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Multiscale problems and homogenization for second-order Hamilton–Jacobi equations

Olivier Alvarez [a], Martino Bardi [b,*], Claudio Marchi [c]

[a] UMR 60-85, Université de Rouen, 76821 Mont-Saint Aignan cedex, France
[b] Dipartimento di Matematica Pura ed Applicata, Università di Padova, via Trieste 63, 35121 Padova, Italy
[c] Dipartimento di Matematica, Università della Calabria, Ponte Bucci 30B, 87036 Rende (CS), Italy

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Abstract

We prove a general convergence result for singular perturbations with an arbitrary number of scales of fully nonlinear degenerate parabolic PDEs. As a special case we cover the iterated homogenization for such equations with oscillating initial data. Explicit examples, among others, are the two-scale homogenization of quasilinear equations driven by a general hypoelliptic operator and the n-scale homogenization of uniformly parabolic fully nonlinear PDEs.

Keywords: Singular perturbations; Viscosity solutions; Nonlinear parabolic equations; Hamilton–Jacobi equations; Bellman–Isaacs equations; Ergodicity; Stabilization; Homogenization; Iterated homogenization; Multiscale problem; Oscillating initial data; Hypoelliptic operators

0. Introduction

This paper is devoted to singular perturbation problems with an arbitrary finite number of scales for fully nonlinear degenerate parabolic PDEs, and to the iterated homogenization of such PDEs with oscillating initial data, in the framework of viscosity solutions. Some new results on

* Corresponding author.
E-mail addresses: olivier.alvarez@univ-rouen.fr (O. Alvarez), bardi@math.unipd.it (M. Bardi), marchi@mat.unical.it (C. Marchi).

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the case of two scales are given for their own interest and because they are needed in the proof of the general n-scale case.

The theory of homogenization of fully nonlinear PDEs by viscosity methods started with the seminal paper by P.-L. Lions, Papanicolaou, and Varadhan [46] on first-order Hamilton–Jacobi equations

$$\partial_t v^\varepsilon + G\left(x, \frac{x}{\varepsilon}, D_x v^\varepsilon\right) = 0 \quad \text{in } (0,T)\times\mathbb{R}^n, \qquad v^\varepsilon(0,x) = h(x), \tag{1}$$

and with the work of L.C. Evans [31,32], who introduced the perturbed test function method for first-order and second-order elliptic equations. It was continued by many authors to cover a number of different issues, such as problems without equicontinuity estimates [15], perforated domains [3,9,36], nonperiodic homogenization [10,20,38,47], elliptic and parabolic equations in divergence form [28,35], Neumann boundary conditions [16]. A recent important addition to the theory is the stochastic homogenization of PDEs in stationary ergodic media [29,48,52,54]. The main motivation of all this theory is understanding the macroscopic properties of models with high oscillations at a microscopic scale, as in the classical homogenization of variational problems, see, e.g., the monographs [19,25,41,56].

Some related asymptotic problems are the singular perturbations of degenerate parabolic equations of the form

$$\begin{aligned} &\partial_t u^\varepsilon + H\left(x, y, D_x u^\varepsilon, \frac{1}{\varepsilon}D_y u^\varepsilon, D^2_{xx}u^\varepsilon, \frac{1}{\varepsilon}D^2_{yy}u^\varepsilon, \frac{1}{\sqrt{\varepsilon}}D^2_{xy}u^\varepsilon\right) = 0 \quad \text{in } (0,T)\times\mathbb{R}^n\times\mathbb{R}^m,\\ &u^\varepsilon(0,x,y) = h(x,y) \quad \text{on } \mathbb{R}^n\times\mathbb{R}^m, \end{aligned} \tag{2}$$

where one seeks a limit $u(t,x)$ of $u^\varepsilon(t,x,y)$ independent of y and solving a suitable Cauchy problem in $(0,T)\times\mathbb{R}^n$. They arise in the optimal control of deterministic or stochastic systems whose state variables evolve on two different time-scales, namely,

$$\begin{aligned} dx_s &= f(x_s, y_s, a_s)\,ds + \sigma(x_s, y_s, a_s)\,dW_s, \quad 0 < s < t,\ x_0 = x \in \mathbb{R}^n,\\ \varepsilon\,dy_s &= g(x_s, y_s, a_s)\,ds + \sqrt{\varepsilon}\tau(x_s, y_s, a_s)\,dW_s, \quad y_0 = y \in \mathbb{R}^m, \end{aligned}$$

where W_s is a multidimensional Brownian motion and a_s an admissible control function. If one minimizes an integral cost functional plus a terminal cost $h(x_t, y_t)$, the corresponding value function $u^\varepsilon(t,x,y)$ solves (2) with a Hamilton–Jacobi–Bellman Hamiltonian H (i.e., the sup of a family of linear degenerate elliptic operators parametrized by a). Here the *macroscopic variable* x in (2) has the meaning of the *slow variable* in the dynamical system, whereas the *microscopic variable* y corresponds to the *fast variable* of the control system. Passing to the limit as $\varepsilon \to 0+$ in this problem amounts to reducing the dimension of a large system by decoupling the behavior of the fast and the slow variables. These problems have a large literature, see the books [13,17,42,44] and the references therein. A viscosity solutions approach to these problems was developed by the first two authors of the present paper [4,5,7], using some of the ideas of the homogenization theory quoted above.

In fact, many homogenization problems can be seen as special cases of singular perturbations. For instance, the PDE in (1) is transformed into

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$$\partial_t u^\varepsilon + G\left(x, y, D_x u^\varepsilon + \frac{D_y u^\varepsilon}{\varepsilon}\right) = 0 \quad \text{in } (0,T) \times \mathbb{R}^n \times \mathbb{R}^n,$$

by setting $v^\varepsilon(t,x) = u^\varepsilon(t,x,\frac{x}{\varepsilon})$ and $y = \frac{x}{\varepsilon}$. In the same way the second-order equation

$$\partial_t v^\varepsilon + F\left(x, \frac{x}{\varepsilon}, D_x v^\varepsilon, D^2_{xx} v^\varepsilon\right) = 0 \quad \text{in } (0,T) \times \mathbb{R}^n \tag{3}$$

becomes

$$u^\varepsilon_t + F\left(x, y, D_x u^\varepsilon + \frac{D_y u^\varepsilon}{\varepsilon}, D^2_{xx} u^\varepsilon + \frac{D^2_{yy} u^\varepsilon}{\varepsilon^2} + \frac{D^2_{xy} u^\varepsilon}{\varepsilon} + \frac{(D^2_{xy} u^\varepsilon)^T}{\varepsilon}\right) = 0$$
$$\text{in } (0,T) \times \mathbb{R}^n \times \mathbb{R}^n.$$

After replacing ε with $\sqrt{\varepsilon}$, this is also of the form (2) in the case F is independent of the first-order terms. The general homogenization problem with first and second-order terms can be written as an equation of the form (2) if H is replaced by a suitable H^ε with

$$H^\varepsilon \to H \quad \text{as } \varepsilon \to 0 \text{ uniformly on the compact sets.}$$

With this motivation, and as a tool for the asymptotic problems with more than two scales, we first prove a convergence theorem for regular perturbations of the singular perturbation problem, that is, for (2) with H is replaced by H^ε and h replaced by h^ε, $h^\varepsilon \to h$ uniformly. The assumptions are only on the unperturbed Hamiltonian H and are the same as in [5], namely, the properties of *ergodicity* and *stabilization to a constant* that we recall in the next sections. These assumptions allow to define the effective PDE and initial data that should be satisfied by the limit of u^ε as $\varepsilon \to 0$. The local uniform convergence of u^ε to the unique solution of the effective Cauchy problem is desired. However, this strong convergence does not hold in general, as shown by the example in Section 11 of [7]. Our result then states the weak convergence of u^ε, in the sense that the relaxed upper (respectively lower) semi-limit is a viscosity sub- (respectively super-) solution of the limit equation. Strong convergence will be shown under suitable additional assumptions on the Hamiltonian that guarantee that the limit equation satisfies the Comparison Principle. We give several examples and refer to [5] and [7] for more details.

This theorem embeds homogenization theory into singular perturbations, at least for Hamilton–Jacobi–Bellman–Isaacs equations. It allows an approach to homogenization that is a fully nonlinear counterpart of the two-scale convergence by Allaire and Nguetseng [1] for variational problems. An immediate consequence is a new general treatment of the degenerate parabolic equations (3) with oscillating initial data, that is, under the initial condition

$$v^\varepsilon(0,x) = h\left(x, \frac{x}{\varepsilon}\right).$$

Of course there is a boundary layer at $t = 0$ and one must find the *effective initial condition* $\overline{h}$ so that v^ε converges as $\varepsilon \to 0$ to the solution of

$$\partial_t v + \overline{F}\left(x, D_x v, D^2_{xx} v\right) = 0, \qquad v(0,x) = \overline{h}(x),$$

where $\overline{F}$ is the *effective Hamiltonian* associated to F. The definitions are recalled in the next sections and some methods for determining $\overline{h}$ and $\overline{F}$ can be found in [5,7,12,32].

We give three examples: the first is the case of uniformly elliptic F, for which we improve the existing theory. The second is a very degenerate case where F satisfies a nonresonance condition introduced by Arisawa and Lions [12]. The third concerns the equation

$$\partial_t v^\varepsilon - \operatorname{tr}\left(\frac{\sigma\sigma^T}{2}\left(\frac{x}{\varepsilon}\right)D^2_{xx}v^\varepsilon\right) + G\left(x, \frac{x}{\varepsilon}, D_x v^\varepsilon\right) = 0, \quad v^\varepsilon(0,x) = h\left(x, \frac{x}{\varepsilon}\right),$$

where tr denotes the trace, under a full rank bracket generating condition on the columns of the matrix σ. We prove the uniform convergence of v^ε over compact sets to the solution of the effective limit problem

$$\partial_t u - \int_{(0,1)^n}\left[\operatorname{tr}\left(\frac{\sigma\sigma^T}{2}(y)D^2_{xx}u\right) - G(x,y,D_x u)\right]d\mu(y) = 0, \quad u(0,x) = \int_{(0,1)^n} h(x,y)\,d\mu(y),$$

where $d\mu(y) = \varphi(y)\,dy$ is the invariant measure associated to the diffusion process generated by σ and $\varphi \in C^\infty(\mathbb{R}^n)$. Although the PDE is quasilinear with a hypoelliptic principal part, this result seems to be completely new; indeed, all the results we know involving subelliptic operators treat only variational and stationary equations on the Heisenberg group and use completely different methods, see Biroli, Mosco and Tchou [21,22] as well as [23,34] and the references therein.

The second and main part of the paper is devoted to singular perturbations with more than two scales. The simplest situation is the three-scale problem

$$\partial_t u^\varepsilon + H^\varepsilon\left(x,y,z,D_x u^\varepsilon, \frac{D_y u^\varepsilon}{\varepsilon}, \frac{D_z u^\varepsilon}{\varepsilon^2}, D^2_{xx}u^\varepsilon, \frac{D^2_{yy}u^\varepsilon}{\varepsilon}, \frac{D^2_{zz}u^\varepsilon}{\varepsilon^2}, \frac{D^2_{xy}u^\varepsilon}{\varepsilon^{1/2}}, \frac{D^2_{xz}u^\varepsilon}{\varepsilon}, \frac{D^2_{yz}u^\varepsilon}{\varepsilon^{3/2}}\right) = 0,$$
$$u^\varepsilon(0,x,y,z) = h^\varepsilon(x,y,z),$$

still with $H^\varepsilon \to H$ and $h^\varepsilon \to h$ locally uniformly. It arises, in the special case $H^\varepsilon = H$, in the study via Dynamic Programming of the value function in optimal control and differential games for multiscale stochastic systems of the form

$$\begin{aligned} dx_s &= f\,ds + \sigma\,dW_s,\\ \varepsilon\,dy_s &= g\,ds + \sqrt{\varepsilon}\,\tau\,dW_s,\\ \varepsilon^2\,dz_s &= \varphi\,ds + \varepsilon\,\upsilon\,dW_s, \end{aligned}$$

where the drift vectors f, g, φ and the dispersion matrices σ, τ, υ depend on the state variables x_s, y_s, z_s and on the control functions of one or two controllers, and W_s is a multidimensional Brownian motion. It applies also, for a suitable choice of $H^\varepsilon \to H$ locally uniformly, to the iterated homogenization problem

$$\partial_t v^\varepsilon + F\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, D_x v^\varepsilon, D^2_{xx}v^\varepsilon\right) = 0, \quad v^\varepsilon(0,x) = h\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right),$$

if $u^\varepsilon(t, x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}) = v^\varepsilon(t, x)$. Here we work out in detail the case of uniformly elliptic F. The applications to first-order equations are made in our paper [8].

Our methods are general and allow to attack singular perturbation and homogenization problems with an arbitrary number of scales. As far as we know, this is the first paper treating problems with more than two scales for fully nonlinear equations. Up to now iterated homogenization was addressed only in the variational setting, starting with the pioneering work of Bensoussan, J.-L. Lions and Papanicolaou [19] for linear equations and, afterwards, for semilinear equations, using the Γ-convergence approach, see [25,49] and the references therein, or G-convergence techniques [2,14,45].

The plan of the paper is as follows. The standing assumptions are listed in Section 1. Section 2 recalls the notions of ergodicity and stabilization for a Hamiltonian. In Section 3, we improve the convergence result of [5] by considering the regular perturbation of a singular perturbation problem. Applications to problems with a noncritical scale factor and to homogenization are given in Section 4. In Section 5, we present the regular–singular perturbation result for three scales, then apply it to iterated homogenization, and finally extend it to an arbitrary number of scales.

1. Standing assumptions

In order to avoid a long list of assumptions on the Hamiltonian and the initial data that may seem technical to the reader nonexpert in viscosity solutions, we shall specialize to the case of Hamilton–Jacobi–Bellman–Isaacs (HJBI) operators. This is an expedient to get easily existence and the Comparison Principle (which implies uniqueness) for a few ancillary problems. We leave it to the reader to extend our results to Hamiltonians in the general form (as in [5]). We therefore assume that the Hamiltonian is given by

$$H(x, y, p_x, p_y, X_{xx}, X_{yy}, X_{xy}) := \min_{\beta\in B}\max_{\alpha\in A} L^{\alpha,\beta}(x, y, p_x, p_y, X_{xx}, X_{yy}, X_{xy}),$$

for the family of linear operators

$$\begin{aligned} L^{\alpha,\beta}(x, y, p_x, p_y, X_{xx}, X_{yy}, X_{xy}) := &- \operatorname{tr}\big(X_{xx}a(x, y, \alpha, \beta)\big) - \operatorname{tr}\big(X_{yy}b(x, y, \alpha, \beta)\big) \\ &- 2\operatorname{tr}\big(X_{xy}c(x, y, \alpha, \beta)\big) - p_x \cdot f(x, y, \alpha, \beta) \\ &- p_y \cdot g(x, y, \alpha, \beta) - \ell(x, y, \alpha, \beta) \end{aligned}$$

with

$$a := \sigma\sigma^T/2, \qquad b := \tau\tau^T/2, \qquad c := \tau\sigma^T/2,$$

where $\operatorname{tr}(M)$ denotes the trace of the matrix M. HJBI operators arise in the dynamic programming approach to stochastic optimal control problems and stochastic differential games (see Section 4.2). But it actually concerns far more general situations, as a wide class of elliptic operators can be represented as HJBI operators [43].

The following *standing assumptions* are very classical, apart perhaps for the last one, which will be discussed below. They will hold throughout this paper.

– The control sets A and B are compact metric spaces.

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– The functions f, g, σ, τ, and ℓ are bounded continuous functions in $\mathbb{R}^n \times \mathbb{R}^m \times A \times B$ with values, respectively, in $\mathbb{R}^n$, $\mathbb{R}^m$, $\mathbb{M}^{n,r}$ (the set of the $n \times r$ real matrices), $\mathbb{M}^{m,r}$, and $\mathbb{R}$.
– The drift vectors f and g and the dispersion matrices σ and τ are Lipschitz continuous in (x, y), uniformly in (α, β).
– The running cost ℓ is uniformly continuous in (x, y), uniformly in (α, β).
– The initial data h is a bounded continuous function in $\mathbb{R}^n \times \mathbb{R}^m$ with values in $\mathbb{R}$.
– The functions f, g, σ, τ, ℓ and h are $\mathbb{Z}^m$-periodic in the fast variable y.

Periodicity in the fast variable y is a simplification that permits to ignore boundary conditions or conditions at infinity since y will remain in a compact manifold without boundary (the torus). Most of the results we prove here extend to the case of a fast variable y living in a compact set with appropriate boundary conditions, such as Neumann conditions or the boundary conditions arising in problems with state constraints (see Section 1 in [4]). The adaptations require suitable assumptions on g and τ near the boundary.

