ASYMPTOTIC ANALYSIS OF MUMFORD-SHAH TYPE ENERGIES IN PERIODICALLY-PERFORATED DOMAINS

Matteo Focardi
Dip. Mat. "U. Dini"
V.le Morgagni, 67/a
I-50134 Firenze
focardi@math.unifi.it

Maria Stella Gelli
Dip. Mat. "L. Tonelli"
L.go Bruno Pontecorvo, 5
I-56127 Pisa
gelli@dm.unipi.it

Abstract. We study the asymptotic limit of obstacle problems for Mumford-Shah type functionals with p-growth in periodically-perforated domains *via* the Γ-convergence of the associated free-discontinuity energies. In the limit a non-trivial penalization term related to the 1-capacity of the reference hole appears if and only if the size of the perforation scales like $\varepsilon^{\frac{n}{n-1}}$, being ε its periodicity. We give two different formulations of the obstacle problem to include also perforations with Lebesgue measure zero.

1. Introduction

The aim of this paper is to study the limiting behaviour of Mumford-Shah type functionals in periodically-perforated domains. We express the obstacle constraint by two different formulations according to the "size" of the perforation, thus including $(n-1)$-dimensional sets. For both cases we identify the meaningful scaling yielding a non trivial limit energy (see Theorem 3.1 and Theorem 4.1). A model case for this kind of problems is the following: studying the asymptotics as ε tends to 0 of

$$\inf\left\{\int_\Omega |\nabla u(x)|^p\,dx + \mathcal{H}^{n-1}(S_u) + \text{lower order terms} : u \in SBV(\Omega),\ u = 0 \text{ on } \mathbf{B}_\varepsilon \cup \partial\Omega\right\}, \tag{1.1}$$

where $\Omega \subset \mathbf{R}^n$ is a given regular bounded open set, ∇u and S_u are, respectively, the approximate gradient and the set of approximate discontinuities of u (see Subsection 2.3), and $\mathbf{B}_\varepsilon = \Omega \cap \cup_{\underline{i}\in\mathbf{Z}^n} B_{r_\varepsilon}(\underline{i}\varepsilon)$, with $B_{r_\varepsilon}(\underline{i}\varepsilon)$ the ball centered in $\underline{i}\varepsilon$ of radius $r_\varepsilon > 0$. This is the first step in studying *obstacle problems for free-discontinuity energies* which we are currently investigating [30].

The case in which the minimum problems (1.1) above are restricted to the Sobolev space $W^{1,p}$, $p > 1$, is classical and it has been object of many researches since the pioneering works of Marchenko and Khruslov [31], Rauch and Taylor [34],[35] and Cioranescu and Murat [14]. A wide literature deals also with Neumann or Robin conditions on the boundary of the set of the perforations (see [15],[13] and the books [12],[16] for a more exhaustive list of references).

A typical phenomenon occurring in this context is that the limit problem is no longer related to an obstacle constraint and the limit energy to be minimized contains an extra term. The latter is a finite penalization keeping track of the local capacity density of the homogenizing obstacles (with the appropriate notion of capacity related to the Dirichlet type energy under consideration).

In order to deal with this *relaxation phenomenon*, De Giorgi, Dal Maso and Longo proposed in [27] an approach which was then carried out by many authors (see [9],[23],[3],[4],[20],[21],[33]). The method is based on abstract Γ-convergence arguments (see Section 2.2 for the definition and main properties of Γ-limits) for the associated Dirichlet energies and needs a deep study of some fine properties of Sobolev functions. It turns out that one can confine the analysis to the range $1 < p \leq n$ since for $p > n$ the convergence result is trivial. Moreover, also in case $1 < p \leq n$ a simple computation shows that there exists only one meaningful scaling of the *radius of the periodic perforation* r_ε depending on the space dimension n and on the exponent p: $r_\varepsilon \sim \varepsilon^{\frac{n}{n-p}}$ if $1 < p < n$, $r_\varepsilon \sim e^{-\varepsilon^{-n}}$ if $p = n$.

A different method using direct Γ-convergence arguments was developed more recently in [2]. The main tool there is a joining lemma in varying domains (see Lemma 3.1 [2]) which allows to modify sequences of functions in the closeness of the perforation set, reminiscent of a method proposed by De Giorgi to match boundary conditions.

Going back to our framework, in order to deal with problems (1.1) we introduce for any $p > 1$ the functionals $\mathcal{F}_\varepsilon : SBV(\Omega) \to [0, +\infty]$ defined as

$$\mathcal{F}_\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega |\nabla u|^p \, dx + \mathcal{H}^{n-1}(S_u) & u \in SBV(\Omega),\ u = 0\ \mathcal{L}^n \text{ a.e. on } \mathbf{B}_\varepsilon \\ +\infty & \text{otherwise in } SBV(\Omega), \end{cases} \tag{1.2}$$

thus neglecting the boundary condition on the fixed boundary $\partial\Omega$ (we refer to Theorem 3.1 and Proposition 3.3 for the exact statement and the right functional framework). In Proposition 3.4 we show how to recover the case in which the boundary datum on $\partial\Omega$ is imposed.

Unlike the Sobolev setting, it turns out that for any $p > 1$ there exists only one meaningful scaling for the radius r_ε which depends only on the space dimension n. This is due to having enlarged the domain of the problem allowing for *fractured configurations*, with a penalization on the site of fracture added. In terms of Γ-convergence a rigorous statement of this fact is the following (see Proposition 3.3): $(\mathcal{F}_\varepsilon)$ Γ-converges to the functional $\mathcal{F}$ given for any $u \in SBV(\Omega)$ by

$$\mathcal{F}(u) = \int_\Omega |\nabla u|^p \, dx + \mathcal{H}^{n-1}(S_u) + n\omega_n \beta^{n-1} \mathcal{L}^n \left(\{x \in \Omega : u(x) \neq 0\}\right) \tag{1.3}$$

w.r.t. the L^1 convergence, where the coefficient β is finite and different from 0 if and only if $r_\varepsilon \sim \varepsilon^{\frac{n}{n-1}}$. This result is achieved by studying the more general case of a unilateral constraint of the same type (see Theorem 3.1).

Similarly to the Sobolev case, the term $n\omega_n$ has a capacitary interpretation and it is related to the *functional capacity of degree 1* studied in details in [29],[10]. Indeed, we prove the convergence result contained in Theorem 3.1 for a generic reference perforation set E replacing $n\omega_n$ in (1.3) with $\mathrm{C}_1(E_+)$, the 1-capacity of a suitable $\mathcal{L}^n$ representant of E (see Subsection 2.5, and Remark 3.2).

An heuristic motivation explaining the appearance of the capacitary term (and also the independence of p in the meaningful threshold) can be given by considering the energy of an optimizing sequence for a constant function $u \equiv \eta < 0$. The latter is obtained modifying u itself in a neighbourhood of the periodic perforation in order to satisfy the constraint. In such a neighbourhood the transition between the values 0 and η is minimal, for Mumford-Shah type energies, on *totally fractured* configurations, being the contribution of the bulk term of order strictly greater than the surface one (see Lemma 3.6). Moreover, since on piecewise constant functions the energy $\mathcal{F}_\varepsilon$ reduces to the perimeter of their level sets, one have to solve locally an obstacle problem for minimal surfaces taking also into account the effect of the vanishing size of the perforation. This is indeed the argument with which an upper bound for the Γ-limit is obtained for a generic SBV function (see Proposition 3.9).

To prove that the latter is actually an optimal bound, one reduces to a local picture and estimates in each ε-cell contained in Ω separately the contribution of the energy far and close to the perforation set. The first term accounts for the Mumford-Shah energy in the limit, while the second for the capacitary contribution (see Step 1 and 2 of Lemma 3.5).

In Section 4 we consider reference perforation sets which may also have Lebesgue measure zero, the so called *thin obstacles* (see Theorem 4.1). In such a case formulation (1.2) of the obstacle condition is trivial and the constraint has to be imposed in a different way. As usual in this kind of problems (see [10]) this can be done by exploiting fine properties of the class of functions under consideration. In particular, for a function u in $BV(\Omega)$ the representant u_+ is defined $\mathcal{H}^{n-1}$ a.e. on Ω. By taking this into account, we prove that the family $(\mathcal{F}_\varepsilon)$, with $\mathcal{F}_\varepsilon : SBV(\Omega) \to [0, +\infty]$ given by

$$\mathcal{F}_\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega |\nabla u|^p \, dx + \mathcal{H}^{n-1}(S_u) & u \in SBV(\Omega), u_+ \geq 0 \ \mathcal{H}^{n-1} \text{ a.e. on } \mathbf{E}_\varepsilon \\ +\infty & \text{otherwise in } SBV(\Omega), \end{cases}$$

where $\mathbf{E}_\varepsilon = \Omega \cap \cup_{\underline{i} \in \mathbf{Z}^n} (\underline{i}\varepsilon + r_\varepsilon E)$, Γ-converges w.r.t. the L^1 convergence to the functional $\mathcal{F}$ equal for any $u \in SBV(\Omega)$ to

$$\mathcal{F}(u) = \int_\Omega |\nabla u|^p \, dx + \mathcal{H}^{n-1}(S_u) + \mathrm{C}_1(E)\beta^{n-1}\mathcal{L}^n \left(\{x \in \Omega : u(x) < 0\}\right) \tag{1.4}$$

(see Theorem 4.1). Due to the occurrence of a relaxation phenomenon the analysis of the capacitary contribution in the statement above requires a delicate argument founded on the theory of obstacle problems in the linear setting [26],[10],[11] (see Lemma 4.4).

This fact led us to distinguish the two formulations of the obstacle problem, being the one in Section 3 more intuitive and less technically demanding than the one of Section 4 (see Remark 4.2 for a comparison between Theorem 3.1 and 4.1).

Eventually, in Section 5 we generalize the results obtained in the model case of the Mumford-Shah functional to a wider class of free-discontinuity energies (see Theorem 5.1).

2. Notation and preliminaries

2.1. **Basic Notation.** In the sequel Ω denotes a bounded open set of $\mathbf{R}^n$ with Lipschitz boundary and $\mathcal{H}^{n-1}(\partial\Omega) < +\infty$, with $n \geq 2$ a fixed integer. Given an open set $A \subseteq \mathbf{R}^n$ the family of its open subsets is denoted by $\mathcal{A}(A)$.
The symbol $B \triangle C$ stands for the symmetric difference $(B \setminus C) \cup (C \setminus B)$ of the sets B and C in $\mathbf{R}^n$. As usual, B_1 denotes the open ball in $\mathbf{R}^n$ of radius 1 centered in the origin, and Q_1 the semi-open unit cube with side 1 centered in the origin, that is $Q_1 = [-1/2, 1/2)^n$. For any set $E \subset \mathbf{R}^n$, $z \in \mathbf{R}^n$ and $r > 0$, we denote by $E_r(z)$ the set $z + rE$, and, in case $z = \underline{0}$ we simply write E_r for $E_r(\underline{0})$.
If $B, C \in \mathcal{A}(\Omega)$ and $\mathrm{dist}(B, C) = L > 0$ we call *cut-off function between B and C* any $\theta \in C^\infty(\overline{\Omega})$ with $0 \leq \theta \leq 1$ such that $\theta \equiv 1$ on B and $\theta \equiv 0$ on C. Moreover, we will assume that $|\nabla\theta| \leq c/L$.
We employ the standard notation $\overline{C}$ for the topological closure in $\mathbf{R}^n$ of the set C.

2.2. **Γ-convergence.** We recall the notion of Γ-convergence introduced by De Giorgi (see [22],[6]) in a generic metric space (X, d) endowed with the topology induced by d. A family of functionals $\mathcal{F}_\varepsilon : X \to [0, +\infty]$ Γ-*converges* to a functional $\mathcal{F} : X \to [0, +\infty]$ in $u \in X$, in short $\mathcal{F}(u) = \Gamma\text{-}\lim_\varepsilon \mathcal{F}_\varepsilon(u)$, if for every sequence (ε_j) of positive numbers decreasing to 0 the following two conditions hold:
(i) (*liminf inequality*) $\forall$ (u_j) converging to u in X, we have $\liminf_j \mathcal{F}_{\varepsilon_j}(u_j) \geq \mathcal{F}(u)$;
(ii) (*limsup inequality*) $\exists$ (u_j) converging to u in X such that $\limsup_j \mathcal{F}_{\varepsilon_j}(u_j) \leq \mathcal{F}(u)$.
We say that $\mathcal{F}_\varepsilon$ Γ-*converges* to $\mathcal{F}$ (or $\mathcal{F}=$ Γ-$\lim_\varepsilon \mathcal{F}_\varepsilon$) if $\mathcal{F}(u) = \Gamma\text{-}\lim_\varepsilon \mathcal{F}_\varepsilon(u)$ $\forall u \in X$. We may also define the *lower* and *upper* Γ-*limits* as

$$\Gamma\text{-}\limsup_{\varepsilon\to0^+} \mathcal{F}_\varepsilon(u) = \inf\{\limsup_{\varepsilon\to0^+} \mathcal{F}_\varepsilon(u_\varepsilon) : \ u_\varepsilon \to u\},$$

$$\Gamma\text{-}\liminf_{\varepsilon\to0^+} \mathcal{F}_\varepsilon(u) = \inf\{\liminf_{\varepsilon\to0^+} \mathcal{F}_\varepsilon(u_\varepsilon) : \ u_\varepsilon \to u\},$$

respectively, so that conditions (i) and (ii) are equivalent to Γ-$\limsup_\varepsilon \mathcal{F}_\varepsilon(u) = \Gamma$-$\liminf_\varepsilon \mathcal{F}_\varepsilon(u) = \mathcal{F}(u)$. Moreover, the functions Γ-$\limsup_\varepsilon \mathcal{F}_\varepsilon(\cdot)$ and Γ-$\liminf_\varepsilon \mathcal{F}_\varepsilon(\cdot)$ are lower semicontinuous.
One of the main reasons for the introduction of this notion is explained by the following fundamental theorem.

