On a Characterisation of Perimeters in $\mathbf{R}^N$ via Heat Semigroup

M. Miranda Jr* D. Pallara* F. Paronetto* M. Preunkert†

Abstract

In this poster we present the results contained in [9]; in that paper we generalise to all sets with finite perimeter an equality concerning the short time behaviour of the heat semigroup proved for balls in [8] and exploited there in connection with the isoperimetric inequality. For sets with smooth boundary a more precise result is shown. The above result for sets with finite perimeter gives also a characterization of the perimeter using the heat semigroup very similar to the one used by De Giorgi in [7] to define the perimeter. We also extend these results to all the function with bounded variation, giving a relation between the jump part of the total variation measure and the small time diffusion of the level sets.

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Introduction

Sets with finite perimeter have been introduced by E. De Giorgi in the fifties (see [6], [7]), as a part of the theory of functions of bounded variation, in order to deal with geometric variational problems and have proved to be very useful in several contexts. The first researches of De Giorgi were connected with the investigations of R. Caccioppoli, and in fact sets with finite perimeter are also called *Caccioppoli sets*. Let us refer to [2] for a comprehensive treatment of BV functions and the properties of sets with finite perimeter. De Giorgi's original definition of the perimeter of a (measurable) set $E \subset \mathbf{R}^N$ was based on the heat semigroup $(T(t))_{t\geq 0}$ in $\mathbf{R}^N$, because of its regularising effects, and can be phrased as follows:

$$P(E) = \lim_{t\to 0} \|\nabla_x T(t)\chi_E\|_{L^1(\mathbf{R}^N)}, \tag{1}$$

where χ_E denotes the characteristic function of E and, for any function $f \in L^1(\mathbf{R}^N)$,

$$T(t)f(x) = \frac{1}{(4\pi t)^{N/2}} \int_{\mathbf{R}^N} f(y) e^{-|x-y|^2/4t} dy \int_{\mathbf{R}^N} f(y) p_N(x,y,t) dy.$$

*Dipartimento di Matematica "Ennio De Giorgi", Università di Lecce, C.P.193, 73100, Lecce, Italy. e-mail: michele.miranda@unile.it, diego.pallara@unile.it, fabio.paronetto@unile.it

†Mathematisches Institut, Universität Tübingen, Auf der Morgenstelle 10, D-72076 Tübingen, Germany. e-mail: mapr@fa.uni-tuebingen.de

In [8] M. Ledoux investigated in a different perspective some connections between the heat semigroup $(T(t))_{t\geq 0}$ on $L^2(\mathbf{R}^N)$ and the isoperimetric inequality, observing that the L^2-inequality

$$\|T(t)\chi_E\|_2 \leq \|T(t)\chi_B\|_2 \tag{2}$$

for all sets with smooth boundary E with the same volume as the ball B implies the isoperimetric inequality. By the self-adjointness of the operators $T(t)$ and

$$\|T(\tfrac{t}{2})\chi_E\|_2^2\langle T(\tfrac{t}{2})\chi_E, T(\tfrac{t}{2})\chi_E\rangle = \langle T(t)\chi_E, \chi_E\rangle,$$

the behaviour of $\langle T(t)\chi_E, \chi_E\rangle$ is related to the L^2-norm of $T(t)\chi_E$. Here we have used the notation $\langle f, g\rangle = \int_{\mathbf{R}^N} fg dx$ whenever the integral is finite. Inequality (2), in turn, had been deduced by the isoperimetric inequality by Baernstein ([3]). In [8] one important point has been the formula

$$\lim_{t\to 0}\sqrt{\frac{\pi}{t}}\,\langle T(t)\chi_B, \chi_{B^c}\rangle = P(B), \tag{3}$$

where B is a ball, and the inequality

$$\sqrt{\frac{\pi}{t}}\langle T(t)\chi_E, \chi_{E^c}\rangle \leq P(E) \qquad \text{for every } t\geq 0, \tag{4}$$

which has been generalised in [10] for all $E \subset \mathbf{R}^N$ such that either E or its complementary set E^c has a finite volume (otherwise, both terms are infinite). If E and B have the same volume, $|E| = |B|$, from the elementary relation

$$\langle T(t)\chi_E, \chi_{E^c}\rangle \;=\; \langle T(t)\chi_E, 1\rangle - \langle T(t)\chi_E, \chi_E\rangle|E| - \langle T(t)\chi_E, \chi_E\rangle \qquad \text{for every } t\geq 0$$

it follows that the L^2-inequality (2) is equivalent to

$$\langle T(t)\chi_E, \chi_{E^c}\rangle \geq \langle T(t)\chi_B, \chi_{B^c}\rangle \qquad \text{for every } t\geq 0,$$

and the semigroup inequality (2) implies the isoperimetric inequality for Caccioppoli sets in $\mathbf{R}^N$. In connection with these results, it seems to be interesting to pursue the investigation of the relationships between the perimeter of a set and the short-time behaviour of the heat semigroup.

