CONVEXITY AND A HORIZONTAL SECOND FUNDAMENTAL FORM FOR HYPERSURFACES IN CARNOT GROUPS

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Abstract. We use a Riemannian approximation scheme to give a characterization for smooth convex functions on a Carnot group (in the sense of Danielli–Garofalo–Nhieu or Lu–Manfredi–Stroffolini) in terms of the positive semidefiniteness of the horizontal second fundamental form of their graph.

1. Introduction

In 1996 Caffarelli proposed a notion of convexity for functions on the Heisenberg group in terms of the standard one dimensional convexity of their restriction to horizontal lines through any fixed point. This notion surfaced again in 2002 when it was independently discovered by Danielli–Garofalo–Nhieu [8] and generalized to arbitrary Carnot groups. At the same time, Lu–Manfredi–Stroffolini [20] (see also [17]) proposed an equivalent definition based on the notion of viscosity sub-solutions. For related work, see also Balogh-Rickly [3], Gutierrez–Montanari [13], [14], Garofalo-Tournier [?], Wang [24] and Magnani [?]. A notion of convexity for sets was introduced in [8], where the relationship between convexity of a function and convexity of its epigraph is studied.

In this paper we propose a notion of convexity for hypersurfaces in Carnot groups, in terms of the *horizontal second fundamental form* II_0 of their graphs (see Definition 3.7). To relate this definition to the previous literature we show that the graphs of (suitably regular) functions have positive definite symmetrized horizontal second fundamental form $(II_0)^*$ if and only if the functions in question are convex (in the sense of Caffarelli, [8], [20], or [17]) and provide quantitative statements for this fact. More precisely, we prove

Theorem 1.1. *Let $\mathbb{G}$ be a Carnot group and denote by $\Gamma^2(\mathbb{G})$ the class of functions twice continuously differentiable along horizontal directions.*

Date: June 2, 2006.

Key words and phrases. Convexity, Carnot group, second fundamental form, Riemannian approximation.

The authors are partially supported by the National Science Foundation: LC NSF DMS-0134318; SDP NSF DMS-0306752; JTT, NSF DMS-0228807; all authors NSF DMS-0503695. Part of the research for this paper was done while LC and JTT were visitors at the Dartmouth College Mathematics Department in Spring 2006. They are grateful to the department for the hospitality.

(i) *Let* $u : \mathbb{G} \to \mathbb{R}$ *be a* Γ^2 *function, and denote by* $(D_0^2 u)^*$ *its symmetrized horizontal Hessian and by* $(\mathrm{II}_0^{\mathscr{G}(u)})^*$ *the symmetrized horizontal second fundamental form of* $\mathscr{G}(u)$*, the graph of* u*. At every point of* $\mathbb{G}$*, we have*

$$\Lambda \geq (D_0^2 u)^* \geq \lambda \ \textit{ if and only if } \ \Lambda' \geq (\mathrm{II}_0^{\mathscr{G}(u)})^* \geq \lambda',$$

where $\nabla_0 u$ *is the horizontal gradient of* u*.*

(ii) *A* Γ^2 *function* $u : \mathbb{G} \to \mathbb{R}$ *is convex if and only if* $(\mathrm{II}_0^{\mathscr{G}(u)})^*$ *is positive semi-definite.*

(iii) *Let* $U = \{(x, s) : u(x) \leq s\}$ *be the epigraph of a* Γ^2 *function* $u : \mathbb{G} \to \mathbb{R}$*. Then* U *is convex (as a set, in the sense of* [8]*) if and only if the symmetrized horizontal second fundamental form of its boundary,* $(\mathrm{II}_0^{\partial U})^*$*, is positive semi-definite.*

In part (i), the bounds Λ' and Λ depend only on each other and on the norm of $\nabla_0 u$. A similar statement holds for λ' and λ. We have $\Lambda < \infty$ if and only if $\Lambda' < \infty$, and $\lambda > 0$ if and only if $\lambda' > 0$. See Definitions 2.5 and Definition 3.7, respectively, for the definitions of the symmetrized horizontal Hessian and symmetrized horizontal second fundamental form.

The results in Theorem 1.1 are new even in the simplest setting of the first Heisenberg group. We feel, however, that the main contribution of this paper does not lie in the results themselves but rather in the method of proof, which is based on a careful study of cancellation properties arising from differentiating certain horizontal tensors in the Riemannian approximation scheme.

There is a big gap in terms of regularity of u between our *differential geometric* definition of convexity and the definition in [8],[20], and [17]. While we require two derivatives along the horizontal directions, the original definition can be applied to any lower-semicontinuous function and eventually yields Lipschitz regularity along the horizontal directions. For more details, see [3], [20], [17] or [24].

Our motivations for providing a more geometric understanding of convexity are twofold:

(i) In the Riemannian setting, the second fundamental form encodes a wealth of critical geometric information on the behavior of the Gauss map, likewise, the study of the horizontal second fundamental form for submanifolds of Carnot groups will allow for a better understanding of the horizontal Gauss map and possibly lead to an approach to the analog of the Alexandrov-Bakelman-Pucci maximum principle for subelliptic linear equations in non-divergence form.

(ii) Convexity of manifolds evolving by curvature flows (for instance mean curvature flow [16]) is based on applications of maximum principles to certain non-linear evolution equations which describe the behavior of the second fundamental form. If one wants to extend such analysis to the sub-Riemannian context (and there are plenty of reasons to do so, see [5]) then it is crucial to link convexity to some notion of second fundamental form.

Sub-Riemannian analogs of the second fundamental form have recently been introduced for level sets by Danielli, Garofalo and Nhieu [7] in terms of restrictions of the defining function to horizontal planes. In this paper we follow a different approach and use the approximation of the sub-Riemannian geometry with a family of Riemannian metrics. The equivalence of the two definitions can be found in [4] and [5]. The horizontal second fundamental form in Definition 3.7 was proposed originally by Hladky and Pauls [15]. Here we relate it for the first time to the Riemannian approximation scheme and use systematically its symmetrization.

In closing we also want to mention related work of Arcozzi and Ferrari [1] (who studied the Hessian of the distance function in $\mathbb{H}^n$), and Calin and Mangione [6] (who studied the second fundamental form in the Riemannian approximants to $\mathbb{H}$).

After a brief section where we recall basic definitions and results concerning Carnot groups, we analyze the relation between the symmetrized horizontal Hessian of a function and the symmetrized horizontal second fundamental form of its level set. Next we study the particular case of graphs and conclude by presenting concrete examples in the Heisenberg group. In our analysis we first consider only smooth objects (functions and hypersurfaces) and then reduce the regularity assumption to the Folland-Stein class Γ^2 using group mollifiers.

2. Notation and setup

The notation and terminology in this paper draws heavily from the forthcoming expository monograph [5].

Definition 2.1. ([11], [23]) Let $\mathbb{G}$ be a Lie group with Lie algebra $\mathfrak{g}$. We say that $\mathbb{G}$ is a graded nilpotent Lie group if there exists $k < \infty$ and vector subspaces $\mathcal{V}_i \subset \mathfrak{g}$ so that

$$\mathfrak{g} = \mathcal{V}_0 \oplus \mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_k \tag{2.1}$$

with the property that $[\mathcal{V}_0, \mathcal{V}_i] = \mathcal{V}_{i+1}$ for $0 \le i < k$ and $[\mathcal{V}_i, \mathcal{V}_k] = 0$ for $0 \le i \le k$.

Given a graded nilpotent Lie group $\mathbb{G}$ equipped with a Riemannian metric g, we say that g is *compatible*[1] with the grading of $\mathfrak{g}$ if (2.1) is an orthogonal splitting of the Lie algebra with respect to g. In this paper, we consider only compatible Riemannian metrics. For a fixed $(\mathbb{G}, g)$, let $\{X_i\}_{i=1}^{m+1} \cup \{Y_j\}_{j=1}^{n+1}$ be an orthonormal set of left invariant vector fields on $\mathbb{G}$ with the following properties:[2]

(1) The span of $\{X_i\}_{i=1}^{m+1}$ is $\mathcal{V}_0$ and the span of $\{Y_j\}_{j=1}^{n+1}$ is $\mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_k$.
(2) For each $1 \le j \le n+1$, there exists an integer $d(j)$, $1 \le d(j) \le k$, so that $Y_j \in \mathcal{V}_{d(j)}$.

We denote by exp the exponential map from $\mathfrak{g}$ to $\mathbb{G}$.