Theorem 2.1. *Let $\mathcal{F} = \Gamma$-$\lim_\varepsilon \mathcal{F}_\varepsilon$, and assume there exists a compact set $K \subset X$ such that $\inf_X \mathcal{F}_\varepsilon = \inf_K \mathcal{F}_\varepsilon$ for all ε. Then there exists $\min_X \mathcal{F} = \lim_\varepsilon \inf_X \mathcal{F}_\varepsilon$. Moreover, if (u_j) is a converging sequence such that $\lim_j \mathcal{F}_{\varepsilon_j}(u_j) = \lim_j \inf_X \mathcal{F}_{\varepsilon_j}$ then its limit is a minimum point for $\mathcal{F}$.*

2.3. **BV functions.** In this section we recall some basic definitions and results of sets of finite perimeter, BV, SBV and GSBV functions. We refer to the book [1] for all the results used throughout the whole paper, for which we will give a precise reference.
Let $A \subseteq \mathbf{R}^n$ be an open set, for every $u \in L^1(A)$ and $x \in A$, we define

$$u_+(x) = \inf\left\{t \in \mathbf{R} : \lim_{r\to0^+} r^{-n}\mathcal{L}^n(\{y \in B_r(x) : u(y) > t)\} = 0\right\}$$

$$u_-(x) = \sup\left\{t \in \mathbf{R} : \lim_{r\to 0^+} r^{-n}\mathcal{L}^n(\{y \in B_r(x) : u(y) < t\}) = 0\right\},$$

with the convention $\inf\emptyset = +\infty$ and $\sup\emptyset = -\infty$. We remark that u_+, u_- are Borel functions uniquely determined by the $\mathcal{L}^n$-equivalence class of u. If $u_+(x) = u_-(x)$ the common value is denoted by $\tilde{u}(x)$ or $\text{ap-}\lim_{y\to x} u(y)$ and it is said to be the *approximate limit* of u in x.
Notice that for every $\mathcal{L}^n$ measurable set $E \subseteq \mathbf{R}^n$ there holds $(\chi_E)_+ = \chi_{E_+}$, where

$$E_+ = \{x \in \mathbf{R}^n : \limsup_{r\to 0^+} r^{-n}\mathcal{L}^n(E \cap B_r(x)) > 0\}.$$

Moreover, we have

$$\mathcal{L}^n(E \setminus D) = 0 \Longleftrightarrow E_+ \subseteq D_+, \tag{2.1}$$

thus, by (2.1) above E_+ is a $\mathcal{L}^n$ representant of E, i.e. $\mathcal{L}^n(E\triangle E_+) = 0$.
The set $S_u = \{x \in A : u_-(x) < u_+(x)\}$ is called the *set of approximate discontinuity points* of u and it is well known that $\mathcal{L}^n(S_u) = 0$. Let $x \in A \setminus S_u$ be such that $\tilde{u}(x) \in \mathbf{R}$, we say that u is *approximately differentiable* at x if there exists $L \in \mathbf{R}^n$ such that

$$\text{ap-}\lim_{y\to x} \frac{|u(y) - \tilde{u}(x) - L(y - x)|}{|y - x|} = 0. \tag{2.2}$$

If u is approximately differentiable at a point x, the vector L uniquely determined by (2.2), will be denoted by $\nabla u(x)$ and will be called the *approximate gradient* of u at x.
A function $u \in L^1(A)$ is said to be of *Bounded Variation* in A, in short $u \in BV(A)$, if its distributional derivative is a $\mathbf{R}^n$-valued finite Radon measure. If $u \in BV(A)$ denote by $D^a u$, $D^s u$ the absolutely and singular part of the Lebesgue decomposition of Du w.r.t. $\mathcal{L}^n \llcorner A$, respectively. Then u turns out to be approximately differentiable a.e. on A (Theorems 3.83 [1]), S_u to be *countably $\mathcal{H}^{n-1}$-rectifiable* (see Theorem 3.78 [1]), and the values $u_+(x)$, $u_-(x)$ are finite and specified $\mathcal{H}^{n-1}$ a.e. in A (see Remark 3.79 [1]). Moreover, there holds

$$D^a u = \nabla u\, \mathcal{L}^n \llcorner A, \quad D^s u \llcorner S_u = (u_+ - u_-)\nu_u\, \mathcal{H}^{n-1} \llcorner S_u,$$

where $\nu_u \in \mathbf{S}^{n-1}$ is an orientation for S_u.
We say that a $\mathcal{L}^n$ measurable set $E \subseteq \mathbf{R}^n$ is of *finite perimeter* in A if $\chi_E \in BV(A)$, and we call the total variation of χ_E in A the *perimeter* of E in A, denoting it by $\text{Per}(E, A)$ and simply by $\text{Per}(E)$ if $A \equiv \mathbf{R}^n$. It is well known that $D\chi_E = D\chi_E \llcorner \partial^* E = \nu_{\partial^* E}\mathcal{H}^{n-1} \llcorner \partial^* E$ (see Theorem 3.59 [1]), where the countably $\mathcal{H}^{n-1}$-rectifiable set $\partial^* E$ is called the *essential boundary* of E and $\nu_{\partial^* E}$ is an orientation for it.
We recall that if A has Lipschitz boundary, any $u \in BV(A)$ leaves an inner boundary trace on ∂A, which we denote by $\text{tr}(u)$, and moreover $\text{tr}(u) \in L^1(\partial A, \mathcal{H}^{n-1})$ (see Theorem 3.87 [1]).
We say that $u \in BV(A)$ is a *Special Function of Bounded Variation* in A if $D^s u \equiv D^j u$ on A, in short $u \in SBV(A)$. Moreover, $u \in SBV_{\text{loc}}(A)$ if $u \in SBV(U)$ for every open subset $U \subset\subset A$.

We say that $u \in L^1(A)$ is a *Generalized Special Function of Bounded Variation* in A, in short $u \in GSBV(A)$, if for every $M > 0$ the truncated function $(u \wedge M) \vee (-M) \in SBV(A)$.

Functions in $GSBV$ inherit from BV ones many properties: they are approximately differentiable a.e. on A, and S_u turns out to be countably $\mathcal{H}^{n-1}$-rectifiable (see Theorem 4.34 [1]).

The space $(G)SBV$ has been introduced by De Giorgi and Ambrosio [25] in connection with the weak formulation of the image segmentation model proposed by Mumford and Shah (see [32]). If $u \in GSBV(A)$ and $p \in (1, +\infty)$ the *Mumford-Shah energy* of u is defined as

$$MS_p(u) = \int_A |\nabla u|^p \, dx + \mathcal{H}^{n-1}(S_u). \tag{2.3}$$

We recall the SBV compactness theorem due to Ambrosio in a form needed for our purposes (see Theorem 4.8 and Theorem 5.22 [1]).

Theorem 2.2. *Let $(u_j) \subset SBV(A)$ and assume that for some $p \in (1, +\infty)$*

$$\sup_j \left(MS_p(u_j) + \|u_j\|_{L^\infty(A)} \right) < +\infty.$$

Then, there exist a subsequence (u_{j_k}) and a function $u \in SBV(A)$ such that $u_{j_k} \to u$ a.e. in A, $\nabla u_{j_k} \to \nabla u$ weakly in $L^p(A; \mathbf{R}^n)$, $D^s u_{j_k} \llcorner S_{u_{j_k}} \to D^s u \llcorner S_u$ weakly $*$ in the sense of measures. Moreover, if $\psi : \mathbf{R}^n \to \mathbf{R}$ is a norm on $\mathbf{R}^n$ satisfying $c_1 \leq \psi(\nu) \leq c_2$ for every $\nu \in \mathbf{S}^{n-1}$, with $c_1, c_2 > 0$, there holds

$$\int_{S_u} \psi(\nu_u) d\mathcal{H}^{n-1} \leq \liminf_k \int_{S_{u_{j_k}}} \psi(\nu_{u_{j_k}}) d\mathcal{H}^{n-1}.$$

Eventually, in case $u \in GSBV(A)$ and $MS_p(u, A) < +\infty$ the values $u_+(x)$, $u_-(x)$ are finite and specified $\mathcal{H}^{n-1}$ a.e. in A (see Theorem 4.40 [1]).

2.4. **Homogenization in SBV.** Here we collect the main results of [8] (see Proposition 2.1, Proposition 2.2 and Theorem 2.3 there) in a form which is convenient for our purposes.

Let $\varphi : \mathbf{R}^{2n} \to [0, +\infty)$ and $\psi : \mathbf{R}^{3n} \times \mathbf{S}^{n-1} \to [0, +\infty)$ be two Borel functions with $\psi(x, a, b, \nu) = \psi(x, b, a, -\nu)$ for every $(x, a, b, \nu) \in \mathbf{R}^{3n} \times \mathbf{S}^{n-1}$. Suppose that φ and ψ satisfy

(i) $\varphi(\cdot, \xi)$ is 1-periodic for every $\xi \in \mathbf{R}^n$, and there exist $c_1, c_2 > 0$ such that for every $\xi \in \mathbf{R}^n$ and a.e. $x \in \mathbf{R}^n$ there holds

$$c_1 |\xi|^p \leq \varphi(x, \xi) \leq c_2(1 + |\xi|^p);$$

(ii) $\psi(\cdot, a, b, \nu)$ is 1-periodic for every $(a, b, \nu) \in \mathbf{R}^{2n} \times \mathbf{S}^{n-1}$, and there exist $c_3, c_4 > 0$ such that for every $(x, a, b, \nu) \in \mathbf{R}^{3n} \times \mathbf{S}^{n-1}$ there holds

$$c_3(1 + |b - a|) \leq \psi(x, a, b, \nu) \leq c_4(1 + |b - a|);$$

(iii) there exists a continuous non-decreasing function $\omega : [0,+\infty) \to [0,+\infty)$, with $\omega(0) = 0$, and $L > 0$ such that $\omega(t) \leq Lt$ for $t \geq 1$ and

$$|\psi(x,a,b,\nu) - \psi(x,a_1,b_1,\nu)| \leq \omega(|a-a_1| + |b-b_1|)$$

for every $(x,a,b,\nu), (x,a_1,b_1,\nu) \in \mathbf{R}^{3n} \times \mathbf{S}^{n-1}$.

For every $\varepsilon > 0$, define $\mathcal{G}_\varepsilon : SBV(A) \times \mathcal{A}(A) \to [0,+\infty)$ by

$$\mathcal{G}_\varepsilon(u,U) = \int_U \varphi\left(\frac{x}{\varepsilon}, \nabla u\right) dx + \int_{S_u \cap U} \psi\left(\frac{x}{\varepsilon}, u_+, u_-, \nu_u\right) d\mathcal{H}^{n-1}, \tag{2.4}$$

then we have

Theorem 2.3. *For every $U \in \mathcal{A}(A)$ the family $(\mathcal{G}_\varepsilon(\cdot,U))$ Γ-converges w.r.t. the L^1-convergence to the functional $\mathcal{G}_{hom} : SBV(A) \times \mathcal{A}(A) \to [0,+\infty)$ defined by*

$$\mathcal{G}_{hom}(u,U) = \int_U \varphi_{hom}(\nabla u)dx + \int_{S_u \cap U} \psi_{hom}(u_+, u_-, \nu_u)\, d\mathcal{H}^{n-1}, \tag{2.5}$$

where

1. *$\varphi_{hom} : \mathbf{R}^n \to [0,+\infty)$ is the convex function given by*

$$\varphi_{hom}(\xi) = \lim_{\varepsilon \to 0^+} \inf \left\{ \int_{Q_1} \varphi\left(\frac{x}{\varepsilon}, \nabla v + \xi\right) dx : v \in W_0^{1,p}(Q_1) \right\}. \tag{2.6}$$

2. *$\psi_{hom} : \mathbf{R}^{2n} \times \mathbf{S}^{n-1} \to [0,+\infty)$ is the function given by*

$$\begin{aligned} \psi_{hom}(a,b,\nu) = \lim_{\varepsilon \to 0^+} \inf \Big\{ & \int_{S_v \cap Q^\nu} \psi\left(\frac{x}{\varepsilon}, v_+, v_-, \nu_v\right) d\mathcal{H}^{n-1} : \\ & v \in SBV(Q^\nu) \text{ with } \nabla v = 0 \text{ a.e.}, \mathrm{tr}(v) = \mathrm{tr}(v_{a,b,\nu}) \text{ on } \partial Q^\nu \Big\}, \end{aligned} \tag{2.7}$$

where Q^ν is any unit cube in $\mathbf{R}^n$ centered in the origin and one face orthogonal to ν, and $v_{a,b,\nu}(x) = a\chi_{\{x:\langle x,\nu\rangle \geq 0\}}(x) + b\chi_{\{x:\langle x,\nu\rangle < 0\}}(x)$.

Remark 2.4. *In case $\varphi(x,\cdot)$ is convex for all $x \in \mathbf{R}^n$ formula (2.6) can be further specialized (see Theorem 14.7* [7]*) and reduces to a cell minimization formula*

$$\varphi_{hom}(\xi) = \min \left\{ \int_{Q_1} \varphi(x, \nabla v + \xi)\, dx : v \in W^{1,p}_{\mathrm{per}}(Q_1) \right\}. \tag{2.8}$$

2.5. **Functional capacity of degree 1.** Let $\mathcal{Y}_1(\mathbf{R}^n)$ be the subspace of $L^{\frac{n}{n-1}}(\mathbf{R}^n)$ of functions with distributional derivative of function type. For any set $E \subseteq \mathbf{R}^n$ consider the quantity

$$\Gamma_1(E) = \inf \left\{ \int_{\mathbf{R}^n} |\nabla u| dx : u \in \mathcal{Y}_1(\mathbf{R}^n),\ E \subset \mathrm{int}(\{x \in \mathbf{R}^n : u(x) \geq 1\}) \right\},$$

according to Federer and Ziemer [29] we call it the *functional capacity of degree 1* of E. Actually, different minimization problems characterize it, in particular it can be expressed in terms of the perimeter of the sets containing E as shows the following proposition which summarizes the results contained in Section 4 [29] and Theorem 2.1 [10].

