A WEIGHTED GRADIENT THEORY OF PHASE TRANSITIONS WITH A POSSIBLY SINGULAR AND DEGENERATE SPATIAL INHOMOGENEITY

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Abstract. We study the asymptotic behaviour for a perturbed variational problem for the Cahn-Hilliard theory of phase transitions in a fluid, with spatial inhomogeneities in the internal free energy term. The standard minimal interface criterion will be recovered even in the case in which the inhomogeneity vanishes or becomes infinite.

Contents

1. Introduction

This paper deals with the Γ-convergence of the functional

$$F_\varepsilon(u) = F_\varepsilon(u,\Omega) := \int_\Omega \left[\varepsilon^{p-1} A^{1-p}(x)|Du|^p + \frac{1}{\varepsilon} A(x) W(u)\right] dx, \tag{1.1}$$

where W is a "double-well" potential, $p \in (1,\infty)$ and $\varepsilon > 0$ is a small parameter.

Differently from the previous literature, we consider here the case in which the spatial inhomogeneity $A : \Omega \to [0,\infty]$ does not need to be either bounded or bounded away from zero.

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In spite of this severe degeneracy or singularity, we will be able to recover the standard Γ-limit properties of the functional, in relation with minimal area interfaces.

As well-known, the functional in (1.1) is related to the Cahn-Hilliard theory of phase transitions in fluids (see, e.g., [23, 15]). When $p = 2$ and $A \equiv 1$, the Γ-convergence of F_ε is a classical topic that goes back to De Giorgi and Franzoni in [10], and which was completely settled by Modica and Mortola in [18].

After that, many important extensions and generalisations have been considered in the literature: see, in particular, [6] which considered the case $p \neq 2$ in (1.1) and also [21] for very general results.

Recently, some attention has been devoted to the case in which A in (1.1) may become arbitrarily close to zero or infinity. In particular, in [14] the case in which A behaves like a power of the distance from the boundary was considered (such case is very important in applications, since it is related to fractional operators and to nonlocal and boundary effects, see [3, 4]).

Here, we will deal with even more general types of degeneracies and singularities of A: namely, in our hypotheses, A may even take value zero or infinity and it does not need to be bounded or bounded from zero away from the boundary. The physical motivation for the model in (1.1) may be thought as follows: the phase segregation between the two pure phases -1 and $+1$ is driven by a double-well potential W which is influenced by the inhomogeneity of the medium (this might be caused, for instance, by some impurities of the material). The kinetic term driven by the gradient may be seen as a penalization which makes the problem consistent.

Now, we fix notation, as well as briefly recalling some definitions related to Sobolev spaces and functions with bounded variation. Then we analyse the asymptotic behaviour of the functional F_ε defined in (1.1), stating the corresponding Γ-convergence result.

1.1. **Notation.** In this work, Ω will be a bounded open set of $\mathbb{R}^N$. We denote by $\partial\Omega$ the boundary of Ω relative to the ambient space; for simplicity, $\partial\Omega$ is assumed to be smooth.

We suppose that W is a non-negative, Lipschitz continuous function on $\mathbb{R}$ with at least linear growth at infinity, vanishing only at $\{-1, 1\}$, convex near ± 1 and such that

$$W(r) \leq C(1 - |r|)^p \tag{1.2}$$

for r close to ± 1, for a suitable constant $C > 0$.
A typical example to keep in mind is the case $W(r) = (1 - r^2)^p$.

For every $u \in L^1_{\mathrm{loc}}(\Omega)$, we denote by Du the derivative of u in the sense of distributions.

We suppose that A^{1-p}, $A \in L^1(\Omega)$ and we consider the space

$$\mathcal{D}_A = \mathcal{D}_A(\Omega) := \big\{ u \in L^1_{\mathrm{loc}}(\Omega) \text{ with } A^{1-p}|Du|^p \in L^1(\Omega) \big\}. \tag{1.3}$$

As usual, $BV(\Omega)$ denotes the space of all $u \in L^1(\Omega)$ with bounded variation; i.e., such that Du is a bounded Borel measure on Ω.

We denote by Su the *jump set* of u; i.e., the complement of the set of Lebesgue points of u. For details and results about the theory of Sobolev spaces and BV functions, we refer to [1], [11] and [7].

We suppose that there exists a bounded open set $\Omega_\star \subseteq \Omega$, with smooth boundary, such that:

$$\begin{aligned} &\forall \gamma > 0, \text{ there exists a finite number of balls } \{B_i\}_{1 \le i \le M(\gamma)} \\ &\qquad \text{of radius } r_i(\gamma) \in (0,1) \text{ such that} \\ &\qquad \sum_{1 \le i \le M(\gamma)} r_i(\gamma)^{N-1} \le \gamma \\ &\qquad \text{and } A \text{ is } C^1 \text{ and positive in } \overline{\Omega_\star \setminus \bigcup_{1 \le i \le M(\gamma)} B_i}. \end{aligned} \tag{1.4}$$

Moreover, we assume that either $\Omega_\star = \Omega$ or that

$$\begin{aligned} &N = 3,\ A(x) \text{ coincides with } \big(\mathrm{dist}(x, \partial\Omega)\big)^a \text{ in } \Omega \setminus \Omega_\star \\ &\text{and one of the following two conditions is met:} \\ &\qquad \text{either } p = 2 \text{ and } a \in (-1, 0), \\ &\qquad \text{or } p \in (2,3) \text{ and } a = (p-2)/(p-1). \end{aligned} \tag{1.5}$$

Simple examples of functions A satisfying (1.4) and (1.5) are suitable powers of the distance from $\partial\Omega$, and functions that go to zero or infinity at a finite number of points. These two type of weights are quite important for applications, since they are related to phase transitions driven by fractional power of the Laplacian (see [9, 25, 8, 14, 22]), to weights of Muckenhoupt type (see [19]), to trace spaces (see [20, 16]) and to quasiconformal maps (see [12]).

Condition (1.5) is taken in order to use, near $\partial\Omega$, the results of [14, 22]. In particular, the dimensional condition $N = 3$ is needed for a slicing argument in [4] which is also used in [14, 22], and the condition $p \in [2, 3)$ is needed for having a trace of the a-power of the distance function along $\partial\Omega$, see [20]. Of course, it would be interesting to extend the results of this paper by relaxing such conditions.

We also note that assumption (1.4) includes all functions A with a countable set of singular or degenerate points $\{x_k\}_{k\in\mathbb{N}}$ with finitely many cluster points.

Indeed, let $\{y_i\}_{1\le i\le J} \subset \Omega$ be the cluster points of the sequence $\{x_k\}_{k\in\mathbb{N}}$.

Fix $\gamma > 0$. For every $1 \le i \le J$, we take the ball B_i of radius $\left(\frac{\gamma}{2J}\right)^{\frac{1}{N-1}}$ centered in y_i:

$$B_i := B(y_i, \left(\frac{\gamma}{2J}\right)^{\frac{1}{N-1}}).$$

Thus, for every ball B_i there are infinitely many natural number k such that $x_k \in B_i$. The remaining singular or degenerate points are in finite number, so we can suppose

$$\{x_k\}_{k\in\mathbb{N}} \cap \left(\Omega \setminus \bigcup_{1\le i\le J} B_i\right) = \{x_i\}_{1\le i\le L}.$$

Now, for any $1 \le i \le L$, we take

$$B_{i+J} := B(x_i, \left(\frac{\gamma}{2L}\right)^{\frac{1}{N-1}}).$$

Clearly, the finite covering set of balls $\{B_i\}_{1\le i\le J+L}$ is such that

$$\sum_{1\le i\le J+L} r_i^{N-1} = \frac{\gamma}{2} + \frac{\gamma}{2} = \gamma$$

and then condition (1.4) is satisfied.

1.2. **The Γ-convergence theorem.** For every $\varepsilon > 0$ we consider the functional F_ε defined in (1.1) acting on $\mathcal{D}_A$ (recall (1.3)).

We analyse the asymptotic behaviour of the functional F_ε in terms of Γ-convergence.

Let (u_ε) be an equi-bounded sequence for F_ε; i.e., there exists a constant C such that $F(u_\varepsilon) \le C$. We observe that the "boundedness" of the term $\frac{1}{\varepsilon}\int_\Omega A(x)W(u_\varepsilon)dx$ forces u_ε to take values close to -1 and 1, while the term $\varepsilon^{p-1}\int_\Omega A^{1-p}(x)|Du_\varepsilon|^p dx$ penalises the oscillations of u_ε.

Then, we expect that the sequence (u_ε) converges (in $L^1(\Omega)$, up to subsequences) to a function u, belonging to $BV(\Omega)$ which takes only the values -1 and 1. Moreover, one expects that each u_ε has a transition from the value -1 to the value 1 in a thin layer close to the surface Su, which separates the phases $\{u = -1\}$ and $\{u = 1\}$.

