Regularity of convex functions on Heisenberg groups *

Zoltán M. Balogh
Institute of Mathematics
University of Bern
Sidlerstrasse 5
3012 Bern, Switzerland
zoltan.balogh@math-stat.unibe.ch

Matthieu Rickly
Institute of Mathematics
University of Bern
Sidlerstrasse 5
3012 Bern, Switzerland
matthieu.rickly@math-stat.unibe.ch

May 23, 2003

Abstract

We discuss differentiability properties of convex functions on Heisenberg groups. We show that the notions of horizontal convexity (h-convexity) and viscosity convexity (v-convexity) are equivalent and that h-convex functions are locally Lipschitz continuous. Finally we exhibit Weierstrass-type h-convex functions which are nowhere differentiable in the vertical direction on a dense set or on a Cantor set of vertical lines.

1 Introduction

Convex functions in Euclidean space play an important role in partial differential equations, especially in the theory of fully non-linear PDE's (see [1], [6], [7], [8]). This fact has motivated the development of a theory of convex functions on Heisenberg groups (cf. [17]) and more generally on Carnot groups (cf. [10]) with applications towards subelliptic fully nonlinear PDE's on such groups.

Lu, Manfredi and Stroffolini have transposed the notion of *convexity in the viscosity sense (v-convexity)* to the sub-Riemannian setting of the Heisenberg group. Using results about viscosity solutions of the subelliptic ∞-Laplacian by Bieske ([5]), they proved that upper semicontinuous, v-convex functions on the Heisenberg group are locally Lipschitz continuous.

A dual approach to convexity on Carnot groups is provided by the work of Danielli, Garofalo and Nhieu [10]. Here the starting point is the more algebraic notion of *weak H-convexity* which we refer to as *horizontal convexity* or *h-convexity*. One of the main results of [10] is that locally bounded h-convex functions are locally Lipschitz continuous.

Two interesting questions arise: The first one concerns the relationship between these two notions of convexity. It has been proved in [17] that upper semicontinuous h-convex functions on the Heisenberg group are v-convex. Does the reverse implication also hold? Our first result gives an affirmative answer to this question.

Theorem 1.1. *Upper semicontinuous v-convex functions on Heisenberg groups are h-convex.*

* **Mathematics Subject Classification (2000):** 43A80, 26B25 (primary), 49L25 (secondary)

Remark 1.1. A different proof of this result, which uses the latest developments of the general PDE machinery for Carnot groups and more generally for sub-Riemannian geometries, has been recently communicated to us by J. Manfredi. We refer to the forthcoming article [15].

The second question is whether the extra assumption of local boundedness from [10] can be removed in the proof of the local Lipschitz continuity of h-convex functions on the Heisenberg groups. The main result of this paper is to show that this is indeed the case.

Theorem 1.2. *h-convex functions on Heisenberg groups are locally Lipschitz continuous.*

Here Lipschitz continuity is understood with respect to the sub-Riemannian Carnot-Carathéodory metric. This is a weaker notion than the Euclidean Lipschitz continuity which definitely fails for h-convex functions on the Heisenberg group (see Theorem 1.3 below). Nevertheless, by Pansu's differentiability theorem ([20]), this implies that h-convex functions are differentiable almost everywhere in horizontal directions.

In the second part of the paper, we address the question of second order differentiability of h-convex functions on Heisenberg groups. Recall that by the celebrated theorem of Alexandrov, convex functions defined on Euclidean spaces are twice differentiable almost everywhere (cf. [11]). A key step in the proof of this theorem is to show that the second order mixed partial derivatives of convex functions are Radon measures. It turns out that for h-convex function $u : \mathbb{H}_n \to \mathbb{R}$, this is the case if and only if the partial derivative in the vertical direction is a Radon measure. Two recent articles by Garofalo-Tournier and Gutiérrez-Montanari on Monge-Ampère measures in Heisenberg groups contain an implicit proof of the fact that the partial derivative in the vertical direction of a (continuous) h-convex function is actually locally square integrable. However, the notion of h-convexity is less rigid than its Euclidean counterpart, and in particular does not imply pointwise almost everywhere differentiability on single vertical lines. In the last section we exhibit interesting examples of h-convex functions with highly irregular behaviour in the vertical direction on sparse sets of vertical lines.

Theorem 1.3.

(i) There exists an h-convex function $u : \mathbb{H}_1 \to \mathbb{R}$ whose restriction to a dense set of vertical lines is nowhere differentiable.

(ii) There exists an h-convex function $u : \mathbb{H}_1 \to \mathbb{R}$ whose restriction to a positive dimensional Cantor set of vertical lines is nowhere differentiable.

The paper is organized as follows. In Section 2 we recall the background and terminology and prove Theorem 1.1. In Section 3 we give the proof of Theorem 1.2. The last section is devoted to the proof of Theorem 1.3.

2 v-convexity implies h-convexity

In this section we recall the basic definitions needed in the rest of the paper and we give a proof of Theorem 1.1.

In the following,

$$\mathbb{H}_n \equiv \mathbb{R}^{2n+1} \equiv \{(x,t) \mid x \in \mathbb{R}^{2n},\, t \in \mathbb{R}\}$$

denotes the n-th Heisenberg group with the group law

$$(x,t)*(x',t') = \left(x+x', t+t'+2\sum_{i=1}^{n}(x'_i x_{n+i} - x_i x'_{n+i})\right).$$

One can check that the unit element is $0 \in \mathbb{R}^{2n+1}$ and that the inverse of $p = (x,t)$ is $p^{-1} = (-x,-t)$. In the setup of the Heisenberg groups, the Euclidean translations and dilations on $\mathbb{R}^{2n+1}$ are replaced by *left translations*

$$\mathrm{l}_p : \mathbb{H}_n \to \mathbb{H}_n;\ \mathrm{l}_p(q) = p*q$$

$(p \in \mathbb{H}_n)$ and *anisotropic dilations*

$$\delta_r : \mathbb{H}_n \to \mathbb{H}_n;\ \delta_r(x,t) := (rx, r^2 t)$$

$(r > 0)$. Clearly, $(\delta_r)_{r>0}$ is a group of automorphisms of $\mathbb{H}_n$.

Metrics on $\mathbb{H}_n$ which are compatible with left translations and dilations are called *left invariant, homogeneous metrics*. It turns out that any two such metrics are comparable. Left invariant, homogeneous distances on $\mathbb{H}_n$ can be obtained in several ways. One possibility is to consider the distance induced by the *Heisenberg gauge* $\|\cdot\|_H$ which is given by

$$\|(x,t)\|_H := \left(\Big(\sum_{i=1}^{n}(x_i^2 + x_{n+i}^2)^2\Big)^2 + t^2\right)^{\frac{1}{4}}. \tag{2.1}$$

$\|\cdot\|_H$ is obviously homogeneous with respect to dilations. It is a well-known (although non-trivial) fact that the gauge is also subadditive in the sense that $\|p*q\|_H \leq \|p\|_H + \|q\|_H$ whenever p, $q \in \mathbb{H}_n$ (cf. [9]). Consequently,

$$d_H(p,q) := \|p^{-1} * q\|_H \tag{2.2}$$

defines a left invariant, homogeneous metric on $\mathbb{H}_n$, the so-called *Heisenberg metric*. By the above definition, it is immediate that

$$\frac{1}{C} d_E(p,q) \leq d_H(p,q) \leq C(d_E(p,q))^{\frac{1}{2}} \tag{2.3}$$

for $B \subseteq \mathbb{H}_n$ bounded, p, $q \in B$ and $0 < C = C(B) < \infty$ (where d_E denotes the Euclidean metric on $\mathbb{H}_n \equiv \mathbb{R}^{2n+1}$). This shows that d_H induces the Euclidean topology on $\mathbb{H}_n$. However, the Hausdorff measures induced by d_E and d_H are quite different; we refer to the recent results in [4] on this subject.

The differential structure on $\mathbb{H}_n$ is determined by the left invariant vector fields

$$X_1, \ldots, X_{2n}, T$$

where

$$X_i = \frac{\partial}{\partial x_i} + 2x_{n+i}\frac{\partial}{\partial t}, \quad X_{n+i} = \frac{\partial}{\partial x_{n+i}} - 2x_i \frac{\partial}{\partial t} \quad (i = 1, \ldots, n)$$

are the so-called *horizontal vector fields* and

$$T = \frac{\partial}{\partial t}.$$

The only non-trivial bracket relation is $[X_i, X_{n+i}] = -4T$ $(i = 1, \ldots, n)$.

