Lusin Type Approximation of BD functions

Ebobisse François B.
Dipartimento di Matematica
Via F. Buonarroti 2, 56127 Pisa, Italy
E-mail:ebobisse@paley.dm.unipi.it

The purpose of this paper is to establish a Lusin type approximation of functions with bounded deformation by Lipschitz or C^1 functions. The main ingredients in the proof of our result are the maximal function of the measure Eu, the "Poincaré type" result by Kohn [14] and the approximate symmetric differentiabily of BD functions by Ambrosio-Coscia-Dal Maso [5].

1 Introduction.

Lusin type results for Sobolev functions have been established by Liu [15], Michael-Ziemer [17] and Acerbi-Fusco [1]. Those results have been successfully applied by Acerbi-Fusco [1] to prove the lower semicontinuity of quasi-convex integral functionals defined in $W^{1,p}(\Omega)$ $(1 \leq p < +\infty)$. Precisely, they proved that it is possible to extract from bounded sequences in $W^{1,p}(\Omega)$, approximating subsequences in the sense of Lusin which are bounded in $W^{1,\infty}(\Omega)$ and for which the $W^{1,\infty}(\Omega)$-weak* lower semicontinuity of quasi-convex integral functionals was already known.
By using the notion of maximal function of Radon measures with finite total variation, Ambrosio [4] was able to extend in the space $BV(\Omega, \mathbb{R}^k)$ the Acerbi-Fusco approximating result and proved the $L^1(\Omega, \mathbb{R}^k)$-strong lower semicontinuity of integral functionals defined in $SBV(\Omega, \mathbb{R}^k)$, the space of *special functions of bounded variation* introduced in De Giorgi-Ambrosio [8].
The Lusin type approximation of Sobolev functions is also useful to study the regularity properties of minimizers of quasi-convex integral functionals. In fact, a partial regularity result of minimizers of the integral functional $\int_\Omega f(\nabla u(x))dx$ with a twice continuously differentiable and uniformly strictly quasi-convex integrand f was first proved by Evans [10] under a more restrictive assumption on the growth of the second derivative of f. Acerbi-Fusco [2] have singled out the Lusin type approximation of Sobolev functions as a powerful tool to prove the Evans's result without no assumption on the growth of $D^2 f$.
In this paper we show that, given $u \in BD(\Omega)$, the space of *functions with bounded deformation*, and using the *maximal function* of the measure Eu, the "Poincaré type"

result by Kohn [14] and the *approximate symmetric differentiability* by Ambrosio-Coscia-Dal Maso [5] for BD functions, we obtain a Lusin type result for the function u. Precisely we prove the following theorem.

Theorem 1.1 *Let Ω be a Lipschitz bounded open subset of $\mathbb{R}^n$ and $u \in BD(\Omega)$. Then for any $\lambda > 0$, there exists a Lipschitz function $v_\lambda : \Omega \to \mathbb{R}^n$ with $lip(v_\lambda) \leq C(n,\Omega)\lambda$ such that:*

$$\mathcal{L}^n(\{x \in \Omega : v_\lambda(x) \neq u(x)\}) \leq \frac{C(n,\Omega)}{\lambda} \|u\|_{BD(\Omega)}, \tag{1.1}$$

where $C(n,\Omega)$ is a positive constant only depending on n and Ω.

From the theorem above, one can easily deduce the *approximate differentiability* of BD functions. This last result has also been proved by Ambrosio-Coscia-Dal Maso [5] and Hajłasz [13]. It is well known (see Federer [12, Theorem 3.1.8]) that for any function u approximately differentiable $\mathcal{L}^n$-almost everywhere and any $\varepsilon > 0$ there is a Lipschitz function v that agrees with u on a set whose complement has Lebesgue measure less than ε. Thus, the approximate differentiability of BD functions implies part of the theorem above. So, the new fact here with respect to the papers [5], [13] is the estimate (1.1) which could also be useful as in the case of Sobolev and SBV spaces to prove the lower semicontinuity of integral functionals defined in the spaces BD or SBD. The space SBD called *space of special functions of bounded deformation* has been introduced in Bellettini-Coscia-Dal Maso [7] and also studied in Ambrosio-Coscia-Dal Maso [5].

