# CONVERGENCE OF EQUILIBRIA OF THIN ELASTIC PLATES UNDER PHYSICAL GROWTH CONDITIONS FOR THE ENERGY DENSITY

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Abstract. The asymptotic behaviour of the equilibrium configurations of a thin elastic plate is studied, as the thickness $h$ of the plate goes to zero. More precisely, it is shown that critical points of the nonlinear elastic functional $\mathcal{E}^h$, whose energies (per unit thickness) are bounded by $Ch^4$, converge to critical points of the $\Gamma$-limit of $h^{-4}\mathcal{E}^h$. This is proved under the physical assumption that the energy density $W(F)$ blows up as $\det F \to 0$.

## 1. Introduction

A *thin plate* is a three-dimensional body, occupying in a reference configuration a region of the form $\Omega_h := S\times(-\frac{h}{2}, \frac{h}{2})$, where the mid-surface $S$ is a bounded domain of $\mathbb{R}^2$ and the small parameter $h > 0$ measures the thickness of the plate.

The elastic behaviour of such bodies is classically described by means of two-dimensional models, which are easier to handle both from an analytical and a computational viewpoint than their three-dimensional counterparts. There exists a large variety of such theories in the literature (see [9, 18] for a survey). However, as their derivation is usually based on a priori assumptions on the form of relevant deformations, their rigorous range of validity is typically not clear. A fundamental question in elasticity is thus to justify rigorously lower dimensional models in relation to the three-dimensional theory.

Recently, a novel variational approach through $\Gamma$-convergence has led to the rigorous derivation of a hierarchy of limiting theories. Among other features, it ensures the convergence of three-dimensional minimizers to minimizers of suitable lower dimensional limit energies.

In this paper we discuss the convergence of (possibly non-minimizing) *stationary points* of the three-dimensional elastic energy, assuming *physical growth conditions* for the energy density. Previous convergence results for critical points have been obtained under unphysical assumptions on the energy density which are incompatible with the requirements of non-interpenetration of matter and preservation of orientation (see [24, 25, 26]). The validity of similar convergence results under physical growth conditions was an open question, raised in [26]. In the present contribution we prove it by combining $\Gamma$-convergence methods with an alternative first-order necessary condition for minimality introduced in [4].

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We first review the main results of the variational approach. Given a thin plate $\Omega_h$, the starting point of the variational analysis is the three-dimensional nonlinear elastic energy (scaled by unit thickness) $\mathcal{E}^h(w, \Omega_h)$ associated to a deformation $w$ of the plate. The limiting behaviour of $\mathcal{E}^h$, as the thickness of the plate tends to zero, can be described by the Γ-limit $\mathcal{I}_\beta$ of the functionals

$$h^{-\beta}\mathcal{E}^h(\cdot, \Omega_h),$$

as $h \to 0$, for a given scaling $\beta \geq 0$. As mentioned above, this implies, roughly speaking, convergence of minimizers $w^h$ of $\mathcal{E}^h(\cdot, \Omega_h)$ (subject to applied forces or boundary conditions) to minimizers of the two-dimensional energy $\mathcal{I}_\beta$, provided $\mathcal{E}^h(w^h, \Omega_h) \leq Ch^\beta$. For the definition and main properties of Γ-convergence we refer to the monographs [7, 13].

In this setting Γ-convergence was first proved by Le Dret and Raoult in [16] for the scaling $\beta = 0$, under additional growth conditions from above on the energy density. This led to a rigorous justification of the *nonlinear membrane theory*. This work was then extended to energy densities satisfying weaker growth conditions in [5]. In the seminal papers [14, 15] Friesecke, James, and Müller established Γ-convergence for all $\beta \geq 2$. The scaling $\beta = 2$ corresponds in the limit to the *Kirchhoff plate theory*, while $\beta = 4$ to the *von Kármán plate theory*. For $\beta > 4$ the usual linear theory is derived, while the intermediate scalings $2 < \beta < 4$ relate to a linear theory with constraints. The case of $0 < \beta < 5/3$ was recently solved by Conti and Maggi [12]. The regime $5/3 \leq \beta < 2$ remains open and is conjectured to be relevant for crumpling of elastic sheets. Analogous results have been proved for thin rods in [1, 21, 22]. Other related results concern derivation of limiting theories for incompressible plates [10, 11, 29], heterogeneous films [27], and multiphase materials [6, 8, 23, 28].

The intent of this paper is to investigate the convergence of stationary points of the three-dimensional nonlinear elastic energy (subject to applied forces and boundary conditions) to stationary points of the Γ-limit functional. The first result concerning convergence of equilibria for thin bodies has been shown in [24], in the case of a thin strip and for the scaling $\beta = 2$. Using the same technique, this work has been extended in [25] to the case of a thin rod in the regime $\beta = 2$, and then in [26] to a thin plate in the von Kármán regime $\beta = 4$ (see also [17] for an extension to thin shells). A crucial assumption in all these papers is that the elastic energy density $W$ is differentiable everywhere and that its derivative satisfies a linear growth condition. This assumption is unsatisfactory from both a physical and a mathematical point of view. Indeed, the bound on $DW$ prevents the blow-up of $W(F)$ as the determinant of the deformation gradient $F$ tends to zero (corresponding to total compression), which is a natural assumption in elasticity. Moreover, it implies, together with the other assumptions on $W$, that $W(F)$ must be essentially of the form $\text{dist}^2(F, SO(3))$. We point out that, instead, the results in [14, 15], as well as the ones in [21, 22], do not require any bound from above on $W$. On the other hand, without assuming a linear growth condition on $DW$, it is not even clear to which extent minimizers satisfy the Euler-Lagrange equations in the usual form (see (2.10) below).

A growth condition on $W$, which is compatible with the blow-up condition as $\det F \to 0$ is:

$$\left|DW(F)F^T\right| \le k(W(F)+1) \tag{1.1}$$

for every $F$ with $\det F > 0$. In [3, 4] Ball has shown that, under assumption (1.1), it is possible to derive an alternative first-order necessary condition for minimizers (Theorem 2.1). We underline that, when minimizers are invertible, this condition is equivalent to the Eulerian (spatial) formulation of the classical equilibrium equations of elasticity (see Remark 2.2).

In this paper we focus on the scalings $\beta \ge 4$ and we consider an elastic energy density $W$ satisfying the physical growth condition (1.1). We call a deformation a stationary point of the three-dimensional energy if it satisfies the first-order necessary condition introduced by Ball in [3, 4] (Definition 2.3). In Theorem 3.1 we prove that any sequence of stationary points $w^h$ of the three-dimensional energy, satisfying $\mathcal{E}^h(w^h,\Omega_h) \le Ch^\beta$, $\beta \ge 4$, converges to a stationary point of the corresponding limiting functional (i.e., to a solution of the classical Euler-Lagrange equations of the von Kármán functional if $\beta = 4$, and of the functional of linear plate theory if $\beta > 4$). This is the first result of convergence of equilibria for thin plates compatible with the physical requirement that $W(F) \to +\infty$ as $\det F \to 0$.

A first key ingredient in the proof of our main result is the quantitative rigidity estimate proved by Friesecke, James, and Müller in [14, Theorem 3.1]. It is first used to deduce compactness of sequences of stationary points from the bound on the elastic energy, and then to define suitable strain-like and stress-like variables $G^h$ and $E^h$ (see (4.9) and (4.13)). In order to derive the limiting Euler-Lagrange equations, some compactness properties for the sequence $(E^h)$ are needed. A new difficulty with respect to the previous works [24, 26] is that the $L^2$-bound on the strains $G^h$ (which is a direct consequence of the rigidity estimate) does not imply an analogous bound on the stresses $E^h$ anymore. Indeed, in our setting the stresses $E^h$ turn out to be naturally defined as

$$E^h = \frac{1}{h^2} DW(\mathrm{Id} + h^2 G^h)(\mathrm{Id} + h^2 G^h)^T \tag{1.2}$$

($E^h$ can be interpreted as a sort of Cauchy stress tensor, read in the undeformed configuration, see also Remark 2.2). Hence, using the growth condition (1.1) and the bound on the elastic energy we can only deduce weak compactness of $E^h$ in $L^1$ and this convergence is not enough to pass to the limit in the three-dimensional Euler-Lagrange equations (see Steps $2-3$ of the proof and the discussion therein).

This difficulty is overcome by identifying a sequence of measurable sets $B_h$, which converge in measure to the whole set $\Omega := S\times(-\frac{1}{2},\frac{1}{2})$ and satisfy the following properties. On $B_h$ the remainder in the first order Taylor expansion of $DW$ around the identity is uniformly controlled with respect to $h$, so that one can deduce an $L^2$ bound for $E^h$ from (1.2) and from the $L^2$ bound on $G^h$. On the complement of $B_h$ one can use the growth condition (1.1) to show that the contribution of $E^h$ on this set is negligible at the limit in the $L^1$ norm. This mixed type of convergence of the stresses is then shown to be sufficient to pass to the limit in the three-dimensional Euler-Lagrange equations.

