=\Big\{\tau\in\Sigma\ :\ \int_\Omega \tau:\varepsilon(v)\,dx=M_N^m(v),\quad \text{for all } v\in V_0\Big\},$$

where we refer to (3.2) and (3.3) for the definition of M_N^m. We note that

$$\tau\in Q_{f_N^m}\Leftrightarrow \operatorname{div}\tau=-f_N^m \text{ in } \Omega,\quad [\tau\nu]=F_N^m \text{ on } \Gamma_1.$$

5.3. **The relaxed problem.** Let us check the conditions (4.4), (4.5) and (4.11)-(4.13). Since the functional G is convex and finite, that is $\operatorname{dom} G=P$, the function $p\mapsto G(Au_1+p)$ is continuous at zero for any $u_1\in\delta w_N^m+V_0$. By the finiteness of the functional M, condition (4.5) is fulfilled. Conditions (4.11) and (4.12) are obviously satisfied.

The conditions (4.4) and (4.13) are guaranteed by the safe-load condition (2.2):

$$\begin{gathered}
I(v)=\frac{K_0}{2}\int_\Omega|\operatorname{div} v+\operatorname{tr} e_N^{m-1}|^2\,dx+\\
+\sup_{\sigma\in\mathcal{K}}\Big\{\int_\Omega\sigma^D:(\varepsilon^D(v)+e_N^{m-1D})-g^*(\sigma^D)\,dx\Big\}-\\
-\int_\Omega\varrho_N^m:(\varepsilon(v)-\varepsilon(\delta w_N^m))\,dx+M(\delta w_N^m)\geq\frac{K_0}{2}\int_\Omega|\operatorname{div} v+\operatorname{tr} e_N^{m-1}|^2\,dx+\\
+\sup_{\sigma\in\mathcal{K}}\Big\{\int_\Omega(\sigma^D-\varrho_N^{mD}):(\varepsilon^D(v)+e_N^{m-1D})-g^*(\sigma^D)\,dx\Big\}-\\
-C\int_\Omega\operatorname{tr}\varrho_N^m\operatorname{div} v\,dx+\int_\Omega\varrho_N^{mD}:e_N^{m-1}\,dx+\\
+\int_\Omega\varrho_N^m:\varepsilon(\delta w_N^m)\,dx+M(\delta w_N^m)\geq C_1\Big[\int_\Omega|\operatorname{div} v|^2\,dx+|\varepsilon^D(v)|_1\Big]-C\to\infty
\end{gathered}\tag{5.10}$$

whenever $\|v\|_V\to\infty$, $v\in\delta w_N^m+V_0$. So the coercivity is established. Finally, the condition (4.4) is provided by the estimate

$$\widehat{C}=\inf\{I(v)\ :\ v\in\delta w_N^m+V_0\}\geq R(\varrho_N^m)>-\infty.$$

Thus, according to Theorem 4.1 we can state that the problem (5.9) has at least one solution $\sigma\in Q_{f_N^m}\cap\mathcal{K}$, that identity (4.9) holds and the statement (4.10) is valid. Due to the non-reflexivity of V the variational problem (5.8) in general has no solutions. We construct relaxations of these variational problems following the scheme described above.

Define the operator $A^*:D(A^*)\to U^*$. As in (4.14), a pair $p^*=\{\sigma,b\}\in D(A^*)$ if and only if there exists $u^*\in U^*=L^n(\Omega;\mathbb{R}^n)$, such that

$$\int_\Omega u^*\cdot v\,dx=\int_\Omega\sigma:\varepsilon(v)\,dx-\int_{\Gamma_1}b\cdot v\ d\mathcal{H}^{n-1}\quad\text{for all } v\in V_0,$$

that is $A^*p^*:=u^*=-\operatorname{div}\sigma\in L^n(\Omega;\mathbb{R}^n)$. Therefore

$$\begin{gathered}D(A^*)=\Big\{p^*=\{\sigma,b\}\in P^*\ :\ \operatorname{div}\sigma\in L^n(\Omega;\mathbb{R}^n),\\
\int_{\Gamma_1}b\cdot v\,d\mathcal{H}^{n-1}=\int_\Omega(\sigma:\varepsilon(v)+v\cdot\operatorname{div}\sigma)\,dx,\quad\text{for all } v\in V_0\Big\}.\end{gathered}$$

According to (4.16) the extension V_+ of the set $\delta w_N^m+V_0$ is

$$\begin{gathered}V_+=\Big\{v\in L^{\frac{n}{n-1}}(\Omega;\mathbb{R}^n):\\
\sup_{\|p^*\|_{P^*}\leq1,\ p^*=\{\sigma,b\}\in D(A^*)}\Big\langle-\int_{\Gamma_1}b\cdot\delta w_N^m\,d\mathcal{H}^{n-1}+\int_\Omega(\sigma:\varepsilon(\delta w_N^m)+(\delta w_N^m-v)\cdot\operatorname{div}\sigma)\,dx\Big\rangle<+\infty\Big\}.\end{gathered}$$

The important properties of this space are summarized below. In particular, the following proposition shows that a triple (u,e,p), constructed from a solution $(\delta u^m,\sigma^m)$ of a relaxed minimax problem in an obvious way (see Theorem 5.4 below), is kinematically admissible for a boundary data δw_N^m.

Proposition 5.1. *The following relations hold:*

$$V_+ \subset BD(\Omega), \tag{5.11}$$

and for every $v\in V_+$

$$\operatorname{div} v \in L^2(\Omega), \tag{5.12}$$

$$(v-\delta w_N^m)\cdot\nu = 0 \quad \text{on } \Gamma_0. \tag{5.13}$$

PROOF: The definition of V_+ implies that

$$\sup_{\sigma\in C_0^\infty(\Omega\cup\Gamma_0)} \left\langle \int_\Omega (\sigma : \varepsilon(\delta w_N^m) + (\delta w_N^m - v)\cdot \operatorname{div}\sigma)\, dx \right\rangle \le C(\|\operatorname{tr}\sigma\|_{L^2(\Omega)} + \|\sigma^D\|_{L^\infty(\Omega)}). \tag{5.14}$$

This estimate and the fact that $\delta w_N^m \in W^{1,2}(\mathbb{R}^n;\mathbb{R}^n)$ ensures the estimate

$$\sup_{\sigma\in C_c^\infty(\Omega;\mathbb{M}^{n\times n}_{sym})} \int_\Omega v\cdot \operatorname{div}\sigma\, dx \le C\|\sigma\|_{L^\infty(\Omega;\mathbb{M}^{n\times n}_{sym})}.$$

So the claim (5.11) is established.

By taking the test vector fields in (5.14) with $\sigma^D = 0$ we conclude, that $\operatorname{div} v\in L^2(\Omega)$, thus (5.12) is proved.

As to the last claim, by taking arbitrary $\varphi\in C_c^\infty(\Omega\cup\Gamma_0)$ and taking $\sigma = \varphi I$ we get by the integration by parts formula the following inequality:

$$\int_{\Gamma_0} \varphi(\delta w_N^m - v)\cdot \nu\, d\mathcal{H}^{n-1} = \int_{\partial\Omega} (\delta w_N^m - v)\cdot [\sigma\nu]\, d\mathcal{H}^{n-1} =$$
$$= \int_\Omega (\delta w_N^m - v)\cdot \operatorname{div}\sigma\, dx + \int_\Omega \operatorname{tr}\sigma \operatorname{div}(\delta w_N^m - v)\, dx \le C\|\varphi\|_{L^2(\Omega)}.$$

This estimate, in its turn, implies that $(\delta w_N^m - v)\cdot\nu = 0$ on Γ_0. □

By the properties of g_0^* and by (5.5) we have that $G^*(p^*) = G^*(\{\tau, b\}) = +\infty$ if $b\ne F$ on Γ_1 or $\tau\notin K$. Introduce the relaxed Lagrangian, as in (4.17):

$$L(v,q^*) = \mathcal{E}(\delta w_N^m, q^*) + \langle A^*q^*, v\rangle - G^*(q^*) + \widehat{M}(v) =$$
$$= -\int_{\Gamma_1} F_N^m\cdot \delta w_N^m\, d\mathcal{H}^{n-1} +$$
$$+ \int_\Omega \Big[\varepsilon(\delta w_N^m) : \tau + (\delta w_N^m - v)\cdot\operatorname{div}\tau - g^*(\tau) - f_N^m\cdot v + \tau : e_N^{m-1}\Big]\, dx$$

for all $v\in V_+$ and $q^* = \{\tau, F_N^m\}\in D(A^*)$, such that $\tau\in\mathcal{K}$. Now we introduce the set

$$Q = \{\tau\in\Sigma \;:\; \{\tau, F_N^m\}\in D(A^*)\} \tag{5.15}$$

and a new Lagrangian on $V_+\times(Q\cap\mathcal{K})$ defined as

$$\widetilde{L}(v,\tau) = L(v,q^*) \tag{5.16}$$

where

$$q^* = \{\tau, F_N^m\}\in D(A^*), \quad \tau\in K.$$

Now, instead of the minimax problem (5.6) we consider its relaxation

$$\begin{cases} \text{find a pair } (u,\sigma)\in V_+\times(Q\cap\mathcal{K}) \text{ such that} \\ \widetilde{L}(u,\tau)\le \widetilde{L}(u,\sigma)\le\widetilde{L}(v,\sigma), \quad \text{for all } v\in V_+,\ \tau\in Q\cap\mathcal{K}. \end{cases} \tag{5.17}$$

For the functional $\Phi : V_+ \to \mathbb{R}$ we have the formula

$$\Phi(v) = \sup_{q^* \in D(A^*)} L(v, q^*) = \sup_{q^*=\{\tau, F_N^m\} \in D(A^*),\ \tau \in \mathcal{K}} L(v, q^*) = \sup_{\tau \in Q \cap \mathcal{K}} \widetilde{L}(v, \tau), \tag{5.18}$$

and the relaxation of the variational problem (5.8) takes the form

$$\text{find } u \in V_+ \text{ such that } \Phi(u) = \inf_{v \in V_+} \Phi(v) \tag{5.19}$$

As in [4, Lemma 1.3.1] one can show, that (4.20) holds, and thus Lemma 4.4 reads as follows.

Lemma 5.2. *We have*

$$\Phi(v) = I(v), \quad \textit{for all } v \in \delta w_N^m + V_0.$$

Finally, we can state Theorem 4.7, which in this case takes the following form.

Theorem 5.3. *Suppose that conditions* (2.2) *and* (5.7) *hold. Then there exists at least one pair* $(\delta u^m, \sigma^m) \in V_+ \times (Q \cap \mathcal{K})$ *being a solution to the minimax problem* (5.17)*. Moreover,* σ^m *is the unique solution to the dual variational problem* (5.9)*,* δu^m *is a solution of the relaxed variational problem* (5.19) *and the identity*

$$\Phi(\delta u^m) = \inf\{I(v) \ : \ v \in \delta w_N^m + V_0\} = \widetilde{L}(\delta u^m, \sigma^m) = R(\sigma^m)$$

holds.

