Weak and strong density results for the Dirichlet energy

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Let $\mathcal{Y}$ be a smooth oriented Riemannian manifold which is compact, connected, without boundary and with second homology group without torsion. In this paper we characterize the sequential weak closure of smooth graphs in $B^n \times \mathcal{Y}$ with equibounded Dirichlet energies, B^n being the unit ball in $\mathbb{R}^n$. More precisely, weak limits of graphs of smooth maps $u_k : B^n \to \mathcal{Y}$ with equibounded Dirichlet integral give rise to elements of the space $\mathrm{cart}^{2,1}(B^n \times \mathcal{Y})$, compare [5], [6], [7]. In this paper we prove that every element T in $\mathrm{cart}^{2,1}(B^n \times \mathcal{Y})$ is the weak limit of a sequence of smooth graphs $\{u_k\}$ with equibounded Dirichlet energies. Moreover, in dimension $n = 2$, we show that the sequence $\{u_k\}$ can be chosen in such a way that the energy of u_k converges to the energy of T.

1 Notation and preliminary results

In this section we recall some facts from the theory of Cartesian currents with finite Dirichlet energy. We refer to [6] and [7] for proofs and details.

Let B^n be the unit ball in $\mathbb{R}^n$ and let $\mathcal{Y}$ be a smooth oriented Riemannian manifold of dimension $M \geq 2$. By Nash theorem we can suppose that $\mathcal{Y}$ is isometrically embedded in $\mathbb{R}^N$ for some $N \geq 3$. We shall assume that $\mathcal{Y}$ is compact, connected, without boundary and that its integral 2-homology group $H_2(\mathcal{Y}, \mathbb{Z})$ has no torsion, so that $H_2(\mathcal{Y}, X) = H_2(\mathcal{Y}, \mathbb{Z}) \otimes X$ for $X = \mathbb{R}, \mathbb{Q}$. Note that the last condition automatically holds if $M = 2$.

$\mathcal{D}_{n,2}$-currents. Every differential n-form $\omega \in \mathcal{D}^n(B^n \times \mathcal{Y})$ splits as a sum $\omega = \sum_{k=0}^{\underline{n}} \omega^{(k)}$, $\underline{n} := \min(n, M)$, where the $\omega^{(k)}$'s are the n-forms that contain exactly k differentials in the vertical $\mathcal{Y}$ variables. We denote by $\mathcal{D}^{n,2}(B^n \times \mathcal{Y})$ the subspace of $\mathcal{D}^n(B^n \times \mathcal{Y})$ of n-forms of the type $\omega = \sum_{k=0}^{2} \omega^{(k)}$, and by $\mathcal{D}_{n,2}(B^n \times \mathcal{Y})$ the dual space of $\mathcal{D}^{n,2}(B^n \times \mathcal{Y})$. Every $(n, 2)$-current $T \in \mathcal{D}_{n,2}(B^n \times \mathcal{Y})$ splits as $T = \sum_{k=0}^{2} T_{(k)}$, where $T_{(k)}(\omega) = T(\omega^{(k)})$. For example, if $u \in W^{1,2}(B^n, \mathcal{Y})$, i.e., $u \in W^{1,2}(B^n, \mathbb{R}^N)$ with $u(x) \in \mathcal{Y}$ for a.e. $x \in B^n$, then $G_u \in \mathcal{D}_{n,2}(B^n \times \mathcal{Y})$, where in an approximate sense $G_u := (Id \bowtie u)_{\#}[\![B^n]\!]$, $(Id \bowtie u)(x) := (x, u(x))$, compare [6].

D-norm. For $\omega \in \mathcal{D}^{n,2}(B^n \times \mathcal{Y})$ we set

$$
\begin{aligned}
\|\omega\|_{\mathbf{D}} &:= \max\left\{ \sup_{x,y} \frac{|\omega^{(0)}(x,y)|}{1+|y|^2}, \int_{B^n} \sup_y |\omega^{(1)}(x,y)|^2 \, dx, \int_{B^n} \sup_y |\omega^{(2)}(x,y)| \, dx \right\}, \\
\|T\|_{\mathbf{D}} &:= \sup\left\{ T(\omega) \mid \omega \in \mathcal{D}^{n,2}(B^n \times \mathcal{Y}), \ \|\omega\|_{\mathbf{D}} \leq 1 \right\}.
\end{aligned}
$$

It is not difficult to show that $\|T\|_{\mathbf{D}}$ is a norm on $\{T \in \mathcal{D}_{n,2}(B^n \times \mathcal{Y}) \mid \|T\|_{\mathbf{D}} < +\infty\}$.

Weak $\mathcal{D}_{n,2}$-convergence. If $\{T_k\} \in \mathcal{D}_{n,2}(B^n \times \mathcal{Y})$, we say that $\{T_k\}$ converges weakly in $\mathcal{D}_{n,2}(B^n \times \mathcal{Y})$, $T_k \rightharpoonup T$, if $T_k(\omega) \to T(\omega)$ for every $\omega \in \mathcal{D}^{n,2}(B^n \times \mathcal{Y})$. Now, the class $\mathcal{D}_{n,2}(B^n \times \mathcal{Y})$ is closed under weak convergence and $\|\cdot\|_{\mathbf{D}}$ is weak lower semicontinuous. Moreover, if $\sup_k \|T_k\|_{\mathbf{D}} < +\infty$, then there is a subsequence which weakly converges to some $T \in \mathcal{D}_{n,2}(B^n \times \mathcal{Y})$ with $\|T\|_{\mathbf{D}} < +\infty$.

Boundaries. The exterior differential d splits into a horizontal and a vertical differential $d = d_x + d_y$. Of course $\partial_x T(\omega) := T(d_x\omega)$ defines a boundary operator $\partial_x : \mathcal{D}_{n,2}(B^n \times \mathcal{Y}) \to \mathcal{D}_{n-1,2}(B^n \times \mathcal{Y})$. Now, for any $\omega \in \mathcal{D}^{n-1,2}(B^n \times \mathcal{Y})$, we have that $d_y\omega$ belongs to $\mathcal{D}^{n,2}(B^n \times \mathcal{Y})$ if and only if $d_y\omega^{(2)} = 0$. Then $\partial_y T$ makes sense only as an element of the dual space of

$$\mathcal{Z}^{n-1,2}(B^n \times \mathcal{Y}) := \{\omega \in \mathcal{D}^{n-1,2}(B^n \times \mathcal{Y}) \mid d_y\omega^{(2)} = 0\}\,.$$

D-graphs. The study of weak limits of sequences of maps with equibounded Dirichlet energy, minimization problems and concentration phenomena, see [6], draw the authors of [5] to introduce the subclass $\mathbf{D}$-graph$(B^n \times \mathcal{Y})$ given by the $(n,2)$-currents $T \in \mathcal{D}_{n,2}(B^n \times \mathcal{Y})$ with $\|T\|_{\mathbf{D}} < +\infty$ and such that

$$T = G_{u_T} + S_T \tag{1.1}$$

for some function $u_T \in W^{1,2}(B^n, \mathcal{Y})$ and some $S_T \in \mathcal{D}_{n,2}(B^n \times \mathcal{Y})$ with $S_{T(0)} = S_{T(1)} = 0$, i.e. S_T is completely vertical, so that

$$\partial_x T = 0 \quad \text{on } \mathcal{D}^{n-1,2}(B^n \times \mathcal{Y})\,, \qquad \partial_y T = 0 \quad \text{on } \mathcal{Z}^{n-1,2}(B^n \times \mathcal{Y})\,.$$

They also showed that:

i) the decomposition (1.1) is unique;

ii) weak limits in $\mathcal{D}_{n,2}$ of sequences of graphs of smooth maps $u_k : B^n \to \mathcal{Y}$, with equibounded Dirichlet energy, belong to $\mathbf{D}$-graph$(B^n \times \mathcal{Y})$;

iii) if $T \in \mathbf{D}$-graph$(B^n \times \mathcal{Y})$, in general $\partial G_{u_T} \neq 0$. However

$$\partial G_{u_T} = 0 \quad \text{on } \mathcal{D}^{n-1,1}(B^n \times \mathcal{Y})\,, \qquad \partial G_{u_T}(\omega^{(2)}) = 0 \quad \text{if } \omega^{(2)} = d\eta \quad \text{and} \quad \operatorname{spt}\eta \subset B^n \times \mathcal{Y}$$

and

$$\partial_y G_{u_T} = 0 \quad \text{on } \mathcal{Z}^{n-1,1}(B^n \times \mathcal{Y})\,, \qquad \partial G_{u_T} = \partial_x G_{u_T} \quad \text{on } \mathcal{Z}^{n-1,2}(B^n \times \mathcal{Y})\,;$$

in particular

$$\partial_y S_T(\omega^{(2)}) = 0 \quad \text{if } \omega^{(2)} = d\eta \quad \text{and} \quad \operatorname{spt}\eta \subset B^n \times \mathcal{Y}\,, \qquad \partial_x S_T = 0 \quad \text{on } \mathcal{D}^{n-1,2}(B^n \times \mathcal{Y})\,;$$

iv) $\|G_{u_T}\|_{\mathbf{D}} = \|u_T\|_{W^{1,2}} \leq \|T\|_{\mathbf{D}}$, consequently $\|S_T\|_{\mathbf{D}} \leq 2\,\|T\|_{\mathbf{D}}$;

v) $\mathbf{D}$-graph$(B^n \times \mathcal{Y})$ is closed under weak convergence in $\mathcal{D}_{n,2}$ with equibounded $\mathbf{D}$-norm.

The 2-dimensional case. If $n = 2$, obviously $\mathcal{D}_{n,2}(B^n \times \mathcal{Y}) = \mathcal{D}_2(B^2 \times \mathcal{Y})$ and ∂T is the usual boundary of currents, whereas $\mathbf{M}(T) \leq c\|T\|_{\mathbf{D}}$ for some absolute constant. Consequently, weak limits of smooth graphs with equibounded Dirichlet energy are integer multiplicity (shortly i.m.) rectifiable currents in $\mathcal{R}_2(B^2 \times \mathcal{Y})$, and $\mathbf{D}$-graph$(B^2 \times \mathcal{Y}) \cap \mathcal{R}_2(B^2 \times \mathcal{Y})$ is closed under weak convergence with equibounded $\mathbf{D}$-norm.

It was proved in [5] and [6] that every T in $\mathbf{D}$-graph$(B^2 \times \mathcal{Y}) \cap \mathcal{R}_2(B^2 \times \mathcal{Y})$ decomposes as

$$T = G_{u_T} + S_T\,, \qquad S_T = \sum_{i=1}^{I} \delta_{x_i} \times C_i + S_{T,sing}\,, \tag{1.2}$$

where δ_x is the Dirac mass in x, $x_i \in B^2$, $C_i \in \mathcal{Z}_2(\mathcal{Y})$ are integral cycles with non trivial homology and $S_{T,sing}$ is a completely vertical, homologically trivial, i.m. rectifiable current supported on a set not containing $\{x_i\} \times \mathcal{Y}$, $i = 1, \ldots, I$. More precisely, for every Borel set $A \subset B^2$ we have $\partial(S_T \llcorner A \times \mathbb{R}^N) = 0$. Moreover, if $\pi : \mathbb{R}^2 \times \mathbb{R}^N \to \mathbb{R}^2$ and $\widehat{\pi} : \mathbb{R}^2 \times \mathbb{R}^N \to \mathbb{R}^N$ denote the orthogonal projection onto the first and the second factor, respectively, then for any bounded Borel function φ in B^2 we have

$$S_{T,sing}(\pi^{\#}\varphi \wedge \widehat{\pi}^{\#}\sigma) = 0$$

for every element $[\sigma]$ in the second De Rham cohomology group $H^2_{dR}(\mathcal{Y})$. Finally

$$\|S_{T,sing}\|(\{x_1,\dots,x_I\}\times\mathcal{Y})=0\,,$$

$\|\cdot\|$ denoting the total variation. As a consequence, we have $S_{T,sing}(\omega)\neq 0$ only on forms $\omega\in\mathcal{D}^2(B^2\times\mathcal{Y})$ such that $d_y\omega^{(2)}\neq 0$. In particular, if $\mathcal{Y}$ has dimension 2, then $S_{T,sing}=0$ whereas, if $\mathcal{Y}=S^2$, the unit 2-sphere in $\mathbb{R}^3$, then $C_i=z_i\,[\![\, S^2\,]\!]$ for some integer z_i.

