ENERGY RELEASE RATE AND STRESS INTENSITY FACTOR IN ANTIPLANE ELASTICITY

GIULIANO LAZZARONI AND RODICA TOADER

Abstract. In the setting of antiplane linearized elasticity, we show the existence of the stress intensity factor and its relation with the energy release rate when the crack path is a $C^{1,1}$ curve. Finally, we show that the energy release rate is continuous with respect to the Hausdorff convergence in a class of admissible cracks.

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Contents

Introduction

The present work is devoted to the study of the stability of cracks in *brittle materials* in the nonsmooth case. We consider bodies with a perfectly elastic behaviour outside the cracked region and we suppose that no force is transmitted across the cracks. The physical model relies on Griffith's principle [8] that the propagation of a crack is the result of the

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competition between the elastic energy released when the crack opens and the energy spent to produce new crack.

In this paper, we extend the class of curves for which the energy release rate can be rigorously computed; hence, we provide a set of admissible cracks where the energy release rate is continuous with respect to the convergence of the curves. Previous contributions in the characterization of the energy release rate were given in [6, 10] for a straight crack and in [13, 14] under strong regularity hypotheses on the crack path. Some alternative approaches were proposed in [2, 4], considering nonsmooth extensions of a straight initial crack. In particular, in [4] the authors provide a generalized notion of energy release rate for any connected add-crack with density $1/2$ at the crack tip, assuming that the initial crack is straight. Unfortunately, none of these methods allows passing to the limit in the energy release rates under a suitable convergence of curves.

The starting point of our analysis is the singular behaviour at crack tips of solutions to linear elasticity problems in brittle materials with cracks: as noticed by Griffith, around the crack tips the strain must take high values tending to infinity. Let us describe in detail the type of singularities observed by Griffith. We consider a cylinder, whose section is a smooth bounded open set $\Omega\subset\mathbb{R}^2$, subject to deformations of the type

$$\Omega\times\mathbb{R}\ni(x_1,x_2,x_3)\mapsto(x_1,x_2,x_3+u(x_1,x_2))\,.$$

This is the case of *antiplane elasticity.* We assume that a cut is present in the domain Ω, lying on a straight line $\Gamma_{s_0}:=\{(x_1,0)\colon -s_0\le x_1\le 0\}$ (we suppose that $0:=(0,0)\in\Omega$ and $(-s_0,0)\in\partial\Omega$). The elasticity equations for the displacement u take the form

$$\begin{cases}-\Delta u=f & \text{in } \Omega\,,\\ u=\psi & \text{on } \partial\Omega\,,\\ \frac{\partial u}{\partial\nu}=0 & \text{on } \Gamma_{s_0}\,,\end{cases}\tag{0.1}$$

where the external volume force f and the boundary condition ψ are given, while ν denotes the normal vector to Γ_{s_0}. The last line of (0.1) says that the crack is traction-free.

Fix a system of polar coordinates (r,θ) around the crack tip 0 (with $r>0$ and $-\pi<\theta<\pi$); then the variational solution $u\in H^1(\Omega\backslash\Gamma_{s_0})$ to (0.1) can be written in the following form:

$$u=u^R+K\,r^{\frac12}\sin\frac{\theta}{2}\,,\tag{0.2}$$

where $u^R\in H^2(\Omega'\backslash\Gamma_{s_0})$ for every $\Omega'\subset\subset\Omega$ and $K\in\mathbb{R}$. This fact can be seen by writing the expansion of u in power series, in the simple case where Ω is a circle centred at 0 and $f=0$; the complete proof requires some finer mathematical arguments, described, e.g., by Grisvard [9, 10]. Since the stress tensor σ is a linear function of ∇u, it is clear that $|\sigma|\to+\infty$ unless $K=0$; hence, the multiplicative coefficient K is called *stress intensity factor.*

This phenomenon, appearing when the equations are linearized and a Neumann condition is prescribed on the crack, leads to a paradox from the physical point of view: a material subject to an infinite stress would immediately break up! Therefore, Griffith's remark permits excluding all models for crack growth based on an a priori bound on the stress intensity in the uncracked region, when the equations are linearized and homogeneous Neumann conditions are imposed on the crack path.

Nevertheless, Griffith proposed to keep the linearity of the problem and allow for the singularity it implies: then one may develop a model where the crack's stability does not depend on a bound on the stress, but it is connected to the energy balance. Indeed, his approach is based on an energy criterion: the stored elastic energy released by crack's increase is completely dissipated in the process of crack's formation; the crack stops growing if equilibrium is reached.

Griffith's criterion is based on the notion of *energy release rate*, that is the opposite of the derivative of the energy associated with the solution when the crack length varies. To be more precise, we define the increasing family of cracks $\Gamma_\ell := \{(x_1, 0)\colon -s_0 \leq x_1 \leq \ell - s_0\}$. For every $\ell \geq s_0$ we consider the variational solution u_ℓ of the problem

$$\begin{cases} -\Delta u_\ell = f & \text{in } \Omega\backslash\Gamma_\ell\,, \\ u_\ell = \psi & \text{on } \partial\Omega\,, \\ \frac{\partial u_\ell}{\partial \nu} = 0 & \text{on } \Gamma_\ell \end{cases} \tag{0.3}$$

and the associated elastic energy

$$\mathcal{E}^{\mathrm{el}}(\ell) := \frac{1}{2}\int_{\Omega\backslash\Gamma_\ell} |\nabla u_\ell(x)|^2 \,\mathrm{d}x - \int_{\Omega\backslash\Gamma_\ell} f(x)\, u_\ell(x)\,\mathrm{d}x\,. \tag{0.4}$$

Then the energy release rate is defined as $-\frac{\mathrm{d}\mathcal{E}^{\mathrm{el}}}{\mathrm{d}\ell}(s_0)$.

Assume now that the external force f and the boundary condition ψ vary in dependence on time, so that the energy becomes a function $\mathcal{E}^{\mathrm{el}}(t,\ell)$ of the instant and the crack length. In what follows, we assume, for such time dependence, all the regularity needed in order to derive the energy and the crack length. The fundamental contribution of Griffith is an energetic criterion to determine the crack length $\ell(t)$ during the evolution process. The energetic cost is related to the toughness $\kappa > 0$, a parameter depending on the material, which represents the energy needed to break atomic bonds along a line of length one.

According to Griffith's criterion, $\ell(t)$ must satisfy:

(a) $\dot{\ell}(t) \geq 0$, i.e., the crack growth is irreversible;
(b) $-\dfrac{\mathrm{d}\mathcal{E}^{\mathrm{el}}}{\mathrm{d}\ell}(t,\ell(t)) \leq \kappa$, i.e., the rate cannot exceed the fracture toughness;
(c) $\left[\dfrac{\mathrm{d}\mathcal{E}^{\mathrm{el}}}{\mathrm{d}\ell}(t,\ell(t)) + \kappa\right]\dot{\ell}(t) = 0$, i.e., the crack grows only if the rate equals κ.

Griffith's theory has been the starting point of variational models for crack growth based on an energetic formulation [1, 16].

We have seen that near the crack tip the model introduces an infinite stress which is not present in the physical process, because of the error coming from linearization when the displacements are not small. However, the linearized system is still a good approximation away from the crack tip, while near the crack tip one may study the singularities and give them a precise physical interpretation when considering the problem from the energetic point of view. Indeed, Irwin [11] observed that the energy release rate is connected to the stress intensity factor K appearing in (0.2), by the relation

$$-\frac{\mathrm{d}\mathcal{E}^{\mathrm{el}}}{\mathrm{d}\ell}(s_0) = \frac{\pi}{4}\,K^2\,; \tag{0.5}$$

we refer to [10, Theorem 6.4.1] for the proof.

Hence, Irwin's remark gives a physical meaning to the singularity of the solution. Moreover, the computation shows the double nature of the energy release rate: on the one hand, it can be expressed by a volume integral of a quantity depending on the elastic coefficients and on the deformation gradient; on the other hand, it is proportional to the stress intensity factor, which can be known from the solution in a neighbourhood of 0.

In this article, we study a bidimensional problem for antiplane linearized elasticity; in particular, we extend the properties described above to curves of class $C^{1,1}$. We prove the existence of the stress intensity factor in this case and show that the relation (0.5) holds also for curvilinear cracks (see Theorems 1.4, 1.9, and 2.1); this allows us to prove the continuity of the energy release rate in a class of admissible cracks. Our arguments are based on the theory developed by Grisvard [9, 10], who studied the singularities of solutions to elliptic problems in polygonal domains.

We suppose that the crack path Γ is parametrized by arc length through a function $\gamma\colon [0,l] \to \overline{\Omega}$, with $\gamma(0) \in \partial\Omega$; then we consider the increasing cracks $\Gamma_s := \gamma([0,s])$ for $s \in (0,l)$. The standard strategy for the computation of the derivative of the energy is to rewrite the energy integrals so that they are defined on a fixed domain. If the crack has a rectilinear path, it is easy to construct a diffeomorphism F_s which coincides with the identity in a neighbourhood of $\partial\Omega$ and transforms $\Omega_s := \Omega\backslash\Gamma_s$ into a fixed domain $\Omega_{s_0} := \Omega\backslash\Gamma_{s_0}$ (see the proof of Theorem 2.1 for the details). This procedure can be followed also if the crack is a curve of class C^2, defining F_s around $\gamma(s_0)$ as the flow of a vector field tangent to Γ (see, e.g., [13, 14]). However, this allows the computation of the energy release rate only if the second derivative of Γ exists at the crack tip.

We present a different method to calculate the derivative of the energy when the crack path Γ is only of class $C^{1,1}$, proving that the derivative exists at all the points, even if the curve has not a second derivative. We reduce the problem to the rectilinear case, thanks to a diffeomorphism Φ that straightens the cut in a neighbourhood of $\gamma(s_0)$; moreover, Φ transforms the elliptic coefficients so that the conormal vector is parallel to the normal (see Section 1.1 for the precise construction). A similar procedure was performed by Mumford and Shah [17] for a slightly different variational problem. The change of variables Φ is used to show the existence of the stress intensity factor in this case, following the lines of a proof by Grisvard [9] for a pure Dirichlet problem. Our theorems have a natural generalization to elliptic operators with variable coefficients of class $C^{0,1}$. Furthermore, they permit extending the results of [4] to the case of an initial crack of class $C^{1,1}$ (instead of a straight one), as noticed by G. A. Francfort.

The computation of the energy release rate at $\gamma(s_0)$ shows that it depends only on the piece of curve $\gamma([0,s_0])$: more precisely, if $\widetilde{\gamma}\colon [0,\widetilde{l}] \to \overline{\Omega}$ is another curve of class $C^{1,1}$ such that $\widetilde{\gamma}([0,s_0]) = \gamma([0,s_0])$, the energy release rate calculated for $\widetilde{\gamma}$ at $\widetilde{\gamma}(s_0) = \gamma(s_0)$ coincides with the one found for γ. Hence, when studying the stability of a crack we need not prescribe a priori its continuation. Moreover, we show that also in the case of $C^{1,1}$ curvilinear cracks the energy release rate is an integral invariant (see Proposition 2.4).

This characterization allows us to prove the continuity of the energy release rate with respect to the Hausdorff convergence of cracks in a suitable class of admissible $C^{1,1}$ curves with bounded curvature (see Theorem 2.12). Actually, this motivates the study of the energy release rate in the $C^{1,1}$ case, because a sequence of C^2 curves with bounded curvature has limit only in the class of $C^{1,1}$ curves. The continuity of the energy release rate will be a basic ingredient for the study of crack evolution in the setting of $C^{1,1}$ curves, without prescribing a priori the crack path [15].

1. Stress intensity factor for curvilinear cracks

We will define the stress intensity factor in the case of elliptic operators with Lipschitz coefficients in domains with $C^{1,1}$ curvilinear cracks.