In [9] we present two different proofs of formula (3); the first one applies when ∂A admits a tubular neighbourghood with the unique projection property, which is exactly the case when ∂A is uniformly $C^{1,1}$. The second proof is based on weak* convergence of measures and applies to every Caccioppoli set. Both proofs use heavily the fact that the heat kernel can be factorised, that is

$$p_N(x, y, t) = p_{N-1}(x', y', t)p_1(x_N, y_N, t), \quad x = (x', x_N), y = (y', y_N),$$

and on the fact that the heat kernel is given by a convolution.

The result given for general Caccioppoli set can be extended also to BV function; this extension gives, among other properties, that the diffusion for graphs of BV functions only occours on the jump set, in the sense that will be clarified in Theorem 2.5.

Let us point out that the same characterisation of finite perimeter sets and of BV functions, though formulated in different terms, is also proved, following a different approach, in the papers [4] and [5] (see also [1]), where convolution kernels more general than the Gauss-Weierstrass one are considered.

1 Diffusion for smooth sets

The aim of this paper is twofold: we prove that equality (3) holds true not only for balls, but for all Caccioppoli sets, and we also prove a more precise result for sets with smooth boundary (see Theorem 1 below). By *smooth* we mean the minimal regularity ensuring the unique projection property in a tubular neighbourhood of the boundary. To this end, the Lipschitz continuity of the unit normal vector field is the natural requirement. We say that $A \subset \mathbf{R}^N$ is *uniformly $C^{1,1}$-regular* if there are $\varrho, L > 0$ such that for every $p \in \partial A$ the set $\partial A \cup B_\varrho(p)$ is the graph of a $C^{1,1}$ function ψ with $\|\nabla\psi\|_\infty \le L$. Setting

$$A^\delta := \{x \in A^c : \mathrm{dist}(x, \partial A) \le \delta\}, \qquad A_\varepsilon := \{x \in A : \mathrm{dist}(x, \partial A) \le \varepsilon\} \tag{5}$$

for $\delta, \varepsilon > 0$, we prove the following Theorem, where we denote by $\mathcal{H}^{N-1}$ the $(N-1)$-dimensional Hausdorff measure.

Theorem 1 *Let $A \subset \mathbf{R}^N$ be $C^{1,1}$-regular. Let A_ε and A^δ be an inner and outer tubular neighbourhood of ∂A defined in (5). Then for every continuous $\varphi : \mathbf{R}^N \to \mathbf{R}$ with compact support the equality*

$$\lim_{t \to 0} \sqrt{\frac{\pi}{t}}\, \langle T(t)\chi_{A_\varepsilon}, \varphi\chi_{A^\delta}\rangle \int_{\partial A} \varphi \, d\mathcal{H}^{N-1}$$

holds.

Equality (3) for bounded uniformly $C^{1,1}$-regular sets follows easily from Theorem 1, since by direct computation

$$\langle T(t)\chi_{A_\varepsilon}, \chi_{A^\delta}\rangle \le \langle T(t)\chi_{A_\varepsilon}, \chi_{A^c}\rangle \le \langle T(t)\chi_A, \chi_{A^c}\rangle$$

and then

$$P(A) = \lim_{t \to 0} \sqrt{\frac{\pi}{t}}\, \langle T(t)\chi_{A_\varepsilon}, \chi_{A^\delta}\rangle \le \liminf_{t \to 0} \sqrt{\frac{\pi}{t}}\, \langle T(t)\chi_A, \chi_{A^c}\rangle \le P(A),$$

where the last inequality has been proved in [10]. It is important to notice that the perimeter of a regular set A can be recovered by the heat amount in the tubular neighbourhoods A^δ and A_ε and not by the heat amount in the interior of A.

Concerning the proof of Theorem 1, it is divided into several steps. The first one starts assuming that A is an halfspace and the function φ doesn't depend on the direction othogonal to ∂A; in this case the computation is very simple, thanks to the factorisation property of the heat kernel. In the second step, we deal again with an halfspace and we consider a general function φ; this step shows in particular that the heat diffusion for the halfspace is orthogonal to the boundary. In the last step, we show that also for a general smooth set A the diffusion is transversal to ∂A; the result follows from a partition of unity argument. In this last step we also use the fact that the transversal diffusion is given by the heat amount contained in a fixed set of the covering of the partition of unity, and the contributions coming from the other elements vanish as t tends to 0.

Remark 1.1 Since for compact subsets $A \subset \mathbf{R}^N$ with smooth boundary ∂A the perimeter $P(A)$ and the $(N-1)$-dimensional Hausdorff measure $\mathcal{H}^{N-1}(\partial A)$ coincide (cf [2, Proposition 3.62]), we also have

$$\lim_{t \to 0} \sqrt{\frac{\pi}{t}}\, \langle T(t)\chi_A, \chi_{A^c}\rangle = \mathcal{H}^{N-1}(\partial A).$$

2 Diffusion for Caccioppoli sets and for BV functions

In the case of Caccioppoli sets, it is possible to prove the following result, which gives the heat amount coming from a given set E into another set F.