Given the above Hamiltonian H, we associate its *recession function*, or homogeneous part, in the fast derivatives (p_y, X_{yy}) given by

$$H'(x, y, p_y, X_{yy}) := \min_{\beta\in B}\max_{\alpha\in A}\bigl\{-\operatorname{tr}\bigl(X_{yy}\, b(x, y, \alpha, \beta)\bigr) - p_y \cdot g(x, y, \alpha, \beta)\bigr\}.$$

We note that H' is positively 1-homogeneous in (p_y, X_{yy}), i.e.

$$H'(x, y, \lambda p_y, \lambda X_{yy}) = \lambda H'(x, y, p_y, X_{yy}), \quad \lambda > 0,$$

and that, for every $\bar{x} \in \mathbb{R}^n$, $\bar{p}_x \in \mathbb{R}^n$, $\overline{X}_{xx} \in \mathbb{S}^n$ (the set of the $n \times n$ symmetric matrices), there is a constant C so that

$$\begin{aligned}&\bigl|H(x, y, p_x, p_y, X_{xx}, X_{yy}, 0) - H'(x, y, p_y, X_{yy})\bigr| \leqslant C\\ &\forall (y, p_y, X_{yy}) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{S}^m,\end{aligned} \tag{4}$$

for every (x, p_x, X_{xx}) in a neighborhood of $(\bar{x}, \bar{p}_x, \overline{X}_{xx})$.

2. Ergodicity and stabilization

2.1. Ergodicity and the effective Hamiltonian

In this subsection we briefly recall the definition of ergodicity of the operator H from [5]. We refer to [6,7] for a discussion of the notion with numerous examples arising from stochastic optimal control and differential games.

We fix a slow variable $(\bar{x}, \bar{p}_x, \overline{X}_{xx})$. By the standard viscosity solution theory, under the current assumptions, the *cell δ-problem* with discounting parameter $\delta > 0$

$$\delta w_\delta + H\bigl(\bar{x}, y, \bar{p}_x, D_y w_\delta, \overline{X}_{xx}, D^2_{yy} w_\delta, 0\bigr) = 0 \quad \text{in } \mathbb{R}^m,\ w_\delta \text{ periodic}, \tag{CP$_\delta$}$$

has a unique viscosity solution. We denote the solution by $w_\delta(y;\overline{x},\overline{p}_x,\overline{X}_{xx})$ so as to display its dependence on the frozen slow variables. We say that the Hamiltonian is (uniquely) *ergodic* in the fast variable at $(\overline{x},\overline{p}_x,\overline{X}_{xx})$ if

$$\delta w_\delta(y;\overline{x},\overline{p}_x,\overline{X}_{xx}) \to \text{const} \quad \text{as } \delta \to 0, \text{ uniformly in } y.$$

We say that the Hamiltonian is ergodic if it is ergodic at every $(\overline{x},\overline{p}_x,\overline{X}_{xx})$. When the operator is ergodic at $(\overline{x},\overline{p}_x,\overline{X}_{xx})$, we put

$$\overline{H}(\overline{x},\overline{p}_x,\overline{X}_{xx}) = -\lim_{\delta\to 0} \delta w_\delta(y;\overline{x},\overline{p}_x,\overline{X}_{xx}).$$

The function $\overline{H}$ is called the *effective operator*, or *effective Hamiltonian*.

In general, there is no explicit formula for the effective Hamiltonian, but it can be proved that it inherits several properties from the Hamiltonian H [5]. In particular, $\overline{H}$ is automatically continuous in $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S}^n$ and degenerate elliptic, that is,

$$\overline{H}(x,p_x,X_{xx}) \leqslant \overline{H}\big(x,p_x,X'_{xx}\big) \quad \text{if } X_{xx} \geqslant X'_{xx}.$$

A more natural definition of effective Hamiltonian, following the classical homogenization theory for Hamilton–Jacobi equation [32,46], is in terms of the *true cell problem*

$$H\big(\overline{x},y,\overline{p}_x,D_y\chi,\overline{X}_{xx},D^2_{yy}\chi,0\big) = \lambda \quad \text{in } \mathbb{R}^m,\ \chi \text{ periodic.} \tag{5}$$

There is at most one $\lambda \in \mathbb{R}$ such that (5) has a continuous solution χ. When it exists, λ is called the effective Hamiltonian and χ a *corrector*. This definition is less general than ours. If the true cell problem has a solution, then the Hamiltonian is ergodic with effective Hamiltonian λ, but in the current generality the converse is false, i.e. there may be no corrector associated to a given ergodic Hamiltonian, see [12] and [7] for a simple example with a linear Hamiltonian. In fact, the solution of (CP_δ) can be used to construct *approximate correctors*, an idea used also by other authors, such as Arisawa [10] and Ishii [38].

There are many papers on sufficient conditions for ergodicity, see [6,7,12] and the references therein; [7] gives also a characterization of ergodicity in terms of the validity of the Strong Maximum Principle. The two most classical examples are the following. The first is the nondegeneracy of the fast diffusion

$$\text{for some } \nu > 0, \quad b(x,y,\alpha,\beta) \geqslant \nu I_m \quad \text{for all } (x,y,\alpha,\beta). \tag{6}$$

It entails the uniform ellipticity of the Hamiltonian in X_{yy}. The second is the existence of $A' \subseteq A$ such that

$$B(0,\nu) \subset \overline{\mathrm{conv}}\big\{g(x,y,\alpha,\beta)\ \big|\ \alpha \in A'\big\}, \quad \tau(x,y,\alpha,\beta) = 0 \quad \text{for all } x,y,\beta \text{ and } \alpha \in A', \tag{7}$$

where $B(0,\nu) \subset \mathbb{R}^m$ denotes the open ball of radius ν centered at the origin. It means the existence of a deterministic fast subsystem with a strong property of small-time controllability by the player acting on α. It implies that the Hamiltonian is of first-order with respect to the fast variable (i.e. it is independent of X_{yy} and X_{xy}) and that it is coercive with respect to the fast gradient p_y.

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We close this subsection with a technical lemma on ergodicity that will be used later. It states that the uniform limit of ergodic Hamiltonians is ergodic. We emphasize that the result is global. Indeed, ergodicity is not preserved by local uniform convergence (any Hamiltonian, whether ergodic or not, is locally uniformly the limit of uniformly elliptic Hamiltonians, all of which are ergodic). We shall write $\|f\|_\infty$ for the uniform norm of the function f.

Lemma 1. *Let $H_k(y, p_y, X_{yy})$ be a sequence of ergodic Hamiltonians with effective Hamiltonian $\overline{H_k}$. Then, for every k and k', we have the inequality*

$$|\overline{H_k} - \overline{H_{k'}}| \leqslant \|H_k - H_{k'}\|_\infty.$$

In particular, if H_k converges uniformly to H, then H is ergodic with effective Hamiltonian $\overline{H} = \lim \overline{H_k}$.

Proof. Let $w_{\delta,k}$ be the solution of (CP_δ) with Hamiltonian H_k. By the Comparison Principle, we immediately get that

$$\|\delta w_{\delta,k} - \delta w_{\delta,k'}\|_\infty \leqslant \|H_k - H_{k'}\|_\infty.$$

Sending $\delta \to 0$, we obtain the inequality stated in the lemma.

If H_k uniformly converges to H, the inequality implies that $\overline{H_k}$ must converge to some constant L. The inequality above applied to H_k and H gives that

$$\|\delta w_{\delta,k} - \delta w_\delta\|_\infty \leqslant \|H_k - H\|_\infty.$$

Hence,

$$\|\delta w_\delta - L\|_\infty \leqslant \|H_k - H\|_\infty + \|\delta w_{\delta,k} - \overline{H_k}\|_\infty + |\overline{H_k} - L|.$$

By choosing k large enough we see that δw_δ converges to L uniformly as $\delta \to 0$. Hence, H is ergodic with effective Hamiltonian L. □

Technical remark. In the study of multiscale problems in Section 5 we need the ergodicity of some intermediate effective Hamiltonian that in general is merely continuous. Therefore the cell problem (CP_δ) for this Hamiltonian may not satisfy the Comparison Principle. As it is well known, this implies that (CP_δ) may have several viscosity solutions, possibly discontinuous (existence follows from Perron's method). We say that such a Hamiltonian is ergodic if, for every collection of solutions $(w_\delta)_{\delta>0}$ of (CP_δ), δw_δ converges uniformly as $\delta \to 0$ to a constant that is independent of the collection. This extended definition allows us to prove the weak convergence for n-scale problems under minimal assumptions. In the applications presented here, however, we shall mainly deal with cases for which all the effective Hamiltonians satisfy the Comparison Principle and therefore we will get strong convergence results. For these examples the standard definition of ergodicity would be enough.

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2.2. Stabilization and the effective initial data

The notion of stabilization to a constant for degenerate equations was introduced in [5]. For $\bar{x}$ fixed, the *cell Cauchy problem* for the homogeneous Hamiltonian H'

$$\partial_t w + H'\big(\bar{x}, y, D_y w, D^2_{yy} w\big) = 0 \quad \text{in } (0,+\infty)\times\mathbb{R}^m, \quad w(0,y) = h(\bar{x}, y), \quad w \text{ periodic,} \tag{CP$'$}$$

has a unique bounded viscosity solution $w(t, y; \bar{x})$. Using the Comparison Principle and the fact that the constants are solutions of the equation by virtue of the positive homogeneity of H', we have the uniform bound $\|w(t,\cdot)\|_\infty \leqslant \|h(\bar{x},\cdot)\|_\infty$ for all $t \geqslant 0$.

We say that the pair (H, h) is *stabilizing* (to a constant) at $\bar{x}$ if

$$w(t, y; \bar{x}) \to \text{const} \quad \text{as } t \to +\infty, \text{ uniformly in } y. \tag{8}$$

In this case, we set

$$\bar{h}(\bar{x}) := \lim_{t\to+\infty} w(t, y; \bar{x}). \tag{9}$$

We say that the Hamiltonian is stabilizing if it is stabilizing at every $\bar{x} \in \mathbb{R}^n$ and for every continuous h. The function $\bar{h}$ is called the *effective initial data*. It can be proved that $\bar{h}$ is continuous and bounded [5].

Sufficient conditions for stabilization are, for instance, uniform ellipticity (6) or coercivity (7) [7,12]; [7] gives other examples and a characterization of stabilization via the parabolic Strong Maximum Principle.

Technical remark. As for ergodicity, we need to define stabilization under the assumption that the Hamiltonian H' is merely continuous. This will imply that (CP$'$) has discontinuous solutions, and they can be nonunique. By stabilization, we mean here that, for every discontinuous viscosity solution w of (CP$'$) , $w(t,\cdot)$ converges as $t \to +\infty$ uniformly in y to a constant that is independent of the solution. This remark will only apply on Section 5.

3. Regular perturbation of singular perturbation problems

In this section, we prove a general convergence result for the regular perturbation of a singular perturbation problem

$$\begin{aligned}&\partial_t u^\varepsilon + H^\varepsilon\left(x, y, D_x u^\varepsilon, \frac{1}{\varepsilon} D_y u^\varepsilon, D^2_{xx} u^\varepsilon, \frac{1}{\varepsilon} D^2_{yy} u^\varepsilon, \frac{1}{\sqrt{\varepsilon}} D^2_{xy} u^\varepsilon\right) = 0 \quad \text{in } (0,T)\times\mathbb{R}^n\times\mathbb{R}^m,\\ &u^\varepsilon(0,x,y) = h^\varepsilon(x,y) \quad \text{on } \mathbb{R}^n\times\mathbb{R}^m.\end{aligned} \tag{HJ$^\varepsilon$}$$

By *regular perturbation*, we mean that

$$H^\varepsilon \to H \quad \text{and} \quad h^\varepsilon \to h \quad \text{as } \varepsilon \to 0 \text{ uniformly on all compact sets.}$$

We assume that H, h and every H^ε, h^ε satisfy the standard assumptions of Section 1. For example, the Hamiltonian H^ε will be a regular perturbation of H if the control sets A and B are independent of ε and if the functions f^ε, g^ε, σ^ε, τ^ε and ℓ^ε converge uniformly on the compact sets to f, g, σ, τ and ℓ. Under these assumptions on the Hamiltonian, and because the scaling generates a Hamiltonian of the same form (with fast drift $\varepsilon^{-1}g^\varepsilon$ and fast diffusion $\varepsilon^{-1/2}\tau^\varepsilon$), the equation (HJ$^\varepsilon$) has a unique bounded viscosity solution.

We suppose also that

$$\left|h^\varepsilon(x,y)\right| \leqslant C \quad \text{for all } (x,y), \tag{10}$$

and

$$\left|H^\varepsilon(x,y,0,\dots,0)\right| \leqslant C \quad \text{for all } (x,y), \tag{11}$$

for some constant C independent of ε small. These assumptions are satisfied for instance if the initial costs h^ε and running costs ℓ^ε are equibounded.

We finally assume that condition (4) holds uniformly in ε, in the following sense. For every $\varepsilon > 0$, there is a function $H^{\varepsilon,\prime}(x,y,p_y,X_{yy})$ that is positively 1-homogeneous in (p_y,X_{yy}), which fulfills the following property: for every $(\bar{x},\overline{p}_x,\overline{X}_{xx})$, there is a constant C so that

$$\left|H^\varepsilon(x,y,p_x,p_y,X_{xx},X_{yy},0) - H^{\varepsilon,\prime}(x,y,p_y,X_{yy})\right| \leqslant C \quad \text{for every } (y,p_y,X_{yy}), \tag{12}$$

for every (x,p_x,X_{xx}) in a neighborhood of $(\bar{x},\overline{p}_x,\overline{X}_{xx})$ and every ε. This is satisfied for instance if the functions f^ε, σ^ε and ℓ^ε are bounded uniformly in ε. The uniformity of the condition in ε implies of course that the recession function H' of H is the uniform limit on the compact sets of $H^{\varepsilon,\prime}$ as $\varepsilon \to 0$.

The convergence result we prove extends the result of [5] which corresponds to the case when $H^\varepsilon = H$. It says roughly that whenever the limit Hamiltonian H is ergodic and stabilizing in the fast variable, u^ε will converge to the solution of the effective equation

$$\partial_t u + \overline{H}\left(x, D_x u, D^2_{xx}u\right) = 0 \quad \text{in } (0,T)\times\mathbb{R}^n, \quad u(0,x) = \bar{h}(x) \quad \text{on } \mathbb{R}^n, \tag{$\overline{\text{HJ}}$}$$

where $\overline{H}$ and $\bar{h}$ are the effective Hamiltonian and data associated to H and h by the ergodicity and stabilization assumptions.

In most cases the convergence is locally uniform. This happens when the limit equation satisfies the Comparison Principle. However, this is not true in the current generality, see [7] for a counterexample. Therefore we state the main result in terms of relaxed semi-limits.

The family $\{u^\varepsilon\}$ is equibounded under (10) and (11) for ε small. Indeed, the Comparison Principle gives the a priori bound

$$\left\|u^\varepsilon(t,\cdot)\right\|_\infty \leqslant \sup_\varepsilon \left\|h^\varepsilon\right\|_\infty + Ct.$$

We can therefore define the upper semi-limit $\bar{u}$ of u^ε as follows

$$\bar{u}(t,x) := \limsup_{\varepsilon\to 0,\,(t',x')\to(t,x)} \sup_y u^\varepsilon(t',x',y) \quad \text{if } t>0,$$
$$\bar{u}(0,x) := \limsup_{(t',x')\to(0,x),\,t'>0} \bar{u}(t',x') \quad \text{if } t=0.$$

It is a bounded u.s.c. function. We define analogously the lower semi-limit $\underline{u}$ by replacing lim sup with lim inf and sup with inf. The two-steps definition of the semi-limit for $t = 0$ is necessary to sweep away an expected initial layer.

Theorem 1. *Assume that H^ε and h^ε converge, respectively, to H and to h uniformly on the compact sets. Assume the equiboundedness conditions* (10), (11), *and* (12). *Assume also that the limit Hamiltonian H is ergodic and stabilizing. Then, the semi-limits $\bar{u}$ and $\underline{u}$ are, respectively, a subsolution and a supersolution of the effective Cauchy problem* ($\overline{\mathrm{HJ}}$).

Before giving the proof, let us stress that from the weak convergence stated by this theorem it is easy to deduce the strong convergence of u^ε if the Comparison Principle holds for the limit equation ($\overline{\mathrm{HJ}}$), i.e.

$$\text{if } u \text{ is a bounded u.s.c. subsolution of } (\overline{\mathrm{HJ}}) \text{ and } v \text{ is a bounded l.s.c. supersolution,}$$
$$\text{then } u \leqslant v \text{ on } [0,T) \times \mathbb{R}^n. \tag{13}$$

This will imply indeed that $\bar{u} \leqslant \underline{u}$. Since the reverse inequality is always true by definition, we deduce that $\underline{u} = \bar{u}$. This implies that u^ε converges locally uniformly to the function $\bar{u} = \underline{u}$ which is the unique continuous viscosity solution of ($\overline{\mathrm{HJ}}$). We thus have the following corollary.