Let $\Omega \subset \mathbb{R}^2$ be a bounded open set, simply connected, with Lipschitz boundary. In Ω we consider a curve $\gamma\colon [0,l] \to \overline{\Omega}$ of class $C^{1,1}$, parametrized by arc length, without self-intersections; let $\Gamma := \gamma([0,l])$.

We suppose that $\gamma(0) \in \partial\Omega$ and $\gamma(s) \in \Omega$ for every $s \in (0,l]$. We fix a point $s_0 \in (0,l)$ and consider the portion of curve $\Gamma_{s_0} := \gamma([0,s_0])$; up to a translation, we may assume also that $\gamma(s_0) = 0$. The set Ω represents the section of an elastic body with a crack, Γ_{s_0}, whose tip is $0 = (0,0)$.

Furthermore, we suppose that $\Omega\backslash\Gamma$ is the union of two Lipschitz open sets. This allows us to employ the Poincaré inequality in $\Omega\backslash\Gamma$, by considering separately the two Lipschitz subdomains.

We denote the two lips of Γ by Γ^+ and Γ^-: Γ^+ has the orientation given by the arc length parametrization, Γ^- the opposite, so that $\partial(\Omega\backslash\Gamma)$ is oriented as usual. Analogously, we denote by $\Gamma^+_{s_0}$ and $\Gamma^-_{s_0}$ the two lips of Γ_{s_0}.

Consider an elliptic operator (with only a principal part, for the simplicity sake)

$$\mathcal{A}u := -\sum_{i,j=1}^{2} \mathrm{D}_i \left(a_{ij}\mathrm{D}_j u\right) , \tag{1.1}$$

where the coefficients $a_{ij} = a_{ji} \in C^{0,1}(\overline{\Omega})$ are uniformly elliptic, i.e., there exists $\alpha > 0$ such that

$$\sum_{i,j=1}^{2} a_{ij}\xi^i\xi^j \geq \alpha \left|\xi\right|^2 \quad \text{for every } x \in \Omega \text{ and every } \xi \in \mathbb{R}^2 .$$

Let A denote the coefficient matrix, $A(x) = (a_{ij}(x))_{ij}$. Applying an affine change of coordinates, we may assume that $A(0) = I$.

Given $f \in L^2(\Omega\backslash\Gamma_{s_0})$ and $\psi \in H^1(\Omega\backslash\Gamma_{s_0})$, we study the problem

$$\begin{cases} \mathcal{A}u = f & \text{in } \Omega\backslash\Gamma_{s_0} , \\ \gamma_\Omega u = \psi & \text{on } \partial\Omega , \\ \gamma^\pm \frac{\partial}{\partial\nu_A^\pm} u = 0 & \text{on } \Gamma^\pm_{s_0} , \end{cases}$$

where $\nu_A^\pm := A\nu^\pm$ denotes the conormal vector to $\Gamma^\pm_{s_0}$, $\nu^\pm$ the normal, and $\gamma_\Omega, \gamma^\pm$ are the trace operators on $\partial\Omega, \Gamma^\pm$, respectively.

We define the space of test functions vanishing on $\partial\Omega$,

$$H_0(\Omega\backslash\Gamma_{s_0}) := \left\{ u \in H^1(\Omega\backslash\Gamma_{s_0})\colon \gamma_\Omega u = 0 \text{ on } \partial\Omega \right\} .$$

Under these hypotheses, we have a result of existence and uniqueness for the variational solution: there is a unique function $u \in H_0(\Omega\backslash\Gamma_{s_0}) + \psi$ such that

$$\sum_{i,j=1}^{2} \int_{\Omega\backslash\Gamma_{s_0}} a_{ij}(x)\,\mathrm{D}_j u(x)\,\mathrm{D}_i w(x)\,\mathrm{d}x = \int_{\Omega\backslash\Gamma_{s_0}} f(x)w(x)\,\mathrm{d}x \tag{1.2}$$

for every $w \in H_0(\Omega\backslash\Gamma_{s_0})$.

By the classical regularity theorems, we see that the variational solution u is of class H^2 inside $\Omega\backslash\Gamma_{s_0}$ and up to the cut Γ_{s_0}, far from 0 and $\gamma(0)$ (where the boundary is not smooth).

Theorem 1.1. *Let u be the variational solution of* (1.2). *Let Ω' and Ω'' be two open sets such that $0 \in \Omega' \subset \Omega'' \subset\subset \Omega$; let $\zeta \in C^\infty_c(\Omega''\backslash\Omega')$. Then $\zeta\, u \in H^2(\Omega\backslash\Gamma_{s_0})$.*

In the following section we characterize the singularity around the crack tip 0: for this purpose it will be enough to restrict our attention to a neighbourhood of 0.

1.1. **A diffeomorphism that straightens the crack.** We construct a diffeomorphism that in a neighbourhood of the origin transforms the curve Γ into a segment and the elliptic operator $\mathcal{A}$ in an operator $\mathcal{B}$ with coefficients near to the Laplacian (since $A(0)=I$): this will allow us to reduce the problem to the one for the Laplacian with rectilinear crack, which was treated in [9, 10]. A similar change of variables was presented in [17, Appendix 1] for a slightly different variational problem.

The construction of such a diffeomorphism is a technical point, necessary for the study of nonsmooth cracks. Alternative approaches, based e.g. on flows of vector fields tangent to the cut, require the existence of the second derivative of the curve at each point.

For the sake of simplicity, from now on we fix a coordinate system such that the tangent vector $\dot\gamma(s_0)$ coincides with the first coordinate vector e_1.

First step. We define a diffeomorphism Φ_1 of class $C^{1,1}$ which induces an isometry of Γ into a segment, at least near the origin.

In a neighbourhood ω of 0, we may write Γ as the graph of a cartesian curve $x_2=\phi(x_1)$, defined for $-\delta\le x_1\le\delta$. In ω we set

$$\Phi_1(x_1,x_2):=(l(x_1,\phi(x_1)),\ x_2-\phi(x_1))\,,$$

where $l(x_1,\phi(x_1)):=\displaystyle\int_0^{x_1}\left(1+\dot\phi(t)^2\right)^{\frac12}\,\mathrm{d}t$ is the signed length of the part of curve between $(x_1,\phi(x_1))$ and $0=(0,0)$. Notice that Φ_1 is of class $C^{1,1}$, $\Phi_1(0)=0$, and $\Gamma\cap\omega$ is mapped into a segment on the line $\{x_2=0\}$.

This transformation can then be extended to the whole of $\mathbb{R}^2$ in such a way that it is $C^{1,1}$ and coincides with the identity in $\mathbb{R}^2\backslash\omega'$, where $\omega\subset\subset\omega'\subset\subset\Omega$.

The change of variables defined by Φ_1 transforms $\mathcal{A}$ in an operator $\mathcal{A}_1$ with uniformly elliptic Lipschitz coefficients $a^{(1)}_{ij}$, whose matrix is denoted by $A_1:=\left(a^{(1)}_{ij}\right)_{ij}$. We have $A_1(0)=A(0)=I$.

Second step. In (a part of) the set $\Phi_1(\omega)$ where the crack path is a segment we apply a diffeomorphism Φ_2 such that $\Phi_2(x_1,0)=(x_1,0)$ and the new coefficient matrix $A_2=\frac{\nabla\Phi_2A_1\nabla\Phi_2^{\mathrm{T}}}{|\det\nabla\Phi_2|}\circ\Phi_2^{-1}$ has the conormal vector proportional to the second coordinate vector e_2, i.e., $A_2(x_1,0)e_2=\lambda_2(x_1)e_2$.

For instance, fixed a neighbourhood ω_1 of 0 contained in $\Phi_1(\omega)$ and a cut-off function ζ supported in ω_1 and equal to one around 0, we may take

$$\Phi_2(x_1,x_2):=\left(x_1-\zeta(x_1,x_2)\int_{x_1}^{x_1+x_2}\frac{a^{(1)}_{12}(s,0)}{a^{(1)}_{22}(s,0)}\,\mathrm{d}s,\ x_2\right)\,.$$

Up to choosing ω_1 small enough, it is possible to see that Φ_2 is a diffeomorphism of class $C^{1,1}$, since the coefficients $a^{(1)}_{ij}$ are Lipschitz and $a^{(1)}_{22}$ is bounded away from 0 by uniform ellipticity. Moreover, it coincides with the identity in $\mathbb{R}^2\setminus\omega_1$ and $A_2(0)=A(0)=I$. The property of the conormals holds with $\lambda_2(x_1)=a^{(1)}_{22}(x_1,0)$.

We now consider the change of variables $\Phi:=\Phi_2\circ\Phi_1$ defined in the whole of Ω. In $\Phi(\Omega)=\Omega$ the equation becomes

$$\mathcal{B}v:=-\Delta v+\sum_{i,j=1}^{2}\mathrm{D}_i(b_{ij}\mathrm{D}_jv)=g\,,\tag{1.3}$$

where $v(y) := u\left(\Phi^{-1}(y)\right)$, $g(y) := f\left(\Phi^{-1}(y)\right)\left|\det \nabla\Phi^{-1}(y)\right|$ is of class L^2, and b_{ij} are Lipschitz coefficients with $b_{ij}(0) = 0$. We denote by $B := (\delta_{ij} - b_{ij})_{ij}$ the new coefficient matrix (uniformly elliptic with a constant $\beta > 0$) and by $\nu_B := B\nu$ the conormal vector, which is proportional to the normal ν to Γ. We have a Dirichlet condition on $\partial\Omega$ and a Neumann condition on the cut.

We point out the properties of the change of variables:

- Φ is a $C^{1,1}$-diffeomorphism,
- it coincides with the identity out of a neighbourhood of the origin,
- $\Phi(0) = 0 = \gamma(s_0)$,
- $\widetilde{\Gamma} := \Phi(\Gamma \cap \omega)$ is a segment on the axis $\{x_2 = 0\}$ in a neighbourhood of 0,
- Φ transforms the conormal to the crack in the normal, in the vicinity of the tip, i.e., ν_B is proportional to ν along $\widetilde{\Gamma}$ near 0;
- the length of the piece of curve from the origin to the current point is preserved if this point belongs to a suitably small neighbourhood of the origin, i.e., for $|s - s_0|$ small enough we have $\mathcal{H}^1(\Phi \circ \gamma([s_0, s])) = s - s_0$ if $s > s_0$ and $\mathcal{H}^1(\Phi \circ \gamma([s, s_0])) = s_0 - s$ if $s < s_0$.

The symbol $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure, coinciding with the usual notion of length on this class of curves.

1.2. **Fredholm property.** Thanks to the change of variables Φ of the previous section, we can compare the problem with the case of the Laplacian with a rectilinear crack, using the abstract theory of Fredholm operators. Indeed, the Fredholm properties of the elliptic operator $\mathcal{B}$ introduced in (1.3) allow us to study the singularity of the solution at the crack tip. We adapt the methods of [9, Section 5.2].

In this section we set $\widetilde{\Gamma}_{s_0} := \Phi(\Gamma_{s_0})$. For our purposes it suffices to restrict our study to a neighbourhood U of the crack tip 0, so we choose U to be an equilateral triangle centred at 0, with a vertex belonging to the rectified crack $\widetilde{\Gamma}_{s_0}$, contained in the zone where the crack is rectified and its normal coincides with ν_B. This choice allows us to employ Grisvard's theory [9, 10] for singularities in polygons: the angles are such that the only singularity appears at 0. We denote by $\widetilde{\Gamma}^{\pm}_{s_0}$ the two lips of the crack $\widetilde{\Gamma}_{s_0}$ lying in U, by $\gamma^{\pm}$ the trace operators on $\widetilde{\Gamma}^{\pm}_{s_0}$, and by $\nu^{\pm}$ the normal vectors to $\widetilde{\Gamma}^{\pm}_{s_0}$, which are proportional to the conormal vectors $\nu^{\pm}_B := B\nu^{\pm}$. Moreover, γ_U is the trace operator on ∂U.