We will prove that the asymptotic behaviour of the functional F_ε is described by the following functional F

$$F(u) = \sigma_p \mathcal{H}^{N-1}(Su), \quad \forall u \in BV(\Omega; \{-1, 1\}), \tag{1.6}$$

where c_p and σ_p are the positive constants defined by

$$c_p := \frac{p}{(p-1)^{\frac{p-1}{p}}} \quad \text{and} \quad \sigma_p := c_p |\mathcal{W}(1) - \mathcal{W}(-1)|, \tag{1.7}$$

$\mathcal{W}$ being an antiderivative of $W^{\frac{p-1}{p}}$, that is

$$\mathcal{W}' = W^{\frac{p-1}{p}}. \tag{1.8}$$

The main convergence result is precisely stated in the following theorem.

Theorem 1.1. *Assume $p > 1$. Let $F_\varepsilon : \mathcal{D}_A \to \mathbb{R}$ and $F : BV(\Omega; \{-1, 1\}) \to \mathbb{R}$ be defined by* (1.1) *and* (1.6)*.*

Then

(i) [Compactness] *If $(u_\varepsilon) \subset \mathcal{D}_A$ is a sequence such that $F_\varepsilon(u_\varepsilon)$ is uniformly bounded, then (u_ε) is pre-compact in $L^1(\Omega)$ and every cluster point belongs to $BV(\Omega; \{-1, 1\})$.*

(ii) [Lower bound inequality] *For every $u \in BV(\Omega; \{-1, 1\})$ and every sequence $(u_\varepsilon) \subset \mathcal{D}_A$ such that $u_\varepsilon \to u$ in $L^1(\Omega)$,*
$$\liminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) \geq F(u).$$

(iii) [Upper bound inequality] *For every $u \in BV(\Omega; \{-1, 1\})$ there exists a sequence $(u_\varepsilon) \subset \mathcal{D}_A$ such that $u_\varepsilon \to u$ in $L^1(\Omega)$ and*
$$\limsup_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) \leq F(u).$$

We can easily rewrite this theorem in terms of Γ-convergence. To this aim, we extend each F_ε to ∞ on $L^1_{\mathrm{loc}}(\Omega) \setminus \mathcal{D}_A$ and, from Theorem 1.1, we deduce the following remark.

Remark 1.2. *F_ε Γ-converges in $L^1(\Omega)$ to $\tilde{F}$, given by*

$$\tilde{F}(u) := \begin{cases} F(u) & \text{if } u \in BV(\Omega; \{-1, 1\}), \\ \infty & \text{elsewhere in } L^1(\Omega). \end{cases}$$

2. Proof of the Γ-convergence result

Whereas the compactness and the lower bound inequality will follow exactly like in the case $A \equiv 1$, the proof of the upper bound inequality requires several technical modifications due to the presence of the spatial inhomogeneity, and it is complicated by the possible singularity or degeneracy of A.

2.1. **Compactness and lower bound inequality.** The key point of the proof of the compactness and the lower bound result relies on the following Young's inequality. For every $X, Y \geq 0$

$$\frac{X^p}{p} + \frac{Y^q}{q} \geq XY, \quad \Big(q : \quad \frac{1}{p} + \frac{1}{q} = 1\Big). \tag{2.1}$$

For every $u \in \mathcal{D}_A$, for almost every $x \in \Omega$, we define

$$X = \big(p\varepsilon^{p-1} A^{1-p}(x)|Du|^p\big)^{\frac{1}{p}}$$

and

$$Y = \Big(\frac{q}{\varepsilon} A(x) W(u)\Big)^{\frac{1}{q}}.$$

As a consequence of the previous choice of X and Y in (2.1), it follows that

$$\begin{aligned} F_\varepsilon(u) &\geq c_p \int_\Omega W(u)^{\frac{p-1}{p}} |Du| dx \\ &= c_p \int_\Omega |D(\mathcal{W}(u))| dx, \end{aligned} \tag{2.2}$$

where c_p and $\mathcal{W}$ are the ones defined in (1.7) and (1.8).

By (2.2), we can obtain the compactness result Theorem 1.1-(i) and the lower bound inequality Theorem 1.1-(ii), using standard arguments (see [17, Proposition 1 and 2] and also [2, Theorem 1] or [5, Theorem 3.10] for a very well written survey).

2.2. **An auxiliary measure theoretic observation.** Now, we show that we can include all the bad points of A into a nice set, by paying a small price in $\mathcal{H}^{N-1}$. This will be useful in the forthcoming argument on page 9.

Lemma 2.1. *Let (1.4) and (1.5) hold. Let U be an open subset of Ω with smooth boundary, with $\overline{U} \subset \Omega$. Then, for any $\gamma > 0$ there exists an open set $U_\gamma \supset U$ with smooth boundary, such that*

$$|U_\gamma \setminus U| \le \gamma, \tag{2.3}$$

$$A \text{ is } C^1 \text{ and positive in } \Omega \setminus U_\gamma \tag{2.4}$$

and

$$\mathcal{H}^{N-1}(\partial U_\gamma) \le \mathcal{H}^{N-1}(\partial U) + \gamma. \tag{2.5}$$

Proof. We can consider γ smaller than the distance from ∂U and $\partial\Omega$, and a set of small balls $\{B_i\}_{1 \le i \le M := M(\gamma)}$, of radius $r_i := r_i(\gamma)$, which includes the bad points of A.
We set

$$E_\gamma := \bigcup_{1 \le i \le M} B_i$$

and so we have

$$A \text{ is } C^1 \text{ and positive in } \overline{\Omega_\star \setminus E_\gamma}. \tag{2.6}$$

Let U be an open subset of Ω with smooth boundary. We define $U_{\gamma,0} := U$ and, recursively, for any $i = 1, \dots, M$, we let $U_{\gamma,i}$ to be an open subset of Ω with smooth boundary which contains $U_{\gamma,i-1} \cup B_i$ and such that

$$\big| U_{\gamma,i} \setminus (U_{\gamma,i-1} \cup B_i) \big| \le \frac{\gamma}{M} \tag{2.7}$$

and

$$\mathcal{H}^{N-1}(\partial U_{\gamma,i}) \le \mathcal{H}^{N-1}\Big(\partial(U_{\gamma,i-1} \cup B_i)\Big) + \frac{\gamma}{M}. \tag{2.8}$$

Let $U_\gamma := U_{\gamma,M}$. Then

$$U_\gamma \supseteq \bigcup_{1 \le i \le M} B_i = E_\gamma,$$

hence $\overline{\Omega \setminus U_\gamma} \subseteq \overline{\Omega \setminus E_\gamma}$ and so (2.4) is a consequence of (2.6).

Furthermore, since

$$\partial(U_{\gamma,i-1} \cup B_i) \subseteq (\partial U_{\gamma,i-1}) \cup (\partial B_i),$$

we deduce from (2.8) that

$$\begin{aligned} \mathcal{H}^{N-1}(\partial U_{\gamma,i}) &\le \mathcal{H}^{N-1}(\partial U_{\gamma,i-1}) + \mathcal{H}^{N-1}(\partial B_i) + \frac{\gamma}{M} \\ &\le \mathcal{H}^{N-1}(\partial U_{\gamma,i-1}) + c_N r_i^{N-1} + \frac{\gamma}{M}, \end{aligned}$$

for a suitable $c_N > 0$. By iterating this formula and recalling (1.4), we obtain

$$\begin{aligned} \mathcal{H}^{N-1}(\partial U_\gamma) &= \mathcal{H}^{N-1}(\partial U_{\gamma,M}) \\ &\le \mathcal{H}^{N-1}(\partial U_{\gamma,0}) + \sum_{i=1}^{M} \left(c_N r_i^{N-1} + \frac{\gamma}{M} \right) \\ &\le \mathcal{H}^{N-1}(\partial U_\gamma) + \tilde{c}_N \gamma, \end{aligned}$$

for a suitable $\tilde{c}_N > 0$, that is (2.5), up to renaming γ.

Now we observe that

$$|U_{\gamma,i} \setminus U_{\gamma,i-1}| \le |B_i| + \frac{\gamma}{M}$$

due to (2.7), and therefore

$$\begin{aligned} |U_\gamma \setminus U| &\le \sum_{i=1}^{M} |U_{\gamma,i} \setminus U_{\gamma,i-1}| \le \gamma + C \sum_{i=1}^{M} r_i^N \\ &\le \gamma + C \sum_{i=1}^{M} r_i^{N-1} \le (C+1)\gamma, \end{aligned}$$

for a suitable $C > 0$. This proves (2.3), up to renaming γ again. □

2.3. **Upper bound inequality.** The main contribution in the proof of the upper bound inequality is essentially contained in forthcoming Proposition 2.2, in which we will be able to construct a recovery sequence for the Γ-limsup of the functional $F_\varepsilon(\cdot, K)$, for any open set K strictly contained in Ω. Then, we need to extend that construction to the whole domain Ω, via a modification to allow the inhomogeneity function A to be singular or degenerate also on the boundary; this last part will exploit condition (1.5), the results of [14, 22] and a fine interpolation technique of [24].