Definition 2.1. The *horizontal plane* at $p = (x,t) \in \mathbb{H}_n$ is

$$\mathrm{H}_p\mathbb{H}_n := p + \mathrm{d}_0\mathrm{l}_p(\mathrm{H}_0\mathbb{H}_n),$$

where $\mathrm{d}_0\mathrm{l}_p$ is the differential of the left translation $\mathrm{l}_p : \mathbb{H}_n \to \mathbb{H}_n$ at 0 and

$$\mathrm{H}_0\mathbb{H}_n := \{(x',0) \mid x' \in \mathbb{R}^{2n}\}$$

is the horizontal plane at 0. Hence

$$\mathrm{H}_p\mathbb{H}_n = \mathrm{l}_p(\mathrm{H}_0\mathbb{H}_n) = \Big\{(x+x', t+2\sum_{i=1}^{n}(x_i' x_{n+i} - x_i x_{n+i}')) \;\Big|\; x' \in \mathbb{R}^{2n}\Big\}.$$

Observe that $\mathrm{H}_p\mathbb{H}_n$ is just the $2n$-dimensional hyperplane $p + \mathrm{span}_{\mathbb{R}}(V_1, \ldots, V_{2n})$, where V_i is the coordinate vector of X_i with respect to the basis $(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{2n}}, \frac{\partial}{\partial t})$.

The main issue in analysis on the Heisenberg groups is that the classical first and second order differential operators are considered only in terms of horizontal vector fields. Likewise, convexity is defined with regard to the horizontal directions. Let us recall the notion of weak H-convexity (we use h-convexity) due to X. Cabré and L. Caffarelli and studied in [10] and [17]:

Definition 2.2. Let $\Omega \subseteq \mathbb{H}_n$ be open. $u : \Omega \to \mathbb{R}$ is said to be *h-convex* if the restriction of u to the segment $[p,q]$ is a convex function whenever $p \in \Omega$ and $[p,q] \subseteq \Omega \cap \mathrm{H}_p\mathbb{H}_n$.

Here $[p,q]$ denotes the convex closure (in the Euclidean sense) of the set $\{p,q\}$.

C^2 smooth convex functions in $\mathbb{R}^n$ are characterized by the positivity of the Hessian. The analog of the Hessian in Heisenberg groups is defined below.

Definition 2.3. Let $\Omega \subseteq \mathbb{H}_n$ be open and $u : \Omega \to \mathbb{R}$. If $X_iX_ju\,(x,t)$ exist for all $1 \le i,j \le 2n$ for some $(x,t) \in \Omega$, then the matrix

$$H_{sym}u(x,t) = \Big(\frac{1}{2}(X_iX_ju\,(x,t) + X_jX_iu\,(x,t))\Big)_{i,j=1,\ldots,2n}$$

is called the *symmetrized horizontal Hessian of u at (x,t)*.

As in the case of Euclidean spaces, it turns out that for sufficiently regular functions, the symmetrized horizontal Hessian characterizes h-convexity. Indeed, by [10, Theorem 5.11], a function $u : \Omega \subseteq \mathbb{H}_n \to \mathbb{R}$ such that X_iX_ju exists and is continuous in Ω for all $1 \le i,j \le 2n$ is h-convex if and only if $H_{sym}u$ is positive semidefinite everywhere in Ω.

The concept of viscosity allows to extend the notion of positive semidefinite symmetrized horizontal Hessian to functions u for which $H_{sym}u$ may fail to exist or to be continuous and to take this notion as a starting point for a theory of convex functions on Heisenberg groups.

Definition 2.4. Let $\Omega \subseteq \mathbb{H}_n$ be open and $u : \Omega \to \mathbb{R}$. u is said to be *convex in the viscosity sense*, or just *v-convex*, if

$$H_{sym}u \ge 0 \quad \text{in } \Omega \text{ in the viscosity sense.}$$

That is, if $p \in \Omega$, $U \subseteq \Omega$ is an open neighbourhood of p and $\phi \in C^2(U)$ touches u from above at p (meaning $\phi(p) = u(p)$ and $\phi(q) \ge u(q)$ for $q \in U$), then

$$H_{sym}\phi(p) \ge 0.$$

Remark 2.1. The above definition doesn't make sense for functions u which are not locally bounded above.

Remark 2.2. The above notions of convexity are compatible with the group structure : If $u : \Omega \to \mathbb{R}$ is h-convex (respectively v-convex) and $p \in \mathbb{H}_n$, $r > 0$ then $u \circ \mathrm{l}_p : \mathrm{l}_{p^{-1}}(\Omega) \to \mathbb{R}$ and $u \circ \delta_r : \delta_{\frac{1}{r}}(\Omega) \to \mathbb{R}$ are h-convex (respectively v-convex). Moreover, given a family $(u_n : \Omega \to \mathbb{R})_{n\in\mathbb{N}}$ of h-convex functions (v-convex functions), $u := \sup_{n\in\mathbb{N}} u_n$ is h-convex (v-convex) as well. Also, if $(u_n : \Omega \to \mathbb{R})_{n\in\mathbb{N}}$ are h-convex functions converging pointwise to some $u : \Omega \to \mathbb{R}$, then u is h-convex. Finally, observe that functions $u : \mathbb{R}^{2n+1} \equiv \mathbb{H}_n \to \mathbb{R}$ which are convex in the Euclidean sense are both h- and v-convex.

We now give a proof of Theorem 1.1. The main idea is the same as in the Euclidean case (cf. [16]) and uses the fact that the differential structure on $\mathbb{H}_n$ does not change in the vertical direction.

Proof of Theorem 1.1. Let us suppose that $u : \Omega \to \mathbb{R}$ is v-convex but not h-convex. After appropriate left translation, dilation, addition of an affine mapping $\mathbb{R}^{2n+1} \to \mathbb{R}$ and multiplication with a positive constant, we can assume that there exist a unit vector v in $\mathrm{H}_0\mathbb{H}_n$ and an open bounded set U such that

$$\{\lambda \cdot v \mid \lambda \in [-1,1]\} \subseteq U \subseteq \overline{U} \subseteq \Omega,$$

$$u(0) = 0, \quad u(-v) \le -2 \quad \text{and} \quad u(v) \le -2.$$

Let $v_1 := v$, $v_{2n+1} := (0,\dots,0,1)$, $(v_1,\dots,v_{2n+1})$ be an orthonormal basis of $\mathbb{R}^{2n+1}$ and

$$M := \max\Big\{u(x,t) \;\Big|\; (x,t) \in \overline{U}\Big\}.$$

By [17, Theorem 3.1] u is continuous and so $M < \infty$. For $a \ge 0$ and $\epsilon > 0$, consider the function $\phi_a : \mathbb{H}_n \to \mathbb{R}$ defined by

$$\phi_a\Big(\sum_{i=1}^{2n+1} \alpha_i v_i\Big) = a - \alpha_1^2 + \sum_{i=2}^{2n} \frac{\alpha_i^2}{\epsilon^2} + \frac{\alpha_{2n+1}^2}{\epsilon}$$

and the domain

$$D_\epsilon = \Big\{ \sum_{i=1}^{2n+1} \alpha_i v_i \in U \;\Big|\; \sum_{i=2}^{2n} \frac{\alpha_i^2}{\epsilon^2} + \frac{\alpha_{2n+1}^2}{\epsilon} < M+2,\, |\alpha_1| < 1\Big\}.$$

The intuitive geometric idea behind the definition of D_ϵ is that D_ϵ will become very thin in the orthogonal complement of the 2-dimensional subspace spanned by v_1 and v_{2n+1} compared to its size in this subspace as $\epsilon \downarrow 0$. In view of

$$\frac{d^2}{d\lambda^2}\phi_a(\lambda v_1 + \alpha_{2n+1} v_{2n+1}) = -2$$

(meaning that ϕ_a is uniformly concave on the horizontal line $\{\lambda v_1 + \alpha_{2n+1} v_{2n+1} \mid \lambda \in \mathbb{R}\}$ independently of a and ϵ), it is reasonable to expect that $H_{sym}\phi_a$ will fail to be positive semidefinite on D_ϵ for sufficiently small ϵ. We now show that this is indeed the case.