To restrict the problem to $U\backslash\widetilde{\Gamma}_{s_0}$, we use a cut-off function equal to one near 0 and supported in $\overline{U}$. Changing the names of v and g, we are led to a problem with the same elliptic operator $\mathcal{B}$ defined in (1.3), homogeneous Dirichlet conditions on ∂U, and homogeneous Neumann conditions on $\widetilde{\Gamma}_{s_0}$:

$$\begin{cases} \mathcal{B}v = g & \text{in } U\backslash\widetilde{\Gamma}_{s_0}\,, \\ \gamma_U v = 0 & \text{on } \partial U\,, \\ \gamma^{\pm}\frac{\partial v}{\partial \nu^{\pm}} = 0 & \text{on } \widetilde{\Gamma}^{\pm}_{s_0}\,, \end{cases}$$

where the new force g is again of class $L^2(\Omega\backslash\widetilde{\Gamma}_{s_0})$ (indeed, it depends just on the first derivatives of v). The variational formulation is

$$\begin{cases} v \in H_0(U\backslash\widetilde{\Gamma}_{s_0})\,, \\ \int_{U\backslash\widetilde{\Gamma}_{s_0}} \nabla v(x)\, B(x)(\nabla w(x))^{\mathrm{T}}\,\mathrm{d}x = \int_{U\backslash\widetilde{\Gamma}_{s_0}} g(x)\, w(x)\,\mathrm{d}x \quad \text{for every } w \in H_0(U\backslash\widetilde{\Gamma}_{s_0})\,, \end{cases} \tag{1.4}$$

where the space of test functions is

$$H_0(U\backslash\widetilde{\Gamma}_{s_0}) := \left\{ w \in H^1(U\backslash\widetilde{\Gamma}_{s_0})\colon \gamma_U w = 0 \text{ on } \partial U \right\} .$$

By convention, gradient vectors are considered as row matrices.

Furthermore, we consider the space of "strong solutions"

$$S^2(U\backslash\widetilde{\Gamma}_{s_0}) := \left\{ w \in H^2(U\backslash\widetilde{\Gamma}_{s_0})\colon \gamma_U w = 0 \text{ on } \partial U\,,\ \gamma^\pm \frac{\partial w}{\partial \nu^\pm} = 0 \text{ on } \widetilde{\Gamma}^\pm_{s_0} \right\}$$

and regard $\mathcal{B}$ as an operator which maps $S^2(U\backslash\widetilde{\Gamma}_{s_0})$ into $L^2(U\backslash\widetilde{\Gamma}_{s_0})$:

$$\mathcal{B}\colon S^2(U\backslash\widetilde{\Gamma}_{s_0}) \to L^2(U\backslash\widetilde{\Gamma}_{s_0})\,.$$

We would like to extend the domain so that $\mathcal{B}$ becomes surjective: the first step is showing that $\operatorname{Rg}\mathcal{B}$ is closed, thanks to an a priori bound; then we will compute its index.

We will use the following estimate on the Laplacian, which can be proven arguing as in [10, Theorem 2.2.3]: for every $w \in S^2(U\backslash\widetilde{\Gamma}_{s_0})$

$$\|w\|_{H^2(U\backslash\widetilde{\Gamma}_{s_0})} \le C_{U\backslash\widetilde{\Gamma}_{s_0}} \|\Delta w\|_{L^2(U\backslash\widetilde{\Gamma}_{s_0})}\,, \tag{1.5}$$

where $C_{U\backslash\widetilde{\Gamma}_{s_0}}$ is the Poincaré constant of $U\backslash\widetilde{\Gamma}_{s_0}$. An analogous estimate holds for the operator $\mathcal{B}$, as we show in the next lemma.

Lemma 1.2. *There is a constant $C>0$ (depending on U) such that*

$$\|w\|_{H^2(U\backslash\widetilde{\Gamma}_{s_0})} \le C\left(\|\mathcal{B}w\|_{L^2(U\backslash\widetilde{\Gamma}_{s_0})} + \|w\|_{L^2(U\backslash\widetilde{\Gamma}_{s_0})} \right) \tag{1.6}$$

for every $w \in S^2(U\backslash\widetilde{\Gamma}_{s_0})$. In particular, $\mathcal{B}$ satisfies the Fredholm property, i.e., it is injective and $\operatorname{Rg}\mathcal{B}$ is closed.

Proof. We have for every $w \in S^2(U\backslash\widetilde{\Gamma}_{s_0})$

$$\begin{aligned}
\|-\Delta w\|_{L^2(U\backslash\widetilde{\Gamma}_{s_0})} &= \left\| \mathcal{B}w - \sum_{i,j=1}^{2} \mathrm{D}_i(b_{ij}\mathrm{D}_j w) \right\|_{L^2(U\backslash\widetilde{\Gamma}_{s_0})} \\
&\le \|\mathcal{B}w\|_{L^2(U\backslash\widetilde{\Gamma}_{s_0})} + M_1 \|w\|_{H^1(U\backslash\widetilde{\Gamma}_{s_0})} + 2M_0 \|w\|_{H^2(U\backslash\widetilde{\Gamma}_{s_0})}
\end{aligned}$$

where $M_0 := \max_{U\backslash\widetilde{\Gamma}_{s_0}} |b_{ij}|$ and $M_1 := \max_{U\backslash\widetilde{\Gamma}_{s_0}} |\nabla b_{ij}|$. Since $b_{ij} \to 0$ as $x \to 0$, we can rescale U so that $C_{U\backslash\widetilde{\Gamma}_{s_0}} M_0 \le \frac14$; recalling (1.5), we find $C>0$ such that for every $w \in S^2(U\backslash\widetilde{\Gamma}_{s_0})$

$$\|w\|_{H^2(U\backslash\widetilde{\Gamma}_{s_0})} \le C\left(\|\mathcal{B}w\|_{L^2(U\backslash\widetilde{\Gamma}_{s_0})} + \|w\|_{H^1(U\backslash\widetilde{\Gamma}_{s_0})} \right) .$$

To pass from $\|w\|_{H^1(U\backslash\widetilde{\Gamma}_{s_0})}$ to $\|w\|_{L^2(U\backslash\widetilde{\Gamma}_{s_0})}$, we integrate by parts, using the Dirichlet and Neumann conditions, and get

$$\left| \langle w, \mathcal{B}w\rangle_{L^2(U\backslash\widetilde{\Gamma}_{s_0})} \right| = \left| \int_{U\backslash\widetilde{\Gamma}_{s_0}} \sum_{i,j=1}^{2} (\delta_{ij} - b_{ij}) \mathrm{D}_i w \mathrm{D}_j w \,\mathrm{d}x \right| \ge \beta \|\nabla w\|^2_{L^2(U\backslash\widetilde{\Gamma}_{s_0})}\,,$$

where we have used the uniform ellipticity of the coefficients. Thanks to the Poincaré inequality we obtain

$$\|w\|^2_{H^1(U\backslash\widetilde{\Gamma}_{s_0})} \le \frac{C_{U\backslash\widetilde{\Gamma}_{s_0}}}{\beta} \left| \langle w, \mathcal{B}w\rangle_{L^2(U\backslash\widetilde{\Gamma}_{s_0})} \right| \le \frac{C_{U\backslash\widetilde{\Gamma}_{s_0}}}{2\beta} \left(\|\mathcal{B}w\|_{L^2(U\backslash\widetilde{\Gamma}_{s_0})} + \|w\|_{L^2(U\backslash\widetilde{\Gamma}_{s_0})} \right)^2 .$$

Hence we deduce (1.6), changing the value of C.

Finally, injectivity is obvious, while the fact that $\operatorname{Rg}\mathcal{B}$ is closed descends from the compact immersion of H^2 in L^2, thanks to (1.6). $\square$

The result about the index of $\mathcal{B}$, regarded as a Fredholm operator, follows. It is based on the nontrivial fact that the Laplacian (as operator acting on $S^2(U\backslash\widetilde{\Gamma}_{s_0})$) has range of codimension 1: this was shown in [10, Section 2.3].

Proposition 1.3. *We have* $\operatorname{codim}\operatorname{Rg}\mathcal{B} = 1$.

Proof. The theorem is an application of the Fredholm theory. By [10, Section 2.3] we deduce that $\operatorname{codim}\operatorname{Rg}(-\Delta) = 1$.

We compare $\mathcal{B}$ and $-\Delta$, so we consider the convex combinations between these two operators: for $\lambda \in [0,1]$ let $\mathcal{B}_\lambda = \lambda\mathcal{B} - (1-\lambda)\Delta$. Repeating the arguments of Lemma 1.2, we find for every $\lambda \in [0,1]$ a constant $C_\lambda > 0$ such that

$$\|w\|_{H^2(U\backslash\widetilde{\Gamma}_{s_0})} \le C_\lambda \left(\|\mathcal{B}_\lambda w\|_{L^2(U\backslash\widetilde{\Gamma}_{s_0})} + \|w\|_{L^2(U\backslash\widetilde{\Gamma}_{s_0})} \right)$$

for every $w \in H^2(U\backslash\widetilde{\Gamma}_{s_0})$. Hence $\mathcal{B}_\lambda$ is a Fredholm operator (injective with closed range) for every $\lambda \in [0,1]$.

As the index ι (i.e., the difference between the dimension of the kernel and the codimension of the range) is invariant under homotopy [12, Chapter 4, Section 5.1], we obtain $\iota(\mathcal{B}) = \iota(-\Delta) = -1$. By injectivity, $\dim\ker\mathcal{B} = \dim\ker(-\Delta) = 0$, so $\operatorname{codim}\operatorname{Rg}\mathcal{B} = \operatorname{codim}\operatorname{Rg}(-\Delta) = 1$. □

1.3. **Singular solutions and stress intensity factor.** We are now able to describe the singularities of a solution near 0. First, we argue in the case where the cut has been rectified by the diffeomorphism Φ of Section 1.1.

Using the notation of the previous section, we introduce in the triangle U a system of polar coordinates (r,θ), where the straight part of the crack coincides with the discontinuity line of the angle (recall that $\dot\gamma(s_0) = e_1$). We define the singular solution

$$S := r^{\frac{1}{2}} \sin\frac{\theta}{2} \in H^1(U\backslash\widetilde{\Gamma}_{s_0}) \setminus H^2(U\backslash\widetilde{\Gamma}_{s_0}). \tag{1.7}$$

Indeed, as shown in [10, Chapter 2], S describes the singularity of the solution to the problem with $b_{ij} = 0$; by comparison with $-\Delta$, we will prove that S is the singular part in the general case (up to a multiplicative constant).

Let ζ be a radial cut-off, equal to one around 0 and with support in U, and consider ζS and $F := \mathcal{B}(\zeta S)$. By uniqueness we have that $F \neq 0$, since ζS satisfies the Neumann and Dirichlet conditions being radial, and that $F \notin \operatorname{Rg}\mathcal{B}$, because $S \notin H^2$. Furthermore, it is possible to see that $F \in L^2$: in fact, from a direct computation we get $-\Delta(\zeta S) \in L^2$ and $|\mathrm{D}_{ij} S| \le C r^{-\frac{3}{2}}$ near 0; on the other hand, the coefficients b_{ij} are Lipschitz and $b_{ij}(0) = 0$, so that $|b_{ij}| \le Cr$ in a neighbourhood of 0 (here, $C > 0$).

Since $\operatorname{Rg}\mathcal{B}$ is a closed subspace of $L^2(U\backslash\widetilde{\Gamma}_{s_0})$ with codimension one, we have the decomposition

$$L^2(U\backslash\widetilde{\Gamma}_{s_0}) = \operatorname{Rg}\mathcal{B} \oplus \langle F\rangle. \tag{1.8}$$

Hence, given $g \in L^2(U\backslash\widetilde{\Gamma}_{s_0})$, there are a unique function $v^R \in S^2(U\backslash\widetilde{\Gamma}_{s_0})$ and a unique constant $K \in \mathbb{R}$, such that

$$g = \mathcal{B}v^R + K\,F.$$

If $v \in H_0(U\backslash\widetilde{\Gamma}_{s_0})$ is the variational solution of (1.4), by uniqueness we obtain

$$v = v^R + K\,\zeta\,S,$$

or equivalently

$$v - K\,S \in H^2(U\backslash\widetilde{\Gamma}_{s_0}),$$

as $K(1-\zeta)S$ is regular.