[1]In contrast with the intrinsic approach used in [21] where the metric is only assigned on the horizontal distribution at first and then extended to a it tame metric on the whole ambient space.

[2]Here we identify left invariant vector fields with elements of $\mathfrak{g}$.

Definition 2.2. Given a graded nilpotent Lie group $\mathbb{G}$ equipped with a compatible Riemannian metric g, and vector fields $\{X_i\}_{i=1}^{m+1} \cup \{Y_j\}_{j=1}^{n+1}$ as above, we define the left invariant frame

$$\mathscr{F}_1^{(\mathbb{G},g)} = \{X_1, \dots, X_{m+1}, Y_1, \dots, Y_{n+1}\}$$

When the metric is clear from context, we will simply write this as $\mathscr{F}_1^{\mathbb{G}}$.

We call the $\{X_i\}$ *horizontal vector fields* and call their span, denoted $H\mathbb{G}$, the *horizontal bundle.* We call the $\{Y_j\}$ *vertical vector fields* and call their span, denoted $V\mathbb{G}$, the *vertical bundle.* Then $T\mathbb{G} = H\mathbb{G} \oplus V\mathbb{G}$.

Definition 2.3. Given a graded nilpotent Lie group $\mathbb{G}$, we define a Carnot-Carathéodory metric on $\mathbb{G}$ by specifying an inner product, $\langle \cdot, \cdot \rangle$, on $\mathcal{V}_0$ and defining the metric by

$$d_{cc}(x,y) = \inf_{\gamma \in \mathscr{A}} \left\{ \int_0^1 \langle \dot\gamma(s), \dot\gamma(s) \rangle^{\frac{1}{2}} \, ds \mid \gamma(0) = x,\ \gamma(1) = y \right\}$$

where $\mathscr{A}$ is the set of all absolutely continuous paths whose derivative, when it exists, lies in $H\mathbb{G}$.

Definition 2.4. A Carnot group is a connected, simply connected graded nilpotent Lie group equipped with a Carnot-Carathéodory metric.

Definition 2.5. Given a Carnot group $\mathbb{G}$ equipped with a compatible Riemannian metric g and basis $\mathscr{F}_1^{\mathbb{G}}$, for $k \in \mathbb{N}$ we will denote by $\Gamma^k(\mathbb{G})$ the Folland-Stein space, i.e. the space of functions on G which are k times continuously differentiable along horizontal directions (see [11] and [23]). We define the *horizontal gradient operator* by

$$\nabla_0 = (X_1, \dots, X_{m+1}).$$

In other words, given a Γ^1 function $u : \mathbb{G} \to \mathbb{R}$,

$$\nabla_0 u = (X_1 u)\ X_1 + \cdots + (X_{m+1} u)\ X_{m+1}.$$

Further, if $u \in \Gamma^2$, we define the *horizontal Hessian* of u to be

$$D_0^2 u = (X_i X_j u)_{i,j=1,\dots,m+1},$$

and the *symmetrized horizontal Hessian* of u to be

$$(D_0^2 u)^* = (\frac{1}{2}(X_i X_j u + X_j X_i u))_{i,j=1,\dots,m+1}.$$

Various definitions for convexity have been proposed and studied in the setting of Carnot groups. For our purposes the following definition is convenient. See [8], [20], [17], [3], [24], [13], [14] and the notes and bibliographies in these papers for a complete history and detailed list of pertinent references.

Definition 2.6. A Γ^2 function $u : \mathbb{G} \to \mathbb{R}$ is *convex* if its symmetrized horizontal Hessian $(D_0^2 u)^*$ is positive semi-definite. A set $A \subset \mathbb{G}$ is *convex* if, for every $x \in A$, the intersection of A with $\exp H_x\mathbb{G}$ is the image under the exponential map of a set in the Lie algebra $\mathfrak{g}$ which is (Euclidean) starlike with respect to x.

Remark 2.7. We note that, as a special case of Proposition 7.6 of [8], a function u on $\mathbb{G}$ is convex if and only if its epigraph

$$\mathrm{epi}(u) = \{(x,s) \in \mathbb{G} \times \mathbb{R} : u(x) \leq s\}$$

is convex in the Carnot group $\mathbb{G} \times \mathbb{R}$ (equipped with a product sub-Riemannian structure as in section 4). See also Corollary 4.7.

Next, we construct a family of compatible Riemannian metrics which approximate the Carnot-Carathéodory metric (see Korányi [18], Korányi and Reimann [19], Pansu [22] and Gromov [12]).

Definition 2.8. Let $(\mathbb{G}, g)$ be a graded nilpotent Lie group with fixed Riemannian metric and with a coordinate frame $\mathscr{F}_1^{(\mathbb{G},g)}$. For each $L > 0$ we define Riemannian metrics g_L, *the anisotropic dilations of the metric* g, characterized by $g_L(X_i, X_j) = g(X_i, X_j) = \delta_{ij}$, $g_L(X_i, Y_j) = g(X_i, Y_j) = 0$ and $g_L(Y_i, Y_j) = L^{2/d(j)}\delta_{ij}$, where $d(j)$ is defined in the discussion following Definition 2.2.[3] We define a new, rescaled frame orthonormal with respect to g_L:

$$\mathscr{F}_1^{(\mathbb{G},g_L)} = \{X_1, \dots, X_{m+1}, \tilde{Y}_1, \dots, \tilde{Y}_{n+1}\}$$

where $X_i, Y_j \in \mathscr{F}_1^{(\mathbb{G},g)}$ and $\tilde{Y}_j = L^{-1/d(j)} Y_j$.

Example 2.9. The simplest non-abelian example is the first Heisenberg group $\mathbb{H} = \mathbb{H}^1$. We use coordinates (x_1, x_2, x_3) in $\mathbb{H}$, and denote the standard frame in $T\mathbb{H}$ by $X_1 = \partial_{x_1} - \frac{1}{2}x_2\partial_{x_3}$, $X_2 = \partial_{x_2} + \frac{1}{2}x_1\partial_{x_3}$, and $Y_1 = \partial_{x_3}$. In these coordinates the metric g_L takes the form

$$g_L(x) = \begin{pmatrix} 1 + \frac{1}{4}x_2^2 L & -\frac{1}{4}x_1 x_2 L & -\frac{1}{2}x_2 L \\ -\frac{1}{4}x_1 x_2 L & 1 + \frac{1}{4}x_1^2 L & \frac{1}{2}x_1 L \\ -\frac{1}{2}x_2 L & \frac{1}{2}x_1 L & L \end{pmatrix}. \tag{2.2}$$

The metrics g_L approximate the sub-Riemannian structure of the Carnot group $\mathbb{G}$ in the sense of the following lemma. Here we denote by d_L the distance function on $\mathbb{G}$ induced by the metric g_L.

Lemma 2.10. *As $L \to \infty$, the metric spaces $(\mathbb{G}, d_L)$ converge in the Gromov-Hausdorff sense to $(\mathbb{G}, d_{cc})$.*

See [21, p. 18], [12, p. 144, 1.4.D], and [5, Theorem 2.9] for related statements.

Our notation for the Levi-Civita connection associated to g_L is $\boldsymbol{\nabla}$. Given a smooth function $u : \mathbb{G} \to \mathbb{R}$, we denote its Riemannian gradient with respect to g_L by

$$\nabla_L u = (X_1 u)\, X_1 + \cdots + (X_{m+1} u)\, X_{m+1} + (\tilde{Y}_1 u)\, \tilde{Y}_1 + \cdots + (\tilde{Y}_{n+1} u)\, \tilde{Y}_{n+1}.$$

[3] A rougher rescaling, with $g_L(Y_i, Y_j) = L\delta_{ij}$, would work as well for our purposes.

In addition we will use two variants on this notation. Given vectors $U, V \in T\mathbb{G}$, we let $\langle U, V\rangle_L = g_L(U,V)$ and $|U|_L^2 = g_L(U,U)$. We remark that if $U \in H\mathbb{G}$ then $|U|_L$ is independent of U. In this case, we will simply write $|U|$ instead of $|U|_L$.