Theorem 2.1 *Let $E, F \subset \mathbf{R}^N$ be sets of finite perimeter; then the following equality holds:*

$$\lim_{t\to 0}\sqrt{\frac{\pi}{t}}\langle \chi_E - T(t)\chi_E, \chi_F\rangle = \int_{\mathcal{F}E\cap\mathcal{F}F} \nu_E(x)\cdot\nu_F(x)d\mathcal{H}^{N-1}(x). \tag{6}$$

The proof of Theorem 2.1 is based on a weak* convergence result: more precisely, for $x \in \mathcal{F}E$, the measures

$$d\mu_{s,x} = \mathcal{L}^N \llcorner \left(\frac{E-x}{\sqrt{s}}\right),$$

are weakly* convergent, as $s \to 0$, to the Lebesgue measure restricted to the halfspace

$$H_{\nu_E(x)} = \left\{z \in \mathbf{R}^N : z\cdot\nu_E(x) \geq 0\right\}.$$

This convergence property implies in particular that one can still consider the heat diffusion as to be transversal to the boundary $\mathcal{F}E$ of E and the result again relies on the factorisation property of the heat kernel.
An important corollary of the previous Theorem is given by the following.

Theorem 2.2 *Let $E \subset \mathbf{R}^N$ be a set of finite perimeter; then the following equality holds*

$$\lim_{t\to 0}\sqrt{\frac{\pi}{t}}\langle T(t)\chi_E, \chi_{E^c}\rangle = P(E).$$

The proof simply follows from Theorem 2.1 by taking $F = E$ and noticing that

$$\langle \chi_E - T(t)\chi_E, \chi_E\rangle\langle \chi_E - T(t)\chi_E, 1-\chi_{E^c}\rangle\langle T(t)\chi_E, \chi_{E^c}\rangle.$$

We point out that this result is more general regarding the sets to which it applies, but is less precise since we cannot conclude that the heat amount comes only from a neighbourhood of any given point of the boundary.

It is also possible to prove that Theorem 2.2 is in some sense optimal, that is, the only sets with a finite diffusion are the Caccioppoli sets. More precisely, the following Theorem holds.

Theorem 2.3 *Let $E \subset \mathbf{R}^N$ be a set such that either E or E^c has finite mesure, and*

$$\liminf_{t\to 0^+}\frac{\langle T(t)\chi_E, \chi_{E^c}\rangle}{\sqrt{t}} < +\infty;$$

then E has finite perimeter.

The proof of this Theorem essentially relies on the fact that a set E with finite diffusion has almost every directional derivative bounded in L^1, whence the fact that E has finite perimeter.

Starting from Theorem 2.2, if E is the level set of a function $u \in BV$ and integrating on these level sets, using the coarea formula, it is possible to prove the following.

Theorem 2.4 *Let $u \in BV(\mathbf{R}^N)$; then the following equality holds:*

$$|Du|(\mathbf{R}^N) = \lim_{t\to 0} \frac{\sqrt{\pi}}{2\sqrt{t}} \int_{\mathbf{R}^N\times\mathbf{R}^N} |u(x) - u(y)| p_N(x, y, t)dxdy.$$

Finally, using Theorem 2.1, when E and F are the level sets of two functions $u, v \in BV$ respectively, it is possible to prove the following Theorem, which says that the diffusion of the graphs of two BV function is concentrated on the jump sets S_u and S_v of u and v.

Theorem 2.5 *Let $u, v \in BV(\mathbf{R}^N)$; then the following formula holds:*

$$\lim_{t\to 0} \sqrt{\frac{\pi}{t}} \langle u - T(t)u, v\rangle = \int_{S_u\cap S_v} (u^\vee - u^\wedge)(v^\vee - v^\wedge)\nu_u \cdot \nu_v \, d\mathcal{H}^{N-1}.$$

As an immediate Corollary of this Theorem, we have an interpolation-like result, that is, if $u \in BV(\mathbf{R}^N) \cap L^2(\mathbf{R}^N)$, the following limit formula holds

$$\lim_{t\to 0} \frac{\sqrt{\pi}}{2\sqrt{t}} \|u - T(t)u\|_{L^2(\mathbf{R}^N)} = \int_{S_u} (u^\vee - u^\wedge)^2 d\mathcal{H}^{N-1}.$$

Remark 2.6 We want to stress out the fact that in the proofs of previous Theorems it was important the precise formulation of the heat kernel and the fact that the representation of the heat semigroup is given by a convolution. In fact, also the definition of the perimeter of a set in an open subset $\Omega \subset \mathbf{R}^N$ using the heat semigroup as in (1) is not clear. In fact, in this case one generally can't write explicitly the heat kernel, and so the same proof can't be repeated.

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