Branching time estimates in quasi-static evolution for the average distance functional

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October 24, 2011

Abstract

We analyze in this paper the discrete quasi-static irreversible with small steps evolution of a connected network related to an average distance functional minimization problem. Our main goal is to determine whether new branches may appear during the evolution, thus changing the topology. We would give conditions on this, and an upper bound for the time at which it must happen for a particular class of configurations. We will use extensively tools belonging to minimizing movements and optimal transportation theory with free Dirichlet regions. Then we will give some explicit examples of quasi-static evolution, whose branching time will be estimated direct computation, by using both pure energy and mixed geometric/energy estimates.

Keywords: optimal transport, Euler scheme, average distance

2010 Mathematics Subject Classification: 49Q10, 49J40, 49L15

1 Introduction

Many evolution schemes, like Euler schemes, arise to model evolution problems with some kind of variational structure. An interesting question, still not deeply studied, is whether and when the evolving set will exhibit a branching behavior.

In this paper we will consider the discrete quasi-static evolution with small steps for connected networks related to an average distance functional, and our main goal is to analyze whether and when optimal sets may exhibit a branching behavior.

Let Ω be a compact subset of $\mathbb{R}^2$, $S \subset \Omega$ a Hausdorff one-dimensional connected set of given Hausdorff measure; we define the main functional of this paper:

$$F(S) := \int_\Omega \text{dist}(x, S)dx.$$

As we will see, the Lebesgue measure choice facilitates relating measure theory quantities with geometrical ones.

Next we introduce a definition, with a slight abuse of name:

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Definition 1.1. *Given a domain Ω, $S \subseteq \Omega$ a subset, $P \in S$ a point, its Voronoi cell is*

$$V(P) := \{x \in \Omega : dist(x, P) = dist(x, S)\}.$$

This is very similar to the classic definition of Voronoi cell, the main difference being not imposing S discrete.

Now we present our evolution models: given a time step $\varepsilon > 0$ and an initial datum $\Sigma_0 \in A$, we consider the following recursive sequence:

$$\begin{cases} \Sigma_0 \in A(\Omega) \\ \Sigma_{n-1} \subset \Sigma_n \\ \Sigma_n \in \text{argmin}_{\mathcal{H}^1(S)=n\varepsilon+\mathcal{H}^1(\Sigma_0)} F(S) \end{cases} \tag{1.1}$$

where

$$A_l(\Omega) := \left\{ \mathcal{X} \subseteq \Omega : \mathcal{X} \text{ compact, connected and } \mathcal{H}^1(\mathcal{X}) \leq l \right\}, \qquad A(\Omega) := \bigcup_{j \geq 0} A_j.$$

This is the so-called *Euler scheme*. In other words we are forcing the set to evolve choosing at every step a set which minimizes the functional F, under some constraints.

This Euler scheme can be written in another way: if we consider the evolution in time interval $[0, T]$, we can define

$$f_\varepsilon : [0, T] \longrightarrow A, \qquad f_\varepsilon(t) := \Sigma_{[t/\varepsilon]},$$

and with an abuse of terminology, we call f_ε "Euler scheme" too, when there will be no risk of confusion.

While definitions of A_l, A and Voronoi cell imply a dependence on the domain Ω, we will omit it when there is no risk of confusion.

We define now what we mean with "branching behavior":

Definition 1.2. *Given a domain Ω, a positive time $T > 0$, a time step $\varepsilon > 0$, consider the Euler scheme*

$$f_\varepsilon : [0, T] \longrightarrow A, \qquad f_\varepsilon(t) := \Sigma_{[t/\varepsilon]}.$$

Then we will say f_ε exhibits a branching behavior at time $t_0 \in [0, T]$ if there exists $Q \in f_\varepsilon(t_0)$ such that for any $t > t_0$ there exists $R(t) > 0$ such that for any $r < R(t)$ $B(Q, r) \cap f_\varepsilon(t_0) \backslash \{Q\}$ has connected components $\{C_i\}_{i \in I}$ while $B(Q, r) \cap f_\varepsilon(t) \backslash \{Q\}$ has at least 3 connected components and at least one connected components not included in $\{C_i\}_{i \in I}$.

Now we present an easy result on F:

Proposition 1.3. *Given a domain Ω, for any $S_1, S_2 \in A$, with $S_1 \subseteq S_2$, we have $F(S_1) \geq F(S_2)$.*

Proof. The proof is easy: as $S_1 \subseteq S_2$, for any $x \in \Omega$ we have

$$\text{dist}(x, S_1) \geq \text{dist}(x, S_2)$$

and integrating on Ω

$$\int_\Omega \text{dist}(x, S_1) dx \geq \int_\Omega \text{dist}(x, S_2) dx$$

which concludes the proof. $\square$

Moreover we see from the proof of Proposition 1.3 that if there exists $\Omega' \subseteq \Omega$ with $\mathcal{L}^2(\Omega') > 0$ and $\mathrm{dist}(x, S_1) > \mathrm{dist}(x, S_2)$ for any $x \in \Omega'$, then $F(S_1) > F(S_2)$. So for any $h > 0$

$$\min_{\mathcal{X} \in A_h} F(\mathcal{X}) = \min_{\mathcal{H}^1(\mathcal{X}')=h} F(\mathcal{X}')$$

and

$$\mathrm{argmin}_{A_h} F = \mathrm{argmin}_{A_h \setminus \bigcup_{0 \leq h' < h} A_{h'}} F.$$

which effectively allows us to use these constraints indifferently.

Our goal is to investigate the topological behavior of the evolution process, as it is not clear whether the optimal set changes topology or not. This doubt arises from the following fact: given a generic $l \geq 0$, for *most* sets $S \in \mathrm{argmin}_{A_l} F$ (further explanation can be found in Proposition 3.1), adding a small piece of curve at an endpoint of its is better than adding it elsewhere, and a closed path is never present in any optimal set (for references, see [6], [7] and [8]). But as we will see in the following sections, there are some configurations in which this kind of argument fails and definitely a branching behavior will arise.

This paper will be structured as follows: in Section 2 we will present some results of optimal transport in presence of Dirichlet regions, in particular the geometrical properties of optimal sets; in Section 3 we will prove our main results; in Section 4 we will present some explicit situations, and we will estimate the time at which a triple (or multiple) point will arise.

Notations

The most used in this paper will be:

- Ω to denote the domain,
- ε, δ, r, ρ to denote small positive number,
- l to denote generic positive number,
- S to denote generic connected compact sets in the domain,
- S_0 to denote the initial datum of an evolution,
- $w(k, \cdot)$, $w(k)$ ($k \in \mathbb{N}$) to denote the $(k+1)$-th set of an Euler scheme.

To avoid using excessive number of different notations, some symbols will be used in more situations: unless explicitly specified, if a notation is used in two different Definitions/ Propositions/ Lemma/ Theorems, there is no connection between them, so there is no risk of confusion.

The only notable exceptions are

- A_l (with $l \geq 0$), and A: if there is a given domain Ω, they always denote the sets defined after (1.1),
- F which always stands for the average distance functional

- $V(\cdot)$ which stands for the Voronoi cell of the point.