Furthermore,

$$\Phi(v) = I(v) \quad \textit{for all } v \in \delta w_N^m + V_0.$$

Finally, any minimizing sequence of the problem (5.8) *converges strongly in* $L^1(\Omega; \mathbb{R}^n)$ *and weakly in* $L^{\frac{n}{n-1}}(\Omega; \mathbb{R}^n)$ *to some solution of problem* (5.19)*.*

5.4. **Saddle points generate solutions to the incremental problem.** Let us show, that if we interpret a saddle point $(\delta u^m, \sigma^m)$ of (5.17) as the increment of u and the updated value of σ, then we get a solution to the incremental problem (5.1).

Theorem 5.4. *Let* $(\delta u^m, \sigma^m) \in V_+ \times (Q \cap \mathcal{K})$ *be a saddle point for the relaxed Lagrangian* $\widetilde{L}$*. Then the triple* (u^m, e^m, p^m)*, constructed as*

$$\begin{aligned}
&u^m = u^{m-1} + \delta u^m,\\
&e^m = \mathbb{A}\sigma^m,\\
&p^m = \varepsilon(u^m) - e^m \quad \textit{in } \Omega,\\
&p^m = (w^m - u^m) \odot \nu\, \mathcal{H}^{n-1} \quad \textit{on } \Gamma_0,
\end{aligned}$$

is admissible for the boundary data w_N^m*, in the sense of Definition 3.1 and is a solution to the incremental problem (5.1).*

Proof: Let $(\delta u^m, \sigma^m) \in V_+ \times (Q \cap \mathcal{K})$ be a saddle point of $\widetilde{L}$:

$$\widetilde{L}(\delta u^m, \tau) \le \widetilde{L}(\delta u^m, \sigma^m) \le \widetilde{L}(v, \sigma^m) \quad \text{for all } v \in V_+,\ \tau \in Q \cap \mathcal{K}.$$

As $\sigma^m \in Q \cap \mathcal{K}$, we have that $\sigma^m \in \mathcal{K}$ and $[\sigma^m \nu] = F$ on Γ_1 and

$$\int_\Omega (v - \delta u^m) \cdot \operatorname{div} \sigma^m \, dx \le -\int_\Omega f_N^m \cdot (v - \delta u^m) \, dx,$$

which is in fact an equality, valid for all $v \in V_+$. Hence,

$$\operatorname{div} \sigma^m = -f_N^m \in L^n. \tag{5.20}$$

The inequality of a saddle point yields

$$\begin{aligned}
&\int_\Omega \Big[\varepsilon(\delta w_N^m) : \sigma^m + (\delta w_N^m - \delta u^m) \cdot \operatorname{div} \sigma^m - g^*(\sigma^m) + \sigma^m : e^{m-1}\Big] \, dx \ge \\
&\ge \int_\Omega \Big[\varepsilon(\delta w_N^m) : \tau + (\delta w_N^m - \delta u^m) \cdot \operatorname{div} \tau - g^*(\tau) + \tau : e^{m-1}\Big] \, dx
\end{aligned} \tag{5.21}$$

On the other hand, by the integration by parts formula (see [7, Theorem 3.2]) for $\delta u^m \in BD(\Omega)$ and $\sigma^m \in \Sigma$ with $-\operatorname{div}\sigma^m = f_N^m$ and $[\sigma^m \nu] = F_N^m$ on Γ_1:

$$\begin{aligned}\int_\Omega (\delta w_N^m - \delta u^m) \operatorname{div}\sigma^m \, dx = -\langle \varepsilon^D(\delta w_N^m - \delta u^m), \sigma^{mD}\rangle - \\ -\frac{1}{n}\int_\Omega \operatorname{div}(\delta w_N^m - \delta u^m) \operatorname{tr}\sigma^m \, dx + \int_{\partial\Omega} [\sigma^m \nu]\cdot(\delta w_N^m - \delta u^m).\end{aligned} \tag{5.22}$$

We note that, strictly speaking, in the boundary term the integrand is just a distribution, an element of $(C^1(\partial\Omega))'$. However, as $(\delta w_N^m - \delta u^m)\cdot \nu = 0$ on Γ_0 and $[\sigma^m \nu] = F_N^m \in L^\infty(\Gamma_1)$ by [7, Proposition 3.4] one has

$$\begin{aligned}\int_{\partial\Omega} [\sigma^m \nu]\cdot(\delta w_N^m - \delta u^m) = \\ \int_{\Gamma_1} (\delta w_N^m - \delta u^m)\cdot F_N^m \, d\mathcal{H}^{n-1} + \int_{\Gamma_0} (\delta w_N^m - \delta u^m)_\tau \cdot [\sigma^m \nu]_\tau \, d\mathcal{H}^{n-1}.\end{aligned}$$

This relation together with (5.21) and (5.22) implies

$$\begin{aligned}\langle \varepsilon^D(\delta u^m), \tau^D - \sigma^{mD}\rangle - \int_\Omega \frac{1}{2}(\mathbb{A}\tau,\tau)\, dx + \\ + \int_\Omega \frac{1}{2}(\mathbb{A}\sigma^m,\sigma^m)\, dx + \frac{1}{n}\int_\Omega \operatorname{div}\delta u^m \operatorname{tr}(\tau - \sigma^m)\, dx + \\ + \int_\Omega (\tau - \sigma^m) : e^{m-1}\, dx + \int_{\Gamma_0} (\delta w_N^m - \delta u^m)_\tau \cdot [\tau - \sigma^m]_\tau \, d\mathcal{H}^{n-1} \le 0,\end{aligned}$$

and hence

$$\begin{aligned}\langle \varepsilon^D(\delta u^m), \tau^D - \sigma^{mD}\rangle + \int_\Omega (\tau - \sigma^m) : e^{m-1}\, dx - \int_\Omega \mathbb{A}\sigma^m : (\tau - \sigma^m)\, dx + \\ + \frac{1}{n}\int_\Omega \operatorname{div}\delta u^m \operatorname{tr}(\tau - \sigma^m)\, dx + \\ + \int_{\Gamma_0} (\delta w_N^{m-1} - \delta u^m)_\tau \cdot [\tau - \sigma^m]_\tau \, d\mathcal{H}^{n-1} - \int_\Omega \frac{1}{2}\mathbb{A}(\sigma^m - \tau) : (\sigma^m - \tau)\, dx \le 0.\end{aligned}$$

Now, taking $\widetilde{\tau} = \sigma^m + \alpha(\tau - \sigma^m) \in \mathcal{K}$ and letting $\alpha \to 0$ one gets

$$\begin{aligned}\langle \varepsilon^D(\delta u^m), \tau^D - \sigma^{mD}\rangle + \int_\Omega (\tau - \sigma^m) : e^{m-1}\, dx + \frac{1}{n}\int_\Omega \operatorname{div}\delta u^m \operatorname{tr}(\tau - \sigma^m)\, dx - \\ - \int_\Omega \mathbb{A}\sigma^m : (\tau - \sigma^m)\, dx + \int_{\Gamma_0} (\delta w_N^m - \delta u^m)_\tau \cdot [\tau - \sigma^m]_\tau \, d\mathcal{H}^{n-1} = \\ = \langle \varepsilon^D(\delta u^m), \tau^D - \sigma^{mD}\rangle + \int_\Omega (\tau^D - \sigma^{mD}) : (e^{m-1D} - e^{mD})\, dx + \\ + \frac{1}{n}\int_\Omega (\operatorname{div}\delta u^m - \operatorname{tr}\delta e^m) \operatorname{tr}(\tau - \sigma^m)\, dx + \int_{\Gamma_0} (\delta w_N^m - \delta u^m)_\tau \cdot [\tau - \sigma^m]_\tau \, d\mathcal{H}^{n-1} \le 0.\end{aligned}$$

for all $\tau \in Q \cup \mathcal{K}$. Taking $\tau \in C_c^\infty(\Omega; \mathbb{M}^{n\times n}_{sym})$ with $\tau^D = 0$ we conclude that

$$\operatorname{tr}(\varepsilon(\delta u^m) - \delta e^m) = \operatorname{div}\delta u^m - \operatorname{tr}\delta e^m = 0 \quad \text{a.e. in } \Omega,$$

and the induction hypothesis $\operatorname{tr}(\varepsilon(u^{m-1}) - e^{m-1}) = 0$ a.e. in Ω implies that $\operatorname{tr}(\varepsilon(u^m) - e^m) = 0$ a.e. in Ω, and thus

$$p^m \in M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D).$$

Therefore we have the following inequality

$$\langle p^m - p_N^{m-1}, \tau - \sigma\rangle \le 0$$

for all $\tau \in Q \cap \mathcal{K}$.

The last relation, in its turn, implies that

$$\mathcal{H}(p^m - p_N^{m-1}) = \langle p^m - p_N^{m-1}, \sigma\rangle,$$

which yields the following

$$\begin{gathered}\mathcal{H}(\varepsilon q+p^m-p_N^{m-1})-\mathcal{H}(p^m-p_N)-\langle\varepsilon q,\sigma\rangle\geq\\ \geq\langle\varepsilon q+p^m-p_N^{m-1},\sigma\rangle-\langle p^m-p_N^{m-1},\sigma\rangle-\langle\varepsilon q,\sigma\rangle\geq 0,\end{gathered}$$

for every triple $(v,\eta,q)\in A(0)$.

The latter inequality and (5.20) imply that $(u^m,e^m,p^m)\in A(w_N^m)$ is a solution to problem (5.1). □

6. Approximations

In this section we will show that some solutions of the relaxed minimax problem (5.17) possess an important property of being approximated by more regular functions in a way that allows us to get the higher regularity of stresses for our evolutionary problem.

Now we consider a family of regularized problems and show that their solutions converge to a saddle point of (5.17) in a suitable weak sense.

6.1. **Regularized problems.** We consider a family of variational problems depending on a parameter $\alpha\in(0,1)$

$$\begin{cases}\text{find } u^\alpha\in V_*\\ I_\alpha(u^\alpha)=\inf\{I_\alpha(v):v\in\delta w_N^m+V_*\},\end{cases}\tag{6.1}$$

where

$$\begin{gathered}V_*=V_0\cap W^{1,2}(\Omega;\mathbb{R}^n),\\ I_\alpha(v)=\frac{\alpha}{2}\int_\Omega|\varepsilon^D(v)+e_N^{m-1D}|^2\,dx+I(v)=\\ \frac{\alpha}{2}\int_\Omega|\varepsilon^D(v)+e_N^{m-1D}|^2\,dx+\int_\Omega g(\varepsilon(v)+e_N^{m-1})\,dx-\int_\Omega f_N^m\cdot v\,dx-\int_{\Gamma_1}F_N^m\cdot v\,d\mathcal{H}^{n-1}.\end{gathered}$$

As it is easy to see, for each $\alpha>0$, the coercivity estimate (5.10) and Korn inequality guarantee that the problem (6.1) has the unique minimizer $u^\alpha\in V_*$ which satisfies a nonlinear system of PDE's of elliptic type:

$$\int_\Omega\sigma^\alpha:\varepsilon(v)\,dx=M_N^m(v)\equiv\int_\Omega f_N^m\cdot v\,dx+\int_{\Gamma_1}F_N^m\cdot v\,d\mathcal{H}^{n-1}\quad\text{for all } v\in V_*,\tag{6.2}$$

where

$$\begin{gathered}\sigma^\alpha=\alpha(\varepsilon^D(u^\alpha)+e_N^{m-1D})+\frac{\partial g}{\partial\varkappa}(\varepsilon(u^\alpha)+e_N^{m-1})=\\ =\alpha(\varepsilon^D(u^\alpha)+e_N^{m-1D})+K_0(\operatorname{div}u^\alpha+\operatorname{tr}e_N^{m-1})\mathbf{1}+\\ +g_0'(|\varepsilon^D(u^\alpha)+e_N^{m-1}|)\frac{\varepsilon^D(u^\alpha)+e_N^{m-1}}{|\varepsilon^D(u^\alpha)+e_N^{m-1}|}\,.\end{gathered}\tag{6.3}$$

Therefore,

$$\operatorname{div}\sigma^\alpha+f_N^m=0\quad\text{in }\Omega.\tag{6.4}$$

Remark 6.1. We note, that the functional $I_\alpha(v)$ is of the form $I(v)+\frac{\alpha}{2}\|\varepsilon^D(v)\|_{L^2}^2$, where the second summand is added to make it coercive in $W^{1,2}$.