Another crucial difference with respect to [24, 26] is that the admissible test functions in the weak formulation of Ball's stationarity condition must be uniformly bounded. This requires to introduce an ad hoc truncation argument, which is completely new (see Step 6 of the proof).

This paper gives a positive answer to a question raised in [26], where the authors suggest to investigate whether the analysis developed in [24, 26] can be extended to the case of energy densities $W$ satisfying physical growth conditions. They actually propose to look at an alternative stationarity condition from the one we consider, also introduced by Ball in [4], which involves the so-called energy-momentum tensor and which is valid if $W$ satisfies an analogous (but stronger) growth condition to (1.1). However, as explained in Remark 2.4, this second stationarity condition does not provide precise information about the boundary behaviour of the limiting quantities and, therefore, is not suitable for the purpose of our analysis.

Convergence results for thin plates in the Kirchhoff regime $\beta = 2$ and in the intermediate scalings $2 < \beta < 4$ are still open, even under the simplifying assumption of linear growth of $DW$. The additional difficulties in the analysis of these regimes are due to the weaker compactness properties arising from the rigidity theorem and to the presence, in the limiting model, of a nonlinear or geometrically linear isometry constraint.

The plan of the paper is the following. In Section 2 we describe the setting of the problem and we discuss the first order necessary condition by Ball. Section 3 contains the statement of the main result, which is proved in Section 4.

## 2. Setting of the problem

We consider a thin plate, whose reference configuration is given by the set $\Omega_h = S\times(-\frac{h}{2}, \frac{h}{2})$, where $S \subset \mathbb{R}^2$ is a bounded domain with Lipschitz boundary and $h > 0$.

Deformations of the plate are described by maps $w\colon \Omega_h \to \mathbb{R}^3$, which are assumed to belong to the space $H^1(\Omega_h;\mathbb{R}^3)$. Moreover, we require the deformations $w$ to satisfy the boundary condition

$$w(z) = z \quad \text{for every } z \in \Gamma\times(-\tfrac{h}{2}, \tfrac{h}{2}), \tag{2.1}$$

where $\Gamma$ is a (non-empty) relatively open subset of $\partial S$.

To any deformation $w \in H^1(\Omega_h;\mathbb{R}^3)$ we associate the total energy (per unit thickness) defined as

$$\mathcal{F}^h(w) = \frac{1}{h}\int_{\Omega_h} W(\nabla w)\, dz - \frac{1}{h}\int_{\Omega_h} f^h \cdot w\, dz, \tag{2.2}$$

where $f^h \in L^2(\Omega_h;\mathbb{R}^3)$ is the density of a body force applied to $\Omega_h$.

On the stored-energy density $W\colon \mathbb{M}^{3\times3} \to [0,+\infty]$ we require the following asssumptions:

$$W \text{ is of class } C^1 \text{ on } \mathbb{M}^{3\times3}_+; \tag{2.3}$$

$$W(F) = +\infty \quad \text{if } \det F \le 0, \qquad W(F) \to +\infty \quad \text{as } \det F \to 0^+; \tag{2.4}$$

$$W(RF) = W(F) \quad \text{for every } R \in SO(3),\ F \in \mathbb{M}^{3\times3} \quad \text{(frame indifference)}. \tag{2.5}$$

Here $\mathbb{M}^{3\times3}_+$ denotes the set of matrices $F \in \mathbb{M}^{3\times3}$ with $\det F > 0$, while $SO(3)$ denotes the set of proper rotations $\{R \in \mathbb{M}^{3\times3}:\ R^TR = \mathrm{Id},\ \det R = 1\}$. Condition (2.4) is related to the physical requirements of non-interpenetration of matter and preservation of orientation. It ensures local invertibility of $C^1$ deformations with finite energy.

We also require $W$ to have a single well at $SO(3)$, namely

$$W = 0 \quad \text{on } SO(3); \tag{2.6}$$

$$W(F) \geq C\text{dist}^2(F, SO(3)); \tag{2.7}$$

$$W \text{ is of class } C^2 \text{ in a } \delta\text{-neighbourhood of } SO(3). \tag{2.8}$$

Finally, we assume the following growth condition:

$$\left|DW(F)F^T\right| \leq k(W(F) + 1) \quad \text{for every } F \in \mathbb{M}^{3\times 3}_+. \tag{2.9}$$

This is a mild growth condition on $W$, introduced by Ball in [3, 4], which is compatible with the physical requirement (2.4), but is nevertheless sufficient to derive a first-order condition for minimizers of $\mathcal{F}^h$. In fact, by performing external variations $w + \varepsilon\phi$ of a minimizer $w$, one is formally led to the Euler-Lagrange equations in the usual form

$$\int_{\Omega_h} DW(\nabla w) \cdot \nabla\phi \, dz = \int_{\Omega_h} f^h \cdot \phi \, dz \quad \forall\phi \text{ smooth with } \phi|_{\Gamma\times(-\frac{h}{2},\frac{h}{2})} = 0. \tag{2.10}$$

To justify rigorously this derivation, one has to require that either $DW$ is Lipschitz continuous or the minimizer $w$ belongs to $W^{1,\infty}$ and satisfies a stronger orientation preserving condition, namely $\det \nabla w \geq c > 0$ a.e. in $\Omega_h$. However, none of these assumptions is satisfactory: the Lipschitz continuity of $DW$ is incompatible with (2.4), while there may exist minimizers that do not belong to $W^{1,\infty}$ or do not satisfy the stronger orientation preserving condition (see the discussion in [4, Section 2.4]). In other words, it is not possible in general to guarantee the integrability of the term $DW(\nabla w)$ and thus, to give a rigorous meaning to equation (2.10).

If instead condition (2.9) is assumed, then it is possible to derive an alternative equilibrium equation for minimizers. More precisely, by considering variations of the form $w + \varepsilon\, \phi \circ w$ one can deduce the following condition.

**Theorem 2.1** ([4, Theorem 2.4])**.** *Assume that $W$ satisfies* (2.3)*,* (2.4)*, and* (2.9)*. Let $U \subset \mathbb{R}^3$ be a bounded domain with a Lipschitz boundary $\partial U = \partial U_1 \cup \partial U_2 \cup N$, where $\partial U_1$, $\partial U_2$ are disjoint relatively open subsets of $\partial U$ and $N$ has zero two-dimensional measure. Let $\bar{w} \in H^{1/2}(\partial U; \mathbb{R}^3)$ and $f \in L^2(U; \mathbb{R}^3)$. Let $w \in H^1(U; \mathbb{R}^3)$ be a local minimizer of the functional*

$$\mathcal{F}(w) := \int_U W(\nabla w) \, dz - \int_U f \cdot w \, dz$$

*subject to the boundary condition $w = \bar{w}$ on $\partial U_1$, that is, there exists $\varepsilon > 0$ such that $\mathcal{F}(w) \leq \mathcal{F}(v)$ for every $v \in H^1(U; \mathbb{R}^3)$ satisfying $\|v - w\|_{H^1} \leq \varepsilon$ and $v = \bar{w}$ on $\partial U_1$. Then*

$$\int_U DW(\nabla w)(\nabla w)^T : \nabla\phi(w) \, dz = \int_U f \cdot \phi(w) \, dz \tag{2.11}$$

*for all $\phi \in C^1_b(\mathbb{R}^3; \mathbb{R}^3)$ such that $\phi \circ w = 0$ on $\partial U_1$ in the sense of trace.*

In the theorem above and in what follows, given a subset $U$ of $\mathbb{R}^n$ we denote by $C^k_b(U)$ the space of functions of class $C^k$ that are bounded in $U$, with bounded derivatives up to the $k$-th order. We also stress that in (2.11) the term $\nabla\phi(w)$ denotes the gradient of $\phi$ computed at the point $w(z)$.

**Remark 2.2.** Under the assumptions of Theorem 2.1, if in addition $w$ is a smooth homeomorphism of $U$ onto $U' := w(U)$, then equation (2.11) reduces by means of a change of variables to

$$\int_{w(U)} T(w^{-1}(x)) : \nabla\phi(x)\,dx = \int_{w(U)} \tilde f(w^{-1}(x)) \cdot \phi(x)\,dx$$

for all $\phi \in C^1(\mathbb{R}^3;\mathbb{R}^3)$ such that $\phi|_{w(\partial U_1)} = 0$. In the formula above $T$ is the Cauchy stress tensor:

$$T(z) = (\det \nabla w(z))^{-1} DW(\nabla w(z))(\nabla w(z))^T, \qquad z \in U$$

and $\tilde f = (\det \nabla w)^{-1} f$ (see [4, Theorem 2.6]). In other words, Theorem 2.1 asserts that the equilibrium equations are satisfied in the deformed configuration.