Corollary 1. *Besides the hypotheses of Theorem* 1, *assume also that the effective Hamiltonian $\overline{H}$ satisfies the Comparison Principle* (13). *Then, u^ε converges uniformly on the compact subsets of $(0,T) \times \mathbb{R}^n \times \mathbb{R}^m$ to the unique viscosity solution of* ($\overline{\mathrm{HJ}}$).

In general, however, ($\overline{\mathrm{HJ}}$) does not satisfy the Comparison Principle without further assumptions on the data. In Section 11 of [7] it is shown that u^ε may have a discontinuous limit under the mere assumptions of Theorem 1. In the next section we give explicit conditions that imply the Comparison Principle, and therefore the uniform convergence, for homogenization problems, see Corollaries 2, 3, 4, and 5. We give next two simple examples that do not come from homogenization and extend the pioneering work of Jensen and P.-L. Lions [40] motivated by stochastic control theory. We refer the reader to the papers [4,5,7] for other singular perturbation problems.

Examples. Consider first the problem

$$\partial_t u^\varepsilon + F^\varepsilon\big(x, y, D_x u^\varepsilon, D^2_{xx} u^\varepsilon\big) - \frac{1}{\varepsilon} \Delta_{yy} u^\varepsilon = 0, \quad u^\varepsilon(0,x,y) = h^\varepsilon(x,y),$$

with $F^\varepsilon \to F$, $h^\varepsilon \to h$ locally uniformly and F^ε, F satisfying the structural conditions for the Comparison Principle [30]. Then the Hamiltonian $H := F(x, y, p_x, X_{xx}) - \operatorname{tr} X_{yy}$ is ergodic because it is uniformly elliptic in the fast variables and the effective Cauchy problem is

$$\partial_t u + \int_{(0,1)^m} F\big(x, y, D_x u, D^2_{xx} u\big)\, dy = 0, \quad u(0,x) = \int_{(0,1)^m} h(x,y)\, dy,$$

see [5,7]. The explicit formula for $\overline{H}$ allows to check easily that it verifies the structural conditions for the Comparison Principle, and then $u^\varepsilon \to u$ locally uniformly on $(0,T) \times \mathbb{R}^n \times \mathbb{R}^m$.

The second example is

$$\partial_t u^\varepsilon + F^\varepsilon\big(x, y, D_x u^\varepsilon, D^2_{xx} u^\varepsilon\big) + \frac{1}{\varepsilon}\big|D_y u^\varepsilon\big| = 0, \quad u^\varepsilon(0, x, y) = h^\varepsilon(x, y),$$

with the same assumptions on F^ε and h^ε. Then the Hamiltonian $H := F(x, y, p_x, X_{xx}) + |p_y|$ is ergodic because it is coercive with respect to the gradient of the fast variables and the effective Cauchy problem is

$$\partial_t u + \max_{y\in[0,1]^m} F\big(x, y, D_x u, D^2_{xx} u\big) = 0, \quad u(0, x) = \min_{y\in[0,1]^m} h(x, y),$$

see [5,7]. The structural conditions for the Comparison Principle are again easy to check and $u^\varepsilon \to u$ locally uniformly on $(0, T) \times \mathbb{R}^n \times \mathbb{R}^m$.

Remark. In the case when the initial data h are independent of y, one easily adapts the above convergence results to show that convergence is uniform on the compact subsets of $[0, T) \times \mathbb{R}^n \times \mathbb{R}^m$. This follows from a simple change in the definition of the semi-limits (see [5, Remark 3]).

Proof of Theorem 1. The proof of the convergence result is close to the main result in [5, Theorem 1]. However, since the ideas will be used later for the multiscale problem, we prefer to give the complete proof instead of referring to the steps that are common with [5, Theorem 1]. A key observation in the proof is that we do not require the correctors to be smooth.

We begin by proving that the upper semi-limit $\bar{u}$ is a subsolution of ($\overline{\text{HJ}}$) in $(0, T) \times \mathbb{R}^n$ by contradiction. We therefore assume that there are a point $(\bar{t}, \bar{x}) \in (0, T) \times \mathbb{R}^n$ and a smooth test function φ such that: $\bar{u}(\bar{t}, \bar{x}) = \varphi(\bar{t}, \bar{x})$, $(\bar{t}, \bar{x})$ is a strict maximum point of $\bar{u} - \varphi$ and there holds

$$\partial_t \varphi(\bar{t}, \bar{x}) + \overline{H}\big(\bar{x}, D_x\varphi(\bar{t}, \bar{x}), D^2_{xx}\varphi(\bar{t}, \bar{x})\big) \geqslant 3\eta$$

for some $\eta > 0$. For every $r > 0$, we define

$$H^\varepsilon_r(y, p_y, X_{yy}) := \min\big\{H^\varepsilon\big(x, y, D_x\varphi(t, x), p_y, D^2_{xx}\varphi(t, x), X_{yy}, 0\big) \;\big|\; |t - \bar{t}| \leqslant r,\ |x - \bar{x}| \leqslant r\big\}.$$

We put $\overline{H} := \overline{H}(\bar{x}, \overline{p}_x, \overline{X}_{xx})$ with $\overline{p}_x = D_x\varphi(\bar{t}, \bar{x})$ and $\overline{X}_{xx} = D^2_{xx}\varphi(\bar{t}, \bar{x})$ and we fix $r_0 > 0$ so that

$$\big|\partial_t\varphi(t, x) - \partial_t\varphi(\bar{t}, \bar{x})\big| \leqslant \eta \quad \text{as } |t - \bar{t}| < r_0,\ |x - \bar{x}| \leqslant r_0.$$

We claim that, for every $r > 0$ small enough, there is a parameter $\varepsilon' > 0$ and an equibounded family of functions $\{\chi^\varepsilon \mid 0 < \varepsilon < \varepsilon'\}$ so that

$$H^\varepsilon_r\big(y, D_y\chi^\varepsilon, D^2_{yy}\chi^\varepsilon\big) \geqslant \overline{H} - 2\eta \quad \text{in } \mathbb{R}^m. \tag{14}$$

The function χ^ε will be referred to as a corrector (by analogy with the true cell problem (5)).

To construct the corrector, we first fix a small parameter $\delta > 0$ so that

$$\|\delta w_\delta + \overline{H}\|_\infty \leqslant \eta,$$

where w_δ is the solution of the cell δ-problem (CP_δ). This is possible by virtue of the ergodicity of H. Then, as

$$H_r^\varepsilon(y, p_y, X_{yy}) \to H(\bar{x}, y, \overline{p}_x, p_y, \overline{X}_{xx}, X_{yy}, 0) \quad \text{as } (\varepsilon, r) \to (0,0)$$

uniformly on the compact sets, we deduce from the stability properties of viscosity solutions that the solution $w_{\delta,r}^\varepsilon$ of

$$\delta w_{\delta,r}^\varepsilon + H_r^\varepsilon\big(y, D_y w_{\delta,r}^\varepsilon, D_{yy}^2 w_{\delta,r}^\varepsilon\big) = 0 \quad \text{in } \mathbb{R}^m,\ w_{\delta,r}^\varepsilon \text{ periodic},$$

converges uniformly to w_δ as $(\varepsilon, r) \to (0,0)$. Thus, for $\varepsilon' > 0$ and $0 < r' < \min\{r_0, \bar{t}\}$, we shall have

$$\big\|\delta w_{\delta,r}^\varepsilon + \overline{H}\big\|_\infty \leqslant 2\eta \quad \text{when } 0 < \varepsilon < \varepsilon' \text{ and } 0 < r < r'.$$

The function $\chi^\varepsilon = w_{\delta,r}^\varepsilon$ is clearly a supersolution of (14). Moreover, by the Comparison Principle, the family $\{\chi^\varepsilon\}$ is equibounded with the bound

$$\|\chi^\varepsilon\|_\infty \leqslant \delta^{-1} \sup\big\{\big|H_r^\varepsilon(y,0,0)\big| \bigm| y \in \mathbb{R}^m,\ 0 < \varepsilon < \varepsilon'\big\}.$$

Once the corrector is constructed, the rest of the proof is like the one of [5, Theorem 1]. We consider the perturbed test function

$$\psi^\varepsilon(t,x,y) = \varphi(t,x) + \varepsilon\chi^\varepsilon(y).$$

In the cylinder $Q_r =]\bar{t} - r, \bar{t} + r[\times B_r(\bar{x}) \times \mathbb{R}^m$, the function ψ^ε is a supersolution of

$$\begin{aligned}
&\partial_t \psi^\varepsilon(t,x,y) + H^\varepsilon\big(x, y, D_x\psi^\varepsilon, \varepsilon^{-1} D_y\psi^\varepsilon, D_{xx}^2\psi^\varepsilon, \varepsilon^{-1} D_{yy}^2\psi^\varepsilon, \varepsilon^{-1/2} D_{xy}^2\psi^\varepsilon\big) \\
&\quad = \partial_t\varphi(t,x) + H^\varepsilon\big(x, y, D_x\varphi(t,x), D_y\chi^\varepsilon(y), D_{xx}^2\varphi(t,x), D_{yy}^2\chi^\varepsilon(y), 0\big) \\
&\quad \geqslant \partial_t\varphi(t,x) + H_r^\varepsilon\big(y, D_y\chi^\varepsilon(y), D_{yy}^2\chi^\varepsilon(y)\big) \\
&\quad \geqslant \partial_t\varphi(t,x) + \overline{H} - 2\eta \\
&\quad \geqslant \partial_t\varphi(\bar{t},\bar{x}) + \overline{H} - 3\eta \\
&\quad \geqslant 0.
\end{aligned}$$

This formal computation was derived as if the corrector were differentiable and the inequalities hold pointwise. Actually, in full generality, the corrector may be nonsmooth; in this case, the above computation is justified by the argument given in [5]. For the sake of completeness, we shall provide the rigorous computation at the end of the proof.

Since $\{\psi^\varepsilon\}$ converges uniformly to φ on $\overline{Q_r}$ because of the equiboundedness of $\{\chi^\varepsilon\}$, we have

$$\limsup_{\varepsilon\to 0,\, t'\to t,\, x'\to x} \sup_y \big(u^\varepsilon - \psi^\varepsilon\big)(t', x', y) = \bar{u}(t,x) - \varphi(t,x).$$

But $(\bar{t}, \bar{x})$ is a strict maximum point of $\bar{u} - \varphi$, so the above relaxed upper limit is < 0 on ∂Q_r. By compactness, one can find $\eta' > 0$ so that $u^\varepsilon - \psi^\varepsilon \leqslant -\eta'$ on ∂Q_r for ε small, i.e., $\psi^\varepsilon \geqslant u^\varepsilon + \eta'$

on ∂Q_r. Since ψ^ε is a supersolution in Q_r, we deduce from the Comparison Principle that $\psi^\varepsilon \geqslant u^\varepsilon + \eta'$ in Q_r for ε small. Taking the upper semi-limit, we get $\varphi \geqslant \bar{u} + \eta'$ in $(\bar{t} - r, \bar{t} + r) \times B(\bar{x}, r)$. This is impossible, for $\varphi(\bar{t}, \bar{x}) = \bar{u}(\bar{t}, \bar{x})$. Thus, we have reached the desired contradiction.

One can show similarly that $\underline{u}$ is a supersolution of ($\overline{\text{HJ}}$) in $(0, T) \times \mathbb{R}^n$.

We now verify the initial condition. We define the homogeneous Hamiltonian in the fast derivatives (p_y, X_{yy})

$$H_r^{\varepsilon,\prime}(y, p_y, X_{yy}) = \min\bigl\{H^{\varepsilon,\prime}(x, y, p_y, X_{yy}) \bigm| |x - \bar{x}| \leqslant r\bigr\}$$

as well as the initial data

$$h_r^\varepsilon(y) = \max\bigl\{h^\varepsilon(x, y) \bigm| |x - \bar{x}| \leqslant r\bigr\}.$$

Let w_r^ε be the unique solution of the following Cauchy problem

$$\partial_t w_r^\varepsilon + H_r^{\varepsilon,\prime}\bigl(y, D_y w_r^\varepsilon, D_{yy}^2 w_r^\varepsilon\bigr) = 0 \quad \text{in } (0, +\infty) \times \mathbb{R}^m, \quad w_r^\varepsilon(0, y) = h_r^\varepsilon(y),\ w_r^\varepsilon \text{ periodic in } y.$$

Since $H_r^{\varepsilon,\prime} \to H'(\bar{x}, \cdot)$ and $h_r^\varepsilon \to h(\bar{x}, \cdot)$ as $(\varepsilon, r) \to (0, 0)$ uniformly on the compact sets, we can show that

$$\limsup_{r \to 0,\, \varepsilon \to 0,\, t \to \infty} \sup_y \bigl|w_r^\varepsilon(t, y) - \bar{h}(\bar{x})\bigr| = 0. \tag{15}$$

Indeed, let w' be the solution of the Cauchy problem

$$\partial_t w + H'\bigl(\bar{x}, y, D_y w, D_{yy}^2 w\bigr) = 0 \quad \text{in } (0, +\infty) \times \mathbb{R}^m,$$
$$w(0, y) = h(\bar{x}, y) \quad \text{on } \mathbb{R}^m,\ w \text{ periodic.}$$

Fix $\eta > 0$. By the definition of $\bar{h}(\bar{x})$, we can find some time $T > 0$ so that

$$\bigl\|w(T, \cdot) - \bar{h}(\bar{x})\bigr\|_\infty \leqslant \eta.$$

By the stability properties of viscosity solutions, we know that $w_r^\varepsilon \to w'$ uniformly on the compact sets as $(\varepsilon, r) \to (0, 0)$. Therefore there are ε' and r' so that

$$\bigl\|w_r^\varepsilon(T, \cdot) - \bar{h}(\bar{x})\bigr\|_\infty \leqslant 2\eta \quad \text{for all } 0 < \varepsilon < \varepsilon',\ 0 < r < r'.$$

Noting that $H_r^{\varepsilon,\prime}(\cdot, 0, 0) \equiv 0$ by positive homogeneity, we deduce from the Comparison Principle that

$$\bigl\|w_r^\varepsilon(t, \cdot) - \bar{h}(\bar{x})\bigr\|_\infty \leqslant 2\eta \quad \text{for all } t \geqslant T,\ 0 < \varepsilon < \varepsilon',\ 0 < r < r'.$$

This gives (15).

The rest of the proof is similar to that of [5, Theorem 1]. We fix $\eta > 0$ arbitrarily and then $r > 0$, $\varepsilon' > 0$, $T > 0$ so that

$$\sup_{0<\varepsilon<\varepsilon'} \sup_{t \geqslant T} \sup_y \bigl|w_r^\varepsilon(t, y) - \bar{h}(\bar{x})\bigr| \leqslant \eta.$$

We put $Q_r^+(\bar{x}) = (0,r) \times B_r(\bar{x}) \times \mathbb{R}^m$ and we fix M so that $M \geqslant \|u^\varepsilon\|_{L^\infty(Q_r^+(\bar{x}))}$ for all $\varepsilon < \varepsilon'$. Then, we construct a bump function ψ_0 that is nonnegative, smooth, with $\psi_0(\bar{x}) = 0$ and $\psi_0 \geqslant 2M$ on $\partial B_r(\bar{x})$. Finally, we choose the constant $C > 0$ given by (12) so that

$$\left| H^\varepsilon\big(x, y, D_x\psi_0(x), p_y, D^2_{xx}\psi_0(x), X_{yy}, 0\big) - H^{\varepsilon,\prime}(x, y, p_y, X_{yy}) \right| \leqslant C$$

for every (y, p_y, X_{yy}), $x \in B_r(\bar{x})$, $0 < \varepsilon < \varepsilon'$. The function

$$\psi^\varepsilon(t,x,y) = w_r^\varepsilon\left(\frac{t}{\varepsilon}, y\right) + \psi_0(x) + Ct$$

is a supersolution of

$$\begin{aligned} &\partial_t\psi^\varepsilon + H^\varepsilon\big(x, y, D_x\psi^\varepsilon, \varepsilon^{-1}D_y\psi^\varepsilon, D^2_{xx}\psi^\varepsilon, \varepsilon^{-1}D^2_{yy}\psi^\varepsilon, \varepsilon^{-1/2}D^2_{xy}\psi^\varepsilon\big) = 0 \quad \text{in } Q_r^+(\bar{x}), \\ &\psi^\varepsilon = h^\varepsilon \quad \text{on } \{0\} \times B_r(\bar{x}) \times \mathbb{R}^m, \qquad \psi^\varepsilon = M \quad \text{on } [0,r) \times \partial B_r(\bar{x}) \times \mathbb{R}^m. \end{aligned}$$

Indeed, the initial and boundary conditions are clearly satisfied by the definition of w_r^ε and by the construction of M (note in particular that $\|w_r^\varepsilon\|_\infty \leqslant \|h^\varepsilon\|_\infty \leqslant M$). Moreover, in $Q_r^+(\bar{x})$, we have that

$$\begin{aligned} &\partial_t\psi^\varepsilon + H^\varepsilon\big(x, y, D_x\psi^\varepsilon, \varepsilon^{-1}D_y\psi^\varepsilon, D^2_{xx}\psi^\varepsilon, \varepsilon^{-1}D^2_{yy}\psi^\varepsilon, \varepsilon^{-1/2}D^2_{xy}\psi^\varepsilon\big) \\ &\quad = \varepsilon^{-1}\partial_t w_r^\varepsilon + C + H^\varepsilon\big(x, y, D_x\psi_0, \varepsilon^{-1}D_y w_r^\varepsilon, D^2_{xx}\psi_0, \varepsilon^{-1}D^2_{yy}w_r^\varepsilon, 0\big) \\ &\quad \geqslant \varepsilon^{-1}\big(\partial_t w_r^\varepsilon + H^{\varepsilon,\prime}\big(x, y, D_y w_r^\varepsilon, D^2_{yy}w_r^\varepsilon\big)\big) \\ &\quad \geqslant \varepsilon^{-1}\big(\partial_t w_r^\varepsilon + H_r^{\varepsilon,\prime}\big(y, D_y w_r^\varepsilon, D^2_{yy}w_r^\varepsilon\big)\big) \\ &\quad = 0. \end{aligned}$$

By the Comparison Principle, we deduce that

$$u^\varepsilon(t,x,y) \leqslant \psi^\varepsilon(t,x,y) = w_r^\varepsilon\left(\frac{t}{\varepsilon}, y\right) + \psi_0(x) + Ct \quad \text{in } Q_r^+(\bar{x}).$$

Taking the supremum over y and sending $\varepsilon \to 0$, we obtain the inequality

$$\bar{u}(t,x) \leqslant \bar{h}(\bar{x}) + \eta + \psi_0(x) + Ct \quad \text{for all } t > 0,\ x \in B_r(\bar{x}).$$

Sending $t \to 0^+$, $x \to \bar{x}$, we get $\bar{u}(0,\bar{x}) \leqslant \bar{h}(\bar{x}) + \eta$. The arbitrariness of η yields $\bar{u}(0,\bar{x}) \leqslant \bar{h}(\bar{x})$.