2 Preliminaries and Notations.

Let $n \geq 1$ be an integer, $M^{n\times n}$ be the space of $n \times n$ matrices and $M^{n\times n}_{sym}$ be the subspace of symmetric matrices in $M^{n\times n}$. For any $A \in M^{n\times n}$, A^T denotes the transposition of A. For any $x \in \mathbb{R}^n$ and $\rho > 0$, we denote by $B_\rho(x)$ the open ball of $\mathbb{R}^n$ centered at x with radius ρ. By $\mathcal{L}^n$ we denote the Lebesgue measure in $\mathbb{R}^n$. Let Ω be an open subset of $\mathbb{R}^n$. If μ is a Radon measure in Ω, we denote by $|\mu|$ its variation.

Definition 2.1 *We say that $u : \Omega \to \mathbb{R}^n$ is a function with bounded deformation in Ω, and we write $u \in BD(\Omega)$ if $u \in L^1(\Omega, \mathbb{R}^n)$ and*

$$Eu := (Du + D^T u)/2 \in M_b(\Omega, M^{n\times n}_{sym});$$

where Du is the distributional gradient of u and $M_b(\Omega, M^{n\times n}_{sym})$ is the space of $M^{n\times n}_{sym}$-valued Radon measures with finite total variation in Ω.

This space has been introduced by Matthies-Strang-Christiansen in [16] and widely studied by Anzellotti-Giaquinta [6], Kohn [14], Suquet [19], Temam [20] etc... in connection

with problems arising in the theory of plastic bodies.
The space $BD(\Omega)$ is a Banach space when equipped with the norm

$$\|u\|_{BD(\Omega)} := \|u\|_{L^1(\Omega,\mathbb{R}^n)} + |Eu|(\Omega). \tag{2.1}$$

Let $\mathcal{R}$ be the class of *rigid motion* in $\mathbb{R}^n$ i.e. affine maps of the form $Ax+d$ with A a skew symmetric $n \times n$ matrix and $d \in \mathbb{R}^n$. We will need the following "Poincaré type" inequality by Kohn [14] (see also Ambrosio-Coscia-Dal Maso [5], Temam [20] etc...) for BD functions.

Theorem 2.2 *Let Ω be a bounded connected open subset of $\mathbb{R}^n$ with Lipschitz boundary. Let $R : BD(\Omega) \to \mathcal{R}$ be a continuous linear map which leaves $\mathcal{R}$ fixed. Then there exists a positive constant $C(\Omega, R)$ such that:*

$$\int_\Omega |u - R(u)|dx \le C(\Omega, R)|Eu|(\Omega) \quad \textit{for any } u \in BD(\Omega). \tag{2.2}$$

When Ω is an open ball of $\mathbb{R}^n$ there is a precise representation of the rigid motion $R(u)$, given in the following theorem.

Theorem 2.3 *Let $u \in BD(\mathbb{R}^n)$, $x \in \mathbb{R}^n$ and $\rho > 0$. Then there exists a vector $d_\rho(u)(x) \in \mathbb{R}^n$ and an $n \times n$ skew symmetric matrix $A_\rho(u)(x)$ such that:*

$$\int_{B_\rho(x)} |u(y) - d_\rho(u)(x) - A_\rho(u)(x)(y-x)|dy \le C(n)\rho|Eu|(B_\rho(x)) \tag{2.3}$$

where $C(n)$ is a positive constant depending only on the dimension n. Moreover, $d_\rho(u)(x)$ and $A_\rho(u)(x)$ are expressed as singular integrals in the following ways:

$$d^i_\rho(u)(x) = \sum_{l,m=1}^{n} \int_{|y-x|\ge\rho} \frac{\bigwedge^i_{lm}(y-x)}{nw_n|y-x|^n} dEu_{lm}(y);$$

$$A^{ij}_\rho(u)(x) = \sum_{l,m=1}^{n} \int_{|y-x|\ge\rho} -\frac{\Gamma^{lm}_{ij}(y-x)}{2w_n|y-x|^{n+2}} dEu_{lm}(y),$$

where $\bigwedge$ and Γ, respectively third and fourth order tensor valued functions, are defined and studied in Kohn [14], Ambrosio-Coscia-Dal Maso [5].

