THE SHARP QUANTITATIVE ISOPERIMETRIC INEQUALITY

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Abstract. A quantitative sharp form of the classical isoperimetric inequality is proved, thus giving a positive answer to a conjecture by Hall.

1. Introduction

The classical *isoperimetric inequality* states that if E is a Borel set in $\mathbb{R}^n$, $n \geq 2$, with finite Lebesgue measure $|E|$, then the ball with the same volume has a lower perimeter, or, equivalently, that

$$n\omega_n^{1/n}|E|^{(n-1)/n} \leq P(E)\,. \tag{1.1}$$

Here $P(E)$ denotes the distributional perimeter of E (which coincides with the classical $(n-1)$-dimensional measure of ∂E when E has a smooth boundary) and ω_n is the measure of the unit ball B in $\mathbb{R}^n$. It is also well known that equality holds in (1.1) if and only if E is a ball.

The history of the various proofs and different formulations of the isoperimetric inequality is definitely a very long and complex one. Therefore we shall not even attempt to sketch it here, but we refer the reader to the many review books and papers (e.g. [3], [18], [5], [21], [7], [13]) available on the subject and to the original paper by De Giorgi [8] (see [9] for the English translation) where (1.1) was proved for the first time in the general framework of sets of finite perimeter.

In this paper we prove a quantitative version of the isoperimetric inequality. Inequalities of this kind have been named by Osserman [19] *Bonnesen type inequalities*, following the results proved in the plane by Bonnesen in 1924 (see [4] and also [2]). More precisely, Osserman calls in this way any inequality of the form

$$\lambda(E) \leq P(E)^2 - 4\pi|E|\,,$$

valid for smooth sets E in the plane $\mathbb{R}^2$, where the quantity $\lambda(E)$ has the following three properties: (i) $\lambda(E)$ is non-negative; (ii) $\lambda(E)$ vanishes only when E is a ball; (iii) $\lambda(E)$ is a suitable measure of the "asymmetry" of E. In particular, any Bonnesen inequality implies the isoperimetric inequality as well as the characterization of the equality case.

The study of Bonnesen type inequalities in higher dimension has been carried on in recent times in [12], [16], [15]. In order to describe these results let us introduce, for any Borel set E in $\mathbb{R}^n$ with $0 < |E| < \infty$, the *isoperimetric deficit of E*

$$D(E) := \frac{P(E)}{n\omega_n^{1/n}|E|^{(n-1)/n}} - 1 = \frac{P(E) - P(rB)}{P(rB)}\,,$$

where r is the radius of the ball having the same volume as E, that is $|E| = r^n|B|$.

The paper [12] by Fuglede deals with convex sets. Namely, he proves that if E is a convex set having the same volume of the unit ball B then

$$\min\{\delta_H(E, x + B) : x \in \mathbb{R}^n\} \leq C(n)D(E)^{\alpha(n)}\,,$$

where $\delta_H(\cdot,\cdot)$ denotes the Hausdorff distance between two sets and $\alpha(n)$ is a suitable exponent depending on the dimension n. This result is sharp, in the sense that in [12] examples are given showing that the exponent $\alpha(n)$ found in the paper cannot be improved (at least if $n \neq 3$).

When dealing with general non convex sets, we cannot expect the isoperimetric deficit to control the Hausdorff distance from E to a ball. To see this it is enough to take, in any dimension, the union of a large ball and a far away tiny one or, if $n \geq 3$, a connected set obtained by adding to a ball an arbitrarily long (and suitably thin) "tentacle". It is then clear that in this case a natural notion of asymmetry is the so-called *Fraenkel asymmetry of E*, defined by

$$\lambda(E) := \min\left\{\frac{d(E, x + rB)}{r^n} : x \in \mathbb{R}^n\right\},$$

where $r > 0$ is again such that $|E| = r^n|B|$ and $d(E, F) = |E\Delta F|$ denotes the measure of the symmetric difference between any two Borel sets E, F.

This kind of asymmetry has been considered by Hall, Hayman and Weitsman in [16] where it is proved that if E is a smooth open set with a sufficiently small deficit $D(E)$, then there exists a suitable straight line such that, denoting by E^* the Steiner symmetral of E with respect to the line (see definition in Section 3), one has

$$\lambda(E) \le C(n)\sqrt{\lambda(E^*)}\,. \tag{1.2}$$

Later on Hall proved in [15] that for any axially symmetric set F

$$\lambda(F) \le C(n)\sqrt{D(F)} \tag{1.3}$$

and thus, combining (1.3) (applied with $F = E^*$) with (1.2), he was able to conclude that

$$\lambda(E) \le C(n)D(E^*)^{1/4} \le C(n)D(E)^{1/4}\,, \tag{1.4}$$

where the last inequality is immediate on recalling that Steiner symmetrization lowers the perimeter, hence the deficit. Though both estimates (1.2) and (1.3) are sharp, in the sense that one cannot replace the square root on the right-hand side by any better power, the exponent $1/4$ appearing in (1.4) does not seem to be optimal. And in fact Hall himself conjectured that the term $D(E)^{1/4}$ should be replaced by the smaller term $D(E)^{1/2}$. If so, the resulting inequality would be optimal, as one can easily check by taking an ellipsoid E with $n-1$ semiaxes of length 1 and the last one of length larger than 1.

We give a positive answer to Hall's conjecture by proving the following estimate.

Theorem 1.1. *Let $n \ge 2$. There exists a constant $C(n)$ such that for every Borel set E in $\mathbb{R}^n$ with $0 < |E| < \infty$*

$$\lambda(E) \le C(n)\sqrt{D(E)}\,. \tag{1.5}$$

A few remarks are in order. As we have already observed, the exponent $1/2$ in the above inequality is optimal and cannot be replaced by any bigger power. Notice also that both $\lambda(E)$ and $D(E)$ are scale invariant, therefore it is enough to prove (1.5) for sets of given measure. Thus, throughout the paper we shall assume that

$$|E| = |B|\,.$$

Moreover, since $\lambda(E) \le 2|B|$, it is clear that one needs to prove Theorem 1.1 only for sets with a small isoperimetric deficit. In fact, if $D(E) \ge \delta > 0$, (1.5) is trivially satisfied by taking a suitably large constant $C(n)$. Finally, a more or less standard truncation argument (see Lemma 5.1) shows that in order to prove Theorem 1.1 it is enough to assume that E is contained in a suitably large cube.

Let us now give a short description of how the proof goes. Apart from the isoperimetric property of the sphere, we do not use any sophisticated technical tool. On the contrary, the underlying idea is to reduce the problem, by means of suitable geometric constructions, to the case of more and more symmetric sets.

To be more precise, let us introduce the following definition, which will play an important role in the sequel. We say that a Borel set $E \subseteq \mathbb{R}^n$ is *n-symmetric* if E is symmetric with respect to n orthogonal hyperplanes $H_1, \ldots, H_n$. A simple, but important property of n-symmetric sets is that the Fraenkel asymmetry $\lambda(E)$ is equivalent to the distance from E to the ball centered at the intersection x_0 of the n hyperplanes H_i. In fact, we have (see Lemma 2.2)

$$\lambda(E) \le d(E, x_0 + B) \le 2^n\lambda(E)\,. \tag{1.6}$$

Coming back to the proof, the first step is to pass from a general set E to a set E' symmetric with respect to a hyperplane, without loosing too much in terms of isoperimetric deficit and asymmetry, namely

$$\lambda(E) \le C(n)\lambda(E') \qquad\qquad D(E') \le C(n)D(E)\,. \tag{1.7}$$

A natural way to do this, could be to take any hyperplane dividing E in two parts of equal measure and then to reflect one of them. In fact, calling E^+ and E^- the two resulting sets (see Figure 1.a), it is easily checked that

$$D(E^+) + D(E^-) \le 2D(E)\,,$$

but, unfortunately, it is not true in general that

$$\lambda(E) \le C(n) \max\{\lambda(E^+), \lambda(E^-)\}\,.$$

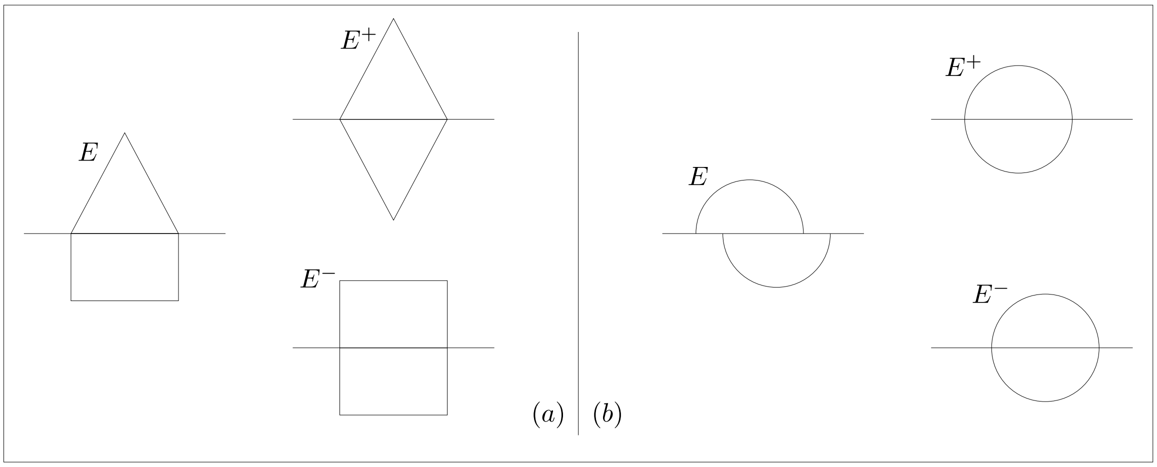

FIGURE 1. The sets E, E^+ and E^-

This is clear if we take, for instance, E equal to the union of two slightly shifted half-balls, as in Figure 1.b. However, if we take, instead, *two* orthogonal hyperplanes, each one dividing E in two parts of equal volume, at least one of the four sets thus obtained by reflection will satisfy (1.7) for a suitable constant $C(n)$ (see Lemma 2.5). Thus, iterating this procedure, we obtain a set with $(n-1)$ symmetries and eventually, using a variant of this argument to get the last symmetry, an n-symmetric set E' satisfying (1.7).

Once we have reduced the proof of Theorem 1.1 to the case of an n-symmetric set E, equivalently to a set symmetric with respect to all coordinate hyperplanes, all we have to do, thanks to (1.6), is to estimate $d(E, B)$ by $\sqrt{D(E)}$ (as $x_0 = 0$).

To this aim we compare E with its Steiner symmetral E^* with respect to one of the coordinate axes, say x_1. Simplifying a bit, the idea is to estimate each one of the two terms appearing on the right-hand side of the triangular inequality

$$d(E, B) \le d(E, E^*) + d(E^*, B) \tag{1.8}$$

by the square root of the isoperimetric deficit. Concerning the first term, by Fubini's theorem we can write

$$d(E, E^*) = \int_{\mathbb{R}} \mathscr{H}^{n-1}(E_t \Delta E_t^*)\, dt\,, \tag{1.9}$$

where, for any set F, F_t stands for $\{x \in F : x_1 = t\}$. Since E_t^* is the $(n-1)$-dimensional ball with the same measure of E_t, centered on the axis x_1, and since E_t is symmetric with respect to the remaining

$(n-1)$ coordinate hyperplanes, by applying (1.6) –in one dimension less– to the set E_t suitably rescaled we get

$$\mathscr{H}^{n-1}(E_t \Delta E_t^*) \leq \frac{2^{n-1}}{\omega_{n-1}} \mathscr{H}^{n-1}(E_t) \lambda_{n-1}(E_t) \leq C \lambda_{n-1}(E_t) , \tag{1.10}$$

where $\lambda_{n-1}(E_t)$ denotes the Fraenkel asymmetry of E_t in $\mathbb{R}^{n-1}$. Then, assuming that Theorem 1.1 holds in dimension $n-1$, we can estimate $\lambda_{n-1}(E_t)$ by the deficit $D_{n-1}(E_t)$ of E_t in $\mathbb{R}^{n-1}$, thus getting from (1.9) and (1.10)

$$d(E, E^*) \leq C \int_{\mathbb{R}} \lambda_{n-1}(E_t)\, dt \leq C \int_{\mathbb{R}} \sqrt{D_{n-1}(E_t)}\, dt .$$

Finally, by a suitable choice of the symmetrization axis x_i, we are able to prove that

$$\int_{\mathbb{R}} \sqrt{D_{n-1}(E_t)}\, dt \leq C \sqrt{D(E)} ,$$

thus concluding from (1.6) and (1.8) that

$$\lambda(E) \leq d(E, B) \leq C\sqrt{D(E)} + d(E^*, B) \leq C\sqrt{D(E)} + 2^n \lambda(E^*) .$$

At this point, in order to control $\lambda(E^*)$ by $\sqrt{D(E^*)}$, which in turn is smaller than $\sqrt{D(E)}$, we could have relied on Hall's inequality (1.3). However, we have preferred to do otherwise. In fact in our case, since we may assume that E is n-symmetric (and thus E^* is n-symmetric too) we can give a simpler, self-contained proof, ultimately reducing the required estimate to the case of two overlapping balls with the same radii (see the proof of Theorem 4.1). And this particular case can be handled by elementary one-dimensional calculations.

The methods developed in this paper, besides giving a positive answer to the question posed by Hall, can also be used to obtain an optimal quantitative version of the Sobolev inequality. This application is contained in the forthcoming paper [14] by the same authors.

2. Reduction to an n-symmetric set

In this section, we aim to reduce ourselves to the case of a set with a wide symmetry, namely n-symmetric. Since, as will shall see in Section 5, we may always reduce the proof of Theorem 1.1 to the case where E is contained in a suitably large cube $Q_l = (-l, l)^n$ and $|E| = |B|$, we shall work here and in the next two sections with uniformly bounded sets contained in

$$X := \{E \subseteq \mathbb{R}^n :\, E \text{ is Borel}, |E| = |B|\} .$$

And thus we shall use the convention that $C = C(n, l)$ denotes a sufficiently large constant, that may change from line to line, and that depends uniquely on the dimension n and on l.

The whole section is devoted to show the following result.