One shows similarly that $\underline{u}(0,\bar{x}) \geqslant \bar{h}(\bar{x})$.

Finally, let us provide the rigorous argument of [5] for showing that ψ^ε is a viscosity supersolution to (HJ^ε) in Q_r. We denote by $J^-\psi^\varepsilon(t,x,y)$ and by $\overline{J^-}\psi^\varepsilon(t,x,y)$, respectively, the parabolic subdifferential of ψ^ε in (t,x,y) and its closure (see the User's guide [30] for the precise definitions); in particular, a vector $(\pi, p, q, \Theta) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{S}^{n+m}$, with $\Theta := \left(\begin{smallmatrix} X & Z \\ Z^T & Y \end{smallmatrix}\right)$ (where $\mathbb{S}^i$ is the set of $i \times i$ symmetric real matrices) belongs to $J^-\psi^\varepsilon(t,x,y)$ if the Taylor inequality

$$\psi^\varepsilon(t+h_t, x+h_x, y+h_y) \geqslant \psi^\varepsilon(t,x,y) + \pi h_t + (p, h_x) + (q, h_y) + \frac{1}{2}(Xh_x, h_x) + \frac{1}{2}(Yh_y, h_y) + (Zh_y, h_x) - o\big(|h_t| + |h_x|^2 + |h_y|^2\big)$$

is fulfilled for every h_t, h_x and h_y sufficiently small.

Our aim is to show that, for every $(t,x,y) \in Q_r$ and $(\pi, p, q, \Theta) \in J^-\psi^\varepsilon(t,x,y)$ there holds

$$\pi + H^\varepsilon\big(x, y, p, \varepsilon^{-1}q, X, \varepsilon^{-1}Y, \varepsilon^{-1/2}Z\big) \geqslant 0. \tag{16}$$

We apply [30, Theorem 8.3] on the characterization of the subdifferential of the sum of two functions with independent variables: for every $\delta > 0$, there are $\tilde{X} \in \mathbb{S}^n$ and $\tilde{Y} \in \mathbb{S}^m$ so that $(\pi, p, \tilde{X}) \in \overline{J^-}\varphi(t,x)$ and $(q, \tilde{Y}) \in \varepsilon\overline{J^-}\chi^\varepsilon(y)$ with $\big(\begin{smallmatrix}\tilde{X} & 0\\ 0 & \tilde{Y}\end{smallmatrix}\big) \geqslant \Theta - \delta\Theta^2$. We set $\Theta^2 =: \big(\begin{smallmatrix}X' & Z'\\ (Z')^T & Y'\end{smallmatrix}\big)$. Since it is regular, the function φ satisfies:

$$\pi = \varphi_t(t,x), \qquad p = D_x\varphi(t,x) \quad \text{and} \quad \tilde{X} \leqslant D^2_{xx}\varphi(t,x).$$

By the properties of $\tilde{X}$ and $\tilde{Y}$ and by the degenerate ellipticity of H^ε, we infer

$$\begin{aligned}
&\pi + H^\varepsilon\big(x, y, p, \varepsilon^{-1}q, X - \delta X', \varepsilon^{-1}(Y - \delta Y'), \varepsilon^{-1/2}(Z - \delta Z')\big)\\
&\quad \geqslant \pi + H^\varepsilon\big(x, y, p, \varepsilon^{-1}q, \tilde{X}, \varepsilon^{-1}\tilde{Y}', 0\big)\\
&\quad \geqslant \varphi_t(t,x) + H^\varepsilon\big(x, y, D_x\varphi(t,x), \varepsilon^{-1}q, D^2_{xx}\varphi(t,x), \varepsilon^{-1}\tilde{Y}', 0\big)\\
&\quad \geqslant \varphi_t(t,x) + H^\varepsilon_r\big(y, \varepsilon^{-1}q, \varepsilon^{-1}\tilde{Y}'\big)
\end{aligned}$$

(where, in the last inequality, the definition of H^ε_r has been used). Since χ^ε is a solution to (14) and $(q, \tilde{Y}) \in \varepsilon\overline{J^-}\chi^\varepsilon(y)$, we deduce

$$H^\varepsilon_r\big(y, \varepsilon^{-1}q, \varepsilon^{-1}\tilde{Y}\big) \geqslant \overline{H} - 2\eta.$$

By the last two inequalities, we conclude

$$\begin{aligned}
&\pi + H^\varepsilon\big(x, y, p, \varepsilon^{-1}q, X - \delta X', \varepsilon^{-1}(Y - \delta Y'), \varepsilon^{-1/2}(Z - \delta Z')\big)\\
&\quad \geqslant \varphi_t(t,x) + \overline{H} - 2\eta \geqslant \varphi(\bar{t}, \bar{x}) + \overline{H} - 3\eta \geqslant 0.
\end{aligned}$$

Letting $\delta \to 0$, we accomplish the proof of our claim (16). □

4. Applications: Noncritical scalings and homogenization

4.1. Singular perturbations with noncritical scalings

Most applications we have in mind for regular perturbations of the Hamiltonian correspond to a singular perturbation problem depending on a scale factor $\gamma > 0$,

$$\begin{aligned}
&\partial_t u^\varepsilon + H\left(x, y, D_x u^\varepsilon, \frac{1}{\varepsilon}D_y u^\varepsilon, D^2_{xx}u^\varepsilon, \frac{1}{\varepsilon^{2\gamma}}D^2_{yy}u^\varepsilon, \frac{1}{\varepsilon^\gamma}D^2_{xy}u^\varepsilon\right) = 0 \quad \text{in } (0,T)\times\mathbb{R}^n\times\mathbb{R}^m,\\
&u^\varepsilon(0,x,y) = h(x,y) \quad \text{on } \mathbb{R}^n\times\mathbb{R}^m,
\end{aligned} \tag{$\mathrm{HJ}^\varepsilon_\gamma$}$$

where the Hamiltonian H fulfills the assumptions of Section 1. To simplify the writing, we take the initial data independent of ε. The purpose of this subsection is to explain what the limit will be according to the values of γ.

The critical value is $\gamma = 1/2$ because the derivatives D_y and D_{yy} are multiplied by the same power of ε so they both appear in the cell problem. This corresponds to the situation studied in [5]. It is the natural scaling in most singular perturbations problems arising in optimal stochastic control theory [4,17,44]. But, noncritical values for the scale factor, i.e. $\gamma \neq 1/2$, are also important for the applications. In this case, the first-order term and the second-order term will not have the same power. One of them will therefore dominate the other and will determine the cell problem. The case $0 < \gamma < 1/2$ appears in optimal control and corresponds to weak diffusion; in the limit, the stochastic fast variable will behave like a deterministic process. This will be considered in Section 4.2. On the contrary, when $\gamma > 1/2$, the fast process will behave like a pure diffusion, with no drift. The case $\gamma = 1$ is most important as it arises in periodic homogenization and in problems in very thin domains. This will be explained in Sections 4.4 and 4.3.

Let us now state precise results. When $0 < \gamma < 1/2$, the leading term in the operator is the first-order term in the fast variable. Therefore we expect the situation be the same as if we had started with the Hamiltonian

$$H_f(x, y, p_x, p_y, X_{xx}) := H(x, y, p_x, p_y, X_{xx}, 0, 0).$$

Note that H_f is first-order with respect to the fast variable, and the subscript f recalls this fact. The proof is obtained by applying Theorem 1 to

$$H^\varepsilon(x, y, p_x, p_y, X_{xx}, X_{yy}, X_{xy}) := H\big(x, y, p_x, p_y, X_{xx}, \varepsilon^{1-2\gamma} X_{yy}, \varepsilon^{1/2-\gamma} X_{xy}\big).$$

Indeed, H^ε satisfies the standing assumptions together with (11) and (12) with recession function $H^{\varepsilon,\prime} = H'(x, y, p_y, \varepsilon^{1-2\gamma} X_{yy})$, and H^ε converges uniformly on the compact sets to H_f. We therefore have the following result.

Proposition 1. *Assume* $0 < \gamma < 1/2$. *Assume that the Hamiltonian* H_f *is ergodic and stabilizing and denote with* $\overline{H_f}$ *and* $\overline{h_f}$ *the corresponding effective Hamiltonian and initial data. Then the semi-limits* $\overline{u}$ *and* $\underline{u}$ *of the solutions* u^ε *of* $(\mathrm{HJ}^\varepsilon_\gamma)$ *are, respectively, a subsolution and a supersolution of the effective Cauchy problem*

$$\partial_t u + \overline{H_f}\big(x, D_x u, D^2_{xx} u\big) = 0 \quad \text{in } (0, T) \times \mathbb{R}^n, \qquad u(0, x) = \overline{h_f}(x) \quad \text{on } \mathbb{R}^n. \tag{$\overline{\mathrm{HJ}_1}$}$$

When $\gamma > 1/2$, the leading term in the operator is the second-order term in the fast variable. The situation is now expected to be the same as if we had started with the Hamiltonian

$$H_s(x, y, p_x, X_{xx}, X_{yy}, X_{xy}) := H(x, y, p_x, 0, X_{xx}, X_{yy}, X_{xy}).$$

Note that H_s only involves second-order derivatives with respect to the fast variable. To prove the claim, we put

$$H^\varepsilon(x, y, p_x, p_y, X_{xx}, X_{yy}, X_{xy}) := H\big(x, y, p_x, \varepsilon^{1-1/(2\gamma)} p_y, X_{xx}, X_{yy}, X_{xy}\big).$$

We note that H^ε satisfies the standing assumptions as well as (11) and (12) with recession function $H^{\varepsilon,\prime} = H'(x, y, \varepsilon^{1-1/(2\gamma)} p_y, X_{yy})$ and that H^ε converges uniformly on the compact sets to H_s. We denote by u^ε the solution of $(\mathrm{HJ}^\varepsilon_\gamma)$ and v^ε the solution of $(\mathrm{HJ}^\varepsilon)$ with $h^\varepsilon = h$. An immediate computation reveals that $v^{(\varepsilon^{2\gamma})}$ solves $(\mathrm{HJ}^\varepsilon_\gamma)$. By uniqueness, we deduce that $u^\varepsilon = v^{(\varepsilon^{2\gamma})}$. Theorem 1 thus gives the following:

Proposition 2. *Assume $\gamma > 1/2$. Assume that the Hamiltonian H_s is ergodic and stabilizing and denote with $\overline{H_s}$ and $\overline{h_s}$ the corresponding effective Hamiltonian and initial data. Then the semi-limits $\bar{u}$ and $\underline{u}$ of the solutions u^ε of $(\mathrm{HJ}^\varepsilon_\gamma)$ are, respectively, a subsolution and a supersolution of the effective Cauchy problem*

$$\partial_t u + \overline{H_s}\big(x, D_x u, D^2_{xx} u\big) = 0 \quad \text{in } (0,T)\times\mathbb{R}^n, \quad u(0,x) = \overline{h_s}(x) \quad \text{on } \mathbb{R}^n. \tag{$\overline{\mathrm{HJ}_2}$}$$

4.2. Example: Systems with weak or strong diffusion in the fast dynamics

In this subsection we provide an interpretation of the preceding results in the context of stochastic control problems and differential games. Consider the controlled stochastic differential equation

$$\begin{aligned} dx_s &= f(x_s, y_s, \alpha_s, \beta_s)\,ds + \sigma(x_s, y_s, \alpha_s, \beta_s)\,dW_s, \quad x_0 = x,\\ dy_s &= \frac{1}{\varepsilon} g(x_s, y_s, \alpha_s, \beta_s)\,ds + \frac{1}{\varepsilon^\gamma}\tau(x_s, y_s, \alpha_s, \beta_s)\,dW_s, \quad y_0 = y, \end{aligned}$$

for $s \geqslant 0$, where W_s is a r-dimensional Brownian motion. The admissible controls α_s and β_s take values, respectively, in the sets A and B. We also define a payoff functional on each time interval $[0,t]$ of the form

$$J^\varepsilon(t,x,y,\alpha,\beta) := E_{(x,y)}\left[\int_0^t \ell(x_s, y_s, \alpha_s, \beta_s)\,ds + h(x_t, y_t)\right],$$

where $E_{(x,y)}$ denotes the expectation, ℓ represents a running cost for the players and h is the terminal payoff depending of the position of the system at the final time t. The first player wants to minimize the criterion by acting on α while the second player wants to maximize it by acting on β. There are two value functions, whose definition depends on the information available to each player. The lower value is

$$u^\varepsilon(t,x,y) := \inf_{\alpha\in\Gamma(t)} \sup_{\beta\in\mathcal{B}(t)} J^\varepsilon\big(t,x,y,\alpha[\beta],\beta\big),$$

where $\mathcal{B}(t)$ denotes the set of admissible controls of the second player in the interval $[0,t]$ and $\Gamma(t)$ denotes the set of admissible strategies of the first player in the same interval (i.e., nonanticipating maps from $\mathcal{B}(t)$ into the admissible controls of the first player), see Fleming and Souganidis [33] or Swiech [55] for the precise definitions. Symmetrically, one can define the upper value function by switching the information pattern and allowing the second player to use strategies, instead of the first. Under the assumptions of Section 1 the lower value is the unique viscosity solution of the Hamilton–Jacobi–Bellman–Isaacs equation $(\mathrm{HJ}^\varepsilon_\gamma)$ [33,55], whereas the upper value solves the same equation with min and max switched in the definition of H.

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As mentioned above, the classical scaling is the critical value $\gamma = 1/2$. It is treated in [4] for a single player. Theorem 1 extends that result to two players, initial data depending also on y, and regular perturbations of the vector fields and the running cost.

If $0 < \gamma < 1/2$, we say that the diffusion in the fast dynamic is weak. Problems of this type were studied for instance by Kabanov and Pergamenshchikov [42] without periodicity assumptions, see also the references therein. Proposition 1 states that the singular perturbation converges if there is ergodicity and stabilization of the corrected Hamiltonian

$$H_f(x, y, p_x, p_y, X_{xx}) \\ = \min_{\beta\in B}\max_{\alpha\in A}\bigl\{-\mathrm{tr}\bigl(X_{xx}a(x, y, \alpha, \beta)\bigr) - p_x \cdot f(x, y, \alpha, \beta) - p_y \cdot g(x, y, \alpha, \beta) - \ell(x, y, \alpha, \beta)\bigr\}.$$

Since H_f does not depend on X_{yy}, the needed assumptions are only on the drift g of the fast variables and correspond to setting $\tau \equiv 0$. In other words, we must look at the ergodic properties of the controlled deterministic system

$$dy_s = g(\bar{x}, y_s, \alpha_s, \beta_s)\,ds$$

for each slow variable $\bar{x}$ fixed. As recalled in Section 2, a simple sufficient condition for H_f to be ergodic and stabilizing is its coercivity in p_y, which corresponds to the assumption (7) on the dynamical system. We refer to [7] for weaker assumptions that guarantee ergodicity and stabilization for H_f. Whenever H_f is ergodic, it admits a representation formula as the lower value of an ergodic differential game for the preceding deterministic system, see [4,6,7].