The following singular integral result have been proved in Ambrosio-Coscia-Dal Maso [5].

Proposition 2.4 *Let $K : \mathbb{R}^n \setminus \{0\} \to \mathbf{R}$ be a 0-homogeneous function, smooth and with mean value zero on the unit sphere S^{n-1}. For any Radon measure μ with finite total variation in $\mathbb{R}^n$, let us define the functions*

$$h_\rho(x) = \int_{|y-x|\ge\rho} \frac{K(y-x)}{|y-x|^n} d\mu(y) \qquad \rho > 0.$$

Then the function $h(x) = \sup_{\rho>0} |h_\rho(x)|$ satisfies the following weak L^1 estimate

$$\mathcal{L}^n(\{x \in \mathbb{R}^n \colon h(x) > t\}) \leq \frac{C(n,K)}{t}|\mu|(\mathbb{R}^n). \tag{2.4}$$

The definition and the property (6.2) in Ambrosio-Coscia-Dal Maso [5] of the function Γ allow us to apply the Proposition 2.4 above to the function

$$A_u(x) = \sup_{\rho>0} |A_\rho(u)(x)|. \tag{2.5}$$

So for any $t > 0$,

$$\mathcal{L}^n(\{x \in \mathbb{R}^n \colon A_u(x) > t\}) \leq \frac{C(n,\Gamma)}{t}|Eu|(\mathbb{R}^n), \tag{2.6}$$

where the positive constant $C(n,\Gamma)$ is independent of u.
Let μ be a Radon measure with finite total variation in $\mathbb{R}^n$. We define the maximal function $M(\mu)$ of μ by:

$$M(\mu)(x) = \sup_{\rho>0} \frac{|\mu|(B_\rho(x))}{\mathcal{L}^n(B_\rho(x))} \qquad \forall x \in \mathbb{R}^n.$$

By covering theorems (see Evans-Gariepy [11], Federer [12] etc ...), we have the following weak type L^1-estimate

$$\mathcal{L}^n(\{x \in \mathbb{R}^n \colon M(\mu)(x) > t\}) \leq \frac{C(n)}{t}|\mu|(\mathbb{R}^n).$$

Therefore, for any $u \in BD(\mathbb{R}^n)$, we have

$$\mathcal{L}^n(\{x \in \mathbb{R}^n \colon M(Eu)(x) > t\}) \leq \frac{C(n)}{t}|Eu|(\mathbb{R}^n). \tag{2.7}$$

The Radon-Nikodym decomposition of the measure Eu gives

$$Eu = E^a u + E^s u = \mathcal{E}u\mathcal{L}^n + E^s u$$

where $E^a u$ and $E^s u$ are respectively the absolute continuous and the singular parts of Eu with respect to the Lebesgue measure $\mathcal{L}^n$. In Ambrosio-Coscia-Dal Maso [5] , Bellettini-Coscia-Dal Maso [7] it has been proved that the measure $E^s u$ can be split into two mutually singular measures called jump and Cantor parts and having precise representations and properties. The density of the absolute continuous part $E^a u$ with respect to the Lebesgue measure, denoted by $\mathcal{E}u$, has the following property proved in Ambrosio-Coscia-Dal Maso [5].

Theorem 2.5 *Let $u \in BD(\Omega)$. Then*

$$\lim_{\rho\to 0}\frac{1}{\rho^n}\int_{B_\rho(x)}\frac{|(u(y)-u(x)-\mathcal{E}u(x)(y-x),y-x)|}{|y-x|^2}dy=0 \tag{2.8}$$

for $\mathcal{L}^n$-almost every $x \in \Omega$.