We will work only domains in $\mathbb{R}^2$ which are closure of a sufficiently regular bounded open connected set. Moreover, when we will write $F(\mathcal{X}_1 \cup \mathcal{X}_2)$ (with $\mathcal{X}_1, \mathcal{X}_2 \in A$), we will assume implicitly that $\mathcal{X}_1 \cup \mathcal{X}_2 \in A$, and exclude every other case.

2 Preliminaries

In this section we present some results concerning the geometrical properties of optimal sets. The proofs we give here are somewhat essential, and for accurate details we refer to [6], [7] and [8].

The following two results are about the regularity of optimal sets in the static case.

Proposition 2.1. *Let be Ω a given domain, $l > 0$ a fixed quantity, and $\Sigma_{opt} \in \mathrm{argmin}_{A_l} F$. Then Σ_{opt} cannot contain a loop (a subset homeomorphic to S^1).*

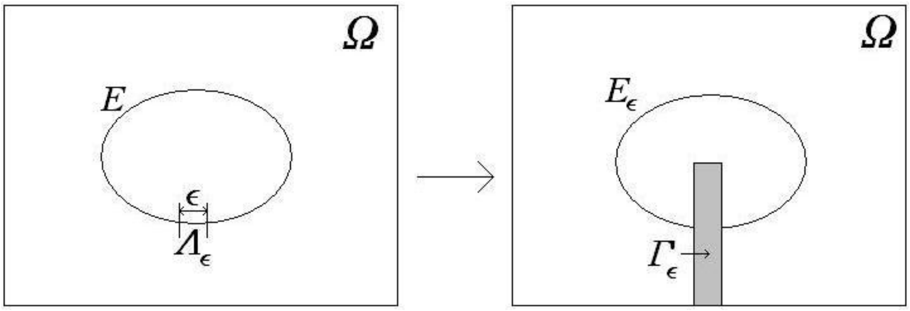

Fig. 1: This is a simple representation of what happens if we remove the portion Λ_ε.

Proof. Suppose that Σ_{opt} contains as subset E homeomorphic to S^1. If we remove the portion Λ_ε from E ($\mathcal{H}^1(\Lambda_\varepsilon) = \varepsilon > 0$), setting $E_\varepsilon := E \backslash \Lambda_\varepsilon$ we have that all the "loss" is concentrated on Γ_ε (the shaded region in Figure 1, which has area no larger than $\varepsilon \mathrm{diam}(\Omega)$), as points belonging to the rest will not change their distance to Σ_ε. For the points in Γ_ε their path can be longer, but it is clear from triangle inequality

$$\mathrm{dist}(x, E_\varepsilon) \leq \mathrm{dist}(x, E) + \mathcal{H}^1(\Lambda_\varepsilon)$$

so we have

$$\int_\Omega \mathrm{dist}(x, \Sigma_\varepsilon) dx \leq \int_\Omega \mathrm{dist}(x, \Sigma) dx + \varepsilon \mathcal{L}^2(\Gamma_\varepsilon)$$

thus the "loss" in energy after removing Λ_ε is upper bounded by $\varepsilon^2 \mathrm{diam}\Omega$. Proposition 2.4, which estimates from below the "gain" in energy by adding such a portion Λ_ε to Σ_{opt}, will conclude the proof. □

Proposition 2.2. *Let Ω be a given domain, $l > 0$ a fixed quantity, and let be $\Sigma_{opt} \in \mathrm{argmin}_{A_l} F$. Then Σ_{opt} cannot contain a cross (a subset homeomorphic to $\{x^2 + y^2 \leq 1 : xy = 0\}$).*

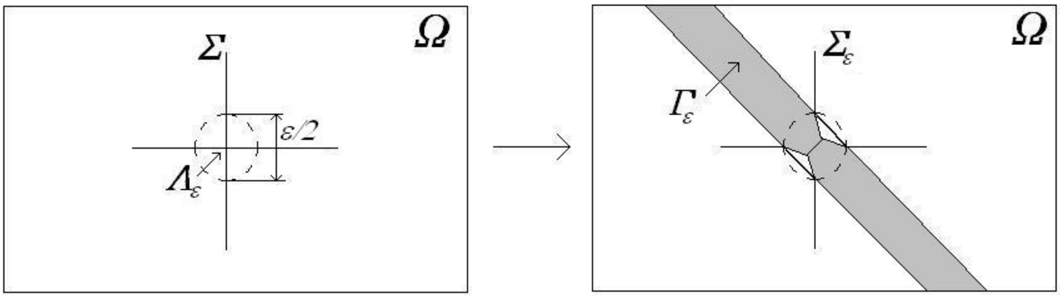

Fig. 2: Σ_ε is obtained from Σ_{opt} by replacing the infinitesimal cross Λ_ε with a slightly shorter Steiner graph.

Proof. Suppose that Σ_{opt} contains as cross Λ_ε ($\mathcal{H}^1(\Lambda_\varepsilon) = \varepsilon > 0$). If we remove the portion Λ_ε from Σ_{opt}, and replacing it with a Steiner graph Z_ε (a direct computation yields the existence of $k > 0$ such that $\mathcal{H}^1(Z_\varepsilon) < k\varepsilon$) in order to keep the connection property, setting $\Sigma_\varepsilon := \Sigma_{opt}\backslash\Lambda_\varepsilon$ we have that all the "loss" is concentrated on Γ_ε (the shaded region in Figure 2, which has area no larger than $\varepsilon\mathrm{diam}(\Omega)$), as points belonging to the rest will not change their distance to Σ_ε. For the points in Γ_ε their path can be longer, but it is clear from triangle inequality

$$\mathrm{dist}(x, \Sigma_\varepsilon) \leq \mathrm{dist}(x, \Sigma_{opt}) + \mathcal{H}^1(\Lambda_\varepsilon)$$

so we have

$$\int_\Omega \mathrm{dist}(x, \Sigma_\varepsilon)dx \leq \int_\Omega \mathrm{dist}(x, \Sigma_{opt})dx + \varepsilon\mathcal{L}^2(\Gamma_\varepsilon)$$

and the "loss" in energy after removing Λ_ε is upper bounded by $\varepsilon^2\mathrm{diam}\Omega$. Again, Proposition 2.4, which estimates from below the "gain" in energy by adding such a portion δ_ε whose length is $\mathcal{H}^1(\Lambda_\varepsilon) - \mathcal{H}^1(Z_\varepsilon) = O(\varepsilon)$ to Σ_{opt}, will conclude the proof. □

The next estimate is a very important one, and widely used in this paper. Some definitions are required first.

Definition 2.3. *Given a domain Ω, $S \in A$ a generic element, a non endpoint $P \in S$ is "smooth" if:*

- *there exists $r > 0$ such that there exists an homeomorphism $f : B(P, r) \cap S \longrightarrow (0, 1)$,*
- *for any sequences $\{X_n\} \subset S$, $\{X_n\} \to P$, the sequence $\{\widehat{X_n P X_{n+1}} \wedge (2\pi - \widehat{X_n P X_{n+1}})\}$ accumulates in $\{0, \pi\}$, and there are sequences $\{X_n^{(1)}\} \subset S$, converging to P such that the sequence $\widehat{X_n^{(1)} P X_{n+1}^{(1)}}$ accumulates in π.*

A subset of S is smooth is all its non endpoints are smooth.