We note directly from the definition that

$$\lim_{L\to\infty} \nabla_L u = \nabla_0 u. \tag{2.3}$$

We recall the Kozul formula for the Levi-Civita connection [10, p. 55]:

$$\langle \boldsymbol{\nabla}_U V, W\rangle_L = \frac{1}{2}\Big\{ U\langle V,W\rangle_L + V\langle W,U\rangle_L - W\langle U,V\rangle_L - \langle W,[V,U]\rangle_L - \langle U,[V,W]\rangle_L - \langle V,[U,W]\rangle_L \Big\} \tag{2.4}$$

for vector fields U, V, W. A standard result in sub-Riemannian geometry (see for instance [5, Proposition 3.1]) states that the restriction to the horizontal bundle of the Levi-Civita connections for the metrics g_L do not depend on L. We will tacitly use this fundamental fact throughout the paper.

The following lemma describes the vertical component of the restriction of $\boldsymbol{\nabla}$ to the horizontal bundle.

Lemma 2.11. *If $U = \sum_{i=1}^{m+1} p_i X_i$, $V = \sum_{i=1}^{m+1} q_i X_i$, $W = \sum_{j=1}^{n+1} r_j \tilde{Y}_j$ and U, V, W are of unit length and mutually g_L-orthogonal, then*

$$\big\langle \boldsymbol{\nabla}_U V, W\big\rangle_L = -\frac{1}{2}\big\langle [V,U], W\big\rangle_L. \tag{2.5}$$

Proof. Observe that

$$\begin{aligned}
[U,W] &= \sum_{i=1}^{m+1}\sum_{j=1}^{n+1}\Big(p_i X_i r_j \tilde{Y}_j + p_i r_j X_i \tilde{Y}_j\Big) - \sum_{i=1}^{m+1}\sum_{j=1}^{n+1}\Big(r_j \tilde{Y}_j p_i X_i + r_j p_i \tilde{Y}_j X_i\Big)\\
&= \sum_{l=1}^{m+1}\sum_{k=1}^{n+1}\Big(p_i (X_i r_j)\tilde{Y}_j + p_i r_j [X_i, \tilde{Y}_j] - r_j(\tilde{Y}_j p_i) X_i\Big),
\end{aligned}$$

while $[V,W]$ is given by the same expression with q_i replacing p_i. A direct computation yields

$$\langle [U,W], V\rangle_L = -\sum_{i=1}^{m+1}\sum_{j=1}^{n+1} r_j q_i (\tilde{Y}_j p_i)$$

Similarly,

$$\langle [V,W], U\rangle_L = -\sum_{i=1}^{m+1}\sum_{j=1}^{n+1} r_j p_i (\tilde{Y}_j q_i).$$

Here we used the fact that $\langle [X_i, Y_j], X_k\rangle_L = 0$. Summing the two terms yields

$$\begin{aligned}\langle [U,W],V\rangle_L + \langle [V,W],U\rangle_L &= -\sum_{i=1}^{m+1}\sum_{j=1}^{n+1} r_j(p_i\tilde{Y}_j q_i + q_i\tilde{Y}_j p_i)\\ &= -\sum_{i=1}^{m+1}\sum_{j=1}^{n+1} r_j\tilde{Y}_j(p_iq_i) = -\sum_{j=1}^{n+1} r_j\tilde{Y}_j\langle U,V\rangle_L = 0.\end{aligned}$$

Now using the orthonormality of $\{U,V,W\}$ and (2.4) we have the desired result. ☐

3. Hypersurfaces in Carnot groups

We fix a Carnot group, $\mathbb{G}$, and a family of metrics g_L associated to a fixed Riemannian metric g as in the previous section. We note that the (topological) dimension of $\mathbb{G}$ is $N = m+n+2$. Let M be a smooth hypersurface in $\mathbb{G}$ given by

$$M = \{x \in \mathbb{G} : u(x) = 0\}$$

where $u : \mathbb{G} \to \mathbb{R}$ is a smooth function with non-vanishing (Riemannian with respect to any g_L metric) gradient in a neighborhood of M.

Denote by $\Sigma(M) = \{x \in M \mid T_xM \supset H_x\mathbb{G}\}$ the *characteristic set* of M. The (Riemannian) surface measure of Σ is zero (see [9] or [2] as well as [5] for further references and more precise statements).

Definition 3.1. For any non-characteristic point, the *unit horizontal normal* to M is defined as the normalized projection of the Riemannian normal to the horizontal subbundle. In the basis $\mathscr{F}_1^{\mathbb{G}}$, it is given as

$$\mathbf{n}_0 = \frac{(X_1u)\ X_1 + \cdots + (X_{m+1}u)\ X_{m+1}}{|(X_1u)\ X_1 + \cdots + (X_{m+1}u)\ X_{m+1}|}. \tag{3.1}$$

Note that the definition of $\mathbf{n}_0$ does not depend on L. Letting

$$\mathbf{n}_L = \frac{\nabla_L u}{|\nabla_L u|_L} \tag{3.2}$$

denote the Riemannian unit normal (with respect to g_L), we note that $\lim_{L\to\infty} \mathbf{n}_L = \mathbf{n}_0$ uniformly in compact subsets of $M \setminus \Sigma(M)$.

We next consider a basis for $T\mathbb{G}|_M$ adapted to the submanifold M:

$$\mathscr{F}_2^{\mathbb{G}} = \{Z_1, \ldots, Z_{m+n+1}, \mathbf{n}_L\}$$

where $\{Z_i\}$ is an orthonormal basis for TM in the metric g_L.

Definition 3.2. We denote the Riemannian second fundamental form of M in the coordinate frame $\mathscr{F}_2^{\mathbb{G}}$ in (M, g_L) by

$$\mathrm{II}_L^{M,\mathscr{F}_2^{\mathbb{G}}} = (h_{ij}^L) = (\langle \boldsymbol{\nabla}_{Z_i}\mathbf{n}_L, Z_j\rangle_L)_{i,j=1,\ldots,m+n+1}$$

and the Riemannian Hessian of a smooth function $u : \mathbb{G} \to \mathbb{R}$ with respect to any orthonormal basis $\mathscr{F}^{\mathbb{G}} = \{W_1, \cdots, W_{m+n+2}\}$ of $T\mathbb{G}$ is

$$\mathrm{Hess}_L^{\mathscr{F}^{\mathbb{G}}}(u) = (\langle \boldsymbol{\nabla}_{W_i}(\nabla_L u), W_j\rangle_L)_{i,j=1,\dots,m+n+2}$$

at any point in $\mathbb{G}$. We will omit the frame from the notation whenever we wish to emphasize coordinate independence.

The following lemma reduces to a simple computation using Definition 3.2 and (3.2).

Lemma 3.3. *Let M be a smooth hypersurface in $\mathbb{G}$ given as a level set of a smooth function u on $\mathbb{G}$. Then for any point in M, and for $i, j = 1, \dots, m+n+1$ we have*

$$|\nabla_L u|_L \left(\mathrm{II}_L^{M,\mathscr{F}_2^{\mathbb{G}}}\right)_{ij} = \left(\mathrm{Hess}_L^{\mathscr{F}_2^{\mathbb{G}}}(u)\right)_{ij}.$$

In the following we view $\mathrm{Hess}_L(u)$ as a bilinear form on $T\mathbb{G}$ by the equation

$$\mathrm{Hess}_L(u)(W,V) = \langle \boldsymbol{\nabla}_W(\nabla_L u), V\rangle_L,$$

for all $V, W \in T\mathbb{G}$. When we want to consider (as we do here) only the restriction of this form to the subspace TM we will use the notation $\mathrm{Hess}_L(u)|_{TM}(W,V)$, $W, V \in TM$. Lemma 3.3 can be rephrased as

$$\mathrm{Hess}_L(u)|_{TM} = |\nabla_L u|_L \ \mathrm{II}_L^M,$$

or more explicitly,[4]

$$\mathrm{Hess}_L(u)(W,V) = \sum_{i,j=1}^{m+n+1} \left(\mathrm{Hess}_L^{\mathscr{F}_2^{\mathbb{G}}}(u)\right)_{ij} a_i b_j = |\nabla_L u|_L \sum_{i,j=1}^{m+n+1} \left(\mathrm{II}_L^{M,\mathscr{F}_2^{\mathbb{G}}}\right)_{ij} a_i b_j = \mathrm{II}_L^M(W,V)$$

if $(a_i), (b_i) \in \mathbb{R}^{m+n+1}$ satisfy $W = \sum_{i=1}^{m+n+1} a_i Z_i$ and $V = \sum_{i=1}^{m+n+1} b_i Z_i$.