Proposition 2.5. *Let $E \subseteq \mathbf{R}^n$ and let*

$$\begin{aligned} \mathrm{C}_1(E) &= \inf\left\{\int_{\mathbf{R}^n} |\nabla u| dx : u \in W^{1,1}(\mathbf{R}^n),\ u_+ \geq 1\ \mathcal{H}^{n-1}\text{ a.e. on } E\right\}, \\ \gamma(E) &= \inf\left\{\|Du\|(\mathbf{R}^n) : u \in BV(\mathbf{R}^n),\ u_+ \geq 1\,\mathcal{H}^{n-1}\text{ a.e. on } E\right\}, \\ \delta(E) &= \inf\left\{\mathrm{Per}(D) : D \text{ is } \mathcal{L}^n \text{ measurable},\ \mathcal{L}^n(D) < +\infty, \mathcal{H}^{n-1}(E \setminus D_+) = 0\right\}. \end{aligned} \tag{2.9}$$

Then $\Gamma_1(E) = \mathrm{C}_1(E) = \gamma(E) = \delta(E)$.

The existence of extremals for the variational problems above fails for many sets E with $\mathrm{C}_1(E) < +\infty$ (e.g. if E is a line segment in $\mathbf{R}^2$). A sufficient condition ensuring existence of minimizers for the formulation (2.9) was proposed in Section 4 [29] (see also Theorem 3.3 and Theorem 3.4 Chapter IV [26]). Here we recall the result and its proof for the readers' convenience.

Proposition 2.6. *For every $\mathcal{L}^n$ measurable set $E \subset \mathbf{R}^n$ with $\mathrm{C}_1(E) < +\infty$ there holds*

(a) $\mathrm{C}_1(E_+) \leq \mathrm{C}_1(E)$;

(b) *problem (2.9) for E_+ has always solution and*

$$\mathrm{C}_1(E_+) = \min\left\{\mathrm{Per}(D) : D \text{ is } \mathcal{L}^n \text{ measurable},\ \mathcal{L}^n(D) < +\infty,\ \mathcal{L}^n(E \setminus D) = 0\right\}. \tag{2.10}$$

Moreover, if $\mathcal{H}^{n-1}(E \setminus E_+) = 0$ then $\mathrm{C}_1(E_+) = \mathrm{C}_1(E)$ and problem (2.9) for E has solution.

Proof. Let (D_j) be a minimizing sequence in problem (2.9) for E, then by the Isoperimetric inequality (see Theorem 3.46 [1]) $\sup_j(\mathcal{L}^n(D_j)+\mathrm{Per}(D_j)) < +\infty$. The BV Compactness Theorem (see Theorem 3.23 [1]) in turn implies the existence of a subsequence (not relabeled for convenience) and a set D with finite perimeter in $\mathbf{R}^n$ such that $\chi_{D_j} \to \chi_D$ in $L^1(\mathbf{R}^n)$. Thus $\mathcal{L}^n(E \setminus D) = 0$, and by taking into account (2.1) we have $E_+ \subseteq D_+$. Hence, D is admissible in problem (2.9) for E_+, i.e. $\mathcal{H}^{n-1}(E_+ \setminus D_+) = 0$, and so (a) is established since

$$\mathrm{C}_1(E) = \liminf_j \mathrm{Per}(D_j) \geq \mathrm{Per}(D) \geq \mathrm{C}_1(E_+).$$

Obviously the same argument applied to a minimizing sequence of $\mathrm{C}_1(E_+)$ provides a set D admissible for such a problem which is then a minimizer. Eventually, characterization (2.10) holds true. □

Sligthly abusing a terminology introduced by De Giorgi in [24],[26] we call *thick* the sets satisfying $\mathcal{H}^{n-1}(E \setminus E_+) = 0$. Indeed, De Giorgi's original definition required the stronger condition $E \subseteq E_+$. In general, one can determine the relaxed problem associated to $\mathrm{C}_1(\cdot)$ by using *De Giorgi's measure* σ introduced in Chapther IV [26] to study non-parametric minimal surfaces problems with obstacles. For any set $E \subseteq \mathbf{R}^n$, σ is the regular Borel measure given by

$$\sigma(E) = \sup_{\varepsilon>0}\left(\inf\left\{\mathrm{Per}(D) + \frac{\mathcal{L}^n(D)}{\varepsilon} : D\ \mathcal{L}^n \text{ measurable},\ \mathcal{H}^{n-1}(E \setminus D_+) = 0\right\}\right). \tag{2.11}$$

We are now able to state the relaxation Theorem 7.1 [10] in a form needed for our purposes (see also Theorem 3.4 Chapter IV [26]).

Theorem 2.7. *For any $\mathcal{L}^n$ measurable set $E \subset \mathbf{R}^n$ there holds*

$$\begin{aligned} \mathrm{C}_1(E) &= \min\left\{\|Du\|(\mathbf{R}^n) + \int_{\mathbf{R}^n} [(\chi_E - u_+) \vee 0] d\sigma : u \in BV(\mathbf{R}^n)\right\} \\ &= \min\left\{\mathrm{Per}(D) + \sigma(E \setminus D_+) : D \text{ is } \mathcal{L}^n \text{ measurable, } \mathcal{L}^n(D) < +\infty\right\}. \end{aligned} \tag{2.12}$$

Eventually, we recall that the set function $\mathrm{C}_1(\cdot)$ is *positively (n-1)-homogeneous*, that is for any set $E \subseteq \mathbf{R}^n$ and $r > 0$ we have $\mathrm{C}_1(E_r) = r^{n-1}\mathrm{C}_1(E)$ (see [36]); and moreover it is such that (see [29])

$$\mathrm{C}_1(E) = 0 \Longleftrightarrow \mathcal{H}^{n-1}(E) = 0.$$

Remark 2.8. *For any bounded set E it is easy to prove that $\mathrm{C}_1(E) < +\infty$. Moreover, if E is contained in the interior of a bounded convex set C, one can restrict the class of competing sets in the capacitary problem for E to those contained in C.*
Indeed, by using the formulation (2.9), given a test set D, consider $D' = D \cap C$, then D' has finite perimeter and, being $E \subset \mathrm{int}(C)$ and $C_+ = \overline{C}$, we have $\mathcal{H}^{n-1}(E \setminus D'_+) = \mathcal{H}^{n-1}(E \setminus (D \cap C)_+) = \mathcal{H}^{n-1}(E \setminus D_+) = 0$. If Π_C denotes the projection on the convex set C, then $\mathcal{H}^{n-1}(\Pi_C(D \cap (\mathbf{R}^n \setminus C))) \leq$ $\mathrm{Per}(D \cap (\mathbf{R}^n \setminus C))$. Hence, we have

$$\begin{aligned} \mathrm{Per}(D') &\leq \mathcal{H}^{n-1}(\Pi_C(D \setminus \mathrm{int}(C))) + \mathcal{H}^{n-1}(\partial^* D \cap \mathrm{int}(C)) \\ &\leq \mathrm{Per}(D \setminus \mathrm{int}(C)) + \mathcal{H}^{n-1}(\partial^* D \cap \mathrm{int}(C)) \leq \mathrm{Per}(D). \end{aligned}$$

3. Obstacle constraint imposed in the $\mathcal{L}^n$ sense

Given a $\mathcal{L}^n$ measurable set $E \subseteq \overline{Q_1}$, for any $\varepsilon > 0$ let $r_\varepsilon \in (0, \varepsilon)$ and $\mathbf{E}_\varepsilon = \Omega \cap \cup_{\mathbf{Z}^n} E_{r_\varepsilon}(\underline{i}\varepsilon)$. Consider the functional $\mathcal{F}_\varepsilon : L^1(\Omega) \to [0, +\infty]$ defined as

$$\mathcal{F}_\varepsilon(u) = \begin{cases} MS_p(u) & u \in GSBV(\Omega),\ u \geq 0\ \mathcal{L}^n \text{ a.e. on } \mathbf{E}_\varepsilon \\ +\infty & \text{otherwise in } L^1(\Omega). \end{cases} \tag{3.1}$$

Moreover, denote by $\mathcal{F}_\varepsilon(\cdot, A)$ its *localized version*, obtained by substituting in definition (3.1) above the domain of integration Ω with any open subset $A \in \mathcal{A}(\Omega)$.
The same convention will be also applied to the localized version of the Mumford-Shah energy (2.3), dropping the set dependence in case $A \equiv \Omega$.

Theorem 3.1. *Let E be a $\mathcal{L}^n$ measurable set and assume that $r_\varepsilon/\varepsilon^{\frac{n}{n-1}} \to \beta \in [0, +\infty)$ as $\varepsilon \to 0^+$. Then, $(\mathcal{F}_\varepsilon)$ Γ-converges to $\mathcal{F} : L^1(\Omega) \to [0, +\infty]$ defined by*

$$\mathcal{F}(u) = \begin{cases} MS_p(u) + \mathrm{C}_1(E_+)\beta^{n-1}\mathcal{L}^n(\{x \in \Omega : u(x) < 0\}) & u \in GSBV(\Omega), \\ +\infty & \text{otherwise in } L^1(\Omega), \end{cases} \tag{3.2}$$

w.r.t. the L^1 convergence.

Remark 3.2. *It is worth noting that definition (3.1) of $\mathcal{F}_\varepsilon$ is not affected by substituting E with any other set G in its $\mathcal{L}^n$ equivalence class. For instance, it would not be restrictive to assume the perforation set E to be thick in the statement of Theorem 3.1, namely to change E with E_+.*
The reason why the representant E_+ is selected in the limit process is given by the minimality property

$$\mathrm{C}_1(E_+) = \min\{\mathrm{C}_1(G) : G\ \mathcal{L}^n \text{ measurable}, \mathcal{L}^n(E \triangle G) = 0\},$$

as follows from (a) of Proposition 2.6.
A further motivation will be discussed in Section 4 (see Theorem 4.1 and Remark 4.2 for details).

Before giving a proof of Theorem 3.1 we state the results mentioned in the introduction concerning the bilateral obstacle case and when a boundary datum on $\partial\Omega$ is imposed. Both their proofs will be addressed after that of Theorem 3.1, since they share many ideas and techniques developed for that as well as using part of those results.

Proposition 3.3. *Let $\mathcal{F}'_\varepsilon$ be defined as $\mathcal{F}_\varepsilon$ by substituting the unilateral positivity condition on $\mathbf{E}_\varepsilon$ in definition (3.1) with $u = 0$ $\mathcal{L}^n$ a.e. on $\mathbf{E}_\varepsilon$. Then, $(\mathcal{F}'_\varepsilon)$ Γ-converges to $\mathcal{F}' : L^1(\Omega) \to [0, +\infty]$ defined by*

$$\mathcal{F}'(u) = \begin{cases} MS_p(u) + \mathrm{C}_1(E_+)\beta^{n-1}\mathcal{L}^n(\{x \in \Omega : u(x) \neq 0\}) & u \in GSBV(\Omega), \\ +\infty & \text{otherwise in } L^1(\Omega), \end{cases} \tag{3.3}$$

w.r.t. the L^1 convergence.

We now consider the case in which a Dirichlet boundary datum is imposed on $\partial\Omega$. For the sake of simplicity we assume in what follows the additional hypothesis that Ω has C^2 boundary, although this condition might be weakened (see for instance Setion 8 [8]).
We introduce for any $\varepsilon > 0$ the "boundary" functionals $\mathcal{D}_\varepsilon : L^1(\Omega) \to [0, +\infty]$ defined as

$$\mathcal{D}_\varepsilon(u) = \begin{cases} \mathcal{F}'_\varepsilon(u) & u \in GSBV(\Omega), \ \mathrm{tr}(u) = 0 \text{ on } \partial\Omega \\ +\infty & \text{otherwise in } L^1(\Omega), \end{cases}$$

and state the following convergence result.

Proposition 3.4. *$(\mathcal{D}_\varepsilon)$ Γ-converges w.r.t. the L^1 convergence to $\mathcal{D} : L^1(\Omega) \to [0, +\infty]$ given by*

$$\mathcal{D}(u) = \begin{cases} \mathcal{F}'(u) + \mathcal{H}^{n-1}(\{x \in \partial\Omega : \mathrm{tr}(u)(x) \neq 0\}) & u \in GSBV(\Omega) \\ +\infty & \text{otherwise in } L^1(\Omega). \end{cases}$$

Notice that if we consider lower order terms converging in a suitable sense (see Proposition 6.20 [22]), for instance fidelity terms or linear perburations, Proposition 3.4 and Theorem 2.1 imply the convergence of problems (1.1) mentioned in the introduction to

$$\min\{\mathcal{D}(u) + \text{lower order terms} : u \in GSBV(\Omega)\}.$$

The result of Theorem 3.1 will be a consequence of Propositions 3.7, 3.9 below in which we show separately the liminf and the limsup inequalities, respectively. Proposition 3.7 will easily follow from Lemma 3.5 below in which we treat the case of sequences bounded in L^∞.

Lemma 3.5. *For every sequence $u_\varepsilon \to u$ in $L^1(\Omega)$ such that $\sup_\varepsilon \|u_\varepsilon\|_{L^\infty(\Omega)} < +\infty$*

$$\liminf_\varepsilon \mathcal{F}_\varepsilon(u_\varepsilon) \geq \mathcal{F}(u).$$

Proof. We may suppose $\mathcal{L}^n(\{x \in \Omega : u(x) < 0\}) > 0$ and $\mathcal{L}^n(E) > 0$, being otherwise the statement trivial. Moreover, it is not restrictive to assume $\liminf_\varepsilon \mathcal{F}_\varepsilon(u_\varepsilon) = \lim_\varepsilon \mathcal{F}_\varepsilon(u_\varepsilon) < +\infty$. Hence, Ambrosio' SBV closure and compactness Theorem 2.2 implies that $u \in SBV(\Omega)$ and also $\liminf_\varepsilon \mathcal{F}_\varepsilon(u_\varepsilon) = \liminf_\varepsilon MS_p(u_\varepsilon) \geq MS_p(u)$.