Definition 1.1 *We say that an integral* 2*-cycle* $C\in\mathcal{Z}_2(\mathcal{Y})$ *is* of spherical type *if its homology class contains a Lipschitz image of the* 2*-sphere* S^2. *More precisely, if there exist* $Z\in\mathcal{Z}_2(\mathcal{Y})$, $R\in\mathcal{R}_3(\mathcal{Y})$ *and a Lipschitz function* $\phi:S^2\to\mathcal{Y}$ *such that*

$$C-Z=\partial R\qquad\text{and}\qquad \phi_\#[\![\, S^2\,]\!]=Z\,.$$

Spherical cycles come into play since, as proved in [5] [6], if T is in the sequential weak closure of smooth graphs with equibounded Dirichlet energies, then every C_i is of spherical type. This facts lead to the following

Definition 1.2 *If* $n=2$, *we denote by* $\mathrm{cart}^{2,1}(B^2\times\mathcal{Y})$ *the class of i.m. rectifiable currents* T *in* $\mathbf{D}$-graph$(B^2\times\mathcal{Y})$ *which decompose as in* (1.2), *where the* C_i*'s are of spherical type.*

It turns out, see [5] [7], that $\mathrm{cart}^{2,1}(B^2\times\mathcal{Y})$ is closed under weak convergence, with equibounded $\mathbf{D}$-norm, and contains the weak limits of sequences of smooth graphs with equibounded $\mathbf{D}$-norm.

The n-dimensional case. As before, let $\pi:\mathbb{R}^n\times\mathbb{R}^N\to\mathbb{R}^n$ denote the orthogonal projection onto the first factor. Let P be an oriented 2-plane in $\mathbb{R}^n$, and $P_t:=P+\sum_{i=1}^{n-2}t_i\nu_i$ the family of oriented 2-planes parallel to P, $t=(t_1,\dots,t_{n-2})\in\mathbb{R}^{n-2}$, $\mathrm{span}(\nu_1,\dots,\nu_{n-2})$ being the orthogonal subspace to P. Similarly to the case of normal currents, for every $T\in\mathcal{D}_{n,2}(B^n\times\mathcal{Y})$ with $\|T\|_{\mathbf{D}}<+\infty$, for $\mathcal{H}^{n-2}$-a.e. t the *slice* $T\llcorner\pi^{-1}(P_t)$ of T over $\pi^{-1}(P_t)$ is a well defined current in $\mathcal{D}_2((B^n\cap P_t)\times\mathcal{Y})$ with finite $\mathbf{D}$-norm. Moreover, if $T_k\rightharpoonup T$ with equibounded $\mathbf{D}$-norm, for $\mathcal{H}^{n-2}$-a.e. t, passing to a subsequence we have $T_k\llcorner\pi^{-1}(P_t)\rightharpoonup T\llcorner\pi^{-1}(P_t)$ with equibounded $\mathbf{D}$-norm. Finally, if $T\in\mathbf{D}$-graph$(B^n\times\mathcal{Y})$, for $\mathcal{H}^{n-2}$-a.e. t we have $T\llcorner\pi^{-1}(P_t)\in\mathbf{D}$-graph$((B^n\cap P_t)\times\mathcal{Y})$. Therefore in any dimension n it was introduced in [7] the following

Definition 1.3 *We say that* T *is in* $\mathrm{cart}^{2,1}(B^n\times\mathcal{Y})$ *if* $T\in\mathbf{D}$-graph$(B^n\times\mathcal{Y})$ *and for any* 2*-plane* P *and for* $\mathcal{H}^{n-2}$*-a.e.* t *the* 2*-dimensional current* $T\llcorner\pi^{-1}(P_t)$ *belongs to* $\mathrm{cart}^{2,1}((B^n\cap P_t)\times\mathcal{Y})$.

It turns out that the class $\mathrm{cart}^{2,1}(B^n\times\mathcal{Y})$ is closed under weak convergence with equibounded $\mathbf{D}$-norm and, in case $\mathcal{Y}=S^2$, that the class $\mathrm{cart}^{2,1}(B^n\times S^2)$ coincides with $\mathbf{D}$-graph$(B^n\times S^2)$, $S_{T,sing}=0$ and

$$T=G_{u_T}+L_T\times[\![\, S^2\,]\!]\,,\tag{1.3}$$

where $L_T\in\mathcal{R}_{n-2}(B^n)$ is an i.m. rectifiable current.

Definition 1.4 *We say that a Sobolev map* $u\in W^{1,2}(B^n,\mathcal{Y})$ *is in* $\mathrm{cart}^{2,1}(B^n,\mathcal{Y})$ *if the current* G_u *associated to its graph is in* $\mathrm{cart}^{2,1}(B^n\times\mathcal{Y})$.

Therefore, a $W^{1,2}$ map u is in $\mathrm{cart}^{2,1}(B^n,\mathcal{Y})$ if its graph has no inner boundary, i.e.,

$$\partial_x G_u=0\quad\text{on }\mathcal{D}^{n-1,2}(B^n\times\mathcal{Y})\,,\qquad \partial_y G_u=0\quad\text{on }\mathcal{Z}^{n-1,2}(B^n\times\mathcal{Y})\,.$$

Remark 1.5 If $u:B^n\to\mathcal{Y}$ is a continuous map in $W^{1,2}(B^n,\mathcal{Y})$, by a standard convolution and projection argument it can be approximated in $W^{1,2}$-strong sense by a smooth sequence in $C^\infty(B^n,\mathcal{Y})$. This yields in particular that $u\in\mathrm{cart}^{2,1}(B^n,\mathcal{Y})$.

The Dirichlet energy in $\mathrm{cart}^{2,1}$. Denote by $\bigwedge_n\mathbb{R}^{n+M}$ the space of n-vectors in $\mathbb{R}^{n+M}$. Moreover, if $G:\mathbb{R}^n\to\mathbb{R}^M$ is a linear transformation, and with the same notation $G:=(G_i^j)_{i,j=1}^{n,M}$ is the $(M\times n)$-matrix associated to it, we let

$$M(G):=(e_1+Ge_1)\wedge\cdots\wedge(e_n+Ge_n)\in\bigwedge\nolimits_n\mathbb{R}^{n+M}\,,$$

$(e_i)_{i=1}^n$ being the canonical basis in $\mathbb{R}^n$. Then $M(G)$ identifies the plane graph of G in $\mathbb{R}^{n+M}$, and in fact orients such an n-plane. If $T \in \mathbf{D}$-graph$(B^n \times \mathcal{Y})$, we define the Dirichlet density as the function of $y \in \mathcal{Y}$, $\xi \in \bigwedge_n \mathbb{R}^{n+M}$ given by

$$F(y,\xi) := \sup \Big\{ \phi(\xi) \mid \phi : \textstyle\bigwedge_n \mathbb{R}^{n+M} \to \mathbb{R} \quad \text{linear, } \phi(M(G)) \le \dfrac{1}{2}|G|^2 \\ \text{for all linear maps } G : \mathbb{R}^n \to T_y\mathcal{Y} \Big\} \,,$$

$T_y\mathcal{Y}$ being the tangent M-space to $\mathcal{Y}$ in y. The Dirichlet integral then extends to $\mathbf{D}$-graphs T, compare [6], as

$$\mathbf{D}(T) := \int F(y, \vec{T})\, d\|T\|_{\mathbf{D}} \,,$$

$\vec{T}$ being the Radon-Nikodym derivative $dT/d\|T\|_{\mathbf{D}}$ and, if (1.1) holds, one has

$$\mathbf{D}(T) = \frac{1}{2}\int_{B^n} |Du_T|^2\, dx + \int_{B^n \times \mathcal{Y}} F(y, \vec{S_T})\, d\|S_T\|_{\mathbf{D}} \,. \tag{1.4}$$

In particular we have

$$\|T\|_{\mathbf{D}} \le c\, \mathbf{D}(T) \tag{1.5}$$

for some absolute constant $c = c(n)$. Finally, if $A \subset B^n$ is a Borel set we will denote

$$\mathbf{D}(T, A \times \mathcal{Y}) := \mathbf{D}(T \llcorner A \times \mathcal{Y})$$

and, if $u \in W^{1,2}(B^n, \mathcal{Y})$,

$$\mathbf{D}(u, A) := \frac{1}{2}\int_A |Du|^2\, dx = \mathbf{D}(G_u, A \times \mathcal{Y}) \,.$$

A part from the case of energy minimizing currents, see [7], if $n \ge 3$ we do not have an explicit formula for the second term on the right hand side of (1.4). However, if $n = 2$ and (1.2) holds, we have

$$\mathbf{D}(T) = \frac{1}{2}\int_{B^2} |Du_T|^2\, dx + \sum_{i=1}^{I} \mathbf{M}(C_i) + \mathbf{M}(S_{T,sing}) \,. \tag{1.6}$$

Finally, if $\mathcal{Y} = S^2$ and (1.3) holds, we have in any dimension

$$\mathbf{D}(T) = \frac{1}{2}\int_{B^n} |Du_T|^2\, dx + 4\pi\, \mathbf{M}(L_T) \,. \tag{1.7}$$

2 Mappings into the sphere

In this section we show that, if $\mathcal{Y}$ is the standard unit sphere S^2 in $\mathbb{R}^3$, every T in $\text{cart}^{2,1}(B^n \times S^2)$ can be approximated weakly as currents by smooth graphs with equibounded Dirichlet energy.

Theorem 2.1 *Let $T \in \text{cart}^{2,1}(B^n \times S^2)$, $n \ge 2$. Then there exists a sequence of smooth maps $u_k : B^n \to S^2$ such that $G_{u_k} \rightharpoonup T$ weakly in $\mathcal{D}_n(B^n \times S^2)$ and*

$$\sup_k \mathbf{D}(u_k, B^n) \le c_n\, \mathbf{D}(T, B^n \times S^2) < +\infty \,,$$

where $c_n > 0$ is an absolute constant.

Proof: By Remark 1.5 it suffices to construct the sequence $\{u_k\}$ in $W^{1,2}(B^n, \mathbb{R}^3) \cap C^0(B^n, S^2)$. Since B^n is bilipschitz homeomorphic to the unit open cube

$$\mathcal{C}^n :=]0,1[^n \,,$$

we will prove the assertion for $T \in \text{cart}^{2,1}(\mathcal{C}^n \times S^2)$. Note that Theorem 2.1 is true for $n = 2$, see Sec. 4.1.2 in [6, vol. II]. Moreover, using the same argument as in Corollary 4.2, with $\mathcal{Y} = S^2$, we have the following

Proposition 2.2 *Let $n = 2$ and $T \in \mathrm{cart}^{2,1}(\mathcal{C}^2 \times S^2)$ be such that $\partial T = G_\varphi$ for some function $\varphi \in W^{1,2}(\partial\mathcal{C}^2, \mathcal{Y})$. Then there exists a sequence of continuous maps $u_k : \overline{\mathcal{C}}^2 \to S^2$, with $\{u_k\} \subset \mathrm{cart}^{2,1}(\mathcal{C}^2, S^2)$ and $\partial G_{u_k} = \partial T$, hence $u_{k|\partial\mathcal{C}^2} = \varphi$, such that $G_{u_k} \rightharpoonup T$ weakly in $\mathcal{D}_2(\mathcal{C}^2 \times S^2)$ and*

$$\lim_k \mathbf{D}(u_k, \mathcal{C}^2) = \mathbf{D}(T, \mathcal{C}^2 \times S^2)\,.$$

Let us fix some notation. If Q is a closed n-cube of $\mathbb{R}^n$ with sides parallel to the coordinate axes, we will denote by $Q_{(j)}$ its j-dimensional skeleton. If Q is contained in the unit open cube $\mathcal{C}^n$, and F is a j-face of $Q_{(j)}$, we will denote

$$T_Q := T \llcorner Q \times \mathbb{R}^3 \qquad \text{and} \qquad T_F := T \llcorner F \times \mathbb{R}^3$$

the restriction of $T \in \mathrm{cart}^{2,1}(\mathcal{C}^n \times S^2)$ to $Q \times \mathbb{R}^3$ and to $F \times \mathbb{R}^3$, respectively. Also, we will set

$$T_{\partial Q} := \sum_{F \in Q_{(n-1)}} \sigma_F\, T \llcorner F \times \mathbb{R}^3,$$

where $\sigma_F = \pm 1$ according to the induced orientation of Q onto its boundary. Finally, if $u \in W^{1,2}(Q, S^2)$ is such that the restriction $u_{|\partial Q} \in W^{1,2}(\partial Q, S^2)$, and F is a j-face of $Q_{(j)}$, we will denote

$$G_{u_{|\partial Q}} := (Id \bowtie u_{|\partial Q})_\# [\![\, \partial Q \,]\!]\,, \qquad G_{u_{|F}} := (Id \bowtie u_{|F})_\# [\![\, F \,]\!]\,.$$

Definition 2.3 *We say that Q is in* generic position *with respect to T if for every $j = 1, \ldots, n-1$ and every j-face F in $Q_{(j)}$ the restriction T_F is a j-dimensional current in $\mathrm{cart}^{2,1}(F \times S^2)$ and moreover for every 1-face F in $Q_{(1)}$ the restriction T_F is the graph G_φ of a Hölder continuous map $\varphi \in W^{1,2}(F, S^2)$.*

We remark that by definition of the class $\mathrm{cart}^{2,1}(\mathcal{C}^n \times S^2)$, by the structure of 2-dimensional currents in $\mathrm{cart}^{2,1}$ and by a slicing argument, it follows that for a.e. choice of the vector $a \in \mathbb{R}^n$ so that $a + Q \subset \mathcal{C}^n$, we have that the n-cube $a + Q$ is in generic position with respect to T. In this case we also have

$$T_{\partial Q} = \partial T_Q\,. \tag{2.1}$$

We will work by induction on the dimension n, making use of the following result, Proposition 2.4, which holds true if $n = 2$ by Proposition 2.2. It will be used in dimension $n - 1$ to prove Theorem 2.1 and will be finally proved in dimension n by means of a readaptation of Theorem 2.1.