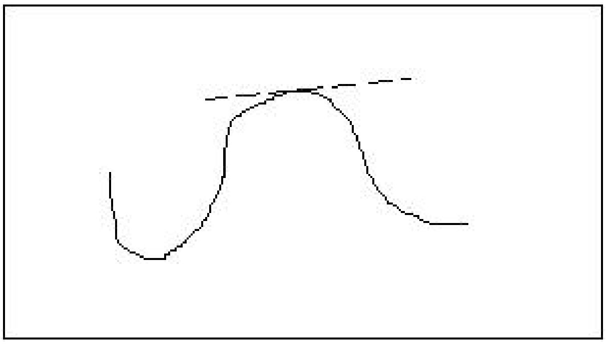

Fig. 3: This is an example of smooth point.

Proposition 2.4. *Given a domain Ω, let be $S \subset \Omega$ be a connected set, if we add a segment λ_ε to a smooth non endpoint of S (with $\mathcal{H}^1(\lambda_\varepsilon) = \varepsilon$), then the "gain" $F(S) - F(S_\varepsilon)$ is at least comparable with $\varepsilon^{3/2}$, where $S_\varepsilon := S \cup \lambda_\varepsilon$.*

Proof. Upon scaling, the configuration can be brought to the following in Figure 4, so all the computations can be done here.

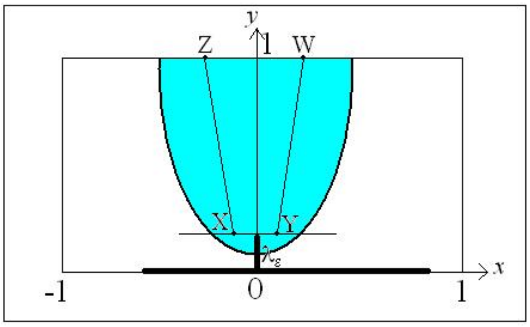

Fig. 4: This is how the scaled case look.

If a point (x, y) can gain in distance, i.e. verifies

$$\text{dist}((x, y), S \cup \lambda_\varepsilon) < \text{dist}((x, y), S)$$

thus

$$(x^2 + (y - \varepsilon)^2)^{1/2} < |y|,$$

this leads to

$$y > \frac{x^2}{2\varepsilon} + \frac{\varepsilon}{2},$$

the shaded parabola of Figure 4.

We have to estimate its area: as we are working in $[-1, 1] \times [0, 1]$, the intersections between $\{(x, y) : y = \frac{x^2}{2\varepsilon} + \frac{\varepsilon}{2}\}$ and $[-1, 1] \times \{1\}$ are

$$x^\pm := \pm\sqrt{2\varepsilon - \varepsilon^2}.$$

We do not need to compute exactly the area of this parabola, it suffices to estimate it. Putting Z, W midpoints between $(0,1)$ and x^-, x^+ respectively, X, Y midpoints between $(0,\varepsilon)$ and $(-\varepsilon,\varepsilon)$, $(\varepsilon,\varepsilon)$ respectively, parabola contains the trapezium $XYWZ$, and $\mathcal{H}^1(XY)=\varepsilon$, $\mathcal{H}^1(WZ)=\sqrt{2\varepsilon-\varepsilon^2}$ and the height is $1-\varepsilon$. The gain in path here is at least $\varepsilon/2$ (this minimum is attained on points X and Y), so the gain for the energy functional is at least

$$\frac{\varepsilon}{2}\frac{1-\varepsilon}{2}(\varepsilon+\sqrt{2\varepsilon-\varepsilon^2})\geq\frac{\varepsilon^{3/2}}{8},$$

and the proof is complete. □

This result can be generalized: it is enough that a point $Q\in S$ verifies

(A) there exists $r>0$ such that there exists an homeomorphism $f: B(Q,r)\cap S\longrightarrow(0,1)$,

(B) there exists $r'\leq r$ and $Q_1, Q_2\in B(Q,r')$ such that the non trivial triangle with vertexes in Q_1, Q, Q_2 verifies $Q_1QQ_2\cap S=\{Q\}$.

The proof is identical to that of Proposition 2.4, with different constants.

For smooth points the gain in energy has exactly order $O(\varepsilon^{3/2})$ (as the area of that parabola in Figure 4 is upper bounded by $2\sqrt{2\varepsilon}$), and this argument is applicable for any points verifying (A) and (B) except for the angular points (see Definition 3.6), for which the gain in energy has order $O(\varepsilon)$.

3 Results

In this section we present the main result of our paper, i.e. conditions sufficient to force a branching behavior for Euler schemes with sufficiently small time step, and an upper bound estimate of the time at which the set must change topology under a particular conditions.

The next result is crucial to the purposes of this paper, and it is an estimate (from below) for the gain in energy when endpoints with non negligible Voronoi center are present.

Proposition 3.1. *Given a domain Ω, let $S\in\mathcal{A}$, and let it have a point O satisfying:*

$(*)$ *there exists $\xi>0$ such that $S\cap B(O,\xi)$ is contained in the circular sector with center O and arc $Q'R'$, with $\widehat{Q'OR'}:=\beta<\pi$ (see Figure 5-I).*

Then we have:

(1) there exist $\rho>0$ and $\theta>0$ and a isosceles triangle $T'\subset V(O)$ with a vertex in O, two sides with length ρ and angle in O measuring θ, that does not intersect S,

(2) there exists ε_0 such for any $\varepsilon<\varepsilon_0$ adding a segment λ_ε at O, with $\mathcal{H}^1(\lambda_\varepsilon)=\varepsilon$ in O leads to a gain for the energy functional comparable with $O(\varepsilon)$.

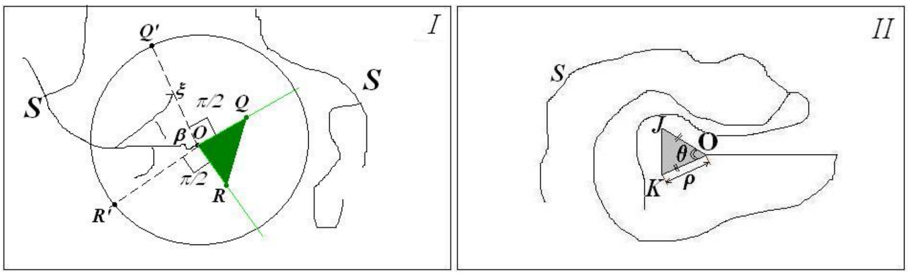

Fig. 5-I: condition $(*)$ is enough to guarantee the existence of a triangle $OQR \subset V(O)$, with sides $OQ = OR = \xi/2$ and $\widehat{ROQ} = \pi - \beta$

Fig. 5-II: the presence of the shaded triangle T' makes adding at this point more convenient than at a smooth non endpoint at least when the added portion has sufficient small length.