To take advantage of this identity in the limit as $L \to \infty$ we need to extract *horizontal data* out of it. To accomplish this, we construct a specific basis of the form of $\mathscr{F}_2^{\mathbb{G}}$ that will be useful for our purposes. To facilitate this, we make several definitions concerning a decomposition of $\mathbf{n}_L$ at non-characteristic points.

Definition 3.4. Let $\mathbf{n}_L$ be the Riemannian unit normal and $\mathbf{n}_0$ be the unit horizontal normal to M. At any non-characteristic point we set

$$\begin{aligned} T_0 &= \frac{\mathbf{n}_L - \langle \mathbf{n}_L, \mathbf{n}_0\rangle_L \mathbf{n}_0}{|\mathbf{n}_L - \langle \mathbf{n}_L, \mathbf{n}_0\rangle_L \mathbf{n}_0|_L}, \\ a_L &= \langle \mathbf{n}_L, \mathbf{n}_0\rangle_L = \frac{|\nabla_0 u|}{|\nabla_L u|_L}, \\ b_L &= \langle \mathbf{n}_L, T_0\rangle_L. \end{aligned}$$

Thus

$$\mathbf{n}_L = a_L\ \mathbf{n}_0 + b_L\ T_0$$

[4]Note that the summation on the left hand side does not extend over the full Hessian matrix.

We may rewrite b_L in a more precise manner as follows:

$$\begin{aligned} b_L^2 = 1 - a_L^2 &= |\nabla_L u|_L^{-2}(|\nabla_L u|_L^2 - |\nabla_0 u|^2) \\ &= |\nabla_L u|_L^{-2}\left(\sum_{j=1}^{n+1}(\tilde{Y}_j u)^2\right) \\ &\le \frac{1}{L^{2/k}|\nabla_L u|_L^2}\left(\sum_{j=1}^{n+1}(Y_j u)^2\right), \end{aligned}$$

where k is as in (2.1). To summarize, using (2.3) we have the following lemma:

Lemma 3.5. *At non-characteristic points we have* $\lim_{L\to\infty} a_L = 1$ *and* $\lim_{L\to\infty} b_L = 0$.

We now define a new basis:

Definition 3.6. Let $(\mathbb{G}, g_L)$ be as above and M be a smooth hypersurface in $\mathbb{G}$ given as a level set of a function $u : \mathbb{G} \to \mathbb{R}$. Then

$$\mathscr{F}_3^{(\mathbb{G},g_L)} = \{e_0, e_1, \dots, e_m, T_1, \dots, T_n, \mathbf{n}_L\}$$

is a basis for $T\mathbb{G}|_{M\setminus\Sigma(M)}$ of the form of $\mathscr{F}_2^{\mathbb{G}}$, where

$$e_0 = b_L\mathbf{n}_0 - a_L T_0,$$

$\{e_1, \dots, e_m\}$ is an orthonormal basis for $HM = TM \cap H\mathbb{G}$, and $\{T_1, \dots, T_n\}$ is an orthonormal basis for $VM = TM \cap V\mathbb{G}$. Again, when the metric is understood, we will suppress it in the notation.

We emphasize that e_0 is not a horizontal vector field. Moreover, as $L \to \infty$, we see that $b_L \to 0$ and hence $e_0 \to -T_0$.

We now introduce a central concept of this paper, the horizontal second fundamental form of a hypersurface in a Carnot group.

Definition 3.7. Given a smooth hypersurface $M \subset (\mathbb{G}, g_L)$, and the adapted basis $\mathscr{F}_3^{\mathbb{G}}$, we define the *horizontal second fundamental form* at any non-characteristic point as

$$\mathrm{II}_0^M = (h_{ij}^0) = (\langle \boldsymbol{\nabla}_{e_i}\mathbf{n}_0, e_j\rangle_L)_{i,j=1}^m$$

Note that the entries in II_0^M are independent of L despite the fact that $\boldsymbol{\nabla}_{e_i}$ and e_j are *a priori* dependent on L. Moreover, note that II_0^M is not necessarily symmetric. In a similar fashion, we define the *vertical second fundamental form* by

$$(v_{ij,L}) = (\langle \boldsymbol{\nabla}_{e_i} T_0, e_j\rangle_L)_{i,j=1}^m .$$

Note that $v_{ij,L}$ does depend on L.

Remark 3.8. While the second fundamental form II_L^M is defined as acting on all the tangent space, we will always tacitly indicate with II_0^M the restriction of the horizontal second fundamental form to HM.

Lemma 3.9. *Given a smooth hypersurface $M \subset (\mathbb{G}, g_L)$, and the adapted basis $\mathscr{F}_3^{\mathbb{G}}$, at any non-characteristic point we have*

(1)
$$\mathrm{II}_L^{M,\mathscr{F}_3^{\mathbb{G}}}\,|_{HM} = (h_{ij}^L)_{i,j=1}^m = (a_L h_{ij}^0 + b_L v_{ij,L})_{i,j=1}^m,$$

(2) *as $L \to \infty$, the restriction of $\mathrm{II}_L^{M,\mathscr{F}_3^{\mathbb{G}}}$ to the horizontal tangent space converges to the symmetrized horizontal second fundamental form, i.e.*
$$\lim_{L\to\infty} h_{ij}^L = \frac{1}{2}\left(h_{ij}^0 + h_{ji}^0\right).$$

Furthermore, given any smooth function $u : \mathbb{G} \to \mathbb{R}$,

(3) $\lim_{L\to\infty} \mathrm{Hess}_L^{\mathscr{F}_1^{\mathbb{G}}}(u)\,|_{H\mathbb{G}} = (D_0^2 u)^*$, *i.e., the limit of the Riemannian Hessians restricted to the horizontal directions is the symmetrized horizontal Hessian.*

Proof. (1) follows from the definition of a_L, b_L, h_{ij}^0 and $v_{ij,L}$. To determine the existence and the value of the limit, we observe that

$$h_{ij}^L = a_L \frac{h_{ij}^0 + h_{ji}^0}{2} + b_L \frac{v_{ij,L} + v_{ji,L}}{2} \tag{3.3}$$

since $h_{ij}^L = h_{ji}^L$. By Lemma 2.11 we have $v_{ij,L} + v_{ji,L} = 0$ and the conclusion follows from (3.3) and from Lemma 3.5.

(3) is a trivial consequence of the definition of the Riemannian Hessian, the fact that it is symmetric and the definition of $D_0^2 u$. □

Definition 3.10. Let $M \subset \mathbb{G}$ be a smooth hypersurface and denote by $(\mathrm{II}_0^M)^*$ its symmetrized horizontal second fundamental form. The *(horizontal) principal curvatures* $k_1, \dots, k_m$ of M at a point $x \in M$ are the eigenvalues of $(\mathrm{II}_0^M)^*(x)$. The *(horizontal) mean curvature* $\mathcal{H}_0^M$ of M at x is the trace of $(\mathrm{II}_0^M)^*(x)$. Finally, the *horizontal Gauss curvature* $\mathcal{G}_0^M$ of M at x is $\det(\mathrm{II}_0^M)^*(x)$. We will also denote by $\mathcal{H}_L^M$ and $\mathcal{G}_L^M$ the mean curvature and the Gauss curvature of M with respect to the metric g_L.

In view of Lemma 3.9(2) we have

Lemma 3.11. *If $M \subset \mathbb{G}$ be a smooth hypersurface, then*
$$\lim_{L\to\infty} \mathcal{H}_L^M = \mathcal{H}_0^M$$
at non-characteristic points.

Remark 3.12. Note that this convergence result is false for the Gauss curvature. Even in the first Heisenberg group, regardless of the choice of the approximating metric g_L, the Gauss curvature is always of the order of L at all non-characteristic points. See [5, Section 3.3].

On the other hand, if M is the zero level set of a smooth function $u : \mathbb{G} \to \mathbb{R}$ then $\mathcal{H}_L^M = \mathrm{div}_{g_L}(\mathbf{n}_L)$ and a direct computation yields

$$|\nabla_L u| \mathcal{H}_L^M = \sum_{i,j=1}^{m+n+2} \left(\delta_{ij} - \frac{Z_i u Z_j u}{|\nabla_L u|^2} \right) Z_i Z_j u, \tag{3.4}$$

where we have denoted by $Z_1, \ldots, Z_{m+n+2}$ any relabelling of the frame $\mathscr{F}_1^{(\mathbb{G},g)}$. As a direct consequence of (3.4) and of Definition 2.8 and (2.3) we recover the familiar expression

$$\mathcal{H}_0^M = \frac{1}{|\nabla_0 u|} \sum_{i,j=1}^{m+1} \left(\delta_{ij} - \frac{X_i u X_j u}{|\nabla_0 u|^2} \right) X_i X_j u$$

on $M \setminus \Sigma$. In fact we have the following convergence result:

$$\lim_{L \to \infty} |\nabla_L u| \mathcal{H}_L^M = \begin{cases} |\nabla_0 u| \mathcal{H}_0^M & \text{on } M \setminus \Sigma, \\ \sum_{i=1}^{m+1} X_i^2 u & \text{on } \Sigma. \end{cases} \tag{3.5}$$

This observation is due to Citti.