Note that the L^1 convergence assumption implies that for $\mathcal{L}^1$ a.e. $\eta < 0$ and for any $A \in \mathcal{A}(\Omega)$

$$\lim_\varepsilon \mathcal{L}^n\left(\{x \in A : u_\varepsilon(x) < \eta\} \triangle \{x \in A : u(x) < \eta\}\right) = 0. \tag{3.4}$$

For every $\eta < 0$ we are going to prove that

$$\liminf_\varepsilon \mathcal{F}_\varepsilon(u_\varepsilon) \geq MS_p(u) + \mathrm{C}_1(E_+)\beta^{n-1}\mathcal{L}^n(\{x \in \Omega \,:\, u(x) < \eta\}). \tag{3.5}$$

Once (3.5) is established the thesis follows by letting $\eta \to 0^-$.

Since by Ambrosio's lower semicontinuity Theorem 2.2, for any $A \in \mathcal{A}(\Omega)$ we have

$$\liminf_\varepsilon \mathcal{F}_\varepsilon(u_\varepsilon, A) \geq \mathcal{H}^{n-1}(S_u \cap A), \tag{3.6}$$

in order to prove (3.5), it suffices to show that for any $A \in \mathcal{A}(\Omega)$ there holds

$$\liminf_\varepsilon \mathcal{F}_\varepsilon(u_\varepsilon, A) \geq \int_A |\nabla u|^p dx + \mathrm{C}_1(E_+)\beta^{n-1}\mathcal{L}^n(\{x \in A \,:\, u(x) < \eta\}). \tag{3.7}$$

Indeed, given (3.7) for granted, inequality (3.5) follows from standard measure theoretic arguments by taking into account that the two quantities on the right hand side of (3.6), (3.7) are mutually orthogonal measures and the left hand side term is a superadditive set function defined on $\mathcal{A}(\Omega)$ (for details see Proposition 1.16 [5]).

Fix $A \in \mathcal{A}(\Omega)$ and choose η for which (3.4) holds for the open set A. Moreover, set $V = \{x \in A : u(x) < \eta\}$, and assume that $\mathcal{L}^n(\{x \in A : u(x) < \eta\}) > 0$ being otherwise (3.7) trivial.

For $k \in \mathbf{N}$ fixed we consider the following splitting of the energies[1]

$$\mathcal{F}_\varepsilon(u_\varepsilon, A) = MS_p(u_\varepsilon, A \setminus \cup_{\mathbf{Z}^n} B_{3\varepsilon/4k}(\underline{i}\varepsilon)) + MS_p(u_\varepsilon, A \cap \cup_{\mathbf{Z}^n} B_{3\varepsilon/4k}(\underline{i}\varepsilon)). \tag{3.8}$$

We will now estimate separately the two terms on the right hand side of (3.8) showing that the first contributes to the gradient energy (Step 1) while the latter provides the capacitary term of (3.7) (Step 2).

[1]The choice of the coefficient 3/4 in the radius of the balls in (3.8) is arbitrary and could be replaced with any $t \in (0,1)$. Indeed, since $r_\varepsilon = o(\varepsilon)$ the set E_{r_ε} is definitively contained in $B_{t\varepsilon}$ for any $t \in (0,1)$.

Step 1. *(Gradient estimate)* We prove that

$$\lim_k \Big(\liminf_\varepsilon MS_p(u_\varepsilon, A \setminus \cup_{\mathbf{Z}^n} B_{3\varepsilon/4k}(\underline{i}\varepsilon))\Big) \geq \int_A |\nabla u|^p dx. \tag{3.9}$$

In order to match the assumptions of Theorem 2.3, let us fix a parameter $\gamma > 0$ and consider the auxiliary (localized) functionals $\mathcal{G}^{\gamma,k}_\varepsilon : SBV(A) \times \mathcal{A}(A) \to [0, +\infty)$ defined as

$$\mathcal{G}^{\gamma,k}_\varepsilon(v, U) = \int_U \varphi^{\gamma,k}\Big(\frac{x}{\varepsilon}, \nabla v\Big)\, dx + \int_{S_v \cap U} \psi^{\gamma,k}\Big(\frac{x}{\varepsilon}, v^+, v^-, \nu_v\Big)\, d\mathcal{H}^{n-1},$$

where $\varphi^{\gamma,k}(x,\xi) = a^{\gamma,k}(x)|\xi|^p$ for $(x,\xi) \in \mathbf{R}^{2n}$, $\psi^{\gamma,k}(x,a,b,\nu) = a^{\gamma,k}(x) + \gamma|b-a|$ for $(x,a,b,\nu) \in \mathbf{R}^{3n} \times \mathbf{S}^{n-1}$, and $a^{\gamma,k}$ is the (Borel) 1-periodic function defined by

$$a^{\gamma,k}(x) = \begin{cases} 1 & x \in Q_1 \setminus B_{3/4k} \\ \gamma & x \in B_{3/4k}. \end{cases}$$

Being $\sup_\varepsilon MS_p(u_\varepsilon) < +\infty$, for a positive constant c we get

$$\limsup_\varepsilon \int_{S_{u_\varepsilon} \cap A} |u^+_\varepsilon - u^-_\varepsilon| d\mathcal{H}^{n-1} \leq 2 \sup_\varepsilon \big(\|u_\varepsilon\|_{L^\infty(\Omega)} \mathcal{H}^{n-1}(S_{u_\varepsilon})\big) \leq c,$$

and

$$\liminf_\varepsilon MS_p(u_\varepsilon, A \setminus \cup_{\mathbf{Z}^n} B_{3\varepsilon/4k}(\underline{i}\varepsilon)) \geq \liminf_\varepsilon \mathcal{G}^{\gamma,k}_\varepsilon(u_\varepsilon, A) - c\gamma. \tag{3.10}$$

For every $U \in \mathcal{A}(A)$ the family $(\mathcal{G}^{\gamma,k}_\varepsilon(\cdot, U))$ satisfies the assumptions of Theorem 2.3, and thus it Γ-converges to the functional $\mathcal{G}^{\gamma,k}_{hom}(\cdot, U)$ defined in (2.5) of Theorem 2.3. Hence, to prove Step 1 it suffices to estimate the volume density $\varphi^{\gamma,k}_{hom}$ of $\mathcal{G}^{\gamma,k}_{hom}$ since (3.10) rewrites as

$$\liminf_\varepsilon MS_p(u_\varepsilon, A \setminus \cup_{\mathbf{Z}^n} B_{3\varepsilon/4k}(\underline{i}\varepsilon)) \geq \mathcal{G}^{\gamma,k}_{hom}(u, A) - c\gamma \geq \int_A \varphi^{\gamma,k}_{hom}(\nabla u) dx - c\gamma. \tag{3.11}$$

We claim that, with fixed $\gamma > 0$, for every $\xi \in \mathbf{R}^n$ we have

$$\lim_k \varphi^{\gamma,k}_{hom}(\xi) = \sup_k \varphi^{\gamma,k}_{hom}(\xi) = |\xi|^p. \tag{3.12}$$

Once (3.12) is established, (3.9) follows from (3.11) by letting first $k \to +\infty$ and using the Monotone convergence theorem, and then $\gamma \to 0^+$.

In order to prove (3.12) we take advantage of (2.4). Indeed, with fixed $\xi \in \mathbf{R}^n$, we prove that the Γ-limit (as $k \to +\infty$) in the L^1 strong topology of the sequence $\mathcal{A}^{\gamma,k} : W^{1,p}_{\mathrm{per}}(Q_1) \to [0, +\infty]$, with

$$\mathcal{A}^{\gamma,k}(v) = \int_{Q_1} a^{\gamma,k}(x)|\nabla v + \xi|^p dx,$$

is given by

$$\mathcal{A}(v) = \int_{Q_1} |\nabla v + \xi|^p dx.$$

Notice that by definition $\min_{W^{1,p}_{\mathrm{per}}(Q_1)} \mathcal{A}^{\gamma,k} = \varphi^{\gamma,k}_{hom}(\xi)$ and by Jensen inequality $\min_{W^{1,p}_{\mathrm{per}}(Q_1)} \mathcal{A} = |\xi|^p$. Moreover, for any fixed $\gamma > 0$ the sequence $(\mathcal{A}^{\gamma,k})$ is equi-coercive in $L^1(Q_1)$, so that we may apply Theorem 2.1 to deduce (3.12).

Eventually, we establish the claimed Γ-limit concerning $(\mathcal{A}^{\gamma,k})$.

The limsup inequality is trivial, being the recovery sequence for any given $v \in W^{1,p}_{\rm per}(Q_1)$ provided by the function itself thanks to Lebesgue dominated convergence theorem. Indeed, $a^{\gamma,k} \to 1$ in $L^1(Q_1)$ and $0 \le a^{\gamma,k}(x) \le 1$ for every $x \in Q_1$.

To accomplish the liminf inequality it suffices to note that for every $(v_k) \subset W^{1,p}_{\rm per}(Q_1)$ such that $v_k \to v$ in $L^1(Q_1)$ and $\liminf_k \mathcal{A}^{\gamma,k}(v_k) < +\infty$, then actually (v_k) converges to v weakly in $W^{1,p}(Q_1)$. Hence, for every $\delta > 0$ we have

$$\begin{aligned}\liminf_k \mathcal{A}^{\gamma,k}(v_k) &\ge \liminf_k \int_{Q_1\setminus\overline{B_\delta}} a^{\gamma,k}(x)|\nabla v_k + \xi|^p dx \\ &= \liminf_k \int_{Q_1\setminus\overline{B_\delta}} |\nabla v_k + \xi|^p dx \ge \int_{Q_1\setminus\overline{B_\delta}} |\nabla v + \xi|^p dx,\end{aligned}$$

and the conclusion by letting $\delta \to 0^+$.

Step 2. *(Capacitary Estimate)* We prove that

$$\liminf_\varepsilon MS_p(u_\varepsilon, A \cap \cup_{\mathbf{Z}^n} B_{3\varepsilon/4k}(\underline{i}\varepsilon)) \ge \mathrm{C}_1(E_+)\beta^{n-1}\left(\mathcal{L}^n\left(V\right) - \frac{1}{k^{n+1}}\right). \tag{3.13}$$

Choose an open set $W \subseteq A$ such that $W \supseteq V$ and $\mathcal{L}^n(W \setminus V) \le 1/(2k^{2(n+1)})$. By (3.4) the set $\{x \in A : u_\varepsilon(x) \ge \eta\} \cap V$ has vanishing $\mathcal{L}^n$ measure so for ε sufficiently small we have

$$\mathcal{L}^n(\{x \in A : u_\varepsilon(x) \ge \eta\} \cap V) \le \frac{1}{2k^{2(n+1)}}.$$

Set $U_\varepsilon = \{x \in W : u_\varepsilon(x) \ge \eta\}$, then for ε small enough there holds

$$\mathcal{L}^n(U_\varepsilon) \le \mathcal{L}^n(U_\varepsilon \cap V) + \mathcal{L}^n(U_\varepsilon \cap (W \setminus V)) \le \frac{1}{k^{2(n+1)}}.$$

Let

$$\mathcal{W}_\varepsilon = \{\underline{i} \in \mathbf{Z}^n : Q_\varepsilon(\underline{i}\varepsilon) \subset\subset W\},$$

and consider

$$\mathcal{I}^k_\varepsilon = \left\{\underline{i} \in \mathcal{W}_\varepsilon \,:\, \mathcal{L}^n(U_\varepsilon \cap Q_\varepsilon(\underline{i}\varepsilon)) \le \frac{\varepsilon^n}{k^{n+1}}\right\}.$$

The set of indices $\mathcal{I}^k_\varepsilon$ identifies those cells for which the contribution to the capacitary term can be estimated up to an error infinitesimal as $k \to +\infty$.

Let us first show that $\mathcal{I}^k_\varepsilon$ nearly exhausts $\mathcal{W}_\varepsilon$, indeed we have

$$\frac{1}{k^{2(n+1)}} \ge \mathcal{L}^n(U_\varepsilon) \ge \sum_{\mathcal{W}_\varepsilon} \mathcal{L}^n(U_\varepsilon \cap Q_\varepsilon(\underline{i}\varepsilon)) \ge \#\left(\mathcal{W}_\varepsilon \setminus \mathcal{I}^k_\varepsilon\right)\frac{\varepsilon^n}{k^{n+1}},$$

from which we deduce $\#\left(\mathcal{W}_\varepsilon \setminus \mathcal{I}^k_\varepsilon\right) \le 1/(k^{n+1}\varepsilon^n)$.

Moreover, setting $\rho_\varepsilon = 3\varepsilon/4k$, the very definition of $\mathcal{I}^k_\varepsilon$ yields also

$$\mathcal{L}^n(U_\varepsilon \cap B_{\rho_\varepsilon}(\underline{i}\varepsilon)) \le \frac{2^n}{w_n k}\mathcal{L}^n(B_{\rho_\varepsilon}(\underline{i}\varepsilon)), \tag{3.14}$$

and a simple translation argument shows that for any such index $\underline{i} \in \mathcal{I}^k_\varepsilon$ we have

$$MS_p(u_\varepsilon, B_{\rho_\varepsilon}(\underline{i}\varepsilon)) \ge m_\varepsilon(\eta) = \inf\{MS_p(v, B_{\rho_\varepsilon}) : v \in SBV(B_{\rho_\varepsilon}),$$

$$v \geq 0 \text{ a.e. on } E_{r_\varepsilon}, \mathcal{L}^n(\{x \in B_{\rho_\varepsilon} : v(x) \geq \eta\}) \leq \frac{2^n}{\omega_n k}\mathcal{L}^n(B_{\rho_\varepsilon})\Big\}.$$

It is clear that if we restrict the class of admissible functions v in the definition of $m_\varepsilon(\eta)$ above to simple functions assuming values in $\{0,\eta\}$, we have by (2.10)

$$m_\varepsilon(\eta) \leq \mathrm{C}_1((E_+)_{r_\varepsilon}) = \mathrm{C}_1(E_+)r_\varepsilon^{n-1}.$$

Next we want to estimate $m_\varepsilon(\eta)$ from below, more precisely we prove

$$\lim_\varepsilon r_\varepsilon^{1-n} m_\varepsilon(\eta) = \mathrm{C}_1(E_+). \tag{3.15}$$

To do that we need the following result.