In our setting it is natural to assume (2.11) as definition of stationary points of $\mathcal{F}^h$. Our aim is to analyse their limit behaviour, as the thickness $h$ goes to 0. To do so, it is convenient to perform a change of variables and to reduce to a fixed domain independent of $h$. Thus, we consider the scaling $(z', z_3) = (x', hx_3)$, $\nabla_h = \big(\nabla', \frac{1}{h}\partial_3\big)$, $y(x) = w(z)$, and $g^h(x) = f^h(z)$, and we introduce the functional

$$\mathcal{J}^h(y) = \mathcal{F}^h(w) = \int_\Omega W(\nabla_h y)\,dx - \int_\Omega g^h \cdot y\,dx, \tag{2.12}$$

where $\Omega = S\times(-\frac12, \frac12)$ and the scaled deformation $y \in H^1(\Omega;\mathbb{R}^3)$ satisfies the boundary condition

$$y(x) = (x', hx_3) \quad \text{for every } x = (x', x_3) \in \Gamma\times(-\tfrac12, \tfrac12). \tag{2.13}$$

According to Theorem 2.1, we give the following definition.

**Definition 2.3.** We say that a deformation $y \in H^1(\Omega;\mathbb{R}^3)$ is a stationary point of $\mathcal{J}^h$, subject to clamped boundary conditions on $\Gamma\times(-\frac12, \frac12)$, if $y(x) = (x', hx_3)$ for every $x \in \Gamma\times(-\frac12, \frac12)$ and the following equation is satisfied:

$$\int_\Omega DW(\nabla_h y)(\nabla_h y)^T : \nabla\phi(y)\,dx = \int_\Omega g^h \cdot \phi(y)\,dx$$

for all $\phi \in C^1_b(\mathbb{R}^3;\mathbb{R}^3)$ satisfying $\phi(x', hx_3) = 0$ for every $x \in \Gamma\times(-\frac12, \frac12)$.

**Remark 2.4.** In [4] Ball has shown that, if $W$ satisfies the growth condition

$$\big|F^T DW(F)\big| \le k(W(F) + 1) \quad \text{for every } F \in \mathbb{M}^{3\times3}_+$$

(which implies, but is not equivalent to (2.9), see [4, Proposition 2.3]), then local minimizers of $\mathcal{F}$ satisfy the equation

$$\int_U \big(W(\nabla w)\,\mathrm{Id} - (\nabla w)^T DW(\nabla w)\big) : \nabla\phi\,dz = \int_U (\nabla w)^T f \cdot \phi\,dz$$

for all $\phi \in C^1_0(U;\mathbb{R}^3)$. This equation is obtained by performing internal variations of the form $w\circ\psi_\varepsilon$, with $\psi_\varepsilon^{-1}(x) = x + \varepsilon\phi(x)$, and can be viewed as a multi-dimensional version of the classical Du Bois-Raymond equation of the one-dimensional calculus of variations. To the purpose of our analysis the use of this equilibrium equation in place of (2.11) seems to be less convenient. Indeed, the requirement of zero boundary values for the test functions suggests that the equation does not provide precise information about the boundary behaviour of the limiting quantities. Moreover, it imposes a severe restriction on the choice of admissible test functions.

**Remark 2.5.** If $W$ satisfies (2.9), then $W$ has polynomial growth, that is, there exists $s > 0$ such that

$$W(F) \leq C(|F|^s + |F^{-1}|^s) \quad \text{for all } F \in \mathbb{M}^{3\times 3}_+$$

(see [4, Proposition 2.7]). In particular, examples of functions satisfying (2.3)–(2.9) are:

$$W(F) = |(F^T F)^{1/2} - \mathrm{Id}|^2 + |\log \det F|^p \qquad \text{for } F \in \mathbb{M}^{3\times 3}_+,$$

or

$$W(F) = |(F^T F)^{1/2} - \mathrm{Id}|^2 + \Big|\frac{1}{\det F} - 1\Big|^p \qquad \text{for } F \in \mathbb{M}^{3\times 3}_+,$$

where $p > 1$ and $W$ is intended to be $+\infty$ if $\det F \leq 0$.

## 3. Statement of the main result

In this paper we focus on the asymptotic study of stationary points $y^h$ of $\mathcal{J}^h$ (according to Definition 2.3) with elastic energy (per unit thickness) of order $h^\beta$ with $\beta \geq 4$, that is,

$$\int_\Omega W(\nabla_h y^h)\, dx \leq C h^\beta, \qquad \beta \geq 4. \tag{3.1}$$

For simplicity we assume that the body forces $g^h$ are independent of the variable $x_3$ and normal to the mid-surface of the plate; more precisely, we assume $g^h(x) = h^{(\beta+2)/2} g(x') e_3$, where $g \in L^2(S)$ is given. The scaling $h^{(\beta+2)/2}$ of the normal force ensures consistency with the elastic energy scaling (3.1).

In [15] Friesecke, James, and Müller have identified the limit of the functionals $h^{-\beta}\mathcal{J}^h$, in the sense of $\Gamma$-convergence, under the assumptions (2.5)–(2.8). For $\beta = 4$ the $\Gamma$-limit $\mathcal{J}_{\mathrm{vK}}$ can be expressed in terms of the averaged in-plane and out-of-plane displacements $u$ and $v$ (see (3.13)) and is given by

$$\mathcal{J}_{\mathrm{vK}}(u, v) = \mathcal{I}_{\mathrm{vK}}(u, v) - \int_S g v \, dx', \tag{3.2}$$

where, for $u \in H^1(S; \mathbb{R}^2)$ and $v \in H^2(S)$, the von Kármán functional $\mathcal{I}_{\mathrm{vK}}$ is defined as

$$\mathcal{I}_{\mathrm{vK}}(u, v) = \frac{1}{2}\int_S Q_2\Big(\mathrm{sym}\, \nabla' u + \frac{1}{2}\nabla' v \otimes \nabla' v\Big)\, dx' + \frac{1}{24}\int_S Q_2((\nabla')^2 v)\, dx'. \tag{3.3}$$

The limit density $Q_2$ is a quadratic form that can be computed through the following minimization procedure. Given the quadratic form of linearized elasticity $Q_3(F) = D^2 W(\mathrm{Id}) F : F$ on $\mathbb{M}^{3\times 3}$, we define the quadratic form $Q_2$ on $\mathbb{M}^{2\times 2}$ as

$$Q_2(G) = \mathcal{L}_2 G : G := \min_{F'' = G} Q_3(F), \tag{3.4}$$

where $F''$ denotes the $2\times 2$ submatrix given by $F''_{ij} = F_{ij}$, $1 \leq i, j \leq 2$.

For $\beta > 4$ the $\Gamma$-limit $\mathcal{J}_{\mathrm{lin}}$ depends only on the averaged out-of-plane displacement $v$ and is given by

$$\mathcal{J}_{\mathrm{lin}}(v) = \mathcal{I}_{\mathrm{lin}}(v) - \int_S g v \, dx', \tag{3.5}$$

where $\mathcal{I}_{\mathrm{lin}}$ is the functional of linear plate theory, defined as

$$\mathcal{I}_{\mathrm{lin}}(v) = \frac{1}{24}\int_S Q_2((\nabla')^2 v)\, dx'. \tag{3.6}$$

for every $v \in H^2(S)$.

The Γ-convergence result guarantees, in particular, that given a minimizing sequence $y^h$ satisfying

$$\limsup_{h\to 0} \frac{1}{h^\beta}\big(\mathcal{J}^h(y^h) - \inf \mathcal{J}^h\big) = 0,$$

the averaged in-plane and out-of-plane displacements associated with $y^h$ converge to a minimizer $(u,v)$ of $\mathcal{J}_{\rm vK}$ if $\beta = 4$. If $\beta > 4$, they converge to a pair of the form $(0,v)$, where $v$ is a minimizer of $\mathcal{J}_{\rm lin}$.

To set the stage for our result on the convergence of equilibria, we derive the Euler-Lagrange equations for a minimizer $(u,v)$ of $\mathcal{J}_{\rm vK}$. First of all, from the clamped boundary conditions (2.13) it follows that the limiting displacement $(u,v)$ satisfies

$$u(x') = 0 \quad \text{and} \quad v(x') = 0,\ \nabla' v(x') = 0 \quad \text{for every } x' \in \Gamma. \tag{3.7}$$

By performing the variations of $\mathcal{J}_{\rm vK}$ in $u$ and $v$, respectively, we deduce the following Euler-Lagrange equations in weak form:

$$\begin{aligned}\int_S \Big(\mathcal{L}_2\big(\mathrm{sym}\,\nabla' u + \frac{1}{2}\nabla' v\otimes\nabla' v\big) : (\nabla' v \otimes \nabla'\varphi) \\ + \frac{1}{12}\mathcal{L}_2((\nabla')^2 v) : (\nabla')^2\varphi - g\varphi\Big)\, dx' = 0\end{aligned} \tag{3.8}$$

for every $\varphi \in H^2(S)$ with $\varphi|_\Gamma = 0$, $\nabla'\varphi|_\Gamma = 0$, and

$$\int_S \mathcal{L}_2\big(\mathrm{sym}\,\nabla' u + \frac{1}{2}\nabla' v\otimes\nabla' v\big) : \nabla'\psi\, dx' = 0 \tag{3.9}$$

for every $\psi \in H^1(S;\mathbb{R}^2)$ with $\psi|_\Gamma = 0$.