Theorem 2.1. *For every $E \in X$, $E \subseteq Q_l$, there exists a set $F \in X$, $F \subseteq Q_{3l}$, symmetric with respect to n orthogonal hyperplanes and such that,*

$$\lambda(E) \leq C(n, l)\, \lambda(F) , \qquad\qquad D(F) \leq 2^n D(E) .$$

This section is divided in two subsections: in the first one, we collect some technical properties that we need later, and in the second one we prove Theorem 2.1.

2.1. Some technical facts.

In this subsection we collect some technical facts that will be used throughout the paper. Even though the ball centered in the center of symmetry is in general not optimal for an n-symmetric set, next lemma states that this is true apart from a constant factor. In the sequel, for any two sets E, $F \subseteq \mathbb{R}^n$ we shall denote by $\lambda(E|F)$ the *Fraenkel asymmetry relative to* F, that is

$$\lambda(E|F) := \min \left\{ \frac{d(E, x + rB)}{r^n} : x \in F \right\} ,$$

again being $|E| = r^n |B|$.

Lemma 2.2. *Let $E \in X$ be a set symmetric with respect to k orthogonal hyperplanes $H_j = \{x \in \mathbb{R}^n : x \cdot \nu_j = 0\}$ for $1 \leq j \leq k$. Then one has*

$$\lambda\Big(E \big| \bigcap\nolimits_{j=1}^k H_j\Big) \leq 2^k \lambda(E).$$

Proof. Let us set

$$Q^- := \big\{x \in \mathbb{R}^n : \; x \cdot \nu_j \leq 0 \quad \forall\, 1 \leq j \leq k\big\}, \qquad Q^+ := \big\{x \in \mathbb{R}^n : \; x \cdot \nu_j \geq 0 \quad \forall\, 1 \leq j \leq k\big\}.$$

By definition and by symmetry, $\lambda(E) = d(E, p + B)$ for some point $p \in \mathbb{R}^n$ belonging to Q^-; as an immediate consequence, denoting by p_0 the orthogonal projection of p on $\bigcap_{j=1}^k H_j$, one has that $(p_0 + B) \cap Q^+ \supseteq (p + B) \cap Q^+$. Hence,

$$\big(E \setminus (p_0 + B)\big) \cap Q^+ \subseteq \big(E \setminus (p + B)\big) \cap Q^+.$$

The conclusion follows just noticing that, since both $p_0 + B$ and E are symmetric with respect to the hyperplanes H_j,

$$\begin{aligned}
\lambda\Big(E \big| \bigcap\nolimits_{j=1}^k H_j\Big) &\leq d\big(E, (p_0 + B)\big) = 2|E \setminus (p_0 + B)| = 2 \cdot 2^k \big|\big(E \setminus (p_0 + B)\big) \cap Q^+\big| \\
&\leq 2 \cdot 2^k \big|\big(E \setminus (p + B)\big) \cap Q^+\big| \leq 2 \cdot 2^k |E \setminus (p + B)| = 2^k d(E, p + B) \\
&= 2^k \lambda(E)\,.
\end{aligned}$$

□

The second result we present shows the stability of the isoperimetric inequality, but without any estimate about the rate of convergence; keep in mind that the goal of this paper is exactly to give a precise and sharp quantitative estimate about this convergence. This weak result is easy and very well known, we present a proof only for the reader's convenience, and to keep this paper self-contained. The proof is based on a simple compactness argument.

Lemma 2.3. *Let $l > 0$. For any $\varepsilon > 0$ there exists $\delta = \delta(n, l, \varepsilon) > 0$ such that if $E \in X$, $E \subseteq Q_l$, and $D(E) \leq \delta$ then $\lambda(E) \leq \varepsilon$.*

Proof. We argue by contradiction. If the assertion were not true, there would be a sequence $\{E_j\} \subseteq X$ with $E_j \subseteq Q_l$, $D(E_j) \to 0$ and $\lambda(E_j) \geq \varepsilon > 0$ for all $j \in \mathbb{N}$. Since each set E_j is contained in the same cube Q_l, thanks to a well-known embedding theorem (see for instance Theorem 3.39 in [1]) we can assume, up to a subsequence, that $\chi_{E_j} \xrightarrow{L^1} \chi_{E_\infty}$ for some set E_∞ of finite perimeter; we deduce that E_∞ is a set with $|E_\infty| = |B|$, and by the lower semicontinuity of the perimeters $P(E_\infty) \leq P(B)$, so E_∞ is a ball. The fact that χ_{E_j} strongly converges in L^1 to χ_{E_∞} immediately implies that $|E_j \Delta E_\infty| \to 0$, against the assumption $\lambda(E_j) \geq \varepsilon$. The contradiction concludes the proof. □

The last result is an estimate about the distance of two sets obtained via translations of half-balls; the proof that we present was suggested by Sergio Conti.

Lemma 2.4. *Let H_1 and H_2 be two orthogonal hyperplanes and let $H_i^{\pm}$ be the corresponding two pairs of half-spaces. Consider two points $x_1, \sigma_1 \in H_1$, $x_2, \sigma_2 \in H_2$ and the sets*

$$B_i := x_i + B\,, \qquad B_i^{\pm} := B_i \cap H_i^{\pm}\,, \qquad D_i := B_i^+ \cup (B_i^- + \sigma_i)\,.$$

There are two constants $\varepsilon = \varepsilon(n)$ and $C = C(n)$ such that, provided $|x_1 - x_2| \leq \varepsilon$ and $|\sigma_1|, |\sigma_2| \leq \varepsilon$ then

$$\max\{|\sigma_1|, |\sigma_2|\} \leq C\, d(D_1, D_2)\,.$$

Proof. For suitable constants $\delta(n)$ and $C(n)$, given two unitary balls F_1 and F_2 with the centers lying at a distance $\delta \leq \delta(n)$, we have

$$\delta \leq C(n)\, d(F_1, F_2)\,.$$

In particular, up to changing $C(n)$, if Q denotes any intersection of two orthogonal half-spaces of $\mathbb{R}^n$, with the property that

$$\min\big\{|F_1\cap Q|,|F_2\cap Q|\big\}\geq\frac{|B|}{8}\,,\tag{2.1}$$

then we have

$$\delta\leq C(n)\,d(F_1\cap Q,F_2\cap Q)\,.$$

We now apply this statement twice to prove the lemma. In the first instance we choose $F_1=B_1$, $F_2=B_2$ and $Q=H_1^+\cap H_2^+$. Note that, provided $\varepsilon(n)$ is small enough with respect to $\delta(n)$, by construction, condition (2.1) is satisfied. Thus

$$d(D_1,D_2)\geq d(D_1\cap Q,D_2\cap Q)=d(B_1\cap Q,B_2\cap Q)\geq C^{-1}|x_1-x_2|\,.$$

In the second instance we choose $F_1=\sigma_1+B_1$, $F_2=B_2$ and $Q=H_1^-\cap H_2^+$, and find similarly that

$$d(D_1,D_2)\geq d(D_1\cap Q,D_2\cap Q)=d\big((\sigma_1+B_1)\cap Q,B_2\cap Q\big)\geq C^{-1}|x_1+\sigma_1-x_2|\,.$$

Thus $|\sigma_1|\leq 2C\,d(D_1,D_2)$, and by symmetry we have the analogous estimate on σ_2. $\square$

2.2. **The proof of Theorem 2.1.** We show first a technique to perform a single symmetrization, and the claim will then be proved by successive applications of this main step. We need also a bit of notation: given a set $E\in X$ and a unit vector $\nu\in\mathbb{S}^{n-1}$, we denote by $H_\nu^+=\{x\in\mathbb{R}^n: x\cdot\nu>t\}$ an open half-space orthogonal to ν where $t\in\mathbb{R}$ is chosen in such a way that

$$|E\cap H_\nu^+|=\frac{|E|}{2}\,;$$

we also denote by $r_\nu:\mathbb{R}^n\to\mathbb{R}^n$ the reflection with respect to $H_\nu=\partial H_\nu^+$, and by $H_\nu^-=r_\nu(H_\nu^+)$ the open half-space complementary to H_ν^+. Finally, we write $E_\nu^\pm=E\cap H_\nu^\pm$.

Lemma 2.5. *There exist two constants C and δ, depending only on n and l such that, given $E\in X$, $E\subseteq Q_l$, and two orthogonal vectors ν_1 and ν_2, there are $i\in\{1,2\}$ and $s\in\{+,-\}$ such that, setting $E'=E^s_{\nu_i}\cup r_{\nu_i}(E^s_{\nu_i})$, one has*

$$\lambda(E)\leq C\lambda(E')\,,\qquad\qquad D(E')\leq 2D(E)\,,\tag{2.2}$$

provided that $D(E)\leq\delta$.

Proof. First of all, given any unit vector ν, let us denote by B_ν^+ a half-ball with center on H_ν that best approximates E_ν^+, i.e. we set $B_\nu^+=(p+B)\cap H_\nu^+$ for some p realizing

$$\min\big\{d(E_\nu^+,(p+B)\cap H_\nu^+):\ p\in H_\nu\big\}\,;$$

analogously, we let B_ν^- be a half-ball with center in H_ν which best approximates E_ν^-.
We consider now the sets

$$F_\nu^{1,2}:=E_\nu^\pm\cup r_\nu(E_\nu^\pm)\,,\qquad\qquad T_\nu:=B_\nu^+\cup B_\nu^-\,,$$

(the first equation must be understood as $F_\nu^1=E_\nu^+\cup r_\nu(E_\nu^+)$ and $F_\nu^2=E_\nu^-\cup r_\nu(E_\nu^-)$). Note that T_ν is the union of two half-balls with (possibly different) centers on H_ν, and that $F_\nu^{1,2}$, with $\nu=\nu_1$ or $\nu=\nu_2$, are the four sets among which we need to select E'. Notice also that by a compactness argument similar to the one used in proving Lemma 2.3 it is clear that for any $\varepsilon>0$ the centers of the four half-balls $B^\pm_{\nu_i}$ are at distance less than ε, provided that $D(E)$ is smaller than a suitable δ.
The following two remarks will be useful. First we note that clearly, by construction of $B_\nu^\pm$ and by symmetry of $F_\nu^{1,2}$, we have

$$\lambda(F_\nu^{1,2}|H_\nu)=d\big(F_\nu^{1,2},B_\nu^\pm\cup r_\nu(B_\nu^\pm)\big)=2d(E_\nu^\pm,B_\nu^\pm)\,.\tag{2.3}$$

It can be easily checked that $P(F_\nu^i|H_\nu)=0$, being $P(F_\nu^i|H_\nu)$ the perimeter of F_ν^i relative to H_ν (see the Appendix), hence

$$P(E)\geq P(E|H_\nu^+)+P(E|H_\nu^-)=\frac{P(F_\nu^1)+P(F_\nu^2)}{2}\,,$$

so that we have always

$$\max\left\{D(F^1_\nu), D(F^2_\nu)\right\} \leq 2\, D(E)\,. \tag{2.4}$$

As shown by (2.4), all the four sets among which we have to choose E' satisfy the estimate on the right in (2.2), so we need only to take care of the one on the left.

Assume now for the moment that, for some constant $K = K(n)$ to be determined later and for some unit vector $\nu \in S^{n-1}$,

$$d\big(B^-_\nu, r_\nu(B^+_\nu)\big) \leq K\left(d(E^+_\nu, B^+_\nu) + d(E^-_\nu, B^-_\nu)\right). \tag{2.5}$$

Then we can easily estimate, also recalling (2.3),

$$\begin{aligned}
\lambda(E) &\leq d\big(E, B^+_\nu \cup r_\nu(B^+_\nu)\big) = d(E^+_\nu, B^+_\nu) + d\big(E^-_\nu, r_\nu(B^+_\nu)\big)\\
&\leq d(E^+_\nu, B^+_\nu) + d(E^-_\nu, B^-_\nu) + d\big(B^-_\nu, r_\nu(B^+_\nu)\big) \leq (K+1)\big(d(E^+_\nu, B^+_\nu) + d(E^-_\nu, B^-_\nu)\big)\\
&= \frac{K+1}{2}\Big(\lambda(F^1_\nu|H_\nu) + \lambda(F^2_\nu|H_\nu)\Big)\,.
\end{aligned}$$

Therefore, up to swap F^1_ν and F^2_ν, we have that

$$\lambda(F^1_\nu|H_\nu) \geq \frac{1}{K+1}\,\lambda(E)\,.$$

Since F^1_ν is symmetric with respect to the hyperplane H_ν, by Lemma 2.2 we conclude

$$\lambda(F^1_\nu) \geq \frac{1}{2(K+1)}\,\lambda(E)\,.$$

Keeping in mind (2.4), the proof of this lemma will be concluded once we show that (2.5) holds either with $\nu = \nu_1$ or with $\nu = \nu_2$.

Suppose this is not true: then, define σ_1 the vector connecting the centers of $B^+_{\nu_1}$ and $B^-_{\nu_1}$. Since $|\sigma_1| < \varepsilon$

$$d\big(B^-_{\nu_1}, r_{\nu_1}(B^+_{\nu_1})\big) \leq C|\sigma_1|\,.$$

Therefore, the assumption that (2.5) does not hold with $\nu = \nu_1$ implies that

$$d(E, T_{\nu_1}) = d(E^+_{\nu_1}, B^+_{\nu_1}) + d(E^-_{\nu_1}, B^-_{\nu_1}) \leq \frac{1}{K}\, d\big(B^-_{\nu_1}, r_{\nu_1}(B^+_\nu)\big) \leq \frac{2\omega_{n-1}}{K}\,|\sigma_1|\,.$$

Analogously, assuming that (2.5) does not hold with $\nu = \nu_2$ yields

$$d(E, T_{\nu_2}) \leq \frac{2\omega_{n-1}}{K}\,|\sigma_2|\,.$$

We deduce, by the triangular inequality, that

$$d(T_{\nu_1}, T_{\nu_2}) \leq \frac{2\omega_{n-1}}{K}\left(|\sigma_1| + |\sigma_2|\right).$$

By Lemma 2.4, this leads to a contradiction provided the constant K is chosen sufficiently large; and, as already noticed, this contradiction completes the proof. □

We can now show the main result of this section.