Proposition 2.2. *Let K be an open bounded subset of Ω such that $\overline{K} \subset \Omega$. Set*

$$F_\varepsilon(u, K) := \int_K \left[\varepsilon^{p-1} A^{1-p}(x) |Du|^p + \frac{1}{\varepsilon} A(x) W(u) \right] dx.$$

Then

$$\begin{aligned} &\forall u \in BV(K; \{-1,+1\}) \text{ there exists a sequence } (w_\varepsilon) \subset \mathcal{D}_A(K) \text{ such that} \\ &\qquad w_\varepsilon \to u \text{ in } L^1(K) \end{aligned} \tag{2.9}$$

and

$$\limsup_{\varepsilon\to 0} F_\varepsilon(w_\varepsilon, K) \le \sigma_p \mathcal{H}^{N-1}(Su \cap K). \tag{2.10}$$

Proof. Consider a function $u \in BV(K; \{-1, +1\})$. We want to construct a sequence of functions w_ε that converges to u in $L^1(K)$.

Our construction will modify the target function u only in a small neighbourhood of Su.

We will play with several positive parameters, namely ε (which is the one of Proposition 2.2), n, δ and γ. We will take limits in these parameters, in the following order:

$$\begin{cases} \text{first, } \varepsilon \to 0; \\ \text{then, } n \to \infty; \\ \text{finally, } \delta,\ \gamma \to 0 \text{ (the order of this last limit will be of no importance).} \end{cases} \tag{2.11}$$

First, we observe that the singular set Su can be approximated by a sequence Su_n of smooth surfaces of dimension $N-1$, so that

$$\mathcal{H}^{N-1}(Su_n) \to \mathcal{H}^{N-1}(Su), \quad \text{as } n \to \infty.$$

This is a well-known approximation result for finite perimeter sets; see [13, Theorem 1.24].

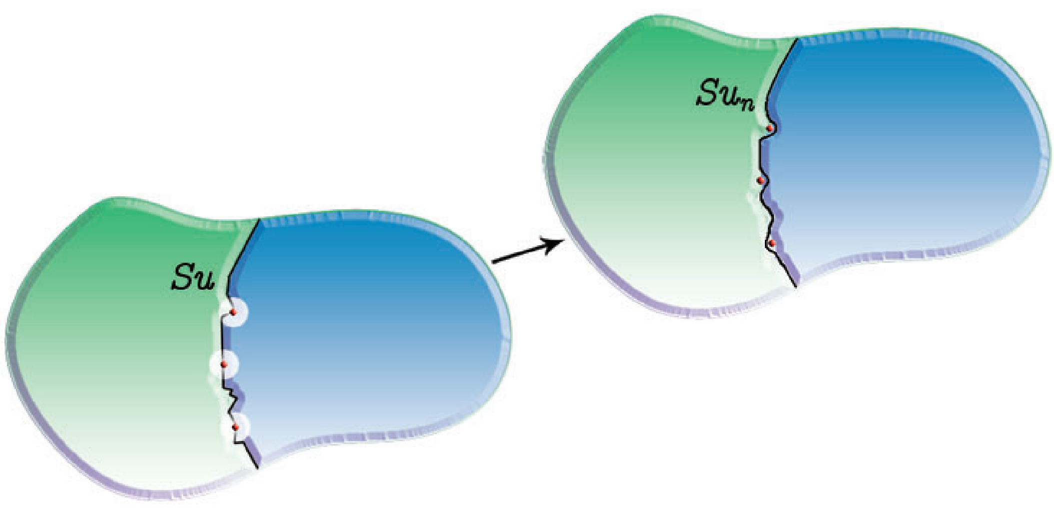

FIGURE 1. An approximating $(N-1)$-dimensional smooth surfaces Su_n.

Second, we should take care of possible intersections of Su_n with the set of points in Ω in which the inhomogeneity A degenerates to zero or to infinity. Thanks to the assumptions on A, we can turn around the balls $\{B_i\}$ losing little volume and boundary measure (recall Lemma 2.1), spreading out Su_n into a new smooth surface, that we denote again[1] by Su_n; see Fig. 1, with the properties that

$$A \text{ is } C^1 \text{ and positive in the vicinity of } Su_n \tag{2.12}$$

and, for every positive γ,

$$\lim_{n\to\infty} |Su_n \setminus Su| + |Su \setminus Su_n| \le \gamma \tag{2.13}$$

and

$$\lim_{n\to\infty} \mathcal{H}^{N-1}(Su_n \cap K) \le \mathcal{H}^{N-1}(Su \cap K) + \gamma, \quad \text{as } n \to \infty. \tag{2.14}$$

We remark that this further approximation step is crucial in our construction, since, in what follows, we will compare the values of A near Su_n with the ones on Su_n: the above approximation allows us to say that these two values are close to each other, in view of (2.12).

Also, we observe that Su_n may be seen as the boundary of a set, say $\mathcal{U}_n$, satisfying

$$\lim_{n\to\infty} \big|\mathcal{U}_n \setminus \{u=1\}\big| + \big|\{u=1\} \setminus \mathcal{U}_n\big| \le \gamma, \tag{2.15}$$

thanks to (2.13).

Now, we provide a new set of coordinates $(d_n(x), \eta)$ in small strips $\mathcal{S}^{k_n}$ around Su_n, such that η parameterises Su_n and $d_n(x) \in \mathbb{R}$ is the signed distance to Su_n, defined in the forthcoming formula (2.16).

More precisely, for $k_n > 0$, we define

$$\mathcal{S}^{k_n} := \{x \in K : \operatorname{dist}(x, Su_n) \le k_n\},$$

where

$$\operatorname{dist}(x, Su_n) := \inf_{y\in Su_n} |x-y|.$$

We also set

$$d_n(x) := \begin{cases} \operatorname{dist}(x, Su_n) & \text{if } x \in \mathcal{U}_n, \\ -\operatorname{dist}(x, Su_n) & \text{otherwise.} \end{cases} \tag{2.16}$$

Then, since Su_n is smooth, if $k_n > 0$ is sufficiently small, for any $x \in \mathcal{S}^{3k_n}$ there exists a unique $\eta \in Su_n$ for which $\operatorname{dist}(x, Su_n) = |x-\eta|:$ we will write $\eta := \pi_n(x)$. So, if $k_n > 0$

[1] We stress that this new Su_n does depend on γ. According to (2.14), γ is taken here to be a fixed parameter. At the end of the proof, we will let $n \to \infty$, for a fixed $\gamma > 0$: recall (2.11). This is why forgetting the dependence on γ in the notation is not too dangerous.

is sufficiently small, the map sending x to the couple given by such $\eta = \pi_n(x) \in Su_n$ and $t := d_n(x)$ is a diffeomorphism[2] on $\mathcal{S}^{3k_n}$.

Notice that such diffeomorphism $x \mapsto (t, \eta) = (d_n(x), \pi_n(x))$ depends on n – as well as on the $\gamma > 0$ fixed at the beginning, recall (2.11) and the footnote on page 10 – but it is independent of ε.

In particular,

$$\frac{1}{c(n,\gamma)} dt\, d\eta \;\leq\; dx \;\leq\; c(n,\gamma)\, dt\, d\eta \tag{2.17}$$

for a suitable $c(n, \gamma) \geq 1$.

With a slight abuse of notation, we will write $\mathcal{S}^{k_n}$ in the system of coordinates given by (t, η) as

$$\mathcal{S}^{k_n} = \{x \in K : x = (t, \eta),\ \eta \in Su_n,\ t = d_n(x) : |t| \leq k_n\}\,. \tag{2.18}$$

Analogously, a function $g = g(x)$ will be often written as $g(t, \eta)$ with the obvious meaning that the above diffeomorphism is omitted and intending that $t = d_n(x)$ and $\eta = \pi_n(x)$.

The idea of the subsequent proof is that we need to capture all the energy of the transition in the recovery sequence: to do this job, we will build a modified version of the optimal profile modelled for the case $A \equiv 1$.

For this, we take the real function $\theta \in W^{1,1}_{\rm loc}(\mathbb{R})$ satisfying the following ODE:

$$\begin{cases} \theta' = \dfrac{1}{(p-1)^{\frac{1}{p}}} W^{\frac{1}{p}}(\theta) \quad \text{a.e.} \\[2ex] \theta(0) = 0, \\[2ex] \lim\limits_{t \to \pm\infty} \theta(t) = \pm 1, \end{cases} \tag{2.19}$$

We notice that θ may be explicitly constructed as follows: for any $|\tau| < 1$, let

$$\Psi(\tau) := \int_0^\tau \frac{(p-1)^{\frac{1}{p}}}{W^{\frac{1}{p}}(r)}\, dr.$$

Then, $\Psi' > 0$ and so Ψ is invertible. Also, $\Psi(\pm 1) = \pm\infty$, due to (1.2). Taking $\theta(t) := \Psi^{-1}(t)$, we have that (2.19) is satisfied.

[2]The choice of having a diffeomorphism on $\mathcal{S}^{3k_n}$ and not only on $\mathcal{S}^{k_n}$ is due to the fact that we will like to interpolate a phase transition outside $\mathcal{S}^{k_n}$ in the forthcoming formula (2.31). Of course, with no loss of generality, we may and do suppose that k_n is so small that A is C^1 and positive in $\mathcal{S}^{3k_n}$: recall (2.12).