In the case of the linear functional $\mathcal{J}_{\rm lin}$ the limit displacement $v$ satisfies the boundary conditions

$$v(x') = 0, \quad \nabla' v(x') = 0 \quad \text{for every } x' \in \Gamma \tag{3.10}$$

and the Euler-Lagrange equations are given by

$$\int_S \Big(\frac{1}{12}\mathcal{L}_2((\nabla')^2 v) : (\nabla')^2\varphi - g\varphi\Big)\, dx' = 0 \tag{3.11}$$

for every $\varphi \in H^2(S)$ with $\varphi|_\Gamma = 0$, $\nabla'\varphi|_\Gamma = 0$,

From now on we will adopt the notation $y = (y', y_3)$.

The main result of the paper is the following.

**Theorem 3.1.** *Assume that the energy density $W$ satisfies* (2.3)–(2.9)*. Let $\beta \geq 4$. Let $(y^h)$ be a sequence of stationary points of $\mathcal{J}^h$ according to Definition 2.3, with $g^h(x) = h^{(\beta+2)/2} g(x') e_3$. Assume further that*

$$\int_\Omega W(\nabla_h y^h)\, dx \leq C h^\beta. \tag{3.12}$$

*Set*

$$\begin{aligned} u^h(x') &:= \frac{1}{h^{\beta/2}} \int_{-\frac12}^{\frac12} \big((y^h)'(x',x_3) - x'\big)\, dx_3, \\ v^h(x') &:= \frac{1}{h^{(\beta-2)/2}} \int_{-\frac12}^{\frac12} y_3^h(x',x_3)\, dx_3 \end{aligned} \tag{3.13}$$

*for every* $x' \in S$*. Then the following assertions hold.*

(i) *(von Kármán regime) Assume* $\beta = 4$*. Then, there exist* $u \in H^1(S;\mathbb{R}^2)$ *and* $v \in H^2(S)$ *such that, up to subsequences,*

$$u^h \rightharpoonup u \quad \textit{weakly in } H^1(S;\mathbb{R}^2) \tag{3.14}$$

*and*

$$v^h \to v \quad \textit{strongly in } H^1(S), \tag{3.15}$$

*as* $h \to 0$*, and the limit displacement* $(u, v)$ *solves* (3.8)–(3.9)*, and satisfies the boundary conditions* (3.7)*.*

(ii) *(linear regime) Assume* $\beta > 4$*. Then,* (3.14) *and* (3.15) *hold with* $u = 0$*, and the limit displacement* $v$ *solves* (3.11) *and satisfies the boundary conditions* (3.10)*.*

**Remark 3.2.** If $y^h$ is a sequence of minimizers of $\mathcal{J}^h$ with $g^h(x) = h^{(\beta+2)/2} g(x')e_3$, then condition (3.12) is automatically satisfied. This can be proved by means of a Poincaré-like inequality related to the rigidity theorem by Friesecke, James, and Müller (see the proof of [15, Theorem 2, part iii]).

**Remark 3.3.** In [19] Mielke used a centre manifold approach to compare solutions in a thin strip to a one-dimensional problem. This method works already for finite $h$, but it requires that the nonlinear strain $(\nabla_h y)^T \nabla_h y$ is close to the identity in $L^\infty$. Applied forces $g$ are also difficult to include. We also mention a more recent result by Monneau [20], based on a careful use of the implicit function theorem. Given a sufficiently smooth and small solution of the von Kármán equations, he proves the existence of a nearby solution of the three-dimensional problem.

**Remark 3.4.** In Theorem 3.1 we assume that a sequence of stationary points $(y^h)$ exists. Under additional assumptions on $W$ (such as, e.g., polyconvexity, see [2]) one can prove existence of minimizers of $\mathcal{J}^h$ and, therefore, of stationary points. For general $W$, proving the existence of stationary points (according to Definition 2.3 or to the classical formulation (2.10)) is a difficult issue in elasticity. We refer to [4, Section 2.7] for a discussion of results in this direction.

## 4. Proof of Theorem 3.1

This section is devoted to the proof of the main result of the paper. As mentioned in the introduction, our result substantially improves the previous result obtained in [26].

For the reader's convenience, we intentionally use the same structure of the proof as used in [26] (and before in [24, 25]). Nevertheless, major differences arise in the proof of every step (except for Steps 1, 2, and 7, where we explicitly refer to previous works). These differences are due to the different notion of Euler-Lagrange equation and to the corresponding different definition of stress.

*Proof of Theorem 3.1.* Let $\beta \geq 4$. For notational convenience we set

$$\alpha := (\beta + 2)/2,$$

so that $\alpha \geq 3$. Let $(y^h)$ be a sequence of stationary points of $\mathcal{J}^h$, i.e., suppose that

$$\int_\Omega DW(\nabla_h y^h)(\nabla_h y^h)^T : \nabla \phi(y^h)\, dx = \int_\Omega h^\alpha g e_3 \cdot \phi(y^h)\, dx \tag{4.1}$$

for all $\phi \in C^1_b(\mathbb{R}^3;\mathbb{R}^3)$ satisfying $\phi(x', hx_3) = 0$ for every $x \in \Gamma\times(-\frac{1}{2}, \frac{1}{2})$. Furthermore, assume that condition (3.12) is fulfilled.

*Step 1. Compactness of the displacements.* The energy bound (3.12) and the coercivity condition (2.7) imply that

$$\int_\Omega \mathrm{dist}^2(\nabla_h y^h, SO(3))\, dx \leq Ch^{2\alpha-2}.$$

Owing to the rigidity estimate [14, Theorem 3.1], this bound guarantees the existence of a sequence of smooth rotations $R^h$, whose $L^2$ distance from $\nabla_h y^h$ is of order $h^{\alpha-1}$. A careful analysis of the increment of $R^h$ in neighbouring squares of side $h$ shows that the gradient of $R^h$ is well controlled in terms of $h$. From this it follows that $\nabla_h y^h$ must converge to a constant rotation (namely, the identity, because of the boundary condition) and that the in-plane and out-of-plane displacements satisfy the compactness properties (3.14) and (3.15), respectively.

More precisely, arguing as in [15, Theorem 6 and Lemma 1], one can construct a sequence $(R^h) \subset C^\infty(S;\mathbb{M}^{3\times3})$ such that $R^h(x') \in SO(3)$ for every $x' \in S$ and

$$\|\nabla_h y^h - R^h\|_{L^2} \leq Ch^{\alpha-1}, \tag{4.2}$$

$$\|\nabla' R^h\|_{L^2} \leq Ch^{\alpha-2}, \tag{4.3}$$

$$\|R^h - \mathrm{Id}\|_{L^2} \leq Ch^{\alpha-2}. \tag{4.4}$$

From (4.2) and (4.4) it follows that $\nabla_h y^h$ converge to Id strongly in $L^2(\Omega;\mathbb{M}^{3\times3})$; in particular, $\nabla y^h \to \mathrm{diag}\{1, 1, 0\}$ strongly in $L^2(\Omega;\mathbb{M}^{3\times3})$. Therefore, by the boundary condition $y^h(x', x_3) = (x', hx_3)$ for every $x \in \Gamma\times(-\frac{1}{2}, \frac{1}{2})$ and the Poincaré inequality, we have that

$$y^h \to (x', 0) \quad \text{strongly in } H^1(\Omega;\mathbb{R}^3). \tag{4.5}$$

By [15, Lemma 1] there exist $u \in H^1(S;\mathbb{R}^2)$ and $v \in H^2(S)$ such that (3.14) and (3.15) hold true, up to subsequences. From the boundary condition satisfied by $y^h$ we obtain immediately that $u(x') = 0$ and $v(x') = 0$ for every $x' \in \Gamma$. Moreover, by [15, Corollary 1] the first moment $\xi^h$ of the in-plane displacement satisfies

$$\xi^h(x') := \frac{1}{h^{\alpha-1}} \int_{-\frac{1}{2}}^{\frac{1}{2}} x_3\big((y^h)'(x', x_3) - x'\big) dx_3 \rightharpoonup -\frac{1}{12}\nabla' v \quad \text{weakly in } H^1(S;\mathbb{R}^2).$$