Proof. (of Theorem 2.1). Let us assume for the moment that $D(E) < \delta$, where δ is the constant appearing in Lemma 2.5. Let us take the standard orthonormal basis $\{e_i\}_{i=1}^n$: we will prove the existence of a set F of volume $|F| = |E|$, symmetric with respect to n hyperplanes $H_1, H_2, \ldots, H_n$ such that each hyperplane H_i is orthogonal to e_i, and with the property that

$$\lambda(E) \leq C\lambda(F)\,, \qquad\qquad D(F) \leq 2^n D(E)\,.$$

We start by the versors e_1 and e_2: thanks to Lemma 2.5, up to a permutation we can find a hyperplane H_1 orthogonal to e_1 and a set F_1 with $|F_1| = |E|$, symmetric with respect to H_1 and with the property that

$$\lambda(E) \leq C\lambda(F_1)\,, \qquad\qquad D(F_1) \leq 2D(E)\,.$$

Consider now the versors e_2 and e_3, and apply Lemma 2.5 to the set F_1: up to a permutation of e_2 and e_3, we find a hyperplane H_2 orthogonal to e_2 and a set F_2 symmetric with respect to H_2 with the property that

$$\lambda(E) \leq C\lambda(F_1) \leq C^2\lambda(F_2)\,, \qquad\qquad D(F_2) \leq 2D(F_1) \leq 4D(E)\,.$$

Moreover, by Lemma 2.5 and by the fact that the hyperplanes H_1 and H_2 are orthogonal, the set F_2 is symmetric also with respect to H_1. By an immediate iteration, we arrive to a set F_{n-1}, symmetric with respect to $n-1$ orthogonal hyperplanes $H_1,\ H_2, \dots\ , H_{n-1}$ and such that

$$\lambda(E) \leq C^{n-1}\lambda(F_{n-1})\,, \qquad\qquad D(F_{n-1}) \leq 2^{n-1}\, D(E)\,. \tag{2.6}$$

To find the last hyperplane of symmetry we need a different argument, since we do not have anymore two different hyperplanes among which to choose. We let then H_n be a hyperplane orthogonal to e_n such that $|F_{n-1}\cap H_n^+| = |F_{n-1}\cap H_n^-|$, being $H_n^{\pm}$ the two half-spaces corresponding to H_n, and we define $q = \cap_{i=1}^n H_i$; q is clearly a point since the H_i's are n orthogonal hyperplanes. Defining now

$$F_n^1 := (F_{n-1}\cap H_n^+) \cup r_\nu(F_{n-1}\cap H_n^+)\,, \qquad\qquad F_n^2 := (F_{n-1}\cap H_n^-) \cup r_\nu(F_{n-1}\cap H_n^-)\,,$$

with $\nu = e_n$, we first notice that, with the same argument used to obtain (2.4), it is

$$\max\big\{D(F_n^1), D(F_n^2)\big\} \leq 2\, D(F_{n-1})\,. \tag{2.7}$$

Moreover, by definition we have

$$d(F_{n-1}, q+B) = \frac{d(F_n^1, q+B) + d(F_n^2, q+B)}{2}\,;$$

applying then Lemma 2.2 to F_n^1 and F_n^2 –which are symmetric with respect to the n orthogonal hyperplanes H_i– we deduce

$$\begin{aligned}
\lambda(F_{n-1}) \leq d(F_{n-1}, q+B) &= \frac{d(F_n^1, q+B) + d(F_n^2, q+B)}{2}\\
&= \frac{\lambda(F_n^1|\cap_{i=1}^n H_i) + \lambda(F_n^2|\cap_{i=1}^n H_i)}{2} \leq 2^{n-1}\Big(\lambda(F_n^1) + \lambda(F_n^2)\Big)\\
&\leq 2^n \max\big\{\lambda(F_n^1), \lambda(F_n^2)\big\}\,.
\end{aligned} \tag{2.8}$$

Putting now together (2.6), (2.7) and (2.8), we may find $F \in \{F_n^1, F_n^2\}$ such that

$$\lambda(E) \leq C^{n-1}\lambda(F_{n-1}) \leq 2^n C^{n-1}\lambda(F) \qquad\qquad D(F) \leq 2D(F_{n-1}) \leq 2^n\, D(E)\,,$$

we conclude the proof, since the inclusion $F \subseteq Q_{3l}$ is obvious by construction.
Finally, when $D(E) \geq \delta$ it is enough to take as F, independently from E, any n-symmetric set such that $0 < D(F) \leq 2^n\delta$. □

Remark 2.6. A quick inspection to the proofs of Lemma 2.5 and Theorem 2.1 shows that if we assume that the set E satisfies the following condition

$$\mathscr{H}^{n-1}\big(\big\{x \in \partial^* E :\, \nu^E(x) = \pm e_i\big\}\big) = 0 \quad \forall\, i = 1, \dots, n\,, \tag{2.9}$$

then the same property is inherited by the set F constructed above (here $\partial^* E$ denotes the reduced boundary of E, see the Appendix). To see this, it is enough to check that if we take a set E satisfying condition (2.9), split it in two parts by a hyperplane H orthogonal with respect to one of the e_i's and reflect with respect to H (these are the only operations we performed in the previous proofs), then we obtain another set E' still satisfying (2.9).

3. Reduction to axially symmetric sets

In this section we show how to reduce the n-symmetric case to the axially symmetric one. The goal is to show Theorem 3.1 below, which will be the starting point for the induction argument over the dimension n used in Section 5 to prove Theorem 1.1.

In the following, given any set E of finite perimeter, for any $t \in \mathbb{R}$, we denote by E_t the $(n-1)$-dimensional section $\{x \in E : x_1 = t\}$ and by E^* the Steiner symmetrization of E with respect to the axis e_1, that is, the set $E^* \subseteq \mathbb{R}^n$ such that for any $t \in \mathbb{R}$ the section E^*_t is the $(n-1)$-dimensional ball centered at $(t, 0, 0, \ldots, 0)$ with $\mathscr{H}^{n-1}(E^*_t) = \mathscr{H}^{n-1}(E_t)$. We also set

$$v_E(t) := \mathscr{H}^{n-1}(E_t)\,,$$

the $(n-1)$-dimensional measure of the section E_t, and denote by $p_E(t)$ the perimeter of E_t in $\mathbb{R}^{n-1}$, i.e., for $\mathscr{H}^1$-a.e. t,

$$p_E(t) := \mathscr{H}^{n-2}(\partial^* E_t)\,.$$

Theorem 3.1. *Let $E \in X$, $E \subseteq Q_l$ be symmetric with respect to the n coordinate hyperplanes and assume that (2.9) holds. If $n = 2$, or $n \geq 3$ and Theorem 1.1 holds in dimension $n-1$, then, up to a rotation of the coordinate axes,*

$$d(E, B) \leq 4d(E^* \cap Z, B \cap Z) + C(n, l)\sqrt{D(E)}\,,$$

where $Z = \{x \in \mathbb{R}^n : |x_1| \leq \sqrt{2}/2\}$.

Our first step will be to select one of the hyperplanes which will have a particular role in the following construction; we need to ensure that the symmetric difference between E and B is not, roughly speaking, too concentrate close to the "poles" of B (i.e., the regions of B having greatest distance from the hyperplane).

Lemma 3.2. *Up to a rotation we can assume*

$$d(E, B) \leq 4d(E \cap Z, B \cap Z)\,.$$

Proof. Define the sets

$$Z_1 := \left\{x \in \mathbb{R}^n : |x_1| \leq \sqrt{2}/2\right\}, \qquad Z_2 := \left\{x \in \mathbb{R}^n : |x_2| \leq \sqrt{2}/2\right\},$$

so that $B \subseteq Z_1 \cup Z_2$; as a consequence,

$$B \setminus E = \big((B \setminus E) \cap Z_1\big) \cup \big((B \setminus E) \cap Z_2\big)\,.$$

Therefore, up to changing the axis e_1 with the axis e_2, we may assume that

$$|(B \setminus E) \cap Z_1| \geq \frac{|B \setminus E|}{2}\,,$$

and finally conclude

$$d(E, B) = 2|B \setminus E| \leq 4|(B \setminus E) \cap Z_1| \leq 4d(E \cap Z_1, B \cap Z_1)\,.$$

□

It is well known that the Steiner symmetrization lowers the perimeter (see e.g. [17]), that is

$$P(E^*) \leq P(E)\,.$$

In turn, the above inequality can be deduced also from the following estimate for the perimeters. This is immediate if E is bounded and condition (3.1) holds, and follows by a simple approximation argument in the general case.

Lemma 3.3. *Let $E \in X$ be a set of finite perimeter such that*

$$\mathscr{H}^{n-1}(\{x \in \partial^* E : \nu^E(x) = \pm e_1\}) = 0 . \tag{3.1}$$

Then $v_E \in W^{1,1}(\mathbb{R})$ and

$$P(E) \geq \int_{-\infty}^{+\infty} \sqrt{p_E(t)^2 + v_E'(t)^2}\, dt . \tag{3.2}$$

Moreover, for an axially symmetric set E the preceding inequality is in fact an equality, and it can be written as

$$P(E) = \int_{-\infty}^{+\infty} \sqrt{\tau\, v_E(t)^{\frac{2n-4}{n-1}} + v_E'(t)^2}\, dt , \tag{3.3}$$

where $\tau = (n-1)^2 \omega_{n-1}^{\frac{2}{n-1}}$.

Proof. Since E satisfies (3.1), from the coarea formula (6.1) we have that

$$P(E) = \mathscr{H}^{n-1}(\partial^* E) = \int_{\partial^* E} \frac{\sqrt{1-|\nu_1^E|^2}}{\sqrt{1-|\nu_1^E|^2}}\, d\mathscr{H}^{n-1} = \int_{-\infty}^{+\infty} \int_{(\partial^* E)_t} \frac{1}{\sqrt{1-|\nu_1^E|^2}}\, d\mathscr{H}^{n-2}\, dt .$$

Let us now recall that, by Theorem 6.2, E_t is a set of finite perimeter for $\mathscr{H}^1$-a.e. t and $\partial^* E_t$ equals $(\partial^* E)_t$ up to a $\mathscr{H}^{n-2}$-negligible set. Thus, from the equality above we get, using also Jensen's inequality,

$$\begin{aligned} P(E) &= \int_{-\infty}^{+\infty} \int_{\partial^* E_t} \frac{1}{\sqrt{1-|\nu_1^E|^2}}\, d\mathscr{H}^{n-2}\, dt = \int_{-\infty}^{+\infty} p_E(t) \fint_{\partial^* E_t} \sqrt{1 + \frac{|\nu_1^E|^2}{1-|\nu_1^E|^2}}\, d\mathscr{H}^{n-2}\, dt \\ &\geq \int_{-\infty}^{+\infty} p_E(t) \sqrt{1 + \left(\fint_{\partial^* E_t} \frac{|\nu_1^E|}{\sqrt{1-|\nu_1^E|^2}} d\mathscr{H}^{n-2} \right)^2}\, dt . \end{aligned}$$

Then, (3.2) immediately follows from the expression of v_E' given in (6.4). Finally, if E is axially symmetric, then ν_1^E is clearly constant on each boundary $\partial^* E_t$. Hence the inequality above is indeed an equality and (3.3) follows. $\square$

The second main ingredient that we give now is an L^∞ estimate for $v_E - v_B$.

Lemma 3.4. *For any $\rho > 0$ there exists $\delta > 0$ such that, if E is as in Theorem 3.1 and $D(E) \leq \delta$, then*

$$\| v_E - v_B \|_{L^\infty} \leq \rho .$$

Proof. Let us fix $\rho > 0$. By Lemma 2.2, since $v_E = v_{E^*}$,

$$\| v_E - v_B \|_{L^1} = d(E^*, B) \leq 2^n \lambda(E^*) . \tag{3.4}$$

Therefore, given $\varepsilon > 0$ (to be chosen later), by Lemma 2.3, if $D(E^*) \leq D(E) \leq \delta$, for δ small, then $\lambda(E^*) \leq \rho\varepsilon / 2^{n+2}$. Thus, from (3.4) we get

$$\mathscr{H}^1\big(\{t : |v_E - v_B| > \rho/4\}\big) < \varepsilon . \tag{3.5}$$

Recall that by assumption (2.9) and by Theorem 6.1 v_E is continuous. Thus, if $\|v_E - v_B\|_{L^\infty} > \rho$, there exists $\bar t$ such that

$$|v_E(\bar t) - v_B(\bar t)| > \rho .$$

By the uniform continuity of v_B, provided ε is sufficiently small, for $|t - \bar t| < \varepsilon$ one has $|v_B(t) - v_B(\bar t)| < \rho/4$. Then, by (3.5), and the continuity of v_E, there exist

$$\bar t - \varepsilon < t^- < \bar t < t^+ < \bar t + \varepsilon ,$$

such that

$$v_E(t^-) = v_E(t^+) , \qquad |v_E(t^\pm) - v_B(\bar t)| = \frac{\rho}{2} .$$

Let us now define an axially symmetric set F by letting $v_F(t) = v_E(t)$ if $t \notin (t^-, t^+)$ and $v_F(t) = v_E(t^-) = v_E(t^+)$ otherwise. Clearly

$$|F| \geq |E^*| - 2^n l^{n-1} \varepsilon , \tag{3.6}$$

and

$$P(F) = P(E^*) + P\big(F|\{x : t^- < x_1 < t^+\}\big) - P\big(E^*|\{x : t^- < x_1 < t^+\}\big)\,. \tag{3.7}$$

Moreover, by (3.3)

$$\begin{aligned} &P\big(F|\{x : t^- < x_1 < t^+\}\big) = (t^+ - t^-)\sqrt{\tau}v_E(t^+)^{\frac{n-2}{n-1}} \le C(n,l)\varepsilon\,,\\ &P\big(E^*|\{x : t^- < x_1 < t^+\}\big) \ge \int_{t^-}^{t^+} |v_E'(t)|\,dt \ge \rho\,. \end{aligned} \tag{3.8}$$

From (3.7) and (3.8) we get

$$P(F) \le P(E^*) + C\varepsilon - \rho \le P(E) + C\varepsilon - \rho\,.$$

On the other hand, by the isoperimetric inequality and (3.6), choosing ε small and recalling that $|E^*| = |B|$,

$$P(F) \ge n\omega_n^{1/n}\big(|E^*| - 2^n l^{n-1}\varepsilon\big)^{\frac{n-1}{n}} \ge P(B) - C(n,l)\varepsilon\,.$$

Therefore,

$$D(E) \ge \frac{\rho - C\varepsilon}{P(B)}\,.$$

Hence, the assertion follows by choosing $\varepsilon < \rho/2C$ and $\delta = \rho/2P(B)$. $\square$

We can finally give the proof of Theorem 3.1.