In order to take into account of the inhomogeneity A, for every fixed η we define the following function

$$\varphi_\eta(t) = \varphi(t,\eta) := \theta\big(A(0,\eta)t\big), \quad \forall t \in \mathbb{R}. \tag{2.20}$$

We note that $(0,\eta)$ is a point of Su_n and A is finite and positive on Su_n, due to (2.12), therefore the definition in (2.20) is well-posed.

Moreover,

$$\lim_{t\to\pm\infty} \varphi_\eta(t) = \pm 1 \tag{2.21}$$

and

$$\varphi'_\eta(t) := \frac{\partial}{\partial t}\varphi_\eta(t) = \frac{1}{(p-1)^{\frac{1}{p}}} W^{\frac{1}{p}}(\varphi_\eta(t))A(0,\eta).$$

As a consequence,

$$\begin{aligned} A^{1-p}(0,\eta)\big(\varphi'_\eta(t)\big)^p + A(0,\eta)W(\varphi_\eta(t)) &= \left(\frac{1}{(p-1)} + 1\right) A(0,\eta)W(\varphi_\eta(t)) \\ &= \left(\frac{p}{p-1}\right)(p-1)^{\frac{1}{p}} W^{\frac{p-1}{p}}(\varphi_\eta(t))\,\varphi'_\eta(t) \\ &= c_p W^{\frac{p-1}{p}}(\varphi_\eta(t))\,\varphi'_\eta(t). \end{aligned} \tag{2.22}$$

In particular, recalling (2.12), we set

$$\Xi_n := c_p \sup_{\eta\in Su_n} A^{p-1}(0,\eta).$$

Note that Ξ_n is finite, since the weight A is finite and positive close to the surfaces Su_n and it may not degenerate nor become singular when approaching the boundary of K, that is far from $\partial\Omega$.

From (2.22), we obtain that

$$\begin{aligned} \frac{1}{\Xi_n}\big(\varphi'_\eta(t)\big)^p &\le c_p^{-1} A^{1-p}(0,\eta)\big(\varphi'_\eta(t)\big)^p \\ &\le W^{\frac{p-1}{p}}(\varphi_\eta(t))\varphi'_\eta(t). \end{aligned} \tag{2.23}$$

We deduce from (1.8) and (2.23) that, for every fixed $\eta \in Su_n$,

$$\begin{aligned} \frac{1}{\Xi_n}\int_{-\infty}^{+\infty} |\varphi_\eta'(t)|^p\,dt &\le \int_{-\infty}^{+\infty} W^{\frac{p-1}{p}}\big(\varphi_\eta(t)\big)\varphi_\eta'(t)\,dt \\ &= \int_{-\infty}^{+\infty} \frac{d}{dt}\mathcal{W}\big(\varphi_\eta(t)\big)\,dt \\ &= \mathcal{W}(1)-\mathcal{W}(-1)<\infty \qquad (2.24)\end{aligned}$$

and so

$$\lim_{r\to\infty}\int_r^{2r} |\varphi_\eta'(t)|^p\,dt \,=\, 0. \qquad (2.25)$$

Now, for every $\varepsilon>0$, we define

$$v_\varepsilon(x) := \varphi_{\pi_n(x)}\left(\frac{d_n(x)}{\varepsilon}\right) = \varphi\left(\frac{d_n(x)}{\varepsilon},\pi_n(x)\right), \qquad \forall x\in\mathcal{S}^{2k_n}. \qquad (2.26)$$

Now, we observe that for every $\delta\in(0,1)$ there exists $c(\delta)>0$ such that, for every X, $Y\ge 0$,

$$(X+Y)^p \le (1+\delta)X^p + c(\delta)Y^p, \qquad (2.27)$$

Indeed, we can suppose that $(1+\delta)^{1/p}<2$ and $Y\neq 0$, and, for any $t\in(0,1]$, we let

$$g_\delta(t) := \frac{(1+t)^p-1-\delta}{t^p}.$$

Then $g_\delta\in C([(1+\delta)^{1/p}-1,1])$ and $g_\delta(t)<0$ if $t\in(0,(1+\delta)^{1/p}-1)$. So, we can define

$$c(\delta) := 2^p + \max_{(0,1]} g_\delta.$$

With this, we have that, if $X\le Y$

$$(X+Y)^p \le (2Y)^p \le (1+\delta)X^p + 2^pY^p \le (1+\delta)X^p + c(\delta)Y^p.$$

On the other hand, if $X>Y$, we set $t:=Y/X\in(0,1]$. We have

$$\begin{aligned}(X+Y)^p &= X^p(1+t)^p = X^p\Big((1+\delta)+t^p g_\delta(t)\Big)\\ &\le X^p\Big((1+\delta)+t^p c(\delta)\Big) = (1+\delta)X^p + c(\delta)Y^p.\end{aligned}$$

In any case, (2.27) is proved.

Making use of (2.27), we see that

$$\begin{aligned} |Dv_\varepsilon(x)|^p &= \left| \frac{\partial\varphi}{\partial t}\left(\frac{d_n(x)}{\varepsilon}, \pi_n(x)\right) D\left(\frac{d_n(x)}{\varepsilon}\right) + \frac{\partial\varphi}{\partial\eta}\left(\frac{d_n(x)}{\varepsilon}, \pi_n(x)\right) D\big(\pi_n(x)\big)\right|^p \\ &\leq (1+\delta)\frac{\left(\varphi'_{\pi_n(x)}\left(\frac{d_n(x)}{\varepsilon}\right)\right)^p}{\varepsilon^p} + c(\delta)\mathcal{R}(x), \qquad \forall x \in \mathcal{S}^{2k_n}, \end{aligned} \tag{2.28}$$

where $\mathcal{R}$ is bounded by a quantity possibly depending on n but independent of ε, hence, recalling (2.11),

$$\lim_{\varepsilon\to 0} \varepsilon^{p-1}\mathcal{R}(x) = 0 \text{ uniformly in } \mathcal{S}^{2k_n}, \text{ for fixed } n,\ \delta \text{ and } \gamma. \tag{2.29}$$

From (2.17), (2.28) and (2.24), we also conclude that

$$\begin{aligned} \int_{\mathcal{S}^{2k_n}} |Dv_\varepsilon(x)|^p\,dx &\leq \int_{\mathcal{S}^{2k_n}} (1+\delta)\frac{\left(\varphi'_{\pi_n(x)}\left(\frac{d_n(x)}{\varepsilon}\right)\right)^p}{\varepsilon^p} + c(\delta)\mathcal{R}(x)\,dx \\ &\leq C_\star\left(\frac{1}{\varepsilon}\int_{k_n}^{2k_n}\int_{Su_n}\left(\varphi'_\eta\left(\frac{t}{\varepsilon}\right)\right)^p dt\,d\eta + 1\right) \\ &\leq C_\star\left(\int_{-\infty}^{+\infty}\int_{Su_n}\big(\varphi'_\eta(\tau)\big)^p\,d\tau\,d\eta + 1\right) < +\infty, \end{aligned} \tag{2.30}$$

for a suitable $C_\star$ possibly depending on all the parameters ε, n, δ and γ and on the size of K.

The function v_ε will be the relevant contribution for the recovery sequence. In order to provide a recovery sequence defined on the whole of K, it has to be extended away from $\mathcal{S}^{2k_n}$: for this, we introduce the signed strips $\mathcal{S}^{k_n}_\pm$ as follows

$$\mathcal{S}^{k_n}_+ := \mathcal{S}^{k_n} \cap \{d_n(x) > 0\}$$

and

$$\mathcal{S}^{k_n}_- := \mathcal{S}^{k_n} \cap \{d_n(x) < 0\}.$$

For every $x \in K$, the recovery sequence w_ε is defined by

$$w_\varepsilon(x) := \begin{cases} 1 & \text{if } x \in K \setminus \mathcal{S}^{2k_n}_+, \\ \big(2 - \frac{d_n(x)}{k_n}\big)v_\varepsilon(x) + \big(\frac{d_n(x)}{k_n} - 1\big) & \text{if } x \in \mathcal{S}^{2k_n}_+ \setminus \mathcal{S}^{k_n}_+, \\ v_\varepsilon(x) & \text{if } x \in \mathcal{S}^{k_n}, \\ \big(2 + \frac{d_n(x)}{k_n}\big)v_\varepsilon(x) + \big(\frac{d_n(x)}{k_n} + 1\big) & \text{if } x \in \mathcal{S}^{2k_n}_- \setminus \mathcal{S}^{k_n}_-, \\ -1 & \text{if } x \in K \setminus \mathcal{S}^{2k_n}_-, \end{cases} \tag{2.31}$$

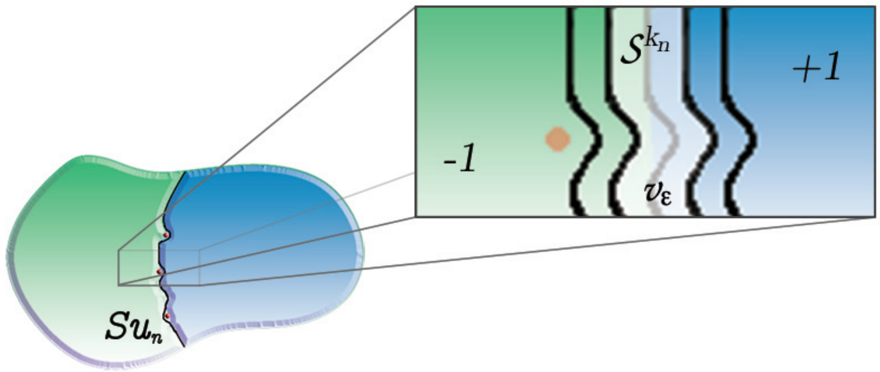

FIGURE 2. Construction of the recovery sequence.

see Fig. 2. We note that

$$|Dw_\varepsilon| \leq 2\left(|Dv_\varepsilon| + 1 + \frac{1}{k_n}\right)\chi_{\mathcal{S}^{2k_n}},$$

and so the sequence w_ε belongs to $\mathcal{D}_A(K)$, because of (2.12) and (2.30).