If $\gamma > 1/2$, we say that the diffusion is strong. Though less studied in the context of singular perturbations, this case is natural in homogenization. In this case we have to consider the ergodicity and stabilization properties of the Hamiltonian H_s, i.e. the ergodic properties of the purely stochastic control system

$$dy_s = \tau(\bar{x}, y_s, \alpha_s, \beta_s)\,dW_s.$$

As recalled in Section 2, a sufficient condition for ergodicity and stabilization for H_s is the uniform nondegeneracy of τ (6). Finally, when H_s is ergodic it can be represented as the lower value of an ergodic stochastic differential game for the preceding controlled diffusion process [4,6,7].

4.3. Example: Thin domains

Proposition 2 applies also to HJBI equations in a *very thin domain*. In the case of homogeneous Neumann boundary conditions, the problem was studied by Arisawa and Giga [11] for operators modeling the propagation of fronts. In the periodic setting, the problem is a slight variant of homogenization. The equation now has the form

$$\partial_t u^\varepsilon + F\left(x, \frac{z}{\varepsilon}, D_x u^\varepsilon, D_z u^\varepsilon, D^2_{xx} u^\varepsilon, D^2_{zz} u^\varepsilon, D^2_{xz} u^\varepsilon\right) = 0 \quad \text{in } (0, T) \times \mathbb{R}^n \times \mathbb{R}^m,$$

$$u^\varepsilon(0, x) = h\left(x, \frac{z}{\varepsilon}\right) \quad \text{on } (x, z) \in \mathbb{R}^n \times \mathbb{R}^m.$$

The functions F and h are periodic in the variable $y = z/\varepsilon$. By uniqueness, the solution $u^\varepsilon(t, x, z)$ is periodic in z with period ε. This models the situation where the state variable (x, z) lies in a very thin strip with width ε (namely $\mathbb{R}^n \times (0, \varepsilon)^m$) and it can be extended periodically in the z direction so as to live in $\mathbb{R}^n \times \mathbb{R}^m$. Then, $v^\varepsilon(t, x, y) = u^\varepsilon(t, x, \varepsilon y)$ is a solution of ($\mathrm{HJ}_\gamma^\varepsilon$) with scale factor $\gamma = 1$. Proposition 2 applies in a trivial manner.

4.4. *Periodic homogenization for uniformly parabolic equations*

In this subsection, we specialize to the case $\gamma = 1$ and explain how it covers homogenization problems. We are given the Cauchy problem

$$\partial_t u^\varepsilon + F\left(x, \frac{x}{\varepsilon}, D_x u^\varepsilon, D_{xx}^2 u^\varepsilon\right) = 0 \quad \text{in } (0, T) \times \mathbb{R}^n, \quad u^\varepsilon(0, x) = h\left(x, \frac{x}{\varepsilon}\right) \quad \text{on } \mathbb{R}^n, \tag{17}$$

where the functions F and h are periodic in the $y = x/\varepsilon$ variable. We assume that F is a HJBI operator

$$F(x, y, p_x, X_{xx}) := \min_{\beta \in B} \max_{\alpha \in A} \bigl\{-\mathrm{tr}\bigl(X_{xx} a(x, y, \alpha, \beta)\bigr) - p_x \cdot f(x, y, \alpha, \beta) - \ell(x, y, \alpha, \beta)\bigr\}$$

where $a = \sigma\sigma^T/2$ and the coefficients σ, f, ℓ, and the initial datum h satisfy the assumptions of Section 1. The recession function needed for the cell Cauchy problem is defined as

$$F'(x, y, X_{xx}) := \min_{\beta \in B} \max_{\alpha \in A} \bigl\{-\mathrm{tr}\bigl(X_{xx} a(x, y, \alpha, \beta)\bigr)\bigr\}.$$

(It is independent of the gradient because we are in the case $\gamma > 1/2$.)

The relationship between homogenization and singular perturbations is the following. In homogenization for differential games, the dynamical system is

$$dx_s = f\left(x_s, \frac{x_s}{\varepsilon}, \alpha_s, \beta_s\right) ds + \sigma\left(x_s, \frac{x_s}{\varepsilon}, \alpha_s, \beta_s\right) dW_s, \quad x_0 = x. \tag{18}$$

Introducing the shadow variable $y = x/\varepsilon$, we see, by uniqueness, that the system can be written as

$$\begin{aligned} dx_s &= f(x_s, y_s, \alpha_s, \beta_s)\, ds + \sigma(x_s, y_s, \alpha_s, \beta_s)\, dW_s, \quad x_0 = x, \\ dy_s &= \frac{1}{\varepsilon} f(x_s, y_s, \alpha_s, \beta_s)\, ds + \frac{1}{\varepsilon}\sigma(x_s, y_s, \alpha_s, \beta_s)\, dW_s, \quad y_0 = y, \end{aligned}$$

for the initial data $y = x/\varepsilon$.

Let us interpret this observation in terms of the HJBI equation and explain how Proposition 2 applies. Consider the solution v^ε of ($\mathrm{HJ}_\gamma^\varepsilon$) with scale factor $\gamma = 1$ and Hamiltonian

$$H(x, y, p_x, p_y, X_{xx}, X_{yy}, X_{xy}) = F\bigl(x, y, p_x + p_y, X_{xx} + X_{yy} + X_{xy} + X_{xy}^T\bigr).$$

It clearly satisfies the assumptions of Section 1. By uniqueness, one sees immediately that

$$u^\varepsilon(t, x) = v^\varepsilon(t, x, x/\varepsilon).$$

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Now define the relaxed semi-limits of v^ε, that is, for $t > 0$,

$$\overline{v}(t,x) := \limsup_{\varepsilon\to 0,\,(t',x')\to(t,x)} \sup_y v^\varepsilon(t',x',y), \qquad \underline{v}(t,x) := \liminf_{\varepsilon\to 0,\,(t',x')\to(t,x)} \inf_y v^\varepsilon(t',x',y),$$

extended for $t = 0$ by taking their u.s.c. and l.s.c. envelopes, respectively, and call them *two-scale semi-limits* associated to the homogenization problem (17). Their connection with the relaxed semi-limits of the solution u^ε to (17), namely $\overline{u}(t,x) := \limsup_{\varepsilon\to 0,\,(t',x')\to(t,x)} u^\varepsilon(t',x')$ and the symmetric definition for $\underline{u}$, is given by the inequalities $\underline{v} \leqslant \underline{u} \leqslant \overline{u} \leqslant \overline{v}$. Consequently, if Proposition 2 applies and the effective Hamiltonian H_s satisfies the Comparison Principle, we shall deduce that $\underline{v} = \overline{v}$ and conclude that u^ε converges uniformly on the compact sets.

Let us give some explicit conditions on the operator F that allow to apply Proposition 2 to periodic homogenization. We need the ergodicity and stabilization of the Hamiltonian

$$H_s(x,y,p_x,X_{xx},X_{yy},0) = F(x,y,p_x,X_{xx}+X_{yy}).$$

The effective Hamiltonian for F is defined by $\overline{F}(x,p_x,X_{xx}) := \overline{H_s}(x,p_x,X_{xx})$. More explicitly,

$$\overline{F}(x,p_x,X_{xx}) = -\lim_{\delta\to 0+} \delta w_\delta(y;x,p_x,X_{xx}), \tag{19}$$

where, for the fixed parameters x, p_x, X_{xx}, w_δ solves

$$\delta w_\delta + \min_{\beta\in B}\max_{\alpha\in A}\bigl\{-\mathrm{tr}\bigl(D^2_{yy} w_\delta a(x,y,\alpha,\beta)\bigr) - L(y,\alpha,\beta)\bigr\} = 0 \quad \text{in } \mathbb{R}^n,\ w_\delta \text{ periodic},$$

and

$$L(y,\alpha,\beta) = L(y,\alpha,\beta;x,p_x,X_{xx}) := \mathrm{tr}\bigl(X_{xx}a(x,y,\alpha,\beta)\bigr) + p_x\cdot f(x,y,\alpha,\beta) + \ell(x,y,\alpha,\beta).$$

Since H_s does not depend on p_y, a natural sufficient condition for ergodicity is the uniform ellipticity

$$\text{for some } \nu > 0, \quad a(x,y,\alpha,\beta) \geqslant \nu I_n \quad \text{for all } (x,y,\alpha,\beta) \tag{20}$$

(this is (6) of course, since $\sigma = \tau$ here).

The effective initial condition is

$$\overline{h_s}(x) := \lim_{t\to\infty} w(t,y;x), \tag{21}$$

where w solves

$$\partial_t w - \max_{\beta\in B}\min_{\alpha\in A}\mathrm{tr}\bigl(D^2_{yy} w\, a(x,y,\alpha,\beta)\bigr) = 0 \quad \text{in } (0,+\infty)\times\mathbb{R}^n, \quad w(0,y) = h(x,y).$$

In order to guarantee that the Comparison Principle holds for the limit Cauchy problem

$$\partial_t u + \overline{F}\bigl(x, D_x u, D^2_{xx} u\bigr) = 0 \quad \text{in } (0,T)\times\mathbb{R}^n, \quad u(0,x) = \overline{h_s}(x) \quad \text{on } \mathbb{R}^n, \tag{22}$$

we can make one of the following assumptions. Either F is concave in X_{xx} and Hölder continuous in y, i.e.

$$a \equiv a(x, y, \beta), \quad \ell \text{ is Hölder continuous in } y, \text{ uniformly in } (x, y, \alpha, \beta), \tag{23}$$

or F is uniformly continuous in x, uniformly in X_{xx} (see (25) below), i.e., in terms of the dynamics,

$$a \equiv a(y, \alpha, \beta). \tag{24}$$

We therefore obtain the following result.

Corollary 2. *If F is uniformly elliptic* (20)*, then the two-scale semi-limits $\underline{v}$ and $\overline{v}$ are, respectively, a supersolution and a subsolution of the effective Cauchy problem* (22) *with $\overline{F}$ and $\overline{h_s}$ given by* (19) *and* (21).

If, in addition, F satisfies either (23) *or* (24)*, then the solution u^ε of the homogenization problem* (17) *converges uniformly on the compact subsets of* $(0, T) \times \mathbb{R}^n$ *as $\varepsilon \to 0$ to the unique solution of* (22).

Proof. The first part of the corollary follows from the ergodicity and stabilization of H_s under (20) and from Proposition 2.

By Corollary 1 we prove the second part once we show that the effective Hamiltonian $\overline{F}$ is regular enough to ensure the Comparison Principle for the effective equation (22). It is classical to check that $\overline{F}$ is uniformly elliptic. Under (23), the regularity of $\overline{F}$ follows from the results of [7]. Under (24), this follows from the inequality

$$\bigl|\overline{F}(x', p_x, X_{xx}) - \overline{F}(x, p_x, X_{xx})\bigr| \leqslant C|x' - x|\bigl(1 + |p_x|\bigr) + \omega\bigl(|x' - x|\bigr).$$

This is a simple consequence of the inequality

$$\begin{aligned} &\bigl|F(x', y, p_x, X_{xx}) - F(x, y, p_x, X_{xx})\bigr| \leqslant C|x' - x|\bigl(1 + |p_x|\bigr) + \omega\bigl(|x' - x|\bigr) \\ &\text{for all } x, x', y, p_x, X_{xx}. \end{aligned} \tag{25}$$

(See e.g. the argument of [4, Proposition 12] for a proof of this implication.) These regularity properties for $\overline{F}$ imply the Comparison Principle for (22), see, e.g., [39]. □

Remark. A different characterization of the effective data $\overline{F}$ and $\overline{h_s}$ can be given in terms of differential games. By the results of Swiech [55], w_δ is the lower value function of the stochastic game for the system

$$dy_s = \sigma(x, y_s, \alpha_s, \beta_s)\, dW_s, \qquad y_0 = y,$$

with infinite horizon cost functional

$$E_y\left[\int_0^{+\infty} L(y_s, \alpha_s, \beta_s) e^{-\delta s}\, ds\right].$$

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Therefore $\overline{F}$ is the lower value of an ergodic game, see [6]. Similarly, w is the lower value function of the stochastic game for the same system with finite horizon cost functional $E_y[h(x, y_t)]$, so we can write a representation formula for $\overline{h_s}$.

Remark. Under the assumption (20) it is possible to prove that also the semi-limits $\bar{u}$ and $\underline{u}$ are a sub- and a supersolution to the effective Cauchy problem.

4.5. Periodic homogenization under a nonresonance condition

In this subsection we give a homogenization theorem where uniform ellipticity is replaced by a nonresonance condition (introduced by Arisawa, Lions [12] for optimal control problems). It concerns the HJB equation

$$\begin{aligned}&\partial_t u^\varepsilon + \max_{\alpha\in A}\left\{-\mathrm{tr}\left(\frac{\sigma\sigma^T}{2}(x,\alpha)D^2_{xx}u^\varepsilon\right) - D_x u^\varepsilon\cdot f\left(x,\frac{x}{\varepsilon},\alpha\right) - \ell\left(x,\frac{x}{\varepsilon},\alpha\right)\right\} = 0\\ &\quad \text{in } (0,T)\times\mathbb{R}^n,\\ &u^\varepsilon(0,x) = h\left(x,\frac{x}{\varepsilon}\right) \quad \text{on } \mathbb{R}^n.\end{aligned} \tag{26}$$

The nonresonance condition is, for each fixed x,

$$\text{for every } k\in\mathbb{Z}^n\setminus\{0\},\quad \text{there is } \alpha\in A \text{ such that } \sigma^T(x,\alpha)k\neq 0. \tag{27}$$

It is the natural counterpart for controlled diffusions of the classical nonresonance condition for the ergodicity of the translations on the torus. It allows for very degenerate diffusions. For instance, if ξ is a vector with rationally independent coordinates and if the diffusion matrix is of the form $a(\alpha) = \xi\otimes\xi$, then the nonresonance condition is satisfied (and the matrix has rank 1). We refer to Arisawa, Lions [12] for a complete discussion of this hypothesis and for a proof of the ergodicity of the associated Hamiltonian H_s, and to [5,7] for the proofs that H_s is stabilizing and the Comparison Principle holds for $\overline{F}$ if σ is independent of x.

Corollary 3. *Assume* (27) *for all $x\in\mathbb{R}^n$. Then there exist a continuous degenerate elliptic $\overline{F}$ and a continuous $\overline{h_s}$ such that the two-scale semi-limits $\underline{v}$ and $\overline{v}$ associated to* (26) *are, respectively, a supersolution and a subsolution of the effective Cauchy problem* (22).

If, in addition, $\sigma = \sigma(\alpha)$ is independent of x, then the solution u^ε of the homogenization problem (26) *converges uniformly on the compact subsets of* $(0,T)\times\mathbb{R}^n$ *as $\varepsilon\to 0$ to the unique solution of* (22).

Remark. The same statement holds if we replace the PDE in (26) with

$$\partial_t u^\varepsilon - \min_{\alpha\in A}\mathrm{tr}\left(\frac{\sigma\sigma^T}{2}(x,\alpha)D^2_{xx}u^\varepsilon\right) + G\left(x,\frac{x}{\varepsilon},D_x u^\varepsilon\right) = 0$$

with

$$G:\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}\quad \text{Lipschitz continuous and } \mathbb{Z}^n \text{ periodic in } y. \tag{28}$$

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No convexity in the p_x variables is needed. Therefore it applies to stochastic differential games for the system (18) if the dispersion matrix σ depends at most on x and on one of the players, and it satisfies the nonresonance condition.

A more precise result can be obtained if σ in independent of the controls. Then we have a quasilinear equation of the form

$$\partial_t u^\varepsilon - \mathrm{tr}\left(\frac{\sigma\sigma^T}{2}(x) D^2_{xx} u^\varepsilon\right) + G\left(x, \frac{x}{\varepsilon}, D_x u^\varepsilon\right) = 0$$

and the nonresonance condition reads

$$\sigma^T(x)k \neq 0 \quad \text{for every } k \in \mathbb{Z}^n \setminus \{0\}. \tag{29}$$

For quasilinear equations the effective data are obtained by averaging with respect to the invariant measure associated to the diffusion process, and it was shown in [5] that the unique invariant probability measure of a nonresonant diffusion is the Lebesgue measure. This leads to the following.