Proposition 2.4 *Let $T \in \mathrm{cart}^{2,1}(\mathcal{C}^n \times S^2)$ and Q be a closed n-cube of $\mathcal{C}^n$ in generic position with respect to T. Then there exists a sequence $u_k : Q \to S^2$ of continuous maps in $\mathrm{cart}^{2,1}(Q, S^2)$ for which the following properties hold:*

i) *for every k the boundary ∂G_{u_k} coincides with the $(n-1)$-dimensional graph $G_{u_{k|\partial Q}}$ where $u_{k|\partial Q}$ is a continuous map in $\mathrm{cart}^{2,1}(\partial Q, S^2)$;*

ii) *for every k and every $(n-1)$-face F of the boundary of Q, the restriction $G_{u_{k|F}}$ of $G_{u_{k|\partial Q}}$ to $F \times \mathbb{R}^3$ only depends on the restriction T_F of T to $F \times \mathbb{R}^3$;*

iii) *$G_{u_k} \rightharpoonup T_Q$ weakly in $\mathcal{D}_n(Q \times S^2)$ as $k \to +\infty$ and*

$$\sup_k \mathbf{D}(u_k, Q) \le \widetilde{c}_n\, \mathbf{D}(T_Q, Q \times S^2) < +\infty\,,$$

where $\widetilde{c}_n := 2\, c_n > 0$ is an absolute constant.

Let now $T \in \mathrm{cart}^{2,1}(\mathcal{C}^n \times S^2)$, $n \ge 3$, and let $(e_1, \ldots, e_n)$ be the canonical basis of $\mathbb{R}^n$. For $i = 1, \ldots, n$ and $t \in [0, 1]$, we denote by $P(t, i)$ the restriction to $\mathcal{C}^n$ of the hyperplane containing the point $t\, e_i$ and orthogonal to e_i, i.e.,

$$P(t, i) := \{x \in \mathcal{C}^n \mid (x - t\, e_i \mid e_i)_{\mathbb{R}^n} = 0\}\,.$$

By slicing theory
$$T \llcorner P(t,i)\times\mathbb{R}^3 = < T, d_i, t > \in \mathrm{cart}^{2,1}(P(t,i)\times S^2) \tag{2.2}$$
for a.e. $t\in[0,1]$, where
$$d_i(x,y) := x_i\,, \qquad x=(x_1,\dots,x_n)\in\mathbb{R}^n, \quad y\in\mathbb{R}^3.$$
For $m\in\mathbb{N}^*$ and $a=(a_1,\dots,a_n)\in[1/4m,3/4m]^n$ we denote by $C^{(n-1)}_{m,a}$ the $(n-1)$-skeleton of the grid of $\mathcal{C}^n$ given by
$$C^{(n-1)}_{m,a} := \bigcup_{i=1}^n \bigcup_{j=0}^{m-1} P(a_i+j/m,i)\,.$$
By (1.3) and (1.7), for every i we have
$$\begin{aligned}\int_{1/4m}^{3/4m}\sum_{j=0}^{m-1}\mathbf{D}(< T, d_i, t+j/m >, P(t+j/m,i)\times S^2)\,dt \quad &\le \quad \sum_{j=0}^{m-1}\mathbf{D}(T,\{j/m\le d_i\le (j+1)/m\}) \\ &= \quad \mathbf{D}(T,\mathcal{C}^n\times S^2)\,.\end{aligned}$$
Setting then
$$T^{(n-1)}_{m,a} := \sum_{i=1}^n\sum_{j=0}^{m-1} < T, d_i, a_i+j/m >\,,$$
there exists a vector $a=a(m)\in[1/4m,3/4m]^n$ such that $< T, d_i, a_i+j/m > \in \mathrm{cart}^{2,1}(P(a_i+j/m,i)\times S^2)$ for every $i\in\{1,\dots,n\}$ and $j\in\{0,\dots,m-1\}$ and
$$\mathbf{D}(T^{(n-1)}_{m,a}, C^{(n-1)}_{m,a}\times S^2) \le \widetilde{c}(n)\, m\, \mathbf{D}(T,\mathcal{C}^n\times S^2)\,, \tag{2.3}$$
where $\widetilde{c}(n)=n$. Let now $\mathcal{Q}_{m,a}$ denote the family of all n-cubes Q of side $1/m$ with boundary contained in the $(n-1)$-grid $C^{(n-1)}_{m,a}$, i.e. $\partial Q\subset C^{(n-1)}_{m,a}$, so that
$$\cup\mathcal{Q}_{m,a} = a(m) + [0,(m-1)/m]^n\,. \tag{2.4}$$
By Definition 2.3 and the following remark, taking e.g. $\widetilde{c}(n)=2n$ in (2.3), we may and do choose $a(m)$ so that each n-cube Q of $\mathcal{Q}_{m,a}$ is in generic position with respect to T.

For every Q in $\mathcal{Q}_{m,a}$ and every $(n-1)$-face F of the boundary of Q, we apply Proposition 2.4, which is supposed to hold true in dimension $n-1$, to the restriction T_F of T to F. Then there exists a sequence $u^F_k : F\to S^2$ of continuous maps in $\mathrm{cart}^{2,1}(F,S^2)$ for which the following properties hold:

i) for every k the boundary $\partial G_{u^F_k}$ coincides with the $(n-2)$-dimensional graph $G_{u^F_{k|\partial F}}$ where $u^F_{k|\partial F}$ is a continuous map in $\mathrm{cart}^{2,1}(\partial F,S^2)$;

ii) for every k and every $(n-2)$-face I of the boundary of F, the restriction $G_{u^F_{k|I}}$ of $G_{u^F_{k|\partial F}}$ to $I\times\mathbb{R}^3$ only depends on the restriction T_I of T to $I\times\mathbb{R}^3$;

iii) $G_{u^F_k}\rightharpoonup T_F$ weakly in $\mathcal{D}_{n-1}(F\times S^2)$ as $k\to+\infty$ and
$$\sup_k \mathbf{D}(u^F_k,F) \le \widetilde{c}_{n-1}\,\mathbf{D}(T_F,F\times S^2) < +\infty\,, \tag{2.5}$$
where $\widetilde{c}_{n-1} := 2\,c_{n-1} > 0$ is an absolute constant.

If $\mathcal{Q}^{(n-1)}_{m,a}$ denotes the $(n-1)$-skeleton of $\cup\mathcal{Q}_{m,a}$, we now define $v_k:\cup\mathcal{Q}^{(n-1)}_{m,a}\to S^2$ by setting
$$v_k(x) := u^F_k(x) \qquad \text{if} \quad x\in F \tag{2.6}$$
for every $(n-1)$-face F of side $1/m$ of some n-cube of $\mathcal{Q}_{m,a}$. Note that if F_1 and F_2 are two $(n-1)$-faces which intersect in a common $(n-2)$-face I, by i) and ii) for every k we have $\partial G_{v^{F_1}_k}\llcorner I\times\mathbb{R}^3 = -\partial G_{v^{F_1}_k}\llcorner I\times\mathbb{R}^3$. Then $\{v_k\}$ is a well defined continuous sequence such that $\partial G_{v_{k|\partial Q}}=0$ for every Q in $\mathcal{Q}_{m,a}$ and
$$G_{v_{k|\partial Q}} \rightharpoonup T_{\partial Q} \qquad \text{in } \mathcal{D}_{n-1}(\partial Q\times S^2) \tag{2.7}$$

as $k \to +\infty$. In particular the graph of G_{v_k} has no boundary, $\partial G_{v_k} = 0$, and from (2.3) and (2.5)

$$\sup_k \mathbf{D}(v_k, \cup \mathcal{Q}_{m,a}^{(n-1)}) \le 2\, \widetilde{c}_{n-1}\, m\, n\, \mathbf{D}(T, \mathcal{C}^n \times S^2)\,. \tag{2.8}$$

We now wish to extend v_k to a map U_k defined in a homogeneous way in the interior of each n-cube Q of $\mathcal{Q}_{m,a}$, but a small sphere about the center where we wish to remove the singularity, see (2.10). To remove the point singularities of the homogeneous extension in the center of each cube, we make use of the following

Proposition 2.5 *For k sufficiently large and for every n-cube $Q \in \mathcal{Q}_{m,a}$, we have*

$$\{v \in W^{1,2}(Q, \mathbb{R}^3) \cap C^0(Q, S^2) \mid v_{|\partial Q} = v_{k|\partial Q}\} \neq \emptyset\,.$$

PROOF: It suffices to prove that $v_{k|\partial Q}$ is homotopic to a constant map in S^2. Arguing as in [2], we recall that the Hurewicz homomorphism $\rho : \pi_2(S^2) \to H_2(S^2, \mathbb{Q})$ is an isomorphism, $\pi_2(S^2)$ and $H_2(S^2, \mathbb{Q})$ being respectively the homotopy group and the rational homology group of order 2 of S^2. As a consequence, it suffices to show that for k sufficiently large the pull-back via $v_{k|\partial Q}$ of the volume 2-form ω_{S^2} in S^2 is a closed form, i.e.,

$$d\, {v_{k|\partial Q}}^{\#} \omega_{S^2} = 0\,, \tag{2.9}$$

which means zero degree if $n = 3$. To this aim, by (2.7) and (2.1) we infer that $G_{v_{k|\partial Q}}$ weakly converges to the boundary of the Cartesian current $T_Q \in \mathrm{cart}^{2,1}(Q \times S^2)$. If $n \ge 4$, for every $(n-4)$-form $\eta \in \mathcal{D}^{n-4}(\partial Q)$ we have

$$\partial T_Q(\pi^{\#} d\eta \wedge \widehat{\pi}^{\#} \omega_{S^2}) = 0$$

(whereas $\partial T_Q(\widehat{\pi}^{\#} \omega_{S^2}) = 0$, i.e., zero degree, if $n = 3$), where $\pi : \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R}^n$ and $\widehat{\pi} : \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R}^N$ denote the orthogonal projection onto the first and the second factor, respectively. Hence, by weak convergence,

$$\int_{\partial Q} d\eta \wedge v_k^{\#} \omega_{S^2} = G_{v_{k|\partial Q}}(\pi^{\#} d\eta \wedge \widehat{\pi}^{\#} \omega_{S^2}) \to \partial T_Q(\pi^{\#} d\eta \wedge \widehat{\pi}^{\#} \omega_{S^2}) = 0$$

($\int_{\partial Q} v_k^{\#} \omega_{S^2} \to 0$ if $n = 3$) as $k \to +\infty$. This clearly yields (2.9). □

Let now $Q \in \mathcal{Q}_{m,a}$. If $\varphi_k \in W^{1,2}(Q, S^2)$ is a continuous extension of $v_{k|\partial Q}$, the existence of which is provided by Proposition 2.5, we fix $\delta \in (0, 1/2m)$ and extend $v_{k|\partial Q}$ to the interior of Q by

$$U_k^{(Q)}(x) := \begin{cases} v_k\left(p + \dfrac{1}{2m} \dfrac{x-p}{\|x-p\|}\right) & \text{if} \quad \delta \le \|x-p\| \le \dfrac{1}{2m} \\ \varphi_k\left(p + \dfrac{1}{2m\delta}(x-p)\right) & \text{if} \quad \|x-p\| \le \delta\,, \end{cases} \tag{2.10}$$

where p is the center of Q and $\|x\| := \max_{1 \le i \le n} |x_i|$, so that $\|x-p\| = 1/2m$ if $x \in \partial Q$.

Trivially $U_k^{(Q)}$ is a continuous function in $W^{1,2}(Q, S^2)$. Moreover, since for $\delta \le \|x-p\| \le 1/2m$

$$|D_x U_k^{(Q)}(x)| \le \frac{1}{2m} \cdot \frac{|x-p|}{\|x-p\|^2} \cdot |D_y v_k(y)|\,, \qquad y := p + \frac{1}{2m} \frac{x-p}{\|x-p\|}\,, \qquad |x| := \sqrt{{x_1}^2 + \ldots + {x_n}^2}\,,$$

by the area formula [3] we estimate

$$\int_{\delta \le \|x-p\| \le 1/2m} |DU_k^{(Q)}|^2\, dx \le \frac{n}{n-2} \cdot \frac{1}{2m} \cdot \int_{\partial Q} |Dv_k|^2\, d\mathcal{H}^{n-1} \tag{2.11}$$

and by changing variable

$$\int_{\|x-p\| \le \delta} |DU_k^{(Q)}|^2\, dx = (2m\, \delta(m, v_k))^{n-2} \int_Q |D\varphi_k|^2\, dx \le \frac{1}{k\, m^3}\,, \tag{2.12}$$

if we choose $\delta = \delta(m, v_k) > 0$ suitably with $\lim_{k \to +\infty} \delta(m, v_k) = 0$ for every m.