We have to estimate the energy of the function w_ε. For this, we observe that

$$\begin{aligned} F_\varepsilon(w_\varepsilon, K) &= F_\varepsilon(w_\varepsilon, \mathcal{S}^{k_n}) + F_\varepsilon(w_\varepsilon, K \setminus \mathcal{S}^{k_n}) \\ &= F_\varepsilon(v_\varepsilon, \mathcal{S}^{k_n}) + F_\varepsilon(w_\varepsilon, K \setminus \mathcal{S}^{k_n}). \end{aligned} \tag{2.32}$$

The second term in the right hand side of (2.32) is zero outside of $\mathcal{S}^{2k_n}$, so we have

$$\begin{aligned} F_\varepsilon(w_\varepsilon, K \setminus \mathcal{S}^{k_n}) &= F_\varepsilon(w_\varepsilon, \mathcal{S}^{2k_n} \setminus \mathcal{S}^{k_n}) \\ &= \varepsilon^{p-1}\int_{\mathcal{S}^{2k_n}\setminus\mathcal{S}^{k_n}} A^{1-p}(x)|Dw_\varepsilon|^p dx + \frac{1}{\varepsilon}\int_{\mathcal{S}^{2k_n}\setminus\mathcal{S}^{k_n}} A(x)W(w_\varepsilon)dx \\ &=: I_{1,\varepsilon} + I_{2,\varepsilon}. \end{aligned} \tag{2.33}$$

Using the definition of w_ε in the strips $\mathcal{S}^{2k_n}$, we see that the integral $I_{1,\varepsilon}$ can be estimated as follows

$$
\begin{aligned}
I_{1,\varepsilon} &\leq 3^p\varepsilon^{p-1}\int_{\mathcal{S}_\pm^{2k_n}\setminus\mathcal{S}_\pm^{k_n}} A^{1-p}(x)\left(\Big|2\pm\frac{d_n(x)}{k_n}\Big||Dv_\varepsilon|\right)^p dx \\
&\quad +3^p\varepsilon^{p-1}\int_{\mathcal{S}_\pm^{2k_n}\setminus\mathcal{S}_\pm^{k_n}} A^{1-p}(x)\left(|v_\varepsilon(x)\pm 1|\Big|D\big(2\pm\frac{d_n(x)}{k_n})\Big|\right)^p dx \\
&\leq 4^p3^p\varepsilon^{p-1}\int_{\mathcal{S}^{2k_n}\setminus\mathcal{S}^{k_n}} A^{1-p}(x)|Dv_\varepsilon|^p dx \\
&\quad +\left(\frac{3^p2^p}{k_n^p}\int_{\mathcal{S}^{2k_n}\setminus\mathcal{S}^{k_n}} A^{1-p}(x)dx\right)\varepsilon^{p-1}, \qquad (2.34)
\end{aligned}
$$

where we also used that $|Dd_n|\leq 1$.

Now, using (2.17) and (2.28), and passing to the the new coordinates (t,η), with k_n small, we have

$$
\begin{aligned}
&\frac{4^p3^p\varepsilon^{p-1}}{c(n,\gamma)}\int_{\mathcal{S}^{2k_n}\setminus\mathcal{S}^{k_n}} A^{1-p}(x)|Dv_\varepsilon|^p dx \\
&\qquad \leq 4^p3^p(1+\delta)\int_{Su_n}\left[\varepsilon^{p-1}\int_{k_n}^{2k_n} A^{1-p}(t,\eta)\frac{|\varphi_\eta'\big(\frac{t}{\varepsilon}\big)|^p}{\varepsilon^p}dt\right]d\eta \\
&\qquad\quad +4^p3^pc(\delta)\int_{k_n}^{2k_n}\int_{Su_n} A^{1-p}(t,\eta)\Big(\varepsilon^{p-1}\mathcal{R}(t,\eta)\Big)dtd\eta. \qquad (2.35)
\end{aligned}
$$

The first term in the right hand side of the above formula can be evaluated by the Change of variable Formula (setting $\tau=t/\varepsilon$ in the integral in square brackets) as follows

$$
\begin{aligned}
&4^p3^p(1+\delta)\int_{Su_n}\left[\varepsilon^{p-1}\int_{k_n}^{2k_n} A^{1-p}(t,\eta)\frac{|\varphi_\eta'\big(\frac{t}{\varepsilon}\big)|^p}{\varepsilon^p}dt\right]d\eta \\
&\qquad = 4^p3^p(1+\delta)\int_{Su_n}\left[\int_{\frac{k_n}{\varepsilon}}^{\frac{2k_n}{\varepsilon}} A^{1-p}(\tau\varepsilon,\eta)|\varphi_\eta'(\tau)|^p d\tau\right]d\eta \qquad (2.36)
\end{aligned}
$$

Therefore, replacing (2.35) and (2.36) into (2.34), we obtain the following estimate for the integral $I_{1,\varepsilon}$. For every $\delta \in (0,1)$

$$\begin{aligned}
\frac{1}{c(n,\gamma)} I_{1,\varepsilon} \;&\leq\; 4^p 3^p (1+\delta)\left(\sup_{\eta \in Su_n} A^{1-p}(0,\eta)\right)\int_{\frac{k_n}{\varepsilon}}^{\frac{2k_n}{\varepsilon}} \int_{Su_n} |\varphi_\eta'(\tau)|^p d\tau d\eta \\
&\quad + 4^p 3^p (1+\delta) \int_{\frac{k_n}{\varepsilon}}^{\frac{2k_n}{\varepsilon}} \int_{Su_n} \Big(A^{1-p}(\tau\varepsilon,\eta) - A^{1-p}(0,\eta) \Big) |\varphi_\eta'(\tau)|^p d\tau d\eta \\
&\quad + 4^p 3^p c(\delta) \int_{k_n}^{2k_n} \int_{Su_n} A^{1-p}(t,\eta) \Big(\varepsilon^{p-1} \mathcal{R}(t,\eta) \Big) dt d\eta \\
&\quad + \left(\frac{3^p 2^p}{k_n^p} \int_{\mathcal{S}^{2k_n} \setminus \mathcal{S}^{k_n}} A^{1-p}(x) dx \right) \varepsilon^{p-1}, \\
&\leq\; C \left(\int_{\frac{k_n}{\varepsilon}}^{\frac{2k_n}{\varepsilon}} \int_{Su_n} |\varphi_\eta'(\tau)|^p d\tau d\eta + \int_{k_n}^{2k_n} \int_{Su_n} \Big(\varepsilon^{p-1} \mathcal{R}(t,\eta) \Big) dt d\eta + \varepsilon^{p-1} \right), \qquad (2.37)
\end{aligned}$$

where C is a constant that depends only on n and δ.

Using the Lebesgue Dominated Convergence Theorem, (2.12), (2.25), and (2.29), we see that every term in the above inequality (2.37) goes to zero as ε goes to zero. Therefore, for fixed n, δ and γ,

$$\lim_{\varepsilon \to 0} I_{1,\varepsilon} = 0. \qquad (2.38)$$

Moreover, it is easily seen from (2.31) that $-1 \leq w_\varepsilon \leq v_\varepsilon$ in $\mathcal{S}_-^{2k_n} \setminus \mathcal{S}_-^{k_n}$ and $v_\varepsilon \leq w_\varepsilon \leq 1$ in $\mathcal{S}_+^{2k_n} \setminus \mathcal{S}_+^{k_n}$, and $|v_\varepsilon|$ is close to 1 in $\mathcal{S}^{2k_n} \setminus \mathcal{S}^{k_n}$, for small ε and fixed n. Accordingly, in view of the behaviour of W near its wells, we have that, for ε small, $W(w_\varepsilon) \leq W(v_\varepsilon)$ in $\mathcal{S}^{2k_n} \setminus \mathcal{S}^{k_n}$, therefore

$$I_{2,\varepsilon} \leq \frac{1}{\varepsilon} \int_{\mathcal{S}^{2k_n} \setminus \mathcal{S}^{k_n}} A(x) W(v_\varepsilon) dx$$

and, as a consequence, we obtain from (2.32) and (2.33) that

$$\begin{aligned}
F_\varepsilon(w_\varepsilon, K) \;&\leq\; F_\varepsilon(v_\varepsilon, \mathcal{S}^{k_n}) + \frac{1}{\varepsilon} \int_{\mathcal{S}^{2k_n} \setminus \mathcal{S}^{k_n}} A(x) W(v_\varepsilon) dx + I_{1,\varepsilon} \\
&\leq\; F_\varepsilon(v_\varepsilon, \mathcal{S}^{2k_n}) + I_{1,\varepsilon}. \qquad (2.39)
\end{aligned}$$

At this moment, the upper bound inequality will follow from a precise estimate of the energy of v_ε on the strip $\mathcal{S}^{2k_n}$, using the CoArea Formula.