Remark 2.6 The theorem above states that any function $u \in BD(\Omega)$ has an *approximate symmetric differential* which coincides $\mathcal{L}^n$ -almost everywhere in Ω with the density $\mathcal{E}u$ of the absolute continuous part $E^a u$ of Eu with respect to $\mathcal{L}^n$.

Proof. For the reader's convenience, we repeat here the proof of Ambrosio-Coscia-Dal Maso [5]. Let $B_\rho(x) \subset \Omega$, $\tau \in (0,1)$ and $v \in \mathcal{C}^1(B_\rho(x), \mathbb{R}^n)$. Using the identity

$$(v(y)-v(x+\tau(y-x)),y-x)=\int_\tau^1(\mathcal{E}v(x+t(y-x))(y-x),y-x)dt$$

and Fubini's theorem, we obtain

$$\int_{B_\rho(x)}\frac{|v(y)-v(x+\tau(y-x)),y-x)|}{|y-x|^2}dy\le\int_\tau^1\frac{|Ev|(B_\rho(x))}{t^n}dt. \tag{2.9}$$

Now, take $v \in BD(\Omega)$ and let $v_\varepsilon = v \star \psi_\varepsilon$ be the corresponding regularization. After writing (2.9) for v_ε, we pass to the limit as $\varepsilon \to 0^+$ using the fact that $|Ev_\varepsilon|(B_{t\rho}(x))$ converges to $|Ev|(B_{t\rho}(x))$ for any $t \in (0,1)$ such that $|Ev|(\partial B_{t\rho}(x)) = 0$. We then obtain (2.9) also for $v \in BD(\Omega)$. If x is a Lebesgue point of v, letting $\tau \to 0^+$ and dividing both sides by ρ^n we obtain

$$\frac{1}{\rho^n}\int_{B_\rho(x)}\frac{|(v(y)-v(x),y-x)|}{|y-x|^2}dy\le\int_0^1\frac{|Ev|(B_{t\rho}(x))}{(t\rho)^n}dt. \tag{2.10}$$

Let us now fix a Lebesgue point x for u and $\mathcal{E}u$ and such that $|E^s u|$ has n-dimensional density zero at x; applying (2.10) to the function $v(y) := u(y) - \mathcal{E}u(x)(y-x)$ we obtain

$$\begin{aligned}&\frac{1}{\rho^n}\int_{B_\rho(x)}\frac{|(u(y)-u(x)-\mathcal{E}u(x)(y-x),y-x)|}{|y-x|^2}dy\\&\le\sup_{t\in(0,1)}\left[\frac{|E^s u|(B_{t\rho}(x))}{(t\rho)^n}+\frac{1}{(t\rho)^n}\int_{B_{t\rho}(x)}|\mathcal{E}u(y)-\mathcal{E}u(x)|dy\right].\end{aligned}$$

By the choice of x, both terms in the right-hand side vanish as $\rho \to 0^+$. $\square$

We will need the following abstract lemma in the proof of our main result.

Lemma 2.7 *Let C be a convex subset of $\mathbb{R}^n$ having a positive measure. If we suppose that C generates all the space $\mathbb{R}^n$, then there exists a positive constant α such that*

$$\int_C |(w,y)|dy \geq \alpha |w| \quad \textit{for any } w \in \mathbb{R}^n \tag{2.11}$$

where $(,)$ denotes the scalar product in $\mathbb{R}^n$.

Proof. Suppose $w \neq 0$. Since $\alpha = \min\limits_{v \in S^{n-1}} \int_C |(v,y)|dy > 0$, then

$$\int_C |(\frac{w}{|w|}, y)|dy \geq \alpha$$

and the lemma follows. □

3 The main result.

For all what follows, we will denote by $C(n)$, or possibly $C(n,\Omega)$, any positive constant only depending on n, or also depending on Ω.