Lemma 6.2. *For any $\alpha\in(0,1)$ the following estimate is true*

$$\sqrt{\alpha}\|\varepsilon^D(u^\alpha)+e_N^{m-1D}\|_{L^2(\Omega)}+\|\operatorname{div}u^\alpha\|_{L^2(\Omega)}+\|\varepsilon^D(u^\alpha)\|_{L^1(\Omega)}+\|u^\alpha\|_{L^{\frac{n}{n-1}}(\Omega)}\leq C,\tag{6.5}$$

where $C=C(\|f_N^m\|_{L^n(\Omega)},\ \|F_N^m\|_{L^\infty(\Gamma_1)},\ \|\delta w_N^m\|_{W^{1,2}(\Omega;\mathbb{R}^n)},\|e_N^{m-1}\|_{L^2})$.

Moreover for a subsequence the following hold:

$$u^\alpha \rightharpoonup u \quad \text{in } L^{\frac{n}{n-1}}(\Omega;\mathbb{R}^n), \tag{6.6}$$

$$u^\alpha \to u \quad \text{in } L^r(\Omega;\mathbb{R}^n) \quad \text{for } r\in[1,n/(n-1)), \tag{6.7}$$

$$\int_\Omega \tau : \varepsilon(u^\alpha)\,dx \to \int_\Omega \tau:\varepsilon(u)\,dx \quad \text{for every } \tau\in C_c^\infty(\Omega;\mathbb{M}^{n\times n}_{sym}), \tag{6.8}$$

$$\operatorname{div} u^\alpha \rightharpoonup \operatorname{div} u \quad \text{in } L^2(\Omega;\mathbb{R}^n), \tag{6.9}$$

$$\alpha\int_\Omega |\varepsilon^D(u^\alpha)+e_N^{m-1D}|^2\,dx \to 0, \tag{6.10}$$

$$\sigma^\alpha \rightharpoonup \sigma \quad \text{in } L^2(\Omega;\mathbb{M}^{n\times n}_{sym}), \tag{6.11}$$

$$\sigma^{\delta D} - \alpha(\varepsilon^D(u^\alpha)+e_N^{m-1D}) \overset{*}{\rightharpoonup} \sigma^D \quad \text{in } L^\infty(\Omega;\mathbb{M}^{n\times n}_{sym}), \tag{6.12}$$

where u is a solution to problem (5.19) *and σ is the unique solution to problem* (5.9).

Proof: From the coercivity estimate (5.10) one immediately obtains (6.5).

It follows from (6.3) that the sequences $\{\sigma^\alpha\}$ and $\{\sigma^{\alpha D}-\alpha(\varepsilon^D(u^\alpha)+e_N^{m-1D})\}$ are bounded in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ and $L^\infty(\Omega;\mathbb{M}^{n\times n}_D)$ respectively.

So we get the estimates (6.6)-(6.9), (6.11) and (6.12). It remains to show, that u and σ are solutions of (5.19) and (5.9) and that (6.10) holds.

As $\tau^\alpha := \sigma^\alpha - \alpha(\varepsilon^D(u^\alpha)+e_N^{m-1D})\in\mathcal{K}$, and since $\mathcal{K}$ is weakly closed in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, it follows that $\sigma\in\mathcal{K}$. Now passing to the limit in (6.2) and using the results of [7, Proposition 3.4] we can extend (6.2) to V_0 and thus $\sigma\in Q_{f_N^m}$.

On the other hand, the duality relations imply that

$$\tau^\alpha : (\varepsilon(u^\alpha)+e_N^{m-1}) - g(\varepsilon(u^\alpha)+e_N^{m-1}) - g^*(\tau^\alpha) = 0 \quad \text{a.e. in } \Omega.$$

But then, by (6.2) and (6.3) one gets

$$\begin{aligned} I_\alpha(u^\alpha) &= \frac{\alpha}{2}\int_\Omega |\varepsilon^D(u^\alpha)+e_N^{m-1D}|^2\,dx + \int_\Omega \Big[\tau^\alpha : (\varepsilon(u^\alpha)+e_N^{m-1}) - g^*(\tau^\alpha)\Big]\,dx - M_N^m(u^\alpha) = \\ &= -\frac{\alpha}{2}\int_\Omega |\varepsilon^D(u^\alpha)+e_N^{m-1D}|^2\,dx + \int_\Omega \Big[\sigma^\alpha : (\varepsilon(u^\alpha)+e_N^{m-1}) - g^*(\tau^\alpha)\Big]\,dx - M_N^m(u^\alpha). \end{aligned}$$

By Theorem 4.1 we get

$$\begin{aligned} &\sup\{R(\tau) \;:\; \tau\in Q_{f_N^m}\cap\mathcal{K}\} = \inf\{I(v) \;:\; v\in\delta w_N^m + V_0\} \le I(u^\alpha) \le I_\alpha(u^\alpha) = \\ &= -\frac{\alpha}{2}\int_\Omega |\varepsilon^D(u^\alpha)+e_N^{m-1D}|^2\,dx - \int_\Omega g^*(\tau^\alpha)\,dx + \int_\Omega \sigma^\alpha : (\varepsilon(u^\alpha)+e_N^{m-1})\,dx - \\ &\quad - M_N^m(u^\alpha) = -\frac{\alpha}{2}\int_\Omega |\varepsilon^D(u^\alpha)+e_N^{m-1D}|^2\,dx - \int_\Omega g^*(\tau^\alpha)\,dx + \\ &\quad + \int_\Omega \sigma^\alpha : (\varepsilon(\delta w_N^m)+e_N^{m-1})\,dx - M_N^m(\delta w_N^M), \end{aligned} \tag{6.13}$$

where the Euler equation (6.2) was used.

Now, by exploiting the convergence (6.11) and (6.12) we go on with (6.13):

$$\lim_{\alpha\to0} I_\alpha(u^\alpha) \le -\int_\Omega g^*(\sigma)\,dx + \int_\Omega \sigma : (\varepsilon(\delta w_N^m)+e_N^{m-1})\,dx - M_N^m(\delta w_N^m) = R(\sigma).$$

Thus, proceeding with (6.13) we obtain

$$\begin{aligned} \sup\{R(\tau) \;:\; \tau\in Q_{f_N^m}\cap\mathcal{K}\} &= \inf\{I(v) \;:\; v\in u_0+V_0\} \le I(u^\alpha) \le I_\alpha(u^\alpha) \le \\ &\le R(\sigma) + \limsup_{\alpha\to0}\; -\frac{\alpha}{2}\int_\Omega |\varepsilon^D(u^\alpha)-p^\alpha|^2\,dx \le R(\sigma), \end{aligned}$$

which gives the relation (6.10) and ensures that σ is a solution to problem (5.9).

Moreover one has the identity

$$\lim_{\alpha\to0} I(u^\alpha) = \inf\{I(v) \;:\; v\in\delta w_N^m + V_0\},$$

which implies that u^α is a minimizing sequence for the problem (5.8), and therefore it converges weakly in $L^{\frac{n}{n-1}}(\Omega;\mathbb{R}^n)$ to a solution of problem (5.19). $\square$

Remark 6.3. One can easily prove the following formula for the derivatives $\sigma^\alpha_{,k}$:

(6.14)
$$\sigma^\alpha_{,k} = \begin{cases} \alpha(\varepsilon^D(u^\alpha_{,k}) + e^{m-1D}_{N,k}) + K_0\,(\operatorname{div} u^\alpha_{,k} + \operatorname{tr} e^{m-1}_{N,k})\mathbf{1} + \\ + \frac{\partial^2 g_0}{\partial \tau^2}(|\varepsilon^D(u^\alpha) + e^{m-1D}_N|)(\varepsilon^D(u^\alpha_{,k}) + e^{m-1D}_{N,k}), & \text{if } |\varepsilon^D(u^\alpha) + e^{m-1D}_N| < \frac{k_*}{\sqrt{2}\mu} \\ \alpha(\varepsilon^D(u^\alpha_{,k}) + e^{m-1D}_{N,k}) + K_0\,(\operatorname{div} u^\alpha_{,k} + \operatorname{tr} e^{m-1}_{N,k})\mathbf{1}, & \text{if } |\varepsilon^D(u^\alpha) + e^{m-1D}_N| \geq \frac{k_*}{\sqrt{2}\mu}. \end{cases}$$

Here and henceforth the subscript $_{,k}$ denotes the partial derivative with respect to x_k.

7. $W^{1,2}_{loc}$-estimates of stresses in the incremental problems

In this section we deduce the iterative estimate of L^2-norms of gradients of functions σ^α, defined by means of (6.3) via the solutions of regularized problems (6.1), and we show, that for every given m and N we have $\sigma^m_N \in W^{1,2}_{loc}(\Omega;\mathbb{M}^{n\times n}_{sym})$. We note, however, that in this section we are concerned only with the problem of regularity of each σ^m_N, that is, we do not care about the uniformity of estimates with respect to m and N. Having these estimates in hand, we conclude, that the convergence of approximate solutions σ^α to σ^m_N, which was known to take place in the weak topology of $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ (see (6.11)) is actually better, and is determined by the critical exponent of the Sobolev embedding.

We note, that, to underline the dependence of σ^α and u^α on m we sometimes write them as σ^α_m and u^α_m. Remark, that in what follows the constant C will denote the constant, which depends upon the data of the problem in a way, as in (3.5), and on the C^3-norm of the cut-off function φ chosen below, that is on the domain $\Omega' \subset\subset \Omega$.

Thus, our objective now is the following estimate

$$\int_{\Omega'} \mathbb{A}\sigma^\alpha_{m,k} : \sigma^\alpha_{m,k}\,dx \leq C(m,N,\Omega'), \tag{7.1}$$

valid for any $\Omega' \subset\subset \Omega$.