We define now $U_k^{(m)} : \cup\mathcal{Q}_{m,a} \to S^2$ by $U_k^{(m)}(x) := U_k^{(Q)}(x)$ if $x \in Q$ for some $Q \in \mathcal{Q}_{m,a}$. Then, by (2.11) and (2.12),

$$\int_{\cup\mathcal{Q}_{m,a}} |DU_k^{(m)}|^2\, dx \le \frac{n}{(n-2)m} \cdot \mathbf{D}(v_k, \cup\mathcal{Q}_{m,a}^{(n-1)}) + \frac{1}{k\,m}\,.$$

Therefore $\{U_k^{(m)}\}_k$ is a continuous sequence in $\mathrm{cart}^{2,1}(\cup\mathcal{Q}_{m,a}, S^2)$ such that by (2.8)

$$\sup_k \mathbf{D}(U_k^{(m)}, \cup\mathcal{Q}_{m,a}) \le c(n)\,\mathbf{D}(T, \mathcal{C}^n \times S^2) + \frac{1}{m} \tag{2.13}$$

for each m, with $c(n) := 2\,\widetilde{c}_{n-1}\, n^2/(n-2)$. Moreover by (2.4) there exists an affine bijective function $\psi_{m,a} : \mathcal{C}^n \to \cup\mathcal{Q}_{m,a}$ such that $\mathrm{Lip}\,\psi_{m,a} = (m-1)/m$ and $\psi_{m,a} \to Id_{\mathcal{C}^n}$ uniformly as $m \to +\infty$. Setting now $V_k^{(m)}(x) := U_k^{(m)}(\psi_{m,a}(x))$, for m fixed, $\{V_k^{(m)}\}_k$ is a continuous sequence in $\mathrm{cart}^{2,1}(\mathcal{C}^n, S^2)$ such that, by (2.13),

$$\sup_k \mathbf{D}(V_k^{(m)}, \mathcal{C}^n) \le \left(\frac{m}{m-1}\right)^{n-2} c(n)\,\mathbf{D}(T, \mathcal{C}^n \times S^2) + \frac{2}{m}\,.$$

Then, by closure-compactness we have both

$$G_{U_k^{(m)}} \rightharpoonup T_m \qquad \text{and} \qquad G_{V_k^{(m)}} \rightharpoonup \widetilde{T}_m$$

as $k \to +\infty$ weakly in $\mathcal{D}_{n,2}(\cup\mathcal{Q}_{m,a} \times S^2)$ and $\mathcal{D}_{n,2}(\mathcal{C}^n \times S^2)$, respectively, for some $T_m \in \mathrm{cart}^{2,1}(\cup\mathcal{Q}_{m,a} \times S^2)$ and $\widetilde{T}_m \in \mathrm{cart}^{2,1}(\mathcal{C}^n \times S^2)$. Moreover $T_m = \Psi_{m,a\#}\widetilde{T}_m$, where $\Psi_{m,a} : \mathcal{C}^n \times S^2 \to \cup\mathcal{Q}_{m,a} \times S^2$ is given by $\Psi_{m,a}(x,y) := (\psi_{m,a}(x), y)$. As a consequence, taking $c_n := 2c(n) = 8\,c_{n-1}\, n^2/(n-2)$, the assertion follows by a diagonal procedure if we prove the following

Proposition 2.6 *Under the previous hypotheses,* $\widetilde{T}_m \rightharpoonup T$ *weakly in* $\mathcal{D}_n(\mathcal{C}^n \times S^2)$ *as* $m \to +\infty$.

PROOF: As before, we fix an n-cube $Q \in \mathcal{Q}_{m,a}$ and let p be its center. Also, denote by $\psi : Q \times S^2 \to \{p\} \times S^2$ the map $\psi(x,y) := (p,y)$, which is constantly equal to p on x and the identity on y. Finally let $h : [0,1] \times (Q \times S^2) \to Q \times S^2$ be the affine homotopy

$$\begin{aligned} h(t,x,y) &:= t\,\psi(x,y) + (1-t)\,Id_{Q\times S^2}(x,y) \\ &= (t\,p + (1-t)\,x, y)\,. \end{aligned}$$

By (2.1), (2.7) and (2.10)

$$\partial(T_m \llcorner Q \times \mathbb{R}^3) = T_m \llcorner \partial Q \times \mathbb{R}^3 = \partial T_Q, \qquad T_Q := T \llcorner Q \times \mathbb{R}^3, \tag{2.14}$$

whereas, since by (2.12) $\mathbf{M}(G_{U_k^{(Q)}} \llcorner \{\|x-p\| < \delta\}) \to 0$ as $k \to +\infty$, we infer

$$h_\#([\![0,1]\!] \times \partial T_Q) = -T_m \llcorner Q \times \mathbb{R}^3\,.$$

As a consequence, setting

$$R_Q^m := h_\#([\![0,1]\!] \times T_Q)\,,$$

by the homotopy formula [10, 26.22] we infer

$$\partial R_Q^m = \psi_\# T_Q - T_Q + T_m \llcorner Q \times \mathbb{R}^3\,.$$

Since $\psi_\# T_Q$ is an n-dimensional i.m. rectifiable current, $n \ge 3$, supported in the 2-dimensional set $\{p\} \times S^2$, we infer

$$T_m \llcorner Q \times \mathbb{R}^3 - T_Q = \partial R_Q^m\,.$$

Moreover, since by [10, 26.23]

$$\mathbf{M}(R_Q^m) \le \frac{c}{m}\,\mathbf{M}(T_Q)\,,$$

setting

$$R_m := \sum_{Q\in\mathcal{Q}_{m,a}} R_Q^m \in \mathcal{R}_{n+1}(\cup\mathcal{Q}_{m,a}\times S^2)\,,$$

by (2.14) we obtain

$$T_m - T \llcorner \cup\,\mathcal{Q}_{m,a}\times\mathbb{R}^3 = \partial R_m \tag{2.15}$$

where

$$\mathbf{M}(R_m) \le \frac{c}{m}\,\mathbf{M}(T) \to 0$$

as $m\to+\infty$. This yields that $T_m - T\llcorner\cup\,\mathcal{Q}_{m,a}\times\mathbb{R}^3 \rightharpoonup 0$ as $m\to+\infty$ in $\mathcal{D}_n(\mathcal{C}^n\times S^2)$, compare [10, 31.2]. Finally the assertion follows since $\Psi^{-1}_{m,a\,\#}T_m = \widetilde{T}_m$ whereas, by uniform convergence $\psi_{m,a}\to Id_{\mathcal{C}^n}$,

$$\Psi^{-1}_{m,a\,\#}T\llcorner\cup\,\mathcal{Q}_{m,a}\times\mathbb{R}^3 \rightharpoonup T$$

as $m\to+\infty$. □

Proof of Proposition 2.4: Without loss of generality we may suppose $Q=\overline{\mathcal{C}}^n := [0,1]^n$. We then modify the proof of Theorem 2.1 as follows.

Let $\widetilde{\mathcal{Q}}_{m,a}$ denote the partition of $\overline{\mathcal{C}}^n$ given by family of all the n-rectangles and n-cubes Q with boundary contained in the $(n-1)$-grid $C^{(n-1)}_{m,a}$ or in the boundary of $\mathcal{C}^n$, i.e. $\partial Q\subset C^{(n-1)}_{m,a}\cup\partial\mathcal{C}^n$. More precisely, $\widetilde{\mathcal{Q}}_{m,a}$ contains all the n-cubes of $\mathcal{Q}_{m,a}$ plus a family of n-rectangles $\widetilde{Q}$, with sides parallel to the coordinate axes, which are contained in $\overline{\mathcal{C}}^n$ and intersect the boundary of $\mathcal{C}^n$. We may and do choose $a(m)$ so that (2.3) holds, with $\widetilde{c}(n)=2n$, and each n-rectangle of $\widetilde{\mathcal{Q}}_{m,a}$ is in generic position with respect to T. For every n-rectangle Q in $\widetilde{\mathcal{Q}}_{m,a}$ and every $(n-1)$-face F of the boundary of Q, we apply Proposition 2.4, in dimension $n-1$, and define the sequence $u_k^F : F\to S^2$ so that in particular (2.5) holds. If $\widetilde{\mathcal{Q}}^{(n-1)}_{m,a}$ denotes the $(n-1)$-skeleton of $\widetilde{\mathcal{Q}}_{m,a}$, we then define the continuous sequence $v_k:\cup\widetilde{\mathcal{Q}}^{(n-1)}_{m,a}\to S^2$ as in (2.6), so that (2.7) holds for every n-rectangle Q of $\widetilde{\mathcal{Q}}_{m,a}$, the graph G_{v_k} has no boundary, $\partial G_{v_k}=0$, and from (2.3) and (2.5)

$$\sup_k \mathbf{D}(v_k,\cup\widetilde{\mathcal{Q}}^{(n-1)}_{m,a}) \le 2\,\widetilde{c}_{n-1}\,m\,n\,\mathbf{D}(T,\mathcal{C}^n\times S^2) + \mathbf{D}(\partial T,\partial\mathcal{C}^n\times S^2)\,. \tag{2.16}$$

As a consequence, Proposition 2.5 holds true for every n-rectangle $Q\in\widetilde{\mathcal{Q}}_{m,a}$. Then, similarly to Theorem 2.1 we extend v_k to a map $U_k^{(Q)}$ in the interior of each element Q of $\widetilde{\mathcal{Q}}_{m,a}$. More precisely, if Q is an n-rectangle of $\widetilde{\mathcal{Q}}_{m,a}$ which intersects the boundary of $\mathcal{C}^n$, since the vector $a(m)$ is chosen in $[1/4m,3/4m]^n$, then Q has sides of length $1/m$ or between $1/4m$ and $3/4m$. As a consequence, Q is bilipschitz homeomorphic to the n-cube $Q_m:=[0,1/m]^n$ for some affine bijective function $\psi_Q:Q\to Q_m$ with $\mathrm{Lip}\,\psi_Q\le 4$ and $\mathrm{Lip}\,\psi_Q^{-1}\le 3/4$. Therefore, if $\widetilde{U}_k^{(Q)}:Q_m\to S^2$ is defined as in (2.10) with $\widetilde{v}_k(x):=v_k(\psi_Q^{-1}(x))$ for $x\in\partial Q_m$, where $\varphi_k\in W^{1,2}(Q_m,S^2)$ is a continuous extension of $\widetilde{v}_{k|\partial Q_m}$, we set $U_k^{(Q)}(x):=\widetilde{U}_k^{(Q)}(\psi_Q(x))$. Since $\cup\widetilde{\mathcal{Q}}_{m,a}=\overline{\mathcal{C}}^n$, we now define $U_k^{(m)}:\overline{\mathcal{C}}^n\to S^2$ by $U_k^{(m)}(x):=U_k^{(Q)}(x)$ if $x\in Q$ for some $Q\in\widetilde{\mathcal{Q}}_{m,a}$. Taking $\delta=\delta(m,v_k)>0$ suitably small in (2.10), with $\lim_{k\to+\infty}\delta(m,v_k)=0$ for every m, it is not difficult to show that every $U_k^{(m)}:\overline{\mathcal{C}}^n\to S^2$ is a continuous map in $\mathrm{cart}^{2,1}(\overline{\mathcal{C}}^n,S^2)$ with

$$\sup_k \mathbf{D}(U_k^{(m)},\overline{\mathcal{C}}^n) \le 2\,c(n)\,\mathbf{D}(T,\mathcal{C}^n\times S^2) + \frac{1}{m} + \widehat{c}(n)\,\frac{1}{m}\,\mathbf{D}(\partial T,\partial\mathcal{C}^n\times S^2)$$

for each m, where again $c(n):=2\,\widetilde{c}_{n-1}\,n^2/(n-2)$ and $\widehat{c}(n)$ is an absolute constant. Then for m sufficiently large

$$\sup_k \mathbf{D}(U_k^{(m)},\overline{\mathcal{C}}^n) \le 4\,c(n)\,\mathbf{D}(T,\mathcal{C}^n\times S^2)$$

and by closure-compactness $G_{U_k^{(m)}}\rightharpoonup T_m$ as $k\to+\infty$ weakly in $\mathcal{D}_{n,2}(\overline{\mathcal{C}}^n\times S^2)$ for some $T_m\in\mathrm{cart}^{2,1}(\overline{\mathcal{C}}^n\times S^2)$. Finally, similarly to Proposition 2.6 we show that $T_m\rightharpoonup T$ weakly in $\mathcal{D}_n(\overline{\mathcal{C}}^n\times S^2)$ as $m\to+\infty$, so that by a diagonal procedure we obtain the assertion with $\widetilde{c}_n:=4\,c(n)=2c_n$. □

3 Mappings into manifolds

In this section we extend Theorem 2.1 to a wide class of target manifolds $\mathcal{Y}$ of dimension larger than or equal to 2.