As $\xi^h = 0$ on $\Gamma$ for every $h$, this implies $\nabla' v = 0$ on $\Gamma$. Finally, [15, Lemma 1] guarantees the following convergence properties for $R^h$:

$$A^h := \frac{R^h - \mathrm{Id}}{h^{\alpha-2}} \rightharpoonup A := -(\nabla' v, 0) \otimes e_3 + e_3 \otimes (\nabla' v, 0) \quad \text{in } H^1(S;\mathbb{M}^{3\times3}) \tag{4.6}$$

and

$$\mathrm{sym}\, \frac{R^h - \mathrm{Id}}{h^{2\alpha-4}} \to \frac{A^2}{2} \quad \text{in } L^q(S;\mathbb{M}^{3\times3}), \quad \forall q < \infty. \tag{4.7}$$

In particular, by the Poincaré-Wirtinger inequality and the equations (4.2) and (4.7), we obtain

$$\Big\| \frac{y_3^h}{h} - x_3 - h^{\alpha-3} v^h \Big\|_{L^2} \leq C \Big\| \frac{\partial_3 y_3^h}{h} - 1 \Big\|_{L^2} \leq Ch^{\alpha-1}. \tag{4.8}$$

*Step 2. Definition of the scaled strain and stress.* The bound (4.2) suggests the following decomposition for the deformation gradients:

$$\nabla_h y^h = R^h(\mathrm{Id} + h^{\alpha-1}G^h). \tag{4.9}$$

By (4.2) the $G^h\colon \Omega \to \mathbb{M}^{3\times3}$ are bounded in $L^2(\Omega;\mathbb{M}^{3\times3})$. Thus, up to subsequences, $G^h \rightharpoonup G$ weakly in $L^2(\Omega;\mathbb{M}^{3\times3})$ for some $G \in L^2(\Omega;\mathbb{M}^{3\times3})$. By [15, Lemma 2] the limiting strain $G$ satisfies

$$G''(x', x_3) = G_0(x') - x_3(\nabla')^2 v, \tag{4.10}$$

where

$$\mathrm{sym}\, G_0 = \mathrm{sym}\, \nabla' u + \frac{1}{2}\nabla' v \otimes \nabla' v \qquad \text{if } \alpha = 3, \tag{4.11}$$

$$\mathrm{sym}\, G_0 = \mathrm{sym}\, \nabla' u \qquad \text{if } \alpha > 3. \tag{4.12}$$

We recall that $G''$ denotes the $2\times2$ submatrix $G''_{ij} = G_{ij}$, $1 \le i, j \le 2$.

Let $E^h\colon \Omega \to \mathbb{M}^{3\times3}$ be the scaled stress defined by

$$E^h := \frac{1}{h^{\alpha-1}} DW(\mathrm{Id} + h^{\alpha-1}G^h)(\mathrm{Id} + h^{\alpha-1}G^h)^T. \tag{4.13}$$

Notice that $E^h$ is symmetric, due to the frame indifference of $W$. Moreover, the following estimate holds true:

$$|E^h| \le C\Big(\frac{W(\mathrm{Id} + h^{\alpha-1}G^h)}{h^{\alpha-1}} + |G^h|\Big). \tag{4.14}$$

Indeed, if $h^{\alpha-1}|G^h| \le \delta/2$, where $\delta$ is the size of the neighbourhood in (2.8), then

$$DW(\mathrm{Id} + h^{\alpha-1}G^h) = h^{\alpha-1}D^2W(F^h)G^h,$$

for some matrix $F^h \in \mathbb{M}^{3\times3}$ with $|F^h - \mathrm{Id}| \le \delta/2$. As $D^2W$ is bounded in this set, we deduce that

$$|DW(\mathrm{Id} + h^{\alpha-1}G^h)| \le Ch^{\alpha-1}|G^h|,$$

which implies

$$|E^h| \le C|G^h| + Ch^{\alpha-1}|G^h|^2 \le C(1+\delta)|G^h|.$$

If $h^{\alpha-1}|G^h| > \delta/2$, by (2.9) we have

$$|E^h| \le \frac{1}{h^{\alpha-1}} k\big(W(\mathrm{Id} + h^{\alpha-1}G^h) + 1\big) \le k\frac{W(\mathrm{Id} + h^{\alpha-1}G^h)}{h^{\alpha-1}} + \frac{2k}{\delta}\,|G^h|.$$

We notice that we are allowed to use the bound (2.9), as $W(\nabla_h y^h)$ is finite a.e. in $\Omega$ by (3.12), hence $\det \nabla_h y^h = \det(\mathrm{Id} + h^{\alpha-1}G^h) > 0$ a.e. in $\Omega$. This concludes the proof of (4.14).

*Step 3. Convergenge properties of the scaled stress.* By the decomposition (4.9) and the frame indifference of $W$, we obtain

$$\begin{aligned} DW(\nabla_h y^h)(\nabla_h y^h)^T &= R^h DW(\mathrm{Id} + h^{\alpha-1}G^h)(\mathrm{Id} + h^{\alpha-1}G^h)^T(R^h)^T \\ &= h^{\alpha-1}R^h E^h (R^h)^T. \end{aligned}$$

Thus, in terms of the stresses $E^h$ the Euler-Lagrange equations (4.1) can be written as

$$\int_\Omega R^h E^h (R^h)^T : \nabla\phi(y^h)\, dx = \int_\Omega hge_3 \cdot \phi(y^h)\, dx \tag{4.15}$$

for all $\phi \in C^1_b(\mathbb{R}^3;\mathbb{R}^3)$ satisfying $\phi(x', hx_3) = 0$ for every $x \in \Gamma\times(-\frac{1}{2}, \frac{1}{2})$.

In order to pass to the limit in (4.15) we are interested in studying the convergence properties of the scaled stresses $E^h$.

By (4.14), (3.12) and the fact that the $G^h$ are bounded in $L^2(\Omega;\mathbb{M}^{3\times 3})$, we deduce that for every measurable set $\Lambda\subset\Omega$

$$\begin{aligned}\int_\Lambda |E^h|\,dx &\le C\int_\Lambda \frac{W(\mathrm{Id}+h^{\alpha-1}G^h)}{h^{\alpha-1}}\,dx + C\int_\Lambda |G^h|\,dx\\ &\le Ch^{\alpha-1}+C|\Lambda|^{1/2}.\end{aligned}\tag{4.16}$$

This bound ensures that the scaled stresses $E^h$ are bounded and equi-integrable in $L^1(\Omega;\mathbb{M}^{3\times 3})$. Therefore, by the Dunford-Pettis theorem

$$E^h \rightharpoonup E \quad \text{weakly in } L^1(\Omega;\mathbb{M}^{3\times 3})\tag{4.17}$$

for some $E\in L^1(\Omega;\mathbb{M}^{3\times 3})$. In particular, since $E^h$ is symmetric, also $E$ is symmetric.

One can immediately realise that weak convergence of $E^h$ in $L^1(\Omega;\mathbb{M}^{3\times 3})$ is not enough to pass to the limit in (4.15). This is due to the fact that, for instance, one cannot guarantee uniform convergence of the term $\nabla\phi(y^h)$ (recall that for $y^h$ we have the convergence (4.5)). Therefore, some more refined convergence properties for $E^h$ are needed. In particular, in contrast with [24, 26] weak compactness of $E^h$ in $L^2(\Omega;\mathbb{M}^{3\times 3})$ is, in general, not satisfied. Nevertheless, it is possible to identify a sequence of sets $B_h$, whose measures converge to the measure of $\Omega$ (and therefore, on $\Omega\setminus B_h$ the sequence $E^h$ converges to 0 in the $L^1$ norm by (4.16)), and such that on $B_h$ the sequence $E^h$ is weakly compact in $L^2$. Using the $C^1_b$ regularity of test functions, we shall show that this mixed type of convergence is sufficient to derive the limit equations.

Let $B_h := \{x\in\Omega : h^{\alpha-1-\gamma}|G^h(x)|\le 1\}$, with $\gamma\in(0,\alpha-2)$, and let $\chi_h$ be its characteristic function. Notice that

$$|\Omega\setminus B_h| \le \int_{\Omega\setminus B_h} h^{\alpha-1-\gamma}|G^h|\,dx \le Ch^{\alpha-1-\gamma}|\Omega\setminus B_h|^{1/2}\|G^h\|_{L^2},$$

hence

$$|\Omega\setminus B_h| \le Ch^{2(\alpha-1-\gamma)}.\tag{4.18}$$

This implies in particular that $\chi_h$ converges to 1 in measure and thus, $\chi_h G^h$ converges to $G$ weakly in $L^2(\Omega;\mathbb{M}^{3\times 3})$.

From (4.16) and (4.18) it follows that

$$\int_{\Omega\setminus B_h} |E^h|\,dx \le Ch^{\alpha-1-\gamma},\tag{4.19}$$

hence

$$(1-\chi_h)E^h \to 0 \quad \text{strongly in } L^1(\Omega;\mathbb{M}^{3\times 3}).\tag{4.20}$$

On the set $B_h$ we have a uniform control of the term $h^{\alpha-1}G^h$, so that we can deduce weak convergence of $\chi_h E^h$ in $L^2(\Omega;\mathbb{M}^{3\times 3})$ from the weak convergence of $G^h$ simply by Taylor expansion. More precisely, let $\mathcal{L}$ be the linear operator defined by $\mathcal{L} := D^2W(\mathrm{Id})$. We claim that

$$\chi_h E^h \rightharpoonup \mathcal{L}G \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times 3}).\tag{4.21}$$

We note that, as $R^h$ converges boundedly in measure to Id, the claim implies also that $\chi_h R^h E^h$ converges to $\mathcal{L}G$ weakly in $L^2(\Omega;\mathbb{M}^{3\times 3})$. This remark will be repeatedly used in the next steps of the proof.