Proof. Let us analyze the two statements separately.

(1) The whole configuration looks like that in Figure 5-I: let us consider the triangle OQR, with $OQ = OR = \xi/2$. It is contained in $V(O)$, as given any point $X \in OQR$, $\mathrm{dist}(X,O) \leq \xi/2$, while $\mathrm{dist}(X, \partial B(O,\xi)) \geq \xi/2$, and the shortest path from X to the circular sector containing $(B(O,\xi) \cap S)\backslash\{O\}$ must cross OQ' or OR', and $\mathrm{dist}(X, OQ') = \mathrm{dist}(X, OR') = \mathrm{dist}(X, O)$. So choices $\rho := \xi/2$ and $\theta := \pi - \beta$ are sufficient to prove the first statement.

(2) Adding a straight segment λ_ε at O and in the triangle JOK along the direction of bisector of angle $\widehat{JOK}$ (see Figure 5-II), with ε small enough, then all points on $JKK'J'$ (where J', K' are midpoints of segment OJ and OK) will have a gain in path to S at least

$$\frac{\rho}{2} - \sqrt{\frac{\rho^2}{4} - \frac{\varepsilon\rho}{2}\cos\frac{\theta}{2}} \approx \varepsilon \cos\frac{\theta}{2} - O(\varepsilon^2) \tag{3.1}$$

as this is the gain of points on OJ and OK , and points inside gain even more. But it is clear that for ε small enough the higher order term $O(\varepsilon^2)$ becomes negligible compared to $\varepsilon\cos\frac{\theta}{2}$, thus (when ε is sufficiently small) we will have $O(\varepsilon^2) \leq \frac{\varepsilon}{2}\cos\frac{\theta}{2}$. So all points of triangle JOK gain in path at least $\frac{\varepsilon}{2}\cos\frac{\theta}{2}$, thus the gain in energy is not lower than

$$\frac{\varepsilon}{2}\cos\frac{\theta}{2}\mathcal{L}^2(JOK)$$

and as $\mathcal{L}^2(JOK) = \rho^2 \sin\frac{\theta}{2}\cos\frac{\theta}{2}$,

$$F(S \cup \lambda_\varepsilon) \leq F(S) - \frac{\varepsilon}{2}\cos\frac{\theta}{2}\mathcal{L}^2(JOK) = F(S) - \frac{\varepsilon}{2}\rho^2 \sin\frac{\theta}{2}\cos^2\frac{\theta}{2}$$

and the proof is complete.

□

3.1 Changing topology

Now we investigate all the situations that may appear during the evolution. Given an initial datum $S_0 \in A$, a positive time $T > 0$, a time step $\varepsilon > 0$, consider

$$\begin{cases} w(0) := S_0 \\ w(k) \in \text{argmin}_{\mathcal{H}^1(\mathcal{X}') \leq \mathcal{H}^1(S_0)+k\varepsilon,\ w(k-1) \subseteq \mathcal{X}'} F(\mathcal{X}') \end{cases} .$$

Putting

$$\Sigma : [0, T] \longrightarrow A, \qquad \Sigma(t) := w([t/\varepsilon]),$$

at any time $T_0 \in [0, T]$, the following (obvious) dichotomy is possible:

(1) $\Sigma(T_0)$ does not branch,

(2) $\Sigma(T_0)$ exhibits a branching behavior.

In order to provide an upper bound to the branching time, we need to establish when choice (2) becomes necessary preferable to choice (1).

As $\Sigma(t)$ is connected for any t, in spite of Proposition 3.1, possible ways are either to admit the existence non endpoints verifying its hypothesis, or negate the existence of endpoints, or negate the existence of endpoints verifying condition $(*)$. The last reads:
"for any endpoint O', for any $\rho, \theta > 0$, for all triangles with a vertex in O' and sides $\rho, \rho, \rho\sqrt{2 - 4\cos\theta}$ the set $\Sigma(t)$ intersects that triangle".

Let us try with this possibility first.

These tools will be used:

Definition 3.2. *Let S be a compact connected set in a given domain Ω,, $P \in S$ a point, and a positive value $\mathcal{R} > 0$. The "the inner radial projection" is the function*

$$\pi_{P,\mathcal{R}} : B(P, \mathcal{R}) \longrightarrow \partial B(P, \mathcal{R}), \qquad \pi_{P,\mathcal{R}}(x) := \partial\overline{B(P, \mathcal{R})} \cap Px$$

where Px denotes the halfline starting from P and passing through x.

The above function is useful to define the equivalent of a loop:

Definition 3.3. *Given a domain Ω, let be Γ a curve, a subset $\gamma \subseteq \Gamma$ is "general loop" around a point $Q \in \Omega$ if it is a closed connected set satisfying:*

(1) There exists a $\mathcal{R}'$ for which $\gamma \subseteq B(Q, \mathcal{R}')$ and $\pi_{Q,\mathcal{R}'}(\gamma \cap B(Q, \mathcal{R}')) = \partial B(Q, \mathcal{R}')$;

(2) No connected proper subsets of γ satisfies the first condition.

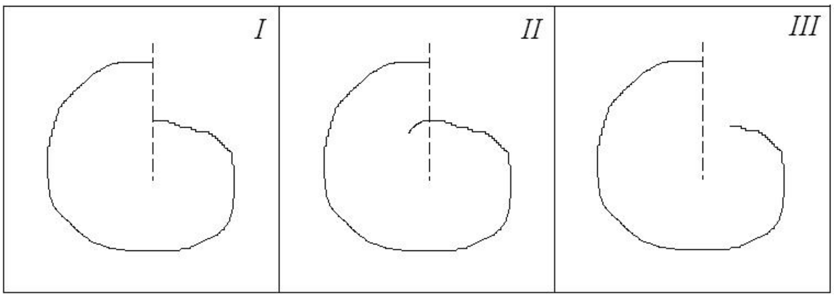

Fig. 6: I is an example of general loop, while II and III are not.

Notice that in the above definitions $B(P, \mathcal{R}')$ is not necessary in the domain, and the exact value $\mathcal{R}'$ in condition (1) is not relevant, and if the condition holds for a suitable $\mathcal{R}_0$, then it holds for every $\mathcal{R} > \mathcal{R}_0$.

Using the above notations, we introduce the following tool:

Definition 3.4. *Given a domain Ω, let be Γ a curve, $P \in \Gamma$ an endpoint, and suppose that there exist a sequence $\{\rho_n\}_{n=0}^{\infty}$ with $\rho_n \downarrow 0$ such that for any n $\pi_{P,\rho_n}(\Gamma \cap B(P, \rho_n)) = \partial B(P, \rho_n)$. Then if there exists a partition of Γ, namely $\Gamma = \bigcup_{n=0}^{\infty} L_n$, $P \notin L_n$ for any n, such that:*

- *for every n, $\partial B(P, \rho_n)$ contains the farthest point of L_n from P,*
- *for every n, $\partial B(P, \rho_{n+1})$ contains the closest point of L_n from P,*

then we call the sequence $\{\rho_n\}_{n=0}^{\infty}$ "distance sequence" for P.