Suppose, by induction, that we have already proved that $\sigma^{m-1}_N \in W^{1,2}_{loc}(\Omega;\mathbb{M}^{n\times n}_{sym})$. To simplify the notation, in this section for the solutions of the incremental problem (5.1) we will omit writing the index N. Let us turn to the regularized problem (6.1). Since u^α_m is a solution of the nonlinear elliptic system (6.4) with $f^m \in L^n(\Omega;\mathbb{R}^n)$ and $e^{m-1}_N \in W^{1,2}_{loc}(\Omega;\mathbb{M}^{n\times n}_{sym})$, one can show, by considering the difference quotients, that

$$u^\alpha_m \in W^{2,2}_{loc}(\Omega;\mathbb{R}^n), \qquad \varepsilon(u^\alpha_m),\ \sigma^\alpha_m \in W^{1,2}_{loc}(\Omega;\mathbb{M}^{n\times n}_{sym}). \tag{7.2}$$

As

$$(\sigma^\alpha_m)_{ij,j} = -f^m_i \quad \text{a.e. in } \Omega, \tag{7.3}$$

one has

$$\int_\Omega \sigma^\alpha_{m,k} : \varepsilon(v)\,dx = -\int_\Omega f^m \cdot v_{,k}\,dx \quad \text{for all } v \in C^\infty_0(\Omega;\mathbb{R}^n),\ k = 1,\dots,n. \tag{7.4}$$

By using formula (6.14) for the gradient of σ^α_m,

$$\begin{aligned}\mathbb{A}\sigma^\alpha_{m,k} : \sigma^\alpha_{m,k} &= \mathbb{A}[\alpha\mathbf{1}^D + D^2 g(\varepsilon(u^\alpha_m) + e^{m-1})]\ (\varepsilon(u^\alpha_{m,k}) + e^{m-1}_{,k}) : \sigma^\alpha_{m,k} \leq \\ &\leq \Big[[\alpha\mathbf{1}^D + D^2 g(\varepsilon(u^\alpha_m) + e^{m-1})]\ (\varepsilon(u^\alpha_{m,k}) + e^{m-1}_{,k}) : (\varepsilon(u^\alpha_{m,k}) + e^{m-1}_{,k})\Big]^{1/2} \cdot \\ &\cdot\Big[[\alpha\mathbf{1}^D + D^2 g(\varepsilon(u^\alpha_m) + e^{m-1})]\,\mathbb{A}^2\ \sigma^\alpha_{m,k} : \sigma^\alpha_{m,k}\Big]^{1/2} = \\ &= \Big[\sigma^\alpha_{m,k} : (\varepsilon(u^\alpha_{m,k}) + e^{m-1}_{,k})\Big]^{1/2} \cdot \\ &\cdot\Big[[\alpha\mathbf{1}^D + D^2 g(\varepsilon(u^\alpha_m) + e^{m-1})]\,\mathbb{A}^2\ \sigma^\alpha_{m,k} : \sigma^\alpha_{m,k}\Big]^{1/2} \leq \\ &\leq \frac{1}{2}\sigma^\alpha_{m,k} : \varepsilon(u^\alpha_{m,k}) + \frac{1}{2}\mathbb{A}\sigma^\alpha_{m,k} : \sigma^{m-1}_{,k} + \frac{1}{2}\mathbb{A}\sigma^\alpha_{m,k} : \sigma^\alpha_{m,k} + C\,\alpha\mathbb{A}\sigma^\alpha_{m,k} : \sigma^\alpha_{m,k}.\end{aligned}$$

By applying again the Cauchy inequality to $\mathbb{A}\,\sigma^\alpha_{m,k} : \sigma^{m-1}_{,k}$, we get

$$\mathbb{A}\,\sigma^\alpha_{m,k} : \sigma^\alpha_{m,k} \leq \mathbb{A}\,\sigma^{m-1}_{,k} : \sigma^{m-1}_{,k} + 2\sigma^\alpha_{m,k} : \varepsilon(u^\alpha_{m,k}) + \alpha\, C\,\mathbb{A}\,\sigma^\alpha_{m,k} : \sigma^\alpha_{m,k},$$

so that

$$(1 - o_\alpha(1))\mathbb{A}\sigma^\alpha_{m,k} : \sigma^\alpha_{m,k} \leq \mathbb{A}\,\sigma^{m-1}_{,k} : \sigma^{m-1}_{,k} + 2\sigma^\alpha_{m,k} : \varepsilon(u^\alpha_{m,k}). \tag{7.5}$$

Thus, it remains to prove the boundedness of the second summand of (7.5), which will be denoted by J^α_m.

Let us introduce the notation

$$\sigma^\alpha := \sigma^\alpha_m, \quad f := f^m, \quad \overline{u}^\alpha := u^\alpha_m,$$

omitting also the index m for further convenience. Let $\varphi \in C^3_0(\Omega)$ be an arbitrary cut-off function. By (7.2) we can put the function

$$v = \varphi^6 \overline{u}^\alpha_{,k}$$

into the identity (7.4).

Here and henceforth we adopt the summation convention over repeated indices (excluding m and α). We start by

$$\begin{aligned}\int_\Omega \sigma^\alpha_{,k} : \varepsilon(\varphi^6\overline{u}^\alpha_{,k})\,dx &= -\int_\Omega f\cdot(\varphi^6\overline{u}^\alpha_{,k})_{,k}\,dx = \\ &= -\int_\Omega f\cdot\varphi^6\Delta\overline{u}^\alpha\,dx - \int_\Omega f\cdot\varphi^6_{,k}\overline{u}^\alpha_{,k}\,dx.\end{aligned} \tag{7.6}$$

As

$$\frac{1}{2}\Delta\overline{u}^\alpha = \operatorname{div}\varepsilon(\overline{u}^\alpha) - \frac{1}{2}\nabla\operatorname{div}\overline{u}^\alpha,$$

we go on

$$\begin{aligned}&-\int_\Omega f\cdot\varphi^6\Delta\overline{u}^\alpha\,dx - \int_\Omega f\cdot\varphi^6_{,k}\overline{u}^\alpha_{,k}\,dx = -2\int_\Omega f\cdot\varphi^6\operatorname{div}\varepsilon(\overline{u}^\alpha)\,dx + \\ &+\int_\Omega f\cdot\varphi^6\nabla\operatorname{div}\overline{u}^\alpha\,dx - \int_\Omega f\cdot\varphi^6_{,k}\overline{u}^\alpha_{,k}\,dx = 2\int_\Omega \varepsilon(f) : \varphi^6\varepsilon(\overline{u}^\alpha)\,dx + \\ &+2\int_\Omega (f\odot\nabla\varphi^6) : \varepsilon(\overline{u}^\alpha)\,dx + \int_\Omega \varphi^6\, f\cdot\nabla\operatorname{div}\overline{u}^\alpha\,dx - \int_\Omega f\cdot\varphi^6_{,k}\overline{u}^\alpha_{,k}\,dx = \\ &= 2\int_\Omega \varphi^6\varepsilon(f) : \varepsilon(\overline{u}^\alpha)\,dx + \int_\Omega \varphi^6\, f\cdot\nabla\operatorname{div}\overline{u}^\alpha\,dx + \int_\Omega f_i\,\varphi^6_{,j}\overline{u}^\alpha_{j,i}\,dx = \\ &= 2\int_\Omega \varphi^6\varepsilon(f) : \varepsilon(\overline{u}^\alpha)\,dx + \\ &+\int_\Omega \varphi^6\, f\cdot\nabla\operatorname{div}\overline{u}^\alpha\,dx - \int_\Omega \nabla\varphi^6\cdot\overline{u}^\alpha\operatorname{div} f\,dx - \int_\Omega (f\odot\overline{u}^\alpha) : \nabla^2\varphi^6\,dx.\end{aligned} \tag{7.7}$$

Thus (7.6) and (7.7) yield

$$J_m^\alpha := \int_\Omega \varphi^6 \sigma^\alpha_{,k} : \varepsilon(\overline{u}^\alpha_k)\, dx = J_1 + J_2 + J_3, \tag{7.8}$$

where

$$J_1 := -2\int_\Omega \sigma^\alpha_{ij,k}\varphi^6_{,k}\varepsilon_{kj}(\overline{u}^\alpha)\, dx, \tag{7.9}$$

$$J_2 := \int_\Omega \sigma^\alpha_{ij,k}\varphi^6_{,i}\overline{u}^\alpha_{k,j}\, dx, \tag{7.10}$$

$$J_3 := \int_\Omega \Big[2\varphi^6 \varepsilon(f) : \varepsilon(\overline{u}^\alpha) + \varphi^6\, f\cdot \nabla \mathrm{div}\, \overline{u}^\alpha\, dx - \\ \nabla\varphi^6 \cdot \overline{u}^\alpha\, \mathrm{div}\, f - (f \odot \overline{u}^\alpha) : \nabla^2 \varphi^6 \Big]\, dx. \tag{7.11}$$

Now, by using the orthogonal decomposition of $\mathbb{M}^{n\times n}_{sym} = \mathbb{M}^D + \mathbb{R}\,\mathbf{1}$:

$$\varepsilon(\overline{u}^\alpha) = \varepsilon^D(\overline{u}^\alpha) + \frac{1}{n}\mathrm{div}\, \overline{u}^\alpha \mathbf{1}, \qquad \sigma^\alpha = \sigma^{\alpha D} + \frac{1}{n}\mathrm{tr}\, \sigma^\alpha \mathbf{1}$$

and the Euler equation (7.3), one gets

$$\begin{aligned} J_1 = -2\int_\Omega \sigma^\alpha_{ij,k}\varphi^6_{,i}\varepsilon^D_{jk}(\overline{u}^\alpha)\, dx - \frac{2}{n}\int_\Omega \sigma^\alpha_{ij,j}\varphi^6_{,i}\mathrm{div}\, \overline{u}^\alpha\, dx = \\ -\frac{2}{n}\int_\Omega \mathrm{tr}\, \sigma^\alpha_{,k}\varphi^6_{,i}\varepsilon^D_{ik}(\overline{u}^\alpha)\, dx - 2\int_\Omega \sigma^{\alpha D}_{ij,k}\varphi^6_{,i}\varepsilon^D_{jk}(\overline{u}^\alpha)\, dx + \\ +\frac{2}{n}\int_\Omega f\cdot\nabla\varphi^6 \mathrm{div}\, \overline{u}^\alpha\, dx = -\frac{2}{n}\int_\Omega (f_k + \sigma^{\alpha D}_{ks,s})\varphi^6_{,i}\varepsilon^D_{ik}(\overline{u}^\alpha)\, dx - \\ -2\int_\Omega \sigma^{\alpha D}_{ij,k}\varphi^6_{,i}\varepsilon^D_{jk}(\overline{u}^\alpha)\, dx + \frac{2}{n}\int_\Omega f\cdot\nabla\varphi^6 \mathrm{div}\, \overline{u}^\alpha\, dx = \\ = 2\int_\Omega (f\odot\nabla\varphi^6) : \varepsilon^D(\overline{u}^\alpha)\, dx + \frac{2}{n}\int_\Omega f\cdot\nabla\varphi^6\mathrm{div}\, \overline{u}^\alpha\, dx + \\ +12\int_\Omega \varphi^5\sigma^{\alpha D}_{ij,k}\Big(-\varphi_{,i}\varepsilon^D_{jk}(\overline{u}^\alpha) + \delta_{ik}\varphi_{,s}\varepsilon^D_{js}(\overline{u}^\alpha)\Big)\, dx = \\ = 2\int_\Omega (f\odot\nabla\varphi^6) : \varepsilon^D(\overline{u}^\alpha)\, dx + 12\int_\Omega \varphi^5\sigma^{\alpha D}_{,k} : S^{(k)}\, dx + \frac{2}{n}\int_\Omega f\cdot\nabla\varphi^6\mathrm{div}\, \overline{u}^\alpha\, dx. \end{aligned} \tag{7.12}$$

where the matrices $S^{(k)}$ are defined by

$$S^{(k)}_{ij} := \Big(\delta_{ik}\varphi_{,s}\varepsilon^D_{js}(\overline{u}^\alpha) - \varphi_{,i}\varepsilon^D_{jk}(\overline{u}^\alpha)\Big).$$