Lemma 3.6. *Let $H \subset \mathbf{R}^n$ be a bounded $\mathcal{L}^n$ measurable thick set, and $v_\varepsilon \in SBV(B_{R_\varepsilon})$, $R_\varepsilon \to +\infty$, be such that*

(i) $v_\varepsilon \geq 0$ *a.e. on* H, $\sup_\varepsilon \|v_\varepsilon\|_{L^\infty(B_{R_\varepsilon})} < +\infty$,

(ii) $\lim_\varepsilon \|\nabla v_\varepsilon\|_{L^p(B_{R_\varepsilon})} = 0$, $\limsup_\varepsilon \mathcal{H}^{n-1}(S_{v_\varepsilon}) \leq \mathrm{C}_1(H)$,

(iii) $\sup_\varepsilon \|Dv_\varepsilon\|(B_{R_\varepsilon}) < +\infty$,

(iv) *there exists* $\zeta < 0$ *such that* $\mathcal{L}^n(\{x \in B_{R_\varepsilon} : v_\varepsilon(x) \geq \zeta\}) < \frac{1}{2}\mathcal{L}^n(B_{R_\varepsilon})$.

Then, $\lim_\varepsilon \mathcal{H}^{n-1}(S_{v_\varepsilon}) = \mathrm{C}_1(H)$.

Moreover, for every subsequence (v_{ε_m}) there exist $(v_{\varepsilon_{m_j}})$ and $v \in SBV_{\mathrm{loc}}(\mathbf{R}^n)$ such that $v_{\varepsilon_{m_j}} \to v$ in $L^1_{\mathrm{loc}}(\mathbf{R}^n)$, $v \geq 0$ a.e. on H, $v = \sum_{s\in I} a_i\chi_{E_i}$, where I is a finite set, E_i has finite perimeter, $a_i \in \mathbf{R}$, and $\mathcal{H}^{n-1}(S_v) = \mathrm{C}_1(H)$.

Proof. (of Lemma 3.6) First note that by assumption (ii) it is sufficient to show that

$$\liminf_\varepsilon \mathcal{H}^{n-1}(S_{v_\varepsilon}) \geq \mathrm{C}_1(H).$$

Denote by (v_{ε_m}) a sequence for which $\liminf_\varepsilon \mathcal{H}^{n-1}(S_{v_\varepsilon}) = \lim_m \mathcal{H}^{n-1}(S_{v_{\varepsilon_m}})$. Ambrosio' SBV compactness and lower semicontinuity Theorem 2.2 applied on every ball B_R, $R > 0$, and an obvious diagonalization argument ensure the existence of an extracted subsequence $(v_{\varepsilon_{m_j}}) \subseteq (v_{\varepsilon_m})$, and of $v \in SBV_{\mathrm{loc}} \cap L^\infty(\mathbf{R}^n)$ such that $v_{\varepsilon_{m_j}} \to v$ in $L^1_{\mathrm{loc}}(\mathbf{R}^n)$, $\nabla v = 0$ a.e. in $\mathbf{R}^n$ and

$$\mathcal{H}^{n-1}(S_v) \leq \lim_j \mathcal{H}^{n-1}(S_{v_{\varepsilon_{m_j}}}) \leq \mathrm{C}_1(H).$$

For the sake of simplicity in the rest of the proof we set $v_j = v_{\varepsilon_{m_j}}$ and $R_j = R_{\varepsilon_{m_j}}$.

The BV Coarea formula (see Theorem 3.40 [1]) and the Mean value theorem provide $t_j \in (\zeta, \zeta/2)$ such that

$$\begin{aligned}
\|Dv_j\|(B_{R_j}) &\geq \int_\zeta^{\zeta/2} \mathrm{Per}(\{x \in B_{R_j} : v_j(x) > t\}, B_{R_j})dt \\
&\geq \frac{|\zeta|}{2}\mathrm{Per}(\{x \in B_{R_j} : v_j(x) > t_j\}, B_{R_j}) \geq \frac{|\zeta|}{2}c\mathcal{L}^n(\{x \in B_{R_j} : v_j(x) > t_j\})^{1-1/n} \\
&\geq \frac{|\zeta|}{2}c\mathcal{L}^n(\{x \in B_{R_j} : v_j(x) > \zeta/2\})^{1-1/n},
\end{aligned}$$

where in the third inequality we have used assumption (iv) and the Relative isoperimetric inequality in balls (see Remark 3.50 [1]). Hence, (iii) gives $\sup_j \mathcal{L}^n(\{x \in B_{R_j} : v_j(x) > \zeta/2\}) < +\infty$, so that the $L^1_{\rm loc}$ convergence implies $\mathcal{L}^n(\{x \in \mathbf{R}^n : v(x) > \zeta/2\}) < +\infty$ as well as $v \geq 0$ a.e. on H.

Being $v \in SBV_{\rm loc} \cap L^\infty(\mathbf{R}^n)$ with $\nabla v = 0$ a.e. on $\mathbf{R}^n$ and $\mathcal{H}^{n-1}(S_v) < +\infty$, the decomposition $v = \sum_{i\geq 0} a_i \chi_{\Sigma_i}$, with Σ_i a set with finite perimeter for every i, and the equality $2\mathcal{H}^{n-1}(S_v) = \sum_{i\geq 0} \mathrm{Per}(\Sigma_i)$ holds true (see Theorem 4.23 [1]).

Since $\{x \in \mathbf{R}^n : v(x) \geq 0\} = \cup_{r=1}^s \Sigma_{i_r}$ for some i_r, then $\mathrm{Per}(\cup_{r=1}^s \Sigma_{i_r}) \leq \mathcal{H}^{n-1}(S_v) \leq \mathrm{C}_1(H)$. Moreover, since H is a thick obstacle, $\cup_{r=1}^s \Sigma_{i_r}$ has finite perimeter and $\cup_{r=1}^s \Sigma_{i_r} \supseteq H$, we have $\mathcal{H}^{n-1}(H \setminus (\cup_{r=1}^s \Sigma_{i_r})_+)$=0. Thus, $\chi_{\cup_{r=1}^s \Sigma_{i_r}}$ is a test function for the capacitary problem on H, which implies $\mathrm{Per}(\cup_{r=1}^s \Sigma_{i_r}) = \mathrm{C}_1(H)$.

Eventually, if $\Sigma = \cup_{i\neq i_r} \Sigma_i$ it is easy to prove that there exists an index $t \geq 1$, with $a_t \neq a_{i_r}$ for every r, such that $v = \sum_{r=1}^s a_{i_r} \chi_{\Sigma_{i_r}} + a_t \chi_\Sigma$. □

Let us go back to the proof of inequality (3.15).

Given w_ε such that $MS_p(w_\varepsilon, B_{\rho_\varepsilon}) \leq m_\varepsilon(\eta) + r_\varepsilon^n$, let us check that the family $v_\varepsilon(x) = w_\varepsilon(r_\varepsilon x)$, $x \in B_{R_\varepsilon}$, where $R_\varepsilon = \rho_\varepsilon / r_\varepsilon$, satisfies the assumptions of Lemma 3.6 above with $H = E_+$. Indeed, (i) is trivially satisfied, while (ii) holds true since by scaling

$$\frac{MS_p(w_\varepsilon, B_{\rho_\varepsilon})}{r_\varepsilon^{n-1}} = r_\varepsilon^{1-p} \int_{B_{R_\varepsilon}} |\nabla v_\varepsilon|^p dx + \mathcal{H}^{n-1}(S_{v_\varepsilon}) \leq \mathrm{C}_1(E_+) + r_\varepsilon. \tag{3.16}$$

Moreover, (3.16) and Hölder's inequality yield

$$\int_{B_{R_\varepsilon}} |\nabla v_\varepsilon| dx \leq R_\varepsilon^{n-n/p} \|\nabla v_\varepsilon\|_{L^p(B_{R_\varepsilon})} \leq \left(\frac{3\varepsilon}{4k r_\varepsilon^{1-1/n}} \right)^{n-n/p} (\mathrm{C}_1(E_+) + r_\varepsilon)^{1/p}, \tag{3.17}$$

so that $\sup_\varepsilon \|Dv_\varepsilon\|(B_{R_\varepsilon}) < +\infty$, and (iii) is satisfied, too. Eventually, (iv) easily follows from (3.14) for $k \geq 2^{n+2}$, hence Lemma 3.6 implies (3.15).

To conclude fix $W' \subset\subset W$ and notice that for ε small $W' \subset \bigcup_{\mathcal{W}_\varepsilon} Q_\varepsilon(\underline{i}\varepsilon)$. Then

$$\begin{aligned} \liminf_\varepsilon \sum_{\mathcal{I}_\varepsilon^k} MS_p(u_\varepsilon, B_{\rho_\varepsilon}(\underline{i}\varepsilon)) &\geq \liminf_\varepsilon m_\varepsilon(\eta) \# \mathcal{I}_\varepsilon^k \\ &\geq \beta^{n-1} \lim_\varepsilon \frac{m_\varepsilon(\eta)}{r_\varepsilon^{n-1}} \left(\varepsilon^n \# \mathcal{W}_\varepsilon - \frac{1}{k^{n+1}} \right) \geq \beta^{n-1} \mathrm{C}_1(E_+) \left(\mathcal{L}^n(W') - \frac{1}{k^{n+1}} \right). \end{aligned} \tag{3.18}$$

To get (3.13), it remains to pass on the supremum on the sets $W' \subset\subset W$ and to recall that $W \supseteq V$.

Step 3. *(Estimate (3.7))* We eventually obtain (3.7) by collecting Step 1 and Step 2, and by passing to the limit as $k \to +\infty$ in (3.8), i.e.

$$\begin{aligned} \liminf_\varepsilon \mathcal{F}_\varepsilon(u_\varepsilon, A) &\geq \liminf_k \left(\liminf_\varepsilon MS_p(u_\varepsilon, A \setminus \cup_{\mathbf{Z}^n} B_{3\varepsilon/4k}(\underline{i}\varepsilon)) \right) \\ &+ \liminf_k \left(\liminf_\varepsilon MS_p(u_\varepsilon, A \cap \cup_{\mathbf{Z}^n} B_{3\varepsilon/4k}(\underline{i}\varepsilon)) \right) \geq \int_A |\nabla u|^p dx + \mathrm{C}_1(E_+) \beta^{n-1} \mathcal{L}^n(V). \end{aligned}$$

□

The lower bound inequality in the general case is an easy consequence of a standard truncation argument.

Proposition 3.7. *Under the hypotheses of Theorem 3.1, for every $u \in L^1(\Omega)$ there holds*

$$\Gamma\text{-}\liminf_{\varepsilon} \mathcal{F}_\varepsilon(u) \geq \mathcal{F}(u),$$

where $\mathcal{F}$ is defined in (3.2).

Proof. The thesis follows straightforward by Lemma 3.5 once one notices that the energies $\mathcal{F}_\varepsilon$, $\mathcal{F}$ are decreasing by truncation and the Mumford-Shah functional is continuos along such sequences. More precisely, if $v \in L^1(\Omega)$ and $M > 0$, denoting $(v \wedge M) \vee (-M)$ by v^M, if v satisfies the constraint v^M does, and there hold $MS_p(v^M) \leq MS_p(v)$ for every $M > 0$, and $MS_p(v^M) \to MS_p(v)$ as $M \to +\infty$. □

Remark 3.8. *As a consequence of Step 2 in Lemma 3.5 above, we have that in case $\varepsilon^{\frac{n}{n-1}} = o(r_\varepsilon)$, that is $\beta = +\infty$, the Γ-limit of $(\mathcal{F}_\varepsilon)$ equals $MS_p(u)$ if $u \in GSBV(\Omega)$, $u \geq 0$ $\mathcal{L}^n$ a.e. on Ω, and $+\infty$ otherwise in $L^1(\Omega)$. This follows straightforward from (3.18).*

Let us now conclude the proof of Theorem 3.1 and prove the upper bound inequality. We introduce the notation

$$\mathcal{U}_\rho(A) = \{x \in \mathbf{R}^n : \operatorname{dist}(x, A) < \rho\}$$

where $\rho > 0$, $A \subseteq \mathbf{R}^n$.

Proposition 3.9. *Under the hypotheses of Theorem 3.1, for every $u \in L^1(\Omega)$ there holds*

$$\Gamma\text{-}\limsup_{\varepsilon} \mathcal{F}_\varepsilon(u) \leq \mathcal{F}(u), \tag{3.19}$$

where $\mathcal{F}$ is defined in (3.2).

Proof. Let $u \in GSBV(\Omega)$ be such that $\mathcal{F}(u) < +\infty$, being otherwise the inequality trivially verified. We first prove the Γ-limsup inequality under the following additional assumptions

(a) $u \in SBV(\Omega)$ such that $\mathcal{H}^{n-1}(\overline{S_u} \setminus S_u) = 0$, $u \in W^{k,\infty}(\Omega \setminus \overline{S_u})$ for any $k \in \mathbf{N}$, and $\overline{S_u} \subseteq \cup_{j=1}^N \Sigma_j$ where Σ_j are $(n-1)$-simplexes;

(b) the set $\{x \in \Omega : u(x) < 0\}$ has finite perimeter in Ω, $\{x \in \Omega \setminus \overline{S_u} : u(x) = 0\}$ is a $(n-1)$-dimensional smooth manifold in $\Omega \setminus \overline{S_u}$.