Notice that from the above definition it is not necessary that every endpoint has a distance sequence, and even if it had one, this is not unique, as if $\{\rho_n\}_{n=0}^{\infty}$ is a distance sequence for an endpoint P, then for any positive integer M_0 $\{\rho_n\}_{n=M_0}^{\infty}$ is a distance sequence for P too.

Now we can present a condition on the branching behavior, for the discrete case first:

Proposition 3.5. *Given a domain Ω, let be $S_0 \in A$ (A defined just after (1.1)) be the initial datum of the following class of Euler schemes (ε' is the time step, a free variable parameter)*

$$\begin{cases} w(0) := S_0 \\ w(k) \in \mathit{argmin}_{\mathcal{H}^1(\mathcal{X}') \leq \mathcal{H}^1(S_0) + k\varepsilon',\ w(k-1) \subseteq \mathcal{X}'} F(\mathcal{X}') \end{cases} .$$

Then, a branching behavior occurs at the very beginning for Euler schemes with sufficiently small time step if the following condition is satisfied:

$(**)$ *any endpoint $P' \in S_0$ has a distance sequence $\{\rho_n^{(P')}\}_{n=0}^{\infty}$ and a constant $Wr(P')$ which verifies*

$$\limsup_{n \to \infty} \log_{\rho_n^{(P')}} \rho_{(n+1)}^{(P')} \leq Wr(P') < 2.$$

Moreover, there exists a constant $r > 0$, depending only on geometric quantities, such that the branching happens outside $B(P, r)$ for ε' sufficiently small.

Proof. We assume first that $\Sigma(0) = S_0$ has an unique endpoint P.

Let us analyze what happens if we add some set $J_{\varepsilon'}$ (with length $\varepsilon' > 0$ small) at P: we have to estimate the gain for the energy. As $J_{\varepsilon'} \subset B(P, \varepsilon')$, the gain is upper bounded by the quantity

$$\varepsilon' |B(P, \rho^{(P)}_{m(\varepsilon')-2})|$$

where $\rho_{m(\varepsilon')-2}$ will be explained in the following.

As the point P satisfies condition $(**)$, there exists a maximum $m(\varepsilon')$ for which $\rho^{(P)}_{m(\varepsilon')} \leq \varepsilon' < \rho^{(P)}_{m(\varepsilon')-1}$, so the total gain can be estimated by $\pi(\rho^{(P)}_{m(\varepsilon')-2})^2 \varepsilon'$, and as

$$\rho^{(P)}_{m(\varepsilon')} \leq \varepsilon' < \rho^{(P)}_{m(\varepsilon')-1} < \rho^{(P)}_{m(\varepsilon')-2}$$

the logarithmic condition in $(**)$ gives

$$\varepsilon'^2 < (\rho^{(P)}_{m(\varepsilon')-2})^2 \leq (\rho^{(P)}_{m(\varepsilon')})^{1/Wr(P)} \leq \varepsilon'^{1/Wr(P)} = o(\varepsilon'^{1/2})$$

and the total gain is an $O(\varepsilon'^{1+1/Wr(P)}) = o(\varepsilon'^{3/2})$.

Adding $J_{\varepsilon'}$ in this way is not optimal for ε' sufficiently small.

But if we add $J_{\varepsilon'}$ at a non endpoint, it can gain $O(\varepsilon'^{\frac{3}{2}})$ (see Proposition 2.4). Moreover, if both $\mathrm{dist}(P, J_{\varepsilon'})$ and ε' are small enough (i.e. every point of $J_{\varepsilon'}$ is close to P), then there exists ρ^* (dependent on $\mathrm{dist}(P, J_{\varepsilon'})$ and ε') and a general loop G_{ρ^*} such that $J_{\varepsilon'} \subset conv(G_{\rho^*}) \subset B(P, \rho^*) \subset conv(G_{\rho^*-1})$. Thus the gain in energy would be smaller than $\varepsilon'^{3/2}\alpha(G_{\rho^*-1})$, where $\alpha(G_{\rho^*-1})$ is a value that depends only on the geometry of the general loop G_{ρ^*-1} and goes to 0 as the area of G_{ρ^*-1} goes to 0. So if we let both $\mathrm{dist}(P, J_{\varepsilon'})$ and ε' go to 0, the maximum ρ^* goes to ∞, thus the gain in energy becomes $\varepsilon'^{3/2}$ multiplied by a factor going to 0.

As $\mathcal{H}^1(\Sigma_{\varepsilon'}(T)) \leq \mathcal{H}^1(S_0) + T + 1$, there will always be point $Z \in S_0$, such that adding $J_{\varepsilon'}$ at Z leads to a gain in energy not lower than $d\varepsilon'^{3/2}$, with d dependent only on geometric quantities (of S_0 near Z).

So even adding positive (small) length too close to the unique endpoint P is not optimal: this means that no set in any Euler scheme (with small time step) can add length too close to P, and a branching behavior appears at the very beginning (the very first step) of this evolution.

This argument can be generalized to S_0 having more endpoints (by applying it to all endpoints of S_0), so the proof is complete. □

3.2 Frequent visiting

In the previous subsection we have given a condition sufficient to force a branching behavior, but it looks quite artificial, and its branching is at the very first step. Here we try to put a weaker one, by exploiting the existence of non endpoints with not negligible Voronoi cell, and by estimating the required "free space" (i.e. the minimum value for Voronoi cells of its endpoints) to evolve without changing topology.

The choice of adjective "frequent" will be clear at the end of this section, in Theorem 3.8.

Proposition 2.4 is too weak, as the gain obtained in that way has order $O(\varepsilon^{3/2})$ for ε small enough. Something stronger is required.

We introduce a new class of points:

Definition 3.6. *Given a domain Ω, $S \in A$ a generic element, a non endpoint $P \in S$ is "angular" if there exists $r > 0$ and $\theta < \pi$ such that:*

(1) there exists an homeomorphism $f : B(P, r) \cap S \longrightarrow (0, 1)$;

(2) given any sequence $\{P_n\}_{n=0}^{\infty} \subset S$ converging to P, the sequence $\widehat{P_nPP_{n+1}} \wedge (2\pi - \widehat{P_nPP_{n+1}})$ accumulates in $\{0, \theta\}$.

An angular point can be imagined as a point around which the tangent vector form an angle or a cuspid.

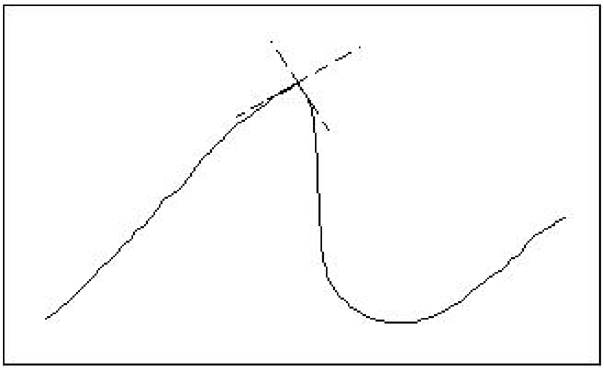

Fig. 7: This is an example of angular point, with the dashed lines indicating tangent directions.