By Taylor expansion we have

$$DW(\mathrm{Id}+h^{\alpha-1}G^h)=h^{\alpha-1}\mathcal{L}G^h+\eta(h^{\alpha-1}G^h),$$

where the remainder $\eta$ satisfies $\eta(F)/|F|\to 0$, as $|F|\to 0$. This identity leads to the following decomposition of $\chi_h E^h$:

$$\begin{aligned}\chi_h E^h &= \chi_h \frac{1}{h^{\alpha-1}}\big(h^{\alpha-1}\mathcal{L}G^h+\eta(h^{\alpha-1}G^h)\big)(\mathrm{Id}+h^{\alpha-1}G^h)^T\\ &= \chi_h\mathcal{L}G^h+\chi_h h^{\alpha-1}\mathcal{L}G^h(G^h)^T+\chi_h\frac{\eta(h^{\alpha-1}G^h)}{h^{\alpha-1}}+\chi_h\eta(h^{\alpha-1}G^h)(G^h)^T. \quad (4.22)\end{aligned}$$

To prove the claim (4.21) we analyse carefully each term on the right-hand side of (4.22). The weak convergence of $\chi_h G^h$ to $G$ in $L^2(\Omega;\mathbb{M}^{3\times3})$ and the linearity of $\mathcal{L}$ yield

$$\mathcal{L}(\chi_h G^h)\rightharpoonup \mathcal{L}G \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times3}). \qquad (4.23)$$

The second term in the right-hand side of (4.22) can be estimated as follows:

$$|\chi_h h^{\alpha-1}\mathcal{L}G^h(G^h)^T|\le \chi_h C h^{\alpha-1}|G^h|^2\le C h^\gamma |G^h|.$$

Therefore, it converges to zero strongly in $L^2(\Omega;\mathbb{M}^{3\times3})$ by the $L^2$ bound of the $G^h$. As for the third term in (4.22), we have the following bound:

$$\left|\chi_h\frac{\eta(h^{\alpha-1}G^h)}{h^{\alpha-1}}\right|\le \omega(h^\gamma)\,|G^h|,$$

where for every $t>0$ we have set

$$\omega(t):=\sup\left\{\frac{|\eta(A)|}{|A|} : |A|\le t\right\}.$$

Since $\omega(t)\to 0$ for $t\to 0^+$, we can conclude as before that $\chi_h\eta(h^{\alpha-1}G^h)/h^{\alpha-1}$ converges to zero strongly in $L^2(\Omega;\mathbb{M}^{3\times3})$. Finally, as

$$|\chi_h\eta(h^{\alpha-1}G^h)(G^h)^T|\le h^{\alpha-1}\chi_h\omega(h^\gamma)\,|G^h|^2\le \omega(h^\gamma)\,h^\gamma|G^h|,$$

also this last term converges to zero strongly in $L^2(\Omega;\mathbb{M}^{3\times3})$. Combining together (4.23) and the previous convergence properties, we obtain the claim (4.21). Notice that by (4.17) and (4.20) this implies $E=\mathcal{L}G\in L^2(\Omega;\mathbb{M}^{3\times3})$.

*Step 4. Consequences of the Euler-Lagrange equations.* We now begin to derive some preliminary information from the Euler-Lagrange equations (4.15).

Let $\phi\in C^1_b(\mathbb{R}^3;\mathbb{R}^3)$ be such that $\phi(x',x_3)=0$ for every $x\in\Gamma\times(-\frac12,\frac12)$, and let us consider a test function of the form $\phi^h(x):=h\phi(x',\frac{x_3}{h})$. We notice that $\phi^h$ is an admissible test function, as $\phi^h\in C^1_b(\mathbb{R}^3;\mathbb{R}^3)$ and $\phi^h(x',hx_3)=h\phi(x',x_3)=0$ for every $x\in\Gamma\times(-\frac12,\frac12)$.

Inserting $\phi^h$ in (4.15) leads to

$$\begin{aligned}&h\int_\Omega\sum_{i=1}^2 R^hE^h(R^h)^Te_i\cdot\partial_i\phi\big((y^h)',\frac{y^h_3}{h}\big)\,dx\\ &\quad+\int_\Omega R^hE^h(R^h)^Te_3\cdot\partial_3\phi\big((y^h)',\frac{y^h_3}{h}\big)\,dx=\int_\Omega h^2ge_3\cdot\phi\big((y^h)',\frac{y^h_3}{h}\big)\,dx.\end{aligned}$$

As $R^hE^h(R^h)^T$ is bounded in $L^1(\Omega;\mathbb{M}^{3\times3})$ and $\nabla'\phi$ is a bounded function, the first integral on the left-hand side converges to zero as $h\to 0$. Since the right-hand side is clearly infinitesimal, we deduce

$$\lim_{h\to 0}\int_\Omega R^h E^h (R^h)^T e_3 \cdot \partial_3\phi\big((y^h)', \frac{y_3^h}{h}\big)\,dx = 0. \tag{4.24}$$

On the other hand, owing to (3.15), (4.5), (4.8), and to the continuity and boundedness of $\partial_3\phi$, we have

$$\partial_3\phi\big((y^h)', \frac{y_3^h}{h}\big) \;\to\; \partial_3\phi(x', x_3 + v(x')) \;\text{ strongly in } L^2(\Omega;\mathbb{R}^3), \quad \text{if } \alpha = 3, \tag{4.25}$$

$$\partial_3\phi\big((y^h)', \frac{y_3^h}{h}\big) \;\to\; \partial_3\phi(x', x_3) \;\text{ strongly in } L^2(\Omega;\mathbb{R}^3), \quad \text{if } \alpha > 3, \tag{4.26}$$

(the convergence is actually strong in $L^p(\Omega;\mathbb{R}^3)$ for every $p < \infty$). Therefore, splitting the integral in (4.24) as

$$\begin{aligned}
&\int_\Omega R^h E^h (R^h)^T e_3 \cdot \partial_3\phi\big((y^h)', \frac{y_3^h}{h}\big)\,dx \\
&\quad= \int_\Omega \chi_h R^h E^h (R^h)^T e_3 \cdot \partial_3\phi\big((y^h)', \frac{y_3^h}{h}\big)\,dx \\
&\qquad+ \int_\Omega (1-\chi_h) R^h E^h (R^h)^T e_3 \cdot \partial_3\phi\big((y^h)', \frac{y_3^h}{h}\big)\,dx
\end{aligned}$$

and using (4.20) and (4.21), we conclude that

$$\int_\Omega E e_3 \cdot \partial_3\phi(x', x_3 + v(x'))\,dx = 0 \qquad \text{if } \alpha = 3, \tag{4.27}$$

$$\int_\Omega E e_3 \cdot \partial_3\phi\,dx = 0 \qquad \text{if } \alpha > 3, \tag{4.28}$$

for every $\phi \in C_b^1(\mathbb{R}^3;\mathbb{R}^3)$ such that $\phi(x', x_3) = 0$ for every $x \in \Gamma\times(-\frac{1}{2}, \frac{1}{2})$.

In the case $\alpha = 3$, let $w_k \in C_b^1(\mathbb{R}^2)$ be a sequence of functions such that the restriction of $w_k$ to $S$ converges to $v$ strongly in $L^2(S)$ and $w_k(x') = 0$ for every $x' \in \Gamma$. Then, given any $\phi \in C_b^1(\mathbb{R}^3;\mathbb{R}^3)$ satisfying $\phi = 0$ on $\Gamma\times(-\frac{1}{2}, \frac{1}{2})$ we can choose $\phi_k(x) := \phi(x', x_3 - w_k(x'))$ as test function in (4.27). Passing to the limit with respect to $k$, we obtain that equation (4.28) holds true also for $\alpha = 3$.

From (4.28) it follows that $Ee_3 = 0$ a.e. in $\Omega$. This property, together with the fact that $E$ is symmetric, entails

$$E = \begin{pmatrix} E_{11} & E_{12} & 0 \\ E_{12} & E_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.29}$$

for any $\alpha \geq 3$.

*Step 5. Zeroth moment of the Euler-Lagrange equations.* Let $\bar{E}\colon S \to \mathbb{M}^{3\times 3}$ be the zeroth moment of the limit stress $E$, defined as

$$\bar{E}(x') := \int_{-\frac{1}{2}}^{\frac{1}{2}} E(x)\,dx_3 \tag{4.30}$$

for every $x' \in S$. In the following we derive the equation satisfied by $\bar{E}$.