By (2.10) and Remark 2.8 we choose a set of finite perimeter $D \subseteq Q_1$ with $\mathrm{C}_1(E_+) = \mathrm{Per}(D)$ and $\mathcal{H}^{n-1}(E_+ \setminus D_+) = 0$, which of course implies $\mathcal{L}^n(E \setminus D) = 0$.

Define $\mathcal{J} = \{\underline{i} \in \mathbf{Z}^n : \mathcal{L}^n(D_{r_\varepsilon}(\underline{i}\varepsilon) \cap \{x \in \Omega : u(x) < 0\}) > 0\}$, $\mathbf{D}_\varepsilon = \cup_{\underline{i} \in \mathcal{J}} D_{r_\varepsilon}(\underline{i}\varepsilon)$, and $u_\varepsilon \in L^1(\Omega)$ as $u_\varepsilon = u\chi_{\Omega \setminus \mathbf{D}_\varepsilon}$. Then, $u_\varepsilon \in SBV(\Omega)$ and by construction $u_\varepsilon \geq 0$ $\mathcal{L}^n$ a.e. on $\mathbf{D}_\varepsilon$, actually $u_\varepsilon = 0$ $\mathcal{L}^n$ a.e.

on $\mathbf{D}_\varepsilon$. Since $\mathcal{L}^n(\mathbf{D}_\varepsilon) \leq \#(\mathcal{J})r_\varepsilon^n \mathcal{L}^n(D) \leq c r_\varepsilon$, we have $u_\varepsilon \to u$ in $L^1(\Omega)$, and a direct computation shows

$$\begin{aligned}\mathcal{F}_\varepsilon(u_\varepsilon) &\leq \int_{\Omega\setminus\mathbf{D}_\varepsilon} |\nabla u|^p\,dx + \mathcal{H}^{n-1}(S_u \setminus \mathbf{D}_\varepsilon) + \mathrm{Per}(\mathbf{D}_\varepsilon)\\ &\leq \int_\Omega |\nabla u|^p\,dx + \mathcal{H}^{n-1}(S_u) + \#(\mathcal{J})r_\varepsilon^{n-1}\mathrm{Per}(D)\\ &\leq MS_p(u) + \mathrm{C}_1(E_+)\frac{r_\varepsilon^{n-1}}{\varepsilon^n}\mathcal{L}^n(\mathcal{U}_{\sqrt{n}\varepsilon}(\{x\in\Omega : u(x)<0\})). \end{aligned} \tag{3.20}$$

In the last inequality we used that $\#(\mathcal{J})\varepsilon^n = \mathcal{L}^n(\cup_{\underline{i}\in\mathcal{J}}Q_\varepsilon(\underline{i}\varepsilon))$ and $\cup_{\underline{i}\in\mathcal{J}}Q_\varepsilon(\underline{i}\varepsilon) \subseteq \mathcal{U}_{\sqrt{n}\varepsilon}(\{x\in\Omega : u(x)<0\})$. To estimate the Lebesgue measure of the last term in (3.20) we use the equality

$$\bigcap_{\varepsilon>0}\mathcal{U}_{\sqrt{n}\varepsilon}(\{x\in\Omega : u(x)<0\}) = \overline{\{x\in\Omega : u(x)<0\}},$$

so that

$$\lim_{\varepsilon\to0^+}\mathcal{L}^n(\mathcal{U}_{\sqrt{n}\varepsilon}(\{x\in\Omega : u(x)<0\})) = \mathcal{L}^n\left(\overline{\{x\in\Omega : u(x)<0\}}\right).$$

By passing to the limsup as $\varepsilon\to0^+$ in (3.20) we get

$$\limsup_{\varepsilon\to0^+}\mathcal{F}_\varepsilon(u_\varepsilon) \leq MS_p(u) + \mathrm{C}_1(E_+)\beta^{n-1}\mathcal{L}^n\left(\overline{\{x\in\Omega : u(x)<0\}}\right).$$

To obtain (3.19) it suffices to notice that

$$\mathcal{L}^n\left(\overline{\{x\in\Omega : u(x)<0\}}\setminus\{x\in\Omega : u(x)<0\}\right) = 0$$

thanks to (a), (b) and the regularity of $\partial\Omega$.

We now remove assumption (b). In order to do that it suffices to note that by applying Sard's lemma to u on $\Omega\setminus\overline{S_u}$ and by the BV Coarea formula (see Theorem 3.40 [1]), we can find a sequence $\eta_k\to0^-$ such that for any $k\in\mathbf{N}$ the functions $u-\eta_k$ satisfy (b). Hence, the previous step implies

$$\Gamma\text{-}\limsup_{\varepsilon}\mathcal{F}_\varepsilon(u-\eta_k) \leq \mathcal{F}(u-\eta_k) \leq \mathcal{F}(u),$$

and the upper bound inequality for u follows by letting $\eta_k\to0^-$ and by taking into account the lower semicontinuity of $\Gamma\text{-}\limsup_\varepsilon\mathcal{F}_\varepsilon$.

For a general function $u\in GSBV(\Omega)$ we use a density result with respect to Mumford-Shah type energies and in $L^1(\Omega)$ with functions satisfying (a) proved in [17] (see also [18] for a more general statement).

This given, consider (u_j) satisfying (a) and such that $u_j\to u$ in $L^1(\Omega)$ and $MS_p(u_j)\to MS_p(u)$, and let $\eta_k\to0^-$ be such that $\mathcal{L}^n(\{x\in\Omega : u_j(x)<\eta_k\})\to\mathcal{L}^n(\{x\in\Omega : u(x)<\eta_k\})$ as $j\to+\infty$ for every $k\in\mathbf{N}$.

Then, by using for every $j\in\mathbf{N}$ the identity $\Gamma\text{-}\limsup_\varepsilon\mathcal{F}_\varepsilon(u_j-\eta_k) = \mathcal{F}(u_j-\eta_k)$, and the lower semicontinuity of $\Gamma\text{-}\limsup_\varepsilon\mathcal{F}_\varepsilon$ we infer

$$\Gamma\text{-}\limsup_{\varepsilon}\mathcal{F}_\varepsilon(u-\eta_k) \leq \lim_j\mathcal{F}(u_j-\eta_k)$$

$$= \lim_j \left(MS_p(u_j) + \mathrm{C}_1(E_+)\beta^{n-1}\mathcal{L}^n(\{x \in \Omega : u_j(x) < \eta_k\})\right)$$
$$= MS_p(u) + \mathrm{C}_1(E_+)\beta^{n-1}\mathcal{L}^n(\{x \in \Omega : u(x) < \eta_k\}) \leq \mathcal{F}(u).$$

Passing to the liminf as $k \to +\infty$ and taking again into account the lower semicontinuity of $\Gamma\text{-}\limsup_\varepsilon \mathcal{F}_\varepsilon$ we finally conclude. □

Remark 3.10. *It is clear from the proof of Proposition 3.9 that in the regime* $r_\varepsilon = o\left(\varepsilon^{\frac{n}{n-1}}\right)$, *that is* $\beta = 0$, *the* Γ*-limit of* $(\mathcal{F}_\varepsilon)$ *is trivial and identically equal to* MS_p.

We now provide the proof of the bilateral obstacle case contained in Proposition 3.3.

Proof. (of Proposition 3.3) Lower Bound: First notice that for every $A \in \mathcal{A}(\Omega)$, $\varepsilon > 0$ and $u \in L^1(\Omega)$ we have

$$\mathcal{F}'_\varepsilon(u, A) \geq \mathcal{F}_\varepsilon(u, A), \quad \mathcal{F}'_\varepsilon(u, A) \geq \mathcal{F}_\varepsilon(-u, A). \tag{3.21}$$

Hence, given (u_ε) converging to u in $L^1(\Omega)$, by applying Proposition 3.7 to the two terms on the right hand sides of the inequalities in (3.21), we get

$$\liminf_\varepsilon \mathcal{F}'_\varepsilon(u_\varepsilon, A) \geq \mathcal{F}(u, A) = MS_p(u, A) + \mathrm{C}_1(E_+)\beta^{n-1}\mathcal{L}^n(\{x \in A : u(x) < 0\}) \tag{3.22}$$

and

$$\liminf_\varepsilon \mathcal{F}'_\varepsilon(u_\varepsilon, A) \geq \mathcal{F}(-u, A) = MS_p(u, A) + \mathrm{C}_1(E_+)\beta^{n-1}\mathcal{L}^n(\{x \in A : u(x) > 0\}). \tag{3.23}$$

In particular, this entails $u \in GSBV(\Omega)$ provided $\liminf_\varepsilon \mathcal{F}'(u_\varepsilon) < +\infty$. Moreover, the usual measure theoretic arguments imply the lower bound inequality. Indeed, the second terms in the sums on the right hand sides of (3.22), (3.23) are mutually orthogonal measures and the left hand side term is a superadditive set function defined on $\mathcal{A}(\Omega)$ (for details see Proposition 1.16 [5]).

Upper Bound: To conclude we construct a recovery sequence for any $u \in GSBV(\Omega)$ such that $\mathcal{F}'(u) < +\infty$. Moreover, we may assume $\mathcal{L}^n(\{x \in \Omega : u(x) \neq 0\}) > 0$, being otherwise the result trivial.

We keep the notation of Proposition 3.9, and first prove the limsup inequality under the additional assumptions (a) and (b) with the set $\{x \in \Omega : u(x) \neq 0\}$ playing the role of $\{x \in \Omega : u(x) < 0\}$ there. Supposing this, we may perform the very same construction of Proposition 3.9 substituting the 0 sub-level set of u with $\{x \in \Omega : u(x) \neq 0\}$. Indeed, the recovery sequence $(u_\varepsilon) \subset SBV(\Omega)$ built up there is such that $u_\varepsilon = 0$ $\mathcal{L}^n$ a.e. on $\mathbf{E}_\varepsilon$. Hence, using the same arguments one achieves

$$\mathcal{F}_\varepsilon(u_\varepsilon) \leq MS_p(u) + \mathrm{C}_1(E_+)\beta^{n-1}\mathcal{L}^n(\{x \in \Omega : u(x) \neq 0\}) + o(1),$$

passing to the limsup as $\varepsilon \to 0^+$ we get the desired inequality.

We now remove the regularity assumption (b) on the set $\{x \in \Omega : u(x) \neq 0\}$.

To do this, argue as in Proposition 3.9 and consider a positive sequence (η_k) such that $\eta_k \to 0^+$ as $k \to +\infty$ and both the sets $\{x \in \Omega : u(x) > \eta_k\}$, $\{x \in \Omega : u(x) < -\eta_k\}$ satisfy (b).

Let $u^k \in SBV(\Omega)$ be defined as $u^k = (u \vee \eta_k) + (u \wedge (-\eta_k))$, then notice that $|u(x)| \leq \eta_k \Leftrightarrow u^k(x) = 0$, $u(x) \geq \eta_k \Rightarrow u^k(x) = u(x) - \eta_k$, $u(x) \leq -\eta_k \Rightarrow u^k(x) = u(x) + \eta_k$, and $\mathcal{H}^{n-1}(S_{u^k} \setminus S_u) = 0$.
Clearly $u^k \to u$ in $L^1(\Omega)$ and

$$\begin{aligned}\Gamma\text{-}\limsup_{\varepsilon} \mathcal{F}'_\varepsilon(u^k) &\leq \mathcal{F}'(u^k) = MS_p(u^k) + \mathrm{C}_1(E_+)\beta^{n-1}\mathcal{L}^n(\{x \in \Omega : u^k(x) \neq 0\}) \\ &\leq MS_p(u) + \mathrm{C}_1(E_+)\beta^{n-1}\mathcal{L}^n(\{x \in \Omega : |u(x)| > \eta_k\}).\end{aligned}$$

Passing to the liminf as $k \to +\infty$ and taking into account the lower semicontinuity of $\Gamma\text{-}\limsup_\varepsilon \mathcal{F}'_\varepsilon$, we get the desired inequality.
Eventually, to finish the proof for any $u \in GSBV(\Omega)$ consider a sequence (u_j) satisfying (a) and such that $u_j \to u$ in $L^1(\Omega)$ and $MS_p(u_j) \to MS_p(u)$ (see Theorem 3.9 [17]), and let $\eta_k \to 0^+$ be such that $\mathcal{L}^n(\{x \in \Omega : |u_j(x)| > \eta_k\}) \to \mathcal{L}^n(\{x \in \Omega : |u(x)| > \eta_k\})$ as $j \to +\infty$ for every $k \in \mathbf{N}$.
Since $(u_j)^k \to u^k$ in $L^1(\Omega)$, arguing as in the last step of Proposition 3.9 we infer

$$\begin{aligned}\Gamma\text{-}\limsup_{\varepsilon} \mathcal{F}'_\varepsilon(u^k) &\leq \liminf_j \mathcal{F}'((u_j)^k) \\ &\leq \lim_j \big(MS_p(u_j) + \mathrm{C}_1(E_+)\beta^{n-1}\mathcal{L}^n(\{x \in \Omega : |u_j(x)| > \eta_k\})\big) \\ &= MS_p(u) + \mathrm{C}_1(E_+)\beta^{n-1}\mathcal{L}^n(\{x \in \Omega : |u(x)| > \eta_k\}) \leq \mathcal{F}'(u).\end{aligned}$$

Passing to the liminf as $k \to +\infty$ and taking again into account the lower semicontinuity of $\Gamma\text{-}\limsup_\varepsilon \mathcal{F}'_\varepsilon$ we finally conclude. □

Eventually, we prove the case in which Dirichlet boundary conditions are imposed.