This class of point is important for the following result:

Lemma 3.7. *Given a domain Ω, let $S \in A$ be an arbitrary element, and suppose there exists $Q \in S$. The Q verifies condition $(*)$ of Proposition 3.1.*

Proof. Introduce a local coordinate system as in Figure 8:

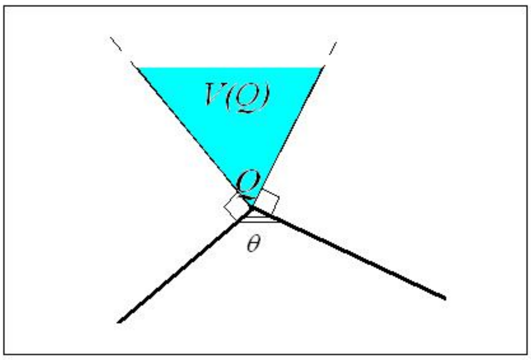

Fig. 8: All the points in the shaded area belong to $V(Q)$, and it contains a triangle.

As Q is angular, we have that by definition there exists $\theta < \pi$ given any sequence $\{Q_n\}_{n=0}^{\infty} \subset S$ converging to Q the sequence $\widehat{Q_nQQ_{n+1}} \wedge (2\pi - \widehat{Q_nQQ_{n=1}})$ accumulates in $\{0, \theta\}$. We claim that there exists $\rho > 0$ such that all points belonging to $S \cap B(Q, \rho)$ are contained in the circular sector (of $B(Q, \rho)$) with center Q and central angle $\dfrac{\theta + \pi}{2}$.

This can be translated by saying that there are at least two sequences $\{X_n\}_{n=0}^{\infty}$, $\{Y_n\}_{n=0}^{\infty}$ in S converging to Q such that the angle between vectors $v(X_nQ)/||v(X_nQ)||$ ($v(X_nQ)$ denotes the vector starting in X_n and pointing towards Q) and $v(Y_nQ)/||v(Y_nQ)||$ tend to be θ for $n \to \infty$. Now if condition $(*)$ has to fail, then there exists $r_n \downarrow 0$ such that for any n there exists a Q_n not in the circular sector (of $B(Q, r_n)$) with center Q and central angle $\dfrac{\theta + \pi}{2}$.

Now it is clear that $Q_n \to Q$, and (upon passing to subsequences) $\{Q_n\}_{n=0}^{\infty}$ converges to Q along a third direction, different from the previous two. So by definition of angular point, if $\theta \neq \dfrac{2}{3}\pi$, the proof is complete.

We have to analyze the last case, when $\{X_n\}_{n=0}^{\infty}$, $\{Y_n\}_{n=0}^{\infty}$ and $\{Q_n\}_{n=0}^{\infty}$ converge to Q and the the value of angle formed between then ($\widehat{X_nQY_n}$, $\widehat{Q_nQY_n}$, $\widehat{X_nQQ_n}$) accumulates all to $\dfrac{2}{3}\pi$.

In this case, as $B(Q, r) \cap S$ is homeomorphic to $(0, 1)$ (by definition of angular point), it cannot contain more than one connected component, thus Q_n must be connected frequently to X_n or Y_n by a path γ_r (and as Q has multiplicity 2, we can assume that Q_n is connected frequently to X_n by a path γ_r not passing in Q). As this is valid for arbitrary small r, we can consider a sequence $r_n \downarrow 0$ and pathes γ_{r_n}: $\gamma_{r_n} \subset B(Q, r_n)$, thus in the Hausdorff metric $\gamma_{r_n} \to Q$, thus there must be another sequence $\{Z_n\}_{n=0}^{\infty}$ which converges to Q along a direction different from that of $\{X_n\}_{n=0}^{\infty}$, $\{Y_n\}_{n=0}^{\infty}$, $\{Q_n\}_{n=0}^{\infty}$, and the proof is complete.

$\square$

The next result is a condition required to avoid branching behaviors for Euler schemes.

Theorem 3.8. *Given a domain Ω, let $S_0^{(1)} \in A$ be a generic element, T a positive time and $\varepsilon > 0$ a (small) positive time step, let us consider the Euler scheme*

$$\begin{cases} w(0) := S_0^{(1)} \\ w(k) \in \mathit{argmin}_{\mathcal{H}^1(\mathcal{X}') \leq \mathcal{H}^1(S_0^{(1)}) + k\varepsilon,\ w(k-1) \subseteq \mathcal{X}'} F(\mathcal{X}') \end{cases}$$

in the time interval $[0, T]$.

Suppose that there exist $P_0 \in S_0^{(1)}$ angular and $\eta > 0$ such that $B(P_0, \eta) \cap (w(k) \backslash w(0)) = \emptyset$ for any k. Then there is an upper bound $T_{\max}^{\varepsilon}$ such that $T > T_{\max}^{\varepsilon}$ causes a branching behavior.

Proof. As P_0 is angular, Lemma 3.7 affirms that Proposition 3.1 is verified for some positive ρ, θ. So from the estimate of Proposition 3.1 there is a constant $K(P_0) > 0$ (depending only on ρ, θ and not on ε) such that for any j

$$\min_{\mathcal{H}^1(\mathcal{X}') \leq \mathcal{H}^1(w(j-1)) + v\varepsilon,\ w(j-1) \subseteq \mathcal{X}'} F(\mathcal{X}') \leq F(w(j-1)) - K(P_0)\varepsilon,$$

as this gain is achieved by simple adding a segment $Seg_\varepsilon \subset T_P$ ($\mathcal{H}^1(Seg_\varepsilon) = \varepsilon$) along the bisector of $\hat{P}_0$, which would create a branching behavior.

In order to avoid this, for any d, $w(d)$ must be obtained from $w(d-1)$ by adding length at points of $ext(w(d-1))$, and the gain in energy must be more than $K(P_0)\varepsilon$, i.e.

$$F(w(d)) \leq F(w(d-1)) - K(P_0)\varepsilon \qquad \forall d = 1, \cdots, \left[\frac{T}{\varepsilon}\right]$$

which leads to

$$F(w(d)) \leq F(w(0)) - dK(P_0)\varepsilon \qquad \forall d = 1, \cdots, \left[\frac{T}{\varepsilon}\right]$$

and finally, for $d = \left[\frac{T}{\varepsilon}\right]$,

$$F(\left[\frac{T}{\varepsilon}\right]) \leq F(w(0)) - \left[\frac{T}{\varepsilon}\right] K(P_0)\varepsilon.$$

As $\frac{T}{\varepsilon} - 1 \leq \left[\frac{T}{\varepsilon}\right] \leq \frac{T}{\varepsilon}$, this leads to

$$0 \leq F(\left[\frac{T}{\varepsilon}\right]) \leq F(w(0)) - (T-\varepsilon)K(P_0),$$

which forces

$$T \leq \varepsilon + \frac{F(S_0^{(1)})}{K(P_0)}$$

and putting $T_{\max}^{\varepsilon} := \varepsilon + \frac{F(S_0^{(1)})}{K(P_0)}$ completes the proof. □

Now some word about the use of adjective “frequent”: indeed Theorem 3.8 states that if some angular point has a neighbor never visited by the evolving set, than soon or later a branching will arise in this point, i.e. for any angular point the latter must pass frequently close to it to avoid branching behaviors.