Theorem 3.1 *Let Ω be a Lipschitz bounded open subset of $\mathbb{R}^n$ and $u \in BD(\Omega)$. Then for any $\lambda > 0$, there exists a Lipschitz function $v_\lambda : \Omega \to \mathbb{R}^n$ with Lipschitz constant lip(v_λ) less than $C(n,\Omega)\lambda$ such that:*

$$\mathcal{L}^n(\{x \in \Omega : v_\lambda(x) \neq u(x)\}) \leq \frac{C(n,\Omega)}{\lambda} \|u\|_{BD(\Omega)}, \tag{3.1}$$

where $C(n,\Omega)$ is a positive constant only depending on n and Ω.

Proof. We will first prove the theorem for $\Omega = \mathbb{R}^n$. Let us define for any $\lambda > 0$,

$$E_\lambda = \{x \in \mathbb{R}^n\colon\ M(Eu)(x) \leq \lambda \ ; \ A_u(x) \leq \lambda\},$$

where $M(Eu)$ and A_u have been defined before. By (2.3), we have for any $x \in E_\lambda$ and $\rho > 0$,

$$\int_{B_\rho(x)} |u(y) - d_\rho(u)(x) - A_\rho(u)(x)(y-x)|dy \leq C(n)\rho|Eu|(B_\rho(x)) \leq C(n)\lambda\rho^{n+1}. \tag{3.2}$$

We also have from (2.10) in the proof of Theorem 2.5

$$\int_{B_\rho(x)} |(u(y)-u(x), y-x)|dy \quad \leq \quad \lambda\rho^{n+2},$$

for any Lebesgue point x of u in E_λ. Now let S_u be the complement of Lebesgue points of u, x and x' two points in $E_\lambda \setminus S_u$ and let $\rho = |x - x'|$. Then

$$\begin{aligned}
&\int_{B_\rho(x)} |(d_\rho(u)(x) + A_\rho(u)(x)(y-x) - u(x), y-x)|dy \\
&\leq \int_{B_\rho(x)} |(u(y) - u(x), y-x)|dy \\
&+ \int_{B_\rho(x)} |(u(y) - d_\rho(u)(x) - A_\rho(u)(x)(y-x), y-x)|dy \\
&\leq C(n)\lambda\rho^{n+2}.
\end{aligned}$$

Therefore

$$\int_{B_\rho(x)} |(d_\rho(u)(x) + A_\rho(u)(x)(y-x) - u(x), y-x)|dy \leq C(n)\lambda\rho^{n+2}$$

and this yields

$$\begin{aligned}
&\int_{B_\rho(x)} |(d_\rho(u)(x) - u(x), y-x)|dy \\
&\leq C(n)\lambda\rho^{n+2} + \int_{B_\rho(x)} |(A_\rho(u)(x)(y-x), y-x)|dy \\
&\leq C(n)\lambda\rho^{n+2}. \qquad (3.3)
\end{aligned}$$

Then

$$|d_\rho(u)(x) - u(x)| \leq C(n)\lambda\rho \qquad (3.4)$$

since the left hand side of (3.3) is equal to $|d_\rho(u)(x) - u(x)|\rho^{n+1}$ up to a positive multiplicative constant depending only on n. Now from

$$\int_{B_\rho(x')} |(u(y) - u(x'), y-x')|dy \leq \lambda\rho^{n+2} \qquad (3.5)$$

and the identity

$$\begin{aligned}
&(d_\rho(u)(x) - A_\rho(u)(x)(x-x') - u(x'), y-x') \\
&= (d_\rho(u)(x) + A_\rho(u)(x)(y-x) - u(y), y-x') + (u(y) - u(x'), y-x') \\
&-(A_\rho(u)(x)(y-x'), y-x'),
\end{aligned}$$

we obtain

$$\int_{B_\rho(x)\cap B_\rho(x')} |(d_\rho(u)(x) - A_\rho(u)(x)(x-x') - u(x'), y-x')|dy$$

$$\leq \int_{B_\rho(x)} |(d_\rho(u)(x) + A_\rho(u)(x)(y-x) - u(y), y - x')|dy$$
$$+ \int_{B_\rho(x')} |(u(y) - u(x'), y - x')|dy$$
$$+ \int_{B_\rho(x)\cap B_\rho(x')} |(A_\rho(u)(x)(y-x'), y - x'|dy. \tag{3.6}$$