To come back to the operator $\mathcal{A}$ defined in $\Omega\backslash\Gamma_{s_0}$, we apply the diffeomorphism Φ^{-1}. Hence, recalling that $u = v\circ\Phi$ is the solution of (1.2) and setting $u^R := v^R\circ\Phi$, we get

$$u = u^R + K(\zeta\, S)\circ\Phi\,.$$

With the aid of Theorem 1.1, this concludes the proof of the following theorem.

Theorem 1.4. *Given $f \in L^2(\Omega\backslash\Gamma_{s_0})$, let $u \in H^1(\Omega\backslash\Gamma_{s_0})$ be the variational solution of* (1.2)*. Then there exists a unique constant K, called* stress intensity factor*, such that*

$$u - K\ S\circ\Phi \in H^2(\Omega'\backslash\Gamma_{s_0}) \tag{1.9}$$

for every $\Omega' \subset\subset \Omega$.

Remark 1.5. The stress intensity factor has been defined as the coefficient of the projection on $\langle F\rangle$ in the decomposition (1.8). Hence, the application which maps the force into the stress intensity factor of the associated solution is linear and continuous with respect to the convergence in L^2.

1.4. **A simpler singular function.** In order to compute the singular solution in (1.9) one has to apply first the change of variables Φ described in Section 1.1, which transforms the crack into a segment (at least near the origin). Here we provide another singular function, whose computation is simpler: indeed, we are not required to straighten the crack. As before, we assume that $A(0) = I$ and $\dot\gamma(s_0) = e_1$.

In Ω we fix a system of polar coordinates (ρ,ϑ), such that, at a point x, $\rho = |x|$ and ϑ is the determination of the angle between e_1 and $x-0$, continuous in $\Omega\backslash\Gamma_{s_0}$ (see Figure 1).

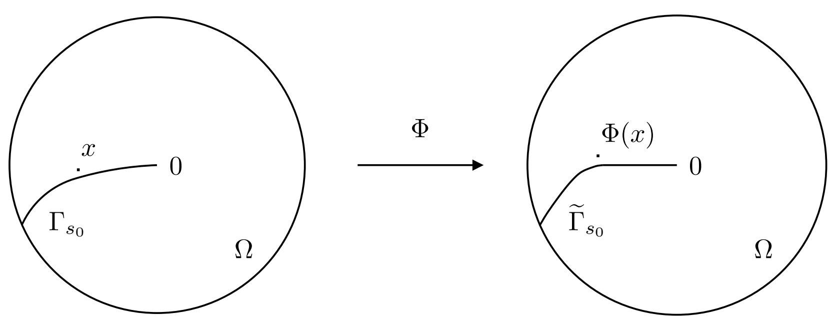

Figure 1. The angle ϑ is continuous in $\Omega\backslash\Gamma_{s_0}$, whilst θ is continuous in $\Omega\backslash\widetilde\Gamma_{s_0}$. Hence, in the figure we have $\vartheta(x) > \pi$, $-\pi < \theta(x) < 0$, and $0 < \theta(\Phi(x)) < \pi$.

Hence, ϑ is the usual angle in the plane, determined so that the discontinuity line lies in Γ_{s_0}. We define in $\Omega\backslash\Gamma_{s_0}$ the singular function

$$\widetilde S := \rho^{\frac12}\sin\frac{\vartheta}{2}\,. \tag{1.10}$$

We prove that $S\circ\Phi$ can be replaced in (1.9) by $\widetilde S$ (with the same stress intensity factor), because their difference is H^2. Using such a function we are not required to compute the diffeomorphism Φ of Section 1.1.

Proposition 1.6. *For every $\Omega' \subset\subset \Omega$ we have*

$$\widetilde S - S\circ\Phi \in H^2(\Omega'\backslash\Gamma_{s_0})\,. \tag{1.11}$$

Proof. As $S, \widetilde{S} \in H^1(\Omega' \backslash \Gamma_{s_0})$, we have only to check the summability of the difference between the second derivatives in a neighbourhood of 0. We have

$$\mathrm{D}_{ij}(S \circ \Phi) - \mathrm{D}_{ij}\widetilde{S} = \mathrm{D}_{hk}S(\Phi)\,\mathrm{D}_i\Phi^h\,\mathrm{D}_j\Phi^k + \mathrm{D}_k S(\Phi)\,\mathrm{D}_{ij}\Phi^k - \delta_i^h\,\delta_j^k\,\mathrm{D}_{hk}\widetilde{S}$$

where δ is the Kronecker symbol. Since $\mathrm{D}_k S(\Phi) \in L^2$ and $\mathrm{D}_{ij}\Phi^k \in L^\infty$, it is enough to estimate

$$\begin{aligned} \left|\mathrm{D}_{hk}S(\Phi)\,\mathrm{D}_i\Phi^h\,\mathrm{D}_j\Phi^k - \delta_i^h\,\delta_j^k\,\mathrm{D}_{hk}\widetilde{S}\right| \le \left|\mathrm{D}_{hk}\widetilde{S}\right| \left|\mathrm{D}_i\Phi^h\,\mathrm{D}_j\Phi^k - \delta_i^h\,\delta_j^k\right| + \\ + \left|\mathrm{D}_{hk}S(\Phi) - \mathrm{D}_{hk}\widetilde{S}\right| \left|\mathrm{D}_i\Phi^h\,\mathrm{D}_j\Phi^k\right| \,. \end{aligned}$$

As for the first summand, we have $\left|\mathrm{D}_i\Phi^h - \delta_i^h\right| \le L\,|x|$, where L is the Lipschitz constant of the derivatives of Φ, so

$$\left|\mathrm{D}_i\Phi^h\mathrm{D}_j\Phi^k - \delta_i^h\delta_j^k\right| \le \left|\mathrm{D}_i\Phi^h - \delta_i^h\right| \left|\mathrm{D}_j\Phi^h\right| + \delta_i^h \left|\mathrm{D}_j\Phi^k - \delta_j^k\right| \le C\,|x|$$

for some $C > 0$, whence

$$\left|\mathrm{D}_{hk}\widetilde{S}\right| \left|\mathrm{D}_i\Phi^h\mathrm{D}_j\Phi^k - \delta_i^h\delta_j^k\right| \le C\,|x|^{-\frac{1}{2}} \,.$$

To estimate the second summand, we fix x such that $x \neq \Phi(x)$ (otherwise, the term is null); in particular, $x \neq 0$. We consider the segment $[x, \Phi(x)]$ between x and $\Phi(x)$; let d be its distance from 0.

Lemma 1.7. *If $|x|$ is sufficiently small, we have $d \ge \frac{1}{2}\,|x|$.*

Proof. As $\Phi \in C^{1,1}$, $\Phi(0) = 0$, and $\nabla\Phi(0) = I$, we get $|x - \Phi(x)| \le \frac{L}{2}\,|x|^2$ (where L is the Lipschitz constant of the derivatives of Φ). Let $y \in [x, \Phi(x)]$ be the point of minimal distance from 0; we have

$$|x| \le |y| + |x - y| \le |y| + |x - \Phi(x)| \le |y| + \frac{L}{2}\,|x|^2 \,,$$

so $d \ge |x| - \frac{L}{2}\,|x|^2$. If $|x| \le \frac{1}{L}$, we obtain $|x| - \frac{L}{2}\,|x|^2 \ge \frac{1}{2}\,|x|$. □

We compare S and $\widetilde{S}$, which are two different determinations of the multifunction $z \mapsto \operatorname{Im} z^{\frac{1}{2}}$. In order to avoid some problems related to the discontinuities of S and $\widetilde{S}$ (see Remark 1.8), we fix two other determinations S^+ and S^- such that their common cut does not meet the segment $[x, \Phi(x)]$ (which passes far from 0 by the lemma): S^+ is chosen to be positive along $\{x_1 \le 0,\, x_2 = 0\}$, S^- negative. Because of the definition of Φ we have

$$\widetilde{S}(x) = S^\pm(x) \quad \text{if and only if} \quad S(\Phi(x)) = S^\pm(\Phi(x))\,,$$

so we can replace both S and $\widetilde{S}$ writing either S^+ or S^-.

By the Mean Value Theorem we find $\overline{x} \in [x, \Phi(x)]$ such that

$$\left|\mathrm{D}_{hk}S^\pm(\Phi(x)) - \mathrm{D}_{hk}S^\pm(x)\right| \le \left|\nabla\mathrm{D}_{hk}S^\pm(\overline{x})\right| |x - \Phi(x)| \;;$$

finally we control the third derivatives with $|\overline{x}|^{-\frac{5}{2}} \le d^{-\frac{5}{2}} \le C\,|x|^{-\frac{5}{2}}$ (by the lemma) and $|x - \Phi(x)|$ with $\frac{L}{2}\,|x|^2$, so the second summand is bounded by $C\,|x|^{-\frac{1}{2}}$ (for some $C > 0$). The proof is concluded. □

Remark 1.8. In the previous proof, we apply the Mean Value Theorem to the determinations of the multifunction $z \mapsto \operatorname{Im} z^{\frac{1}{2}}$, along the segment $[x, \Phi(x)]$. If we considered $\widetilde{S}$, this argument could fail because its discontinuity line Γ_{s_0} could intersect $[x, \Phi(x)]$. This is why we have to pass to the determinations S^+ and S^-, continuous along $[x, \Phi(x)]$.

The next theorem follows as a corollary.

Theorem 1.9. *Given $f \in L^2(\Omega \backslash \Gamma_{s_0})$, let $u \in H^1(\Omega \backslash \Gamma_{s_0})$ be the variational solution of the problem* (1.2). *Let*

$$\widetilde{S} := \rho^{\frac{1}{2}} \sin \frac{\vartheta}{2} ,$$

where ρ and ϑ are polar coordinates such that ϑ is continuous in $\Omega \backslash \Gamma_{s_0}$. Then there exists a unique constant K such that

$$u - K\ \widetilde{S} \in H^2(\Omega' \backslash \Gamma_{s_0}) \tag{1.12}$$

for every $\Omega' \subset\subset \Omega$.

2. Energy release rate and stress intensity factor

2.1. **Computing the energy release rate in terms of the stress intensity factor.** In this section we study the connection between the stress intensity factor and the energy release rate, that is the opposite of the derivative of the energy with respect to crack length. The case of the Poisson equation in a domain with a rectilinear cut was treated in [6] and [10, Section 6.4]; our result is an extension to curvilinear cuts of class $C^{1,1}$ and operators with Lipschitz coefficients.

In the geometrical setting of Section 1, we define for $s \in (0, l]$ the increasing family of cracks

$$\Gamma_s := \{\gamma(t) \colon 0 \le t \le s\} ,$$

the cut domains

$$\Omega_s := \Omega \backslash \Gamma_s ,$$

and the spaces of test functions

$$H_s := \{w \in H^1(\Omega_s) \colon \gamma_\Omega w = 0 \text{ in } \partial\Omega\} .$$

We consider the variational problem for the operator $\mathcal{A}$ defined in (1.1)

$$\begin{cases} u_s - \psi \in H_s , \\ \int_{\Omega_s} \nabla u_s(x)\, A(x) (\nabla w(x))^{\mathrm{T}}\, \mathrm{d}x = \int_{\Omega_s} f(x)\, w(x)\, \mathrm{d}x \quad \text{for every } w \in H_s , \end{cases} \tag{2.1}$$

where we assigned a force $f \in L^2(\Omega_{s_0})$ and a boundary datum $\psi \in H^1(\Omega_{s_0})$, which is identically zero in a neighbourhood of $\gamma(s_0) = 0$.