First of all, we need to keep safe from the degenerate points of the spatial inhomogeneity A. We have

$$\begin{aligned} F_\varepsilon(v_\varepsilon, \mathcal{S}^{2k_n}) &= \int_{\mathcal{S}^{2k_n}} \Big[\varepsilon^{p-1} A^{1-p}(\pi_n(x))|Dv_\varepsilon|^p + \frac{1}{\varepsilon} A(\pi_n(x)) W(v_\varepsilon)\Big] dx \\ &\quad + \int_{\mathcal{S}^{2k_n}} \Big[\varepsilon^{p-1}\left(A^{1-p}(x) - A^{1-p}(\pi_n(x))\right)|Dv_\varepsilon|^p \\ &\qquad + \frac{1}{\varepsilon}\left(A(x) - A(\pi_n(x))\right) W(v_\varepsilon)\Big] dx \\ &=: J_{1,\varepsilon} + J_{2,\varepsilon}. \end{aligned} \tag{2.40}$$

The integral $J_{1,\varepsilon}$ will capture the energy of the recovery sequence, while the integral $J_{2,\varepsilon}$ can be estimated thanks to the continuity of the function A on Su_n. As in the previous computations, we also use the Change of variable Formula, (2.17) and (2.28): for every $\delta \in (0,1)$, we have

$$\begin{aligned} \frac{1}{c(n,\gamma)} J_{2,\varepsilon} &\leq (1+\delta) \int_{Su_n} \int_{\frac{k_n}{\varepsilon}}^{\frac{2k_n}{\varepsilon}} \Big[\left(A^{1-p}(\tau\varepsilon, \eta) - A^{1-p}(0,\eta)\right)|\varphi_\eta'(\tau)|^p \\ &\qquad + \big(A(\tau\varepsilon,\eta) - A(0,\eta)\big) W(\varphi_\eta(\tau))\Big] d\tau d\eta \\ &\quad + c(\delta) \int_{k_n}^{2k_n} \int_{Su_n} \left(A^{1-p}(t,\eta) - A^{1-p}(0,\eta)\right) \Big(\varepsilon^{p-1}\mathcal{R}(t,\eta)\Big) dt d\eta. \end{aligned}$$

Since, when ε goes to zero, $(A(\tau\varepsilon,\eta) - A(0,\eta))$ goes to zero, thanks to (2.12), we deduce, recalling (2.25) and (2.29), that

$$\lim_{\varepsilon \to 0} J_{2,\varepsilon} = 0 \quad \text{for fixed } n,\ \delta \text{ and } \gamma. \tag{2.41}$$

It remains to estimate the integral $J_{1,\varepsilon}$. For this, we use (2.26) and, once more, (2.28): for every $\delta \in (0,1)$, we have

$$
\begin{aligned}
J_{1,\varepsilon} &= \int_{\mathcal{S}^{2k_n}} \Big[\varepsilon^{p-1} A^{1-p}(\pi_n(x))|Dv_\varepsilon|^p + \frac{1}{\varepsilon} A(\pi_n(x))W(v_\varepsilon)\Big] dx \\
&\le \int_{\mathcal{S}^{2k_n}} \Big[(1+\delta)\varepsilon^{p-1} A^{1-p}(\pi_n(x)) \frac{\Big(\varphi'_{\pi_n(x)}\big(\frac{d_n(x)}{\varepsilon}\big)\Big)^p}{\varepsilon^p} \\
&\quad + c(\delta) A^{1-p}(\pi_n(x))\Big(\varepsilon^{p-1}\mathcal{R}(x)\Big) + \frac{1}{\varepsilon} A(\pi_n(x))W(v_\varepsilon)\Big] dx \\
&\le (1+\delta)\int_{\mathcal{S}^{2k_n}} \frac{1}{\varepsilon}\Big[A^{1-p}(\pi_n(x))\Big(\varphi'_{\pi_n(x)}\Big(\frac{d_n(x)}{\varepsilon}\Big)\Big)^p \\
&\quad + A(\pi_n(x))W\Big(\varphi_{\pi_n(x)}\Big(\frac{d_n(x)}{\varepsilon}\Big)\Big)\Big] dx \\
&\quad + c(\delta)\int_{\mathcal{S}^{2k_n}} A^{1-p}(\pi_n(x))\Big(\varepsilon^{p-1}\mathcal{R}(x)\Big)\, dx \\
&=: L_{1,\varepsilon} + L_{2,\varepsilon}.
\end{aligned}
\tag{2.42}
$$

By (2.29), we deduce that

$$
\lim_{\varepsilon\to 0} L_{2,\varepsilon} = 0 \quad \text{for fixed } n,\ \delta \text{ and } \gamma. \tag{2.43}
$$

In order to estimate $L_{1,\varepsilon}$ we argue as follows. We observe that if $g : \mathcal{S}^{2k_n} \to [0,\infty]$ is any measurable function, the CoArea Formula says that

$$
\begin{aligned}
\frac{1}{\varepsilon}\int_{\mathcal{S}^{2k_n}} g(x)\, dx &= \frac{1}{\varepsilon}\int_{-2k_n}^{2k_n} \left(\int_{\{d_n = r\}} g(\zeta)\, d\mathcal{H}^{N-1}(\zeta)\right) dr \\
&\le \int_{-\infty}^{+\infty} \left(\int_{\{d_n = \varepsilon s\}} g(\zeta)\, d\mathcal{H}^{N-1}(\zeta)\right) ds.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
L_{1,\varepsilon} &= (1+\delta)\int_{\mathcal{S}^{2k_n}} \frac{1}{\varepsilon}\Bigg[A^{1-p}(\pi_n(x))\left(\varphi'_{\pi_n(x)}\Big(\frac{d_n(x)}{\varepsilon}\Big)\right)^p \\
&\qquad + A(\pi_n(x))W\Big(\varphi_{\pi_n(x)}\Big(\frac{d_n(x)}{\varepsilon}\Big)\Big)\Bigg]dx \\
&\le (1+\delta)\int_{-\infty}^{\infty}\int_{\{d_n=\varepsilon s\}}\Big[A^{1-p}(\pi_n(\zeta))\Big(\varphi'_{\pi_n(\zeta)}(s)\Big)^p \\
&\qquad + A(\pi_n(\zeta))W(\varphi_{\pi_n(\zeta)}(s))\Big]d\mathcal{H}^{N-1}(\zeta)ds. \qquad (2.44)
\end{aligned}
$$

Thus, from (2.22) and (2.44),

$$L_{1,\varepsilon} \le (1+\delta)\int_{-\infty}^{\infty}\int_{\{d_n=\varepsilon s\}} c_p W^{\frac{p-1}{p}}\Big(\varphi_{\pi_n(\zeta)}(s)\Big)\varphi'_{\pi_n(\zeta)}(s)\, d\mathcal{H}^{N-1}(\zeta)\, ds.$$

Hence, there exists $T_n > 0$ such that

$$L_{1,\varepsilon} \le (1+\delta)\int_{-T_n}^{T_n}\int_{\{d_n=\varepsilon s\}} c_p W^{\frac{p-1}{p}}\Big(\varphi_{\pi_n(\zeta)}(s)\Big)\varphi'_{\pi_n(\zeta)}(s)\, d\mathcal{H}^{N-1}(\zeta)\, ds + \frac{1}{n}.$$

We note that when ε goes to zero (for fixed n, δ and γ: recall (2.11)), the level set $\{d_n = \varepsilon s\}$ converges to $Su_n \cap K$. Also, π_n is the identity on Su_n. Therefore,

$$
\begin{aligned}
\limsup_{\varepsilon\to 0} L_{1,\varepsilon} &\le (1+\delta)\int_{-T_n}^{T_n}\int_{Su_n\cap K} c_p W^{\frac{p-1}{p}}\Big(\varphi_\zeta(s)\Big)\varphi'_\zeta(s)d\mathcal{H}^{N-1}(\zeta)\, ds + \frac{1}{n} \\
&\le (1+\delta)\int_{Su_n\cap K}\int_{-1}^{1} c_p W^{\frac{p-1}{p}}(r)drd\mathcal{H}^{N-1}(\zeta) + \frac{1}{n} \\
&= (1+\delta)\sigma_p\mathcal{H}^{N-1}(Su_n\cap K) + \frac{1}{n}, \qquad (2.45)
\end{aligned}
$$

where we used (2.21).