We will consider any smooth oriented Riemannian manifold $\mathcal{Y}$ of dimension $M \geq 2$, isometrically embedded in $\mathbb{R}^N$ for some $N \geq 3$. As in Sec. 1, we assume that $\mathcal{Y}$ is compact, connected, without boundary and that its integral 2-homology group $H_2(\mathcal{Y}, \mathbb{Z})$ has no torsion. Moreover, we shall also assume that the Hurewicz homomorphism $\pi_2(\mathcal{Y}) \to H_2(\mathcal{Y}, \mathbb{Q})$ is injective. We observe that, by Hurewicz theorem [4], if in particular $\mathcal{Y}$ is 1-connected, i.e., $\pi_1(\mathcal{Y}) = 0$, then the last condition actually follows from the others.

Theorem 3.1 *Let $T \in \mathrm{cart}^{2,1}(B^n \times \mathcal{Y})$, $n \geq 2$. Then there exists a sequence of smooth maps $u_k : B^n \to \mathcal{Y}$ such that $G_{u_k} \rightharpoonup T$ weakly in $\mathcal{D}_{n,2}(B^n \times \mathcal{Y})$ and*

$$\sup_k \mathbf{D}(u_k, B^n) \leq c_n \, \mathbf{D}(T, B^n \times \mathcal{Y}) < +\infty \,,$$

where $c_n > 0$ is an absolute constant.

We notice that our result does not answer the problem, raised in [1], whether every $u \in W^{1,2}(B^n, \mathcal{Y})$ is the weak limit in $W^{1,2}(B^n, \mathcal{Y})$ of a sequence of smooth maps $u_k : B^n \to \mathcal{Y}$.

Proof of Theorem 3.1: Since the result is true for $n = 2$ by Theorem 4.1, it suffices to readapt Theorem 2.1, and therefore the inductive argument based on Proposition 2.4, with $\mathcal{Y}$, $\mathbb{R}^N$, weak convergence and boundary in $\mathcal{D}_{n,2}$ instead of S^2, $\mathbb{R}^3$, weak convergence and boundary in $\mathcal{D}_n$, respectively, taking account of the following facts: Proposition 2.4 holds for $n = 2$ by Corollary 4.2. Similarly to the case of normal currents, the slice (2.2) is well defined in $\mathrm{cart}^{2,1}(P(t, i) \times \mathcal{Y})$. Moreover, by definition of the class $\mathrm{cart}^{2,1}(\mathcal{C}^n, \mathcal{Y})$, by the structure of 2-dimensional currents in $\mathrm{cart}^{2,1}$ and by a slicing argument, we may again and choose $a(m)$ so that each n-cube Q of $\mathcal{Q}_{m,a}$ is in generic position with respect to T. Then $T^{(n-1)}_{m,a}$ is well defined and (2.3) holds. In Proposition 2.5, to prove that $v_{k|\partial Q}$ is homotopic to a constant map in $\mathcal{Y}$, since by assumption the Hurewicz homomorphism $\pi_2(\mathcal{Y}) \to H_2(\mathcal{Y}, \mathbb{Q})$ is injective, it suffices to show that for every closed 2-form ω in $\mathcal{Y}$, or in a basis of $\mathcal{Z}^2(\mathcal{Y})$, we have $d \, {v_{k|\partial Q}}^{\#}\omega = 0$. This follows from the same computation. In fact, (2.7) holds again, where ∂T_Q is by (2.1) the boundary, in $\mathcal{D}_{n,2}$ sense, of the Cartesian current $T_Q \in \mathrm{cart}^{2,1}(Q \times \mathcal{Y})$. Since for every $(n-4)$-form $\eta \in \mathcal{D}^{n-4}(\partial Q)$ we have that $\pi^{\#} d\eta \wedge \widehat{\pi}^{\#}\omega$ is both d_x-closed and d_y-closed, we obtain $\partial T_Q(\pi^{\#} d\eta \wedge \widehat{\pi}^{\#}\omega) = 0$ and by weak convergence in $\mathcal{D}_{n,2}$ the assertion. As to Proposition 2.6, by (2.12) we have that $\mathbf{D}(G_{U_k^{(Q)}}, \{\|x - p\| < \delta\}) \to 0$. Since the map h does not move the vertical directions, the homotopy formula holds again. Moreover, $\psi_{\#} T_Q = 0$ in $\mathcal{D}_{n-2}(\mathcal{C}^n \times \mathcal{Y})$, for $n \geq 3$, since it can take non zero values only on forms with more than two differentials in the vertical direction, hence (2.15) holds in $\mathcal{D}_{n,2}$, where

$$\|R_m\|_{\mathbf{D}} \leq \frac{c}{m} \mathbf{D}(T, \mathcal{C}^n \times S^2) \to 0$$

by (1.5). This yields again that $T_m - T \llcorner \cup \mathcal{Q}_{m,a} \times \mathbb{R}^N \rightharpoonup 0$ in $\mathcal{D}_{n,2}(\mathcal{C}^n \times \mathcal{Y})$ and hence Theorem 3.1. Finally, Proposition 2.4 follows from a readaptation of Theorem 3.1 similar to the one in Sec. 2. □

4 A strong density result

In this section we prove the following strong density result for the Dirichlet energy of maps from 2-dimensional domains into general target manifolds. Compare Sec. 4.1.2 in [6, vol. II] for the case $\mathcal{Y} = S^2$. As before, $\mathcal{C}^2 :=]0, 1[^2$, the unit open square in $\mathbb{R}^2$.

Theorem 4.1 *Let $n = 2$. Let $\mathcal{Y}$ be a smooth, compact, connected, oriented Riemannian manifold of dimension $M \geq 2$, isometrically embedded in $\mathbb{R}^N$, $N \geq 3$. Assume that the integral 2-homology group $H_2(\mathcal{Y}, \mathbb{Z})$ has no torsion. Then, for every $T \in \mathrm{cart}^{2,1}(\mathcal{C}^2 \times \mathcal{Y})$ there exists a sequence of smooth maps $u_k : \mathcal{C}^2 \to \mathcal{Y}$ such that $G_{u_k} \rightharpoonup T$ weakly in $\mathcal{D}_2(\mathcal{C}^2 \times \mathcal{Y})$ and*

$$\lim_{k \to +\infty} \mathbf{D}(u_k, \mathcal{C}^2) = \mathbf{D}(T, \mathcal{C}^2 \times \mathcal{Y}) \,. \tag{4.1}$$

As a consequence, if the boundary of T coincides with the graph of a $W^{1,2}$ map, we obtain the following

Corollary 4.2 *Under the hypotheses of Theorem 4.1, if $\partial T = G_\varphi$ for some function $\varphi \in W^{1,2}(\partial\mathcal{C}^2, \mathcal{Y})$, then there exists a sequence of continuous maps $u_k : \overline{\mathcal{C}}^2 \to \mathcal{Y}$, with $\{u_k\} \subset \mathrm{cart}^{2,1}(\mathcal{C}^2, \mathcal{Y})$ and $\partial G_{u_k} = \partial T$, hence $u_{k|\partial\mathcal{C}^2} = \varphi$, such that $G_{u_k} \rightharpoonup T$ weakly in $\mathcal{D}_2(\mathcal{C}^2 \times \mathcal{Y})$ and (4.1) holds.*

We recall by Sec. 1 that every $T \in \mathrm{cart}^{2,1}(\mathcal{C}^2 \times \mathcal{Y})$ has the form

$$T = G_{u_T} + \sum_{i=1}^{I} \delta_{x_i} \times C_i + S_{T,sing} \tag{4.2}$$

for some $x_i \in \mathcal{C}^2$, where the non trivial 2-cycles C_i are of spherical type, compare Definition 1.1, and $S_{T,sing}$ is a completely vertical, homologically trivial, i.m. rectifiable current supported on a set not containing $\{x_i\} \times \mathcal{Y}$, $i = 1, \ldots, I$; moreover

$$\mathbf{D}(T) = \frac{1}{2} \int_{\mathcal{C}^2} |Du_T|^2 \, dx + \sum_{i=1}^{I} \mathbf{M}(C_i) + \mathbf{M}(S_{T,sing}) \,.$$

Proposition 4.3 (Approximation of spherical cycles) *Let u be a smooth map from B^2 into $\mathcal{Y}$. Let $C \in \mathcal{Z}_2(\mathcal{Y})$ be a 2-cycle of spherical type. Then there exist a sequence $\{u_k\}$ of smooth maps from B^2 into $\mathcal{Y}$ and a sequence of radii $\delta_k \searrow 0$ such that $u_k = u$ outside $B^2_{\delta_k}$ and $G_{u_k} \rightharpoonup G_u + \delta_0 \times C$ weakly in $\mathcal{D}_2(B^2 \times \mathcal{Y})$ with*

$$\lim_{k \to +\infty} \mathbf{D}(u_k, B^2) = \mathbf{D}(u, B^2) + \mathbf{M}(C) \,.$$

Proposition 4.4 (Approximation of the singular vertical part) *Under the hypotheses of Theorem 4.1, if (4.2) holds, then for every smooth map u from $\mathcal{C}^2$ into $\mathcal{Y}$ there exists a sequence $\{u_h\}$ of smooth maps from $\mathcal{C}^2$ into $\mathcal{Y}$ such that $G_{u_h} \rightharpoonup G_u + S_{T,sing}$ weakly in $\mathcal{D}_2(\mathcal{C}^2 \times \mathcal{Y})$ and*

$$\lim_{h \to +\infty} \mathbf{D}(u_h, \mathcal{C}^2) = \mathbf{D}(u, \mathcal{C}^2) + \mathbf{M}(S_{T,sing}) \,.$$

We postpone the proof of these results and first prove Theorem 4.1 and Corollary 4.2.

Proof of Theorem 4.1: Since $n = 2$, by Schoen-Uhlenbeck's density theorem [9] we can find a sequence of smooth maps $u_k : \mathcal{C}^2 \to \mathcal{Y}$ such that $u_k \to u_T$ strongly in $W^{1,2}(\mathcal{C}^2, \mathcal{Y})$. On small disks around each x_i and contained in $\mathcal{C}^2$, we first apply Proposition 4.3 to each u_k and find a sequence of smooth maps $\{u_{k,h}\}_h$ from $\mathcal{C}^2$ into $\mathcal{Y}$ and a sequence of radii $\delta_{k,h} \searrow 0$ as $h \to +\infty$ such that $u_{k,h} = u_k$ outside $B^2_{\delta_{k,h}}(x_i)$,

$$G_{u_{k,h}} \rightharpoonup G_{u_k} + \sum_{i=1}^{I} \delta_{x_i} \times C_i$$

weakly in $\mathcal{D}_2(\mathcal{C}^2 \times \mathcal{Y})$ and

$$\lim_{h \to +\infty} \mathbf{D}(u_{k,h}, \mathcal{C}^2) = \mathbf{D}(u_k, \mathcal{C}^2) + \sum_{i=1}^{I} \mathbf{M}(C_i) \,.$$

Secondly, we apply Proposition 4.4 to each $u_{k,h}$ and find a sequence of smooth maps $\{u_{k,h,l}\}_l$ from $\mathcal{C}^2$ into $\mathcal{Y}$ such that $G_{u_{k,h,l}} \rightharpoonup G_{u_{k,h}} + S_{T,sing}$ weakly in $\mathcal{D}_2(\mathcal{C}^2 \times \mathcal{Y})$ as $l \to +\infty$ and

$$\lim_{l \to +\infty} \mathbf{D}(u_{k,h,l}, \mathcal{C}^2) = \mathbf{D}(u_{k,h}, \mathcal{C}^2) + \mathbf{M}(S_{T,sing}) \,.$$

The claim follows by a diagonal procedure. □

Proof of Corollary 4.2: Since φ is Hölder continuous in $\partial\mathcal{C}^2$, it suffices to apply Schoen-Uhlenbeck's density theorem requiring in particular that $\partial G_{u_k} = G_\varphi$. Moreover, since the points x_i in (4.2) are distant from the boundary of $\mathcal{C}^2$, we apply Proposition 4.3 requiring that $B^2_{\delta_{k,h}}(x_i) \subset\subset \mathcal{C}^2$, so that in particular $u_{k,h}$ coincides with u_k in a small neighborhood of $\partial\mathcal{C}^2$. Finally, since in the proof of Proposition 4.4 we modify the functions $u_{k,h}$ near points which have positive distance from the boundary of $\mathcal{C}^2$, the functions $u_{k,h,l}$ coincide with $u_{k,h}$ in a small neighborhood of $\partial\mathcal{C}^2$, whence $\partial G_{u_{k,h,l}} = G_\varphi$, as required. □

To prove Proposition 4.3 we use of the following result.