Proof. (of Theorem 3.1). Since $d(E,B) \le 2|B|$, by choosing $C(n,l)$ sufficiently large, we may assume $D(E)$ as small as we wish.
Thanks to Lemma 3.2, we can estimate

$$\begin{aligned} d(E,B) \le 4\,d(E\cap Z, B\cap Z) &= 4\int_{-\sqrt{2}/2}^{\sqrt{2}/2} \mathscr{H}^{n-1}(E_t \Delta B_t)\,dt\\ &= 4\int_I \mathscr{H}^{n-1}(E_t \Delta B_t)\,dt + 4\int_J \mathscr{H}^{n-1}(E_t \Delta B_t)\,dt\,, \end{aligned} \tag{3.9}$$

where we divide the interval $\big[-\sqrt{2}/2, \sqrt{2}/2\big]$ in two subsets I and J defined as

$$I := \left\{ t \in \left[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right] : \ |v_E'(t)| < M \right\}, \qquad J := \left\{ t \in \left[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right] : \ |v_E'(t)| \ge M \right\}.$$

for a constant M to be determined later.
We will consider separately the situation in the sets I and J. Let us then start working on I. By the triangular inequality we have immediately

$$\int_I \mathscr{H}^{n-1}(E_t \Delta B_t)\,dt \le \int_I \mathscr{H}^{n-1}(E_t^* \Delta B_t)\,dt + \int_I \mathscr{H}^{n-1}(E_t \Delta E_t^*)\,dt\,. \tag{3.10}$$

Concerning $\mathscr{H}^{n-1}(E_t^*\Delta B_t)$ we have easily

$$\int_I \mathscr{H}^{n-1}(E_t^* \Delta B_t)\,dt \le \int_{-\sqrt{2}/2}^{\sqrt{2}/2} \mathscr{H}^{n-1}(E_t^* \Delta B_t)\,dt = d(E^*\cap Z, B\cap Z)\,. \tag{3.11}$$

On the other hand, since E_t^* is a $(n-1)$-dimensional ball of $(n-1)$-dimensional volume $v_E(t)$, its perimeter is

$$p_{E^*}(t) = \mathscr{H}^{n-2}(\partial E_t^*) = \sqrt{\tau v_E(t)^{\frac{2n-4}{n-1}}}\,.$$

Of course, $p_E(t) \ge p_{E^*}(t)$ for every t. Thus we can give the following definition, which will be very useful in the sequel,

$$d(t) := p_E(t)^2 - p_{E^*}(t)^2 = p_E(t)^2 - \tau v_E(t)^{\frac{2n-4}{n-1}} \ge 0\,.$$

We now claim the inequality

$$\mathscr{H}^{n-1}(E_t \Delta E_t^*) \leq C(n,l)\sqrt{p_E(t) - \sqrt{\tau v_E(t)^{\frac{2n-4}{n-1}}}}\,.$$

If $n \geq 3$, this can be proved recalling again Lemma 2.2 and the symmetries of E_t, and applying Theorem 1.1 in dimension $n-1$ to E_t (whose $\mathscr{H}^{n-1}$ measure is bounded by $2^{n-1}l^{n-1}$). On the other hand, for $n = 2$ the inequality is true since when $\mathscr{H}^1(E_t \Delta E_t^*) > 0$, we have $\mathscr{H}^1(E_t \Delta E_t^*) \leq 2l$ and $\sqrt{p_E(t) - p_{E^*}(t)} \geq 1$. Therefore,

$$\begin{aligned}
\int_I \mathscr{H}^{n-1}(E_t \Delta E_t^*) &\leq C \int_I \sqrt{p_E(t) - \sqrt{\tau v_E(t)^{\frac{2n-4}{n-1}}}} \\
&= C \int_I \sqrt{\sqrt{\tau v_E(t)^{\frac{2n-4}{n-1}} + d(t)} - \sqrt{\tau v_E(t)^{\frac{2n-4}{n-1}}}} \\
&= C \int_I \sqrt{\frac{d(t)}{\sqrt{\tau v_E(t)^{\frac{2n-4}{n-1}} + d(t)} + \sqrt{\tau v_E(t)^{\frac{2n-4}{n-1}}}}} \\
&\leq C \left(\int_I \frac{d(t)}{\sqrt{\tau v_E(t)^{\frac{2n-4}{n-1}} + d(t)} + \sqrt{\tau v_E(t)^{\frac{2n-4}{n-1}}}} \right)^{1/2},
\end{aligned} \tag{3.12}$$

where the last inequality comes from Hölder inequality recalling that the length of I is of course a priori bounded by $\sqrt{2}$.
Let us estimate now the difference $P(E) - P(E^*)$. By applying formula (3.3) to the axially symmetric set E^*, we have

$$\begin{aligned}
P(E) - P(E^*) &\geq \int_{-\infty}^{+\infty} \sqrt{p_E(t)^2 + v_E'(t)^2} - \sqrt{p_{E^*}(t)^2 + v_E'(t)^2} \\
&\geq \int_I \sqrt{\tau v_E(t)^{\frac{2n-4}{n-1}} + v_E'(t)^2 + d(t)} - \sqrt{\tau\, v_E(t)^{\frac{2n-4}{n-1}} + v_E'(t)^2} \\
&= \int_I \frac{d(t)}{\sqrt{\tau v_E(t)^{\frac{2n-4}{n-1}} + v_E'(t)^2 + d(t)} + \sqrt{\tau\, v_E(t)^{\frac{2n-4}{n-1}} + v_E'(t)^2}}\,.
\end{aligned} \tag{3.13}$$

We use now the following fact: there exists a constant C, depending only on n, l, M, such that for any $t \in I$ one has

$$\frac{\sqrt{\tau v_E(t)^{\frac{2n-4}{n-1}} + v_E'(t)^2 + d(t)} + \sqrt{\tau\, v_E(t)^{\frac{2n-4}{n-1}} + v_E'(t)^2}}{\sqrt{\tau v_E(t)^{\frac{2n-4}{n-1}} + d(t)} + \sqrt{\tau\, v_E(t)^{\frac{2n-4}{n-1}}}} \leq C(n,l,M)\,. \tag{3.14}$$

To show this estimate, we first remark that $v_E(t)$ is bounded from below by a strictly positive constant inside $[-\sqrt{2}/2, \sqrt{2}/2] \supseteq I$ (indeed, $v_B(t) \geq \omega_{n-1}/2^{(n-1)/2}$ inside $[-\sqrt{2}/2, \sqrt{2}/2]$, so we apply Lemma 3.4). Then, we recall that by definition $|v_E'| \leq M$ inside I, and this immediately concludes (3.14). Finally, putting together the estimates (3.12) and (3.13) and making use of (3.14) we obtain

$$\begin{aligned}
\int_I \mathscr{H}^{n-1}(E_t \Delta E_t^*) &\leq C(n,l,M) \left(\int_I \frac{d(t)}{\sqrt{\tau v_E(t)^{\frac{2n-4}{n-1}} + d(t)} + \sqrt{\tau v_E(t)^{\frac{2n-4}{n-1}}}} \right)^{1/2} \\
&\leq C \left(\int_I \frac{d(t)}{\sqrt{\tau v_E(t)^{\frac{2n-4}{n-1}} + v_E'(t)^2 + d(t)} + \sqrt{\tau\, v_E(t)^{\frac{2n-4}{n-1}} + v_E'(t)^2}} \right)^{1/2} \\
&\leq C\sqrt{P(E) - P(E^*)} \leq C(n,l,M)\sqrt{D(E)}\,,
\end{aligned} \tag{3.15}$$

where the last inequality (up to a redefinition of the constant C) is true since $P(E^*) \geq P(B)$. Putting now together (3.9), (3.10), (3.11) and (3.15), we finally obtain

$$d(E,B) \leq 4d(E^* \cap Z, B \cap Z) + C(n,l,M)\sqrt{D(E)} + 4\int_J \mathscr{H}^{n-1}(E_t \Delta B_t)\,dt\,. \tag{3.16}$$

Let us now consider the situation in the set J. Notice that the above argument requires an a priori upper bound on $|v_E'|$. On the other hand, it is clear that a region where $|v_E'|$ is very large is far from being optimal from the point of view of the perimeter. Therefore is not surprising that an even stronger estimate than (3.15) holds in J, namely

$$\int_J \mathscr{H}^{n-1}(E_t \Delta B_t)\,dt \leq CD(E^*)\,. \tag{3.17}$$

Notice that this inequality together with (3.16) concludes the proof of the theorem.
To prove (3.17) observe that, since $\mathscr{H}^{n-1}(E_t \Delta B_t) \leq \mathscr{H}^{n-1}(E_t) + \mathscr{H}^{n-1}(B_t) = v_E(t) + v_B(t)$ and $E \subseteq Q_l$, we have

$$\int_J \mathscr{H}^{n-1}(E_t \Delta B_t)\,dt \leq C\mathscr{H}^1(J).$$

Thus, (3.17) will be established once we prove that there exists a constant K depending only on n such that

$$\mathscr{H}^1(J) \leq \frac{K}{M} D(E^*)\,. \tag{3.18}$$

To this aim, let us first assume that v_E' is continuous and $J = (a, a+\varepsilon)$ is an interval.

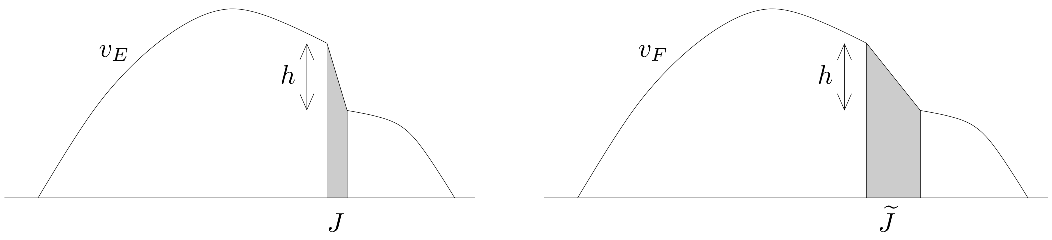

FIGURE 2. Construction of F

As in the proof of Lemma 3.4, we introduce a new set F, not necessarily of the same volume of E^* and B, and prove (3.18) by applying the isoperimetric inequality $P(F) \geq P(\widetilde{B})$, where $\widetilde{B}$ is the ball having the same volume of F. As in Figure 2, let h be the jump of v_E inside J, and let F be an axially symmetric set with respect to the axis e_1. Thus, in order to define F it is sufficient to define v_F. To this aim, we consider the interval $\widetilde{J} = (a, a + h/N)$, where N is an integer to be determined later depending only on the dimension n of the ambient space. We set

$$v_F(t) := \begin{cases} v_E(t) & \text{if } t \leq a; \\ v_E(a) \pm N(t-a) & \text{if } a \leq t \leq a + h/N; \\ v_E(t + \varepsilon - h/N) & \text{if } t \geq a + h/N. \end{cases}$$

In the above definition, the sign in the second row coincides with the sign of v_E' inside J: since $|v_E'| > M$ in J, by continuity this sign is well-defined. For instance, in the situation of the figure this sign is negative. Notice that, by definition, v_F is a continuous function. Let us start by recalling the information we have so-far:

$$\mathscr{H}^1(J) = \varepsilon\,, \qquad h \geq M\varepsilon\,, \qquad \mathscr{H}^1(\widetilde{J}) = \frac{h}{N}\,. \tag{3.19}$$

In order to evaluate both $|F|$ and $P(F)$, we set $J_* = \{x \in \mathbb{R}^n : x_1 \in J\}$ and $\widetilde{J}_* = \{x \in \mathbb{R}^n : x_1 \in \widetilde{J}\}$. Notice that $\omega_{n-1}/2^{(n-1)/2} \leq v_B(t) \leq \omega_{n-1}$ for $-\sqrt{2}/2 \leq t \leq \sqrt{2}/2$, thus by Lemma 3.4 and by construction we have

$$\frac{\omega_{n-1}}{2^{n/2}} \leq v_E(t) \leq 2\omega_{n-1}\,, \qquad\qquad \frac{\omega_{n-1}}{2^{n/2}} \leq v_F(t) \leq 2\omega_{n-1}\,.$$

This immediately ensures

$$|E^* \cap J_*| \leq 2\omega_{n-1}\mathscr{H}^1(J) = 2\omega_{n-1}\varepsilon\,, \qquad\qquad |F \cap \widetilde{J}_*| \geq \frac{\omega_{n-1}}{2^{n/2}}\mathscr{H}^1(\widetilde{J}) = \frac{h\omega_{n-1}}{2^{n/2}N}\,,$$

from which we deduce

$$|F| = |E^*| + |F \cap \widetilde{J}_*| - |E^* \cap J_*| \geq |E^*| + \frac{h\omega_{n-1}}{2^{n/2}N} - 2\omega_{n-1}\varepsilon \geq |E^*| + \frac{h\omega_{n-1}}{2 \cdot 2^{n/2}N}\,. \tag{3.20}$$

The last inequality holds provided $h \geq 4 \cdot 2^{n/2}N\varepsilon$, and this can be achieved with a suitable choice of M in view of the second estimate in (3.19) –recall that the constant M can be chosen as big as we need, while the integer N (still to be determined) depends only on the dimension of the space.
We turn now to the evaluation of the perimeter $P(F)$ of F. Clearly, we have

$$P(F) = P(E^*) - P(E^*|J_*) + P(F|\widetilde{J}_*)\,. \tag{3.21}$$

Moreover, since both E^* and F are axially symmetric, (3.3) allows to calculate their perimeters in terms of v_E and v_F. Keep in mind that our aim is to show that F has a low perimeter, because this will ensure us that F has an isoperimetric deficit quite lower than E^*, and this means that E^* has a quite high deficit. Therefore, first of all we estimate $P(F|\widetilde{J}_*)$ from above, recalling again that $\|v_F\|_{L^\infty} \leq 2\omega_{n-1}$, that $v_F' \equiv N$ inside $\widetilde{J}$ and (3.19),

$$\begin{aligned} P(F|\widetilde{J}_*) &= \int_{\widetilde{J}} \sqrt{\tau\, v_F(t)^{\frac{2n-4}{n-1}} + v_F'(t)^2} = \int_{\widetilde{J}} \sqrt{\tau\, v_F(t)^{\frac{2n-4}{n-1}} + N^2} \\ &\leq \int_{\widetilde{J}} \sqrt{\tau\, (2\omega_{n-1})^{\frac{2n-4}{n-1}} + N^2} = \int_{\widetilde{J}} \sqrt{\tau' + N^2} \leq \int_{\widetilde{J}} N + \frac{\tau'}{2N} \\ &= \Big(N + \frac{\tau'}{2N}\Big)\mathscr{H}^1(\widetilde{J}) = h\Big(1 + \frac{\tau'}{2N^2}\Big)\,, \end{aligned} \tag{3.22}$$

where we have denoted by τ' the new constant (also depending only on n),

$$\tau' := \tau(2\omega_{n-1})^{\frac{2n-4}{n-1}} = 2^{\frac{2n-4}{n-1}}\,(n-1)^2\omega_{n-1}^2\,.$$