Claim 2.1. *There exists $\epsilon_0 > 0$ such that $H_{sym}\phi_a$ fails to be positive semidefinite on D_ϵ for all $0 < \epsilon < \epsilon_0$ and all $a \ge 0$.*

Proof of Claim 2.1. Let $p = \sum_{i=1}^{2n+1} \alpha_i v_i \in D_\epsilon$. Then $|\alpha_i| \leq \sqrt{M+2}\,\epsilon$ for $i = 2, \ldots, 2n$. Observe that

$$p * \delta_\lambda v_1 = (\alpha_1 + \lambda) v_1 + \sum_{i=2}^{2n} \alpha_i v_i + (\alpha_{2n+1} + 2\lambda y) v_{2n+1}$$

with $|y| \leq c\epsilon$ for some constant c not depending on ϵ, a or p. One then computes

$$\phi_a(p * \delta_\lambda v_1) = a - (\alpha_1 + \lambda)^2 + \sum_{i=2}^{2n} \frac{\alpha_i^2}{\epsilon^2} + \frac{t^2 + 4\lambda t y + 4\lambda^2 y^2}{\epsilon}.$$

Differentiating twice with respect to λ, we obtain

$$\frac{d^2}{d\lambda^2} \phi_a(p * \delta_\lambda v_1) = -2 + \frac{8y^2}{\epsilon} \leq -2 + 8c^2\epsilon.$$

This shows that there exists ϵ_0 s.t. $0 < \epsilon < \epsilon_0$ implies

$$\frac{d^2}{d\lambda^2} \phi_a(p * \delta_\lambda v_1) \leq -1$$

for $\lambda \in [0, 1]$, $a \geq 0$ and $p \in D_\epsilon$. By [10, Proposition 5.2], it follows that $H_{sym}\phi_a$ fails to be positive semidefinite everywhere in D_ϵ. □

It remains to show that we can choose $0 < \epsilon < \epsilon_0$ and $a_0 \geq 0$ such that ϕ_{a_0} touches u from above at some point $p_0 \in D_\epsilon$. The following claim is the main step toward this goal.

Claim 2.2. *There exists* $0 < \epsilon < \epsilon_0$ *such that for* $a \geq 0$ *we have the inequality*

$$\phi_a\Big(\sum_{i=1}^{2n+1} \alpha_i v_i\Big) > u\Big(\sum_{i=1}^{2n+1} \alpha_i v_i\Big)$$

on the boundary ∂D_ϵ.

Proof of Claim 2.2. To see this, we divide ∂D_ϵ in two parts:

$$\partial_1 D_\epsilon = \Big\{ \sum_{i=1}^{2n+1} \alpha_i v_i \in \overline{U} \;\Big|\; \sum_{i=2}^{2n} \frac{\alpha_i^2}{\epsilon^2} + \frac{\alpha_{2n+1}^2}{\epsilon} \leq M + 2,\, |\alpha_1| = 1 \Big\}.$$

and

$$\partial_2 D_\epsilon = \Big\{ \sum_{i=1}^{2n+1} \alpha_i v_i \in \overline{U} \;\Big|\; \sum_{i=2}^{2n} \frac{\alpha_i^2}{\epsilon^2} + \frac{\alpha_{2n+1}^2}{\epsilon} = M + 2,\, |\alpha_1| < 1 \Big\}.$$

Consider first the set $\partial_1 D_\epsilon$. Notice that $-v_1 = -v$, $v_1 = v \in \partial_1 D_\epsilon$. Observe that

$$\phi_a \geq a - 1 \geq -1$$

on $\partial_1 D_\epsilon$ independently of $a \geq 0$ and $\epsilon > 0$, while $u(-v) \leq -2$ and $u(v) \leq -2$. By continuity of u, the inequality follows on $\partial_1 D_\epsilon$ for $0 < \epsilon < \epsilon_1$, where $\epsilon_1 < \epsilon_0$ does not depend on a. For the points of $\partial_2 D_\epsilon$ on the other hand, we have

$$\phi_a\Big(\sum_{i=1}^{2n+1} \alpha_i v_i\Big) = a - \alpha_1^2 + M + 2 > a + M + 1 \geq M + 1 > u\Big(\sum_{i=1}^{2n+1} \alpha_i v_i\Big).$$

□

Let us now observe that for $a = 0$ $\phi_a(0) = u(0) = 0$, while for large values of a clearly

$$\phi_a(p) > u(p) \text{ on } \overline{D_\epsilon}.$$

Therefore

$$a_0 := \inf\{a > 0 \mid \phi_a(p) > u(p) \text{ on } \overline{D_\epsilon}\}$$

is well defined and satisfies $a_0 \geq 0$. By definition of a_0, there exists $p_0 \in \overline{D_\epsilon}$ s.t.

$$\phi_{a_0}(p_0) = u(p_0)$$

and

$$\phi_{a_0}(p) \geq u(p)$$

for all $p \in \overline{D_\epsilon}$. Notice that since $\phi_{a_0} > u$ on ∂D_ϵ, we have $p_0 \in D_\epsilon$. This means that ϕ_{a_0} touches u from above at p_0.

In view of Claim 2.1, $H_{sym}\phi_{a_0}(p_0)$ fails to be positive semidefinite, contradicting the v-convexity of u and concluding the proof. □

Remark 2.3. As a consequence of this theorem, all the results we prove for h-convex functions hold for upper semicontinuous v-convex functions as well. This includes the first and second order regularity results presented in the remaining of this paper.

3 Lipschitz continuity of h-convex functions

We start with a few preparatory lemmas. We include the short proofs for the sake of completeness.

Lemma 3.1. *Let $a > 0$, $f : [-4a, 4a] \to \mathbb{R}$ convex.*

- *(i) If $x \in [-a, a]$ and $M := \max\{f(-4a), f(4a), |f(x)|\}$, then $-5M$ is a lower bound for f on $[-4a, -2a] \cup [2a, 4a]$.*
- *(ii) If $M := \max\{f(-4a), |f(-3a)|, |f(3a)|, f(4a)\}$, then $-11M$ is a lower bound for f on $[-2a, 2a]$.*

Proof.

(i) Consider $y \in [-4a, -2a]$. Then $x = (1-\lambda)y + \lambda 4a$ with some $\lambda \leq 5/8$. Hence

$$(1-\lambda)f(y) + \lambda f(4a) \geq f(x)$$

and

$$f(y) \geq \frac{8}{3}(f(x) - \lambda f(4a)) \geq \frac{8}{3}\Big(-\frac{13}{8}M\Big) \geq -\frac{13}{3}M \geq -5M.$$

A similar computation works for $y \in [2a, 4a]$.

(ii) Consider $x \in [-2a, 0]$. $3a = (1-\lambda)x + \lambda 4a$ for some $\lambda \leq 5/6$. Thus

$$(1-\lambda)f(x) + \lambda f(4a) \geq f(3a),$$

so

$$f(x) \geq 6(f(3a) - \lambda f(4a)) \geq 6\Big(-\frac{11}{6}M\Big) \geq -11M.$$

A similar computation works for $x \in [0, 2a]$.

□

Lemma 3.2. *Let $\Omega \subseteq \mathbb{H}_n$ be open, $u : \Omega \to \mathbb{R}$ be h-convex, $p \in \Omega$, $(x, 0) \in \mathrm{H}_0\mathbb{H}_n$ and define $p' = p * (x, 0)$ and $p'' = p * (-x, 0)$. Suppose that $[p', p''] \subseteq \Omega$. Then u is convex on $[p', p'']$.*

Proof. We have $p'' = p' * (-x, 0) * (-x, 0)$. Now observe that $(-x, 0) * (-x, 0) \in \mathrm{H}_0\mathbb{H}_n$. □

Lemma 3.3. *Let $p_1 \neq p_2 \in \mathbb{R}^m$ and define $q_1 := p_1 + \frac{1}{4}(p_2 - p_1)$ and $q_2 := p_1 + \frac{3}{4}(p_2 - p_1)$. Let $f : [p_1, p_2] \to \mathbb{R}$ be convex and bounded by some M on $[p_1, p_2]$. Then u satisfies a $\frac{8M}{|p_2 - p_1|}$- Lipschitz condition on $[q_1, q_2]$.*

Proof. Let $\rho := \frac{|q_2 - q_1|}{2} = \frac{|p_2 - p_1|}{4}$, $p \neq p' \in [q_1, q_2]$ and consider $p'' := p' + \rho \frac{p' - p}{|p' - p|} \in [p_1, p_2]$. By definition,

$$p' = \frac{|p' - p|}{\rho + |p' - p|} p'' + \frac{\rho}{\rho + |p' - p|} p.$$

By convexity of f, we have

$$f(p') - f(p) \leq \frac{|p' - p|}{\rho + |p' - p|} (f(p'') - f(p)) \leq \frac{|p' - p|}{\rho} 2M.$$

Interchanging p and p' in the above estimate gives the claim. □

We now give the proof of Theorem 1.2. We start by giving a detailed proof in the case $n = 1$, $\Omega = \mathbb{H}_1$ in order to illustrate the main ideas in the most simple case, and then briefly indicate how the proof extends to higher dimensions. The main point in the proof is to show the local boundedness of h-convex functions. The first result is formulated as:

Proposition 3.4. *h-convex functions $u : \mathbb{H}_1 \to \mathbb{R}$ are locally bounded.*

The proof is divided in two parts. In the first part we prove that h-convex functions on $\mathbb{H}_1$ are locally bounded above and we use this result in the second part to prove that h-convex functions on $\mathbb{H}_1$ are locally bounded below. In each part the local boundedness is extended successively to sets of increasing topological dimension.