From (3.2) , (3.5) one can easily see that the right hand side of (3.6) is estimated by from above $C(n)\lambda\rho^{n+2}$. The left hand side ycan always be estimated from below by

$$\rho^{n+1}|d_\rho(u)(x) - A_\rho(u)(x)(x-x') - u(x')|$$

up to a multiplicative positive constant. In fact, after rescalation and changes of variables, the left hand side of (3.6) is equal to

$$\rho^{n+1}\int_{B_1(0)\cap B_1((x-x')/\rho)} |(d_\rho(u)(x) - A_\rho(u)(x)(x-x') - u(x'), y)|dy. \tag{3.7}$$

Applying the abstract lemma 2.7 with $w = d_\rho(u)(x) - A_\rho(u)(x)(x-x') - u(x')$ and C is the closure of $B_1(0) \cap B_1((x-x')/\rho)$, we obtain the result. Then from (3.6), (3.7), and Lemma 2.7, the following inequality holds

$$|d_\rho(u)(x) - A_\rho(u)(x)(x-x') - u(x')| \leq C(n)\lambda\rho.$$

Therefore

$$|d_\rho(u)(x) - u(x')| \leq C(n)\lambda\rho. \tag{3.8}$$

So from (3.4), (3.8) and our choice of ρ we have

$$|u(x) - u(x')| \leq C(n)\lambda|x - x'|. \tag{3.9}$$

Then we have proved that the function $u|_{E_\lambda\setminus S_u}$ is Lipschitz continuous with its Lipschitz constant less or equal to $C(n)\lambda$. The Kirszbraun's theorem (see Federer [12, Theorem 2.10.43]) will then provide an extension v_λ in $\mathbb{R}^n$ of the function $u|_{E_\lambda\setminus S_u}$ with $\mathrm{lip}(v_\lambda) \leq C(n)\lambda$.
Let us now prove (3.1) in the case $\Omega = \mathbb{R}^n$. It follows from the weak type estimates (2.6) and (2.7) that

$$\begin{aligned}
\mathcal{L}^n(\{x \in \mathbb{R}^n\colon\ v_\lambda(x) \neq u(x)\}) &\leq \mathcal{L}^n(\mathbb{R}^n \setminus (E_\lambda \setminus S_u)) \\
&\leq \mathcal{L}^n(\{x \in \mathbb{R}^n\colon\ M(Eu)(x) > \lambda\}) \\
&+ \mathcal{L}^n(\{x \in \mathbb{R}^n\colon\ A_u(x) > \lambda\}) \\
&\leq \frac{C(n)}{\lambda}|Eu|(\mathbb{R}^n).
\end{aligned}$$

Now let $u \in BD(\Omega)$, where Ω is a bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$. Then, the extension $\bar{u}$ of u by 0 outside Ω is in $BD(\mathbb{R}^n)$ (see e.g., Temam [20], Ambrosio-Coscia-Dal Maso [5]) and satisfies

$$|E\bar{u}|(\mathbb{R}^n) \leq C(n,\Omega)\, \|u\|_{BD(\Omega)} \, .$$

Applying now what we have done above to $\bar{u}$, we obtain a Lipschitz function v_λ in $\mathbb{R}^n$ with $\mathrm{lip}(v_\lambda) \leq C(n)\lambda$ and

$$\begin{aligned} \mathcal{L}^n(\{x \in \Omega\colon\, v_\lambda(x) \neq u(x)\}) &\leq \mathcal{L}^n(\{x \in \mathbb{R}^n\colon\, v_\lambda(x) \neq \bar{u}(x)\}) \\ &\leq \frac{C(n)}{\lambda}|E\bar{u}|(\mathbb{R}^n) \\ &\leq \frac{C(n,\Omega)}{\lambda}\, \|u\|_{BD(\Omega)} \, . \end{aligned}$$

□

Remark 3.2

(i) Let us observe that the Lusin type result proved above does not require all the approximate symmetric differentiability property of BD-functions. But as we have seen, it only needs the inequality (2.10) used in the proof of Theorem 2.5. As we have said in the introduction, our main result is an other way of proving the full *approximate differentiability* of BD functions. In fact, one can easily see that any density point of the set $\{u = v_\lambda\}$ in which v_λ is differentiable in a classical sense is a point of *approximate differentiability* of the function u.