It follows immediately from the definition that

$$\mathrm{tr}\,(S^{(k)}) = \delta_{ik}\varphi_{,s}\varepsilon^D_{is}(\overline{u}^\alpha) - \varphi_{,i}\varepsilon^D_{ik}(\overline{u}^\alpha) = 0. \tag{7.13}$$

Now let us turn to J_2:

$$J_2 = -\int_\Omega \left[\sigma^\alpha_{ij}\varphi^6_{,ik}\overline{u}^\alpha_{k,j} + \sigma^\alpha_{ij}\varphi^6_{,j}\mathrm{div}\,\overline{u}^\alpha_{,i}\right] =$$

$$= \int_\Omega \sigma^\alpha_{ij,j}\varphi^6_{,ik}\overline{u}^\alpha_k\,dx + \int_\Omega \sigma^\alpha_{ij}\varphi^6_{,ijk}\overline{u}^\alpha_k\,dx + \int_\Omega \sigma^\alpha_{ij,j}\varphi^6_{,i}\mathrm{div}\,\overline{u}^\alpha\,dx +$$

$$+ \int_\Omega \sigma^\alpha_{ij}\varphi^6_{,ij}\mathrm{div}\,\overline{u}^\alpha\,dx = -\int_\Omega (f\odot\overline{u}^\alpha) : \nabla^2\varphi^6\,dx -$$

$$-\int_\Omega f\cdot\nabla\varphi^6\mathrm{div}\,\overline{u}^\alpha\,dx + \int_\Omega \sigma^\alpha_{ij}\varphi^6_{,ijk}\overline{u}^\alpha_k\,dx + \int_\Omega \sigma^\alpha : \nabla^2\varphi^6\mathrm{div}\,\overline{u}^\alpha\,dx.$$

The latter inequality and (7.8)-(7.12) give the estimate:

$$\begin{aligned} J^\alpha_m &\le I^\alpha_0 + 12\int_\Omega \varphi^5\sigma^{\alpha D}_{,k} : S^{(k)}\,dx + \int_\Omega \sigma^\alpha_{ij}\varphi^6_{,ijk}\overline{u}^\alpha_k\,dx + \\ &+ \int_\Omega \sigma^\alpha : \nabla^2\varphi^6\mathrm{div}\,\overline{u}^\alpha\,dx =: I^\alpha_0 + I^\alpha_1 + I^\alpha_2 + I^\alpha_3, \end{aligned} \tag{7.14}$$

with

$$\begin{aligned} I^\alpha_0 = 2\int_\Omega &\Big[(f\odot\nabla\varphi^6) : \varepsilon^D(\overline{u}^\alpha) + \varphi^6\varepsilon(f) : \varepsilon(\overline{u}^\alpha) - \\ &-(f\odot\overline{u}^\alpha) : \nabla^2\varphi^6 - f\cdot\nabla\varphi^6\mathrm{div}\,\overline{u}^\alpha\Big]\,dx - \\ &- \int_\Omega \Big(\varphi^6\mathrm{div}\,f\,\mathrm{div}\,\overline{u}^\alpha + \nabla\varphi^6\cdot\overline{u}^\alpha\mathrm{div}\,f\Big)\,dx. \end{aligned} \tag{7.15}$$

Estimate of I^α_0. By using the convergence $u^\alpha_m \overset{*}{\rightharpoonup} \delta u^m$ in $BD(\Omega)$, I^α_0 can be estimated as

$$|I^\alpha_0| \le C(\|f^m\|_{C^1(\Omega')},\ \|\varphi\|_{C^1(\Omega)})\sup_\alpha \|u^\alpha_m\|_{BD} \le C(m,N,\Omega'). \tag{7.16}$$

Estimate of I^α_1. We have

$$\begin{aligned} |I^\alpha_1| &\le C\left|\int_\Omega \varphi^5[\delta\mathbf{1}^D + D^2g(\varepsilon(u^\alpha_m) + e^{m-1})](\varepsilon(u^\alpha_{m,k}) + e^{m-1}_{,k}) : S^{(k)}\,dx\right| \le \\ &\le C\left[\int_\Omega \varphi^6\sigma^\alpha_{m,k} : (\varepsilon(u^\alpha_{m,k}) + e^{m-1}_{,k})\,dx\right]^{1/2}\cdot \\ &\cdot\left[\int_\Omega (\alpha + 2\mu\,\chi_{\{|\varepsilon^D(u^\alpha_m)+e^{m-1D}|<\frac{k_*}{\sqrt{2}\mu}\}})\,S^{(k)} : S^{(k)}\,dx\right]^{1/2} \le \\ &\le \frac{1}{100}\left(J^\alpha_m + \int_\Omega \varphi^6\mathbb{A}\sigma^\alpha_{m,k} : \sigma^\alpha_{m,k}\,dx + \|\varphi^3\nabla\sigma^{m-1}\|^2_{L^2}\right) + C\alpha\int_\Omega |\varepsilon^D(u^\alpha_m)|^2\,dx + \\ &+C\int_{\Omega\cap\{|\varepsilon^D(u^\alpha_m)+e^{m-1D}|<\frac{k_*}{\sqrt{2}\mu}\}} |\varepsilon^D(u^\alpha_m)|^2\,dx. \end{aligned} \tag{7.17}$$

Estimate of I^α_2. As to the second summand, the embedding $W^{1,2}_0(\Omega;\mathbb{M}^{n\times n}_{sym}) \hookrightarrow L^n(\Omega;\mathbb{M}^{n\times n}_{sym})$ for $n = 2,3$ allows one to make the following estimates

$$\begin{aligned} &C\Big(\int_\Omega |\varphi^3\sigma^\alpha_m|^n\Big)^{1/n}\Big(\int_\Omega |u^\alpha_m|^{\frac{n}{n-1}}\Big)^{\frac{n-1}{n}} \le C\|\nabla(\varphi^3\,\sigma^\alpha_m)\|_{L^2}\Big(\int_\Omega |u^\alpha_m|^{\frac{n}{n-1}}\Big)^{\frac{n-1}{n}} \le \\ &\le C\left[\int_\Omega \varphi^6\mathbb{A}\,\sigma^\alpha_{m,k} : \sigma^\alpha_{m,k}\,dx + \int_\Omega \varphi^4|\nabla\varphi|^2|\sigma^\alpha_m|^2\,dx\right]^{1/2}\|u^\alpha_m\|_{L^{\frac{n}{n-1}}}. \end{aligned} \tag{7.18}$$

Estimate of I_3^α. As

$$\operatorname{div} u_m^\alpha = \frac{1}{nK_0}\operatorname{tr}\sigma_m^\alpha - \operatorname{tr} e^{m-1},$$

we can bound I_3^α as

$$|I_3^\alpha| \le C(\|\sigma_m^\alpha\|_{L^2}^2 + \|\sigma^{m-1}\|_{L^2}) \tag{7.19}$$

So, (7.5), (7.14), (7.16)-(7.19) and the regularity of σ_N^{m-1}, proved at the previous step, allow us to conclude that (7.1) holds, and thus

$$\limsup_{\alpha\to 0}\|\nabla\sigma_m^\alpha\|_{L^2(\Omega')} \le C(m, N, \Omega'), \tag{7.20}$$

where this constant depends on the domain Ω', the step m and the data of the problem.

Remark 7.1. The inequality (7.20) and the convergence $\sigma_m^\alpha \rightharpoonup \sigma_N^m$ in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, see (6.11), imply that

$$\begin{aligned} &\sigma_N^m \in W^{1,2}_{loc}(\Omega;\mathbb{M}^{n\times n}_{sym}),\\ &\sigma_m^\alpha \rightharpoonup \sigma_N^m \quad \text{in } W^{1,2}_{loc}(\Omega),\\ \text{and}\quad &\sigma^\alpha \to \sigma_N^m \quad \text{in } L^n_{loc}(\Omega;\mathbb{M}^{n\times n}_{sym}), \end{aligned} \tag{7.21}$$

where the strong convergence in $L^n_{loc}(\Omega;\mathbb{M}^{n\times n}_{sym})$ is guaranteed by the Sobolev embedding for $n = 2, 3$.

8. Uniform $W^{1,2}_{loc}$-estimates of stresses

To carry out the proof of the uniform boundedness of $\|\sigma_N\|_{L^\infty((0,T);W^{1,2}_{loc}(\Omega;\mathbb{M}^{n\times n}_{sym}))}$ we will make use of the iterative estimate (7.5), deduced in the previous section, which results in a discrete analogue of the Gronwall inequality. To this aim, we need to have the estimate of the last term of (7.5). To make the estimates uniform, we will use the convergence of u_m^α to δu^m as in (6.6)-(6.10), and the convergence of σ_m^α to σ^m as in (7.21).

So, the goal of this section is to prove the following inequality

$$\Big(1 - \frac{C}{N}\Big)\int_\Omega \varphi^6 \mathbb{A}\sigma_N^m : \sigma_N^m\,dx \le \Big(1 + \frac{C}{N}\Big)\int_\Omega \varphi^6\mathbb{A}\sigma_N^{m-1} : \sigma_N^{m-1}\,dx + \frac{C}{N}, \tag{8.1}$$

with C independent of N.

We will deduce it from (7.5). Recall (7.14):

$$\begin{aligned} J_m^\alpha \le I_0^\alpha + 12\int_\Omega \varphi^5\sigma^{\alpha D}_{,k} : S^{(k)}\,dx + \int_\Omega \sigma^\alpha_{ij}\varphi^6_{,ijk}\overline{u}^\alpha_k\,dx+\\ + \int_\Omega \sigma^\alpha : \nabla^2\varphi^6 \operatorname{div}\overline{u}^\alpha\,dx =: I_0^\alpha + I_1^\alpha + I_2^\alpha + I_3^\alpha, \end{aligned} \tag{8.2}$$

with I_0^α defined in (7.15).