Proof. In the following, $M > 0$ is a positive generic constant whose exact value is not important and can change in different instances.

Step I: Local upper bound for u on a vertical line.

Consider the horizontal segments

$$L^+ := \{(1, -s, 2s) \mid -1 \leq s \leq 1\} \quad \text{and} \quad L^- := \{(-1, s, 2s) \mid -1 \leq s \leq 1\}.$$

By Lemma 3.2, u is convex on L^+ and L^-. Consequently, there exists an upper bound M for u on $L^+ \cup L^-$.

For any $-1 \leq t \leq 1$ consider the horizontal line segment $S(t)$ passing through $(0, 0, 2t)$ and connecting the points $(1, -t, 2t)$ and $(-1, t, 2t)$. u is convex on $S(t)$ by Lemma 3.2. Since the endpoints of $S(t)$ are contained in $L^+ \cup L^-$, it follows that M is an upper bound for u on the set $\{(0, 0, t) \mid |t| \leq 2\}$.

Let us observe that the invariance of h-convexity by dilations and left translations implies that u is bounded above on any vertical segment (although the bound may depend on the segment under consideration).

Step II: Local upper bound for u on a vertical plane.

We want to prove the upper boundedness of u on the 2-dimensional square

$$Q^2 = \{(x_1, x_2, t) \in \mathbb{H}_1 \mid |x_1| \le 1,\ x_2 = 0, |t| \le 1\}.$$

By step I, u is bounded above by some constant M on the segments

$$Q_1^1 = \{(-1, 0, t) \mid -1 \le t \le 1\} \quad \text{and} \quad Q_2^1 = \{(1, 0, t) \mid -1 \le t \le 1\}.$$

For any $-1 \le t \le 1$ consider the horizontal line segment $S(t)$ passing through $(0, 0, t)$ and connecting the points $(-1, 0, t)$ and $(1, 0, t)$. u is convex on $S(t)$. It follows that M is an upper bound for u on Q^2.

Observe that a similar construction yields an upper bound for u on the square

$$\{(x_1, x_2, t) \in \mathbb{H}_1 \mid x_1 = 0,\ |x_2| \le 1, |t| \le 1\}.$$

Step III: Local upper bound for u on a full dimensional set.

Consider the 3-dimensional cube

$$Q^3 = \{(x_1, x_2, t) \in \mathbb{H}_1 \mid \max\{|x_1|, |x_2|, |t|\} \le 1\}.$$

By step II and invariance of h-convexity by left translations and dilations, it follows that u is bounded above by some constant M on the faces of Q^3 which are parallel to the vertical axis. Now given any point (x, y, t) lying inside the cube, we consider the segment obtained by intersecting the line through $(0, 0, t)$ and (x, y, t) with the faces of the cube. The convexity of u on this segment and the fact that the boundary values are bounded above by M show that $u(x, y, t) \le M$. Hence M is an upper bound for u in the whole cube.

In order to obtain a local lower bound for u, we give an argument which is similar to the above and makes use of Lemma 3.1 and of the existence of a local upper bound.

Step I: Local lower bound for u on a vertical line segment.

Consider again the square

$$Q^2 = \{(x_1, x_2, t) \in \mathbb{H}_1 \mid |x_1| \le 1,\ x_2 = 0, |t| \le 1\}$$

contained in a vertical plane and the segment

$$S := \{(s/2, 2, s) \mid -1 \le s \le 1\}$$

which is parallel to the horizontal segment $\{(s/2, 1, s) \mid -1 \le s \le 1\}$ and lies in the horizontal plane at $(0, 1, 0)$ (but is not itself horizontal). u is bounded above by some constant M on Q^2 and S. Consider the family of horizontal rays starting at some $p \in S$ and passing through $(0, 1, 0)$. This family of rays intersects Q^2 precisely in the segment

$$\widetilde{S} := \{(s/2, 0, s) \mid -1 \le s \le 1\}.$$

Since u is convex on each such ray, Lemma 3.1 (i) yields a lower bound $M' = M'(M, |u(0, 1, 0)|)$ for u on $\widetilde{S}$. Consider now the horizontal segments

$\{(s,0,t) \mid -1 \le s \le 1\}$ for $|t| \le \frac{1}{2}$ which intersect $\widetilde{S}$ in $(t/2,0,t)$. A second application of Lemma 3.1 (i) gives $u(-1,0,t), u(1,0,t) \ge M''$, where $M'' = M''(M,M')$. Hence M'' is a lower bound for u on the vertical segments

$$Q_1^1 = \{(-1,0,t) \mid |t| \le 1/2\} \quad \text{and} \quad Q_2^1 = \{(1,0,t) \mid |t| \le 1/2\}.$$

Let us remark again that the invariance of h-convexity by dilations and left translations implies that u is now bounded on any vertical segment (although the bound may depend on the segment under consideration).

Step II: Local lower bound for u on a vertical plane.

By step I, we can assume that $|u|$ is bounded by some M on the vertical segments

$$Q_1^1 = \{(2,0,t) \mid |t| \le 1\}, \quad Q_2^1 = \{(-2,0,t) \mid |t| \le 1\},$$
$$Q_3^1 = \{(3/2,0,t) \mid |t| \le 1\}, \quad \text{and} \quad Q_4^1 = \{(-3/2,0,t) \mid |t| \le 1\},$$

Considering the horizontal segments $\{(s,0,t) \mid -2 \le s \le 2\}$ for $|t| \le 1$ and applying Lemma 3.1 (ii), we obtain a lower bound for u on Q^2.

Observe that a similar construction yields a lower bound for u on the square

$$\{(x_1,x_2,t) \in \mathbb{H}_1 \mid x_1 = 0,\ |x_2| \le 1, |t| \le 1\}.$$

Step III: Local lower bound for u on a full dimensional set.

Consider the cubes

$$Q^3(4) = \{(x_1,x_2,t) \in \mathbb{H}_1 \mid \max\{|x_1|,\ |x_2|,\ |t|\} \le 4\}$$

and

$$Q^3(3) = \{(x_1,x_2,t) \in \mathbb{H}_1 \mid \max\{|x_1|,\ |x_2|,\ |t|\} \le 3\}.$$

By step II and the invariance of h-convexity by left translations and dilations, $|u|$ is bounded by some constant M on the faces of the cubes which are parallel to the vertical axis. Given any

$$(x_1,x_2,t) \in Q^3(2) = \{(x_1,x_2,t) \in \mathbb{H}_1 \mid \max\{|x_1|,\ |x_2|,\ |t|\} \le 2\},$$

consider the horizontal segment obtained by intersecting the horizontal line through $(0,0,t)$ and (x,y,t) with the faces of $Q^3(4)$ and $Q^3(3)$. From the convexity of u on such segments and from Lemma 3.1 (ii), it follows that u is bounded below on $Q^3(2)$ by some constant $M' = M'(M)$.

This proof in fact shows that $|u|$ is bounded on a cube of arbitrary size. We are done. □

Proof of Theorem 1.2. Without loss of generality, assume that $u : \Omega \to \mathbb{R}$ is an h-convex function whose domain is $\Omega = \mathbb{H}_n$. The strategy is the same as in the proof of Proposition 3.4. We first prove that an h-convex function $u : \mathbb{H}_n \to \mathbb{R}$ is locally bounded above. We then use the upper bound to prove that u is locally bounded below.