Corollary 4. *Assume that in* (26) $\sigma = \sigma(x)$ *and* (29) *holds for all* $x \in \mathbb{R}^n$. *Then the solution* u^ε *converges uniformly on the compact subsets of* $(0, T) \times \mathbb{R}^n$ *as* $\varepsilon \to 0$ *to the unique solution of*

$$\partial_t u - \mathrm{tr}\left(\frac{\sigma\sigma^T}{2}(x) D^2_{xx} u\right) + \int_{(0,1)^n} G(x, y, D_x u)\, dy = 0 \quad \textit{in } (0, T) \times \mathbb{R}^n,$$
$$u(0, x) = \int_{(0,1)^n} h(x, y)\, dy \quad \textit{on } \mathbb{R}^n.$$

4.6. Periodic homogenization under a hypoellipticity condition

In this subsection we give a homogenization theorem for a quasilinear equation where uniform ellipticity is replaced by a hypoellipticity assumption. We consider the problem

$$\begin{aligned} &\partial_t u^\varepsilon - \mathrm{tr}\left(\frac{\sigma\sigma^T}{2}\left(x, \frac{x}{\varepsilon}\right) D^2_{xx} u^\varepsilon\right) + G\left(x, \frac{x}{\varepsilon}, D_x u^\varepsilon\right) = 0 \quad \text{in } (0, T) \times \mathbb{R}^n, \\ &u^\varepsilon(0, x) = h\left(x, \frac{x}{\varepsilon}\right) \quad \text{on } \mathbb{R}^n, \end{aligned} \tag{30}$$

where G is Lipschitz continuous and periodic in y, and the principal part of the operator is *hypoelliptic in the oscillating variables*, which means the following. Denote with σ^i the ith column of the matrix σ. For each frozen x, consider the operator $X_i := \sigma^i(x, y) \cdot \nabla_y$ associated to the vector field $\sigma^i(x, \cdot)$, $i = 1, \ldots, r$. We assume these vector fields are C^∞ and, for all $x \in \mathbb{R}^n$,

$$\begin{cases} X_1, \ldots, X_r \text{ and their commutators} \\ \text{up to a certain fixed order } \bar{r} \\ \text{span } \mathbb{R}^n \text{ at each point of } \mathbb{R}^n. \end{cases} \tag{31}$$

Under this Hörmander-type condition it is known that there is a unique probability measure μ_x invariant for the diffusion process $dy_s = \sigma(x, y_s)\, dW_s$, see [37] for a probabilistic proof and [7] for an analytic one. Moreover μ_x is the unique solution in the sense of distributions of

$$\sum_{i,j,k} \frac{\partial^2}{\partial y_i \partial y_j}\big(\sigma_{ik}(x,y)\sigma_{jk}(x,y)\mu_x\big) = 0 \quad \text{in } \mathbb{R}^n,\ \mu_x \text{ periodic}, \qquad \int_{(0,1)^n} d\mu_x(y) = 1,$$

and μ_x has a density $\varphi(x,\cdot) \in C^\infty(\mathbb{R}^n)$, $d\mu_x(y) = \varphi(x,y)\,dy$. Therefore the Hamiltonian H_s associated to our homogenization problem is ergodic and stabilizing and the effective data are obtained by averaging with respect to μ_x [7].

Corollary 5. *Assume* (28), $\sigma(x,\cdot)$ *is* C^∞ *and satisfies* (31) *for all* $x \in \mathbb{R}^n$*. Then the two-scale semi-limits* $\underline{v}$ *and* $\overline{v}$ *associated to* (30) *are, respectively, a supersolution and a subsolution of the effective Cauchy problem*

$$\partial_t u - \int_{(0,1)^n} \left[\operatorname{tr}\left(\frac{\sigma\sigma^T}{2}(x,y) D^2_{xx} u\right) - G(x,y,D_x u)\right]\varphi(x,y)\,dy = 0 \quad \textit{in } (0,T)\times\mathbb{R}^n,$$

$$u(0,x) = \int_{(0,1)^n} h(x,y)\varphi(x,y)\,dy \quad \textit{on } \mathbb{R}^n.$$

If, in addition, $\sigma = \sigma(\frac{x}{\varepsilon})$ *is independent of* x*, then* $\varphi = \varphi(y)$ *is independent of* x *and the solution* u^ε *of the homogenization problem* (30) *converges uniformly on the compact subsets of* $(0,T)\times\mathbb{R}^n$ *as* $\varepsilon \to 0$ *to the unique solution of the effective Cauchy problem.*

4.7. Bibliographical remarks on periodic homogenization

We outline here the main differences of our results from the literature on periodic homogenization for nonlinear nonvariational elliptic or parabolic equations. Our main improvement is the general treatment of oscillating initial data. To our knowledge they were considered before us only in the linear case [24,41]. Our related papers [5,50] anticipate some special nonlinear cases (and the third author [50] treats also operators oscillating in t).

The first papers for HJB equations [15,18,31], deal with the Dirichlet problem for quasilinear uniformly elliptic equations in bounded domains. Evans' seminal paper [32] for the fully nonlinear case considers uniformly elliptic equations under structural assumptions on F more restrictive than ours for Bellman–Isaacs operators. Our methods apply to the Dirichlet problem for uniformly elliptic equations, at least for boundary data depending only on x, as soon as at each point of the boundary there exist barriers uniform in ε. So the results of [18,31,32] can be extended to operators satisfying the milder conditions of this paper, at least in the uniformly elliptic case.

The paper [4] by the first two authors contains the results of Sections 4.4 and 4.5 in the context of optimal control problems with nonoscillating initial data, $h = h(x)$. The articles by Pardoux [51] and Buckdahn et al. [26,27] concern probabilistic methods for the homogenization of nonlinear parabolic equations. Finally, the recent paper of Lions and Souganidis [47] studies degenerate elliptic equations with coefficients of the second derivatives vanishing at the same rate as the space oscillations in almost periodic environment.

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5. Singular perturbations with multiple scales

5.1. The three-scale problem

In this subsection, we consider the three-scale Cauchy problem

$$\partial_t u^\varepsilon + H^\varepsilon\left(x, y, z, D_x u^\varepsilon, \frac{D_y u^\varepsilon}{\varepsilon}, \frac{D_z u^\varepsilon}{\varepsilon^2}, D^2_{xx} u^\varepsilon, \frac{D^2_{yy} u^\varepsilon}{\varepsilon}, \frac{D^2_{zz} u^\varepsilon}{\varepsilon^2}, \frac{D^2_{xy} u^\varepsilon}{\varepsilon^{1/2}}, \frac{D^2_{xz} u^\varepsilon}{\varepsilon}, \frac{D^2_{yz} u^\varepsilon}{\varepsilon^{3/2}}\right) = 0 \tag{32}$$

in $(0, T) \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p$, with the initial condition

$$u^\varepsilon(0, x, y, z) = h^\varepsilon(x, y, z) \quad \text{on } \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p.$$

The problem is assumed to be 1-periodic in y and z. Each variable corresponds to a certain scale of the problem. To avoid any confusion in the name of the variables, we prefer the space scale interpretation to the time scale interpretation. We shall therefore call x the *macroscopic variable* (or the slow variable), y is the *mesoscopic variable* (or the not so fast variable) and z is the *microscopic variable* (or the fast variable).

Let us briefly explain the idea of how we take the limit in (32). In a first approximation the problem can be viewed as a singular perturbation problem in the microscopic variable z, because the power of ε in front of the derivatives with respect to z dominates. Under suitable ergodicity and stabilization assumptions in the microscopic variable giving rise to a mesoscopic effective Hamiltonian H_1 and initial data h_1, this suggests that the solution $u^\varepsilon(t, x, y, z)$ of (32) should be close to the solution $v^\varepsilon(t, x, y)$ of the mesoscopic problem

$$\begin{aligned} &\partial_t v^\varepsilon + H_1\left(x, y, D_x v^\varepsilon, \frac{D_y v^\varepsilon}{\varepsilon}, D^2_{xx} v^\varepsilon, \frac{D^2_{yy} v^\varepsilon}{\varepsilon}, \frac{D^2_{xy} v^\varepsilon}{\varepsilon^{1/2}}\right) = 0 \quad \text{in } (0, T) \times \mathbb{R}^n \times \mathbb{R}^m, \\ &v^\varepsilon(0, x, y) = h_1(x, y) \quad \text{on } \mathbb{R}^n \times \mathbb{R}^m. \end{aligned} \tag{33}$$

But this singular perturbation problem falls within the theory of [5]. If the mesoscopic Hamiltonian H_1 is ergodic and stabilizing and if we call $\overline{H}$ and $\overline{h}$ the effective macroscopic Hamiltonian and initial data, we know that v^ε will converge to the solution u of the limit problem ($\overline{\text{HJ}}$). In conclusion, we expect that $u^\varepsilon(t, x, y, z)$ will converge to $u(t, x)$ where the effective quantities are defined iteratively. This viewpoint that consists in regarding the three-scale problem (32) as the singular perturbation of a two-scale problem is the key idea in our proof of convergence.

Let us list the precise assumptions of the subsection. As before, we assume that

$$H^\varepsilon \to H \quad \text{and} \quad h^\varepsilon \to h \quad \text{as } \varepsilon \to 0 \text{ uniformly on the compact sets.}$$

We also suppose that H, h and every H^ε, h^ε satisfy the standard assumptions of Section 1, i.e. they are 1-periodic in (y, z) and H^ε, H are HJBI operators with the regularity in the coefficients suitably extended to the additional variable z. In order to keep the notations reasonable, we shall not write down the explicit form of the HJBI operator for the three-scale problem in terms of the underlying differential game problem. This will be done however for iterated homogenization in the next subsection.

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We also assume that the functions H^ε and h^ε are equibounded in the sense of (10) and (11). We finally suppose that there is a recession function with respect to the variables (y,z), i.e. a function $H^{\varepsilon,\prime} = H^{\varepsilon,\prime}(x,y,z,p_y,p_z,X_{yy},X_{zz},X_{yz})$ positively 1-homogeneous in $(p_y,p_z,X_{yy},X_{zz},X_{yz})$, which satisfies for some constant C

$$\left|H^\varepsilon(x,y,z,p_x,p_y,p_z,X_{xx},X_{yy},X_{zz},0,0,X_{yz}) - H^{\varepsilon,\prime}(x,y,z,p_y,p_z,X_{yy},X_{zz},X_{yz})\right| \leqslant C \tag{34}$$

for every $(y,z,p_y,p_z,X_{yy},X_{zz},X_{yz}) \in \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{S}^m \times \mathbb{S}^p \times \mathbb{M}^{m,p}$, for every (x,p_x,X_{xx}) in a neighborhood of $(\overline{x},\overline{p}_x,\overline{X}_{xx})$ and for every ε. Let us recall that this uniformity in ε implies that H has a recession function H' and that this is the uniform limit on the compact sets of $H^{\varepsilon,\prime}$ as $\varepsilon \to 0$. As mentioned in Section 3, a sufficient condition for (34) is the equiboundedness in ε of the data of the differential game.

These assumptions guarantee that the problem (32) admits a unique bounded viscosity solution $u^\varepsilon \in C([0,T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p)$, that it is periodic in (y,z) and that the family $\{u^\varepsilon\}$ is equibounded. We can therefore define the upper and lower semi-limit $\overline{u}$ and $\underline{u}$. For instance, we shall have

$$\overline{u}(t,x) := \limsup_{\varepsilon\to 0,\,(t',x')\to(t,x)} \sup_{y,z} u^\varepsilon(t',x',y,z) \quad \text{if } t>0.$$

The second set of assumptions is necessary to define the effective Hamiltonian. They require the ergodicity of suitable problems.

– *Microscopic ergodicity.* The Hamiltonian H is ergodic with respect to the microscopic variable z. Namely, for every macro- and mesoscopic variables $(\overline{x},\overline{y},\overline{p}_x,\overline{p}_y,\overline{X}_{xx},\overline{X}_{yy})$, the unique periodic viscosity solution of the microscopic cell problem

$$\delta w_{\delta,2} + H\big(\overline{x},\overline{y},z,\overline{p}_x,\overline{p}_y,D_z w_{\delta,2},\overline{X}_{xx},\overline{X}_{yy},D^2_{zz}w_{\delta,2},0,0,0\big) = 0$$

is such that $\delta w_{\delta,2}$ converges uniformly to a constant as $\delta \to 0$. The constant is denoted by $-H_1(\overline{x},\overline{y},\overline{p}_x,\overline{p}_y,\overline{X}_{xx},\overline{X}_{yy})$ and the function H_1 is called the *effective mesoscopic Hamiltonian*.

– *Mesoscopic ergodicity of the effective mesoscopic Hamiltonian.* The Hamiltonian H_1 is ergodic with respect to the mesoscopic variable y. Namely, for every macroscopic variables $(\overline{x},\overline{p}_x,\overline{X}_{xx})$, all periodic viscosity solutions of the mesoscopic cell problem

$$\delta w_{\delta,1} + H_1\big(\overline{x},y,\overline{p}_x,D_y w_{\delta,1},\overline{X}_{xx},D^2_{yy}w_{\delta,1}\big) = 0 \tag{35}$$

are such that $\delta w_{\delta,1}$ converges uniformly to a constant as $\delta \to 0$. The constant is denoted by $-\overline{H}(\overline{x},\overline{p}_x,\overline{X}_{xx})$ and the function $\overline{H}$ is called the *effective* (*macroscopic*) *Hamiltonian*.

We recall that the Comparison Principle may not hold for the mesoscopic cell problem, because in general H_1 is merely continuous. This is the reason why we extended the definition of ergodicity to all discontinuous viscosity solutions, as explained in the Technical Remark of Section 2. An alternative way would be to keep the classical assumption of ergodicity and suppose that the Comparison Principle holds for the mesoscopic problem. This would give a less general result, though suitable for most applications.

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In order to define the effective initial condition, we have to construct the recession functions of suitable Hamiltonians. We first note that

$$H''(\bar{x}, \bar{y}, z, p_z, X_{zz}) = H'(\bar{x}, \bar{y}, z, 0, p_z, 0, X_{zz}, 0)$$

is the recession function of H with respect to the microscopic variable for frozen macro- and mesoscopic variables. It is indeed positively homogeneous in (p_z, X_{zz}) and, because H is an HJBI operator with bounded coefficients, we have that: for every $(\bar{x}, \bar{y}, \bar{p}_x, \bar{p}_y, \overline{X}_{xx}, \overline{X}_{yy})$, there is a constant C so that

$$\begin{aligned} &\left|H(x, y, z, p_x, p_y, p_z, X_{xx}, X_{yy}, X_{zz}, 0, 0, 0) - H''(x, y, z, p_z, X_{zz})\right| \leqslant C \\ &\quad \text{for every } (z, p_z, X_{zz}), \end{aligned} \tag{36}$$

for every $(x, y, p_x, p_y, X_{xx}, X_{yy})$ in a neighborhood of $(\bar{x}, \bar{y}, \bar{p}_x, \bar{p}_y, \overline{X}_{xx}, \overline{X}_{yy})$.

We shall also need the recession function of the effective mesoscopic Hamiltonian H_1. Its existence is guaranteed by the following lemma.

Lemma 2. *Assume that the Hamiltonian H is ergodic with respect to the microscopic variable z. Then, so is the recession Hamiltonian H'. Moreover, the associated effective Hamiltonian H_1' is the recession function of the mesoscopic effective Hamiltonian H_1.*

Proof. We first show that the recession Hamiltonian H' is ergodic in z. For every $\lambda > 0$, define

$$H_\lambda(x, y, z, p_y, p_z, X_{yy}, X_{zz}, X_{yz}) = \lambda^{-1} H(x, y, z, 0, \lambda p_y, \lambda p_z, 0, \lambda X_{yy}, \lambda X_{zz}, 0, 0, \lambda X_{yz}).$$

One deduces easily from the ergodicity of H in the microscopic variable that H_λ is ergodic with effective Hamiltonian

$$\overline{H}_\lambda(x, y, p_y, X_{yy}) = \lambda^{-1} H_1(x, y, 0, \lambda p_y, 0, \lambda X_{yy}).$$

On the other hand, by assumption (34) written for H and H', we have that, for every x bounded and every $(p_y, p_z, X_{yy}, X_{zz}, X_{yz})$,

$$\left|H_\lambda(x, y, z, p_y, p_z, X_{yy}, X_{zz}, X_{yz}) - H'(x, y, z, p_y, p_z, X_{yy}, X_{zz}, X_{yz})\right| \leqslant C/\lambda. \tag{37}$$

Thus, H_λ converges to H' uniformly (for x bounded) as $\lambda \to +\infty$. We deduce easily from Lemma 1 that H' is ergodic with effective Hamiltonian $H_1' = \lim_{\lambda\to 0} \overline{H}_\lambda$.

It remains to show that H_1' is the recession function of H_1. The positive homogeneity of H_1' in (p_y, X_{yy}) is inherited from that of H' by a standard argument. Since estimate (34) for H and H' is uniform in the microscopic variable, we deduce from Lemma 1 that, for every (x, p_x, X_{xx}) bounded and every (p_y, X_{yy}),

$$\left|H_1(x, y, p_x, p_y, X_{xx}, X_{yy}) - H_1'(x, y, p_y, X_{yy})\right| \leqslant C.$$

Hence, H_1' is the recession function of H_1 in the mesoscopic variable y. □

We are now in a position to state the assumptions for micro and mesoscopic stabilization.

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– *Microscopic stabilization.* The pair (H'', h) is stabilizing for the microscopic variable z at each point $(\bar{x}, \bar{y})$. Namely, for every macro- and mesoscopic variables $(\bar{x}, \bar{y})$, the unique periodic viscosity solution of the microscopic Cauchy cell problem

$$\partial_t w_2 + H''\big(\bar{x}, \bar{y}, z, D_z w_2, D^2_{zz} w_2\big) = 0 \quad \text{in } (0, +\infty) \times \mathbb{R}^p,$$
$$w_2(0, z) = h(\bar{x}, \bar{y}, z), \quad w_2 \text{ periodic},$$

is such that $w_2(t, \cdot)$ converges uniformly to a constant as $t \to +\infty$. The constant is denoted by $h_1(\bar{x}, \bar{y})$ and is called the *effective mesoscopic initial data*.