By Theorem 1.4, the variational solution $u := u_{s_0}$ for $s = s_0$ can be written as

$$u = u^R + K\ S \circ \Phi , \tag{2.2}$$

where $u^R \in H^2(\Omega' \backslash \Gamma_{s_0})$ for every open set $\Omega' \subset\subset \Omega$, $K \in \mathbb{R}$, $S = r^{\frac{1}{2}} \sin \frac{\theta}{2}$ (in polar coordinates around 0, with $\theta = 0$ on the semiaxis determined by $\dot\gamma(s_0)$), and Φ is the change of variable of Section 1.1.

Following the steps of [10, Theorem 6.4.1], we compute the derivative at s_0 of the elastic energy

$$\mathcal{E}(s) := \frac{1}{2} \int_{\Omega_s} \nabla u_s(x)\, A(x) (\nabla u_s(x))^{\mathrm{T}}\, \mathrm{d}x - \int_{\Omega_s} f(x)\, u_s(x)\, \mathrm{d}x .$$

As before, we assume that $A(0) = I$ (the general situation can be recovered through an affine change of variables).

Theorem 2.1. *$\mathcal{E}$ is differentiable at s_0 and*

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}s}(s_0) = -\frac{\pi}{4} K^2 . \tag{2.3}$$

Proof. At a first stage we suppose that $\Gamma = \overline{\Omega} \cap \{x_2 = 0\}$ and the conormal unit vector coincides with e_2 on Γ. In this first part of the proof we assume also that the force is null in a neighbourhood of 0. This case can be treated with standard arguments [10]; however, we must be careful in some passages because of the weak regularity assumptions on the coefficients. Hence, we present the details of the computation for the reader's convenience.

Fixed $\delta > 0$ small enough, for $s \in (s_0 - \delta, s_0 + \delta)$ we consider a family of perturbations of the identical diffeomorphism

$$F_s := I + s\,V\,,$$

where V is a smooth vector field with compact support such that $V^1 \equiv 1$ around 0, $V^2 \equiv 0$, and

$$\operatorname{supp}\,\psi \cap \operatorname{supp}\,V = \emptyset = \operatorname{supp}\,f \cap \operatorname{supp}\,V\,. \tag{2.4}$$

We change variables through F_s and set $U_s := u_s \circ F_s$. By (2.1), for every $w \in H_s$ we have

$$\begin{aligned}\int_{\Omega_{s_0}} fW\,\mathrm{d}x &= \int_{\Omega_s} fw\,\mathrm{d}x = \int_{\Omega_s} \nabla u_s\, A(\nabla w)^{\mathrm{T}}\,\mathrm{d}x \\ &= \int_{\Omega_{s_0}} \nabla U_s\left[\nabla F_s^{-1}(F_s)\, A(F_s)\,(\nabla F_s^{-1}(F_s))^{\mathrm{T}}\,\det\nabla F_s\right](\nabla W)^{\mathrm{T}}\,\mathrm{d}x\,,\end{aligned}$$

with $W := w \circ F_s$. Hence we have recast (2.1) into an integral equation over a fixed domain, with operator

$$C(x,s) := \nabla F_s^{-1}(F_s(x))\, A(F_s(x))\,(\nabla F_s^{-1}(F_s(x)))^{\mathrm{T}}\,\det\nabla F_s(x)\,.$$

We need some facts about elliptic operators depending on a parameter.

Remark 2.2. Let us denote by $s \mapsto c_{ij}(\cdot,s) \in L^\infty(\Omega_{s_0})$ the coefficients of $C(\cdot,s)$, which satisfy:

- $s \mapsto c_{ij}(x,s)$ is continuous for a.e. $x \in \Omega_{s_0}$,
- $\sum_{ij} c_{ij}(x,s)\,\xi^i\xi^j \geq C_0\,|\xi|^2$ for every $\xi \in \mathbb{R}^2$, for every s, and a.e. x,
- $|c_{ij}(x,s)| \leq C_1$ for every s and a.e. x,

where $C_0, C_1 > 0$ are two constants. For the sake of generality, in this remark we consider a force term $s \mapsto f_s \in H'_{s_0}$, with continuous dependence on s, in the dual space H'_{s_0} of H_{s_0}, endowed with the usual norm. In a neighbourhood of s_0 we define the operator

$$T\colon s \mapsto U_s\,,$$

where U_s solves

$$\begin{cases} U_s - \psi \in H_{s_0}\,, \\ -\sum_{ij} \mathrm{D}_i(c_{ij}(x,s)\,\mathrm{D}_j U_s) = f_s \quad \text{in } H'_{s_0}\,. \end{cases} \tag{2.5}$$

Then T is continuous.

Indeed, given a sequence $s_n \to s$, the functions $U_{s_n} - \psi$ are uniformly bounded in H_{s_0}, by the uniform ellipticity of the coefficients and the Poincaré inequality. Hence, up to a subsequence, U_{s_n} converges to some u^* weakly in $H^1(\Omega_{s_0})$. Using the pointwise convergence and the uniform bound on the coefficients c_{ij}, one can conclude by uniqueness that $u^* = U_s$ and thus the whole sequence converges. Moreover, using again the uniform ellipticity of the coefficients and the Poincaré inequality, it is possible to show that the convergence of U_{s_n} to U_s is also strong. This proves the continuity of T.

With a similar argument, one can see the existence of the partial derivative $\dot{U} := \mathrm{D}_s U_{s_0}$ at s_0. Indeed, the coefficients satisfy also:

- $s \mapsto c_{ij}(x,s)$ is differentiable at s_0 for a.e. $x \in \Omega_{s_0}$,

- there exists $C_2 > 0$ such that $|\mathrm{D}_s c_{ij}(x, s_0)| \le C_2$ for a.e. x.

We will denote by $\mathrm{D}_s C(x, s_0)$ the matrix whose components are $\mathrm{D}_s c_{ij}(x, s_0)$. Let us assume also that $s \mapsto f_s$ is differentiable in H'_{s_0}.

By (2.5) we have the following equation for the difference quotient $\frac{U_s - u}{s - s_0}$:

$$\begin{aligned}\sum_{i,j=1}^{2} \int_{\Omega_{s_0}} c_{ij}(x, s_0)\, \mathrm{D}_j \left(\frac{U_s - u}{s - s_0} \right) \mathrm{D}_i W \, \mathrm{d}x = \\ = \left\langle \frac{f_s - f_{s_0}}{s - s_0}, W \right\rangle - \sum_{i,j=1}^{2} \int_{\Omega_{s_0}} \frac{c_{ij}(x, s) - c_{ij}(x, s_0)}{s - s_0} \mathrm{D}_j U_s \, \mathrm{D}_i W \, \mathrm{d}x\end{aligned} \tag{2.6}$$

for every $W \in H_{s_0}$. We can apply the previous argument, since the coefficients of the left-hand side have not been changed, while the force in the right-hand side is continuous. Hence, $\frac{U_s - u}{s - s_0}$ converges in $H^1(\Omega_{s_0})$ to a function $\dot{U}$ that solves the equation obtained by deriving formally (2.5):

$$\int_{\Omega_{s_0}} \nabla \dot{U}(x)\, C(x, s_0) (\nabla W(x))^{\mathrm{T}} \, \mathrm{d}x = \langle \mathrm{D}_s f_{s_0}, W \rangle - \int_{\Omega_{s_0}} \nabla u(x)\, \mathrm{D}_s C(x, s_0) (\nabla W(x))^{\mathrm{T}} \, \mathrm{d}x\,. \tag{2.7}$$

Here, the right-hand side is the extension by continuity at $s = s_0$ of the force term of (2.6), while $\langle \cdot, \cdot \rangle$ denotes the duality between H'_{s_0} and H_{s_0}. In particular, we have strong convergence for the difference quotients:

$$\frac{U_s - u}{s - s_0} \to \dot{U} \quad \text{in } H^1(\Omega_{s_0}) \quad \text{as } s \to s_0\,.$$

We shall apply these general facts to our case, where the force term f_s is actually independent of s since f and V have disjoint supports.

Since F_s is regular and the coefficients a_{ij} are Lipschitz continuous, the map $s \mapsto C(x, s)$ is continuous. Moreover, the derivative $\mathrm{D}_s C(x, s_0)$ exists for a.e. $x \in \Omega$ and is bounded. A direct computation gives

$$\mathrm{D}_s C(x, s_0) = -\nabla V A - A (\nabla V)^{\mathrm{T}} + A \operatorname{div} V + \mathrm{D}_1 A\, V^1\,,$$

where $\mathrm{D}_1 A$ indicates the matrix $(\mathrm{D}_1 a_{ij})_{ij}$. Then by Remark 2.2 the map $s \mapsto U_s$ has a derivative $\dot{U}$ at s_0. Since f and V have disjoint supports, (2.7) reads

$$\begin{aligned}\int_{\Omega_{s_0}} \nabla \dot{U}\, A (\nabla W)^{\mathrm{T}} \, \mathrm{d}x = \int_{\Omega_{s_0}} \big[(\nabla u\, \nabla V) A (\nabla W)^{\mathrm{T}} + \nabla u\, A (\nabla W\, \nabla V)^{\mathrm{T}} \\ - \nabla u\, A (\nabla W)^{\mathrm{T}} \operatorname{div} V - \nabla u\, \mathrm{D}_1 A (\nabla W)^{\mathrm{T}} V^1 \big] \, \mathrm{d}x\end{aligned}$$

for every $W \in H_{s_0}$.

Using $u_s - \psi$ as test function and recalling (2.4), we have

$$\begin{aligned}\mathcal{E}(s) &= \frac{1}{2} \int_{\Omega_s} \nabla u_s\, A (\nabla \psi)^{\mathrm{T}} \, \mathrm{d}x - \frac{1}{2} \int_{\Omega_s} f (u_s + \psi) \, \mathrm{d}x \\ &= \frac{1}{2} \int_{\Omega_{s_0}} \nabla U_s\, A (\nabla \psi)^{\mathrm{T}} \, \mathrm{d}x - \frac{1}{2} \int_{\Omega_{s_0}} f U_s \, \mathrm{d}x - \frac{1}{2} \int_{\Omega_{s_0}} f \psi \, \mathrm{d}x\,.\end{aligned}$$

Therefore, using $\dot U$ and $u-\psi$ as test functions, we obtain that $\mathcal{E}$ is differentiable in s_0 with derivative given by

$$\begin{aligned}\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}s}(s_0) =&\frac{1}{2}\int_{\Omega_{s_0}} \nabla\dot U\, A(\nabla\psi)^{\mathrm{T}}\,\mathrm{d}x - \frac{1}{2}\int_{\Omega_{s_0}} f\dot U\,\mathrm{d}x = \frac{1}{2}\int_{\Omega_{s_0}} \nabla\dot U\, A\,\nabla(\psi-u)^{\mathrm{T}}\,\mathrm{d}x\\ =&-\int_{\Omega_{s_0}} (\nabla u\nabla V)A(\nabla u)^{\mathrm{T}}\,\mathrm{d}x + \frac{1}{2}\int_{\Omega_{s_0}} \nabla u\, A(\nabla u)^{\mathrm{T}}\,\mathrm{div}\, V\,\mathrm{d}x\\ &+\frac{1}{2}\int_{\Omega_{s_0}} \nabla u\,\mathrm{D}_1 A(\nabla u)^{\mathrm{T}} V^1\,\mathrm{d}x\,,\end{aligned}$$

since the terms containing the derivatives of ψ are null by (2.4). An explicit componentwise computation gives

$$\begin{aligned}\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}s}(s_0) =& -\int_{\Omega_{s_0}} \mathrm{D}_1 u\,(a_{11}\mathrm{D}_1 u + a_{12}\mathrm{D}_2 u)\,\mathrm{D}_1 V^1\,\mathrm{d}x - \int_{\Omega_{s_0}} \mathrm{D}_1 u\,(a_{12}\mathrm{D}_1 u + a_{22}\mathrm{D}_1 u)\,\mathrm{D}_2 V^1\,\mathrm{d}x\\ &+\frac{1}{2}\int_{\Omega_{s_0}}\sum_{i,j=1}^{2} a_{ij}\,\mathrm{D}_j u\,\mathrm{D}_i u\,\mathrm{D}_1 V^1\,\mathrm{d}x + \frac{1}{2}\int_{\Omega_{s_0}}\sum_{i,j=1}^{2}\mathrm{D}_1 a_{ij}\,\mathrm{D}_j u\,\mathrm{D}_i u\,V^1\,\mathrm{d}x\\ =& -\int_{\Omega_{s_0}}\left[\mathrm{D}_1 V^1\frac{a_{11}(\mathrm{D}_1 u)^2 - a_{22}(\mathrm{D}_2 u)^2}{2} + \mathrm{D}_2 V^1\left(a_{12}(\mathrm{D}_1 u)^2 + a_{22}\mathrm{D}_1 u\,\mathrm{D}_2 u\right)\right]\mathrm{d}x\\ &+\frac{1}{2}\int_{\Omega_{s_0}} V^1\sum_{i,j=1}^{2}\mathrm{D}_1 a_{ij}\,\mathrm{D}_j u\,\mathrm{D}_i u\,\mathrm{d}x\,.\end{aligned}$$