Estimates of I_0^α: Since $f \in C^1_{loc}(\Omega;\mathbb{R}^n)$, one can employ (6.6)-(6.9) to pass to the limit in (7.15), and use estimates (3.8) of $\|\delta u^m\|_{BD} = \|u^m - u^{m-1}\|_{BD}$ to get

$$|I_0^\delta| \le C(\|f(t)\|_{L^\infty([0,T];C^1(\Omega;\mathbb{R}^n))},\ \|\varphi\|_{C^1(\Omega)})\,\frac{1}{N}. \tag{8.3}$$

Estimates of I_1^α: Taking into account (6.14) and (7.13)

$$\int_\Omega \varphi^5 \sigma^{\alpha D}_{,k} : S^{(k)}\,dx = \int_\Omega \varphi^5 \sigma^{\alpha}_{,k} : S^{(k)}\,dx =$$

$$= \int_\Omega \varphi^5 [\alpha \mathbf{1}^D + D^2 g(\varepsilon(u_m^\alpha) + e^{m-1})]\ (\varepsilon(u^\alpha_{m,k}) + e^{m-1}_{,k}) : S^{(k)}\,dx \le$$

$$\le \left[\int_\Omega \varphi^6 [\alpha \mathbf{1}^D + D^2 g(\varepsilon(u_m^\alpha) + e^{m-1})](\varepsilon(u^\alpha_{m,k}) + e^{m-1}_{,k}) : (\varepsilon(u^\alpha_{m,k}) + e^{m-1}_{,k})\right]^{1/2} \cdot \tag{8.4}$$

$$\cdot \left[\int_\Omega \varphi^4 [\alpha \mathbf{1}^D + D^2 g(\varepsilon(u_m^\alpha) + e^{m-1})]\ S^{(k)} : S^{(k)}\,dx\right]^{1/2} \le \frac{1}{100N} B_1 + C\,N B_2.$$

Let us estimate B_1

$$\int_\Omega \varphi^6 [\alpha \mathbf{1}^D + D^2 g(\varepsilon(u_m^\alpha) + e^{m-1})]\ (\varepsilon(u^\alpha_{m,k}) + e^{m-1}_{,k}) : (\varepsilon(u^\alpha_{m,k}) + e^{m-1}_{,k})\,dx =$$

$$= \int_\Omega \varphi^6 \sigma^\alpha_{m,k} : (\varepsilon(u^\alpha_{m,k}) + e^{m-1}_{,k})\,dx \le$$

$$\le \int_\Omega \varphi^6 \sigma^\alpha_{m,k} : \varepsilon(u^\alpha_{m,k})\,dx + \frac{1}{2}\int_\Omega \varphi^6 \mathbb{A}\sigma^\alpha_{m,k} : \sigma^\alpha_{m,k}\,dx + \frac{1}{2}\int_\Omega \varphi^6 \mathbb{A}\sigma^{m-1}_{,k} : \sigma^{m-1}_{,k}\,dx =$$

$$= J_m^\alpha + \frac{1}{2}\int_\Omega \varphi^6 \mathbb{A}\sigma^\alpha_{m,k} : \sigma^\alpha_{m,k}\,dx + \frac{1}{2}\int_\Omega \varphi^6 \mathbb{A}\sigma^{m-1}_{,k} : \sigma^{m-1}_{,k}\,dx.$$

Let us estimate B_2. As $|S^{(k)}| \le C|\varepsilon^D(u_m^\alpha)|$ we proceed with

$$\int_\Omega \varphi^6 [\alpha \mathbf{1}^D + D^2 g(\varepsilon(u_m^\alpha) + e^{m-1})]\ S^{(k)} : S^{(k)}\,dx \le$$

$$C\,\alpha \int_\Omega |\varepsilon^D(u_m^\alpha)|^2\,dx + \int_\Omega \varphi^6 [D^2 g(\varepsilon(u_m^\alpha) + e^{m-1})]\ S^{(k)} : S^{(k)}\,dx$$

Now we study the second summand carefully. Taking into account (7.13), the properties of g, (6.1) and (7.21) we have

$$\int_\Omega \varphi^6 [D^2 g(\varepsilon(u_m^\alpha) + e^{m-1})]\ S^{(k)} : S^{(k)}\,dx =$$

$$= \int_{\Omega \cap \{|\varepsilon^D(u_m^\alpha) + e^{m-1D}| < \frac{k_*}{\sqrt{2}\mu}\}} 2\mu \varphi^6 S^{(k)} : S^{(k)}\,dx \le$$

$$\le 2\mu\, C \int_{\Omega \cap \{|\varepsilon^D(u_m^\alpha) + e^{m-1D})| < \frac{k_*}{\sqrt{2}\mu}\}} \varphi^6 |\varepsilon^D(u_m^\alpha)|^2\,dx =$$

$$= \frac{C}{2\mu}\int_{\Omega \cap \{|\varepsilon^D(u_m^\alpha) + e^{m-1D}| < \frac{k_*}{\sqrt{2}\mu}\}} \varphi^6 \Big| 2\mu\Big(\varepsilon^D(u_m^\alpha) + e^{m-1D} - e^{m-1D}\Big)\Big|^2\,dx \le$$

$$\le C \int_{\Omega \cap \{|\varepsilon^D(u_m^\alpha) + e^{m-1D}| < \frac{k_*}{\sqrt{2}\mu}\}} \varphi^6 \left| \Big(\frac{\partial g}{\partial \tau}(\varepsilon(u_m^\alpha) + e^{m-1})\Big)^D - \sigma^{m-1D}\right|^2 dx =$$

$$= C \int_{\Omega \cap \{|\varepsilon^D(u_m^\alpha) + e^{m-1D}| < \frac{k_*}{\sqrt{2}\mu}\}} \varphi^6 \Big|\sigma_m^{\alpha D} - \sigma^{m-1D} - \alpha(\varepsilon^D(u_m^\alpha) + e^{m-1D})\Big|^2\,dx \le$$

$$\le C\|\sigma^{mD} - \sigma^{m-1D}\|^2_{L^2} + o_\alpha(1).$$

Finally, by applying the Cauchy inequality to (8.4) and using (3.8), one obtains

$$
\begin{aligned}
&\int_\Omega \varphi^5 \sigma^{\alpha D}_{,k} : S^{(k)}\, dx \le \frac{J^\alpha_m}{5N} + \frac{1}{N}\int_\Omega \varphi^6 \mathbb{A}\, \sigma^\alpha_{m,k} : \sigma^\alpha_{m,k}\, dx + \\
&+\frac{1}{N}\int_\Omega \varphi^6 \mathbb{A}\, \sigma^{m-1}_{,k} : \sigma^{m-1}_{,k}\, dx + CN\|\sigma_m - \sigma_{m-1}\|^2_{L^2(\Omega)} + o_\alpha(1) \le \\
&\le \frac{J^\alpha_m}{5N} + \frac{1}{N}\int_\Omega \varphi^6 \mathbb{A}\, \sigma^\alpha_{m,k} : \sigma^\alpha_{m,k}\, dx + \frac{1}{N}\int_\Omega \varphi^6 \mathbb{A}\, \sigma^{m-1}_{,k} : \sigma^{m-1}_{,k}\, dx + \frac{C}{N} + o_\alpha(1).
\end{aligned} \tag{8.5}
$$

Estimates of I_2^α: To pass to the limit in I_2^α, we exploit (6.6) and (7.21):

$$
\lim_{\alpha\to 0} I_2^\alpha = \lim_{\alpha\to 0}\int_\Omega \sigma^\alpha_{ij}\varphi^6_{,ijk}\overline{u}^\alpha_k\, dx = \int_\Omega \sigma^m_{ij}\varphi^6_{,ijk}(u^m - u^{m-1})_k\, dx =: I_2.
$$

Now let us use the embedding $W^{1,2}_0(\Omega;\mathbb{M}^{n\times n}_{sym}) \hookrightarrow L^n(\Omega;\mathbb{M}^{n\times n}_{sym})$ for $n = 2,3$ and (3.8)

$$
\begin{aligned}
&|I_2| \le C\|\varphi^3\sigma^m\|_{L^n(\Omega)}\|u^m - u^{m-1}\|_{L^{\frac{n}{n-1}}} \le \\
&\le C\Big(\int_\Omega \varphi^6\sigma^m_{,k} : \sigma^m_{,k}\, dx + \|\varphi\sigma^m\|_{L^2}\Big)^{1/2}\|u^m - u^{m-1}\|_{L^{\frac{n}{n-1}}} \le \\
&\le \frac{1}{N}\Big(\int_\Omega \varphi^6\mathbb{A}\sigma^m_{,k} : \sigma^m_{,k}\, dx + C\Big) + CN\|u^m - u^{m-1}\|^2_{L^{\frac{n}{n-1}}} \le \\
&\le \frac{1}{N}\int_\Omega \varphi^6\mathbb{A}\sigma^m_{,k} : \sigma^m_{,k}\, dx + \frac{C}{N}.
\end{aligned} \tag{8.6}
$$

Estimates of I_3^α: The relations (6.9) and (7.21) allow one to pass to the limit in I_3^α:

$$
\lim_{\alpha\to 0} I_3^\alpha = \lim_{\alpha\to 0}\int_\Omega \sigma^\alpha_m : \nabla^2\varphi^6 \mathrm{div}\, u^\alpha_m\, dx = \int_\Omega \sigma^m : \nabla^2\varphi^6\mathrm{div}\,(u^m - u^{m-1})\, dx =: I_3,
$$

so in view of the equality

$$
\mathrm{div}\,(u^m - u^{m-1}) = \mathrm{tr}\,(e^m - e^{m-1})
$$

by (3.8) we conclude that

$$
|I_3| \le C\|\sigma^m\|_{L^2}\|\mathrm{tr}\,(e^m - e^{m-1})\|_{L^2} \le \frac{C}{N}. \tag{8.7}
$$

Proof of Theorem 2.1: The estimates (8.2), (8.3) and (8.5)-(8.7) yield

$$
J^\alpha_m \le \frac{C}{N}\left[\int_\Omega \varphi^6\mathbb{A}\sigma^\alpha_m : \sigma^\alpha_m\, dx + \int_\Omega \varphi^6\mathbb{A}\sigma^m_N : \sigma^m_N\, dx + \int_\Omega \varphi^6\mathbb{A}\sigma^{m-1}_N : \sigma^{m-1}_N\, dx + 1\right] + o_\alpha(1),
$$

where $o_\alpha(1)$ depends upon m, N. Now (7.5) implies

$$
\begin{aligned}
&\Big(1 - \frac{C}{N}\Big)\int_\Omega \varphi^6\mathbb{A}\sigma^\alpha_m : \sigma^\alpha_m\, dx \le \\
&\le \Big(1 + \frac{C}{N}\Big)\int_\Omega \varphi^6\mathbb{A}\sigma^{m-1}_N : \sigma^{m-1}_N\, dx + \frac{C}{N}\int_\Omega \varphi^6\mathbb{A}\sigma^m_N : \sigma^m_N\, dx + \frac{C}{N} + o_\alpha(1).
\end{aligned}
$$

To deduce (8.1) it remains to pass to the limit with respect to α, to use (7.21) and the lower semicontinuity of the norm. By applying the discrete version of Gronwall lemma we obtain (1.2). Now the conclusion follows from convergence of $\sigma_N(t) \rightharpoonup \sigma(t)$ in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ for every $t \in [0,T]$. $\square$

9. Examples

Below we give two examples that show that we cannot hope to get regularity results in the spirit of [4], that is, the existence of the elastic zone, where the equations of linear elasticity are satisfied and where the stress and strain are as regular, as the data of the problem permits.

We consider two particular cases of the periodic problem in dimension two, where not only the stress tensor $\sigma(t)$ is unique, but so are the displacement $u(t)$ and a plastic part of the strain $p(t)$. In this case the problem is reduced to a one-dimensional one.