We conclude this section with an explicit relation between the symmetrized horizontal Hessian and the second fundamental form.

Proposition 3.13. *Let $u : \mathbb{G} \to \mathbb{R}$ be a smooth function and $M = \{u = 0\}$. Let $A = [\mathscr{F}_3^{\mathbb{G}} \to \mathscr{F}_1^{\mathbb{G}}]$ be the change of basis matrix from $\mathscr{F}_3^{\mathbb{G}}$ to $\mathscr{F}_1^{\mathbb{G}}$. We have the following identity of bilinear forms:*

$$|\nabla_0 u| (II_0^M)^* = \left(A^t (D_0^2 u)^* A \right) |_{HM}$$

at non-characteristic points. More explicitly, for all $V = \sum_{i=1}^m v_i e_i$ and $W = \sum_{i=1}^m w_i e_i$, one has

$$\sum_{i,j=1}^m |\nabla_0 u| (\mathrm{II}_0^M)_{ij}^* v_i w_j = \sum_{i,j=1}^m \left(A^t (D_0^2 u)^* A \right)_{ij} v_i w_j.$$

Proof. As both the Hessian and the second fundamental form are bilinear, the result follows immediately from Lemma 3.3 and Lemma 3.9 (2), (3). □

Corollary 3.14. *If $u : \mathbb{G} \to \mathbb{R}$ is a smooth convex function and $M = \{x \in \mathbb{G} : u(x) = 0\}$ a smooth hypersurface, then $(II_0^M)^*$ is positive semi-definite at any non-characteristic point.*

Proof. Since a smooth convex function satisfies $(D_0^2 u)^* \geq 0$ the conclusion follows directly from Proposition 3.13. □

Example 3.15. The converse of Corollary 3.14 is false. Consider the first Heisenberg group $\mathbb{H}$, identified with $\mathbb{R}^3$ using coordinates (x_1, x_2, x_3). Let $u(x_1, x_2, x_3) = x_3 - \frac{x_1 x_2}{2}$ and $M = \{u = 0\}$. The symmetrized horizontal Hessian of u

$$(D_0^2 u)^* = \begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}$$

has eigenvalues $\{\pm\frac{1}{2}\}$ and hence is not positive semi-definite. However, in this case II_0^M and $\mathrm{II}_L^{M,\mathscr{F}_3^{\mathbb{G}}} |_{HM}$ are 1×1 matrices whose only entry is zero. To see this, we will create the basis $\mathscr{F}_3^{\mathbb{H}}$ for M. Using the notation of Example 2.9, we compute $X_1 u = -x_2$, $X_2 u = 0$, and $\tilde{Y}_1 u = \frac{1}{\sqrt{L}}$. Then we have

$$\mathbf{n}_0 = \operatorname{sign}(-x_2) X_1$$

and so

$$\mathbf{n}_L = a_L\ \mathbf{n}_0 + b_L\ T_0$$

where $T_0 = \tilde{Y}_1$, $a_L = |x_2|/\sqrt{1/L + x_2^2}$ and $b_L = 1/\sqrt{1 + L x_2^2}$. Finally,

$$e_0 = b_L\ \mathbf{n}_0 - a_L\ T_0 \quad \text{and} \quad e_1 = X_2$$

So,

$$\begin{aligned} \mathrm{II}_L^{M,\mathscr{F}_3^{\mathbb{G}}} |_{HM} &= (\langle \boldsymbol{\nabla}_{e_1} \mathbf{n}_L, e_1 \rangle_L) \\ &= (-\langle \boldsymbol{\nabla}_{e_1} e_1, \mathbf{n}_L \rangle_L) \\ &= (-\langle \boldsymbol{\nabla}_{X_2} X_2, \mathbf{n}_L \rangle_L) = (0). \end{aligned}$$

We note that $(D_0^2 u)^*_{1,1} = 0$, verifying Proposition 3.13 in this case.

4. Graphs in $\mathbb{G} \times \mathbb{R}$

Let $\mathbb{G}$ be a Carnot group which is equipped with a family of approximating metrics g_L as in the previous section. Consider the Carnot group $\mathbb{G} \times \mathbb{R}$ with coordinates (x, s), $x \in \mathbb{G}$, $s \in \mathbb{R}$. On the level of the Lie algebra, this corresponds to adding a single vector field, $S = \frac{\partial}{\partial s}$, to the first layer of the grading. If $u : \mathbb{G} \to \mathbb{R}$ is a smooth function, we consider its graph in $\mathbb{G} \times \mathbb{R}$:

$$\mathscr{G}(u) = \{(x, s) \in \mathbb{G} \times \mathbb{R} : u(x) - s = 0\} \tag{4.1}$$

Such hypersurface has no characteristic points, hence we will use the results from the previous section without any restriction. At first we will consider only smooth graph, then we extend our result to graphs of Γ^2 functions.

For $(\mathbb{G} \times \mathbb{R}, g_L \oplus ds^2)$ we construct the three bases introduced in the previous section. We write[5]

$$\mathscr{F}_1^{\mathbb{G} \times \mathbb{R}} = \mathscr{F}_1^{\mathbb{G}} \cup \{S\}$$

[5]With a slight abuse of notation we denote by the same symbol the vectors in $\mathbb{G}$ and their lift in $\mathbb{G} \times \mathbb{R}$.

and

$$\mathscr{F}_2^{\mathbb{G}\times\mathbb{R}} = \{\zeta_1, \ldots, \zeta_{m+n+2}, \boldsymbol{\nu}_L\},$$

where $\zeta_1, \ldots, \zeta_{m+n+2}$ are a basis for $T\mathscr{G}(u)$ and $\boldsymbol{\nu}_L$ is the unit normal in the $g_L \oplus ds^2$ metric. Here, we are using a notational convention that will persist through the balance of the paper. Given an object that is defined for both $\mathbb{G}$ and $\mathbb{G}\times\mathbb{R}$, we will use roman letters to denote the object for $\mathbb{G}$ and greek letters to denote the object for $\mathbb{G}\times\mathbb{R}$.

Lemma 3.3 applied to this case yields

$$\sqrt{1+|\nabla_L u|_L^2}\,\mathrm{II}_L^{\mathscr{G}(u)} = \left(\mathrm{Hess}_L(u(x)-s)\right)|_{T\mathscr{G}(u)}. \tag{4.2}$$

As before, we construct the third basis by introducing

$$\begin{aligned}
\boldsymbol{\nu}_L &= \frac{|\nabla_L u|_L \mathbf{n}_L - S}{\sqrt{1+|\nabla_L u|_L^2}},\\
\boldsymbol{\nu}_0 &= \frac{|\nabla_0 u|\mathbf{n}_0 - S}{\sqrt{1+|\nabla_0 u|^2}},\\
\tau_0 &= \frac{\tilde{Y}_1 u\,\tilde{Y}_1 + \cdots + \tilde{Y}_{n+1} u\,\tilde{Y}_{n+1}}{|\tilde{Y}_1 u\,\tilde{Y}_1 + \cdots + \tilde{Y}_{n+1} u\,\tilde{Y}_{n+1}|_L}.
\end{aligned}$$