As usual in this kind of computation [10], we first integrate on the subset $\Omega^\varepsilon_{s_0} := \Omega_{s_0}\backslash\overline{B_\varepsilon(0)}$, where ε is chosen so that $V^1\equiv 1$ in $B_\varepsilon(0)$, and then we pass to the limit as $\varepsilon\to 0$. We integrate by parts the first two summands, taking into account the last term, containing the derivatives of a_{ij}. We obtain as volume integral

$$\int_{\Omega^\varepsilon_{s_0}} V^1\,\mathrm{D}_1 u\sum_{i,j=1}^{2}\mathrm{D}_i\left(a_{ij}\mathrm{D}_j u\right)\,\mathrm{d}x = 0\,,$$

null because of (2.4). The contribution of $\partial\Omega$ is null, too, since V has compact support, while on the cut we have $\nu^1 = 0$ and $(a_{12}\mathrm{D}_1 u + a_{22}\mathrm{D}_2 u)\nu^2 = 0$ by the Neumann condition (here, ν denotes the normal to the cut). The only positive term is the one in ∂B_ε, where $V^1\equiv 1$: we obtain

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}s}(s_0) = \lim_{\varepsilon\to 0}\int_{\partial B_\varepsilon}\left[\frac{a_{11}(\mathrm{D}_1 u)^2 - a_{22}(\mathrm{D}_2 u)^2}{2}\nu^1 + \mathrm{D}_1 u\,(a_{12}\mathrm{D}_1 u + a_{22}\mathrm{D}_2 u)\,\nu^2\right]\mathrm{d}\mathcal{H}^1\,,$$

where $(-\nu^1,-\nu^2) := (-\cos\theta,-\sin\theta)$ is the interior normal vector to B_ε.

Recalling (2.2), we get

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}s}(s_0) = \lim_{\varepsilon\to 0}\left(a_\varepsilon + b_\varepsilon + c_\varepsilon\right)\,,$$

where the first summand contains only quadratic terms in the derivatives of S,

$$a_\varepsilon = K^2\int_0^{2\pi}\left(\frac{a_{11}(\mathrm{D}_1 S)^2 - a_{22}(\mathrm{D}_2 S)^2}{2}\cos\theta + \mathrm{D}_1 S\,(a_{12}\mathrm{D}_1 S + a_{22}\mathrm{D}_2 S)\sin\theta\right)\varepsilon\,\mathrm{d}\theta\,,$$

the second one contains mixed terms,

$$\begin{aligned}b_\varepsilon = K\int_{\partial B_\varepsilon}\Big[&\left(a_{11}\mathrm{D}_1 u^R\,\mathrm{D}_1 S - a_{22}\mathrm{D}_2 u^R\,\mathrm{D}_2 S\right)\cos\theta\\ &+\left(2a_{12}\mathrm{D}_1 u^R\,\mathrm{D}_1 S + a_{22}\mathrm{D}_1 u^R\,\mathrm{D}_2 S + a_{22}\mathrm{D}_2 u^R\,\mathrm{D}_1 S\right)\sin\theta\Big]\,\mathrm{d}\mathcal{H}^1\,,\end{aligned}$$

and the third is given by the derivatives of u^R,

$$c_\varepsilon = \int_{\partial B_\varepsilon} \left[\frac{a_{11}(\mathrm{D}_1 u^R)^2 - a_{22}(\mathrm{D}_2 u^R)^2}{2} \cos\theta + \mathrm{D}_1 u^R \left(a_{12}\mathrm{D}_1 u^R + a_{22}\mathrm{D}_2 u^R \right) \sin\theta \right] \mathrm{d}\mathcal{H}^1 .$$

Now we show that b_ε and c_ε vanish as $\varepsilon \to 0$, so the only term for the derivative of the energy is a_ε. As for b_ε, since $|\mathrm{D}_k S| \le \frac{1}{2}\varepsilon^{-\frac{1}{2}}$ in ∂B_ε, using the Hölder inequality in L^2 we get

$$|b_\varepsilon| \le C_1 \varepsilon^{-\frac{1}{2}} \int_{\partial B_\varepsilon} \left|\nabla u^R\right| \mathrm{d}\mathcal{H}^1(x) \le C_1 \varepsilon^{-\frac{1}{2}} \left\|\nabla u^R\right\|_{L^2(\partial B_\varepsilon)} |\partial B_\varepsilon|^{\frac{1}{2}} = C_2 \left\|\nabla u^R\right\|_{L^2(\partial B_\varepsilon)} ,$$

where $C_1, C_2 > 0$. On the other hand, with the Hölder inequality in L^1 we obtain

$$|c_\varepsilon| \le C_3 \int_{\partial B_\varepsilon} \left|\nabla u^R\right|^2 \mathrm{d}\mathcal{H}^1(x) + C_3 \int_{\partial B_\varepsilon} \left|\mathrm{D}_1 u^R\right| \left|\mathrm{D}_2 u^R\right| \mathrm{d}\mathcal{H}^1(x) \le C_4 \left\|\nabla u^R\right\|^2_{L^2(\partial B_\varepsilon)} ,$$

where $C_3, C_4 > 0$. Hence, we are left to prove that $\left\|\nabla u^R\right\|_{L^2(\partial B_\varepsilon)} \to 0$ as $\varepsilon \to 0$.

We employ the change of variables $y := \frac{x}{\varepsilon}$ and define $v(y) := u^R(\varepsilon y)$; thanks to the continuity of the trace operator, we have for $C > 0$

$$\begin{aligned}
\int_{\partial B_\varepsilon} \left|\nabla u^R\right|^2 \mathrm{d}\mathcal{H}^1(x) &= \frac{1}{\varepsilon} \int_{\partial B_1} |\nabla v|^2 \, \mathrm{d}\mathcal{H}^1(y) \\
&\le \frac{C}{\varepsilon} \int_{B_1} \left|\nabla^2 v\right|^2 \mathrm{d}y + \frac{C}{\varepsilon} \int_{B_1} |\nabla v|^2 \, \mathrm{d}y \\
&= C\varepsilon \int_{B_\varepsilon} \left|\nabla^2 u^R\right|^2 \mathrm{d}x + \frac{C}{\varepsilon} \int_{B_\varepsilon} \left|\nabla u^R\right|^2 \mathrm{d}x .
\end{aligned}$$

The Hölder inequality in $L^{\frac{p}{2}}$, with $p > 1$, gives

$$\int_{\partial B_\varepsilon} \left|\nabla u^R\right|^2 \mathrm{d}\mathcal{H}^1(x) \le C\varepsilon \left\|\nabla^2 u^R\right\|^2_{L^2(B_\varepsilon)} + \frac{C}{\varepsilon} \left\|\nabla u^R\right\|^2_{L^p(B_\varepsilon)} |B_\varepsilon|^{1-\frac{2}{p}} .$$

For $p = 4$, using the absolute continuity of integral we get for $C' > 0$

$$\int_{\partial B_\varepsilon} \left|\nabla u^R\right|^2 \mathrm{d}\mathcal{H}^1(x) \le C\varepsilon \left\|\nabla^2 u^R\right\|^2_{L^2(B_\varepsilon)} + C' \left\|\nabla u^R\right\|^2_{L^4(B_\varepsilon)} \to 0 \quad \text{as } \varepsilon \to 0 .$$

Passing to the limit as $\varepsilon \to 0$ and recalling that $A(0) = I$, through a direct computation we find

$$\lim_{\varepsilon \to 0} a_\varepsilon = -\frac{\pi}{4} K^2 ,$$

so we conclude the proof in the case that $\Gamma = \overline{\Omega} \cap \{x_2 = 0\}$, the conormal unit vector coincides with e_2 on Γ, and the force is null in a neighbourhood of 0.

If the domain and the operator have the general form, we deduce the result by applying the diffeomorphism Φ of Section 1.1. After the change of variables it is enough to choose $V = (V^1, 0)$ having support in the neighbourhood of the origin where the crack is rectified and the conormal unit vector coincides with the normal: then one repeats the computations above.

Finally, the case of a general force is treated by approximation in L^2 with a sequence of forces whose supports are disjoint from 0: indeed, the stress intensity factor is continuous with respect to the convergence of the force in L^2 (see Remark 1.5). For the general form of the derivative in this case, see also (2.9). □

Remark 2.3. The previous proof was done prescribing a priori the crack path. However, assume that only Γ_{s_0} is given, while Γ_s and $\widetilde{\Gamma}_s$ are two increasing families of simple curves of class $C^{1,1}$, both containing Γ_{s_0} for $s > s_0$. Arguing as before, we find for Γ_s and $\widetilde{\Gamma}_s$

at $s = s_0$ the same energy release rate, which depends only on Γ_{s_0}. Therefore, we have a notion of energy derivative common to the whole class of $C^{1,1}$ continuations of Γ_{s_0}: indeed, the energy release rate is a volume integral on the domain with fixed crack Γ_{s_0}, as we will explain in the following section.

2.2. **The energy release rate as integral invariant.** The previous theorem suggests that the energy release rate can be characterized as a volume integral of a quantity depending on the elastic coefficients and on the deformation gradient. We show this characterization considering the problem

$$\begin{cases} u - \psi \in H_{s_0}\,, \\ \int_{\Omega_{s_0}} \nabla u(x)\, A(x) (\nabla w(x))^{\mathrm{T}}\, \mathrm{d}x = \int_{\Omega_{s_0}} f(x)\, w(x)\, \mathrm{d}x \quad \text{for every } w \in H_{s_0}\,, \end{cases} \tag{2.8}$$

where $\Omega_{s_0} := \Omega \backslash \Gamma_{s_0}$, H_{s_0}, A, f, and ψ are as before; we recall that $A(0) = I$. Let u be its variational solution: by Theorem 1.9, u can be written as

$$u = u^R + K\, \widetilde{S}\,,$$

with $u^R \in H^2(\Omega' \backslash \Gamma_{s_0})$ for every open set $\Omega' \subset\subset \Omega$, $K \in \mathbb{R}$, and $\widetilde{S} = \rho^{\frac{1}{2}} \sin \frac{\vartheta}{2}$, where ρ and ϑ are polar coordinates such that ϑ is continuous in $\Omega \backslash \Gamma_{s_0}$ and $\vartheta = 0$ on the semiaxis determined by $\dot\gamma(s_0)$.