In the first example we consider the situation, where the data of the problem is of class C^∞, but the solution of the reduced one-dimensional problem $u^R(t)$ develops a jump after some time t^*. The second example shows, that even when the displacement u^R is continuous, we cannot expect any kind of regularity of u^R or p^R. Namely, it shows, that given any diffuse measure $\mu^R \in M_b^+(0,1)$, we can choose the data of the problem to be C^∞ in such a way, that at time $t = 1$ the plastic strain is precisely measure μ^R, while for almost every time t we have $\sigma(t,x) \in \operatorname{int}\mathbb{K}$ for all x, except one point.

This is in contrast with the case of Hencky plasticity (see [4]), where it was proved the existence of the elastic zone – an open set $\Omega_0 \subset \Omega$, such that

$$\begin{gathered}\sigma(x) \in \operatorname{int}\mathbb{K} \quad \text{for all } x \in \Omega_0,\\ \operatorname{div}\Big(\frac{\partial g}{\partial \varkappa}(\varepsilon(u))\Big) + f = 0 \quad \text{in } \Omega_0,\\ \sigma(x) \in \partial\mathbb{K} \quad \text{for a.e. } x \in \Omega\setminus\Omega_0.\end{gathered}$$

We consider the case of simple shear in Dirichlet-periodic case in dimension $n = 2$. Similar examples can be easily constructed also in higher dimensions. We consider the unit cube $\Omega = (0,1)\times(0,1)$ and x_1-periodic solutions with boundary data of the form

$$\begin{aligned} u(t,x_1,0) &= (0,0),\\ u(t,x_1,1) &= (\sqrt{2}\varphi(t),0),\\ u(t,0,x_2) &= u(t,1,x_2).\end{aligned} \tag{9.1}$$

Let us introduce a linear isometry $M : \mathbb{R} \to \mathbb{M}^{2\times 2}_{sym}$ as

$$M(\alpha) = \begin{pmatrix} 0 & \frac{\alpha}{\sqrt{2}} \\ \frac{\alpha}{\sqrt{2}} & 0 \end{pmatrix}. \tag{9.2}$$

Assume, that the volume force has the form

$$f(t,x) = \frac{1}{\sqrt{2}}(f^R(t,x_2),0), \tag{9.3}$$

where we require the safe-load assumption to hold, and the initial conditions $(u_0, e_0, 0)$ are

$$u_0(x_1,x_2) = \begin{pmatrix} \sqrt{2}u_0^R(x_2) \\ 0 \end{pmatrix} \quad \text{and} \quad e_0(x_1,x_2) = M(e_0^R(x_2)), \tag{9.4}$$

for some functions u_0^R, e_0^R.

First, we will show, that in this particular situation all solutions of the quasistatic problem can be obtained from the solutions of a suitable one-dimensional problem. The definition of quasistatic evolution in dimension one can be obtained from Definitions 3.1 and 3.6 by replacing the spaces $\mathbb{M}^{n\times n}_{sym}$ and $\mathbb{M}^{n\times n}_D$ by $\mathbb{R}$, the compliance tensor $\mathbb{A}$ with a multiplication by $\frac{1}{2\mu}$ and the set $\mathbb{K}$ by $\mathbb{K}^R = [-\sqrt{2}k_*, \sqrt{2}k_*]$.

Let us introduce a space $W \subset \mathbb{M}^{2\times 2}_{sym}$ as follows

$$W = \Big\{ \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix} : a \in \mathbb{R}\Big\}.$$

Given $p^R \in M_b([0,1])$, the measure $M(p^R) \in M_b([0,1]\times[0,1];\mathbb{M}^{2\times 2}_{sym})$ is defined by

$$M(p^R)(A\times B) = M(p^R(B))\,\mathcal{L}^1(A)$$

for every pair of Borel sets $A, B \subset [0,1]$, that is

$$\langle M(p^R), \psi\rangle = \sqrt{2}\int_0^1 \langle p^R, \psi_{12}(x_1,\cdot)\rangle\, dx_1 \tag{9.5}$$

for every $\psi \in C^0([0,1]\times[0,1];\mathbb{M}^{2\times 2}_{sym})$.

Theorem 9.1. *Suppose, that we are given a boundary conditions as in* (9.1) *and the load as in* (9.3) *with* $f \in AC([0,T];L^2(\Omega))$. *Suppose, that the triple* $(u_0, e_0, 0)$ *is kinematically admissible and satisfies the stability condition. Then every solution* (u,e,p) *of the quasistatic problem with the initial conditions* (9.4) *has the form:*

$$u(t,x_1,x_2) = \begin{pmatrix} \sqrt{2}u^R(t,x_2) \\ 0 \end{pmatrix}, \quad e(t,x_1,x_2) = M(e^R(t,x_2)), \quad p(t) = M(p^R(t))$$

with $M(p^R(t))$ *defined in* (9.5), *where* $(u^R(t), e^R(t), p^R(t))$ *is a solution to a one-dimensional quasistatic problem, solved on a domain* $\Omega^R = (0,1)$ *with the initial data* $(u_0^R, e_0^R, 0)$, *Dirichlet boundary conditions* $u^R(t,0) = 0$, $u^R(t,1) = \varphi(t)$ *and the load* $f^R(t,y)$.

Proof: Consider the quasistatic problem with initial data $(u_0^R, e_0^R, 0)$ in dimension one with domain $\Omega^R = (0,1)$, the compliance tensor $\mathbb{A}^R = \frac{1}{2\mu}$, volume force $f^R(t,y)$ and the Dirichlet boundary data $u^R(t,0) = 0$, $u^R(t,1) = \varphi(t)$. Let $(u^R(t,y), e^R(t,y), p^R(t,y))$ be its solution.

Now we show, that the function (u,e,p) defined as follows

$$u(t,x_1,x_2) = (\sqrt{2}u^R(t,x_2), 0), \quad e(t,x_1,x_2) = M(e^R(t,x_2)), \quad p(t) = M(p^R(t)), \tag{9.6}$$

with M defined in (9.2) and (9.5), is a quasistatic evolution in dimension two.

To this aim, let us check conditions (qs1) and (qs2) of Definition 3.6 with $w(t,x_1,x_2) = (\sqrt{2}\varphi(t)x_2, 0)$.

(qs1): The kinematic admissibility condition for (u,e,p) in dimension two (see Definition 3.1) follows from the corresponding condition for (u^R, e^R, p^R) in dimension one.

As the minimality condition in (qs1) is equivalent to $-\operatorname{div}\sigma = f$ and $\sigma \in \mathcal{K}$, and these properties follow form the properties of σ^R. **(qs2):** Since M is an isometry, the energy balance for (u,e,p) follows from the analogous property of (u^R, e^R, p^R), .

Thus, the function $(u(t), e(t), p(t))$, defined in (9.6) is a quasistatic evolution in dimension two.

By the uniqueness of the strain σ, we know, that for any quasistatic evolution in dimension two, the stress field $\sigma(t)$ is given by

$$\sigma(t,x_1,x_2) = \begin{pmatrix} 0 & \sigma_{12}(t,x_2) \\ \sigma_{12}(t,x_2) & 0 \end{pmatrix}. \tag{9.7}$$

By the pointwise formulation of the flow rule, proved in [2, Theorem 6.4], and taking into account the fact, that $\sigma(t)$ is continuous, we have that for a.e. $t \in [0,T]$

$$g(x) := \frac{d\dot p(t)}{d|\dot p(t)|} \in W \quad \text{for } |\dot p(t)|\text{-a.e. } x \in [0,1]\times[0,1].$$

As $\dot p(t) = g(x)\,|\dot p(t)|$ for a.e. $t \in [0,T]$, it follows that

$$\dot p(t) \in M_b([0,1]\times[0,1];W).$$

Thus, as

$$\langle p(t), \varphi\rangle = \int_0^t \langle \dot p(s), \varphi\rangle\, ds,$$

for every $\varphi \in C([0,T]\times[0,T];\mathbb{M}^{n\times n}_{sym})$, we conclude that $p(t) \in M_b([0,1]\times[0,1];W)$ for a.e. $t \in [0,T]$.

So, from (9.7) and the last relation we deduce by the additive decomposition, that

$$\varepsilon(u) \in M_b(\Omega; W).$$

In particular, it implies, that

$$u_{1,1}(t,x) = 0, \quad \text{and} \quad u_{2,2}(t,x) = 0,$$

that is,

$$u_1(t,x) = u_1(t,x_2) \quad \text{and} \quad u_2(t,x) = u_2(t,x_1).$$

However, from the relaxed form of boundary conditions (9.1), which take the form

$$u_2(t,x_1,x_2) = u_2(t,x_1,0) = (-u(t,x_1,0) \odot \nu(x_1))_{22} = 0,$$

we have that $u_2(t,x_1,x_2) \equiv 0$.

Thus, $u(t,x_1,x_2) = (u_1(t,x_2),0)$, the elastic part $e(t)$ has the form, as in the statement of the Theorem, and hence $p \in M_b([0,1]\times[0,1]; W)$ and the Theorem is proved. □

9.1. **Example 1.** In this situation the data of the problem is the following: the domain Ω^R is $(0,1)$, the time interval is $[0,T] = [0,\frac{3}{2}]$, the constraint set $\mathbb{K}^R = [-1,1]$ and the elasticity tensor $\mathbb{A}^R$ is the identity. Taking the initial data to be $(u_0,e_0,p_0) = (0,0,0)$ we show that there exists a unique quasistatic evolution in dimension one, and that the displacement u of the solution has a jump at a point $x = \frac{1}{2}$ after time $t^* = 1$.

We choose a function $G \in C_0^\infty(0,1)$ such that

$$\int_0^1 G(y)\,dy = 0,$$
$$G(1/2) = 1, \quad G(y) < 1 \quad \text{for } y \neq \tfrac{1}{2},$$
$$G(y) > -\frac{1}{10} \quad \text{for } y \in [0,1],$$

and denote by $g(y)$ its derivative.

So, we consider the one-dimensional quasistatic problem with the following C^∞ data:

$$\begin{aligned} &(u_0^R, e_0^R, p_0^R) = (0,0,0), \\ &u^R(t,0) = u^R(t,1) = 0, \\ &f^R(t,y) = -t\,g(y). \end{aligned} \tag{9.8}$$

According to Theorem 9.1 all solutions of the corresponding two-dimensional Dirichlet-periodic problem are generated by the solutions of one-dimensional problem (9.8).

Consider the functions $(u^R(t), e^R(t), p^R(t))$ as follows:

$$u^R(t,y) = \begin{cases} t\displaystyle\int_0^y G(z)\,dz, & \text{for } t \le 1; \\ (1-t)y + (t-1)\chi_{(\frac{1}{2},1)}(y) + t\displaystyle\int_0^y G(z)\,dz, & \text{for } 1 < t \le \frac{3}{2}; \end{cases}$$

$$e^R(t,y) = \sigma^R(t,y) = \begin{cases} tG(y), & \text{for } t \le 1; \\ tG(y) + 1 - t, & \text{for } 1 < t \le \frac{3}{2}; \end{cases}$$

$$p^R(t) = \begin{cases} 0, & \text{for } t \le 1; \\ (t-1)\delta_{1/2}, & \text{for } 1 < t \le \frac{3}{2}. \end{cases}$$

It is easy to see, that this triple satisfies

$$(u(t), e(t), p(t)) \in A^R(0) \quad \text{for all } t \in [0,T].$$

In view of Remark 3.8 the global minimality condition is ensured by the fact that $|\sigma^R(t,y)| \le 1$ and $\sigma^R_y(t,y) = tg(y) = -f(t,y)$.