For $R > 0$, consider the $(2n+1)$-dimensional cube

$$Q^{2n+1}(R) := \{(x,t) \in R^{2n+1} \mid \max\{|x_1|, \ldots, |x_{2n}|, |t|\} \le R\}.$$

For $k = 1, \ldots, 2n$, define inductively $\mathcal{F}^{2n+1}(R) := Q^{2n+1}(R)$, and $\mathcal{F}^{2n+1-k}(R)$ to be the set of facets of elements of $\mathcal{F}^{2n+1-(k-1)}(R)$ that are contained in a hyperplane parallel to the vertical axis. For $F \in \mathcal{F}^d(R)$ $(1 \le d \le 2n+1)$, d is the affine dimension of the facet. The proof now proceeds by induction on d.

Claim 3.1. *If $u : \mathbb{H}_n \to \mathbb{R}$ is h-convex, then u is bounded above in a neighbourhood of 0.*

Proof of Claim 3.1. In order to get an upper bound on a vertical segment of the vertical axis, we perform the same construction as in the proof of Proposition 3.4, simply observing that

$$\{x_1 \cdot e_1 + x_{n+1} \cdot e_{n+1} + t \cdot e_{2n+1} \mid x_1, x_{n+1}, t \in \mathbb{R}\} \cong \mathbb{H}_1 .$$

Following the method in the proof of Proposition 3.4, we can inductively extend the upper boundedness of u on higher dimensional facets of $Q^{2n+1}(R')$, finally reaching the full dimensional set $Q^{2n+1}(R')$. Here R' depends on R and n and is allowed to decrease in every step. □

Claim 3.2. *If $u : \mathbb{H}_n \to \mathbb{R}$ is h-convex, then u is bounded below in a neighbourhood of 0.*

Proof of Claim 3.2. To obtain a lower bound on a vertical segment of the vertical axis, we perform the same construction as in the proof of Proposition 3.4, making again use of

$$\{x_1 \cdot e_1 + x_{n+1} \cdot e_{n+1} + t \cdot e_{2n+1} \mid x_1, x_{n+1}, t \in \mathbb{R}\} \cong \mathbb{H}_1 .$$

As in the proof of Proposition 3.4, we proceed by induction and use the upper bound for u on $Q^{2n+1}(R)$ and Lemma 3.1 to obtain a lower bound for u on facets of $Q^{2n+1}(R')$ whose dimension increase in each induction step. As above, R' depends on R and n and may decrease as the dimension of the facets increases. □

The local boundedness of u follows immediately from Claim 3.1 and Claim 3.2 via left translations.

The local Lipschitz continuity of an h-convex function $u : \Omega \to \mathbb{R}$ is obtained from a short calculation which has already been performed in [10]. We include the argument here for the sake of completeness.

Recall that we want to show the local Lipschitz continuity with respect to the Heisenberg metric d_H introduced in Section 2. Thus we have to show that given $p \in \Omega$ we have $|u(q) - u(p)| \leq L\, d_H(p, q)$ for some constant L and all q in a neighbourhood of p. We can assume without loss of generality that $p = 0$ and that u is bounded on $Q^{2n+1}(R) \subseteq \Omega$ for some $0 < R < 1$. Given $(x, t) \in Q^{2n+1}(R^2/4)$ with $t \geq 0$, consider the four points

$$p_1 := (-\sqrt{t}/2, 0, \ldots, 0), \quad p_2 := (-\sqrt{t}/2, 0, \ldots, 0, \sqrt{t}, 0, \ldots, 0, t),$$
$$p_3 := (0, \ldots, 0, t) \quad \text{and} \quad p_4 := (x, t).$$

We notice that $[p_1, p_2] \subseteq \mathrm{H}_{p_1}\mathbb{H}_n \cap Q^{2n+1}(R/2)$, $[p_2, p_3] \subseteq \mathrm{H}_{p_3}\mathbb{H}_n \cap Q^{2n+1}(R/2)$ and $[p_3, p_4] \subseteq \mathrm{H}_{p_3}\mathbb{H}_n \cap Q^{2n+1}(R/2)$. By Lemma 3.3, u is Lipschitz on $[p_i, p_{i+1}]$ with a constant L depending only on R and on the bound M for u on $Q^{2n+1}(R)$ since the line $\{\lambda \cdot (p_{i+1} - p_i) \mid \lambda \in \mathbb{R}\}$ intersects the boundary of $Q^{2n+1}(R)$ in two points. We obtain the following estimate:

$$\begin{aligned}
|u(x,t) - u(0)| &\leq |u(p_4) - u(p_3)| + |u(p_3) - u(p_2)| + |u(p_2) - u(p_1)| + |u(p_1) - u(0)| \\
&\leq L(|x| + \sqrt{5}\sqrt{t}/2 + \sqrt{t + t^2} + \sqrt{t}/2) \\
&\leq L(|x| + (\sqrt{5}/2 + \sqrt{2} + 1/2)\sqrt{t}) \\
&\leq 5L\|(x,t)\|_H.
\end{aligned}$$

□

4 Weierstrass-type h-convex functions

The question of second order differentiability of h-convex functions has been addressed in [10] and [17]. In the setting of the Heisenberg groups (and more generally in Carnot groups), one considers the second order partial derivatives in the horizontal directions (see [3], [10], [17]).

Definition 4.1. Let $\Omega \subseteq \mathbb{H}_n$ be open and let $u : \Omega \to \mathbb{R}$. We say that u belongs to $BV^2_{loc,H}(\Omega)$ if the second order horizontal derivatives $X_i X_j u$, $1 \leq i, j \leq 2n$, of u exist as finite Radon measures on Ω' whenever $\Omega' \subseteq \mathbb{H}_n$ is open with $\Omega' \subset\subset \Omega$ (meaning that $\overline{\Omega'}$ is compact and that $\overline{\Omega'} \subseteq \Omega$).

It follows by standard considerations that the symmetrized mixed horizontal derivatives $X_i X_j u + X_j X_i u$ of a (locally integrable) h-convex function u are Radon measures (cf. [10, Theorem 8.1] and [17, Theorem 4.1]). Consequently, $u \in BV^2_{loc,H}(\Omega)$ iff Tu exists as a finite Radon measure on every $\Omega' \subset\subset \Omega$ as can easily be seen from the equalities

$$2X_i X_j u = (X_i X_j u + X_j X_i u) + [X_i, X_j]u$$

and

$$[X_i, X_j]u = \begin{cases} -4Tu & 1 \leq i \leq n,\ j = n+i \\ 4Tu & 1 \leq j \leq n,\ i = n+j \\ 0 & \text{otherwise} \end{cases}.$$

In two recent preprints by Garofalo-Tournier (cf. [13]) and Gutiérrez-Montanari (cf. [14]), the authors prove the following remarkable result (cf. [13, Theorem 2.4] and [14, Proposition 7.2]): Given a weakly h-convex function $u : \Omega \to \mathbb{R}$ defined on an open set Ω, with continuous second order mixed horizontal derivatives and $\Omega' \subset\subset \Omega$, there exists a constant $0 < C = C(\Omega, \Omega') < \infty$ such that

$$\int_{\Omega'} \det H_{sym} u + \frac{3}{4}(Tu)^2 \, d\mathcal{L}^3 \leq C(\underset{\Omega}{osc}\, u)^2. \tag{4.1}$$