Now, we easily estimate $P(E^*|J_*)$ from below

$$P(E^*|J_*) = \int_J \sqrt{\tau\, v_E(t)^{\frac{2n-4}{n-1}} + v_E'(t)^2} \geq \int_J |v_E'(t)| = h\,. \tag{3.23}$$

Finally, gathering (3.21), (3.22) and (3.23) we obtain

$$P(F) \leq P(E^*) + h\,\frac{\tau'}{2N^2}\,. \tag{3.24}$$

Now we compare the volume estimate (3.20) with (3.24) thus getting, by the isoperimetric inequality, a perimeter estimate

$$\begin{aligned} P(F) &\geq n\omega_n^{1/n}|F|^{\frac{n-1}{n}} \geq n\omega_n^{1/n}\Big(|E^*| + \frac{h\omega_{n-1}}{2 \cdot 2^{n/2}N}\Big)^{\frac{n-1}{n}} \\ &= n\omega_n^{1/n}|E^*|^{\frac{n-1}{n}}\Big(1 + \frac{h\omega_{n-1}}{2 \cdot 2^{n/2}N\,|E^*|}\Big)^{\frac{n-1}{n}} = P(B)\Big(1 + \frac{h\omega_{n-1}}{2 \cdot 2^{n/2}N\,|E^*|}\Big)^{\frac{n-1}{n}} \\ &\geq P(B) + C_0(n)\,\frac{h}{N}\,; \end{aligned} \tag{3.25}$$

notice that the last inequality holds since h is bounded from above (as a consequence of Lemma 3.4).

Finally, comparing (3.24) with (3.25) we get

$$P(B)D(E^*) = P(E^*) - P(B) \geq P(F) - h\,\frac{\tau'}{2N^2} - P(B)$$
$$\geq C_0(n)\,\frac{h}{N} - h\,\frac{\tau'}{2N^2} = h\Big(\frac{C_0(n)}{N} - \frac{\tau'}{2N^2}\Big)\,.$$

If we choose N so that the expression in parentheses is bounded from below by a positive constant, the constant M, which as we noticed before depends only N, will ultimately depend only on n and

$$D(E^*) \geq \frac{1}{K}\,h\,.$$

In view of (3.19), we then get

$$\mathscr{H}^1(J) = \varepsilon \leq \frac{h}{M} \leq \frac{K}{M}D(E^*)\,,$$

where the constant K again depends only on n. This proves (3.18) when v'_E is continuous and J is an interval. When v'_E is continuous, in general J is a countable union of pairwise disjoint open intervals and it is easily checked that the above proof still works, due to the additivity of both perimeter and volume. If v'_E is not continuous, by assumptions (2.9) and $E \subseteq Q_l$, we have from Lemma 6.1 that v_E is a nonnegative function in $W^{1,1}(\mathbb{R})$ with support in $[-l,l]$. Thus we may construct a sequence of smooth nonnegative functions v_h converging to v_E in $W^{1,1}(\mathbb{R})$, whose supports are all contained in $[-l,l]$, and such that

$$\int_{-\infty}^{+\infty} v_h\,dt = \int_{-\infty}^{+\infty} v_E\,dt \quad \text{for all } h\,, \qquad v_h \to v_E \quad \text{uniformly}\,, \qquad v'_h \to v'_E \quad \text{a.e.}\,. \tag{3.26}$$

Let us denote by E_h the axially symmetric set defined by $v_{E_h} = v_h$. Clearly, from (3.26) and Lemma 3.3 we get easily that $|E_h| = |E^*|$ for all h and $P(E_h) \to P(E^*)$, hence $D(E_h) \to D(E^*)$. The result then follows by applying the above estimate to each set $J_h = \{t \in [-\sqrt{2}/2, \sqrt{2}/2] : |v'_h(t)| \geq M\}$ and observing that $\mathscr{H}^1(J_h) \to \mathscr{H}^1(J)$.

This proves (3.18), hence the result follows. □

4. The axially symmetric case

This section is entirely devoted to the proof of the following estimate, which refers to the particular case of an axially symmetric set.

Theorem 4.1. *Let $E \in X$ be axially symmetric with respect to the first axis and symmetric with respect to the hyperplane $\{x_1 = 0\}$. Let us assume also that $E \subseteq Q_l$ and that (3.1) holds. Then,*

$$d(E \cap Z, B \cap Z) \leq C(n,l)\sqrt{D(E)}\,, \tag{4.1}$$

where $Z = \{x \in \mathbb{R}^n : |x_1| \leq \sqrt{2}/2\}$.

Notice that the claim above states, for an axially symmetric set, an inequality similar to the one we aim to prove, but only the volume of the internal part of the symmetric difference (that is, inside Z) is estimated.

Let us explain now the strategy we will adopt to show Theorem 4.1. The first step will be to show the existence of a section of E such that the $(n-1)$-dimensional symmetric difference between this section of E and the corresponding section of B (in the sense of Lemma 4.2 below) gives a bound to the left term in (4.1). The second step will be to prove that it is not restrictive to assume that this section is the central one. Finally, the last step will be a careful but not difficult comparison between the symmetric difference of the central sections of E and B and the isoperimetric deficit of E.

Lemma 4.2. *There exist $\delta(n,l), C(n) > 0$, with the property that if E is as in Theorem 4.1 and $D(E) < \delta$, there exists a number $0 \leq \bar{x} \leq \sqrt{2}/2$ such that, if x' is defined through the equality*

$$\int_0^{x'} v_B(t)\,dt = \int_0^{\bar{x}} v_E(t)\,dt\,, \tag{4.2}$$

then

$$\big|v_E(\bar{x}) - v_B(x')\big| \geq \frac{1}{C} d(E\cap Z, B\cap Z)\,. \tag{4.3}$$

Notice that the number x' is chosen in such a way that the volume of the set E in the half-space $\{x_1 \geq \bar{x}\}$ coincides with the volume of the ball B in the half-space $\{x_1 \geq x'\}$.

Proof. (of Lemma 4.2). Recalling that both E and B are axially symmetric, we can write

$$d(E\cap Z, B\cap Z) = \int_{-\sqrt{2}/2}^{\sqrt{2}/2} \mathscr{H}^{n-1}\big(E_t \Delta B_t\big)\,dt = \int_{-\sqrt{2}/2}^{\sqrt{2}/2} |v_E(t) - v_B(t)|\,dt\,,$$

so that by symmetry there exists $0 \leq x_0 \leq \sqrt{2}/2$ with the property that

$$|v_E(x_0) - v_B(x_0)| \geq \frac{\sqrt{2}}{2}\, d(E\cap Z, B\cap Z)\,. \tag{4.4}$$

Let us assume for the moment that $v_B(x_0) > v_E(x_0)$. In this case, let us first assume that the inequality

$$\int_0^{x_0} v_B(t)\,dt \geq \int_0^{x_0} v_E(t)\,dt \tag{4.5}$$

holds. Then, defining $\bar{x} = x_0$, by the symmetry of E and B with respect to $\{x_1 = 0\}$, we have that $0 \leq x' \leq \bar{x} \leq 1$ –with x' defined according to (4.2)– so that

$$v_B(x') \geq v_B(\bar{x}) \geq v_E(\bar{x}) + \frac{\sqrt{2}}{2}\, d(E\cap Z, B\cap Z)\,,$$

thus (4.3) follows, and so the thesis is obtained by choosing $C(n) \geq \sqrt{2}$.
When (4.5) does not hold, we consider two cases, depending on whether or not the following inequality holds

$$\int_0^{x_0} v_B(t)\,dt \geq \int_0^{x_0} v_E(t)\,dt - \frac{1}{2\cdot 2^{n/2}(n-1)}\, d(E\cap Z, B\cap Z)\,. \tag{4.6}$$

<u>*Case I.*</u> *If the inequality (4.6) holds.*
In this case, let again $\bar{x} = x_0$ and let x' be defined according to (4.2). Assuming $\delta(n,l)$ sufficiently small, by Lemmas 2.2 and 2.3 we have that $d(E,B)$ is small, and since $v_B \geq \omega_{n-1}/2^{(n-1)/2}$ in $[-\sqrt{2}/2, \sqrt{2}/2]$, we obtain that $|x' - \bar{x}|$ is small; therefore, provided δ is sufficiently small, we have $v_B(t) \geq \omega_{n-1}/2^{n/2}$ for t in $[\bar{x}, x']$. This implies, by (4.6)

$$\begin{aligned}|x' - \bar{x}| &= \int_{\bar{x}}^{x'} 1\,dt \leq \frac{2^{n/2}}{\omega_{n-1}} \int_{\bar{x}}^{x'} v_B(t)\,dt = \frac{2^{n/2}}{\omega_{n-1}}\bigg(\int_0^{x'} v_B(t)\,dt - \int_0^{\bar{x}} v_B(t)\,dt\bigg)\\ &= \frac{2^{n/2}}{\omega_{n-1}} \int_0^{\bar{x}} v_E(t) - v_B(t)\,dx \leq \frac{1}{2(n-1)\,\omega_{n-1}}\, d(E\cap Z, B\cap Z)\,.\end{aligned}$$

Notice now that, in $[0, \sqrt{2}/2]$,

$$0 \geq v_B'(t) = -(n-1)\,\omega_{n-1} t(1-t^2)^{\frac{n-3}{2}} \geq -(n-1)\,\omega_{n-1}\,;$$

therefore, recalling also (4.4), the assumption $v_B(\bar{x}) > v_E(\bar{x})$ and the fact that $|\bar{x} - x'|$ is small enough if $\delta \ll 1$, we have

$$\begin{aligned}v_B(x') &= v_B(\bar{x}) + \int_{\bar{x}}^{x'} v_B'(t)\,dt \geq v_B(\bar{x}) - \frac{6}{5}|x' - \bar{x}|(n-1)\,\omega_{n-1}\\ &\geq v_B(\bar{x}) - \frac{3d(E\cap Z, B\cap Z)}{5} \geq v_E(\bar{x}) + \Big(\frac{\sqrt{2}}{2} - \frac{3}{5}\Big)\, d(E\cap Z, B\cap Z)\,;\end{aligned}$$

hence also in this case (4.3), and so the thesis, is obtained, by choosing $C(n)$ large enough.
<u>*Case II.*</u> *If the inequality (4.6) does not hold.*
In this case we have

$$\int_0^{x_0} v_E(t) - v_B(t)\,dt > \frac{1}{2\cdot 2^{n/2}(n-1)}\, d(E\cap Z, B\cap Z)\,.$$

Since $0 \le x_0 \le \sqrt{2}/2$, this implies

$$\fint_0^{x_0} v_E(t) - v_B(t)\,dt > \frac{\sqrt{2}}{2\cdot 2^{n/2}(n-1)}\, d(E\cap Z, B\cap Z)\,.$$

Let now $\bar{x}$ be the largest number in $[0, x_0]$ such that

$$v_E(\bar{x}) - v_B(\bar{x}) \ge \frac{\sqrt{2}}{2\cdot 2^{n/2}(n-1)}\, d(E\cap Z, B\cap Z)\,.$$

Therefore, we deduce

$$\int_{\bar{x}}^{x_0} v_E(t) - v_B(t) \le (x_0 - \bar{x})\,\frac{\sqrt{2}}{2\cdot 2^{n/2}(n-1)}\, d(E\cap Z, B\cap Z) \le \frac{1}{2\cdot 2^{n/2}(n-1)}\, d(E\cap Z, B\cap Z)\,;$$

the fact that (4.6) does not hold together with this estimate ensures

$$\int_0^{\bar{x}} v_E(t) - v_B(t)\,dt > 0\,,$$

and then $x' \ge \bar{x}$, defining as usual x' according to (4.2). Then

$$v_B(x') \le v_B(\bar{x}) \le v_E(\bar{x}) - \frac{\sqrt{2}}{2\cdot 2^{n/2}(n-1)}\, d(E\cap Z, B\cap Z)\,;$$

so (4.3) is established also in this case, by choosing a suitable constant $C(n)$.

Finally, it remains to consider what happens if $v_B(x_0) \le v_E(x_0)$: the proof in this case is very similar to the preceding one, it suffices to replace (4.5) with

$$\int_0^{x_0} v_B(t)\,dt \le \int_0^{x_0} v_E(t)\,dt$$

and (4.6) with

$$\int_0^{x_0} v_B(t)\,dt \le \int_0^{x_0} v_E(t)\,dt + \frac{1}{2\cdot 2^{n/2}(n-1)}\, d(E\cap Z, B\cap Z),$$

and to argue in the very same way as before. □

Lemma 4.3. *Let E, $\delta(n,l)$ and $\bar{x}$ be as in Lemma 4.2 and in Theorem 4.1; then there exists a set $E' \in X$, axially symmetric with respect to the first axis and symmetric with respect to the hyperplane $\{x_1 = 0\}$, with the property that*

$$P(E') \le P(E)\,, \qquad \big|v_{E'}(0) - v_B(0)\big| \ge \frac{1}{C}\big|v_E(\bar{x}) - v_B(x')\big|$$

for some constant $C = C(n)$.