By [2, Theorem 6.4] the energy balance is equivalent to the pointwise formulation of the flow rule. Since $\dot p(t) = \delta_{\frac12}$ for $t > 1$ and as $\sigma(t, \frac12) = 1$, and $|\sigma(t, y)| < 1$ for $y \neq \frac12$ or $t < 1$ we have that

$$1 = \frac{d\dot p^R(t)}{d|\dot p^R(t)|} \in N_{\mathbb{K}}(\sigma^R(t, y)) \quad \text{for } |\dot p^R(t)|\text{-a.e. } y \in [-1, 1],$$

which is precisely the pointwise expression of the flow rule. Thus, (u, e, p) constructed above is a quasistatic evolution in dimension one.

Now we show that the solution constructed is the unique one. Let us suppose, that there exists another quasistatic evolution $(v(t), \eta(t), q(t))$. By the uniqueness of the stress, $\eta \equiv e$. Now let us show that $q \equiv p$. As the energy balance (qs2) is satisfied for (v, η, q), the pointwise formulation of the flow rule yields

$$\frac{d\dot q(t)}{d|\dot q(t)|} \in N_{\mathbb{K}}(\sigma(t, x)) \quad \text{for } |\dot q(t)|\text{-a.e. } x \in [-1, 1].$$

By the properties of $\sigma(t)$ it follows, that $\operatorname{supp} \dot q(t) \subset \{\frac12\}$ for a.e. $t \in [1, \frac32]$, while $\dot q(t) = 0$ for a.e. $t \in [0, 1]$.

Thus the formula

$$\langle q(t), \varphi\rangle_{M_b;C_0} = \int_0^t \langle \dot q(s), \varphi\rangle_{M_b;C_0}\, ds \quad \text{for any } \varphi \in C_0(0, 1),$$

yields that $q(t) = \beta(t)\delta_{\frac12}$ with $\beta \geq 0$, and the boundary conditions (9.8) imply that $\beta(t) = t - 1$, that is $q \equiv p$. This yields also that $v(t) = u(t)$, and we obtain the uniqueness of $u(t)$.

9.2. **Example 2.** We are given the domain $\Omega^R = (0, 1)$, the time interval $[0, 1]$, the constraint set $\mathbb{K}^R = [-1, 1]$ and the elasticity tensor $\mathbb{A}^R = 1$. Let $\mu^R \in M_b^+(0, 1)$ be a diffuse measure, that is $\mu^R(\{x\}) = 0$. Suppose, that $\mu^R([0, 1]) = 1$.

We will choose the data of the problem to be C^∞ and such that for the unique solution of the quasistatic problem $p^{Rs}(t, \cdot) = \mu^{Rs}$ for $t = 1$.

Let us take the continuous nondecreasing function $\Phi(s) = \mu^R([0, s])$. We consider the left-continuous inverse

$$X(t) := \sup\{s : \Phi(s) < t\},$$

so that $\Phi(X(t)) \equiv t$. Let us take the set $\{(t, X(t)) : t \in [0, 1]\}$ and denote its closure by E:

$$E := \operatorname{cl}\{(t, X(t)) : t \in [0, 1]\} = \{(t, X(t + 0)),\ (t, X(t - 0)) : t \in [0, 1]\}.$$

Then there exists a function $\phi(t, y)$, such that

$$\begin{gathered} \phi \in C_0^\infty(\mathbb{R}^2),\ 0 < \phi \leq 1, \\ \phi^{-1}(\{1\}) = E. \end{gathered}$$

The data of the one-dimensional problem we would like to solve is the following:

$$\begin{gathered} u_0^R(y) = \int_0^y \phi(0, z)\, dz, \quad e_0^R(y) = \phi(0, y), \quad p_0^R = 0, \\ u^R(t, 0) = 0, \quad u^R(t, 1) = \int_0^1 \phi(t, y)\, dy + t, \\ f^R(t, y) = -\phi_y(t, y). \end{gathered} \tag{9.9}$$

Now consider a function $\mu^R : [0, T] \to M_b^+([0, 1])$, defined as

$$\mu^R(t)(B) = \mu^R(B \cap [0, X(t)])$$

for every Borel set $B \subset [0, 1]$. The estimate

$$\|\mu^R(t) - \mu^R(s)\|_1 = \mu^R((x(s), x(t)]) = \Phi(X(t)) - \Phi(X(s)) = t - s,$$

shows that $\mu^R \in AC([0, T]; M_b^+([0, 1]))$. Moreover, the very definition of $X(t)$ yields

$$\dot\mu^R(t) = \delta_{X(t)}.$$

Consider the following functions:

$$\begin{aligned} u(t,y) &= \int_0^y \phi(t,z)\,dz + \mu^R(t)(0,y), \\ e(t,y) &= \sigma(t,y) = \phi(t,y), \\ p(t) &= \mu^R(t), \end{aligned} \tag{9.10}$$

and let us show that (u, e, p) defined in this way is the unique solution of the quasistatic problem (9.9).

First of all, it is obvious that the initial conditions are satisfied and the triple (u_0^R, e_0^R, p_0^R) satisfies the stability condition. Let us check the conditions (qs1) and (qs2) as in the Definition 3.6.

(qs1): As

$$(\mu^R(t)(0,y))_y = \mu^R(t) \quad \text{in } \mathcal{D}'(0,1)\,,$$

the kinematic admissibility condition in $(0,1)$ is trivially satisfied by (9.10). As the boundary conditions hold in the strong sense and $p = 0$ on $\partial\Omega$, we have that the triple $(u(t), e(t), p(t))$ is kinematically admissible for its boundary data.

What about the global stability, it follows from the equivalent condition (see Remark 3.8)

$$-\sigma_y(t,y) = f^R(t,y) \quad \text{and} \quad |\sigma(t,y)| \leq 1.$$

(qs2): As $\sigma(t, X(t+0)) = \sigma(t, X(t-0)) = 1$ and $|\sigma(t,x)| < 1$ otherwise, the pointwise formulation of the flow rule, which is equivalent to the energy balance, is satisfied:

$$\dot{p}^R(t) = \delta_{X(t-0)}, \quad 1 = \frac{d\dot{p}^R(t)}{d|\dot{p}^R(t)|} \in N_{\mathbb{K}}(\sigma^R(t,y)), \quad \text{for } |\dot{p}^R(t)|\text{-a.e. } y \in (0,1)\,.$$

So, (9.10) is a solution to (9.9).

Now let us take any solution $(v(t), \eta(t), q(t))$ to quasistatic problem (9.9). As the stress is unique, we have $\eta(t) \equiv e(t)$. Now, the pointwise formulation of the flow rule implies

$$\frac{d\dot{q}(t)}{d|\dot{q}(t)|} \in N_{\mathbb{K}}(\sigma(t,x)), \quad \text{for } |\dot{q}(t)|\text{-a.e. } x \in (0,1)\,,$$

that is $\operatorname{supp}\dot{q}(t) \subset \{X(t), X(t+0)\}$. As $X(t)$ is a monotone function, it has at most countable number of discontinuities, that is for a.e. $t \in [0,1]$ we have

$$\operatorname{supp}\dot{q}(t) \subset \{X(t)\}. \tag{9.11}$$

The boundary conditions for $v(t)$ yield:

$$v(t,1) = \int_0^1 \phi(t,y)\,dy + q(t)(0,1) = \int_0^1 \phi(t,y)\,dy + t,$$

which in its turn implies that $q(t)(0,1) = t,$ and from (9.11) it follows, that

$$\dot{q}(t) = \delta_{X(t)}.$$

That is, $q(t) \equiv p(t)$ and (9.10) is the unique solution to (9.9).

Acknowledgements. I would like to thank Gianni Dal Maso for very interesting and useful discussions. I am extremely grateful to Gregory Seregin for his helpful and stimulating suggestions.

References

[1] Bensoussan A., Frehse J.: Asymptotic behaviour of the time-dependent Norton-Hoff law in plasticity theory and H^1 regularity. *Comment. Math. Univ. Carolinae* **37** (1996), no. 2, 285-304.

[2] Dal Maso G., DeSimone A., Mora M.G.: Quasistatic evolution problems for linearly elastic-perfectly plastic materials. *Arch. Ration. Mech. Anal.* **180** (2006), no. 2, 237-291.

[3] Frehse J., Málek J.: Boundary regularity results for models of elasto-perfect plasticity. *Math. Meth. Appl. Sci.* **9** (1999), 1307–1321.

[4] Fuchs M., Seregin G.A.: Variational methods for problems from plasticity theory and for generalized Newtonian fluids. Springer-Verlag, Berlin, 2000.

[5] Johnson C.: Existence theorems for plasticity problems. *J. Math. Pures Appl.* **55** (1976), 431-444.

[6] Knees D.: Global regularity of the elastic fields of a power-law model on Lipschitz domains. *Math. Meth. Appl. Sci* **29** (2006), 1363-1391.

[7] Kohn R.V., Temam R.: Dual spaces of stresses and strains, with applications to Hencky plasticity. *Appl. Math. Optim.* **10** (1983), 1-35.

[8] Mielke A., *Evolution of rate-independent systems.* In: Evolutionary equations. Vol. II. Edited by M. Dafermos and E. Feireisl, 461–559, Handbook of Differential Equations. Elseveir/North-Holland, Amsterdam, 2005.

[9] Seregin G.A.: Remarks on regularity up to the boundary for solutions to variational problems in plasticity theory. *J. Math. Sci.* **93** (1999), no. 5, 779-783.

[10] Seregin G.A., Twodimensional variational problems in plasticity theory. *Izvestiya: Mathematics* **60** (1996) no.1, 179–216.

[11] Seregin G.A., On regularity of minimizers of certain variational problems in plasticity theory. *St. Petersburg Math. J.* **4** (1993), 1257–1272.

[12] Seregin G.A., Diferentiability properties of weak solutions of certain variational problems in the theory of perfect elasticplastic plates. *Appl. Math. Optim.* **28** (1993), 307–335.

[13] Seregin G.A., On differentiability properties of the stress-tensor in the Coulomb-Mohr theory of plasticity. *St. Petersburg Math. J.* **4** (1993), no.6, 1257–1272.

[14] Seregin G.A., Differential properties of solution of evolution variational inequalities in the theory of plasticity. *J. Math. Sci* **72** (1994), no.6, 3449-3458.

[15] Suquet, P.: Sur les équations de la plasticité: existence et regularité des solutions. *J. Mécanique*, **20** (1981), 3-39.

[16] Temam R.: Mathematical problems in plasticity. Gauthier-Villars, Paris, 1985. Translation of Problèmes mathématiques en plasticité. Gauthier-Villars, Paris, 1983.

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