This result actually implies that for any h-convex function $u : \Omega \to \mathbb{R}$ defined on an open set Ω and any $\Omega' \subset\subset \Omega$, Tu exists in the weak sense on Ω' and $Tu \in L^2(\Omega')$. Let us briefly indicate the argument: Fix $\Omega' \subset \Omega'' \subset \Omega$. By Theorem 1.2, u is bounded on Ω''. In particular, $\underset{\Omega''}{osc}\, u < \infty$. For sufficiently small $\epsilon > 0$, the regularization u_ϵ of u is defined in some open neighbourhood of Ω'' in Ω and converges uniformly to u on Ω'' since u is continuous in Ω (Theorem 1.2). In particular, $\underset{\Omega''}{osc}\, u_\epsilon$ is bounded by some constant times $\underset{\Omega''}{osc}\, u$, and this constant does not depend on ϵ. Now since u_ϵ is smooth and h-convex (one easily checks that the smoothing preserves the h-convexity), we have $\det H_{sym} u_\epsilon \geq 0$ by [10, Theorem 5.11], and therefore (4.1) gives

$$\int_{\Omega'} (Tu_\epsilon)^2 \, d\mathcal{L}^3 \leq C$$

for some constant $0 < C = C(\Omega, \Omega') < \infty$. By weak compactness in $L^2(\Omega')$, there exists a sequence $(\epsilon_k)_{k \in \mathbb{N}}$ decreasing to 0 and a function $g \in L^2(\Omega')$ s.t. $Tu_{\epsilon_k} \rightharpoonup g$. Hence all that is left to show is that g is the weak derivative of u in the vertical direction in Ω'. Let $\phi \in C^1_c(\Omega')$ and compute

$$\begin{aligned}\int_{\Omega'} T\phi \cdot u \, d\mathcal{L}^3 &= \lim_{k\to\infty} \int_{\Omega'} T\phi \cdot u_{\epsilon_k} \, d\mathcal{L}^3 \\ &= -\lim_{k\to\infty} \int_{\Omega'} \phi \cdot T u_{\epsilon_k} \, d\mathcal{L}^3 \\ &= -\int_{\Omega'} \phi \cdot g \, d\mathcal{L}^3.\end{aligned}$$

In particular, by the preceding observations, it follows that any h-convex function $u : \Omega \to \mathbb{R}$ is in $BV^2_{loc,H}(\Omega)$, and this makes the following result of Ambrosio and Magnani (see [3, Theorem 3.9]) available for h-convex functions: If $u \in BV^2_{loc,H}(\Omega)$, then for a.e. $x \in \Omega$ there exists a polynomial $P_{[x]}$ of homogeneous degree 2 s.t.

$$\lim_{r\downarrow 0} \frac{1}{r^2} \frac{1}{|B(x,r)|} \int_{B(x,r)} |u - P_{[x]}| = 0.$$

However, as already mentioned in the introduction, the notion of h-convexity is less rigid than its Euclidean counterpart. The results from Section 3 show the local Lipschitz continuity of h-convex functions with respect to the Heisenberg metric. Notice that this only implies a Hölder condition with exponent $\frac{1}{2}$ with respect to the Euclidean metric on vertical lines. In particular, the h-convexity does not imply the almost everywhere differentiability of the function restricted to (sparse sets of) vertical lines. In the remaining of this section, we illustrate this statement with examples, which are, in our opinion, both interesting and amusing.

¿From now on, we confine ourselves to the first Heisenberg group $\mathbb{H} = \mathbb{H}_1$, and in order to achieve a more readable notation, we write (x, y, t) instead of (x_1, x_2, t) to denote elements of $\mathbb{H}$. Thus the group operation is given by

$$(x, y, t) * (x', y', t') = (x + x', y + y', t + t' + 2(x'y - xy')).$$

We will now construct h-convex functions which have a highly irregular pointwise behaviour in the vertical direction. The first step consists in exhibiting an h-convex function whose restriction to the vertical axis is periodic. As a starting point, we consider functions of the type

$$h(x, y, t) = ((x^2 + y^2)^2 + g(t))^{\frac{1}{4}},$$

where $g : \mathbb{R} \to \mathbb{R}$ is assumed to be twice continuously differentiable and positive, and try to obtain conditions on g which ensure that the symmetrized horizontal Hessian of h is positive semidefinite. After some rather lengthy calculations with partial derivatives, we obtain

$$\operatorname{tr}(H_{sym} h) = h^{-7}\Big((1 + g'')(x^2 + y^2)^3 + (4g - 3g'^2/4 + gg'')(x^2 + y^2)\Big)$$

and

$$\det(H_{sym} h) = 3h^{-10}\Big((x^2 + y^2)^2 g(1 + g'') - 3(x^2 + y^2)^2 g'^2/4\Big).$$

Consequently, a sufficient condition for $\mathrm{tr}(H_{sym}h) \geq 0$ to hold is that

$$1 + g'' \geq 0 \quad \text{and} \quad 4g(4 + g'') \geq 3g'^2, \tag{4.2}$$

and a necessary and sufficient condition for $\det(H_{sym}h) \geq 0$ to hold is that

$$4g(1 + g'') \geq 3g'^2. \tag{4.3}$$

Summing up, we see that the following conditions on g are sufficient to guarantee that $h : \mathbb{H} \to \mathbb{R};\ h(x, y, t) = ((x^2 + y^2)^2 + g(t))^{\frac{1}{4}}$ is h-convex:

(i) $g \in C^2(\mathbb{R})$, $g > 0$ on $\mathbb{R}$.

(ii) $1 + g'' \geq 0$ on $\mathbb{R}$.

(iii) $4g(1 + g'') \geq 3g'^2$ on $\mathbb{R}$.

Observe that the periodic function

$$g : \mathbb{R} \to \mathbb{R};\ g(t) = 2 + \frac{1}{2}\sin(t)$$

satisfies these conditions.

In the following, we use the h-convex function

$$h : \mathbb{H} \to \mathbb{R};\ h(x, y, t) = \left((x^2 + y^2)^2 + 2 + \frac{1}{2}\sin(t)\right)^{\frac{1}{4}}$$

as building block for our constructions. Here is our first result:

Proposition 4.1. *There exists an h-convex function $w : \mathbb{H} \to \mathbb{R}$ which is invariant under rotations that fix the vertical axis and whose restriction to the vertical axis is nowhere differentiable.*

Proof. The idea is to perform a Weierstrass-type construction as described e.g. in [12, §11.1]. For fixed $1/2 < \beta < 1$, choose $\lambda > 2$ in such a way that

$$\frac{\lambda^{\beta-1}}{1 - \lambda^{\beta-1}} + \frac{\lambda^{-\beta}}{1 - \lambda^{-\beta}} < \epsilon, \tag{4.4}$$

where $\epsilon > 0$ is to be specified later. Let

$$f_k(x, y, t) := h \circ \delta_{(\sqrt{\lambda})^k}(x, y, t) = \left(\lambda^{2k}(x^2 + y^2)^2 + 2 + \frac{1}{2}\sin(\lambda^k t)\right)^{\frac{1}{4}}.$$

The function w is defined by

$$w(x, y, t) := \sum_{k \in \mathbb{N}} (\lambda^{-\beta})^k f_k(x, y, t).$$

It follows from the h-convexity of h and from Remark 2.1 that w is h-convex. In order to prove that w is nowhere differentiable on the vertical axis, we estimate the modulus of continuity of w there. The calculation is similar to the one in [12]. Given $t \in \mathbb{R}$, $0 < \tau < \frac{1}{\lambda}$, let $N \in \mathbb{N}$ s.t.

$$\lambda^{-(N+1)} \leq \tau < \lambda^{-N}.$$

Then

$$\Big|w(0,0,t+\tau) - w(0,0,t) - (\lambda^{-\beta})^N (f_N(0,0,t+\tau) - f_N(0,0,t))\Big| \le$$
$$\sum_{k=1}^{N-1} (\lambda^{-\beta})^k |f_k(0,0,t+\tau) - f_k(0,0,t)| + \sum_{k=N+1}^{\infty} (\lambda^{-\beta})^k |f_k(0,0,t+\tau) - f_k(0,0,t)| \le$$
$$\sum_{k=1}^{N-1} (\lambda^{-\beta})^k \lambda^k \tau + \sum_{k=N+1}^{\infty} (\lambda^{-\beta})^k \le \tau \frac{(\lambda^{1-\beta})^N}{\lambda^{1-\beta}-1} + \frac{(\lambda^{-\beta})^{N+1}}{1-\lambda^{-\beta}} \le$$
$$(\lambda^{-\beta})^N \Big(\frac{\lambda^{\beta-1}}{1-\lambda^{\beta-1}} + \frac{\lambda^{-\beta}}{1-\lambda^{-\beta}} \Big) \le (\lambda^{-\beta})^N \epsilon$$

by (4.4). On the other hand we have

$$|f_N(0,0,t+\tau) - f_N(0,0,t)| \ge c\,|\sin(\lambda^N(t+\tau)) - \sin(\lambda^N t)|$$

for some $c > 0$ not depending on t. Since $1 - 1/\lambda \ge 1/2$ and $\lambda^{-(N+1)} \le \tau < \lambda^{-N}$, τ can be chosen in this interval in such a way that

$$|f_N(0,0,t+\tau) - f_N(0,0,t)| \ge c/10.$$

Let $\epsilon := c/20$. Then, given $\lambda^{-N} \le \delta < \lambda^{-N+1}$, we can choose $\lambda^{-(N+1)} \le \tau < \lambda^{-N}$ in such a way that

$$|w(0,0,t+\tau) - w(0,0,t)| \ge \epsilon(\lambda^{-\beta})^N > \epsilon\lambda^{-\beta}\delta^\beta > C\delta^\beta$$

with some $C > 0$ independent of t and δ. In particular, the derivative of $w(0,0,\cdot)$ does not exist at any t. ☐

Remark 4.1. One can verify that

$$\sum_{k\in\mathbb{N}} (\lambda^{-\beta})^k \frac{\partial}{\partial t} f_k(x,y,t)$$

is locally uniformly convergent away from the vertical axis. This implies that the function $w(x,y,\cdot)$ is in $C^1(\mathbb{R})$ for any $(x,y) \ne (0,0)$.