Proof. Let us write $E = E^l \cup E^c \cup E^r$, where

$$E^l := \big\{x\in E:\, x_1 \le -\bar{x}\big\}\,, \qquad E^c := \big\{x\in E:\, |x_1| < \bar{x}\big\}\,, \qquad E^r := \big\{x\in E:\, x_1 \ge \bar{x}\big\}\,.$$

Notice that there exists a unique pair of numbers $r\in\mathbb{R}$ and $x_c > 0$ such that the intersection of the ball of radius $1+r$ with the strip $\{-x_c \le x_1 \le x_c\}$,

$$B^c := \{x\in\mathbb{R}^n:\ |x| \le 1+r,\, |x_1| \le x_c\}\,,$$

satisfies

$$|B^c| = |E^c|, \qquad \mathscr{H}^{n-1}\big(B^c_{x_c}\big) = \mathscr{H}^{n-1}\big(E_{\bar{x}}\big) = v_E(\bar{x})\,. \tag{4.7}$$

Let us now define $\widetilde{E}^l$ and $\widetilde{E}^r$ the following two horizontal translations of E^l and E^r:

$$\widetilde{E}^l := (\bar{x} - x_c)e_1 + E^l\,, \qquad \widetilde{E}^r := (x_c - \bar{x})e_1 + E^r\,,$$

and finally set $E' = \widetilde{E}^l \cup B^c \cup \widetilde{E}^r$. Notice that the translations of E^l and E^r are made in such a way that the three parts $\widetilde{E}^l$, B^c and $\widetilde{E}^r$ match along their boundaries. We will show that E' has the desired properties. It is immediate from the construction that $|E'| = |E|$ –so that $E' \in X$– and that E' is axially symmetric and symmetric with respect to the hyperplane $\{x_1 = 0\}$. We show now that $P(E') \le P(E)$. Define

$$B^l := \{x \in \mathbb{R}^n : \ |x| \le 1+r,\, x_1 < -x_c\}\,, \qquad\qquad B^r := \{x \in \mathbb{R}^n : \ |x| \le 1+r,\, x_1 > x_c\}\,,$$

so that $B^l \cup B^c \cup B^r$ is the ball of radius $1+r$; let moreover $\widetilde{B}^l$ and $\widetilde{B}^r$ be the horizontal translations of B^l and B^r

$$\widetilde{B}^l := (x_c - \bar{x})e_1 + B^l\,, \qquad\qquad \widetilde{B}^r := (\bar{x} - x_c)e_1 + B^r\,,$$

and notice that $\widetilde{B}^l \cup E^c \cup \widetilde{B}^r$ is a set of the same volume as the ball $B^l \cup B^c \cup B^r$; this implies $P(\widetilde{B}^l \cup E^c \cup \widetilde{B}^r) \ge P(B^l \cup B^c \cup B^r)$, hence $P(E^c) \ge P(B^c)$ and so $P(E) \ge P(E')$. To conclude the thesis, we need then only to show that

$$\big|v_{E'}(0) - v_B(0)\big| \ge \frac{1}{C}\big|v_E(\bar{x}) - v_B(x')\big|\,.$$

Let us suppose, just to fix the ideas, that $v_E(\bar{x}) > v_B(x')$ –in the opposite case, the proof works in a very similar way. We claim that $r > 0$ and $0 < x_c < x'$: indeed, if $r \le 0$ then by (4.2) and the fact that $|E^c| = |B^c|$ we would infer $x_c \ge x'$, so that $\mathscr{H}^{n-1}(B^c_{x_c}) \le \mathscr{H}^{n-1}(B^c_{x'}) \le v_B(x') < v_E(\bar{x})$ against (4.7). This implies $r > 0$, and again by the fact that $|E^c| = |B^c|$ and by (4.2) we deduce $x_c < x'$.
Being $r > 0$, it is

$$\big|v_{E'}(0) - v_B(0)\big| = \big|\mathscr{H}^{n-1}(B^c_0) - \mathscr{H}^{n-1}(B_0)\big| = \omega_{n-1}\big((1+r)^{n-1} - 1\big) \ge (n-1)\omega_{n-1}\, r\,.$$

Hence, to obtain the thesis it is sufficient to show that

$$\big|v_E(\bar{x}) - v_B(x')\big| = \big|v_{E'}(x_c) - v_B(x')\big| \le C\, r\,. \tag{4.8}$$

To this aim, consider Figure 3, which shows the two balls of radii 1 and $1+r$, the points $X_C \equiv (x_c, 0, \dots, 0)$

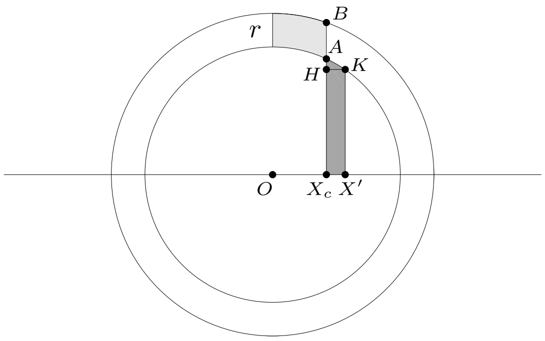

FIGURE 3. Situation in the proof of Lemma 4.3

and $X' \equiv (x', 0, \dots, 0)$ on the first axis, and the points A, B, H and K as drawn. By construction, the two shaded regions have the same volume; moreover, since $0 \le \bar{x} \le \sqrt{2}/2$, $|x' - \bar{x}| = O(D(E))$ and $0 \le x_c \le x'$, there exists a constant $\alpha = \alpha(n)$ such that $\overline{AB} \le \alpha\, r$. As an easy consequence, the volume of the lightly shaded region is less than $C(n)r$. Recalling again that $x' \le \sqrt{2}/2 + O(D(E))$ we deduce that the $(n-1)$-dimensional measures $\mathscr{H}^{n-1}(B_t)$ of the sections B_t of the ball B with $x_c \le t \le x'$ are all greater than $\big((\sqrt{2}/2)^{n-1}/2\big)\omega_{n-1} = 2^{-(n+1)/2}\omega_{n-1}$. Therefore, the volume of the darker region (which equals that of the lighter one) is greater than $2^{-(n+1)/2}\omega_{n-1}|x' - x_c|$. Thus,

$$|x' - x_c| < C r\,.$$

Since the angle $\widehat{AKH}$ is less than $\pi/4 + O(D(E))$, it is

$$\overline{BH} = \overline{AB} + \overline{AH} \le \alpha\, r + 2\overline{HK} = \alpha\, r + 2|x' - x_c| \le C\, r\,.$$

Finally, keeping in mind that

$$v_{E'}(x_c) = \omega_{n-1}\overline{BX_c}^{\,n-1}\,, \qquad\qquad v_B(x') = \omega_{n-1}\overline{KX'}^{\,n-1} = \omega_{n-1}\overline{HX_c}^{\,n-1}$$

and that $\overline{HX_c} \geq \sqrt{2}/2 + O(D(E))$, we finally get (4.8), thus the thesis is concluded. □

Thanks to the above lemmas, the last step to conclude is to perform a precise estimate of the isoperimetric deficit of an axially symmetric set as in the claim of Lemma 4.3: having reduced ourselves to a very specific case, the calculation will be involved but elementary, while an analogous calculation for a general set as in Theorem 4.1 would be extremely complicate.

Proof. (of Theorem 4.1). Thanks to Lemmas 4.2 and 4.3, the proof of Theorem 4.1 will be achieved once we show that, for any set E as in Theorem 4.1, one has

$$\big|v_E(0) - v_B(0)\big| \leq C\sqrt{D(E)}\,. \tag{4.9}$$

First of all, let $\widetilde{B}$ be the ball centered in the first axis such that $\widetilde{B}_0 = E_0$ and

$$\Big|\widetilde{B} \cap \big\{x \in \mathbb{R}^N : x_1 < 0\big\}\Big| = \Big|E \cap \big\{x \in \mathbb{R}^N : x_1 < 0\big\}\Big| = \frac{|B|}{2} \ :$$

in words, in the open half-space $R^n_- = \{x_1 < 0\}$ the sets $\widetilde{B}$ and E have the same volume, and their sections at $x_1 = 0$ coincide. As an immediate consequence, with the same argument we already used in Lemma 4.3, we obtain $P(\widetilde{B}|\mathbb{R}^n_-) \leq P(E|\mathbb{R}^n_-)$.

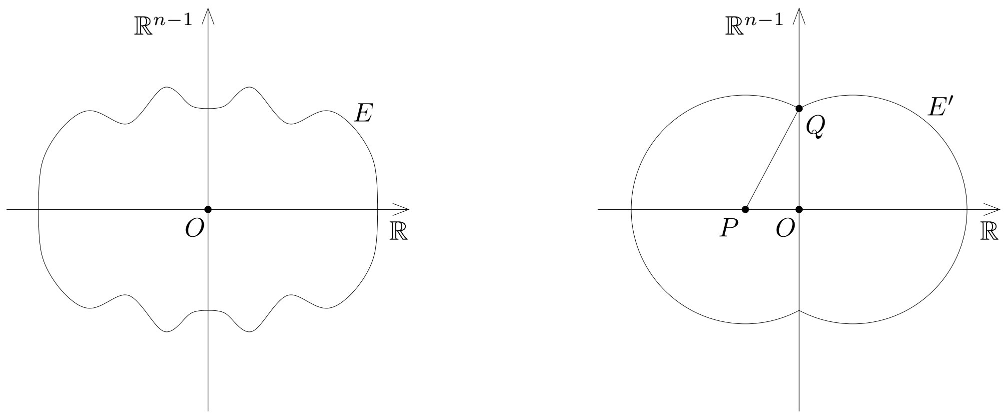

FIGURE 4. The sets E and E'

Define then E' the set which coincides with the ball $\widetilde{B}$ in $\mathbb{R}^n_-$ and which is symmetric with respect to the hyperplane $\{x_1 = 0\}$, as shown in Figure 4. By construction,

$$v_{E'}(0) = v_E(0)\,, \qquad\qquad P(E') \leq P(E)\,,$$

so to show the claim of this Lemma we can reduce to consider the set E' instead of E; more precisely, the proof will be concluded once we establish

$$\big|v_{E'}(0) - v_B(0)\big| \leq C\sqrt{P(E') - P(B)}\,.$$

Let us set

$$\varepsilon := \frac{v_{E'}(0) - v_B(0)}{\omega_{n-1}}\,, \tag{4.10}$$

and let us denote the radius of the ball $\widetilde{B}$ by $1 + r$ and the center of the ball $\widetilde{B}$ by $(-\delta, 0, \ldots, 0)$. Notice that the set E' is fully determined by ε, then $r = r(\varepsilon)$ and $\delta = \delta(\varepsilon)$; moreover, of course when $\varepsilon = 0$ one has $r = \delta = 0$ since $E' \equiv \widetilde{B} \equiv B$. We will now evaluate r and δ in terms of ε.

To do that we first write the following two conditions:

$$\begin{cases} \Big((1+r)^2-\delta^2\Big)^{\frac{n-1}{2}} = 1+\varepsilon\,; \\ \displaystyle\int_{-(1+r)}^{\delta} \Big((1+r)^2-t^2\Big)^{\frac{n-1}{2}}\,dt = \int_{-1}^{0} \Big(1-t^2\Big)^{\frac{n-1}{2}}\,dt\,. \end{cases} \tag{4.11}$$

Let us prove these equations: concerning the first one, if we denote, as in Figure 4, by P the center of $\widetilde{B}$ and by Q some point of $\partial E'\cap\{x_1=0\}$, one has $\overline{PQ}^2=\overline{PO}^2+\overline{OQ}^2$ by Pitagora's Theorem, thus $\overline{OQ}=\sqrt{(1+r)^2-\delta^2}$. Since E'_0 is a ball in $\mathbb{R}^{n-1}$ of radius $\overline{OQ}$, its volume $v_{E'}(0)$ is $\omega_{n-1}\overline{OQ}^{\,n-1}$; recalling (4.10) and the fact that $v_B(0)=\omega_{n-1}$ by definition, we recover the first equation. Concerning the second one, it comes from the fact that $|E'|=|B|$ and E' is symmetric with respect to the hyperplane $\{x_1=0\}$: indeed, the first and the second integral are the volume of E' and the volume of B inside $\mathbb{R}^{n-1}_-$ respectively.

Thus, (4.9) follows once we show that

$$P(E')-P(B)\geq \frac{1}{C}\,\varepsilon^2\,. \tag{4.12}$$

To this aim, let us set

$$a:=\int_{-1}^{0}\big(1-t^2\big)^{\frac{n-3}{2}}\,dt=\int_0^{\pi/2}\sin^{n-2}\theta\,d\theta\,.$$

Evaluating the first equation in (4.11) in r and δ at the second order, we get

$$\varepsilon=(n-1)r+\frac{(n-1)(n-2)}{2}\,r^2-\frac{n-1}{2}\,\delta^2+o(r^2+\delta^2)\,. \tag{4.13}$$

The second order Taylor expansion in r and δ of the second equation in (4.11) yields

$$0=(n-1)ra+\delta+\frac{(n-1)^2a}{2}\,r^2+(n-1)r\delta+o(r^2+\delta^2)\,. \tag{4.14}$$

Finally, (4.13) and (4.14) allow to reach a first order estimate of r and δ:

$$\begin{cases} r=\dfrac{1}{n-1}\,\varepsilon+o(r+\delta),\\ \delta=-a\varepsilon+o(r+\delta)\,. \end{cases}$$

This ensures that both r and δ are of order ε, so in particular $o(r+\delta)=o(\varepsilon)$ and $o(r^2+\delta^2)=o(\varepsilon^2)$. However, we need a more precise estimate, namely an estimate of r and δ up to the second order in ε. Setting

$$r=\frac{1}{n-1}\,\varepsilon+\tilde{r}, \qquad\qquad \delta=-a\varepsilon+\tilde{\delta}\,,$$

and plugging these expressions into (4.13) and (4.14), we find the second order expansion of $\tilde{r}$ and $\tilde{\delta}$, thus concluding that

$$\begin{cases} r=\dfrac{1}{n-1}\,\varepsilon+\dfrac{\varepsilon^2}{2(n-1)}\Big((n-1)\,a^2-\dfrac{n-2}{n-1}\Big)+o(\varepsilon^2)\,,\\ \delta=-a\varepsilon-\dfrac{a\varepsilon^2}{2}\Big((n-1)\,a^2-\dfrac{n-2}{n-1}\Big)+\dfrac{a\varepsilon^2}{2}+o(\varepsilon^2)\,. \end{cases}$$

Once we know r and δ with a second order precision, we pass to the calculation of the perimeter of E' –keep in mind that our aim is to show (4.12). Since E' is a part of a ball, it is immediate to write down its perimeter in terms of r and δ, that is

$$\frac{P(E')}{2(n-1)\omega_{n-1}}=\int_{-(1+r)}^{\delta}\Big((1+r)^2-t^2\Big)^{\frac{n-3}{2}}(1+r)\,dt=:\xi(r,\delta)\,.$$

The second order Taylor expansion of ξ gives

$$\begin{aligned}\xi(r,\delta) &= \xi(0,0) + r(n-1)\,a + \delta + \frac{(n-2)(n-1)\,a}{2}\,r^2 + (n-2)r\delta + o(r^2+\delta^2)\\ &= \xi(0,0) + \frac{a}{2(n-1)}\,\varepsilon^2 + o(\varepsilon^2)\,.\end{aligned}$$

Since

$$\xi(0,0) = \frac{P(B)}{2(n-1)\omega_{n-1}}\,,$$

we conclude that

$$P(E') - P(B) = a\omega_{n-1}\varepsilon^2 + o(\varepsilon^2)\,.$$

So (4.12) follows. □

5. The proof of the main Theorem

We are now ready to give the proof of Theorem 1.1.