Proposition 2.4. *Let V be a vector field of class $C^{0,1}$ with compact support in Ω. Assume that on Γ we have $V(\gamma(s)) = \zeta(\gamma(s))\, \dot\gamma(s)$, where ζ is a cut-off function, equal to one in a neighbourhood of 0. Then*

$$\begin{aligned} \frac{\pi}{4} K^2 = & \int_{\Omega_{s_0}} \frac{a_{11}(\mathrm{D}_1 u)^2 - a_{22}(\mathrm{D}_2 u)^2}{2} (\mathrm{D}_1 V^1 - \mathrm{D}_2 V^2)\, \mathrm{d}x \\ & + \int_{\Omega_{s_0}} \big[a_{12}(\mathrm{D}_1 u)^2 + a_{22} \mathrm{D}_1 u\, \mathrm{D}_2 u\big]\, \mathrm{D}_2 V^1\, \mathrm{d}x - \frac{1}{2} \int_{\Omega_{s_0}} V^1 \sum_{i,j=1}^{2} \mathrm{D}_1 a_{ij}\, \mathrm{D}_j u\, \mathrm{D}_i u\, \mathrm{d}x \\ & + \int_{\Omega_{s_0}} \big[a_{12}(\mathrm{D}_2 u)^2 + a_{11} \mathrm{D}_1 u\, \mathrm{D}_2 u\big]\, \mathrm{D}_1 V^2\, \mathrm{d}x - \frac{1}{2} \int_{\Omega_{s_0}} V^2 \sum_{i,j=1}^{2} \mathrm{D}_2 a_{ij}\, \mathrm{D}_j u\, \mathrm{D}_i u\, \mathrm{d}x \\ & - \int_{\Omega_{s_0}} (\mathrm{D}_1 u\, V^1 + \mathrm{D}_2 u\, V^2)\, f\, \mathrm{d}x\,. \end{aligned} \tag{2.9}$$

Proof. The computations done in the previous proof lead us to consider the following integral over $\Omega^\varepsilon_{s_0} := \Omega_{s_0} \backslash \overline{B_\varepsilon(0)}$ (where $\varepsilon > 0$):

$$\begin{aligned} \mathcal{I}_\varepsilon := & \int_{\Omega^\varepsilon_{s_0}} \frac{a_{11}(\mathrm{D}_1 u)^2 - a_{22}(\mathrm{D}_2 u)^2}{2} (\mathrm{D}_1 V^1 - \mathrm{D}_2 V^2)\, \mathrm{d}x \\ & + \int_{\Omega^\varepsilon_{s_0}} \big[a_{12}(\mathrm{D}_1 u)^2 + a_{22} \mathrm{D}_1 u\, \mathrm{D}_2 u\big]\, \mathrm{D}_2 V^1\, \mathrm{d}x - \frac{1}{2} \int_{\Omega^\varepsilon_{s_0}} V^1 \sum_{i,j=1}^{2} \mathrm{D}_1 a_{ij}\, \mathrm{D}_j u\, \mathrm{D}_i u\, \mathrm{d}x \\ & + \int_{\Omega^\varepsilon_{s_0}} \big[a_{12}(\mathrm{D}_2 u)^2 + a_{11} \mathrm{D}_1 u\, \mathrm{D}_2 u\big]\, \mathrm{D}_1 V^2\, \mathrm{d}x - \frac{1}{2} \int_{\Omega^\varepsilon_{s_0}} V^2 \sum_{i,j=1}^{2} \mathrm{D}_2 a_{ij}\, \mathrm{D}_j u\, \mathrm{D}_i u\, \mathrm{d}x\,. \end{aligned}$$

This quantity can be rewritten as

$$\begin{aligned}\mathcal{I}_\varepsilon = &-\int_{\Omega^\varepsilon_{s_0}} (\mathrm{D}_1 u\, V^1 + \mathrm{D}_2 u\, V^2) \sum_{i,j=1}^{2} \mathrm{D}_i\,(a_{ij}\,\mathrm{D}_j u)\;\mathrm{d}x \\ &+\int_{\partial\Omega^\varepsilon_{s_0}} \left[\frac{a_{11}(\mathrm{D}_1 u)^2 - a_{22}(\mathrm{D}_2 u)^2}{2}\,\nu^1 + \big(a_{12}(\mathrm{D}_1 u)^2 + a_{22}\mathrm{D}_1 u\,\mathrm{D}_2 u\big)\,\nu^2\right] V^1\,\mathrm{d}\mathcal{H}^1 \\ &-\int_{\partial\Omega^\varepsilon_{s_0}} \left[\frac{a_{11}(\mathrm{D}_1 u)^2 - a_{22}(\mathrm{D}_2 u)^2}{2}\,\nu^2 - \big(a_{12}(\mathrm{D}_2 u)^2 + a_{11}\mathrm{D}_1 u\,\mathrm{D}_2 u\big)\,\nu^1\right] V^2\,\mathrm{d}\mathcal{H}^1\,,\end{aligned}$$

where ν denotes the exterior normal to $\Omega^\varepsilon_{s_0}$. This can be seen integrating by parts: indeed, the classical version of the Divergence Theorem can be applied to a sequence of regular vector fields approximating V uniformly with uniformly bounded derivatives. Finally, one recalls that $-\sum_{i,j=1}^2 \mathrm{D}_i\,(a_{ij}\,\mathrm{D}_j u) = f$ in $L^2(\Omega^\varepsilon_{s_0})$.

The boundary integral is made up of three terms, over $\partial\Omega$, Γ_{s_0}, and $\partial B_\varepsilon(0)$, respectively. The contribution of $\partial\Omega$ is zero because V has compact support; the same holds for the part on Γ_{s_0}, as one can easily check using the Neumann condition and the fact that V is tangent to Γ on Γ. Arguing as in the previous proof, we can compute the integral over $\partial B_\varepsilon(0)$ passing to the limit as $\varepsilon \to 0$. We deduce that

$$\lim_{\varepsilon\to 0} \mathcal{I}_\varepsilon = \int_{\Omega_{s_0}} (\mathrm{D}_1 u\, V^1 + \mathrm{D}_2 u\, V^2)\, f\,\mathrm{d}x + \frac{\pi}{4}\,K^2\,.$$

This concludes the proof. □

Remark 2.5. In the case where the elliptic operator is the Laplacian, formula (2.9) becomes

$$\begin{aligned}\frac{\pi}{4}\,K^2 = &\int_{\Omega_{s_0}} \left[\frac{(\mathrm{D}_1 u)^2 - (\mathrm{D}_2 u)^2}{2}(\mathrm{D}_1 V^1 - \mathrm{D}_2 V^2) + \mathrm{D}_1 u\,\mathrm{D}_2 u\,(\mathrm{D}_2 V^1 + \mathrm{D}_1 V^2)\right]\mathrm{d}x \\ &-\int_{\Omega_{s_0}} (\mathrm{D}_1 u\, V^1 + \mathrm{D}_2 u\, V^2)\, f\,\mathrm{d}x\,.\end{aligned}$$

Remark 2.6. Formula (2.9) is independent of the choice of the coordinate system. Indeed, let $W(x,\xi) := \frac{1}{2}\,\xi^{\mathrm{T}} A(x)\xi$ be the bulk energy density. Then (2.9) can be written in the following way:

$$\begin{aligned}\frac{\pi}{4}\,K^2 = &\int_{\Omega_{s_0}} \big[\nabla u(x)^{\mathrm{T}}\,\mathrm{D}_\xi W(x,\nabla u(x)) - W(x,\nabla u(x))\,I\big] : \nabla V(x)\,\mathrm{d}x \\ &-\int_{\Omega_{s_0}} \mathrm{D}_x W(x,\nabla u(x))\cdot V(x)\,\mathrm{d}x - \int_{\Omega_{s_0}} \nabla u(x)\cdot V(x)\,f(x)\,\mathrm{d}x\,,\end{aligned}$$

where the symbols $:$ and $\cdot$ denote the scalar products between matrices and between vectors, respectively. An analogous result is given in [13, Theorem 3.2] when the crack set is smooth and the bulk energy W is a convex function of ξ, independent of x.

The integrand $\nabla u(x)^{\mathrm{T}}\,\mathrm{D}_\xi W(x,\nabla u(x)) - W(x,\nabla u(x))\,I$ in the last equation is the Eshelby or Hamilton tensor. Hence, in Proposition 2.4 we recover the same formula for the derivative of the energy that can be obtained via the slightly different method of inner variations [7, Chapter 3, Section 1, Lemma 1].

2.3. **Continuity of the energy release rate with respect to the crack sets.** Thanks to Proposition 2.4, we are able to show the continuity of the energy release rate with respect to the Hausdorff convergence in a suitable class of admissible cracks. In this section we consider the equations of antiplane elasticity for a homogeneous material, so we set $A(x) = I$ for every $x \in \Omega$. For the sake of simplicity, we suppose that no external volume force is applied on the body.

We assume that an initial crack Γ_0 is present in the domain and we define a class of cracks all containing Γ_0. More precisely, we assume that Γ_0 is a closed arc of curve of class $C^{1,1}$, of length $l_0 > 0$, without self-intersections, contained in Ω except for the initial point, which belongs to $\partial\Omega$, and that $\Omega \backslash \Gamma_0$ is the union of two Lipschitz open sets.

For $\eta > 0$, we define $\mathcal{R}_\eta$ to be the set of all closed arcs of curve Γ of class $C^{1,1}$ in $\overline{\Omega}$, such that the following hold:

(a) $\Gamma \supset \Gamma_0$ and $\Gamma \backslash \Gamma_0 \subset\subset \Omega$;
(b) for every point $x \in \Gamma \backslash \Gamma_0$ there exist two open balls $C_1, C_2 \subset \Omega$ of radius η, such that $(C_1 \cup C_2) \cap (\Gamma \cup \partial\Omega) = \emptyset$ and $\overline{C}_1 \cap \overline{C}_2 = \{x\}$.

Since Γ_0 is of class $C^{1,1}$ we can fix $\eta > 0$ so small that the curvature of Γ_0 is controlled from above by $\frac{1}{\eta}$ at a.e. point and the class $\mathcal{R}_\eta$ is not empty. These technical requirements ensure for any curve $\Gamma \in \mathcal{R}_\eta$ that there are no self-intersections and the curvature is everywhere controlled from above; moreover, these features are stable under Hausdorff convergence (see Proposition 2.9).

Remark 2.7. Every $\Gamma \in \mathcal{R}_\eta$ has length larger than or equal to the length l_0 of Γ_0. Moreover, one can easily prove that under these assumptions there exist two quantities $L, D > 0$, depending only on η, Ω, and Γ_0, such that for every $\Gamma \in \mathcal{R}_\eta$

- $\mathcal{H}^1(\Gamma) \leq L$,
- $\operatorname{dist}(\Gamma \backslash \Gamma_0, \partial\Omega) \geq D$.

Notice that every C^1 curve satisfying (b) is actually of class $C^{1,1}$. Furthermore, condition (b) can be expressed in terms of the normal unit vector to Γ. Indeed, let $\gamma\colon [0,l] \to \overline{\Omega}$ be the arc-length parametrization of Γ; then, (b) is equivalent to requiring for every $s \in]l_0, l]$ that $B(\gamma(s) \pm \eta\, \nu_\Gamma(s), \eta) \cap (\Gamma \cup \partial\Omega) = \emptyset$, where $\nu_\Gamma(s)$ denotes the normal unit vector to Γ at $\gamma(s)$ and $B(x,\eta)$ is the open ball centred at x with radius η.

In the following proposition, we prove the sequential compactness of $\mathcal{R}_\eta$ under Hausdorff convergence; beforehands, let us recall the definition of this convergence.