We consider as test function in (4.15) a map independent of the variable $x_3$. More precisely, let $\psi \in C_b^1(\mathbb{R}^2;\mathbb{R}^2)$ be such that $\psi(x') = 0$ for every $x' \in \Gamma$. Choosing

$\phi(x) = (\psi(x'), 0)$ in (4.15), we have

$$\int_\Omega [R^h E^h (R^h)^T]'' : \nabla' \psi((y^h)')\, dx = 0, \tag{4.31}$$

where $[R^h E^h (R^h)^T]''$ denotes the $2\times 2$ submatrix of $R^h E^h (R^h)^T$, whose entries are given by $[R^h E^h (R^h)^T]''_{ij} = R^h E^h (R^h)^T e_i \cdot e_j$, $1 \le i, j \le 2$.

As in the previous step, it is convenient to split the integral in (4.31) as

$$\begin{aligned}\int_\Omega [R^h E^h (R^h)^T]'' : \nabla' \psi((y^h)')\, dx &= \int_\Omega \chi_h [R^h E^h (R^h)^T]'' : \nabla' \psi((y^h)')\, dx \\ &\quad + \int_\Omega (1-\chi_h) [R^h E^h (R^h)^T]'' : \nabla' \psi((y^h)')\, dx. \end{aligned} \tag{4.32}$$

By (4.5) and the continuity and boundedness of $\nabla'\psi$, the sequence $\nabla'\psi((y^h)')$ converges to $\nabla'\psi$ strongly in $L^2(\Omega; \mathbb{M}^{2\times 2})$. Thus, by (4.21) we obtain

$$\lim_{h\to 0} \int_\Omega \chi_h [R^h E^h (R^h)^T]'' : \nabla' \psi((y^h)')\, dx = \int_\Omega E'' : \nabla'\psi\, dx,$$

while, using the boundedness of $\nabla'\psi$ and (4.20), we have that the last integral in (4.32) converges to 0, as $h \to 0$. Therefore, by (4.31) we conclude that

$$\int_\Omega E'' : \nabla'\psi\, dx = 0$$

for every $\psi \in C^1_b(\mathbb{R}^2; \mathbb{R}^2)$ such that $\psi|_\Gamma = 0$. In terms of the zeroth moment of the stress defined in (4.30), the previous equation yields

$$\int_S \bar{E}'' : \nabla'\psi\, dx' = 0 \tag{4.33}$$

for every $\psi \in C^1_b(\mathbb{R}^2; \mathbb{R}^2)$ such that $\psi|_\Gamma = 0$, and by approximation for every $\psi \in H^1(S; \mathbb{R}^2)$ with $\psi|_\Gamma = 0$.

*Step 6. First moment of the Euler-Lagrange equations.* We now derive the equation satisfied by the first moment of the stress, that is defined as

$$\hat{E}(x') := \int_{-\frac{1}{2}}^{\frac{1}{2}} x_3 E(x)\, dx_3 \tag{4.34}$$

for every $x' \in S$.

Let $\varphi \in C^1_b(\mathbb{R}^2)$ be such that $\varphi|_\Gamma = 0$ and let us consider $\phi(x) = (0, \frac{1}{h}\varphi(x'))$ in (4.15). Since (4.5) and the continuity and boundedness of $\varphi$ entail

$$\lim_{h\to 0} \int_\Omega g\varphi((y^h)')\, dx = \int_\Omega g\varphi\, dx = \int_S g\varphi\, dx',$$

we deduce that

$$\lim_{h\to 0} \int_\Omega \frac{1}{h} \sum_{i=1}^{2} [R^h E^h (R^h)^T]_{3i}\, \partial_i \varphi((y^h)')\, dx = \int_S g\varphi\, dx' \tag{4.35}$$

for every $\varphi \in C^1_b(\mathbb{R}^2)$ such that $\varphi|_\Gamma = 0$.

We now want to identify the limit in (4.35) in terms of the first moment $\hat{E}$. In [26] this is done by first considering in the Euler-Lagrange equations a test function of the form $\phi(x) = (x_3\eta(x'), 0)$, and then passing to the limit with respect to $h$. Using the symmetry of the stress, this leads to an identity relating the first moment

$\hat{E}$ with the limit in (4.35) and, by comparison with (4.35), the limiting equation for $\hat{E}$.

In the present setting the simple choice $\phi(x) = (x_3\eta(x'), 0)$ is not allowed, since this test function is not bounded in $\mathbb{R}^3$. This issue can be solved by means of the following careful truncation argument. We consider a truncation function $\theta^h$, which coincides with the identity in an interval $(-\omega_h, \omega_h)$, for a suitable $\omega_h \to +\infty$, and a corresponding test function $\phi^h$ of the form (4.40) below. The rate of convergence of $\omega_h$ has to be chosen in such a way to match two requirements. On one hand, we need to show that the limiting contribution due to the region where $\theta^h$ does not coincide with the identity is negligible. This can be done by means of the estimate (4.16), once we prove that the measure of the set $D_h$ where $|y_3^h/h| \geq \omega_h$ is sufficiently small. This is guaranteed if the rate of convergence of $\omega_h$ is fast enough (see proof of (4.47) below). On the other hand, because of this choice, the $L^\infty$-norm of the test functions $\phi^h$ is not bounded, but blows up as $\omega_h$. Therefore, the convergence rate of $\omega_h$ has to be carefully chosen to ensure that the integral on $\Omega \setminus B_h$ remains irrelevant, as usual. This is possible owing to the choice of $B_h$ and the estimate (4.19) (see proof of (4.43) below).

To be definite, let $\omega_h$ be a sequence of positive numbers such that

$$h\,\omega_h \to \infty, \qquad h^{\alpha-1-\gamma}\omega_h \to 0, \tag{4.36}$$

where $\gamma$ is the exponent introduced in the definition of $B_h$. This is possible since $\gamma < \alpha - 2$ (for instance, one can choose $\omega_h := h^{-(\alpha-\gamma)/2}$). Let $\theta^h \in C_b^1(\mathbb{R})$ be a truncation function satisfying

$$\theta^h(t) = t \quad \text{for } |t| \leq \omega_h, \tag{4.37}$$

$$|\theta^h(t)| \leq |t| \quad \text{for every } t \in \mathbb{R}, \tag{4.38}$$

$$\|\theta^h\|_{L^\infty} \leq 2\omega_h, \quad \Big\|\frac{d\theta^h}{dt}\Big\|_{L^\infty} \leq 2. \tag{4.39}$$

Let $\eta \in C_b^1(\mathbb{R}^2;\mathbb{R}^2)$ be such that $\eta(x') = 0$ for every $x' \in \Gamma$. We define $\phi^h \colon \mathbb{R}^3 \to \mathbb{R}^3$ as

$$\phi^h(x) := \Big(\theta^h\Big(\frac{x_3}{h}\Big)\eta(x'), 0\Big). \tag{4.40}$$

Owing to the assumptions on $\theta^h$ and $\eta$, the $\phi^h$ are admissible test functions in (4.15); then inserting $\phi^h$ in (4.15) leads to

$$\int_\Omega \theta^h\Big(\frac{y_3^h}{h}\Big)[R^hE^h(R^h)^T]'' : \nabla'\eta((y^h)')\,dx + \int_\Omega \frac{1}{h}\sum_{i=1}^{2}[R^hE^h(R^h)^T]_{i3}\,\eta_i((y^h)')\Big(\frac{d\theta^h}{dt}\Big(\frac{y_3^h}{h}\Big)\Big)\,dx = 0. \tag{4.41}$$

We now compute the limit of each term in (4.41) separately, starting with the first. We consider the usual splitting $\Omega = B_h \cup (\Omega \setminus B_h)$ and we carefully analyse the contributions of the integral in the two subdomains.

If $\alpha = 3$, we have that

$$\lim_{h\to0}\int_\Omega \chi_h\theta^h\Big(\frac{y_3^h}{h}\Big)[R^hE^h(R^h)^T]'' : \nabla'\eta((y^h)')\,dx = \int_S (\hat{E}'' + v\bar{E}'') : \nabla'\eta\,dx'. \tag{4.42}$$

Indeed, by (4.8) and (3.15) the sequence $y_3^h/h$ converges to $x_3 + v$ a.e. in $\Omega$ and is dominated by an $L^2$ function. From (4.37) and (4.38) it follows that the sequence $\theta^h(y_3^h/h)$ converges to $x_3 + v$ a.e. in $\Omega$ and is dominated by an $L^2$ function. Owing to the convergence (4.5) of $y^h$ and to the continuity and boundedness of $\nabla'\eta$, we conclude that

$$\theta^h\Big(\frac{y_3^h}{h}\Big)\nabla'\eta((y^h)') \to (x_3 + v)\nabla'\eta(x') \quad \text{strongly in } L^2(\Omega;\mathbb{R}^2).$$

Therefore, by (4.21) we deduce

$$\lim_{h\to 0}\int_\Omega \chi_h\theta^h\Big(\frac{y_3^h}{h}\Big)[R^hE^h(R^h)^T]''\, g\, \nabla'\eta((y^h)')\, dx = \int_\Omega (x_3 + v)E'' : \nabla'\eta(x')\, dx.$$

Integration with respect to $x_3$ yields (4.42).