Proof. (of Theorem 1.1). As we already observed, it is enough to prove the theorem under the assumption that $|E| = |B|$. Moreover we may also assume that $D(E) \leq 1$ (otherwise the assertion is trivial).
From Lemma 5.1 below it is clear that (with no loss of generality) we may assume that E is contained in a fixed cube Q_l, where l is a constant depending only on n. Let n be either 2 or greater than 2 and such that Theorem 1.1 is true in dimension $n-1$.
Let us fix a set of finite perimeter $E \subseteq Q_l$. Since there are only countably many directions ν such that $\mathscr{H}^{n-1}(\{x \in \partial^*E : \nu^E(x) = \pm\nu\}) > 0$, by rotating E and slightly increasing l if necessary, we may assume without loss of generality that for all $i = 1, \dots, n$

$$\mathscr{H}^{n-1}(\{x \in \partial^*E : \nu^E(x) = \pm e_i\}) = 0\,. \tag{5.1}$$

By Theorem 2.1 and Remark 2.6, we can find a set F, symmetric with respect to the n orthogonal coordinate hyperplanes, satisfying (5.1) and contained in the cube Q_{3l}, such that

$$\lambda(E) \leq C\lambda(F), \qquad\qquad D(F) \leq 2^n D(E)\,. \tag{5.2}$$

Thus, combining Theorems 3.1 and 4.1 we get that, up to a rotation of the coordinate axes,

$$\lambda(F) \leq d(F,B) \leq 4d(F^*\cap Z, B\cap Z) + C\sqrt{D(F)} \leq C\sqrt{D(F)}\,.$$

From this inequality and (5.2), the assertion immediately follows by a trivial induction argument, for any dimension $n \geq 2$. □

Next Lemma is a variant of a similar result proved in [10] (see also [11]).

Lemma 5.1. *There exist two constants, $l = l(n)$ and $C = C(n)$ such that, if $E \in X$, then there is a set $E' \subseteq Q_l$, such that $|E'| = |B|$ and*

$$D(E') \leq CD(E)\,, \qquad\qquad \lambda(E) \leq \lambda(E') + CD(E)\,. \tag{5.3}$$

Proof. First of all we observe that, by rotating E if necessary, we may assume that

$$\mathscr{H}^{n-1}\big(\{x \in \partial^*E : \nu^E(x) = \pm e_i\}\big) = 0 \tag{5.4}$$

for all $i = 1, \dots, n$. By Lemma 6.1 we deduce that $v_E \in W^{1,1}(\mathbb{R})$, so that in particular v_E is continuous. Now, setting $E_t^- = \{x \in E : x_1 < t\}$ for all $t \in \mathbb{R}$, we claim that

$$P(E_t^-) \leq P(E|\{x_1 < t\}) + v_E(t)\,, \qquad\qquad P(E \setminus E_t^-) \leq P(E|\{x_1 > t\}) + v_E(t)\,. \tag{5.5}$$

To prove these inequalities we first notice that, if E is a smooth open set, then by (5.4) the inequalities (5.5) hold as equalities. Assuming now that E is any set of finite perimeter satisfying (5.4), we may take a sequence of smooth open sets E_j such that $|E_j \Delta E| \to 0$, $P(E_j) \to P(E)$ and (5.4) holds for each E_j. Since the sets E_j are smooth, we already know that

$$P\big((E_j)_t^-\big) = P(E_j|\{x_1 < t\}) + v_{E_j}(t), \qquad P\big(E_j \setminus (E_j)_t^-\big) = P(E_j|\{x_1 > t\}) + v_{E_j}(t)\,.$$

Since $\|v_{E_j} - v_E\|_{L^1} \to 0$ and v_E is continuous, one easily obtains by construction that $v_{E_j}(t) \to v_E(t)$ for every $t \in \mathbb{R}$; moreover, $P(E_j|\Omega) \to P(E|\Omega)$ for every open set $\Omega \subseteq \mathbb{R}^n$. Clearly, for any $t \in \mathbb{R}$, $\big|(E_j)_t^- \Delta E_t^-\big| \to 0$, hence by the lower semicontinuity of the perimeters we get

$$P(E_t^-) \leq \liminf_{j\to\infty} P\big((E_j)_t^-\big) = \lim_{j\to\infty} P(E_j|\{x_1 < t\}) + v_{E_j}(t) = P(E|\{x_1 < t\}) + v_E(t) \ :$$

this proves the first inequality in (5.5), and the second one is fully analogous.
Let us now define the function $g : \mathbb{R} \to \mathbb{R}^+$ as

$$g(t) := \frac{|E_t^-|}{|B|}\,;$$

g is a non-decreasing absolutely continuous function such that $g'(t) = v_E(t)/|B|$ for $\mathscr{H}^1$-a.e. $t \in \mathbb{R}$. Let $-\infty \leq a < b \leq +\infty$ be the numbers such that $\{t : \ 0 < g(t) < 1\} = (a,b)$; pick now any $a < t < b$ and notice that by definition

$$\big|g(t)^{-\frac{1}{n}} E_t^-\big| = |B|\,;$$

therefore, $P(g(t)^{-1/n} E_t^-) \geq P(B)$, which gives

$$P(E_t^-) = g(t)^{\frac{n-1}{n}} P\big(g(t)^{-\frac{1}{n}} E_t^-\big) \geq g(t)^{\frac{n-1}{n}}\, P(B)\,.$$

Similarly,

$$P(E \setminus E_t^-) \geq \big(1 - g(t)\big)^{\frac{n-1}{n}}\, P(B)\,.$$

Therefore, from (5.5) and assumption (5.4) we get that

$$P(E) + 2v_E(t) \geq P(B)\Big(g(t)^{\frac{n-1}{n}} + \big(1 - g(t)\big)^{\frac{n-1}{n}}\Big)$$

for all $t \in (a,b)$. On the other hand, by definition of isoperimetric deficit we have $P(E) = P(B)\big(1 + D(E)\big)$, and then

$$v_E(t) \geq \frac{1}{2}\, P(B)\Big(g(t)^{\frac{n-1}{n}} + \big(1 - g(t)\big)^{\frac{n-1}{n}} - 1 - D(E)\Big)\,. \tag{5.6}$$

Let us now define the function $\psi : [0,1] \to \mathbb{R}^+$, whose graph is drawn in Figure 5, as

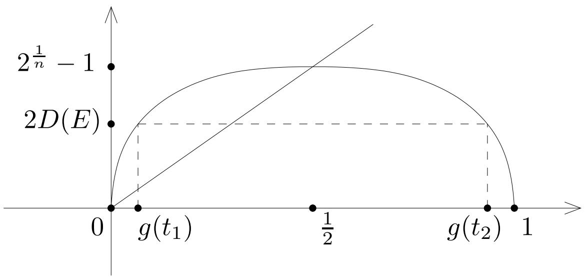

FIGURE 5. The graph of ψ

$$\psi(t) := t^{\frac{n-1}{n}} + \big(1 - t\big)^{\frac{n-1}{n}} - 1\,;$$

notice that $\psi(0) = \psi(1) = 0$, that $\psi(1/2) = 2^{1/n} - 1$ is the maximum, and that ψ is concave, so that

$$\psi(t) \geq 2(2^{1/n} - 1)\, t \quad \forall\, 0 \leq t \leq \frac{1}{2}\,. \tag{5.7}$$

We may assume, without loss of generality, that $2D(E) < \psi(1/2)$, let $a < t_1 < t_2 < b$ be two numbers such that $g(t_1) = 1 - g(t_2)$ and $\psi(g(t_1)) = \psi(g(t_2)) = 2D(E)$; therefore,

$$\psi(g(t)) \geq 2D(E) \quad \forall t \in (t_1, t_2) \tag{5.8}$$

and, by (5.7),

$$g(t_1) = 1 - g(t_2) \leq \frac{D(E)}{2^{1/n} - 1}\,. \tag{5.9}$$

Thanks to (5.6), for any $t_1 \leq t \leq t_2$ we have

$$\begin{aligned} v_E(t) &\geq \frac{1}{2}\,P(B)\Big(\psi(g(t)) - D(E)\Big) \geq \frac{1}{4}\,P(B)\psi(g(t)) + \frac{1}{4}\,P(B)\Big(\psi(g(t)) - 2D(E)\Big) \\ &\geq \frac{n\omega_n}{4}\psi(g(t))\,. \end{aligned} \tag{5.10}$$

Therefore, recalling that $v_E(t) = |B|g'(t)$,

$$t_2 - t_1 \leq \frac{4}{n}\int_{t_1}^{t_2} \frac{g'(t)}{\psi(g(t))}\,dt = \frac{4}{n}\int_{g(t_1)}^{g(t_2)} \frac{1}{\psi(s)}\,ds \leq \frac{4}{n}\int_0^1 \frac{1}{\psi(s)}\,ds = \alpha \tag{5.11}$$

for some dimensional constant $\alpha = \alpha(n)$. Let us now set

$$\tau_1 = \max\left\{t \in (a, t_1] : v_E(t) \leq \frac{n\omega_n D(E)}{2}\right\}, \qquad \tau_2 = \min\left\{t \in [t_2, b) : v_E(t) \leq \frac{n\omega_n D(E)}{2}\right\}.$$

Notice that τ_1 and τ_2 are well defined since v_E is continuous and $v_E(a) = v_E(b) = 0$; moreover, by (5.8) and (5.10), $v_E(\tau_1) = v_E(\tau_2) = (n\omega_n D(E))/2$. Moreover, from (5.9), we have

$$t_1 - \tau_1 \leq \frac{2}{n\omega_n D(E)}\int_{\tau_1}^{t_1} v_E(t)\,dt = \frac{2}{nD(E)}\int_{\tau_1}^{t_1} g'(t)\,dt \leq \frac{2g(t_1)}{nD(E)} \leq \frac{2}{n(2^{1/n} - 1)}$$

and a similar estimate for $\tau_2 - t_2$.

Let us now set $\widetilde{E} = E \cap \{x : \tau_1 < x_1 < \tau_2\}$. From the above estimate and (5.11), we have that $\tau_2 - \tau_1 < \beta(n)$. Moreover, (5.9), the definition of τ_1, τ_2 and (5.5) immediately yield

$$|\widetilde{E}| \geq |B|\Big(1 - 2\,\frac{D(E)}{2^{1/n} - 1}\Big), \qquad\qquad P(\widetilde{E}) \leq P(E) + n\omega_n D(E)\,. \tag{5.12}$$

Let us now assume that $D(E) < (2^{1/n} - 1)/4$ and set

$$\sigma = \left(\frac{|B|}{|\widetilde{E}|}\right)^{1/n}, \qquad\qquad E' = \sigma\widetilde{E}\,.$$

Clearly, $|E'| = |B|$ and E' is contained in a strip $\{\tau_1' < x_1 < \tau_2'\}$, with $\tau_2' - \tau_1' \leq \beta'$, where β' is a constant depending only on n. Let us now show that E' satisfies (5.3) for a suitable constant C depending only on n.

To this aim, notice that since we are assuming $D(E)$ small, from (5.12) we get that $1 \leq \sigma \leq 1 + C_0 D(E)$, with $C_0 = C_0(n)$. Thus, from (5.12), we get

$$\begin{aligned} P(E') = \sigma^{n-1}P(\widetilde{E}) &\leq \sigma^{n-1}\big(P(E) + P(B)D(E)\big) = \sigma^{n-1}P(B)\big(1 + 2D(E)\big) \\ &\leq P(B)(1 + C(n)D(E))\,. \end{aligned}$$

Hence, the first inequality in (5.3) follows. To prove the second inequality, let us denote by $p + B$ a ball such that $\lambda(E') = |E'\Delta(p + B)|$. From the first inequality in (5.12), we then get

$$\begin{aligned} \lambda(E) &\leq |E\Delta((p/\sigma) + B)| \\ &\leq |E\Delta\widetilde{E}| + |\widetilde{E}\Delta((p/\sigma) + B_{1/\sigma})| + |((p/\sigma) + B_{1/\sigma})\Delta((p/\sigma) + B)| \\ &= |E \setminus \widetilde{E}| + \frac{\lambda(E')}{\sigma^n} + |B \setminus B_{1/\sigma}| \\ &\leq C(n)D(E) + \lambda(E') + C(n)(1 - \sigma) \leq \lambda(E') + CD(E)\,. \end{aligned}$$

Thus, the set E' satisfies (5.5). Starting from this set, we may repeat the same construction with respect to the x_2 axis, thus getting a new set, still denoted by E', uniformly bounded with respect to the first two coordinate directions and satisfying (5.5) with a new constant, still depending only on n. Thus, the assertion follows by repeating this argument with respect to all remaining coordinate directions. □

6. Appendix: some definitions and technical facts

In this section we first briefly recall the notion of distributional perimeter, then we collect a couple of technical facts concerning sets of finite perimeter. These properties are certainly known to the experts in the field, but we have not been able to find a precise reference in the literature. Therefore, for the benefit of the non expert reader and for the sake of completeness, we have decided to present here a complete proof of these facts.

6.1. **Distributional perimeter.** A Borel set E is said to be a *set of finite perimeter* if the distributional derivative $D\chi_E$ of its characteristic function χ_E is an $\mathbb{R}^n$-valued Radon measure with finite total variation. If E is a set of finite perimeter then the total variation $|D\chi_E|(\mathbb{R}^n)$ is denoted by $P(E)$, and it is called *perimeter of* E. Moreover, if A is any Borel set in $\mathbb{R}^n$, the *perimeter of* E *in* A is defined by setting $P(E|A) = |D\chi_E|(A)$.