Let us now restate Theorem 1.3 slightly more precisely in the following

Theorem 4.2.

(i) There exists an h-convex function $u : \mathbb{H}_1 \to \mathbb{R}$ and a set of vertical lines whose projection to the (x,y)-plane is dense in the unit square, such that the restriction of u to any of these lines is nowhere differentiable.

(ii) For any $0 < s < 1$, there exists an h-convex function u_s and a set of vertical lines whose projection to the (x,y)-plane has positive s-dimensional Hausdorff measure, such that the restriction of u_s to any of these lines is nowhere differentiable.

We now use the function given by Proposition 4.1 to prove Theorem 4.2 (i).

Proof of Theorem 4.2 (i). Let w be as in Proposition 4.1. For each $k \in \mathbb{N}$, consider the partition of the unit square Q in the (x, y)-plane in 2^{2k} closed squares $Q_{k,l}$ of side length $\frac{1}{2^k}$ each. Let $p_{k,l} = (x_{k,l}, y_{k,l}, 0)$ denote the center of $Q_{k,l}$. Clearly

$$\{p_{k,l} \mid k \in \mathbb{N},\, l \in \{1, \dots, 2^{2k}\}\}$$

is dense in the unit square. Let

$$g_{k,l} := c_{k,l} \, \|(x - x_{k,l}, y - y_{k,l})\|,$$

where $c_{k,l} > 0$ is a constant chosen in order to ensure that $g_{k,l}(x, y, t) \geq \|w \circ l_{-p_{k,l}}\|_{L^\infty(Q)}$ when $(x, y) \in Q$ and $\|(x - x_{k,l}, y - y_{k,l})\| \geq \frac{1}{2^{k+1}}$. Finally define

$$f_{k,l}(x, y, t) := \sup\{w \circ l_{-p_{k,l}}, g_{k,l}\} = \frac{1}{2}(|w \circ l_{-p_{k,l}} - g_{k,l}| + w \circ l_{-p_{k,l}} + g_{k,l}).$$

By definition of $g_{k,l}$, $f_{k,l} = g_{k,l}$ in $Q \setminus \operatorname{interior}(Q_{k,l})$.

Define u by

$$u(x, y, t) := \sum_{k \in \mathbb{N}} \frac{1}{k^2 2^{2k}} \sum_{l=1}^{2^{2k}} \frac{f_{k,l}(x, y, t)}{c_{k,l}}.$$

For fixed $K \in \mathbb{N}$ and $L \in \{1, \dots, 2^{2K}\}$, we have

$$u(x_{K,L}, y_{K,L}, t) = \sum_{k \leq K} \frac{1}{k^2 2^{2k}} \sum_{l=1}^{2^{2k}} \frac{f_{k,l}(x_{K,L}, y_{K,L}, t)}{c_{k,l}}) + \sum_{k \geq K+1} \frac{1}{k^2 2^{2k}} \sum_{l=1}^{2^{2k}} \frac{f_{k,l}(x_{K,L}, y_{K,L}, t)}{c_{k,l}}.$$

For $k \leq K$, $l \in \{1, \dots, 2^{2k}\}$, $l \neq L$, the one-sided derivatives of $f_{k,l}(x_{K,L}, y_{K,L}, \cdot)$ exist everywhere. The second sum does not depend on t, because $(x_{K,L}, y_{K,L}, 0)$ is always outside of the interior of $Q_{k,l}$. Finally, the derivative of $f_{K,L}(x_{K,L}, y_{K,L}, \cdot)$ from the right does not exist anywhere since $f_{K,L}(x_{K,L}, y_{K,L}, \cdot)$ coincides with $w(0, 0, \cdot)$. This shows that the restriction of u to $\{(x_{k,l}, y_{k,l}, t) \mid t \in \mathbb{R}\}$ is nowhere differentiable for $k \in \mathbb{N}$, $l \in \{1, \dots, 2^{2k}\}$. □

In order to obtain the family of functions $(u_s)_{0<s<1}$ appearing in the statement of Theorem 4.2 (ii), we proceed in the following way: We define a Cantor set of positive s-dimensional Hausdorff measure as a countable intersection of finite unions of closed squares. We then use left translations to the centers of these squares together with dilations to perform a Weierstrass-type construction on the whole Cantor set. The argument involves substantially more technicalities than the one used in the proof of Proposition 4.1. Let us indicate the main steps:

Proof of Theorem 4.2 (ii). Let $0 < s < 1$, $\alpha := \frac{s}{2}$ and $\frac{1}{2} + \alpha < \beta < 1$. Choose $\lambda > 2$ in such a way that

(i) $\lambda^\alpha \in \mathbb{N}$ and

(ii) $\frac{\lambda^{\beta-1}}{1-\lambda^{\beta-1}} + \frac{\lambda^{-\beta}}{1-\lambda^{-\beta}} < 10^{-3}$.

Suppose that for $k \in \mathbb{N}$ we have $(\lambda^\alpha)^k$ pairwise disjoint closed squares $Q_{k,l}$, of side length $\lambda^{-k/2}$ each, distributed in the unit square. For fixed $1 \leq l \leq (\lambda^\alpha)^k$, distribute λ^α closed squares $Q_{k+1,l'}$, of side length $\lambda^{-(k+1)/2}$ each, in $Q_{k,l}$. Clearly, when λ is sufficiently big, this can be done in such a way that the squares with the same centers as the $Q_{k+1,l'}$ but twice their side length are pairwise disjoint. Write

$$\mathcal{C}_k := \bigcup_{l=1}^{(\lambda^\alpha)^k} Q_{k,l} \quad \text{and} \quad \mathcal{C} := \bigcap_{k\in\mathbb{N}} \mathcal{C}_k.$$

Using standard arguments (cf. [18, §4.12]), one can prove that the s-dimensional Hausdorff measure of $\mathcal{C}$ is positive and finite.

Let $p_{k,l} = (x_{k,l}, y_{k,l}, 0)$ denote the center of each of the squares $Q_{k,l}$. Let

$$f(x,y,t) := \sup\left\{\left(\|(x,y)\|^4 + 2 + \frac{1}{2}\sin(t)\right)^{\frac{1}{4}}, c\,\|(x,y)\|\right\},$$

$$f_{k,l}(x,y,t) := f \circ \delta_{(\sqrt{\lambda})^k} \circ l_{-p_{k,l}}(x,y,t).$$

Here $c > 0$ is chosen in order to ensure that

$$f_{k,l}(x,y,t) = \left(\lambda^{2k}\|(x-x_{k,l}, y-y_{k,l})\|^4 + 2 + \frac{1}{2}\sin(\lambda^k(t + 2x_{k,l}y - 2xy_{k,l}))\right)^{\frac{1}{4}}$$

in $Q_{k,l}$ and

$$f_{k,l}(x,y,t) = \lambda^{k/2}\, c\,\|(x-x_{k,l}, y-y_{k,l})\|$$

outside of the square of side length $2\lambda^{-k/2}$ with center $p_{k,l}$. ($c = (\frac{7}{2})^{\frac{1}{4}}$ will do). Thus $f_{k,l}(x,y,t) = \lambda^{k/2}c\,\|(x-x_{k,l}, y-y_{k,l})\|$ on $Q_{k,l'}$ for $l' \neq l$ by choice of the $Q_{k,l}$.