– *Mesoscopic stabilization.* The pair (H'_1, h_1) is stabilizing for the mesoscopic variable y at each point $\bar{x}$. Namely, all periodic viscosity solutions of the mesoscopic Cauchy cell problem

$$\partial_t w_1 + H'_1\big(\bar{x}, y, D_y w_1, D^2_{yy} w_1\big) = 0 \quad \text{in } (0, +\infty) \times \mathbb{R}^m,$$
$$w_1(0, y) = h_1(\bar{x}, y), \quad w_1 \text{ periodic}, \tag{38}$$

are such that $w_1(t, \cdot)$ converges uniformly to a constant as $t \to +\infty$ (and the constant is independent of the solution). The constant is denoted by $\overline{h}(\bar{x})$ and is called the *effective* (*macroscopic*) *initial data*.

We recall that, since the Comparison Principle may not hold for the mesoscopic Cauchy cell problem, stabilization has to be defined for all discontinuous viscosity solutions.

Examples and references about sufficient conditions for micro- and mesoscopic ergodicity and stabilization are in Section 2. As an example, in the next subsection on three-scale homogenization we will assume the uniform ellipticity of the operator.

Convergence for the three-scale singular perturbation problem is given in the following result.

Theorem 2. *Under the above assumptions, the semi-limits $\bar{u}$ and $\underline{u}$ are, respectively, a subsolution and a supersolution of the effective Cauchy problem* ($\overline{\text{HJ}}$). *Moreover, if $\overline{H}$ satisfies the Comparison Principle, then u^ε converges uniformly on the compact subsets to the viscosity solution of* ($\overline{\text{HJ}}$).

Proof. The proof adapts the arguments of Theorem 1 in Section 3. Consequently, we shall only stress the main differences.

Let us first show that $\bar{u}$ is a subsolution to the effective equation. Fix a point $(\bar{t}, \bar{x})$ with $\bar{t} > 0$. Let φ be a smooth test function such that: $\bar{u}(\bar{t}, \bar{x}) = \varphi(\bar{t}, \bar{x})$, $(\bar{t}, \bar{x})$ is a strict maximum point of $\bar{u} - \varphi$ and

$$\partial_t \varphi(\bar{t}, \bar{x}) + \overline{H}\big(\bar{x}, D_x\varphi(\bar{t}, \bar{x}), D^2_{xx}\varphi(\bar{t}, \bar{x})\big) \geqslant 3\eta. \tag{39}$$

From now on, we put $\overline{p}_x = D_x\varphi(\bar{t}, \bar{x})$, $\overline{X}_{xx} = D^2_{xx}\varphi(\bar{t}, \bar{x})$ and $\overline{H} := \overline{H}(\bar{x}, \overline{p}_x, \overline{X}_{xx})$. We claim that, for every $r > 0$ small enough, there is a parameter $\varepsilon' > 0$ and an equibounded family of continuous correctors $\{\chi^\varepsilon \mid 0 < \varepsilon < \varepsilon'\}$ so that

$$H^\varepsilon\left(x, y, z, D_x\varphi(t, x), D_y\chi^\varepsilon, \frac{D_z\chi^\varepsilon}{\varepsilon}, D^2_{xx}\varphi(t, x), D^2_{yy}\chi^\varepsilon, \frac{D^2_{zz}\chi^\varepsilon}{\varepsilon}, 0, 0, \frac{D^2_{yz}\chi^\varepsilon}{\varepsilon^{1/2}}\right) \geqslant \overline{H} - 2\eta$$

in $Q_r(\bar t,\bar x)=(\bar t-r,\bar t+r)\times B_r(\bar x)\times\mathbb{R}^m\times\mathbb{R}^p$ for all $\varepsilon<\varepsilon'$. This will imply that the perturbed test function

$$\psi^\varepsilon(t,x,y,z)=\varphi(t,x)+\varepsilon\chi^\varepsilon(y,z)$$

is a supersolution of (32) in $Q_r(\bar t,\bar x)$ for all $\varepsilon<\varepsilon'$. Since $\{\psi^\varepsilon\}$ uniformly converges to φ on $\overline{Q_r}$, we obtain the desired contradiction by the Comparison Principle, as in the proof of Theorem 1.

To construct the family of the equibounded correctors χ^ε, we first consider the solution $w_\delta(y)$ of the mesoscopic cell problem

$$\delta w_\delta+H_1\big(\bar x,y,\overline{p}_x,D_yw_\delta,\overline{X}_{xx},D^2_{yy}w_\delta\big)=0. \tag{40}$$

It has a maximal u.s.c. subsolution $w_{\delta,+}$ and a minimal l.s.c. supersolution $w_{\delta,-}$. By Perron's method, they are actually discontinuous viscosity solutions given by

$$w_{\delta,+}=\sup\big\{w\;\big|\;w\text{ is a periodic subsolution of (40)}\big\}$$

and a symmetric formula for $w_{\delta,-}$. By the ergodicity of H_1, there exists a parameter $\delta>0$ sufficiently small such that

$$\|\delta w_{\delta,-}+\overline{H}\|_\infty\leqslant\eta,\qquad \|\delta w_{\delta,+}+\overline{H}\|_\infty\leqslant\eta. \tag{41}$$

For every $\varepsilon>0$ and $r>0$, we introduce the perturbed Hamiltonian

$$\begin{aligned}&H_r^\varepsilon(y,z,p_y,p_z,X_{yy},X_{zz},X_{yz})\\&\quad:=\min_{(t,x)\in(\bar t-r,\bar t+r)\times B_r(\bar x)}H^\varepsilon\big(x,y,z,D_x\varphi(t,x),p_y,p_z,D^2_{xx}\varphi(t,x),X_{yy},X_{zz},0,0,X_{yz}\big).\end{aligned}$$

We note that

$$H_r^\varepsilon(y,z,p_y,p_z,X_{yy},X_{zz},X_{yz})\to H(\bar x,y,z,\overline{p}_x,p_y,p_z,\overline{X}_{xx},X_{yy},X_{zz},0,0,X_{yz})$$

uniformly on the compact sets as $(\varepsilon,r)\to(0,0)$. Moreover, by the continuity of H_r^ε, the singular perturbation problem

$$\delta w^\varepsilon_{\delta,r}+H_r^\varepsilon\left(y,z,D_yw^\varepsilon_{\delta,r},\frac{D_zw^\varepsilon_{\delta,r}}{\varepsilon},D^2_{yy}w^\varepsilon_{\delta,r},\frac{D^2_{zz}w^\varepsilon_{\delta,r}}{\varepsilon},\frac{D^2_{yz}w^\varepsilon_{\delta,r}}{\varepsilon^{1/2}}\right)=0$$

has a unique viscosity solution $w^\varepsilon_{\delta,r}$ that is periodic both in y and in z. By assumption, the limit Hamiltonian H is ergodic in the microscopic variable z and its effective Hamiltonian is H_1. Hence, arguing as in Theorem 1, we deduce that the semi-limits

$$\underline{w}_\delta(y)=\liminf_{\varepsilon\to0,\,r\to0,\,y'\to y}\;\inf_z w^\varepsilon_{\delta,r}(y',z),\qquad \overline{w}_\delta(y)=\limsup_{\varepsilon\to0,\,r\to0,\,y'\to y}\;\sup_z w^\varepsilon_{\delta,r}(y',z)$$

are, respectively, a supersolution and a subsolution of (40). In particular,

$$w_{\delta,-}\leqslant\underline{w}_\delta\leqslant\overline{w}_\delta\leqslant w_{\delta,+}.$$

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By compactness and (41), we deduce that there are small ε' and r' so that

$$\left\|\delta w^{\varepsilon}_{\delta,r} + \overline{H}\right\|_{\infty} \leqslant 2\eta \quad \text{for all } 0<\varepsilon<\varepsilon',\ 0<r<r'.$$

For every $0<r<r'$ fixed, we define the corrector

$$\chi^{\varepsilon}(y,z) = w^{\varepsilon}_{\delta,r}(y,z).$$

Clearly, the correctors are equibounded with the bound

$$\left\|\chi^{\varepsilon}\right\|_{\infty} \leqslant \delta^{-1}\sup\left\{\left|H^{\varepsilon}_{r}(y,z,0,\dots,0)\right| \,\middle|\, y,z,\varepsilon\right\}$$

(this follows from testing χ^{ε} against constants). Moreover, using the above mentioned properties of $w^{\varepsilon}_{\delta,r}$, we get that

$$H^{\varepsilon}_{r}\left(y,z,D_y\chi^{\varepsilon},\frac{D_z\chi^{\varepsilon}}{\varepsilon},D^2_{yy}\chi^{\varepsilon},\frac{D^2_{zz}\chi^{\varepsilon}}{\varepsilon},\frac{D^2_{yz}\chi^{\varepsilon}}{\varepsilon^{1/2}}\right) = -\delta w^{\varepsilon}_{\delta,r} \geqslant \overline{H} - 2\eta.$$

The definition of H^{ε}_{r} yields

$$H^{\varepsilon}\left(x,y,z,D_x\varphi(t,x),D_y\chi^{\varepsilon},\frac{D_z\chi^{\varepsilon}}{\varepsilon},D^2_{xx}\varphi(t,x),D^2_{yy}\chi^{\varepsilon},\frac{D^2_{zz}\chi^{\varepsilon}}{\varepsilon},0,0,\frac{D^2_{yz}\chi^{\varepsilon}}{\varepsilon^{1/2}}\right) \geqslant \overline{H} - 2\eta$$

in $Q_r(\bar{t},\bar{x})$ for all $\varepsilon<\varepsilon'$. Therefore, we have constructed a family of correctors with the desired properties. This completes the proof that $\bar{u}$ is a subsolution of the effective equation in $(0,T)\times\mathbb{R}^n$.

Now, let us turn to the proof that $\bar{u}$ is a subsolution to the effective initial condition. We introduce the following notations:

$$H^{\varepsilon,\prime}_{r}(y,z,p_y,p_z,X_{yy},X_{zz},X_{yz}) := \min_{|x-\bar{x}|\leqslant r} H^{\varepsilon,\prime}(x,y,z,p_y,p_z,X_{yy},X_{zz},X_{yz}),$$
$$h^{\varepsilon}_{r}(y,z) := \max_{|x-\bar{x}|\leqslant r} h^{\varepsilon}(x,y,z).$$

It can be easily checked that, as $(\varepsilon,r)\to(0,0)$, $H^{\varepsilon,\prime}_{r}$ and h^{ε}_{r} converge locally uniformly, respectively, to $H'(\bar{x},\cdot)$ and to $h(\bar{x},\cdot)$. Let w^{ε}_{r} be the unique solution of the Cauchy problem

$$\partial_t w^{\varepsilon}_{r} + H^{\varepsilon,\prime}_{r}\left(y,z,D_y w^{\varepsilon}_{r},\frac{D_z w^{\varepsilon}_{r}}{\varepsilon},D^2_{yy}w^{\varepsilon}_{r},\frac{D^2_{zz}w^{\varepsilon}_{r}}{\varepsilon},\frac{D^2_{yz}w^{\varepsilon}_{r}}{\varepsilon^{1/2}}\right) = 0 \quad \text{in } (0,+\infty)\times\mathbb{R}^m\times\mathbb{R}^p,$$
$$w^{\varepsilon}_{r}(0,y,z) = h^{\varepsilon}_{r}(y,z), \quad \text{on } \mathbb{R}^m\times\mathbb{R}^p,\ w^{\varepsilon}_{r} \text{ periodic in } y \text{ and in } z.$$

By assumptions and by Lemma 2, the limit Hamiltonian $H'(\bar{x},\cdot)$ is ergodic and stabilizing with respect to the microscopic variable z. We infer from Theorem 1 that the semi-limits $\overline{w}$ and $\underline{w}$ of w^{ε}_{r} as $(\varepsilon,r)\to(0,0)$ are, respectively, a subsolution and a supersolution of the effective problem

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$$\partial_t w + H_1'\big(\bar{x}, y, D_y w, D^2_{yy} w\big) = 0 \quad \text{in } (0, +\infty) \times \mathbb{R}^m,$$
$$w(0, y) = h_1(\bar{x}, y) \quad \text{on } \mathbb{R}^m,\ w \text{ periodic in } y.$$

We denote by w_+ and w_- the maximal u.s.c. subsolution and minimal l.s.c. supersolution of the cell problem (38). We therefore have that

$$w_- \leqslant \underline{w} \leqslant \overline{w} \leqslant w_+.$$

Since w_+ and w_- are discontinuous solutions and since H_1' is stabilizing, we know that for every $\eta > 0$, there exists $T > 0$ such that

$$\big\| w_-(T, \cdot) - \bar{h}(\bar{x}) \big\|_\infty, \big\| w_+(T, \cdot) - \bar{h}(\bar{x}) \big\|_\infty \leqslant \eta/2.$$

By compactness, the previous two relations entail that, for every $\eta > 0$ and for T sufficiently large, there exist (ε', r') so small that

$$\big\| w_r^\varepsilon(T, \cdot,\cdot) - \bar{h}(\bar{x}) \big\|_\infty \leqslant \eta \quad \text{for every } \varepsilon \leqslant \varepsilon',\ r \leqslant r'.$$

Therefore, by the Comparison Principle, we obtain:

$$\big\| w_r^\varepsilon(t, \cdot,\cdot) - \bar{h}(\bar{x}) \big\|_\infty \leqslant \eta \quad \text{for every } \varepsilon \leqslant \varepsilon',\ r \leqslant r',\ t \geqslant T$$

(here, the relation $H_r^{\varepsilon,\prime}(y, z, 0, 0, 0, 0, 0) = 0$ has been used). In other words,

$$w_r^\varepsilon(t, \cdot,\cdot) \to \bar{h}(\bar{x}) \quad \text{uniformly as } (t, \varepsilon, r) \to (+\infty, 0, 0).$$

The rest of the proof is similar to the proof of Theorem 1. For each $\eta > 0$, we consider (ε', r') and T so that

$$\sup_{0<\varepsilon<\varepsilon',\, 0<r<r'} \sup_{t \geqslant T} \sup_{y,z} \big| w_r^\varepsilon(t, y, z) - \bar{h}(\bar{x}) \big| \leqslant \eta.$$

Then, we construct the function ψ_0 as in the proof of Theorem 1 and choose the constant $C > 0$ so that

$$\Big| H^\varepsilon\big(x, y, z, D\psi_0(x), p_y, p_z, D^2_{xx}\psi_0(x), X_{yy}, X_{zz}, 0, 0, X_{yz}\big) \\ - H^{\varepsilon,\prime}(x, y, z, p_y, p_z, X_{yy}, X_{zz}, X_{yz}) \Big| \leqslant C$$

for every $(y, z, p_y, p_z, X_{yy}, X_{zz}, X_{yz})$, $x \in B_r(\bar{x})$, $0 < \varepsilon < \varepsilon'$. The function

$$\psi^\varepsilon(t, x, y, z) := w_r^\varepsilon\left(\frac{t}{\varepsilon}, y, z\right) + \psi_0(x) + Ct$$

is readily seen to be a supersolution of (32) in $(0, r) \times B_r(\bar{x}) \times \mathbb{R}^m \times \mathbb{R}^p$, with boundary condition $\psi^\varepsilon \geqslant u^\varepsilon$ on $(0, r) \times \partial B_r(\bar{x}) \times \mathbb{R}^m \times \mathbb{R}^p$. By the Comparison Principle, we conclude that

$$u^\varepsilon(t, x, y, z) \leqslant \psi^\varepsilon(t, x, y, z) \quad \text{in } (0, r) \times B_r(\bar{x}) \times \mathbb{R}^m \times \mathbb{R}^p.$$

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Taking the supremum over (y,z), sending $\varepsilon \to 0$ and then $t \to 0+$, we obtain the inequality $\overline{u}(0,\bar{x}) \leqslant \overline{h}(\bar{x}) + \eta$. The arbitrariness of η yields $\overline{u}(0,\bar{x}) \leqslant \overline{h}(\bar{x})$.

The proof for $\underline{u}$ is similar so we shall omit it. □

5.2. *Example: Iterated homogenization*

Most applications we have in mind can be written as problems depending on a scalar factor $\gamma > 0$:

$$\partial_t u^\varepsilon + H\left(x, y, z, D_x u^\varepsilon, \frac{D_y u^\varepsilon}{\varepsilon}, \frac{D_z u^\varepsilon}{\varepsilon^2}, D^2_{xx} u^\varepsilon, \frac{D^2_{yy} u^\varepsilon}{\varepsilon^{2\gamma}}, \frac{D^2_{zz} u^\varepsilon}{\varepsilon^{4\gamma}}, \frac{D^2_{xy} u^\varepsilon}{\varepsilon^{\gamma}}, \frac{D^2_{xz} u^\varepsilon}{\varepsilon^{2\gamma}}, \frac{D^2_{yz} u^\varepsilon}{\varepsilon^{3\gamma}}\right) = 0, \tag{42}$$

in $(0,T) \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p$, with initial data

$$u^\varepsilon(0,x,y,z) = h(x,y,z) \quad \text{on } \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p.$$

As in Section 4, $\gamma = 1/2$ is the critical value; indeed, for $\gamma < 1/2$ the first-order terms are the leading ones, while for $\gamma > 1/2$ the second-order terms become the most relevant. This gives rise to three different possible effective Hamiltonians.