(ii) Since any Lipschitz function is approximated in the sense of Lusin by C^1 functions, from Theorem 3.1 one easily deduces the Lusin type approximation of BD-functions by C^1 ones.

Acknowledgement:
I thank L. Ambrosio for having suggested this problem and for many stimulating discussions.

References

[1] E. Acerbi-N. Fusco: Semicontinuity Problems in the Calculus of Variations, *Arch. Rational Mech. Anal.*, **86** (1984), pp. 125-145.

[2] E. Acerbi-N. Fusco: A regularity theorem for minimizers of quasi-convex integrals, *Arch. Rational Mech. Anal.*, **99** (1987), pp. 261-281.

[3] G. Alberti: A Lusin type property of gradients, *J. Funct. Analysis*, **100** (1991), pp. 110-118.

[4] L. Ambrosio: On the lower semicontinuity of quasi-convex integrals in $SBV(\Omega, \mathbb{R}^k)$, *Nonlinear Analysis, Theory, Methods & Applications*, **23** (1994), pp. 405-425.

[5] L. Ambrosio-A. Coscia-G. Dal Maso: Fine properties of functions with bounded deformation, *Arch. Rat. Mech. Anal.*, **139** (1997), pp. 201-238.

[6] G. Anzellotti-M. Giaquinta: Existence of the Displacement Field for an Elasto-plastic Body Subject to the Henky's Law and Von Mises Yield Condition. *Manuscripta Math.* **32** (1980), pp. 101-136.

[7] G. Bellettini-A. Coscia-G. Dal Maso: Special Functions of Bounded Deformation. Preprint SISSA, Trieste (1996).

[8] E. De Giorgi-L. Ambrosio: Un nuovo tipo di funzionale del calcolo delle variazioni. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Mat. Natur.* **82** (1988), pp. 199-210.

[9] F. Ebobisse: On Lower Semicontinuity of Integral Functionals in $LD(\Omega)$, to appear.

[10] L. C. Evans: Quasi-convexity and Partial Regularity in the Calculus of Variations, *Arch. Rat. Mech. Anal.*, **95** (1986), pp. 227-252.

[11] L. C. Evans-R. Gariepy: *Measure Theory and fine properties of functions*, CRC Press, Boca Raton, 1992.

[12] H. Federer: *Geometric Measure Theory*, Springer, Berlin (1969).

[13] P. Hajłasz: On Approximate Differentiability of Functions with Bounded Deformations, *Manuscripta Math.*, **91**, (1996), pp. 61-72.

[14] R.V. Kohn: *New Estimates for Deformations in Terms of Their Strains*, Ph.D. Thesis, Princeton Univ., 1979.

[15] F.C. Liu: A Lusin type property of Sobolev functions, *Indiana Univ. Math. J.*, **26** (1977), pp.645-651.

[16] H. Matthies-G. Strang-E. Christiansen: The saddle Point of a Differential Program, *Energy Methods in Finite Elements Analysis*, Wiley, New York 1979.

[17] J.H. Michael-W.P. Ziemer: A Lusin type approximation of Sobolev functions by smooth functions, *Contemporary Mathematics*, AMS **42** (1985), pp. 135-167.

[18] E.M. Stein: *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton 1970.

[19] P.M. Suquet: Un espace fonctionel pour les équations de la plasticité. *Ann. Fac. Toulouse*, **1** (1979), pp. 77-87.

[20] R.Temam: *Mathematical Problems in Plasticity*, Bordas, Paris, 1985.