Hence, exploiting (2.14) and (2.45), sending now n to infinity (recall (2.11)), we get, for every $\delta \in (0,1)$ and for every $\gamma > 0$

$$\limsup_{\varepsilon\to 0} L_{1,\varepsilon} \le (1+\delta)\sigma_p\left(\mathcal{H}^{N-1}(Su\cap K) + 2\gamma\right). \qquad (2.46)$$

All in all, from (2.39), (2.40), (2.42) and (2.46),

$$\begin{aligned}\limsup_{\varepsilon\to 0} F_\varepsilon(w_\varepsilon,K) &\le \limsup_{\varepsilon\to 0}\Big(F_\varepsilon(v_\varepsilon,\mathcal{S}^{2k_n})+I_{1,\varepsilon}\Big)\\ &= \limsup_{\varepsilon\to 0}\,(J_{1,\varepsilon}+J_{2,\varepsilon})\\ &\le \limsup_{\varepsilon\to 0}\,(L_{1,\varepsilon}+L_{2,\varepsilon})\\ &= \limsup_{\varepsilon\to 0} L_{1,\varepsilon}\\ &\le (1+\delta)\sigma_p\left(\mathcal{H}^{N-1}(Su)+2\gamma\right).\end{aligned}$$

By the arbitrariness of δ and γ, this concludes the proof of (2.10) (up to renaming subsequences). Then, the convergence in (2.9) follows from (2.15) and (2.31). □

It is worth pointing out that a recovery sequence in a given domain is also a recovering sequence in a sub-domain:

Lemma 2.3. *Let the notation of Proposition 2.2 hold and consider a sub-domain $\tilde K \subset K$. Then, w_ε also satisfies (2.10) with $\tilde K$ instead of K, i.e.*

$$\limsup_{\varepsilon\to 0} F_\varepsilon(w_\varepsilon,\tilde K) \le \sigma_p \mathcal{H}^{N-1}(Su\cap \tilde K).$$

Proof. Suppose not, then

$$\limsup_{\varepsilon\to 0} F_\varepsilon(w_\varepsilon,\tilde K) \ge a+\sigma_p \mathcal{H}^{N-1}(Su\cap \tilde K),$$

for some $a>0$. Then, making use of Theorem 1.1-(ii) in $K\setminus \tilde K$,

$$\begin{aligned}\limsup_{\varepsilon\to 0} F_\varepsilon(w_\varepsilon,K) &\ge a+\sigma_p\mathcal{H}^{N-1}(Su\cap\tilde K)+\liminf_{\varepsilon\to 0} F_\varepsilon(w_\varepsilon,K\setminus\tilde K)\\ &\ge a+\sigma_p\mathcal{H}^{N-1}(Su\cap\tilde K)+\sigma_p\mathcal{H}^{N-1}(Su\cap(K\setminus\tilde K))\\ &\ge a+\sigma_p\mathcal{H}^{N-1}(Su\cap K).\end{aligned}$$

This is in contradiction with (2.10) and so it proves the desired result. □

Completion of the proof of Theorem 1.1-(iii). Now, we are ready to complete the proof of the Γ-limsup inequality. For any fixed function u in $BV(\Omega;\{-1,1\})$, we want to construct a recovery sequence u_ε as in Theorem 1.1-(iii). If $\Omega_\star=\Omega$, then one can apply directly Proposition 2.2, taking $u_\varepsilon := w_\varepsilon$, so we suppose $\Omega_\star\subset\Omega$. In this case, we make use of condition (1.5) and we obtain a recovering sequence, say μ_ε, in $\Omega\setminus\Omega_\star$, thanks to the results of [14, 22]. We will glue w_ε in $\Omega_\star$ with μ_ε in $\Omega\setminus\Omega_\star$ by using a fine convex interpolation argument of [24].

In further detail, we argue like this. For any small, positive r_0 and any $r \in (r, r_0)$, we set

$$\Omega_r := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > r\}.$$

For any $\varepsilon > 0$, we take the sequence w_ε constructed in Proposition 2.2, with $K = \Omega_r$; i.e., $w_\varepsilon : \Omega_r \to \mathbb{R}$ is such that

$$w_\varepsilon \text{ converges to } u \text{ in } L^1(\Omega_r) \tag{2.47}$$

and

$$\limsup_{\varepsilon \to 0} F_\varepsilon(w_\varepsilon, \Omega_r) \leq \sigma_p \mathcal{H}^{N-1}(S_u \cap \Omega_r). \tag{2.48}$$

Now, we take v to be the trace of u on $\partial\Omega$, which is well-defined, since u belongs to $BV(\Omega; \{-1, +1\})$ (see [13, Chapter 2]); we take V equal to δW; and we use [14, Theorem 1.1] when $p = 2$, or [22, Theorem 2.1] when $p \in (2, 3)$. In this way, we obtain a recovering sequence

$$\mu_\varepsilon \text{ that converges to } u \text{ in } L^1(\Omega \setminus \Omega_{2r}) \tag{2.49}$$

such that

$$\limsup_{\varepsilon \to 0} F_\varepsilon(\mu_\varepsilon, \Omega \setminus \Omega_{2r}) \leq \sigma_p \mathcal{H}^{N-1}(S_u \cap (\Omega \setminus \Omega_{2r})) + \kappa \mathcal{H}^{N-2}(S_v), \tag{2.50}$$

where (see (5.5) and the Remark after Proposition 5.5 in [14] and (2.4) in [22])

$$\kappa \leq C_\star \inf \left\{ H_\delta(w) : w \in L^1_{\mathrm{loc}}(\mathbb{R} \times (0, \infty)) \text{ such that } \lim_{x \to \pm\infty} w(x, 0) = \pm 1 \right\}, \tag{2.51}$$

with $C_\star$ a suitable constant, the functional H_δ defined by

$$H_\delta(w) := \int_0^\infty \left[\int_{\mathbb{R}} y^\varpi |Dw(x, y)|^p \, dx \right] dy + \delta \int_{\mathbb{R}} W(w(x, 0)) \, dx \quad (\forall \delta > 0),$$

and the power ϖ, depending only on p and s, given by

$$\varpi := \begin{cases} 2 - p & \text{if } p \in (2, 3), \\ 1 - 2s & \text{if } p = 2 \text{ and } s \in (1/2, 1). \end{cases}$$

We can prove that the minimum in (2.51) is achieved (see for instance [22, Proposition 4.7]). In particular, in the case $\delta = 1$ (i.e., the double-well potential on the boundary V coincides with W), we deduce the existence of a function $\theta \in L^1_{\mathrm{loc}}(\mathbb{R} \times (0, \infty))$, with

$$\lim_{x \to \pm\infty} \theta(x, 0) = \pm 1, \tag{2.52}$$

such that

$$H_1(\theta) \le C_1, \tag{2.53}$$

for a suitable positive constant C_1.

For any $\delta > 0$, we take the function

$$w_\delta(x,y) := \theta(\delta^\alpha(x,y)), \quad \forall (x,y) \in \mathbb{R} \times (0,\infty),$$

in which

$$\alpha = \alpha(p,s) := \begin{cases} \dfrac{1}{2p-3} & \text{if } p \in (2,3), \\ \\ \dfrac{1}{2s} & \text{if } p = 2 \text{ and } s \in (1/2,1). \end{cases}$$

We remark that

$$\alpha(p - 2 - \varpi) = 1 - \alpha \tag{2.54}$$

and

$$\alpha \in (0,1). \tag{2.55}$$

Note that

$$\lim_{x\to\pm\infty} w_\delta(x,0) = \lim_{x\to\pm\infty} \theta(\delta^\alpha x, 0) = \pm 1,$$

thanks to (2.52), hence w_δ is a candidate in (2.51) and then

$$\kappa \le C_* H_\delta(w_\delta). \tag{2.56}$$

We want to estimate the energy $H_\delta(w_\delta)$.

First, we have

$$|Dw_\delta(x,y)|^p \le \delta^{\alpha p} |D\theta\left(\delta^\alpha(x,y)\right)|^p, \quad \forall (x,y) \in \mathbb{R} \times (0,\infty).$$

Then, by the Change of variable Formula, setting $(x',y') = \delta^\alpha(x,y)$, it follows that

$$\int_0^\infty \left[\int_{\mathbb{R}} y^\varpi |Dw_\delta(x,y)|^p \, dx\right] dy \le \delta^{1-\alpha} \int_0^\infty \left[\int_{\mathbb{R}} (y')^\varpi |D\theta(x',y')|^p \, dx'\right] dy' \tag{2.57}$$

due to (2.54), and

$$\delta \int_{\mathbb{R}} W\left(w_\delta(x,0)\right) dx = \delta^{1-\alpha} \int_{\mathbb{R}} W(\theta(x',0) dx'. \tag{2.58}$$

Therefore, combining (2.53), (2.57) and (2.58), we get

$$H_\delta(w_\delta) \le \delta^{1-\alpha} H_1(\theta) \le C_1 \delta^{1-\alpha}, \quad (\forall \delta > 0).$$

This and (2.56) imply

$$\kappa \le \bar{C}\delta^{1-\alpha}, \tag{2.59}$$

for a suitable constant $\bar{C} > 0$.