Proposition 4.5 *Let $C \in \mathcal{Z}_2(\mathcal{Y})$ be a 2-cycle of spherical type and $P \in \mathcal{Y}$ be a given point. Then there exists a sequence of Lipschitz functions $f_k : B^2 \to \mathcal{Y}$ such that $f_{k|\partial B^2} \equiv P$, $f_{k\#}[\![B^2]\!] \rightharpoonup C$ weakly in $\mathcal{D}_2(\mathcal{Y})$ and*

$$\lim_{k\to+\infty} \mathbf{D}(f_k, B^2) = \mathbf{M}(C)\,.$$

We postpone its proof and we give the

Proof of Proposition 4.3: If $P := u(0)$, we denote by (U, φ) a local chart centered at P. More precisely, let U be a relatively open and connected subset of $\mathcal{Y}$, containing P, and let $\varphi : U \to V$ be a bilipschitz homeomorphism of U onto an open subset V of $\mathbb{R}^M$ with $\varphi(P) = 0$; finally let $r > 0$ be such that $u(B^2_r) \subset U$. We define, for $k \in \mathbb{N}$ and $\delta \in (0, r)$,

$$u_{k,\delta}(x) := \begin{cases} u(x) & \text{if} \quad \delta < |x| < 1 \\ v_\delta(x) & \text{if} \quad \delta/2 \le |x| \le \delta \\ f_k(2x/\delta) & \text{if} \quad |x| < \delta/2\,, \end{cases}$$

where f_k is given by Proposition 4.5, with $P = u(0)$, and

$$v_\delta(x) := \varphi^{-1}\biggl(\Bigl(\frac{2}{\delta}\,|x| - 1\Bigr) \cdot \varphi \circ u\Bigl(\delta\,\frac{x}{|x|}\Bigr)\biggr)\,.$$

Now, since $v_\delta(x) = u(x)$ for $|x| = \delta$ and $v_\delta(x) \equiv \varphi^{-1}(0) = P$ for $|x| = \delta/2$, it follows that $u_{k,\delta}$ is Lipschitz continuous. Moreover by Proposition 4.5 and a change of variables

$$\mathbf{D}(u_{k,\delta}, B^2_{\delta/2}) = \mathbf{D}(f_k, B^2) \to \mathbf{M}(C)$$

as $k \to +\infty$, so that the claim holds if we show that

$$\liminf_{\delta\to 0^+} \mathbf{D}(v_\delta, B^2_\delta \setminus B^2_{\delta/2}) = 0\,, \tag{4.3}$$

by taking $u_k := u_{k,\delta_k}$ for a suitable sequence $\delta_k \searrow 0$. Now we estimate

$$\mathbf{D}(v_\delta, B^2_\delta \setminus B^2_{\delta/2}) \le c\,\|D\varphi^{-1}\|^2_\infty \biggl(\|\varphi \circ u\|^2_{\infty,\partial B^2_\delta} + \|D\varphi\|^2_\infty \cdot \delta \int_{\partial B^2_\delta} |D_\tau u|^2\, d\mathcal{H}^1\biggr)\,,$$

where $c > 0$ is an absolute constant and τ is the tangential direction to ∂B^2_δ. By continuity we have $\|\varphi \circ u\|_{\infty,\partial B^2_\delta} \to 0$ as $\delta \to 0^+$. Moreover, if $F(\delta) := \int_{\partial B^2_\delta} |D_\tau u|^2\, d\mathcal{H}^1$, by the coarea formula [3]

$$\int_0^r F(\delta)\, d\delta \le \int_{B^2_r} |Du|^2\, dx < +\infty,$$

so that F is a non-negative function in $L^1(0, r)$. As a consequence, $\liminf_{\delta\to 0^+} \delta\, F(\delta) = 0$ and then (4.3) holds, as required. □

Remark 4.6 For future use, we set

$$\mathcal{Y}_\varepsilon := \overline{U_\varepsilon(\mathcal{Y})}\,,$$

where $U_\varepsilon(A) := \{y \in \mathbb{R}^N \mid \mathrm{dist}(y, A) < \varepsilon\}$ is the ε-neighborhood of $A \subset \mathbb{R}^N$, and we observe that, since $\mathcal{Y}$ is smooth, there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ the nearest point projection of $\mathcal{Y}_\varepsilon$ onto $\mathcal{Y}$ is a well defined Lipschitz map with Lipschitz constant $L_\varepsilon \to 1^+$ as $\varepsilon \to 0^+$.

Proof of Proposition 4.5: We divide the proof in four steps. According to Definition 1.1, $C - Z = \partial R$ and $Z = \varphi_{\#}[\![S^2]\!]$. First, we approximate C, Z and R with polyhedral chains C_ε, Z_ε and R_ε so that (4.5) holds for some Lipschitz function ϕ_ε of S^2; secondly, we find a Lipschitz map f_ε with $f_{\varepsilon\#}[\![S^2]\!] = C_\varepsilon$; then we show that we can choose f_ε so that in particular its mapping area is equal to the mass of C_ε; finally, we project f_ε onto $\mathcal{Y}$ and prove the assertion by means of Morrey's ε-conformality theorem.

Step 1: We first apply Federer's strong approximation theorem [3, 4.2.20] to C and find, for every $\varepsilon > 0$, an integral polyhedral cycle C_ε in $\mathbb{R}^N$ and a C^1-diffeomorphism $\psi_\varepsilon : \mathbb{R}^N \to \mathbb{R}^N$ such that

$$\mathbf{M}(C_\varepsilon - \psi_{\varepsilon\#}C) \le \varepsilon\,.$$

Moreover, $\operatorname{Lip}\psi_\varepsilon, \operatorname{Lip}{\psi_\varepsilon}^{-1} \le 1 + \varepsilon$ and $\operatorname{spt} C_\varepsilon \subset \mathcal{Y}_\varepsilon$, so that in particular

$$C_\varepsilon \rightharpoonup C \qquad \text{and} \qquad \mathbf{M}(C_\varepsilon) \to \mathbf{M}(C) \tag{4.4}$$

as $\varepsilon \to 0^+$. Now, since $C_\varepsilon - \psi_{\varepsilon\#}C$ strongly converges to zero, for ε small C_ε is of spherical type in $\mathcal{Y}_\varepsilon$. Then, there exist $\widetilde{Z}_\varepsilon \in \mathcal{Z}_2(\mathcal{Y}_\varepsilon)$ and $\widetilde{R}_\varepsilon \in \mathcal{R}_3(\mathcal{Y}_\varepsilon)$ such that

$$C_\varepsilon - \widetilde{Z}_\varepsilon = \partial \widetilde{R}_\varepsilon \qquad \text{and} \qquad \widetilde{\phi}_{\varepsilon\#}[\![S^2]\!] = \widetilde{Z}_\varepsilon$$

for some Lipschitz function $\widetilde{\phi}_\varepsilon : S^2 \to \mathcal{Y}_\varepsilon$.

As in [11], regarding $\mathbb{R}^N$ as the subspace $\mathbb{R}^N \times \{0_{\mathbb{R}^3}\}$ of $\mathbb{R}^{N+3}$, we now define a Lipschitz homotopy $h_\varepsilon : [0,1] \times S^2 \to \mathbb{R}^{N+3}$, with $h_\varepsilon(0,\cdot) \equiv \widetilde{\phi}_\varepsilon$, such that if $\phi_\varepsilon := h_\varepsilon(1,\cdot)$, then ϕ_ε is a Lipschitz embedding and $Z_\varepsilon := \phi_{\varepsilon\#}[\![S^2]\!]$ is polyhedral (and $h_{\varepsilon\#}([\![0,1]\!] \times [\![S^2]\!])$ has arbitrarily small mass). Moreover, we may and do define h_ε so that $h_\varepsilon([0,1] \times S^2) \subset \widetilde{\mathcal{Y}}_\varepsilon$, where $\widetilde{\mathcal{Y}}_\varepsilon := \mathcal{Y}_{2\varepsilon} \times \mathbb{R}^3$, and $\operatorname{spt} Z_\varepsilon \cap \operatorname{spt} C_\varepsilon = \emptyset$.

Setting now $T_\varepsilon := \widetilde{R}_\varepsilon - h_{\varepsilon\#}([\![0,1]\!] \times [\![S^2]\!])$, since $\partial h_{\varepsilon\#}([\![0,1]\!] \times [\![S^2]\!]) = Z_\varepsilon - \widetilde{Z}_\varepsilon$, we have that T_ε is an i.m. rectifiable current with compact support in $\widetilde{\mathcal{Y}}_\varepsilon$ and polyhedral boundary

$$\partial T_\varepsilon = C_\varepsilon - Z_\varepsilon\,.$$

As a consequence of [3, 4.2.19], we find an integral current $S_\varepsilon \in \mathcal{R}_4(\mathbb{R}^{N+3})$, with compact support, and a C^1-diffeomorphism $g_\varepsilon : \mathbb{R}^{N+3} \to \mathbb{R}^{N+3}$ such that

$$R_\varepsilon := g_{\varepsilon\#}T_\varepsilon - \partial S_\varepsilon$$

is polyhedral, $\mathbf{M}(S_\varepsilon) + \mathbf{M}(\partial S_\varepsilon) \le \varepsilon$, $\operatorname{spt} S_\varepsilon \subset \overline{U_\varepsilon(\operatorname{spt} T_\varepsilon)}$, $\operatorname{Lip} g_\varepsilon, \operatorname{Lip} {g_\varepsilon}^{-1} \le 1 + \varepsilon$ and $g_\varepsilon(x) = x$ if $x \in \operatorname{spt} \partial T_\varepsilon$. Then, since $\operatorname{spt} \partial T_\varepsilon = \operatorname{spt} C_\varepsilon \cup \operatorname{spt} Z_\varepsilon$ and $\operatorname{spt} C_\varepsilon \cap \operatorname{spt} Z_\varepsilon = \emptyset$, we finally infer

$$\partial R_\varepsilon = g_{\varepsilon\#}\partial T_\varepsilon = C_\varepsilon - Z_\varepsilon \qquad \text{and} \qquad \phi_{\varepsilon\#}[\![S^2]\!] = Z_\varepsilon\,. \tag{4.5}$$

Step 2: By a readaptation of the argument in [11], we construct a suitable Lipschitz homotopy $H_\varepsilon : [0,1] \times S^2 \to \widetilde{\mathcal{Y}}_\varepsilon$, see (4.7), with $H_\varepsilon(0,\cdot) \equiv \phi_\varepsilon$, such that if $\varphi_\varepsilon := H_\varepsilon(1,\cdot)$, then

$$\varphi_{\varepsilon\#}[\![S^2]\!] = C_\varepsilon\,.$$

To this aim, let $\sum_{i=1}^r S_i$ be a simplicial decomposition of R_ε, that is, a triangulation of R_ε into oriented 3-simpleces S_i such that any two S_i's that do not coincide either are disjoint or intersect along a common lower-dimensional edge. Since ϕ_ε is a Lipschitz embedding, it satisfies:

$$\begin{gathered}\text{There is some simplicial decomposition } \Delta \text{ of } S^2 \text{ such that } \phi_\varepsilon \text{ maps}\\ \text{each curvilinear 2-simplex } D \text{ of } \Delta \text{ bijectively onto a 2-face of one of the } S_i\text{'s}.\end{gathered} \tag{4.6}$$

Now, choose a 3-simplex, say S_1, one of whose faces is $\phi_{\varepsilon\#}[\![D]\!]$ for some D of Δ. Then clearly there is a Lipschitz homotopy $h_\varepsilon^{(1)} : [0,1] \times S^2 \to \widetilde{\mathcal{Y}}_\varepsilon$, with $h_\varepsilon^{(1)}(0,\cdot) \equiv \phi_\varepsilon$, for which $h_\varepsilon^{(1)}(t,x) = \phi_\varepsilon(x)$ for every $t \in [0,1]$, if $x \notin D$, and such that if $\varphi_\varepsilon^{(1)} := h_\varepsilon^{(1)}(1,\cdot)$, then $h_\varepsilon^{(1)}$ sweeps out S_1 once, with the right orientation, so that $h_{\varepsilon\#}^{(1)}([\![0,1]\!] \times [\![S^2]\!]) = S_1$ and hence by (4.5)

$$\varphi_{\varepsilon\#}^{(1)}[\![S^2]\!] = -\partial \sum_{i=2}^r S_i + C_\varepsilon\,.$$

Moreover, by taking the barycentric subdivision of D we define $h_\varepsilon^{(1)}$ so that $\varphi_\varepsilon^{(1)}$ satisfies (4.6). Finally, iterating the process r times, and gluing together the homotopies $h_\varepsilon^{(i)}$, for $i=1,\dots r$, we then define H_ε as required. In particular, taking into account (4.6),

$$\begin{gathered}\text{There is a simplicial decomposition of } S^2, \text{ say } \widetilde\Delta, \text{ such that}\\ \varphi_\varepsilon \text{ maps each curvilinear 2-simplex } D \text{ of } \widetilde\Delta \text{ bijectively onto a 2-face of the 2-skeleton of } T_\varepsilon.\end{gathered} \tag{4.7}$$

Step 3: We construct a Lipschitz map $g_\varepsilon : S^2 \to \widetilde{\mathcal{Y}}_\varepsilon$ such that g_ε takes the given value P and maps S^2 into C_ε with mapping area equal to the mass of C_ε, i.e.,

$$g_{\varepsilon\#}[\![S^2]\!] = C_\varepsilon \qquad \text{and} \qquad A(g_\varepsilon, S^2) = \mathbf{M}(C_\varepsilon)\,. \tag{4.8}$$

By (4.7) let $\{\widetilde D_i\}$ be a subfamily of the simpleces of $\widetilde\Delta$ such that φ_ε maps each $\widetilde D_i$ bijectively onto a 2-face of C_ε (with multiplicity and orientation), so that if $\widetilde W := \cup \widetilde D_i$, then

$$g_{\varepsilon\#}[\![\widetilde W]\!] = C_\varepsilon \qquad \text{and} \qquad A(g_\varepsilon, \widetilde W) = \mathbf{M}(C_\varepsilon)\,.$$

For every i, let D_i be the 2-simplex obtained by contracting $\widetilde D_i$ from its barycenter with an homothetic factor $1/2$, so that $\mathrm{dist}(D_{i_1}, D_{i_2}) > 0$ if $i_1 \neq i_2$. Finally, let $W := \cup D_i$. We first define g_ε on each D_i by contracting $\varphi_{\varepsilon|\widetilde D_i}$ from the barycenter of $\widetilde D_i$. Then, since the mapping area is invariant under reparametrizations of the domain, (4.8) clearly holds if we are able to find a Lipschitz extension of g_ε to the whole 2-sphere so that the image of $g_{\varepsilon|S^2\setminus W}$ is 1-dimensional.