This distributional notion of perimeter, due to Caccioppoli and De Giorgi, extends the usual one in the sense that $P(E) = \mathscr{H}^{n-1}(\partial E)$ whenever ∂E is a Lipschitz manifold. Notice that, by definition of distributional derivative, for any smooth function f with compact support we have

$$\int_E \nabla f(x)dx = -\int_{\mathbb{R}^n} f(x)d[D\chi_E](x)\,.$$

Moreover, De Giorgi's Rectifiability Theorem (see Theorem 3.59 in [1]) states that $D\chi_E = \nu^E d\mathscr{H}^{n-1} \llcorner \partial^* E$, i.e. that

$$\int_E \nabla f(x)dx = -\int_{\partial^* E} f(x)\nu^E(x)d\mathscr{H}^{n-1}\,, \qquad\qquad P(E|A) = \mathscr{H}^{n-1}(\partial^* E \cap A)\,.$$

Here $\partial^* E$ is the *reduced boundary of* E, i.e. the set of those points $x \in \mathbb{R}^n$ such that there exists

$$\nu^E(x) = \lim_{r\to 0} \frac{D\chi_E(x+rB)}{|D\chi_E|(x+rB)}$$

and $|\nu^E(x)| = 1$. The density ν^E is called the *generalized inner normal to* E, and for all $x \in \partial^* E$

$$\lim_{r\to 0} \frac{|E\cap(x+rB)|}{r^n|B|} = \frac{1}{2}\,, \qquad\qquad \lim_{r\to 0} \frac{\big|\{y \in E\cap(x+rB) :\ (y-x)\cdot\nu^E(x) < 0\}\big|}{r^n} = 0\,.$$

The validity of the isoperimetric inequality (1.1) in the class of sets of finite perimeter, together with the characterization of the equality case mentioned in the introduction, are due to De Giorgi ([8], see also [9] and [13]).

6.2. **Some technical facts about sets of finite perimeter.** In the following, we shall denote the generic point x of $\mathbb{R}^n$ also by (t,z), where $t \in \mathbb{R}$, $z \in \mathbb{R}^{n-1}$. Given a set of finite perimeter E, we denote by $v_E(t)$ the $\mathscr{H}^{n-1}$ measure of its section $E_t = \{(t,z) \in E\}$. The same symbol F_t will be used to denote the sections of any set F.

Recalling that the reduced boundary $\partial^* E$ of a set of finite perimeter is a $\mathscr{H}^{n-1}$-rectifiable set (see [1, Theorem 3.59]), from the coarea formula for rectifiable sets ([1, Theorem 2.39 and Remark 2.40]) we get that if $g : \mathbb{R}^n \to [0,+\infty]$ is a Borel function, then

$$\int_{\partial^* E} g(x)\sqrt{1-|\nu_1^E(x)|^2}\, d\mathscr{H}^{n-1}(x) = \int_{-\infty}^{+\infty} \int_{(\partial^* E)_t} g(t,z)\, d\mathscr{H}^{n-2}(z)\, dt\,. \tag{6.1}$$

The first result we present is the version in codimension $n-1$ of a similar result proved in codimension 1 in [6] (see Lemmas 3.1, 3.2 and Proposition 1.2).

To this aim, we recall that by definition of generalized inner normal to a set of finite perimeter we have that for all $\psi \in C^1_0(\mathbb{R}^n)$ and $i = 1, \dots, n$

$$\int_E \frac{\partial \psi}{\partial x_i}\,dx = -\int_{\partial^* E} \psi \nu_i^E \, d\mathscr{H}^{n-1} \,. \tag{6.2}$$

Theorem 6.1. *Let E be a set of finite perimeter, with finite measure. Then $v_E \in BV(\mathbb{R})$. Moreover, if*

$$\mathscr{H}^{n-1}(\{x \in \partial^* E : \nu_1^E = \pm 1\} = 0 \,, \tag{6.3}$$

then $v_E \in W^{1,1}(\mathbb{R})$ and for $\mathscr{H}^1$-a.e. $t \in \mathbb{R}$

$$v_E'(t) = \int_{(\partial^* E)_t} \frac{\nu_1^E(t,z)}{\sqrt{1 - |\nu_1^E(t,z)|^2}} \, d\mathscr{H}^{n-2}(z) \,. \tag{6.4}$$

Proof. The fact that $v_E \in L^1(\mathbb{R})$ is a simple consequence of Fubini's theorem and of the assumption $|E| < \infty$. To prove that the distributional derivative of v_E is a Radon measure, let us fix $\varphi \in C^1_0(\mathbb{R})$ and take a sequence ψ_h of nonnegative functions from $C^1_0(\mathbb{R}^{n-1})$, such that $\psi_h(z) \nearrow 1$. Then, by Fubini's theorem and (6.2), we get

$$\begin{aligned} \int_{-\infty}^{+\infty} v_E(t)\varphi'(t)\,dt &= \int_{-\infty}^{+\infty} \varphi'(t) \int_{E_t} dz\,dt = \lim_{h\to\infty} \int_E \varphi'(t)\psi_h(z)\,dx \\ &= -\lim_{h\to\infty} \int_{\partial^* E} \varphi(t)\psi_h(z)\nu_1^E(t,z)\,d\mathscr{H}^{n-1} = -\int_{\partial^* E} \varphi(t)\nu_1^E(t,z)\,d\mathscr{H}^{n-1} \,. \end{aligned} \tag{6.5}$$

Taking the supremum of the left-hand side integral among all φ such that $\|\varphi\|_\infty \le 1$,we get immediately that Dv_E is a Radon measure. Notice that once we know this, by using (6.5) and the definition of distributional derivative, we obtain that for all $\varphi \in C^1_0(\mathbb{R})$

$$\int_{-\infty}^{+\infty} \varphi(t)\,d[Dv_E](t) = \int_{\partial^* E} \varphi(t)\nu_1^E(t,z)\,d\mathscr{H}^{n-1} \,. \tag{6.6}$$

From this equation we get immediately that if G is an open subset in $\mathbb{R}$ then

$$|Dv_E|(G) \le \int_{\partial^* E \cap (G \times \mathbb{R}^{n-1})} |\nu_1^E(x)| \, d\mathscr{H}^{n-1}(x) \,. \tag{6.7}$$

Again, we can deduce by approximation that the same inequality holds for any Borel set G in $\mathbb{R}$.
Let us now assume that (6.3) holds and let us apply (6.7) to a null Borel set G. Since now $\sqrt{1 - |\nu_1^E(x)|^2} > 0$ for $\mathscr{H}^{n-1}$-a.e. $x \in \partial^* E$, from coarea formula (6.1) we get

$$\begin{aligned} |Dv_E|(G) &\le \int_{\partial^* E \cap (G \times \mathbb{R}^{n-1})} |\nu_1^E| \, d\mathscr{H}^{n-1} = \int_{\partial^* E \cap (G \times \mathbb{R}^{n-1})} \frac{|\nu_1^E|}{\sqrt{1 - |\nu_1^E|^2}} \sqrt{1 - |\nu_1^E|^2} \, d\mathscr{H}^{n-1} \\ &= \int_G dt \int_{(\partial^* E)_t} \frac{|\nu_1^E|}{\sqrt{1 - |\nu_1^E|^2}} \, d\mathscr{H}^{n-2} = 0 \,. \end{aligned}$$

thus proving that $v_E \in W^{1,1}(\mathbb{R})$. To conclude the proof, let us fix $\varphi \in C^1_0(\mathbb{R})$ and use (6.6) and coarea formula again to get

$$\begin{aligned} \int_{-\infty}^{+\infty} \varphi(t) v_E'(t)\,dt &= \int_{\partial^* E} \varphi(t)\nu_1^E(t,z)\,d\mathscr{H}^{n-1} = \int_{\partial^* E} \frac{\varphi(t)|\nu_1^E(t,z)|}{\sqrt{1 - |\nu_1^E|^2}} \sqrt{1 - |\nu_1^E|^2} \, d\mathscr{H}^{n-1} \\ &= \int_{-\infty}^{+\infty} \varphi(t) \int_{(\partial^* E)_t} \frac{\nu_1^E(t,z)}{\sqrt{1 - |\nu_1^E(t,z)|^2}} \, d\mathscr{H}^{n-2}(z)\,dt \,. \end{aligned}$$

Hence (6.4) follows, by the arbitrariness of φ. □

Next result is an extension to our situation of a well known result by Vol'pert, concerning one-dimensional sections of sets of finite perimeter (see [22] and also [1, Theorem 3.108]).

Theorem 6.2. *Let E be a set of finite perimeter in $\mathbb{R}^n$. Then, for $\mathscr{H}^1$-a.e. $t \in \mathbb{R}$ the section E_t is a set of finite perimeter in $\mathbb{R}^{n-1}$. Moreover, $\partial^* E_t$ coincide with $(\partial^* E)_t$ up to a set of zero $\mathscr{H}^{n-2}$ measure.*

Proof. Let us denote by u_h a sequence of mollifiers of χ_E. Then, $u_h \to \chi_E$ in $L^1_{\rm loc}(\mathbb{R}^n)$ and it is well known that

$$\int_{\mathbb{R}^n} |\nabla u_h|\, dx \to |D\chi_E|(\mathbb{R}^n) = P(E)\,, \qquad \int_{\mathbb{R}^n} |\nabla_z u_h|\, dx \to |D_z\chi_E|(\mathbb{R}^n)\,, \tag{6.8}$$

where $\nabla_z = (\nabla_{x_2}, \dots, \nabla_{x_n})$ and D_z is defined similarly. Moreover, by Fubini's theorem we get that for any compact subset $K \subseteq \mathbb{R}^{n-1}$ and any $R > 0$

$$\lim_{h\to\infty} \int_K \int_{-R}^{R} |u_h(t,z) - \chi_E(t,z)|\, dt\, dz = 0\,.$$

From this equality, by a diagonalization argument, it follows that up to a (not relabeled) subsequence $u_h(t,\cdot) \to \chi_E(t,\cdot) = \chi_{E_t}$ in $L^1_{\rm loc}(\mathbb{R}^{n-1})$ for $\mathscr{H}^1$-a.e. $t \in \mathbb{R}$. Hence, by the lower semicontinuity of the total variation, we have

$$P_{n-1}(E_t) \le \liminf_{h\to\infty} \int_{\mathbb{R}^{n-1}} |\nabla_z u_h(t,z)|\, dz\,,$$

where by P_{n-1} we denote the $(n-1)$-dimensional perimeter in $\mathbb{R}^{n-1}$. Integrating this inequality and using Fatou's lemma we have, from (6.8),

$$\begin{aligned}\int_{\mathbb{R}} P_{n-1}(E_t)\, dt &\le \int_{\mathbb{R}} \liminf_{h\to\infty} \int_{\mathbb{R}^{n-1}} |\nabla_z u_h(t,z)|\, dz\, dt \\ &\le \liminf_{h\to\infty} \int_{\mathbb{R}^n} |\nabla_z u_h(x)|\, dx = |D_z\chi_E|(\mathbb{R}^n) < \infty\,.\end{aligned} \tag{6.9}$$

Hence, we get in particular that, for a.e. $t \in \mathbb{R}$, $P_{n-1}(E_t) < \infty$, i.e. E_t has finite perimeter.
Let us fix $\varphi \in C^1_0(\mathbb{R}^n;\mathbb{R}^{n-1})$ with $\|\varphi\| \le 1$. We have

$$\begin{aligned}\int_{\mathbb{R}^n} \varphi(x)\, d[D_z\chi_E](x) &= -\int_{\mathbb{R}^n} \chi_E \,{\rm div}_z\varphi(x)\, dx \\ &= -\int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \chi_{E_t}(z) {\rm div}_z \varphi(t,z)\, dz\, dt \le \int_{\mathbb{R}} P_{n-1}(E_t)\, dt\end{aligned}$$

and, passing to the supremum over all φ, we get that

$$|D_z\chi_E|(\mathbb{R}^n) \le \int_{\mathbb{R}} P_{n-1}(E_t)\, dt\,. \tag{6.10}$$

Therefore from this inequality and from (6.9) we conclude that in (6.10) the equality holds. With exactly the same argument it can be proved that if Ω is any open set in $\mathbb{R}^n$, then

$$|D_z\chi_E|(\Omega) = \int_{\mathbb{R}} P_{n-1}(E_t|\Omega_t)\, dt$$

and, by a simple approximation argument, that this equality still holds if we replace Ω by any Borel set $G \subseteq \mathbb{R}^n$

$$|D_z\chi_E|(G) = \int_{\mathbb{R}} P_{n-1}(E_t|G_t)\, dt\,.$$

Since $\sqrt{1-|\nu_1^E|^2} = |\nu_z^E| = \dfrac{d|D_z\chi_E|}{d|D\chi_E|} = \dfrac{d|D_z\chi_E|}{d\mathscr{H}^{n-1} \llcorner \partial^* E}$, this equality can be written in the form

$$\int_{\partial^*E\cap G} \sqrt{1-|\nu_1^E(x)|^2}\, d\mathscr{H}^{n-1}(x) = \int_{\mathbb{R}} \mathscr{H}^{n-2}(\partial^* E_t \cap G)\, dt\,.$$

Since this equality holds for any Borel set we may conclude that if $f : \mathbb{R}^n \to \mathbb{R}$ is any bounded Borel function, then

$$\int_{\partial^*E} f(x)\sqrt{1-|\nu_1^E(x)|^2}\, d\mathscr{H}^{n-1}(x) = \int_{\mathbb{R}} \int_{\partial^* E_t} f(t,z)\, d\mathscr{H}^{n-2}(z)\, dt\,. \tag{6.11}$$

On the other hand, from the coarea formula (6.1) we have also that

$$\int_{\partial^*E} f(x)\sqrt{1-|\nu_1^E(x)|^2}\, d\mathscr{H}^{n-1}(x) = \int_{\mathbb{R}} \int_{(\partial^* E)_t} f(t,z)\, d\mathscr{H}^{n-2}(z)\, dt\,. \tag{6.12}$$

Choosing $f(t,z)=\varphi(t)\psi(z)$, and comparing the two equations (6.11), (6.12) we then easily get that for $\mathscr{H}^1$-a.e. $t\in\mathbb{R}$

$$\mathscr{H}^{n-2}\llcorner\partial^*E_t=\mathscr{H}^{n-2}\llcorner(\partial^*E)_t\,.$$

Hence, the result follows. □

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