Proof. (of Proposition 3.4) Lower Bound: The lower bound inequality can be easily derived from Proposition 3.3. Given $v \in GSBV(\Omega)$ denote by $\tilde{v}$ the function obtained extending v to 0 on $\mathbf{R}^n \setminus \Omega$. Then $\tilde{v} \in GSBV(\mathbf{R}^n)$ and fixed any open set $\Omega' \supset\supset \Omega$ with Lipschitz boundary, we have

$$MS_p(\tilde{v}, \Omega') = MS_p(v, \Omega) + \mathcal{H}^{n-1}(\{x \in \partial\Omega : \mathrm{tr}(v)(x) \neq 0\})$$

(see Theorem 3.84 and 3.87 [1]).
Given $(u_\varepsilon) \in GSBV(\Omega)$ with $\mathrm{tr}(u_\varepsilon) = 0$ on $\partial\Omega$ and converging to u in $L^1(\Omega)$, $(\tilde{u}_\varepsilon)$ converges to $\tilde{u}$ in $L^1(\Omega')$, and applying Proposition 3.3 with Ω replaced by Ω', we have

$$\begin{aligned}\liminf_{\varepsilon} \mathcal{D}_\varepsilon(u_\varepsilon) &= \liminf_{\varepsilon} \mathcal{F}'_\varepsilon(\tilde{u}_\varepsilon, \Omega') \geq \mathcal{F}'(\tilde{u}, \Omega') \\ &= \mathcal{F}'(u, \Omega) + \mathcal{H}^{n-1}(\{x \in \partial\Omega : \mathrm{tr}(u)(x) \neq 0\}) = \mathcal{D}(u).\end{aligned}$$

Upper Bound: It remains to prove the upper bound inequality. First note that a recovery sequence for $u \in GSBV(\Omega)$ with $\mathrm{tr}(u) = 0$ on $\partial\Omega$ is given by the one constructed when no boundary condition is imposed in Proposition 3.3 above.

Given a generic function $u \in GSBV(\Omega)$, it is possible to find a sequence $(u_j) \subset GSBV(\Omega)$ with $\mathrm{tr}(u_j) = 0$ on $\partial\Omega$ and converging to u in $L^1(\Omega)$ such that $\lim_j \mathcal{D}(u_j) = \mathcal{D}(u)$. Taken this into account the result follows by the lower semicontinuity of Γ-$\limsup \mathcal{D}_\varepsilon$.

This sequence can be obtained by modifying u in a suitable neighbourhood of the boundary in which the distance function is regular. Fixed a sequence of positive numbers r_j tending to 0 and denoted $d(x) = \mathrm{dist}(x, \partial\Omega)$, consider

$$u_j(x) = \begin{cases} u(x) & \text{if } 2r_j < d(x) \\ u(x + (d(x) - 2r_j)\nabla d(x)) & \text{if } r_j < d(x) < 2r_j \\ 0 & \text{if } 0 < d(x) < r_j. \end{cases}$$

It can be easily checked that

$$\begin{aligned} MS_p(u_j, \Omega) &\leq MS_p(u, \Omega) + cMS_p(u, \{x \in \Omega : r_j < d(x) < 2r_j\}) \\ &+\mathcal{H}^{n-1}(\{x \in \Omega : d(x) = r_j, \mathrm{tr}(u)(x - r_j\nabla d(x)) \neq 0\}) \end{aligned}$$

for a positive constant c not depending on j.

Defining $\varphi_j : \partial\Omega \to \Omega$ as $\varphi_j(y) = y + r_j\nabla d(y)$ (with a slight abuse of notation $\nabla d(y)$ denotes the inner normal to $\partial\Omega$ in y), we have that

$$\{x \in \Omega : d(x) = r_j, \mathrm{tr}(u)(x - r_j\nabla d(x)) \neq 0\} = \varphi_j(\{y \in \partial\Omega : \mathrm{tr}(u)(y) \neq 0\}).$$

The conclusion then follows using the fact that $\mathcal{H}^{n-1}(\varphi_j(H)) \leq (\mathrm{Lip}\varphi_j)^{n-1}\mathcal{H}^{n-1}(H)$ for any set H, and $\mathrm{Lip}\varphi_j \to 1$ as $j \to +\infty$. □

4. The case of thin obstacles

In this section we show how to deal with a general reference perforation set thus including *thin obstacles*, i.e. sets with Lebesgue measure zero. To consider (non-trivial) thin obstacles problems it is clear the need to express the constraint in a different form. To do that it suffices to recall that if $u \in GSBV(\Omega)$ and $MS_p(u) < +\infty$ the values $u_+(x)$, $u_-(x)$ are finite and specified for $\mathcal{H}^{n-1}$ a.e. $x \in \Omega$ (see Theorem 4.40 [1]).

Given a $\mathcal{H}^{n-1}$ measurable set $T \subseteq \overline{Q_1}$, for any $\varepsilon > 0$ let $r_\varepsilon \in (0, \varepsilon)$ and let $\mathbf{T}_\varepsilon = \Omega \cap \cup_{\mathbf{Z}^n} T_{r_\varepsilon}(\underline{i}\varepsilon)$. Consider the functional $\mathcal{F}_\varepsilon : L^1(\Omega) \to [0, +\infty]$ defined as

$$\mathcal{F}_\varepsilon(u) = \begin{cases} MS_p(u) & u \in GSBV(\Omega),\ u_+ \geq 0\ \mathcal{H}^{n-1} \text{ a.e. on } \mathbf{T}_\varepsilon \\ +\infty & \text{otherwise in } L^1(\Omega). \end{cases} \tag{4.1}$$

The asymptotic analysis of $(\mathcal{F}_\varepsilon)$ takes advantage of the ideas and techniques developed in Section 3. The main difference is in the proof of Lemma 4.4, the counterpart of Lemma 3.6 in this framework, for which substantial changes are required. This is not accidental and a mere technical fact, we want to point out that Lemma 4.4 relies on the deep relaxation results contained in Chapter IV [26] and in [10] (see Theorem 2.7).

Theorem 4.1. *Let T be a $\mathcal{H}^{n-1}$ measurable set, assume that $r_\varepsilon/\varepsilon^{\frac{n}{n-1}} \to \beta \in [0,+\infty)$ as $\varepsilon \to 0^+$. Then, $(\mathcal{F}_\varepsilon)$ Γ-converges to $\mathcal{F} : L^1(\Omega) \to [0,+\infty]$ defined by*

$$\mathcal{F}(u) = \begin{cases} MS_p(u) + \mathrm{C}_1(T)\beta^{n-1}\mathcal{L}^n\left(\{x \in \Omega : u(x) < 0\}\right) & u \in GSBV(\Omega), \\ +\infty & \text{otherwise in } L^1(\Omega), \end{cases} \tag{4.2}$$

w.r.t. the L^1 convergence.

Remark 4.2. *Let us point out how Theorem 3.1 can be recovered from Theorem 4.1 above. To this aim we notice that given a set E the following equivalence holds*

$$u \geq 0 \ \mathcal{L}^n \ \text{ a.e. on } \ E \Longleftrightarrow u_+ \geq 0 \ \mathcal{H}^{n-1} \ \text{ a.e. on } \ E_+, \tag{4.3}$$

so that one can rephrase the unilateral obstacle condition in sense $\mathcal{L}^n$ on E with the more precise $\mathcal{H}^{n-1}$ meaning exactly on E_+. Roughly speaking, the equivalence in (4.3) means that the constraint for u intended in the $\mathcal{L}^n$ sense is active, for a suitable representant, only on the $\mathcal{L}^n$ measure theoretic closure of E, and thus it is neglected on lower dimensional parts of the set.
By taking this into account, the functionals in the statement of Theorem 3.1 can be rewritten as

$$\mathcal{F}_\varepsilon(u) = \begin{cases} MS_p(u) & u \in GSBV(\Omega),\, u_+ \geq 0 \ \mathcal{H}^{n-1} \text{ a.e. on } (\mathbf{E}_+)_\varepsilon \\ +\infty & \text{otherwise in } L^1(\Omega), \end{cases}$$

Theorem 3.1 then follows by applying Theorem 4.1 with $T = E_+$.

Remark 4.3. *It is worth noting that a priori the functional $\mathcal{F}_\varepsilon$ in (4.1) could be not L^1 lower semicontinuous. More generally, given a $\mathcal{H}^{n-1}$ measurable set $H \subseteq \Omega$, one can study the lower semicontinuity properties of the functional $G : L^1(\Omega) \to [0,+\infty]$ defined as*

$$G(u) = \begin{cases} MS_p(u) & u \in GSBV(\Omega),\, u_+ \geq 0 \ \mathcal{H}^{n-1} \text{ a.e. on } H \\ +\infty & \text{otherwise in } L^1(\Omega). \end{cases}$$

In a forthcoming paper (see [30]) we will prove that the lower semicontinuous envelope of G in the L^1 topology is given by

$$\mathrm{sc}^-(G)(u) = \begin{cases} MS_p(u) + \frac{1}{2}\sigma\left(\{x \in H \cap S_u : u_+(x) < 0\}\right) + \sigma\left(\{x \in H \setminus S_u : u_+(x) < 0\}\right) \\ \hfill u \in GSBV(\Omega) \\ +\infty \hfill \text{otherwise in } L^1(\Omega), \end{cases}$$

where σ is the measure defined in (2.11). Thus, by taking into account Theorem 4.1, well known results (see Proposition 6.11 [22]*) yield the Γ-convergence to the functional $\mathcal{F}$ in (4.2) of the energies*

$$\mathrm{sc}^-(\mathcal{F}_\varepsilon)(u) = \begin{cases} MS_p(u) + \frac{1}{2}\sigma\left(\{x \in \mathbf{T}_\varepsilon \cap S_u : u_+(x) < 0\}\right) + \sigma\left(\{x \in \mathbf{T}_\varepsilon \setminus S_u : u_+(x) < 0\}\right) \\ \hfill u \in GSBV(\Omega) \\ +\infty \hfill \text{otherwise in } L^1(\Omega). \end{cases}$$

As a further development of this research we are investigating the compactness and integral representation properties of Mumford-Shah type energies with general obstacle constraints, as those established by Dal Maso in the Sobolev setting ([20]*, see also* [10],[11]*).*

We now turn to the proof of Theorem 4.1. As already stated, we will point out only the changes needed in the proof of Theorem 3.1 in order to reach the conclusion. The set T in Theorem 4.1 plays the same role of E_+ in Lemma 3.5. Moreover, we keep the same notation used in Section 3 to which obviously we refer.

Proof. (of Theorem 4.1) Lower bound: The proof of Lemma 3.5 goes through until the capacitary estimate (3.13) of Step 2, having assumed $\mathcal{H}^{n-1}(E) > 0$ being otherwise the statement trivial. The latter is now a consequence of Lemma 4.4 below. Given this for granted, to prove the lower bound inequality for sequences bounded in L^∞ it suffices to verify that the blow-up functions v_ε satisfy the assumptions of Lemma 4.4. The same arguments used in Theorem 3.1 assure (i) and (iii), while (ii) follows by (3.17) taking into account that the right hand side in that formula is bounded as a function of ε and infinitesimal as $k \to +\infty$.
Eventually, the truncation argument of Proposition 3.7 needs no change, so that the lower bound is established.
Upper Bound: The same argument of Proposition 3.9 works by substituting in the construction of the recovery sequence a minimizing set D with a minimizing sequence for the capacitary problem for T.
□

The statement of Lemma 4.4 below is given in a sligthly more general framework than needed in our context.

Lemma 4.4. *Let H be a bounded $\mathcal{H}^{n-1}$ measurable set with $\mathcal{H}^{n-1}(H) > 0$, $N \in \mathbf{N}$, $N \geq 4$, and $v_\varepsilon \in BV(B_{R_\varepsilon})$, $R_\varepsilon \to +\infty$, be such that*

(i) $v_\varepsilon^+ \geq 0$ $\mathcal{H}^{n-1}$ *a.e. on* H, $\sup_\varepsilon \|v_\varepsilon\|_{L^\infty(B_{R_\varepsilon})} < +\infty$,
(ii) $\sup_\varepsilon \|Dv_\varepsilon\|(B_{R_\varepsilon} \setminus S_{v_\varepsilon}) < \frac{1}{N}$,
(iii) *there exists* $\zeta < 0$ *such that* $\mathcal{L}^n(\{x \in B_{R_\varepsilon} : v_\varepsilon(x) \geq \zeta\}) < \frac{1}{2}\mathcal{L}^n(B_{R_\varepsilon})$.

Then, there exists a positive constant $c = c(\zeta)$ such that $\liminf_\varepsilon \mathcal{H}^{n-1}(S_{v_\varepsilon}) \geq \mathrm{C}_1(H) - \frac{c}{\sqrt{N}}$.

Proof. It is not restrictive to assume $\liminf_\varepsilon \mathcal{H}^{n-1}(S_{v_\varepsilon}) < +\infty$, being otherwise the statement trivial. Let (v_{ε_j}) be such that $\lim_j \mathcal{H}^{n-1}(S_{v_{\varepsilon_j}}) = \liminf_\varepsilon \mathcal{H}^{n-1}(S_{v_\varepsilon}) < +\infty$, for the sake of simplicity for the rest of the proof we set $v_j = v_{\varepsilon_j}$ and $R_j = R_{\varepsilon_j}$.

Step 1. *For any open set A, $v \in BV(A)$ satisfying $v^+ \geq 0$ $\mathcal{H}^{n-1}$ a.e. on H, with $H \subseteq A$, and for any $\delta < 0$, there exists $\eta \in (\delta, 0)$ for which $\mathcal{L}^n(\{x \in A : v(x) \geq \eta\}) > 0$, $\mathrm{Per}(\{x \in A : v(x) \geq \eta\}, A) < +\infty$ and $\mathcal{H}^{n-1}(H \setminus \{x \in A : v(x) \geq \eta\}_+) = 0$.*

Let us first prove that there exists $t \in (\delta, 0)$ for which the corresponding super-level set has positive measure. Arguing by contradiction, if there was $\delta_o < 0$ such that $\mathcal{L}^n(\{x \in A : v(x) \geq t\}) = 0$ for every $t \in (\delta_o, 0)$, then the very definition of v^+ would give $v^+(x) \leq \delta_o$ $\mathcal{H}^{n-1}$ a.e. on H, which is clearly a contradiction since $v^+ \geq 0$ $\mathcal{H}^{n-1}$ a.e. on H and $\mathcal{H}^{n-1}(H) > 0$.