Let us work in detail a special important example, namely iterated homogenization for the Cauchy problem

$$\begin{aligned} &\partial_t v^\varepsilon + F\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, D_x v^\varepsilon, D^2_{xx} v^\varepsilon\right) = 0 \quad \text{in } (0,T) \times \mathbb{R}^n, \\ &v^\varepsilon(0,x) = h\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right) \quad \text{on } \mathbb{R}^n. \end{aligned} \tag{43}$$

In order to distinguish the various scales, we introduce the periodic (shadow) variables $y = \varepsilon^{-1} x$ and $z = \varepsilon^{-2} x$.

Our goal in this subsection is simply to give an illustration of how to apply the abstract convergence result Theorem 2. We shall therefore restrict to the simple yet natural setting of uniformly parabolic equations that are concave in the Hessian. We recall that, even in this favorable context (the solutions are classical), the convergence for the value function with iterated homogenization is entirely new.

The precise assumptions are as follows. In addition to the standing assumption that F is an HJBI operator

$$\begin{aligned} &F(x,y,z,p_x,X_{xx}) \\ &\quad := \min_{\beta \in B} \max_{\alpha \in A} \bigl\{-\operatorname{tr}\bigl(X_{xx} a(x,y,z,\alpha,\beta)\bigr) - p_x \cdot f(x,y,z,\alpha,\beta) - \ell(x,y,z,\alpha,\beta)\bigr\} \end{aligned}$$

with coefficients σ ($a = \sigma\sigma^T/2$), f, ℓ and h satisfying the assumptions of Section 1, we suppose that F is uniformly elliptic, i.e.,

$$\text{for some } \nu > 0, \quad a(x,y,z,\alpha,\beta) \geqslant \nu I_n \quad \text{for all } (x,y,z,\alpha,\beta), \tag{44}$$

concave with respect to the Hessian X_{xx} and Hölder continuous in (y,z), i.e.,

$$a \equiv a(x,y,z,\beta), \quad \ell \text{ is Hölder continuous in } (y,z), \text{ uniformly in } (x,y,z,\alpha,\beta). \tag{45}$$

These assumptions are particularly well adapted to stochastic optimal control. As recalled in Section 4, the solution to (43) is the value function

$$v^\varepsilon(t,x) := \inf_{\alpha\in\Gamma(t)} \sup_{\beta\in\mathcal{B}(t)} E_x\left\{\int_0^t \ell\left(x_s, \frac{x_s}{\varepsilon}, \frac{x_s}{\varepsilon^2}, \alpha_s, \beta_s\right) ds + h\left(x_t, \frac{x_t}{\varepsilon}, \frac{x_t}{\varepsilon^2}\right)\right\},$$

where x_s is the solution of the controlled stochastic differential equation

$$dx_s = f\left(x_s, \frac{x_s}{\varepsilon}, \frac{x_s}{\varepsilon^2}, \alpha_s, \beta_s\right) ds + \sigma\left(x_s, \frac{x_s}{\varepsilon}, \frac{x_s}{\varepsilon^2}, \alpha_s, \beta_s\right) dW_s, \quad x_0 = x,$$

$\mathcal{B}(t)$ is the set of admissible controls for the second player, and $\Gamma(t)$ is the set of admissible strategies for the first (see Section 4 and [33,55]).

Corollary 6. *Assume that F is uniformly elliptic* (44) *and satisfies* (45). *Then, there exist a continuous and uniformly elliptic $\overline{F}$ and a continuous $\overline{h_s}$, such that the solution v^ε of the homogenization problem* (43) *converges uniformly on the compact subsets of* $(0,T)\times\mathbb{R}^n$ *as* $\varepsilon\to 0$ *to the unique solution of* (22).

Proof. Iterated homogenization (43) corresponds to a special case of the singular perturbation problem (42) with scale factor $\gamma = 1$ for the Hamiltonian

$$\begin{aligned} &H(x,y,z,p_x,p_y,p_z,X_{xx},X_{yy},X_{zz},X_{xy},X_{xz},X_{yz}) \\ &\quad = F\big(x,y,z,p_x+p_y+p_z, X_{xx}+X_{yy}+X_{zz}+X_{xy}+X_{xz}+X_{yz}+X_{xy}^T+X_{xz}^T+X_{yz}^T\big). \end{aligned}$$

Therefore, to apply the convergence result Theorem 2, we have to verify that the associated pure second-order Hamiltonian

$$H_s(x,y,z,p_x,X_{xx},X_{yy},X_{zz}) := F(x,y,z,p_x,X_{xx}+X_{yy}+X_{zz})$$

is ergodic and stabilizing at both the micro and the mesoscopic scales and that the limit equation (22) satisfies the Comparison Principle. We shall only justify the multiscale ergodicity of the Hamiltonian since the proof of its stabilization follows from similar arguments.

By the uniform ellipticity of the Hamiltonian, we know that the microscopic true cell problem

$$F\big(\bar{x},\bar{y},z,\bar{p}_x,\overline{X}_{xx}+D^2_{zz}\chi_2\big) = F_1(\bar{x},\bar{y},\bar{p}_x,\overline{X}_{xx}) \quad \text{in } \mathbb{R}^n$$

has a solution (see [5,7]). In particular, F is ergodic at the microscopic scale. Moreover, one can show that the corrector χ_2 is in $C^{2,\rho}$ for some $\rho > 0$ (depending only on the ellipticity constants and the Hölder exponent of the running cost) with the a priori bound

$$\big\|\chi_2-\chi_2(0)\big\|_{C^{2,\rho}} \leqslant C\big(1+|\bar{p}_x|+|\overline{X}_{xx}|\big) \tag{46}$$

(see [7] as well as the theory of classical solutions for HJBI equations as exposed for instance in Safonov [53]).

Using this information, one can verify that the mesoscopic effective Hamiltonian has the following properties

- $F_1(x, y, p_x, X_{xx})$ is concave in X_{xx} and jointly continuous with respect to all variables;
- F_1 is uniformly elliptic in X_{xx}, i.e. $-C|\xi|^2 \leqslant F_1(x, y, p_x, X_{xx} + \xi \otimes \xi) - F_1(x, y, p_x, X_{xx}) \leqslant -\nu|\xi|^2$;
- F_1 is Lipschitz continuous in (p_x, X_{xx}), i.e.

$$\left|F_1(x, y, p_x, X_{xx}) - F_1\left(x, y, p'_x, X'_{xx}\right)\right| \leqslant C\left(\left|p_x - p'_x\right| + \left|X_{xx} - X'_{xx}\right|\right);$$

- F_1 is locally bounded, i.e. $|F_1(x, y, 0, 0)| \leqslant C$;
- F_1 is Hölder continuous in y, i.e.

$$\left|F_1(x, y, p_x, X_{xx}) - F_1(x, y', p_x, X_{xx})\right| \leqslant C\left(1 + |p_x| + |X_{xx}|\right)|y - y'|^\rho$$

(see [4] and [7] for similar arguments). In other words, F_1 has the same structural properties as F.

This guarantees that the mesoscopic true cell problem

$$F_1\left(\bar{x}, y, \bar{p}_x, \overline{X}_{xx} + D^2_{yy}\chi_1\right) = \overline{F}(\bar{x}, \bar{p}_x, \overline{X}_{xx}) \quad \text{in } \mathbb{R}^n$$

has a solution. This implies the mesoscopic ergodicity of F_1 because the cell problem

$$\delta w_{\delta,1} + F_1\left(\bar{x}, y, \bar{p}_x, \overline{X}_{xx} + D^2_{yy} w_{\delta,1}\right) = 0$$

has a unique solution and satisfies the Comparison Principle. Moreover, the corrector χ_1 is in $C^{2,\rho}$ for some $\rho > 0$ with a priori estimates of the form (46). This implies that $\overline{F}$ enjoys the same structural properties as F_1. In particular, the Comparison Principle holds for the limit equation (22) (see e.g. [39]). □

5.3. *Example: Iterated averaging*

Let us also briefly mention another interesting application of our results that concerns iterate averaging. We consider the following Cauchy problem

$$\partial_t v^\varepsilon + F\left(\frac{t}{\varepsilon}, \frac{t}{\varepsilon^2}, x, D_x v^\varepsilon, D^2_{xx} v^\varepsilon\right) = 0 \quad \text{in } (0, T) \times \mathbb{R}^n, \quad v^\varepsilon(0, x) = h(x) \quad \text{on } \mathbb{R}^n, \tag{47}$$

where the Hamiltonian $F = F(y, z, x, p_x, X_{xx})$ is periodic both in y and in z and degenerate elliptic with respect to X_{xx}. Taking into account the uniqueness of the solution, one can easily establish the following relation: $v^\varepsilon(t, x) \equiv u^\varepsilon(t, t/\varepsilon, t/\varepsilon^2, x)$, where $u^\varepsilon = u^\varepsilon(t, y, z, x)$ solves the problem (42) with $H(x, y, z, p_x, p_y, p_z, X_{xx}) = p_y + p_z + F(y, z, x, p_x, X_{xx})$. Moreover, taking into account the linearity of H with respect to the microscopic gradient p_z, one can explicitly solve the true microscopic cell problem. We deduce that the Hamiltonian is ergodic with respect to the microscopic variable with effective mesoscopic Hamiltonian

$$H_1(x, y, p_x, p_y, X_{xx}) = p_y + \int_0^1 F(y, z, x, p_x, X_{xx})\,dz.$$

By analogous arguments, one can prove that the effective Hamiltonian $\overline{H}$ can be written as:

$$\overline{H}(x, p_x, X_{xx}) = \int_0^1 \left(\int_0^1 F(y, z, x, p_x, X_{xx})\,dz \right) dy.$$

Hence, Theorem 2 implies that v^ε converges locally uniformly to the solution of ($\overline{\mathrm{HJ}}$) with initial data $\overline{h} = h$. The point here is that we have an (elementary) explicit formula for the effective Hamiltonian because of the very special structure of the problem.

5.4. The multiscale problem

In this subsection, we briefly explain how to generalize the preceding results to the singular perturbation problem with an arbitrary number of scales. Let us consider the Cauchy problem with $j + 1$ scales

$$\begin{aligned} &\partial_t u^\varepsilon + H^\varepsilon\big(x, y_1, \dots, y_j, D_x u^\varepsilon, \varepsilon^{-1} D_{y_1} u^\varepsilon, \dots, \varepsilon^{-j} D_{y_j} u^\varepsilon, D^2_{xx} u^\varepsilon, \varepsilon^{-1} D^2_{y_1 y_1} u^\varepsilon, \dots, \\ &\quad \varepsilon^{-j} D^2_{y_j y_j} u^\varepsilon, \big(\varepsilon^{-i/2-k/2} D^2_{y_i y_k} u^\varepsilon\big)_{1 \leqslant i < k \leqslant j}, \big(\varepsilon^{-i/2} D^2_{x y_i} u^\varepsilon\big)_{1 \leqslant i \leqslant j}\big) = 0 \end{aligned} \tag{48}$$

in $(0, T) \times \mathbb{R}^n \times \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_j}$, with the initial condition

$$u^\varepsilon(0, x, y_1, \dots, y_j) = h^\varepsilon(x, y_1, \dots, y_j) \quad \text{on } \mathbb{R}^n \times \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_j}.$$

As usual, we assume that H^ε and h^ε converge locally uniformly to H and h as $\varepsilon \to 0$. We suppose that the functions H, h and H^ε, h^ε are periodic in $(y_1, \dots, y_j)$, that they satisfy the standard assumptions of Section 1 and that they are equibounded in the sense of (10) and (11). We also assume that, for every ε, there exists a recession function

$$H^{\varepsilon,\prime} = H^{\varepsilon,\prime}\big(x, y_1, \dots, y_j, p_{y_1}, \dots, p_{y_j}, (X_{y_i y_k})_{1 \leqslant i \leqslant k \leqslant j}\big),$$

positively 1-homogeneous in $(p_{y_1}, \dots, p_{y_j}, (X_{y_i y_k})_{1 \leqslant i \leqslant k \leqslant j})$, which satisfies, for some constant $C > 0$

$$\begin{aligned} &\big| H^\varepsilon\big(x, y_1, \dots, y_j, p_x, p_{y_1}, \dots, p_{y_j}, X_{xx}, X_{y_1 y_1}, \dots, X_{y_j y_j}, (X_{y_i y_k})_{1 \leqslant i < k \leqslant j}, 0\big) \\ &\quad - H^{\varepsilon,\prime}\big(x, y_1, \dots, y_j, p_{y_1}, \dots, p_{y_j}, (X_{y_i y_k})_{1 \leqslant i \leqslant k \leqslant j}\big)\big| \leqslant C \end{aligned} \tag{49}$$

for every $(y_i, p_{y_i}, X_{y_i y_i}) \in \mathbb{R}^{m_i} \times \mathbb{R}^{m_i} \times \mathbb{S}^{m_i}$ $(i = 1, \dots, j)$, $X_{y_i y_k} \in \mathbb{M}^{m_i \times m_k}$ $(1 \leqslant i < k \leqslant j)$, for every (x, p_x, X_{xx}) in a neighborhood of $(\overline{x}, \overline{p}_x, \overline{X}_{xx})$ and for every ε. This implies, we recall, that the Hamiltonian H has a recession function H' which is the uniform limit on the compact sets of $H^{\varepsilon,\prime}$ as $\varepsilon \to 0$.

Our ergodicity and stabilization assumptions are defined by induction. We set $H_j = H$ and $h_j = h$.

- *Iterated ergodicity.* For every $i = j, \dots, 1$, H_i is ergodic with respect to y_i. We denote by H_{i-1} its effective Hamiltonian and we put $\overline{H} = H_0$.

Arguing as for the three-scale problem (Lemma 2), one can deduce from (49) that every effective Hamiltonian H_i has a recession function H_i', that every H_i' is ergodic and that its effective Hamiltonian is H_{i-1}'.

- *Iterated stabilization.* For every $i = j, \dots, 1$, we set

$$H_i'' := H_i'(x, y_1, \dots, y_i, 0, \dots, 0, p_i, 0, \dots, 0, X_{y_i y_i}, 0).$$

  The pair (H_i'', h_i) is stabilizing with respect to y_i at each point $(x, y_1, \dots, y_{i-1})$. We denote by h_{i-1} its effective initial data and we put $\overline{h} = h_0$.

As usual, the Cauchy problem (48) has exactly one continuous solution u^ε and the family $\{u^\varepsilon\}$ is equibounded; so the weak semi-limits $\overline{u}$ and $\underline{u}$ are well defined.

Theorem 3. *Under the above assumptions, the semi-limits $\overline{u}$ and $\underline{u}$ are, respectively, a subsolution and a supersolution to problem* $(\overline{\text{HJ}})$. *Furthermore, u^ε converges locally uniformly to the solution of* $(\overline{\text{HJ}})$ *provided that the effective Hamiltonian $\overline{H}$ satisfies the Comparison Principle.*

The proof of the Theorem follows simply by induction and by using the arguments introduced for the three-scale problem (Theorem 2). Since the modifications are only routine, we leave the detailed proof to the reader.

Remark. The convergence result can be immediately extended to nonpower-like scales

$$\partial_t u^\varepsilon + H^\varepsilon\big(x, y_1, \dots, y_j, D_x u^\varepsilon, \varepsilon_1^{-1} D_{y_1} u^\varepsilon, \dots, \varepsilon_j^{-1} D_{y_j} u^\varepsilon, D_{xx}^2 u^\varepsilon, \varepsilon_1^{-1} D_{y_1 y_1}^2 u^\varepsilon, \dots,$$
$$\varepsilon_j^{-1} D_{y_j y_j}^2 u^\varepsilon, \big(\varepsilon_i^{-1/2} \varepsilon_k^{-1/2} D_{y_i y_k}^2 u^\varepsilon\big)_{1 \leqslant i < k \leqslant j}, \big(\varepsilon_i^{-1/2} D_{x y_i}^2 u^\varepsilon\big)_{1 \leqslant i \leqslant j}\big) = 0.$$

The assumption is that $\varepsilon := (\varepsilon_1, \dots, \varepsilon_j)$ tends to 0 in the sense that $\varepsilon_1 \to 0$ and $\varepsilon_i/\varepsilon_{i-1} \to 0$ $(i = 2, \dots, j)$. Equation (48) is a particular case of the above equation, choosing $\varepsilon_i = \varepsilon^i$.

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