To do this, we first make a list of the 1-simpleces of the 1-skeleton of C_ε, each one with a fixed orientation. Then, for every i, we label each 1-face I of the boundary of D_i with $\pm j$, according to the property that g_ε maps I, with the orientation induced by D_i, onto the j^{th} 1-simplex of C_ε with orientation $\pm$.

Let now C_ε^k, $k=1,\dots,l$, be the connected components of C_ε, so that $C_\varepsilon = \sum_{k=1}^l C_\varepsilon^k$, $\partial C_\varepsilon^k = 0$, $\mathbf{M}(C_\varepsilon) = \sum_{k=1}^l \mathbf{M}(C_\varepsilon^k)$ and $\mathrm{spt}\, C_\varepsilon^k$ is connected. At the first step, we consider the simpleces D_i corresponding to the faces of C_ε^1, say $D_1,\dots,D_m$. Possibly reordering the D_i's, for every $i=1,\dots,m-1$ we connect D_i with D_{i+1} by a rectifiable arc γ_i with suitably chosen initial and final points IP_i and FP_i in the 0-skeleton of D_i and D_{i+1}, respectively, so that $g_\varepsilon(IP_i) = g_\varepsilon(FP_i)$, the interior of γ_i lies in $S^2 \setminus W$ and γ_i does not intersect γ_j for $j=1,\dots,i-1$. Also, we slightly modify g_ε on the D_i's so that it is constant near the end points of γ_i. Then, by taking a small tubular neighborhood Γ_i of γ_i in S^2, we extend g_ε on Γ_i as the constant map equal to $g_\varepsilon(IP_i) = g_\varepsilon(FP_i)$.

As a consequence, if $O_1 := \bigcup_{i=1}^m (D_i \cup \Gamma_i)$, with $\Gamma_m = \emptyset$, then

i) O_1 has positive distance from each of the remaining simpleces D_i;

ii) $g_{\varepsilon\#}[\![O_1]\!] = C_\varepsilon^1$ and $A(g_\varepsilon, O_1) = \mathbf{M}(C_\varepsilon^1)$.

Actually, since $\partial C_\varepsilon^1 = 0$, and O_1 is a topological disk in S^2, by using the labels $\pm j$ of the 1-faces of the D_i's, everything can be done in such a way that

iii) $g_{\varepsilon|\partial O_1}$ is contractible.

By induction on the connected components of C_ε, for $k=2,\dots,l$, at the k^{th} step we repeat the previous argument for C_ε^k, defining g_ε on O_k so that i), ii) and iii) hold, with k instead of 1. Moreover, we define the arcs γ_i so that in particular the Γ_i's do not intersect any of the O_j's, for $j=1,\dots,k-1$. Then we can also require that

iv) O_k has positive distance from O_j, for every $j=1,\dots,k-1$.

As a consequence, by iii) and iv) we can find for every k a small neighbourhood $\widetilde O_k$ of O_k in S^2, with $\mathrm{dist}(\widetilde O_{k_1}, \widetilde O_{k_2}) > 0$ if $k_1 \neq k_2$, and a Lipschitz extension of $g_{\varepsilon|O_k}$ to $\widetilde O_k$, so that the image of $\widetilde O_k \setminus O_k$ is a 1-dimensional subset of $\widetilde{\mathcal{Y}}_\varepsilon$ and g_ε takes a constant value, say P_k, in the boundary of $\widetilde O_k$.

Now, for every $k = 1, \dots, l-1$, we connect one point of the boundary of $\widetilde{O}_k$ with one point of the boundary of $\widetilde{O}_{k+1}$ by a rectifiable arc $\widetilde{\gamma}_k$ such that the interior of $\widetilde{\gamma}_k$ lies inside $S^2 \setminus \cup_{k=1}^l \widetilde{O}_k$ and $\widetilde{\gamma}_k$ does not intersect any of the $\widetilde{\gamma}_j$, for $j = 1, \dots, k-1$. Then define g_ε on each $\widetilde{\gamma}_k$ by parametrizing a Lipschitz continuous arc connecting the points P_k and P_{k+1}, so that $g_\varepsilon(\widetilde{\gamma}_k) \subset \widetilde{\mathcal{Y}}_\varepsilon$. As before, we take in S^2 small neighborhoods $\widehat{O}_k$ of the $\widetilde{O}_k$'s, and $\widetilde{\Gamma}_k$ of the $\widetilde{\gamma}_k$'s (with $\widetilde{\Gamma}_l = \emptyset$), so that $\operatorname{dist}(\widehat{O}_{k_1}, \widehat{O}_{k_2}) > 0$ and $\operatorname{dist}(\widetilde{\Gamma}_{k_1}, \widetilde{\Gamma}_{k_2}) > 0$ if $k_1 \neq k_2$. Since the arc connecting P_1 with P_l via the $\widetilde{\gamma}_k$'s is contractible, we then find a Lipschitz extension of g_ε to $O := \bigcup_{k=1}^l (\widehat{O}_k \cup \widetilde{\Gamma}_k)$ such that the image of $O \setminus \bigcup_{k=1}^l ((\widehat{O}_k \cup \widetilde{\Gamma}_k) \setminus \widetilde{O}_k)$ is a 1-dimensional subset of $\widetilde{\mathcal{Y}}_\varepsilon$ and g_ε takes the given constant value P in the boundary of O. Finally set $g_\varepsilon \equiv P$ on $S^2 \setminus O$.

Step 4: By Step 3, fix $Q_\varepsilon \in S^2$ such that $g_\varepsilon(Q_\varepsilon) = P$. Let $\varphi_\varepsilon : B^2 \to S^2$ be a Lipschitz function between the unit disk and the 2-sphere such that $\varphi_{\varepsilon|\partial B^2} \equiv Q_\varepsilon$ and φ_ε maps the interior of B^2 bijectively onto $S^2 \setminus \{Q_\varepsilon\}$, so that $\varphi_{\varepsilon\#}[\![B^2]\!] = [\![S^2]\!]$. Of course, it can be obtained by first asking $\varphi_{\varepsilon|\partial B^2} \equiv$ South Pole, and then by rotating S^2. Moreover, let $\Pi : \mathbb{R}^{N+3} \to \mathbb{R}^N$ be the orthogonal projection onto the first N coordinates and $\Pi_\varepsilon : \mathcal{Y}_{2\varepsilon} \to \mathcal{Y}$ be the nearest point projection, see Remark 4.6.

Set $f_\varepsilon := \Pi_\varepsilon \circ \Pi \circ g_\varepsilon \circ \varphi_\varepsilon : B^2 \to \mathcal{Y}$; for $\varepsilon > 0$ small f_ε is a Lipschitz continuous function, with $f_{\varepsilon|\partial B^2} \equiv P$, and by (4.8) and (4.4), since $(\Pi_\varepsilon \circ \Pi)_\# C = C$,

$$f_{\varepsilon\#}[\![B^2]\!] = (\Pi_\varepsilon \circ \Pi \circ g_\varepsilon)_\#[\![S^2]\!] = (\Pi_\varepsilon \circ \Pi)_\# C_\varepsilon \rightharpoonup C$$

weakly in $\mathcal{D}_2(\mathcal{Y})$, as $\varepsilon \to 0^+$. Moreover, since $\operatorname{Lip}\Pi = 1$, we also have

$$A(f_\varepsilon, B^2) = A(\Pi_\varepsilon \circ \Pi \circ g_\varepsilon, S^2) \le (\operatorname{Lip}\Pi_\varepsilon)^2 A(g_\varepsilon, S^2)\,. \tag{4.9}$$

Finally we apply Morrey's ε-conformality theorem [8, Thm. 2.1] and define an orientation preserving diffeomorphism $\phi_\varepsilon : B^2 \to B^2$ such that

$$\mathbf{D}(f_\varepsilon \circ \phi_\varepsilon, B^2) \le (1+\varepsilon)\, A(f_\varepsilon \circ \phi_\varepsilon, B^2) = (1+\varepsilon)\, A(f_\varepsilon, B^2)\,.$$

Then, by (4.9), (4.8) and (4.4), since $\operatorname{Lip}\Pi_\varepsilon \to 1^+$ as $\varepsilon \to 0^+$, see Remark 4.6, we obtain $\lim_{\varepsilon\to 0}\mathbf{D}(f_\varepsilon \circ \phi_\varepsilon, B^2) = \mathbf{M}(C)$ and then the assertion. □

PROOF OF PROPOSITION 4.4: We divide it in four steps.

Step 1: We first show that there exists a sequence $\{S_j\}$ of i.m. rectifiable cycles in $\mathcal{C}^2 \times \mathcal{Y}$ such that $S_j \rightharpoonup S_{T,sing}$ weakly as currents, $\mathbf{M}(S_j) \to \mathbf{M}(S_{T,sing})$ and each S_j has the following structure

$$S_j := \sum_{k=1}^{I_j} \delta_{p_k^j} \times \partial R_k^j$$

for some distinct points $p_k^j \in \mathcal{C}^2$ and for some i.m. rectifiable currents $R_k^j \in \mathcal{R}_3(\mathcal{Y})$, with $\partial R_k^j \in \mathcal{R}_2(\mathcal{Y})$.

To this aim, fix $0 < \rho << 1$ and let $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$ be such that $|t_k - t_{k-1}| < \rho$ for every $k = 1, \dots, m+1$. Also, let $d_i(x,y) := x^i$, $x = (x^1, x^2)$, $y \in \mathbb{R}^N$. We recall by Sec. 1 that $S_{T,sing}(\omega) \neq 0$ only on forms $\omega \in \mathcal{D}^2(\mathcal{C}^2 \times \mathcal{Y})$ such that $d_y\omega^{(2)} \neq 0$. Then by slicing theory we infer that $< S_{T,sing}, d_i, t >= 0$, and hence $\partial(S_{T,sing} \llcorner \{d_i < t\}) = 0$, for a.e. t and for $i = 1,2$. As a consequence, we may and do choose the t_k's so that $< S_{T,sing}, d_1, t_k >= 0$ and hence $\partial(S_{T,sing} \llcorner \{d_1 < t_k\}) = 0$ for every k. If $p_\rho : \mathcal{C}^2 \times \mathbb{R}^N \to \mathcal{C}^2 \times \mathbb{R}^N$ is the map given by $p_\rho(x,y) := (q_\rho(x_1), x_2, y)$, where

$$q_\rho(t) := \begin{cases} \min\{t_k \mid t_k > t\} & \text{if} \quad t < t_m \\ t_m & \text{if} \quad t \ge t_m\,, \end{cases}$$

we define

$$p_{\rho\#} S_{T,sing} := \lim_{j\to+\infty} p_{\rho\#}(S_{T,sing} \llcorner \{x \in \mathcal{C}^2 \mid |x^1 - t_k| > r_j \quad \forall\, k = 1, \dots, m\} \times \mathbb{R}^N)\,, \tag{4.10}$$

where $r_j \searrow 0$ is such that $\partial(S_{T,sing} \llcorner \{x \in \mathcal{C}^2 \mid |x^1 - t_k| < r_j\} \times \mathbb{R}^N) = 0$ for all k and j. Then, since $|Dp_\rho| \leq 1$ a.e., by Federer-Fleming's closure-compactness theorem we infer that the limit in (4.10) exists and is an i.m. rectifiable current in $\mathcal{R}_2(\mathcal{C}^2 \times \mathcal{Y})$ with $\mathbf{M}(p_{\rho\#}S_{T,sing}) \leq \mathbf{M}(S_{T,sing})$. Moreover, setting $h_\rho(t,x,y) := t\, p_\rho(x,y) + (1-t)\,(x,y)$, $t \in [0,1]$, and defining $h_{\rho\#}([\![0,1]\!] \times S_{T,sing})$ in a way similar to (4.10), since $\mathbf{M}(h_{\rho\#}([\![0,1]\!] \times S_{T,sing})) \leq \rho\, \mathbf{M}(S_{T,sing})$, we infer that $p_{\rho\#}S_{T,sing} \rightharpoonup S_{T,sing}$ as $\rho \to 0^+$. By slicing a second time $p_{\rho\#}S_{T,sing}$ with respect to d_2, and arguing in a similar way, by a diagonal procedure we define a sequence S_j converging to $S_{T,sing}$ weakly with the mass, such that

$$\operatorname{spt} S_j \subset \left(\bigcup_{k=1}^{I_j} \{p_k^j\} \right) \times \mathcal{Y}\,.$$

Finally, due to the trivial homology of $S_{T,sing}$, by weak convergence, for j sufficiently large $S_j(\omega)$ is non zero only on forms $\omega \in \mathcal{D}^2(\mathcal{C}^2 \times \mathcal{Y})$ such that $d_y\omega^{(2)} \neq 0$. We then infer that each component $S_j \llcorner \{p_k^j\} \times \mathbb{R}^N$ of S_j has the form $\delta_{p_k^j} \times \partial R_k^j$ for some $R_k^j \in \mathcal{R}_3(\mathcal{Y})$ with $\partial R_k^j \in \mathcal{R}_2(\mathcal{Y})$, as required.