Definition 2.8. Given two compact subsets $\Gamma_1, \Gamma_2 \subset \overline{\Omega}$, their *Hausdorff distance* is given by

$$d_H(\Gamma_1, \Gamma_2) := \max\left\{ \sup_{x \in \Gamma_1} \operatorname{dist}(x, \Gamma_2)\,,\, \sup_{x \in \Gamma_2} \operatorname{dist}(x, \Gamma_1) \right\},$$

with the conventions $d_H(x, \emptyset) = \operatorname{diam}(\Omega)$ and $\sup \emptyset = 0$. A sequence Γ_n of compact subsets of $\overline{\Omega}$ *converges to* Γ_∞ *in the Hausdorff metric* if $d_H(\Gamma_n, \Gamma_\infty) \to 0$.

Proposition 2.9. *Every sequence $\Gamma_n \in \mathcal{R}_\eta$ admits a limit $\Gamma_\infty \in \mathcal{R}_\eta$ in the Hausdorff metric (up to a subsequence).*

Proof. Let $\gamma_n\colon [0, l_n] \to \overline{\Omega}$ be the arc-length parametrization of Γ_n (with $\gamma_n([0,l_0]) = \Gamma_0$). We may define a regular parametrization $\widetilde{\gamma}_n\colon [0, L] \to \overline{\Omega}$ of Γ_n by setting $\widetilde{\gamma}_n(s) := \gamma_n(p_n s)$, where $p_n := l_n/L \in [l_0/L, 1]$. Using (b), we get a uniform control from above on the curvature of Γ_n, so $|\ddot{\gamma}_n(s)| \leq 1/\eta$ for a.e. s. This implies that the sequence $\widetilde{\gamma}_n$ is bounded in $W^{2,\infty}([0,L]; \mathbb{R}^2)$.

Thanks to the Banach-Alaoglu Theorem, we can find a subsequence, still denoted by $\widetilde{\gamma}_n$, converging weakly* in $W^{2,\infty}([0,L]; \mathbb{R}^2)$; we will denote by $\widetilde{\gamma}_\infty$ the continuous representative of its limit, which is an element of $C^{1,1}([0,L]; \overline{\Omega})$. Using the Fundamental Theorem of

Calculus, one can prove that $\dot{\widetilde{\gamma}}_n$ and $\widetilde{\gamma}_n$ converge pointwise to $\dot{\widetilde{\gamma}}_\infty$ and $\widetilde{\gamma}_\infty$, respectively. The pointwise convergence of $\widetilde{\gamma}_n$ implies in particular that Γ_n converges in the Hausdorff metric to the support Γ_∞ of $\widetilde{\gamma}_\infty$.

In order to show that $\Gamma_\infty \in \mathcal{R}_\eta$, we are left to check point (b) of the definition. By contradiction, assume that there exist a point $\widetilde{\gamma}_\infty(t_1)$, an open ball C_∞ of radius η tangent to Γ_∞ at $\widetilde{\gamma}_\infty(t_1)$ (see Remark 2.7), and a point $\widetilde{\gamma}_\infty(t_2)$ which is contained in C_∞. Thanks to the pointwise convergence of $\widetilde{\gamma}_n$ and of $\dot{\widetilde{\gamma}}_n$, we find a sequence of open balls C_n of radius η, tangent to Γ_n at $\widetilde{\gamma}_n(t_1)$, converging to C_∞ in the Hausdorff distance. Hence, there exists n such that $\widetilde{\gamma}_n(t_2)$ is contained in C_n: this violates (b) for Γ_n and concludes the proof. □

Remark 2.10. In the setting of the previous proof, $\widetilde{\gamma}_\infty$ is a regular parametrization of Γ_∞: indeed, $\left|\dot{\widetilde{\gamma}}_\infty\right| \geq l_0/L$; therefore, condition (b) ensures global injectivity. Hence,

$$\mathcal{H}^1(\Gamma_\infty) = \int_0^L \left|\dot{\widetilde{\gamma}}_\infty(s)\right| \, \mathrm{d}s$$

and $l_n \to l_\infty := \mathcal{H}^1(\Gamma_\infty)$. We set $\gamma_\infty(s) := \widetilde{\gamma}_\infty(s/p_\infty)$, where $p_\infty := l_\infty/L \in [l_0/L, 1]$. One can easily see that γ_∞ coincides with the arc-length parametrization of Γ_∞.

Given a sequence $\Gamma_n \in \mathcal{R}_\eta$ converging to a set $\Gamma_\infty \in \mathcal{R}_\eta$ in the Hausdorff metric, we consider the variational problems

$$\begin{cases} u_n - \psi \in H_n \,, \\ \int_{\Omega_n} \nabla u_n(x) (\nabla w(x))^{\mathrm{T}} \, \mathrm{d}x = 0 \quad \text{for every } w \in H_n \,, \end{cases} \tag{2.10}$$

where $\Omega_n := \Omega \backslash \Gamma_n$, $H_n := \{w \in H^1(\Omega_n) \colon \gamma_\Omega w = 0 \text{ in } \partial\Omega\}$, and $\psi \in H^1(\Omega\backslash\Gamma_0)$. Let u_n be the variational solution: by Theorem 1.9, u_n can be written as

$$u_n = u_n^R + K_n \, \widetilde{S}_n \,,$$

with $u_n^R \in H^2(\Omega'\backslash\Gamma_n)$ for every open set $\Omega' \subset\subset \Omega$, $K_n \in \mathbb{R}$, and $\widetilde{S}_n = \rho_n^{\frac{1}{2}} \sin\frac{\vartheta_n}{2}$, where ρ_n and ϑ_n are polar coordinates around the crack tip such that ϑ_n is continuous in Ω_n and $\vartheta_n = 0$ on the semiaxis determined by the tangent at the crack tip. Analogously we define $\Omega_\infty := \Omega\backslash\Gamma_\infty$, $H_\infty := \{w \in H^1(\Omega_\infty) \colon \gamma_\Omega w = 0 \text{ in } \partial\Omega\}$, the corresponding solution u_∞, and its stress intensity factor K_∞.

Henceforth, we extend the functions ∇u_n and ∇u_∞ to the whole of Ω by setting $\nabla u_n = 0$ in Γ_n and $\nabla u_\infty = 0$ in Γ_∞, respectively: this allows us to regard ∇u_n and ∇u_∞ as elements of $L^2(\Omega;\mathbb{R}^2)$ and to study their convergence.

Remark 2.11. It is possible to show that, if Γ_n converges to Γ_∞ in the Hausdorff metric, then the sequence ∇u_n converges to ∇u_∞ strongly in $L^2(\Omega;\mathbb{R}^2)$. This fact was proven in [5, Theorem 5.1], using a duality method due to [3], in the general case of closed cracks with bounded length and a bounded number of connected components. The arguments of [3, 5] can be simplified in our situation, since the curves in $\mathcal{R}_\eta$ are sufficiently regular and the Hausdorff convergence reduces to the weak* convergence of the parametrizations in $W^{2,\infty}$.

For the reader's convenience, we sketch here the proof of the strong convergence of the gradients. Since the functions ∇u_n are uniformly bounded in $L^2(\Omega;\mathbb{R}^2)$, it is possible to find a displacement $u^* \in H_\infty + \psi$ such that $\nabla u_n \rightharpoonup \nabla u^*$ weakly in $L^2(\Omega;\mathbb{R}^2)$ (up to subsequences). We will see that $u^* = u_\infty$ and that the whole sequence of gradients converges strongly.

For every u_n we consider a corresponding harmonic conjugate, i.e., a function $v_n \in H^1(\Omega)$ such that $\nabla v_n = R\nabla u_n$ a.e. in Ω, where R is the rotation defined by $R(x_1, x_2) := (-x_2, x_1)$.

We fix v_n by setting $\int_\Omega v_n \,\mathrm{d}x = 0$. By the Poincaré inequality we find a function v^* such that $v_n \rightharpoonup v^*$ weakly in $H^1(\Omega)$ and $\nabla v = R\nabla u^*$ a.e. in Ω.

Moreover, by the regularity of the curves and the properties of the traces of Sobolev functions, it is easy to check that each v_n is constant on Γ_n, so that also v^* is constant on Γ_∞. This is sufficient to conclude that u^* coincides with the solution u_∞ in Ω_∞. As a consequence, the whole sequence ∇u_n converges to ∇u_∞ weakly in $L^2(\Omega;\mathbb{R}^2)$.

Finally, using (2.10) with $w = u_n - \psi$, the analogous equation for u_∞ with $w = u_\infty - \psi$, and the weak convergence of ∇u_n, we obtain that $\|\nabla u_n\|_{L^2(\Omega;\mathbb{R}^2)} \to \|\nabla u_\infty\|_{L^2(\Omega;\mathbb{R}^2)}$. This implies that the convergence of ∇u_n to ∇u_∞ is also strong in $L^2(\Omega;\mathbb{R}^2)$.

In the following theorem we show that also the energy release rate is continuous with respect to the Hausdorff convergence.

Theorem 2.12. *Let Γ_n be a sequence in $\mathcal{R}_\eta$, converging in the Hausdorff metric to a curve $\Gamma_\infty \in \mathcal{R}_\eta$ and let u_n, u_∞ be the corresponding solutions to* (2.10). *Let K_n and K_∞ be the stress intensity factors of u_n and u_∞, respectively. Then $K_n^2 \to K_\infty^2$.*

Proof. We will deduce the continuity of the energy release rate employing the representation formula of Remark 2.5, so we have to construct for every n a vector field V_n of class $C^{0,1}$ with compact support in Ω, satisfying the hypotheses of Proposition 2.4.

Arguing as in Proposition 2.9 and in Remark 2.10, we define a sequence of parametrizations $\widetilde{\gamma}_n\colon [0,L] \to \overline{\Omega}$ of Γ_n, such that $\widetilde{\gamma}_n$ converges to a parametrization $\widetilde{\gamma}_\infty$ of Γ_∞ weakly* in $W^{2,\infty}([0,L];\mathbb{R}^2)$. We also extend each curve Γ_n adding a segment which follows the tangent direction to the tip $\widetilde{\gamma}_n(L)$; the same is done for Γ_∞. This allows us, using the Implicit Function Theorem, to find a neighbourhood ω of $\widetilde{\gamma}_\infty(L)$ where all these extended curves are graphs of some $C^{1,1}$ functions ϕ_n, ϕ_∞. We fix in ω two coordinate axes such that the extension of Γ_n is described by $(x_1, \phi_n(x_1))$ and the extension of Γ_∞ is described by $(x_1, \phi_\infty(x_1))$. Given a point $x = (x_1, x_2) \in \omega$, we define $\widetilde{V}_n(x) := (1, \dot{\phi}_n(x_1))$; then we set $V_n := \zeta \widetilde{V}_n$ on Ω, where ζ is a cut-off function supported in ω, equal to one near $\widetilde{\gamma}_\infty(L)$. Analogous definitions hold for $\widetilde{V}_\infty$ and V_∞. As $\widetilde{\gamma}_n$ converges to $\widetilde{\gamma}_\infty$ weakly* in $W^{2,\infty}([0,L];\mathbb{R}^2)$, we obtain that ∇V_n converges to ∇V_∞ weakly* in $L^\infty(\Omega;\mathbb{R}^4)$.

By Remark 2.5 we get

$$\frac{\pi}{4} K_n^2 = \int_{\Omega_n} \left[\frac{(\mathrm{D}_1 u_n)^2 - (\mathrm{D}_2 u_n)^2}{2} (\mathrm{D}_1 V_n^1 - \mathrm{D}_2 V_n^2) + \mathrm{D}_1 u_n \,\mathrm{D}_2 u_n \,(\mathrm{D}_2 V_n^1 + \mathrm{D}_1 V_n^2) \right] \mathrm{d}x$$

and the same for K_∞. As ∇V_n converges to ∇V_∞ weakly* in $L^\infty(\Omega;\mathbb{R}^4)$ and ∇u_n converges to ∇u_∞ strongly in $L^2(\Omega;\mathbb{R}^2)$ (see Remark 2.11), this formula shows that K_n^2 has limit K_∞^2. □

The continuity of the energy release rate under Hausdorff convergence of cracks will be used in a forthcoming paper [15] to study the evolution problem without prescribing a priori the crack path.

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