The function u_s is defined as

$$u_s(x,y,t) := \sum_{k\in\mathbb{N}} \sum_{l=1}^{(\lambda^\alpha)^k} (\lambda^{-\beta})^k f_{k,l}(x,y,t).$$

We show that for a given $p = (x,y,0) \in \mathcal{C}$, the restriction of u_s to the vertical line $\{(x,y,t) \mid t \in \mathbb{R}\}$ is nowhere differentiable: Given $t \in \mathbb{R}$, $p = (x,y,0) \in \mathcal{C}$ and $0 < \tau < \frac{1}{\lambda}$, let $N \in \mathbb{N}$ s.t.

$$\lambda^{-(N+1)} \leq \tau < \lambda^{-N}.$$

Then

$$\left|u_s(x,y,t+\tau) - u_s(x,y,t) - \sum_{l=1}^{(\lambda^\alpha)^N} (\lambda^{-\beta})^N (f_{N,l}(x,y,t+\tau) - f_{N,l}(x,y,t))\right| \leq$$

$$\sum_{k=1}^{N-1}\sum_{l=1}^{(\lambda^\alpha)^k} (\lambda^{-\beta})^k |f_{k,l}(x,y,t+\tau) - f_{k,l}(x,y,t)| +$$

$$+ \sum_{k=N+1}^{\infty}\sum_{l=1}^{(\lambda^\alpha)^k} (\lambda^{-\beta})^k |f_{k,l}(x,y,t+\tau) - f_{k,l}(x,y,t)|.$$

Notice now that for fixed $k \in \mathbb{N}$, by construction,

$$|f_{k,l}(x,y,t+\tau) - f_{k,l}(x,y,t)| \neq 0$$

precisely for one $l \in \{1, \dots, (\lambda^\alpha)^k\}$. A calculation shows that this expression is bounded by 1 for $k \geq N+1$ while for $1 \leq k \leq N-1$ we obtain the bound $\lambda^k \tau$ from the mean value theorem. Whence

$$\Big| u_s(x,y,t+\tau) - u_s(x,y,t) - \sum_{l=1}^{(\lambda^\alpha)^N} (\lambda^{-\beta})^N (f_{k,l}(x,y,t+\tau) - f_{k,l}(x,y,t)) \Big| \leq$$

$$\sum_{k=1}^{N-1} (\lambda^{-\beta})^k \lambda^k \tau + \sum_{k=N+1}^{\infty} (\lambda^{-\beta})^k \leq \tau \frac{(\lambda^{1-\beta})^N}{\lambda^{1-\beta} - 1} + \frac{(\lambda^{-\beta})^{N+1}}{1 - \lambda^{-\beta}} \leq$$

$$(\lambda^{-\beta})^N \Big(\frac{\lambda^{\beta-1}}{1 - \lambda^{\beta-1}} + \frac{\lambda^{-\beta}}{1 - \lambda^{-\beta}} \Big) < (\lambda^{-\beta})^N 10^{-3}$$

by our choice of λ. Finally, we have

$$\sum_{l=1}^{(\lambda^\alpha)^N} (\lambda^{-\beta})^N (f_{N,l}(x,y,t+\tau) - f_{N,l}(x,y,t)) = (\lambda^{-\beta})^N (f_{N,l}(x,y,t+\tau) - f_{N,l}(x,y,t))$$

for some $l \in \{1, \dots, (\lambda^\alpha)^N\}$, and a computation gives the estimate

$$(\lambda^{-\beta})^N |f_{N,l}(x,y,t+\tau) - f_{N,l}(x,y,t)| \geq$$

$$(\lambda^{-\beta})^N \frac{1}{16} \cdot \Big| 1/2 \Big(\sin(\lambda^N((t+\tau) + 2x_{N,l}y - 2xy_{N,l})) - \sin(\lambda^N(t + 2x_{N,l}y - 2xy_{N,l})) \Big) \Big|.$$

Since $1 - 1/\lambda \geq 1/2$ and $\lambda^{-(N+1)} \leq \tau < \lambda^{-N}$, τ can be chosen in this interval in such a way that

$$\frac{1}{2} \Big(\sin(\lambda^N((t+\tau) + 2x_{N,l}y - 2xy_{N,l})) - \sin(\lambda^N(t + 2x_{N,l}y - 2xy_{N,l})) \Big) \geq \frac{1}{20}.$$

This yields

$$\sum_{l=1}^{(\lambda^\alpha)^N} (\lambda^{-\beta})^N (f_{N,l}(x,y,t+\tau) - f_{N,l}(x,y,t)) > 2 \cdot 10^{-3} \cdot (\lambda^{-\beta})^N.$$

Hence, given $\lambda^{-N} \leq \delta < \lambda^{-N+1}$, we can choose $\lambda^{(N+1)} \leq \tau < \lambda^{-N}$ in such a way that

$$|u_s(x,y,t+\tau) - u_s(x,y,t)| \geq 10^{-3} (\lambda^{-\beta})^N > 10^{-3} \lambda^{-\beta} \delta^\beta.$$

In particular, the derivative of $u_s(x,y,\cdot)$ does not exist at any t. □

References

[1] O. ALVAREZ – J.-M. LASRY – P.-L. LIONS, *Convex viscosity solutions and state constraints*, J. Math. Pures Appl. **9 (76)** (1997), no. 3, 265-288.

[2] L. AMBROSIO – N. FUSCO – D. PALLARA, "Functions of Bounded Variation and Free Discontinuity Problems", Oxford Mathematical Monographs, Oxford Science Publications, Clarendon Press, Oxford, 2000.

[3] L. AMBROSIO – V. MAGNANI, *Weak differentiability of BV functions on stratified groups*, to appear in Mathematische Zeitschrift.

[4] Z. BALOGH – M. RICKLY – F. SERRA CASSANO, *Comparison of Hausdorff measures with respect to the Euclidean and the Heisenberg metric*, Publicacions Matemàtiques **47** (2003), no. 1, 237-259.

[5] T. BIESKE, *On ∞-harmonic functions on the Heisenberg group*, Comm. in Partial Differential Equations **27** (2002), no. 3 & no. 4, 727-762.

[6] X. CABRE – L. CAFFARELLI, *Fully nonlinear elliptic equations*, AMS colloquium publications **43**, AMS, Providence, RI, 1995.

[7] M. CRANDALL – C. EVANS – P.-L. LIONS, *Some properties of viscosity solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc. **282 (2)** (1984), 487-502.

[8] M. CRANDALL – H. ISHII – P.-L. LIONS, *User's guide to viscosity solutions of second order partial differential equations*, Bull. of Amer. Math. Soc. **27 (1)** (1992), 1-67.

[9] J. CYGAN, *Subadditivity of homogeneous norms on certain nilpotent Lie groups*, Proc. Amer. Math. Soc. **83** (1981), 69-70.

[10] D. DANIELLI – N. GAROFALO – D.-M. NHIEU, *Notions of convexity in Carnot groups*, to appear in Communications in Analysis and Geometry.

[11] L. EVANS – R. GARIEPY, "Measure Theory and Fine Properties of Functions", Studies in Advanced Mathematics, CRC Press, 1992.

[12] K. J. FALCONER, "Fractal Geometry: Mathematical Foundations and Applications", John Wiley & Sons, 1990.

[13] N. GAROFALO – F. TOURNIER, *Monotonicity and estimates of the supremum for Monge-Ampère measures in the Heisenberg group*, Preprint.

[14] C. E. GUTIERREZ – A. MONTANARI, *Maximum and comparison principles for convex functions on the Heisenberg group*, Preprint.

[15] P. JUUTINEN – J. MANFREDI, in preparation.

[16] P. LINDQVIST – J. MANFREDI – E. SAKSMAN, *Superharmonicity of nonlinear ground states*, Rev. Math. Iberoam. **16** (2000), no. 1, 17-27.

[17] G. LU – J. MANFREDI – B. STROFFOLINI, *Convex functions on the Heisenberg group*, to appear in Calculus of Variations.

[18] P. MATTILA, "Geometry of Sets and Measures in Euclidean Spaces. Fractals and Rectifiability", Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1995.

[19] J. MANFREDI – B. STROFFOLINI, *A version of the Hopf-Lax formula in the Heisenberg group*, Comm. in Partial Differential Equations **27** (2002), 1139-1159.

[20] P. PANSU, *Métriques de Carnot-Carathéodory et quasiisométries des espaces symmétriques de rang un*, Ann. Math., **129** (1989), 1-60.