Step 2: Fix a 3-dimensional integral current R in $\mathcal{Y}$, i.e., an i.m. rectifiable current $R \in \mathcal{R}_3(\mathcal{Y})$ with $\partial R \in \mathcal{R}_2(\mathcal{Y})$, and a point $P \in \mathcal{Y}$. We show existence of a sequence of Lipschitz functions $f_h : B^2 \to \mathcal{Y}$ such that $f_{h|\partial B^2} \equiv P$, $f_{h\#}[\![B^2]\!] \rightharpoonup \partial R$ weakly in $\mathcal{D}_2(\mathcal{Y})$ and

$$\lim_{h\to+\infty} \mathbf{D}(f_h, B^2) = \mathbf{M}(\partial R)\,. \tag{4.11}$$

To this aim, as in Step 1 of Proposition 4.5, we first apply Federer's strong approximation theorem and find, for every $\varepsilon > 0$, a 3-dimensional i.m. polyhedral chain R_ε in $\mathbb{R}^N$ and a C^1-diffeomorphism $\psi_\varepsilon : \mathbb{R}^N \to \mathbb{R}^N$ such that

$$\mathbf{M}(R_\varepsilon - \psi_{\varepsilon\#}R) + \mathbf{M}(\partial R_\varepsilon - \partial\psi_{\varepsilon\#}R) \leq \varepsilon\,.$$

Moreover, $\operatorname{Lip}\psi_\varepsilon, \operatorname{Lip}{\psi_\varepsilon}^{-1} \leq 1+\varepsilon$ and $\operatorname{spt} R_\varepsilon \subset \mathcal{Y}_\varepsilon$, where $\mathcal{Y}_\varepsilon$ is defined in Remark 4.6, so that in particular

$$R_\varepsilon \rightharpoonup R\,, \qquad \mathbf{M}(R_\varepsilon) \to \mathbf{M}(R) \qquad \text{and} \qquad \mathbf{M}(\partial R_\varepsilon) \to \mathbf{M}(\partial R) \tag{4.12}$$

as $\varepsilon \to 0^+$. Let now R_ε^1 be the first connected component of R_ε and let $\{D_i\}$ be the 3-simpleces of a triangulation of R_ε^1. By an argument similar to Step 2 of Proposition 4.5, starting from the constant map $\varphi : S^2 \to \mathcal{Y}_\varepsilon$, $\varphi \equiv Q$ for some vertex Q in the 0-skeleton of R_ε^1, and covering with multiplicity and orientation each one of the D_i's, we can define a Lipschitz function $\varphi_1 : S^2 \to \mathcal{Y}_\varepsilon$ such that $\varphi_{1\#}[\![S^2]\!] = \partial R_\varepsilon^1$ and $\varphi_1(S^2)$ is 2-dimensional. Moreover, as in Step 3 of Proposition 4.5, we can define φ_1 so that the mapping area $A(\varphi_1, S^2) = \mathbf{M}(\partial R_\varepsilon^1)$. Connecting R_ε^1, by means of a loop in $\mathcal{Y}_\varepsilon$, with the second component R_ε^2 of R_ε, repeating the previous argument for each component of R_ε, and finally connecting the last component of R_ε with the given point P, we define a Lipschitz function $\phi_\varepsilon : S^2 \to \mathcal{Y}_\varepsilon$ such that

$$\phi_{\varepsilon\#}[\![S^2]\!] = \partial R_\varepsilon\,, \tag{4.13}$$

$\phi_\varepsilon(S^2)$ is 2-dimensional, $\phi_\varepsilon(Q_\varepsilon) = P$ for some point $Q_\varepsilon \in S^2$, and the mapping area

$$A(\phi_\varepsilon, S^2) = \mathbf{M}(\partial R_\varepsilon)\,. \tag{4.14}$$

We then proceed as in Step 4 of Proposition 4.5. More precisely, let $\varphi_\varepsilon : B^2 \to S^2$ be a Lipschitz function from the unit disk to the 2-sphere such that $\varphi_{\varepsilon|\partial B^2} \equiv Q_\varepsilon$ and φ_ε maps the interior of B^2 bijectively onto $S^2 \setminus \{Q_\varepsilon\}$, so that $\varphi_{\varepsilon\#}[\![B^2]\!] = [\![S^2]\!]$. Moreover, let $\Pi_\varepsilon : \mathcal{Y}_\varepsilon \to \mathcal{Y}$ be the nearest point projection.

Taking $f_\varepsilon := \Pi_\varepsilon \circ \phi_\varepsilon \circ \varphi_\varepsilon : B^2 \to \mathcal{Y}$, for $\varepsilon > 0$ small f_ε is a Lipschitz continuous function, with $f_{\varepsilon|\partial B^2} \equiv P$, for which by (4.13) and (4.12), since ${\Pi_\varepsilon}_\#\partial R = \partial R$,

$$f_{\varepsilon\#}[\![B^2]\!] = (\Pi_\varepsilon \circ \phi_\varepsilon)_\#[\![S^2]\!] = \Pi_{\varepsilon\#}\partial R_\varepsilon \rightharpoonup \partial R$$

weakly in $\mathcal{D}_2(\mathcal{Y})$, as $\varepsilon \to 0^+$. Moreover

$$A(f_\varepsilon, B^2) = A(\Pi_\varepsilon \circ \phi_\varepsilon, S^2) \leq (\operatorname{Lip}\Pi_\varepsilon)^2\, A(\phi_\varepsilon, S^2)$$

and hence, by (4.14) and by Morrey's ε-conformality theorem, modulo componing with an orientation preserving diffeomorphism of B^2 onto itself, we obtain

$$\mathbf{D}(\widetilde{f}_\varepsilon, B^2) \le (1+\varepsilon)\, A(f_\varepsilon, B^2) \le (1+\varepsilon)\, (\mathrm{Lip}\,\Pi_\varepsilon)^2\, \mathbf{M}(\partial R_\varepsilon)$$

and finally (4.11), since $\mathrm{Lip}\,\Pi_\varepsilon \to 1^+$ as $\varepsilon \to 0^+$, compare Remark 4.6.

Step 3: For every $p \in \mathcal{C}^2$ and every 3-dimensional integral current R in $\mathcal{Y}$, we prove the existence of a sequence $\{u_h\}$ of smooth maps from $\mathcal{C}^2$ into $\mathcal{Y}$ such that $G_{u_h} \rightharpoonup G_u + \delta_p \times \partial R$ weakly in $\mathcal{D}_2(\mathcal{C}^2 \times \mathcal{Y})$ and

$$\lim_{h\to+\infty} \mathbf{D}(u_h, \mathcal{C}^2) = \mathbf{D}(u, \mathcal{C}^2) + \mathbf{M}(\partial R)\,.$$

Moreover, we show that u_h can be chosen so that $u_h = u$ outside $B^2_{\delta_h}(p)$, for a sequence of radii $\delta_h \searrow 0$.

Let $P := u(p)$, (U, φ) be a local chart centered at P, so that $\varphi(P) = 0$, and $r > 0$ be such that $B^2_r(p) \subset \mathcal{C}^2$ and $u(B^2_r(p)) \subset U$, compare Proposition 4.3. We define, for $h \in \mathbb{N}$ and $\delta \in (0, r)$,

$$u_{h,\delta}(x) := \begin{cases} u(x) & \text{if} \quad |x-p| > \delta \\ v_\delta(x) & \text{if} \quad \delta/2 \le |x-p| \le \delta \\ f_h(2(x-p)/\delta) & \text{if} \quad |x-p| < \delta/2\,, \end{cases}$$

where f_h is given by Step 2 and

$$v_\delta(x) := \varphi^{-1}\biggl(\Bigl(\frac{2}{\delta}\,|x-p| - 1\Bigr) \cdot \varphi \circ u\Bigl(p + \delta\, \frac{x-p}{|x-p|}\Bigr)\biggr)\,.$$

Similarly to Proposition 4.3, it is not difficult to show that $u_{h,\delta}$ is Lipschitz continuous, by (4.11) and a change of variables

$$\mathbf{D}(u_{h,\delta}, B^2_{\delta/2}(p)) = \mathbf{D}(f_h, B^2) \to \mathbf{M}(\partial R)$$

as $h \to +\infty$ and

$$\liminf_{\delta\to 0^+} \mathbf{D}(v_\delta, B^2_\delta(p) \setminus B^2_{\delta/2}(p)) = 0$$

which yields the assertion, by a diagonal procedure.

Step 4: We fix j and prove Proposition 4.4 with S_j instead of $S_{T,sing}$, where the S_j's are as in Step 1.

To this aim, we iterate the argument of Step 3 working by induction on $k = 1, \dots, I_j$. More precisely, we first apply Step 3 with $p = p_1^j$ and $R = R_1^j$, obtaining a sequence $\{u_h^{(1)}\}$ of smooth maps from $\mathcal{C}^2$ into $\mathcal{Y}$ such that $G_{u_h^{(1)}} \rightharpoonup G_u + \delta_{p_1^j} \times \partial R_1^j$ weakly in $\mathcal{D}_2(\mathcal{C}^2 \times \mathcal{Y})$ as $h \to +\infty$ and

$$\lim_{h\to+\infty} \mathbf{D}(u_h^{(1)}, \mathcal{C}^2) = \mathbf{D}(u, \mathcal{C}^2) + \mathbf{M}(\partial R_1^j)\,.$$

Recall that $u_h^{(1)} = u$ outside $B^2_{\delta_h^{(1)}}(p_1^j)$ for some sequence $\delta_h^{(1)} \searrow 0$. For h large enough so that $p_k^j \notin B^2_{\delta_h^{(1)}}(p_1^j)$ for $k = 2, \dots, I_j$, we then repeat the argument with $u = u_h^{(1)}$, $p = p_2^j$ and $R = R_2^j$, to obtain a sequence $\{u_l^{(2,h)}\}$ of smooth maps from $\mathcal{C}^2$ into $\mathcal{Y}$ such that $G_{u_l^{(2,h)}} \rightharpoonup G_{u_h^{(1)}} + \delta_{p_2^j} \times \partial R_2^j$ weakly in $\mathcal{D}_2(\mathcal{C}^2 \times \mathcal{Y})$ as $l \to +\infty$ and

$$\lim_{l\to+\infty} \mathbf{D}(u_l^{(2,h)}, \mathcal{C}^2) = \mathbf{D}(u_h^{(1)}, \mathcal{C}^2) + \mathbf{M}(\partial R_2^j)\,.$$

Moreover, since $u_l^{(2,h)} = u_h^{(1)}$ outside $B^2_{\delta_l^{(2,h)}}(p_2^j)$ for some sequence $\delta_l^{(2,h)} \searrow 0$ as $l \to +\infty$, by a diagonal procedure we define a sequence $\{u_h^{(2)}\}$ of smooth maps from $\mathcal{C}^2$ into $\mathcal{Y}$ such that

$$G_{u_h^{(2)}} \rightharpoonup G_u + \delta_{p_1^j} \times \partial R_1^j + \delta_{p_2^j} \times \partial R_2^j$$

weakly in $\mathcal{D}_2(\mathcal{C}^2 \times \mathcal{Y})$ and

$$\lim_{h\to+\infty} \mathbf{D}(u_h^{(2)}, \mathcal{C}^2) = \mathbf{D}(u, \mathcal{C}^2) + \mathbf{M}(\partial R_1^j) + \mathbf{M}(\partial R_2^j)\,.$$

Iterating in a similar way the argument on $k = 3, \ldots I_j$, and by a diagonal procedure, due to the strong convergence in energy we construct for every j a smooth sequence $\{u_h\} : \mathcal{C}^2 \to \mathcal{Y}$ such that $G_{u_h} \rightharpoonup G_u + S_j$ weakly in $\mathcal{D}_2(\mathcal{C}^2 \times \mathcal{Y})$ and

$$\lim_{h\to+\infty} \mathbf{D}(u_h, \mathcal{C}^2) = \mathbf{D}(u, \mathcal{C}^2) + \mathbf{M}(S_j)\,.$$

Finally, since by Step 1 we have that $S_j \rightharpoonup S_{T,sing}$ weakly as currents and $\mathbf{M}(S_j) \to \mathbf{M}(S_{T,sing})$, again by a diagonal procedure we prove the claim. □

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