Then we define $\alpha_L = \langle \boldsymbol{\nu}_L, \boldsymbol{\nu}_0\rangle_L$ and $\beta_L = \langle \boldsymbol{\nu}_L, \tau_0\rangle_L$ so that $\boldsymbol{\nu}_L = \alpha_L \boldsymbol{\nu}_0 + \beta_L \tau_0$. From this we define $\epsilon_0 = \beta_L \boldsymbol{\nu}_0 - \alpha_L \tau_0$ and the full frame for $\mathbb{G}\times\mathbb{R}$:

$$\mathscr{F}_3^{\mathbb{G}\times\mathbb{R}} = \{\epsilon_0, \epsilon_1, \ldots, \epsilon_{m+1}, \tau_1, \ldots, \tau_n, \boldsymbol{\nu}_L\},$$

where $\{\epsilon_1, \ldots, \epsilon_{m+1}\}$ is an orthonormal basis for $H\mathscr{G}(u)$ and $\{\tau_1, \ldots, \tau_n\}$ is an orthonormal basis for $V\mathscr{G}(u)$. Applying Lemma 3.3 gives for $i, j = 1, \ldots, m+1$

$$\begin{aligned}
\left(A^t\,\mathrm{Hess}_L^{\mathscr{F}_1^{\mathbb{G}\times\mathbb{R}}}(u(x)-s)A\right)_{ij} &= \left(\mathrm{Hess}_L^{\mathscr{F}_3^{\mathbb{G}\times\mathbb{R}}}(u(x)-s)\right)_{ij}\\
&= \sqrt{1+|\nabla_L u|_L^2}\left(\mathrm{II}_L^{\mathscr{G}(u),\mathscr{F}_3^{\mathbb{G}\times\mathbb{R}}}\right)_{ij}
\end{aligned} \tag{4.3}$$

where $A = [\mathscr{F}_3^{\mathbb{G}\times\mathbb{R}} \to \mathscr{F}_1^{\mathbb{G}\times\mathbb{R}}]$ is the (orthogonal) change of coordinates matrix from $\mathscr{F}_3^{\mathbb{G}\times\mathbb{R}}$ to $\mathscr{F}_1^{\mathbb{G}\times\mathbb{R}}$.

Lemma 4.1. *Let $(\mathbb{G}, g_L)$ be a Riemannian graded nilpotent Lie group and $u : \mathbb{G} \to \mathbb{R}$ a smooth function. Then,*

$$\mathrm{Hess}_L^{\mathscr{F}_1^{\mathbb{G}\times\mathbb{R}}}(u(x)-s)|_{H(\mathbb{G}\times\mathbb{R})} = \begin{pmatrix} (D_0^2 u)^* & 0 \\ 0 & 0 \end{pmatrix} \tag{4.4}$$

Proof. Denote by Γ^j_{ik} the Christoffel symbols for g_L. By the definition of the Hessian one has

$$\begin{aligned}(\mathrm{Hess}_L^{\mathscr{F}_1^{\mathbb{G}\times\mathbb{R}}}(u(x)-s))_{X_i,X_j} &= \langle \boldsymbol{\nabla}_{X_i}\nabla_L(u(x)-s), X_j\rangle_L \\ &= \langle \boldsymbol{\nabla}_{X_i}(\sum_k X_k u\, X_k + \sum_l \tilde{Y}_l u\, \tilde{Y}_l - S), X_j\rangle_L \\ &= \sum_k X_i X_k u \langle X_k, X_j\rangle_L + \sum_k X_k u \langle \boldsymbol{\nabla}_{X_i} X_k, X_j\rangle_L + \sum_l \tilde{Y}_l u \langle \boldsymbol{\nabla}_{X_i}\tilde{Y}_l, X_j\rangle_L \\ &= X_i X_j u + \sum_k X_k u \Gamma^j_{ik} + \sum_l \tilde{Y}_l \Gamma^j_{i,l+m+1}.\end{aligned}$$

To simplify the latter, we note that since $[X_i, X_j] \notin H(\mathbb{G}\times\mathbb{R})$, $\Gamma^j_{i,k} = 0$ for $1 < i, j, k < m+1$. The symmetry of the (Riemannian) Hessian yields

$$(\mathrm{Hess}_L^{\mathscr{F}_1^{\mathbb{G}\times\mathbb{R}}}(u(x)-s))_{X_i,X_j} = (\mathrm{Hess}_L^{\mathscr{F}_1^{\mathbb{G}\times\mathbb{R}}}(u(x)-s))_{X_j,X_i}$$

which, coupled the anti-symmetry of the Γ^j_{ik} in i and j [10, Corollary 3.3, p. 54], yields

$$\begin{aligned}X_i X_j u + \sum_l \tilde{Y}_l \Gamma^j_{i,l+m+1} &= \frac{1}{2}(X_i X_j u + X_j X_i u) + \sum_l \tilde{Y}_l u(\Gamma^j_{i,l+m+1} + \Gamma^i_{j,l+m+1}) \\ &= \frac{1}{2}(X_i X_j u + X_j X_i u)\end{aligned}$$

This shows that the upper left hand block of the Hessian, written with respect to the basis $\mathscr{F}_1^{\mathbb{G}\times\mathbb{R}}$, is equal to the symmetrized horizontal Hessian. Moreover, the Kozul formula implies that $\boldsymbol{\nabla}_S U = 0 = \boldsymbol{\nabla}_U S$ for $U \in \{X_1, \ldots, X_{m+1}, \tilde{Y}_1, \ldots, \tilde{Y}_{n+1}\}$, showing that the remaining entries are zero. $\square$

Remark 4.2. Note that, in contrast with the Euclidean case, the derivation of (4.4) for general Carnot groups relies on significant cancellation properties stemming from the underlying symmetry of the Christoffel symbols.

If M is a symmetric matrix we will write $\lambda \le M \le \Lambda$ if $\lambda \le \langle MW, W\rangle \le \Lambda$ for all unit vectors W. We emphasize that the quantities Λ and λ denote *functions* on $\mathbb{G}$. We can now prove our main theorem.

Theorem 4.3. *Let $\mathbb{G}$ be a Carnot group equipped with a family of Riemannian approximating metrics, $\{g_L\}$, and let $u : \mathbb{G} \to \mathbb{R}$ be a smooth function.*

(a) *If $\Lambda \ge (D_0^2 u)^* \ge \lambda \ge 0$, then*

$$\frac{\lambda}{(1+|\nabla_L u|_L^2)^{3/2}} \le \mathrm{II}_L^{\mathscr{G}(u)}\ |_{H\mathscr{G}(u)} \le \frac{\Lambda}{(1+|\nabla_L u|_L^2)^{1/2}}$$

for all $L > 0$, and

$$\frac{\lambda}{(1+|\nabla_0 u|^2)^{3/2}} \le (\mathrm{II}_0^{\mathscr{G}(u)})^* \le \frac{\Lambda}{(1+|\nabla_0 u|^2)^{1/2}}.$$

(b)(i) *If* $\lambda \le \mathrm{II}_L^{\mathscr{G}(u)} |_{H\mathscr{G}(u)} \le \Lambda$ *then*

$$\lambda(1+|\nabla_L u|_L^2)^{1/2} \le (D_0^2 u)^* \le \Lambda(1+|\nabla_L u|_L^2)^{3/2}.$$

(b)(ii) *If* $\lambda \le (\mathrm{II}_0^{\mathscr{G}(u)})^* \le \Lambda$ *then*

$$\lambda(1+|\nabla_0 u|^2)^{1/2} \le (D_0^2 u)^* \le \Lambda(1+|\nabla_0 u|^2)^{3/2}.$$

(c) *The horizontal Gauss curvature of $\mathscr{G}(u)$ is given by*

$$\det((\mathrm{II}_0^{\mathscr{G}(u)})^*) = \left(\sqrt{1+|\nabla_L u|_L^2}\right)^{-(m+3)} \det[(D_0^2 u)^*].$$

Here $\mathscr{G}(u)$ denotes the graph of u in $\mathbb{G} \times \mathbb{R}$.

In particular,

Corollary 4.4. *A smooth function $u : \mathbb{G} \to \mathbb{R}$ is convex if and only if $(\mathrm{II}_0^{\mathscr{G}(u)})^*$ is positive semi-definite.*

Remark 4.5. Compare part (c) in Theorem 4.3 with Definition 10.3 in [8], where the horizontal Gauss curvature has been first introduced in the literature. Although our definition differs from the one in [8], Theorem 4.3 shows that they are in fact equivalent.

For the proof of Theorem 4.3, we begin with a simple linear algebraic lemma.