Moreover, since $\{x \in A : v(x) \geq \eta\} \supseteq \{x \in A : v(x) \geq t\}$ if $\eta < t$ and $\mathrm{Per}(\{x \in A : v(x) \geq \eta\}, A) < +\infty$ for $\mathcal{L}^1$ a.e. $\eta \in \mathbf{R}$ we get $\{x \in A : v(x) \geq \eta\}_+ \supseteq \{x \in A : v^+(x) \geq t\}$ for $\mathcal{L}^1$ a.e. $\eta \in \mathbf{R}, \eta < t$. Since $\{x \in A : v(x) \geq \eta\}_+ \supseteq \{x \in A : v^+(x) \geq t\} \supseteq \{x \in A : v^+(x) \geq 0\}$, it is clear that we can find $\eta \in (\delta, t)$ for which all the required conditions are satisfied.

Step 2. *There exist $w_j \in SBV(B_{R_j})$ and $\eta_j \in (\zeta, 0)$, with ζ as in assumption (iii), such that $\nabla w_j = 0$ $\mathcal{L}^n$ a.e. on B_{R_j}, $\mathcal{H}^{n-1}(H \setminus \{x \in B_{R_j} : w_j(x) \geq \eta_j\}_+) = 0$, $\sup_j \mathcal{L}^n(\{x \in B_{R_j} : w_j(x) \geq \eta_j\}) < +\infty$, $\sup_j \mathrm{Per}(\{x \in B_{R_j} : w_j(x) \geq \eta_j\}, B_{R_j}) < +\infty$, and*

$$\liminf_j \mathcal{H}^{n-1}(S_{v_j}) \geq \liminf_j \mathcal{H}^{n-1}(S_{w_j}) - \frac{c}{\sqrt{N}}.$$

Let $C = \sup_j \|v_j\|_{L^\infty(B_{R_j})}$ and $k_N = [\sqrt{N}]$, then apply the BV Coarea formula to get

$$\|Dv_j\|(B_{R_j} \setminus S_{v_j}) = \sum_{i=0}^{k_N - 1} \int_{\alpha_i}^{\alpha_{i+1}} \mathrm{Per}(\{x \in B_{R_j} : v_j(x) \geq t\}, B_{R_j} \setminus S_{v_j}) dt,$$

where $\alpha_0 = -C$, $\alpha_{i+1} = \alpha_i + 2C/k_N$ for $0 \leq i \leq k_N - 1$.

Let $0 \leq r \leq k_N - 1$ be such that $\zeta \in (\alpha_{r-1}, \alpha_r]$; and first assume that $\alpha_{r+1} \leq 0$.

For every $0 \leq i \leq k_N - 1$ by the Mean value theorem we may find $t_i^j \in (\alpha_i, \alpha_{i+1})$ such that

$$\frac{2C}{k_N} \mathrm{Per}(\{x \in B_{R_j} : v_j(x) \geq t_i^j\}, B_{R_j} \setminus S_{v_j}) \leq \int_{\alpha_i}^{\alpha_{i+1}} \mathrm{Per}(\{x \in B_{R_j} : v_j(x) \geq t\}, B_{R_j} \setminus S_{v_j}) dt. \tag{4.4}$$

Let $0 \leq s \leq k_N - 1$ be such that $0 \in (t_s^j, t_{s+1}^j]$, and note that $t_s^j \in (\zeta, 0)$ since $\alpha_{r+1} \leq 0$ implies $t_s^j \geq t_r^j > \alpha_r$. Consider $\eta_j \in (t_s^j, 0)$ provided by Step 1 and the sets $\Sigma_i^j = \{x \in B_{R_j} : t_i^j \leq v_j(x) < t_{i+1}^j\}$, then define the function $w_j : B_{R_j} \to \mathbf{R}$ as $w_j(x) = \eta_j$ if $x \in \Sigma_s^j$ and $w_j(x) = t_i^j$ if $x \in \Sigma_i^j$, $0 \leq i \leq k_N - 1$ and $i \neq s$.

Clearly, being Σ_i^j of finite perimeter in B_{R_j}, we have $w_j \in SBV(B_{R_j})$ with $\nabla w_j = 0$ $\mathcal{L}^n$ a.e. on B_{R_j}, $S_{w_j} \subseteq \cup_{i=0}^{k_N - 1} \partial^* \Sigma_i^j$, and $\mathcal{H}^{n-1}(H \setminus \{x \in B_{R_j} : w_j(x) \geq \eta_j\}_+) = 0$ since by the choice of $\eta_j \in (t_s^j, 0)$

$$\{x \in B_{R_j} : w_j(x) \geq \eta_j\} = \{x \in B_{R_j} : v_j(x) \geq t_s^j\} \supseteq \{x \in B_{R_j} : v_j(x) \geq \eta_j\}.$$

Moreover, by assumption (ii) and the definition of k_N there holds

$$\mathcal{H}^{n-1}(S_{w_j}) \leq \mathcal{H}^{n-1}(S_{v_j}) + \sum_{i=0}^{k_N-1} \mathrm{Per}(\{x \in B_{R_j} : v_j(x) \geq t_i^j\}, B_{R_j} \setminus S_{v_j})$$
$$\leq \mathcal{H}^{n-1}(S_{v_j}) + \frac{k_N}{2C}\|Dv_j\|(B_{R_j} \setminus S_{v_j}) \leq \mathcal{H}^{n-1}(S_{v_j}) + \frac{c}{\sqrt{N}}.$$

Eventually, the Relative isoperimetric inequality in balls (see Remark 3.50 [1]), the choices $\zeta < t_s^j < \eta_j < 0$ and assumption (iii) imply that $\sup_j \mathcal{L}^n(\{x \in B_{R_j} : w_j(x) \geq \eta_j\}) < +\infty$.
In case $\zeta \in (\alpha_{r-1}, \alpha_r]$ with $\alpha_{r+1} > 0$, this construction fails since t_s^j might not satisfy $t_s^j > \zeta$. Nevertheless, this case can be handled by slightly modifying the choice of the t_i^j's.
Indeed, choose t_i^j as in (4.4) for $i \notin \{r-1, r\}$, choose $t_r^j \in (\zeta, 0)$ be such that

$$|\zeta|\mathrm{Per}(\{x \in B_{R_j} : v_j(x) \geq t_r^j\}, B_{R_j} \setminus S_{v_j}) \leq \int_{\alpha_{r-1}}^{\alpha_{r+1}} \mathrm{Per}(\{x \in B_{R_j} : v_j(x) \geq t\}, B_{R_j} \setminus S_{v_j})dt,$$

and set $t_{r-1}^j = t_r^j$. Let $\eta_j \in (t_r^j, 0)$ be provided by Step 1, and define $w_j : B_{R_j} \to \mathbf{R}$ as $w_j(x) = \eta_j$ if $x \in \Sigma_r^j$, and $w_j(x) = t_i^j$ if $x \in \Sigma_i^j$, $0 \leq i \leq k_N - 1$ and $i \notin \{r-1, r\}$. Notice that $\Sigma_{r-1}^j = \emptyset$.
The same arguments exploited before entail that (w_j) still satisfies the statement of Step 2.
Step 3. *Conclusion.* Set $\Sigma_j = \{x \in B_{R_j} : w_j(x) \geq \eta_j\}$, then $\mathcal{H}^{n-1}(S_{w_j}) \geq \mathrm{Per}(\Sigma_j, B_{R_j})$ (see Theorem 4.23 [1]), which, together with Step 2, yield

$$+\infty > \lim_j \mathcal{H}^{n-1}(S_{v_j}) \geq \liminf_j \mathcal{H}^{n-1}(S_{w_j}) - \frac{c}{\sqrt{N}} \geq \liminf_j \mathrm{Per}(\Sigma_j, B_{R_j}) - \frac{c}{\sqrt{N}}. \tag{4.5}$$

By applying the BV compactness theorem we may extract a subsequence (not relabeled for convenience) and find a set Σ with locally finite perimeter in $\mathbf{R}^n$, such that $\chi_{\Sigma_j} \to \chi_\Sigma$ in $L^1_{\mathrm{loc}}(\mathbf{R}^n)$. By Theorem 6.1 [10] (see also Chapter IV [26]) for every $R > 0$ we get

$$\liminf_j \mathrm{Per}(\Sigma_j, B_R) \geq \mathrm{Per}(\Sigma, B_R) + \sigma((H \setminus \Sigma_+) \cap B_R), \tag{4.6}$$

so that by combining (4.5) and (4.6), and by passing to the supremum on R, Σ has finite perimeter in $\mathbf{R}^n$ and moreover

$$\lim_j \mathcal{H}^{n-1}(S_{v_j}) \geq \mathrm{Per}(\Sigma) + \sigma(H \setminus \Sigma_+) - \frac{c}{\sqrt{N}}.$$

The thesis eventually follows by taking into account (2.12) of Theorem 2.7. □

5. Further results

In the previous sections we have described the asymptotic behaviour of the Mumford-Shah energy in periodically perforated domains. In the present section we extend the results of Sections 3 and 4 to more general free-discontinuity energies. We limit ourselves to state and give the hints of the proof of the generalization of Theorem 4.1 in this setting, being then the analogous of Proposition 3.3 and 3.4 trivial.
Let $p > 1$ and $\varphi, \psi : \mathbf{R}^n \to \mathbf{R}$ be continuous functions such that

(a) φ is convex, and there exist constants $c_1, c_3 > 0$ and $c_2 \in \mathbf{R}$ such that for every $\xi \in \mathbf{R}^n$

$$c_1|\xi|^p - c_2 \leq \varphi(\xi) \leq c_3(|\xi|^p + 1);$$

(b) ψ is a norm on $\mathbf{R}^n$, and there exist constants $c_4, c_5 > 0$ such that for every $\nu \in \mathbf{S}^{n-1}$

$$c_4 \leq \psi(\nu) \leq c_5.$$

Analogously to the case in which ψ is the euclidean norm one can define an anisotropic capacity as follows: For any set $E \subseteq \mathbf{R}^n$ let

$$C_\psi(E) = \inf\left\{\int_{\partial^* D} \psi(\nu_{\partial^* D}) d\mathcal{H}^{n-1} : D \text{ is } \mathcal{L}^n \text{ measurable}, \mathcal{L}^n(D) < +\infty, \mathcal{H}^{n-1}(E \setminus D_+) = 0\right\}.$$

Different characterizations of C_ψ, similar to those of Proposition 2.5, can be given as follows from Theorem 6.1 [10].

Given a $\mathcal{H}^{n-1}$ measurable set $T \subseteq \overline{Q_1}$, for any $\varepsilon > 0$ let $r_\varepsilon \in (0, \varepsilon)$ and let $\mathbf{T}_\varepsilon = \Omega \cap \cup_{\mathbf{Z}^n} T_{r_\varepsilon}(\underline{i}\varepsilon)$. Consider the functional $\mathcal{F}_\varepsilon : L^1(\Omega) \to [0, +\infty]$ defined as

$$\mathcal{F}_\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega \varphi(\nabla u) dx + \int_{S_u} \psi(\nu_u) d\mathcal{H}^{n-1} & u \in GSBV(\Omega),\ u_+ \geq 0\ \mathcal{H}^{n-1} \text{ a.e. on } \mathbf{T}_\varepsilon \\ +\infty & \text{otherwise in } L^1(\Omega). \end{cases} \tag{5.1}$$

We are now in a position to state the following result whose proof is just a technical adjustment of those of Theorem 3.1 and 4.1.

Theorem 5.1. *Let T be a $\mathcal{H}^{n-1}$ measurable set, assume that $r_\varepsilon/\varepsilon^{\frac{n}{n-1}} \to \beta \in [0, +\infty)$ as $\varepsilon \to 0^+$. Suppose that φ and ψ satisfy assumptions (a) and (b) above, then $(\mathcal{F}_\varepsilon)$ Γ-converges to $\mathcal{F} : L^1(\Omega) \to [0, +\infty]$ defined by*

$$\mathcal{F}(u) = \begin{cases} \displaystyle\int_\Omega \varphi(\nabla u) dx + \int_{S_u} \psi(\nu_u) d\mathcal{H}^{n-1} + C_\psi(T)\beta^{n-1}\mathcal{L}^n(\{x \in \Omega : u(x) < 0\}) & u \in GSBV(\Omega), \\ +\infty & \text{otherwise in } L^1(\Omega) \end{cases}$$

w.r.t. the L^1 convergence.

Proof. Lower bound: We first point out that estimate (3.6) in Lemma 3.5 follows directly by Theorem 2.2.

Moreover, in order to get a gradient estimate as that in (3.9) of Step 1, the same argument developed there can be repeated replacing the function $|\cdot|^p$ with φ, and taking into account the convexity of φ and assumption (a).

Furthermore, a capacitary estimate as that in (3.13) of Step 2 follows by substituting in the statement of Lemma 4.4 the total variation of a BV function with the anisotropic variation

$$\int_\Omega \psi\left(\frac{dDu}{d\|Du\|}\right) d\|Du\|, \tag{5.2}$$

and in its conclusion C_1 with C_ψ. Indeed, thanks to assumption (b), one can use for the anisotropic variation in (5.2) suitable versions of the BV Coarea formula (see Lemma 2.4 [19]) and of the relaxation result for energies with linear growth with obstacles (see Theorem 7.1 [10]).

Eventually, the truncation argument of Proposition 3.7 can be carried out with only minor changes.

Upper bound: The proof of Proposition 3.9 works by substituting in the construction of the recovery sequence a minimizing set D for the usual capacity with a minimizing sequence for the anisotropic capacitary problem for T related to ψ. $\square$

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