Now, for any $i \in \mathbb{N}$, $0 \le i \le r/(4\varepsilon)$, we define

$$\Lambda_i := \Omega_{(3/2)r+i\varepsilon} \setminus \Omega_{(3/2)r+(i+1)\varepsilon}.$$

We remark that

$$\Lambda_i \subseteq \Omega_r \setminus \Omega_{2r}. \tag{2.60}$$

We observe that

$$\sum_{j=0}^{+\infty} \mathcal{H}^{N-1}\Big(S_u \cap (\Omega_{1/2^{j+1}} \setminus \Omega_{1/2^j})\Big) \;\le\; \mathcal{H}^{N-1}(S_u \cap \Omega) \;<\; +\infty.$$

As a consequence, we have that

$$\lim_{j\to\infty} \mathcal{H}^{N-1}\Big(S_u \cap (\Omega_{1/2^{j+1}} \setminus \Omega_{1/2^j})\Big) = 0. \tag{2.61}$$

We let

$$A_\star(x) := \max\Big\{\big(\mathrm{dist}(x,\partial\Omega)\big)^{a(1-p)}, \big(\mathrm{dist}(x,\partial\Omega)\big)^{a}\Big\}.$$

We observe that

$$A_\star(x) \le \big(\mathrm{dist}(x,\partial\Omega)\big)^{a_\star}$$

for a suitable $a_\star \in (0,1)$, due to (1.5).

Also, we set

$$\vartheta_\varepsilon := \min_{\substack{i\in\mathbb{N}\\ 0\le i\le r/(4\varepsilon)}} \left(\varepsilon\Big(F_\varepsilon(w_\varepsilon,\Lambda_i) + F_\varepsilon(\mu_\varepsilon,\Lambda_i)\Big) + \int_{\Lambda_i} A_\star(x)|w_\varepsilon - \mu_\varepsilon|\,dx\right).$$

From (2.47), (2.48), (2.49) and (2.50), we may suppose that

$$F_\varepsilon(w_\varepsilon, \Omega_r \setminus \Omega_{2r}) + F_\varepsilon(\mu_\varepsilon, \Omega_r \setminus \Omega_{2r}) \le C,$$

for some C independent of ε, and that, fixed any $\delta > 0$, if ε is suitably small, possibly in dependence of δ and r, we have that

$$\begin{aligned}\int_{\Omega_r\setminus\Omega_{2r}} A_\star(x)|w_\varepsilon-\mu_\varepsilon|\,dx &\le r^{a_\star}\int_{\Omega_r\setminus\Omega_{2r}}|w_\varepsilon-\mu_\varepsilon|\,dx\\ &\le r^{a_\star}\left[\int_{\Omega_r\setminus\Omega_{2r}}|w_\varepsilon-u|\,dx+\int_{\Omega_r\setminus\Omega_{2r}}|u-\mu_\varepsilon|\,dx\right]\\ &\le \frac{\delta r}{16}.\end{aligned}$$

Hence, by (2.60), if ε is suitably small, we have that

$$\begin{aligned}\frac{\delta r}{8} &\ge C\varepsilon+\frac{\delta r}{16}\\ &\ge \sum_{\substack{i\in\mathbb{N}\\ 0\le i\le r/(4\varepsilon)}}\left[\varepsilon\Big(F_\varepsilon(w_\varepsilon,\Lambda_i)+F_\varepsilon(\mu_\varepsilon,\Lambda_i)\Big)+\int_{\Lambda_i}A_\star(x)|w_\varepsilon-\mu_\varepsilon|\,dx\right]\\ &\ge \frac{r\vartheta_\varepsilon}{8\varepsilon}\end{aligned}$$

and therefore there exists an index $\underline{i}$ for which

$$\varepsilon\Big(F_\varepsilon(w_\varepsilon,\Lambda_{\underline{i}})+F_\varepsilon(\mu_\varepsilon,\Lambda_{\underline{i}})\Big)+\int_{\Lambda_{\underline{i}}}|w_\varepsilon-\mu_\varepsilon|\,dx\;=\;\vartheta_\varepsilon\;\le\;\delta\varepsilon. \tag{2.62}$$

Now, we take ξ to be a cut-off function on such $\Lambda_{\underline{i}}$, that is we suppose

$$\xi\in C_0^\infty(\Omega_{(3/2)r+\underline{i}\varepsilon};[0,1]),$$

with $\xi(x)=1$ for any $x\in\Omega_{(3/2)r+(\underline{i}+1)\varepsilon}$ and we can take $|\nabla\xi|\le\hat{C}/\varepsilon$ for a suitable $\hat{C}>0$, independent of ε. We define

$$u_\varepsilon := \xi w_\varepsilon+(1-\xi)\mu_\varepsilon.$$

We remark that

$$\begin{aligned}W(u_\varepsilon) &= W(w_\varepsilon+(u_\varepsilon-w_\varepsilon))\\ &\le W(w_\varepsilon)+C'|u_\varepsilon-w_\varepsilon|\\ &\le W(w_\varepsilon)+C'|\mu_\varepsilon-w_\varepsilon|,\end{aligned} \tag{2.63}$$

for some $C' > 0$. On the other hand,

$$
\begin{aligned}
|Du_\varepsilon|^p &= |\xi Dw_\varepsilon + (1-\xi)D\mu_\varepsilon + D\xi(w_\varepsilon - \mu_\varepsilon)|^p \\
&\le \Big(|Dw_\varepsilon| + |D\mu_\varepsilon| + (\hat{C}/\varepsilon)|w_\varepsilon - \mu_\varepsilon|\Big)^p \\
&\le 3^p|Dw_\varepsilon|^p + 3^p|D\mu_\varepsilon|^p + \frac{3^p\hat{C}^p}{\varepsilon^p}|w_\varepsilon - \mu_\varepsilon|^p \\
&\le 3^p|Dw_\varepsilon|^p + 3^p|D\mu_\varepsilon|^p + \frac{6^p\hat{C}^p}{\varepsilon^p}|w_\varepsilon - \mu_\varepsilon|. \qquad (2.64)
\end{aligned}
$$

Therefore, making use of (2.63) and (2.64) and then recalling (2.62),

$$
\begin{aligned}
F_\varepsilon(u_\varepsilon, \Lambda_{\underline{i}}) &\le \tilde{C}\left[F_\varepsilon(w_\varepsilon, \Lambda_{\underline{i}}) + F_\varepsilon(\mu_\varepsilon, \Lambda_{\underline{i}}) + \frac{1}{\varepsilon}\int_{\Lambda_{\underline{i}}} A_\star(x)|w_\varepsilon - \mu_\varepsilon|^p\right] \\
&\le \tilde{C}\delta,
\end{aligned}
$$

for a suitable $\tilde{C} > 0$. Consequently,

$$
\begin{aligned}
F_\varepsilon(u_\varepsilon, \Omega) &= F_\varepsilon(w_\varepsilon, \Omega_{(3/2)r+(\bar{i}+1)\varepsilon}) + F_\varepsilon(\mu_\varepsilon, \Omega \setminus \Omega_{(3/2)r+\bar{i}\varepsilon}) + F_\varepsilon(u_\varepsilon, \Lambda_{\underline{i}}) \\
&\le F_\varepsilon(w_\varepsilon, \Omega_r) + F_\varepsilon(\mu_\varepsilon, \Omega \setminus \Omega_{2r}) + \tilde{C}\delta.
\end{aligned}
$$

Therefore, recalling (2.48), (2.59), (2.50) and Lemma 2.3,

$$
\begin{aligned}
\limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, \Omega) &\le \sigma_p\mathcal{H}^{N-1}\Big(S_u \cap \Omega_r\Big) + \sigma_p\mathcal{H}^{N-1}\Big(S_u \cap (\Omega \setminus \Omega_{2r})\Big) + \kappa\mathcal{H}^{N-2}(S_v) + \tilde{C}\delta \\
&\le \sigma_p\mathcal{H}^{N-1}(S_u \cap \Omega) + \sigma_p\mathcal{H}^{N-1}\Big(S_u \cap (\Omega_r \setminus \Omega_{2r})\Big) + \bar{C}\delta^{1-\alpha} + \tilde{C}\delta.
\end{aligned}
$$

Since δ may be taken arbitrarily close to 0 and r may be taken of the form $1/2^j$ with j arbitrarily large, the latter estimate, (2.55) and (2.61) prove the Γ-$\limsup$ inequality.

The fact that u_ε approaches u in $L^1(\Omega)$ is a consequence of (2.47) and (2.49), and so we have completed the proof of Theorem 1.1-(iii). □

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