4 Examples

In this section we give two examples of branching behavior, and two ways to estimate this.

4.1 Energy estimate

In Theorem 3.8 we have given an upper bound estimate for the branching time under that particular configuration: now we present an explicit example.

In order to apply this result, its hypothesis must be verified: so given a domain Ω, let S_0^{ini} be the initial datum, and suppose there exist $P_0 \in S_0^{ini}$ angular and $\xi > 0$ such that $B(P_0, \xi) \cap S_0^{ini}$ is homeomorphic to $(0, 1)$.

Moreover, we must ensure that this ball $B(P_0, \xi)$ is not visited, and one way to do this is imposing that any visiting here must cause a branching behavior. So we choose a particular S_0^{ini}.

Suppose that

- there exist $P_0 \in S_0^{ini}$ angular and let be $\xi > 0$ such that $B(P_0, \xi) \cap S_0^{ini}$ is homeomorphic to $(0, 1)$;
- there exist a closed injective path $\gamma : [0, 1] \longrightarrow \Omega$ such that $\gamma([0, 1]) \subseteq S_0^{ini}$: the domain Ω is now divided in two regions, Ω^+ and Ω^- with $\Omega = \Omega^+ \cup \Omega^-$ (they are the two connected components of $\Omega \backslash \gamma([0, 1])$, and they correspond to the "interior" and the "exterior" part of $\gamma([0, 1])$ – the order is not relevant – as given by the Jordan Curve Theorem);
- triangle $T_{P_0} \subset V(P_0) \cap B(P_0, \xi)$ (whose existence is given by Lemma 3.7) verifies $|T_{P_0} \cap \Omega^+| > 0$, and $ext(S_0^{ini}) \subset \Omega^-$.

The main estimate we are going to present here is Theorem 4.2, whose proof requires a series of preliminary lemma.

In the rest of this subsection we will suppose that Ω^- is large enough (both in diameter and in measure) so that all computations can be done without considering constraints imposed by $\text{diam}(\Omega^-)$, $|\Omega^-|$ (otherwise a branching behavior is exhibited even sooner).

Consider

$$\begin{cases} w(0) = w(0) := S_0^{ini} \\ w(k) \in \text{argmin}_{\mathcal{H}^1(\mathcal{X}') \leq \mathcal{H}^1(S_0^{ini}) + k\varepsilon} F(\mathcal{X}') \end{cases},$$

$$\Sigma_\varepsilon(t) := w\left(\left[\frac{t}{\varepsilon}\right]\right).$$

The notations introduced (except mute counters) will have the same meaning in all this subsection.

Lemma 4.1. *If there exist k such that $(w(k) \backslash S_0^{ini}) \cap \Omega^+ \neq \emptyset$, but $(w(k-1) \backslash S_0^{ini}) \cap \Omega^+ = \emptyset$, this means $w(k)$ is not homeomorphic to $w(k-1)$.*

Proof. The hypothesis force $w(k) \cap \gamma([0, 1]) \neq \emptyset$, as $w(k) \cap \Omega^-$ and $w(k) \cap \Omega^-$ are both non empty.

Moreover at least one point $\gamma(s)$ ($s \in [0, 1]$) belonging to $w(k, n) \cap \gamma([0, 1])$ is connected by a path $\gamma' : [0, 1] :\longrightarrow (w(k) \cap \Omega^+) \cup \{\gamma(s)\}$ to $w(k) \cap \Omega^+$.

Removing the entire set $\gamma([0, 1])$ the following situation may arise:

- a new connected component (previously not present in $w(k-1) \backslash \gamma([0, 1])$) given by $(w(k) \backslash \gamma([0, 1])) \cap \Omega^+$ arises, thus $w(k)$ and $w(k-1)$ are not homeomorphic;
- the set $((w(k) \backslash \gamma([0, 1]))) \cap \Omega^+$ is not a new connected component, so there exists a subset $\sigma \subset w(k-1) \backslash \gamma([0, 1])$ such that σ and $((w(k) \backslash \gamma([0, 1]))) \cap \Omega^+$ are in the same connected component.

Let us analyze the second case: as $w(k-1)$ is connected, this would mean that there exists a sequence $\{O_n\} \subset \sigma$ converging to a point on $\gamma([0, 1])$, and as a similar sequence is present in $(w(k) \backslash \gamma([0, 1])) \cap \Omega^+$, definitely $\sigma \cup ((w(k) \backslash \gamma([0, 1])) \cap \Omega^+)$ is a new closed path connecting two points on $\gamma([0, 1])$, thus the homotopy class changes, and the proof is complete. $\square$

Now we can present an upper bound estimate for the branching time.

Theorem 4.2. *There exists a time $\bar{T}$ such that if $T > \bar{T}$, then there exists al least two sets in $\{\Sigma_\varepsilon(t)\}_{t\in[0,T]}$ which are not homeomorphic, and the branching time is not larger than $\bar{T}$.*

Proof. From Lemma 4.1 we see that for any k, $w(k)\backslash w(0)$ must be in $\Omega^- \cup \gamma([0,1])$, while $T_{P_0} \cap \Omega^+ \subseteq V(P_0)$ has positive measure, so for Theorem 3.8 we have that there exists a constant $K(P_0)$ such that it is not possible to evolve beyond time

$$\frac{F(S_0^{ini})}{K(P_0)}$$

without branching.

Now we estimate $K(P_0)$ from geometric quantities: we will use an argument similar to that found in the proof of Proposition 3.1. Let us call P_1, P_2 the other two vertex of T_{P_0}: by reducing the measure of the triangle we can suppose that $\mathcal{H}^1(P_0P_1) = \mathcal{H}^1(P_0P_2) := \rho$, and let be ϕ the value of $\widehat{P_1P_0P_2}$:

using the same argument in the proof of the second statement of Proposition 3.1, we have that the gain in energy is at least

$$\frac{\varepsilon}{2}\rho^2 \sin\frac{\phi}{2}\cos^2\frac{\phi}{2}$$

thus the choice

$$K(P_0) := \frac{1}{2}\rho^2 \sin\frac{\phi}{2}\cos^2\frac{\phi}{2}$$

is acceptable, upper bound estimate for the branching time is in this case

$$\bar{T} := \frac{F(S_0^{ini})}{K(P_0)} = \frac{2F(S_0^{ini})}{\rho^2 \sin\frac{\phi}{2}\cos^2\frac{\phi}{2}}.$$

□

The above methods relies on the fact that in this configuration there is a lower bound for the gain (for the functional F) at each step in each Euler scheme.

4.2 Geometric-energy estimate

Now we present a more stringent upper bound estimate for the branching time, arising from a finer estimate based on both geometrical arguments and energy considerations. The notations used in the previous subsection are null here.