Lemma 4.6. *Let Π be a codimension one linear subspace of $\mathbb{R}^{d+1}$ which is transverse to the x_{d+1}-axis. Let $\theta \in [0, \pi/2)$ be the angle between the normal to Π and the $d+1$ axis. Let M be an $n \times n$ symmetric matrix and denote by $\tilde{M}$ the restriction to Π of the bilinear form in $\mathbb{R}^{d+1}$ associated to the matrix*

$$\begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix}.$$

One has

$$\lambda \le M \le \Lambda \quad \Rightarrow \quad \lambda \cos^2\theta \le \tilde{M} \le \Lambda$$

while

$$\lambda \le \tilde{M} \le \Lambda \quad \Rightarrow \quad \lambda \le M \le \Lambda \sec^2\theta.$$

Moreover, if we define the determinant of a bilinear form to be the determinant of a matrix representation (in any orthonormal frame) then we have

$$\det(\tilde{M}) = \cos^2\theta \det(M). \tag{4.5}$$

Proof. Since M is symmetric and the conclusions are invariant under conjugation with an orthonormal matrix in $\mathbb{R}^d$ (i.e., a rotation around the x_{d+1} axis), we can assume without loss of generality that the unit normal $\vec{\nu}$ to Π is given by the vector $(\sin\theta U_1, \cos\theta)$, where U_1 is the first vector in the canonical orthonormal basis $\{U_1, \dots, U_d\}$ of $\mathbb{R}^d$. Set $\tilde{U}_1 = (-\cos\theta U_1, \sin\theta)$, and $\tilde{U}_i = (U_i, 0) \in \mathbb{R}^{d+1}$, $i = 2, \dots, d$. The frame $\{\tilde{U}_1, \dots, \tilde{U}_d\}$ is an orthonormal frame for Π. If $w = (w', w_{d+1}) \in \Pi$ then $|w_{d+1}| \le \tan\theta |w'|$ whence

$$|w| \le \sec\theta |w'|.$$

Indeed, $0 = \langle w, \vec{\nu}\rangle = \sin\theta \langle w', U_1\rangle + w_{d+1}\cos\theta$.

If $\lambda \le M \le \Lambda$ and $w = (w', w_{d+1}) \in \Pi$, then

$$\lambda \cos^2\theta |w|^2 \le \lambda |w'|^2 \le \langle Mw', w'\rangle = \langle \tilde{M}w, w\rangle \le \Lambda |w'|^2 \le \Lambda |w|^2.$$

On the other hand, if $\lambda \le \tilde{M} \le \Lambda$ and $u \in \mathbb{R}^d$, choose w_{d+1} so that $w = (u, w_{d+1}) \in \Pi$. (This is possible by the transversality assumption.) Then

$$\lambda |u|^2 \le \lambda |w|^2 \le \langle \tilde{M}w, w\rangle = \langle Mu, u\rangle \le \Lambda |w|^2 \le \Lambda \sec^2\theta |u|^2.$$

In order to prove (4.5) we observe that $\det \tilde{M}$ can be computed by evaluating the determinant of the matrix $(\langle \tilde{M}\tilde{U}_i, \tilde{U}_j\rangle)_{ij}$. The latter coincides with $\langle MU_i, U_j\rangle$ with the first row and the first column both multiplied by $-\cos\theta$. Consequently $\det \tilde{M} = \cos^2\theta \det(M)$. This completes the proof of the lemma. □

Proof of Theorem 4.3. By (4.3), the estimates

$$\Lambda \ge \sqrt{1 + |\nabla_L u|_L^2}\ \mathrm{II}_L^{\mathscr{G}(u), \mathscr{F}_3^{\mathbb{G}\times\mathbb{R}}}\ |_{H\mathscr{G}(u)} \ge \lambda$$

hold if and only if the estimates

$$\Lambda \ge \Big(A^T \operatorname{Hess}_L^{\mathscr{F}_1^{\mathbb{G}\times\mathbb{R}}}(u(x) - s) A\Big)_{i,j=1,\dots,m+1} \ge \lambda \tag{4.6}$$

hold (here A is as in (4.3)).

By Lemma 4.1 we see that (4.6) holds if and only if

$$\Lambda \ge \left(A^T \begin{pmatrix} (D_0^2 u)^* & 0 \\ 0 & 0 \end{pmatrix} A\right) |_{H\mathscr{G}(u)} \ge \lambda. \tag{4.7}$$

To finish the proof of the first part of (a) and part (b)(i), we apply Lemma 4.6 with $\Pi = (H\mathscr{G}(u))$ (represented in the basis $\mathcal{F}_1^{\mathbb{G}\times\mathbb{R}}$), $M = (D_0^2 u)^*$ and $d = m + 1$. Note that the vectors $X_i + (X_i u)S$, $i = 1, \dots, m+1$, form a basis for Π, and the cosine of θ is precisely the component of the unit normal $\boldsymbol{\nu}_L$ in the direction $-x_{d+1}$, i.e.,

$$\cos\theta = \frac{1}{\sqrt{1 + |\nabla_L u|_L^2}}.$$

The results in the first part of (a) as well as (b)(i) then follow from (4.7) and Lemma 4.6.

The second part of (a), as well as (b)(ii) now follow by passing to the limit as $L \to \infty$.

Finally, (c) follows from 4.3 and 4.5. □

Combining Corollary 4.4 and Remark 2.7 gives the following additional corollary.

Corollary 4.7. *Let $U = \{(x, s) : u(x) \le s\}$ be the epigraph of a smooth function $u : \mathbb{G} \to \mathbb{R}$. Then U is convex (as a set, in the sense of* [8]*) if and only if the symmetrized horizontal second fundamental form of its boundary $(\mathrm{II}_0^{\partial U})^*$ is positive semi-definite.*

Next we reduce the smoothness assumptions on the function u and its level sets M from C^∞ to Γ^2. Let $u \in \Gamma^2(\mathbb{G})$ and let f be a standard mollifier in $\mathbb{G}$ (see e.g. [11]). For every $\epsilon > 0$ set $f_\epsilon(x) = \epsilon^{-Q} f(\delta_{\epsilon^{-1}}(x))$, where $Q = \sum_{i=1}^k i \dim(\mathcal{V}_i)$ is the *homogeneous dimension* of $\mathbb{G}$ and $\delta_s : \mathbb{G} \to \mathbb{G}$, $s > 0$, are the non-isotropic dilations given by

$$\delta_s(\exp(\sum_{i=1}^{m+1} a_i X_i + \sum_{j=1}^{n+1} b_j Y_j) = \exp(\sum_{i=1}^{m+1} s a_i X_i + \sum_{j=1}^{n+1} s^{d(j)} b_j Y_j)$$

Let $u_\epsilon(x) = f_\epsilon * u(x) = \int_{\mathbb{G}} f_\epsilon(y) u(y^{-1}x) dy$ denote the group convolution. Clearly u_ϵ is a smooth function and $u_\epsilon \to u$, $X_i u_\epsilon = (X_i u) * f_\epsilon \to X_i u$, and $X_i X_j u_\epsilon = (X_i X_j u) * f_\epsilon \to X_i X_j u$ as $\epsilon \to 0$, uniformly on compact subsets of $\mathbb{G}$.

Note that while at the level of the g_L metrics it is not possible to compute the full Hessian of a function $u \in \Gamma^2$ or the second fundamental form of its graph (as there is no *a priori* differentiability along the higher layers of the stratification), both the horizontal Hessian of u and the horizontal second fundamental form of $\mathscr{G}(u)$ are meaningful for functions $u \in \Gamma^2$ (outside of the characteristic set). Applying the previous results to u_ϵ and its graph for $L = 0$, we finally obtain the Γ^2 (and $L = 0$) versions of Theorem 4.3 and Corollaries 4.4 and 4.7 as stated in Theorem 1.1.

Theorem 4.8. *Let $\mathbb{G}$ be a Carnot group.*

(i) *If $u : \mathbb{G} \to \mathbb{R}$ is a Γ^2 function and $\mathscr{G}(u)$ denotes the graph of u in $\mathbb{G} \times \mathbb{R}$, then*

$$\Lambda \ge (D_0^2 u)^* \qquad \text{for some functions } 0 \le \lambda \le \Lambda < \infty$$

if and only if

$$\Lambda' \ge (\mathrm{II}_0^{\mathscr{G}(u)})^* \ge \lambda' \qquad \text{for some functions } 0 \le \lambda' \le \Lambda' < \infty$$

at every point of $\mathbb{G}$.

(ii) *A Γ^2 function $u : \mathbb{G} \to \mathbb{R}$ is convex if and only if $(\mathrm{II}_0^{\mathscr{G}(u)})^*$ is positive semi-definite.*

(iii) *Let $U = \{(x,s) : u(x) \le s\}$ be the epigraph of a Γ^2 function $u : \mathbb{G} \to \mathbb{R}$. Then U is convex (as a set, in the sense of* [8]*) if and only if the symmetrized horizontal second fundamental form of its boundary, $(\mathrm{II}_0^{\partial U})^*$, is positive semi-definite.*

The choices of Λ', λ' in terms of Λ, λ coincide with those in the statement of Theorem 4.3.