As for the integral on $\Omega \setminus B_h$, by the estimate (4.39) on $\theta^h$ and (4.19) it can be bounded by

$$\begin{aligned}\int_\Omega (1-\chi_h)\Big|\theta^h\Big(\frac{y_3^h}{h}\Big)[R^hE^h(R^h)^T]'' : \nabla'\eta((y^h)')\Big|\, dx \\ \le\ 2\omega_h\|\nabla'\eta\|_{L^\infty}\int_{\Omega\setminus B_h}|E^h| \ \le\ Ch^{\alpha-1-\gamma}\omega_h; \qquad (4.43)\end{aligned}$$

therefore, it is infinitesimal as $h \to 0$ by the second property in (4.36). We conclude that, if $\alpha = 3$,

$$\lim_{h\to 0}\int_\Omega \theta^h\Big(\frac{y_3^h}{h}\Big)[R^hE^h(R^h)^T]'' : \nabla'\eta((y^h)')\, dx = \int_S (\hat{E}'' + v\bar{E}'') : \nabla'\eta\, dx'. \qquad (4.44)$$

Analogously, for $\alpha > 3$, since $y_3^h/h$ converges to $x_3$ strongly in $L^2(\Omega)$, we deduce that

$$\lim_{h\to 0}\int_\Omega \theta^h\Big(\frac{y_3^h}{h}\Big)[R^hE^h(R^h)^T]'' : \nabla'\eta((y^h)')\, dx = \int_S \hat{E}'' : \nabla'\eta\, dx'. \qquad (4.45)$$

In order to analyse the second integral in (4.41), it is convenient to split it as follows:

$$\begin{aligned}&\int_\Omega \frac{1}{h}\sum_{i=1}^2 [R^hE^h(R^h)^T]_{i3}\,\eta_i((y^h)')\Big(\frac{d\theta^h}{dt}\Big(\frac{y_3^h}{h}\Big)\Big)\, dx \\ &\quad = \int_\Omega \frac{1}{h}\sum_{i=1}^2 [R^hE^h(R^h)^T]_{i3}\,\eta_i((y^h)')\, dx \\ &\qquad + \int_\Omega \frac{1}{h}\sum_{i=1}^2 [R^hE^h(R^h)^T]_{i3}\,\eta_i((y^h)')\Big(\frac{d\theta^h}{dt}\Big(\frac{y_3^h}{h}\Big) - 1\Big)\, dx. \qquad (4.46)\end{aligned}$$

We claim that the second term on the right-hand side is infinitesimal, as $h \to 0$, that is,

$$\lim_{h\to 0}\int_\Omega \frac{1}{h}\sum_{i=1}^2 [R^hE^h(R^h)^T]_{i3}\,\eta_i((y^h)')\Big(\frac{d\theta^h}{dt}\Big(\frac{y_3^h}{h}\Big) - 1\Big)\, dx = 0. \qquad (4.47)$$

If $\alpha = 3$, combining (4.41), (4.44), and (4.46), the claim implies that

$$\lim_{h\to 0}\int_\Omega \frac{1}{h}\sum_{i=1}^2 [R^hE^h(R^h)^T]_{i3}\,\eta_i((y^h)')\, dx = -\int_S (\hat{E}'' + v\bar{E}'') : \nabla'\eta\, dx'. \qquad (4.48)$$

If $\alpha > 3$, combining (4.41), (4.45), (4.46), and the claim (4.47), we obtain

$$\lim_{h\to 0}\int_\Omega \frac{1}{h}\sum_{i=1}^{2}[R^hE^h(R^h)^T]_{i3}\,\eta_i((y^h)')\,dx = -\int_S \hat{E}'' : \nabla'\eta\,dx'. \tag{4.49}$$

It remains to prove (4.47). To this aim we introduce the set $D_h := \{x \in \Omega : |y_3^h(x)|/h \geq \omega_h\}$. Since the sequence $y_3^h/h$ is bounded in $L^2(\Omega)$ by (4.8) and (3.15), we have

$$|D_h| \leq \omega_h^{-1}\int_{D_h}\frac{|y_3^h|}{h}\,dx \leq c\,\omega_h^{-1}|D_h|^{1/2},$$

which implies

$$|D_h| \leq C\omega_h^{-2}. \tag{4.50}$$

Since the derivative of $\theta^h$ is equal to 1 on $(-\omega_h, \omega_h)$ by (4.37), the integral in (4.47) reduces to

$$\int_\Omega \frac{1}{h}\sum_{i=1}^{2}[R^hE^h(R^h)^T]_{i3}\,\eta_i((y^h)')\Big(\frac{d\theta^h}{dt}\Big(\frac{y_3^h}{h}\Big)-1\Big)\,dx$$
$$= \int_{D_h} \frac{1}{h}\sum_{i=1}^{2}[R^hE^h(R^h)^T]_{i3}\,\eta_i((y^h)')\Big(\frac{d\theta^h}{dt}\Big(\frac{y_3^h}{h}\Big)-1\Big)\,dx.$$

By (4.16), (4.39), and (4.50), we have

$$\Big|\int_{D_h} \frac{1}{h}\sum_{i=1}^{2}[R^hE^h(R^h)^T]_{i3}\,\eta_i((y^h)')\Big(\frac{d\theta^h}{dt}\Big(\frac{y_3^h}{h}\Big)-1\Big)\,dx\Big|$$
$$\leq \frac{C}{h}\Big(1+\Big\|\frac{d\theta^h}{dt}\Big\|_{L^\infty}\Big)\|\eta\|_{L^\infty}\int_{D_h}|E^h|\,dx$$
$$\leq Ch^{\alpha-2} + \frac{C}{h}|D_h|^{1/2} \leq Ch^{\alpha-2} + \frac{C}{h\,\omega_h}.$$

By (4.36) this proves the claim (4.47).

*Step 7. Limit equations.* Let $\varphi \in C_b^2(\mathbb{R}^2)$ be such that $\varphi(x') = 0$, $\nabla'\varphi(x') = 0$ for every $x' \in \Gamma$. Since $R^hE^h(R^h)^T$ is symmetric, due to the symmetry of $E^h$, we can compare equation (4.35) with (4.48), if $\alpha = 3$, or (4.49), if $\alpha > 3$ (where we specify $\eta = \nabla'\varphi$). In this way we deduce that, if $\alpha = 3$

$$-\int_S (\hat{E}'' + v\bar{E}'') : (\nabla')^2\varphi\,dx' = \int_S g\varphi\,dx', \tag{4.51}$$

while, if $\alpha > 3$,

$$-\int_S \hat{E}'' : (\nabla')^2\varphi\,dx' = \int_S g\varphi\,dx'. \tag{4.52}$$

Applying the relation (4.33) with $\psi = v\nabla'\varphi$ we conclude that equation (4.51) can be rewritten as

$$\int_S \bar{E}'' : (\nabla' v \otimes \nabla'\varphi)\,dx' - \int_S \hat{E}'' : (\nabla')^2\varphi\,dx' = \int_S g\varphi\,dx', \tag{4.53}$$

By approximation the equations (4.52) and (4.53) hold for every $\varphi \in H^2(S)$ with $\varphi|_\Gamma = 0$ and $\nabla'\varphi|_\Gamma = 0$.

In order to express the limiting equations (4.33), (4.53), and (4.52) in terms of the limit displacements, an explicit characterization of $\bar{E}''$ and $\hat{E}''$ is needed. Since

$E = \mathcal{L}G$ and $E$ is of the form (4.29), we have $E'' = \mathcal{L}_2 G''$ (see [26, Proposition 3.2]). Therefore, by (4.10) and (4.11) we obtain, for $\alpha = 3$,

$$\bar{E}'' = \mathcal{L}_2\big(\operatorname{sym}\nabla' u + \frac{1}{2}\nabla' v \otimes \nabla' v\big), \quad \hat{E}'' = -\frac{1}{12}\mathcal{L}_2(\nabla')^2 v.$$

These identities, together with (4.33) and (4.53), provide us with the Euler-Lagrange equations (3.8)–(3.9).

By (4.10) and (4.12) we obtain, for $\alpha > 3$,

$$\bar{E}'' = \mathcal{L}_2(\operatorname{sym}\nabla' u), \quad \hat{E}'' = -\frac{1}{12}\mathcal{L}_2(\nabla')^2 v.$$

The first identity, together with (4.33) and the boundary condition $u = 0$ on $\Gamma$, implies that $u = 0$, while the second identity, together with (4.45), provide us with the Euler-Lagrange equation (3.11). This concludes the proof. □

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