Lemma 4.3. *Given a domain Ω, an element $S_1 \in A$, and suppose that there exists $Q \in \Omega$ and $R > 0$ such that the ball $B(Q,R) \cap S = \emptyset$. Then*

$$F(S_1) \geq \frac{4\pi R^3}{27}.$$

Proof. The proof is easy: as $B(Q,R)\cap S_1 = \emptyset$, for any $r < R$ all points $x \in B(Q,r)$ verify dist$(x, S_1) \geq R - r$, so

$$F(S_1) = \int_\Omega \text{dist}(x, S_1)dx \geq \int_{B(Q,r)} \text{dist}(x, S_1)dx \geq (R-r)\pi r^2.$$

Differentiating the expression $(R-r)\pi r^2$, its maximum value is attained by $r = \frac{2}{3}R$, which corresponds to

$$F(S_1) \geq \frac{4\pi}{27}R^3$$

and the proof is complete. □

Lemma 4.4. *Given a domain Ω, an element $S_2 \in A$, a point $Q' \in S_2$ and suppose that its Voronoi cell $V(Q')$ has $|V(Q')| > 0$. Then there exists $\bar{Q} \in \Omega$ such that $B(\bar{Q}, \frac{1}{2} diam(V(Q'))) \cap S_2 = \emptyset$.*

Proof. For $V(Q')$ we have $|V(Q')| \leq \frac{\pi}{4}\text{diam}(V(Q'))^2$. Let be $X_1, X_2 \in V(Q')$ points such that dist$(X_1, X_2) = \text{diam}(V(Q'))$:

$$\text{dist}(X_1, X_2) = \text{diam}(V(Q')) \leq \text{dist}(X_1, Q') + \text{dist}(Q', X_2)$$

so $\min\{\text{dist}(X_1, Q'), \text{dist}(Q', X_2)\} \geq \frac{1}{2}\text{diam}(V(Q'))$.

Assume that dist$(X_1, Q') \geq \frac{1}{2}\text{diam}(V(Q'))$: $X_1 \in V(Q')$ means for any $i < \frac{1}{2}\text{diam}(V(Q'))$, $B(X_1, i) \cap S_2 = \emptyset$ to avoid dist$(X_1, B(X_1, i) \cap S_2) \leq i < \frac{1}{2}\text{diam}(V(Q'))$.

So we can choose $\bar{Q} := X_1$, and considering that diam$(V(Q')) \geq \sqrt{\frac{4}{\pi}|V(Q')|}$, the proof is complete.

□

Now we consider a configuration similar to the one in the previous subsection:
given a domain Ω, let S_0^{dat} be the initial datum, and there exist

- $P_0' \in S_0^{dat}$ angular and let be $\xi' > 0$ such that $B(P_0', \xi') \cap S_0^{dat}$ is homeomorphic to $(0,1)$;
- a closed injective path $\gamma^* : [0,1] \longrightarrow \Omega$ such that $\gamma^*([0,1]) \subseteq S_0^{dat}$: the domain Ω is now divided in two regions, Ω^+ and Ω^- with $\Omega = \Omega^+ \cup \Omega^-$ (they are the two connected components of $\Omega \backslash \gamma^*([0,1])$, and they correspond to the "interior" and the "exterior" part of $\gamma * ([0,1])$ – the order is not relevant – given by the Jordan Curve Theorem);
- triangle $T_{P_0'} \subset V(P_0') \cap B(P_0', \xi')$ (its existence is given by Lemma 3.7) verifies $|T_{P_0'} \cap \Omega^+| > 0$, and $ext(S_0^{dat}) \subset \Omega^-$.

Notice that S_0^{dat} is very similar to S_0^{ini}, and results as Lemma 4.1 holds.

In the rest of this subsection we will suppose that Ω^- is large enough (both in diameter and in measure) so that all computations can be done without considering constraints imposed by diam(Ω^-), $|\Omega^-|$ (otherwise branching behaviors will be exhibited sooner).

Consider

$$\begin{cases} w(0) = w(0) := S_0^{dat} \\ w(k) \in \text{argmin}_{\mathcal{H}^1(S'') \leq \mathcal{H}^1(S_0^{dat}) + k\varepsilon'} F(S'') \end{cases},$$

$$\Sigma_{\varepsilon'} : [0,T] \longrightarrow A, \Sigma_{\varepsilon'}(t) := w\left(\left[\frac{t}{\varepsilon'}\right]\right).$$

The main estimate here is Theorem 4.5.

The notations introduced (except mute counters like k and n) will have the same meaning in the following of this subsection.

Again we have a positive constant $K(P_0')$ (depending only on geometric quantities, not on ε' and estimable with the same argument found in Theorem 4.2) such that for any k

$$\min_{\mathcal{H}^1(\mathcal{X}'') \leq w(k-1) + k\varepsilon',\ w(k-1) \subset \mathcal{X}''} F(\mathcal{X}'') \leq F(w(k-1)) - K(P_0')\varepsilon'$$

thus

$$F(w(k)) \leq F(w(0)) - kK(P_0')\varepsilon' \tag{4.1}$$

i.e. $\forall t \in [0,T]$

$$F(\Sigma_{\varepsilon'}(t)) := F(w(\left[\frac{t}{\varepsilon'}\right])) \leq F(S_0^{dat}) - \left[\frac{t}{\varepsilon'}\right] K(P_0')\varepsilon' \leq F(S_0^{dat}) - (\frac{t}{\varepsilon'} + 1)K(P_0')\varepsilon'.$$

To avoid a branching behavior, there exists an endpoint P^* of $\Sigma_{\varepsilon'}(t)$ with $|V(P^*)| \geq K(P_0')$, then for Lemma 4.4 there exists a point $X \in \Omega^-$ such that the ball

$$B(X,v) \cap \Sigma(t) = \emptyset, \qquad v = \sqrt{\frac{K(P_0')}{\pi}},$$

and Lemma 4.3 gives

$$F(\Sigma_{\varepsilon'}(t)) \geq \frac{4}{27}\sqrt{\frac{K(P_0')^3}{\pi}}.$$

But we must have

$$F(\Sigma_{\varepsilon'}(t)) \leq F(w(0)) - tK(P_0')$$

and combining the above inequalities,

$$F(w(0)) - tK(P_0') \geq \frac{4}{27}\sqrt{\frac{K(P_0')^3}{\pi}}$$

which gives $t \leq \dfrac{F(w(0))}{K(P_0')} - \dfrac{4}{27}\sqrt{\dfrac{K(P_0')^3}{\pi}}$. So we have proved the following result:

Theorem 4.5. *For this configuration, with the above notations, an upper bound for the branching time is given by*

$$T_{\max} := \frac{F(S_0^{dat})}{K(P_0')} - \frac{4}{27}\sqrt{\frac{K(P_0')^3}{\pi}}.$$

Notice that the partition $\Omega^+ \cup \Omega^-$ is crucial as Lemma 4.1 makes impossible passing from one region to another without changing topology, so it prevents $\Sigma(t)$ from ever visit $T(P_0') \cap \Omega^+$ without exhibiting branching behaviors.

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