5. Examples and further discussion

In this section, we illustrate these results with some explicit computations and examples in the Heisenberg group. First, we give an explicit choice of the frame $\mathscr{F}_3^{\mathbb{H}\times\mathbb{R}}$ relative to which the horizontal component of the second fundamental form has a particularly nice formulation in terms of the symmetrized horizontal Hessian. Then, we present an explicit example showing the possibility of a difference between the horizontal second fundamental form and the limit of the Riemannian second fundamental forms.

Example 5.1. We use the standard vector fields $\{X_1, X_2, Y_1, S\}$ on the product space $\mathbb{H} \times \mathbb{R} = \{(x_1, x_2, x_3, s)\}$, as in Example 2.9. Here $S = \partial/\partial s$ is the vector field governing motion in the s-direction. Let $u \in C^\infty(\mathbb{H})$. We construct an adapted frame $\mathscr{F}_3^{\mathbb{H}\times\mathbb{R}} = \{\epsilon_0, \epsilon_1, \epsilon_2, \boldsymbol{\nu}_L\}$ for the graph $\mathscr{G}(u)$ following the procedure described in section 4: setting $p = X_1 u$, $q = X_2 u$, $r = \tilde{Y}_1 u$,

$$l = |\nabla_0 u| = \sqrt{p^2+q^2}, \qquad m = \sqrt{1+p^2+q^2},$$

and

$$l_L = |\nabla_L u| = \sqrt{p^2+q^2+r^2}, \qquad m_L = \sqrt{1+p^2+q^2+r^2},$$

we choose tangent vectors

$$\epsilon_0 = \frac{pr}{mm_L}X_1 + \frac{qr}{mm_L}X_2 - \frac{m}{m_l}\tilde{Y}_1 - \frac{r}{mm_L}S, \qquad \epsilon_1 = \overline{q}X_1 - \overline{p}X_2,$$

and

$$\epsilon_2 = \frac{\overline{p}}{m}X_1 + \frac{\overline{q}}{m}X_2 + \frac{l}{m}S,$$

orthogonal to the normal vector

$$\boldsymbol{\nu}_L = \frac{p}{m_L}X_1 + \frac{q}{m_L}X_2 + \frac{r}{m_L}\tilde{Y}_1 - \frac{1}{m_L}S.$$

Here we use the convenient shorthand $\overline{p} = p/l$ and $\overline{q} = q/l$.

Then $A = [\mathscr{F}_3^{\mathbb{H}\times\mathbb{R}} \to \mathscr{F}_1^{\mathbb{H}\times\mathbb{R}}]$ is the matrix whose transpose has entries which are precisely the X_1, X_2, $\tilde{Y}_1$, and S coefficients of $\epsilon_0, \epsilon_1, \epsilon_2, \boldsymbol{\nu}_L$. In the basis $\mathscr{F}_1^{\mathbb{H}\times\mathbb{R}}$, the matrix for $\mathrm{Hess}_L^{\mathscr{F}_1^{\mathbb{H}\times\mathbb{R}}}(u(x)-s)$ has principal 2×2 minor

$$H^* := (D_0^2 u)^*,$$

and final column and row consisting entirely of zeros. A simple computation shows that the horizontal second fundamental form, which is the central 2×2 minor of

$$\frac{1}{m} A^t \cdot \mathrm{Hess}_L^{\mathscr{F}_1^{\mathbb{H}\times\mathbb{R}}}(u(x) - s) \cdot A,$$

takes the form

$$II_0^{\mathscr{G}(u)}\Big|_{H\mathscr{G}(u)} = \begin{pmatrix} \frac{1}{m}\langle H^* v^\perp, v^\perp\rangle & \frac{1}{m^2}\langle H^* v, v^\perp\rangle \\ \frac{1}{m^2}\langle H^* v, v^\perp\rangle & \frac{1}{m^3}\langle H^* v, v\rangle \end{pmatrix}, \tag{5.1}$$

where $v = (\overline{p}, \overline{q})$ and $v^\perp = (\overline{q}, -\overline{p})$. Then

$$\det II_0^{\mathscr{G}(u)}\Big|_{H\mathscr{G}(u)} = \frac{\det(D_0^2 u)^*}{(1 + |\nabla_0 u|^2)^2}$$

is the horizontal Gauss curvature of $\mathscr{G}(u)$ (see Theorem 4.3 in the previous section), and

$$\mathrm{Trace}\, II_0^{\mathscr{G}(u)}\Big|_{H\mathscr{G}(u)} = \frac{\mathrm{Trace}(D_0^2 u)^* + |\nabla_0 u|^3 \mathcal{H}_0^M}{(1 + |\nabla_0 u|^2)^{3/2}} = \mathcal{H}_0^{\mathscr{G}(u)}$$

is the horizontal mean curvature of $\mathscr{G}(u)$, see Definition 3.10. Here we wrote M for the level set of u.

Example 5.2. In this example, we show that the limit of the Riemannian second fundamental forms may differ from the horizontal second fundamental form. With notation as in the previous example, we consider the graph $\mathscr{G}(u)$ of $u(x_1, x_2, x_3) = \frac{x_1 x_2}{2} - x_3$ in $\mathbb{H} \times \mathbb{R}$. Then $X_1(u - s) = y$, $X_2(u - s) = 0$, $\tilde{Y}_1(u - s) = -1/\sqrt{L}$, and $S(u - s) = -1$. The Riemannian unit normal to $\mathscr{G}(u)$ is

$$\boldsymbol{\nu}_L = \frac{x_2}{\sqrt{1 + \frac{1}{L} + x_2^2}}\, X_1 - \frac{1}{\sqrt{L}\sqrt{1 + \frac{1}{L} + x_2^2}}\, \tilde{Y}_1 - \frac{1}{\sqrt{1 + \frac{1}{L} + x_2^2}}\, S$$

and the unit horizontal normal is

$$\boldsymbol{\nu}_0 = \frac{y}{\sqrt{1 + x_2^2}}\, X_1 - \frac{1}{\sqrt{1 + x_2^2}}\, S.$$

We construct ϵ_0 as described in section 4, and choose two additional horizontal tangent vector fields $\epsilon_1 = X_2$ and

$$\epsilon_2 = \frac{1}{\sqrt{1 + x_2^2}}\, X_1 + \frac{x_2}{\sqrt{1 + x_2^2}}\, S$$

tangent to $\mathscr{G}(u)$, and write $\mathcal{F} = \{\epsilon_0, \epsilon_1, \epsilon_2, \boldsymbol{\nu}_L\}$ and $\mathcal{F}_0 = \{\epsilon_0, \epsilon_1, \epsilon_2, \boldsymbol{\nu}_0\}$. Next, we compute the horizontal component of the Riemannian second fundamental form, as well as the entire horizontal second fundamental form, using the observations

$$\nabla_{\epsilon_1}\epsilon_2 = \left(\frac{1}{\sqrt{1 + x_2^2}}\right)_{x_2} X_1 + \left(\frac{x_2}{\sqrt{1 + x_2^2}}\right)_{x_2} X_4 - \frac{\sqrt{L}}{2\sqrt{1 + x_2^2}} \tilde{Y}_1$$

and

$$\nabla_{\epsilon_2}\epsilon_1 = \frac{\sqrt{L}}{2\sqrt{1+x_2^2}}\tilde{Y}_1.$$

Taking the inner product with $\boldsymbol{\nu}_L$ and $\boldsymbol{\nu}_0$ respectively gives

$$\mathrm{II}_L^{\mathscr{G}(u),\mathcal{F}}\Big|_{H\mathscr{G}(u)} = \begin{pmatrix} 0 & -\frac{1}{2\sqrt{1+\frac{1}{L}+x_2^2}} \\ -\frac{1}{2\sqrt{1+\frac{1}{L}+x_2^2}} & 0 \end{pmatrix}$$

and

$$\mathrm{II}_0^{\mathscr{G}(u),\mathcal{F}_0}\Big|_{H\mathscr{G}(u)} = \begin{pmatrix} 0 & -\frac{1}{\sqrt{1+x_2^2}} \\ 0 & 0 \end{pmatrix}.$$

Observe that $\lim_{L\to\infty} II_L = II_0^*$.

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