Curvature theory of boundary phases: the two-dimensional case

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Abstract

We describe the behaviour of minimum problems involving non-convex surface integrals in 2D singularly perturbed by a curvature term. We show that their limit is described by functionals which take into account energies concentrated on vertices of polygons. Non-locality and non-compactness effects are highlighted.

Key Words: surface energies, curvature functionals, phase transitions, Γ-convergence, non convex problems

1 Introduction

The starting point of the analysis in this work is the study of minimum problems related to the equilibrium of elastic crystals (see e.g. [16], [15] for the variational formulation, [8] [9] for a derivation of the model from statistical considerations, [3] for its links with Ising systems and [20] [25] for a analogous derivation as a singular perturbation of the Allen Cahn model). The model problem we have in mind is that of finding sets minimizing a (possibly highly anisotropic) perimeter functional, of the form

$$\min\Big\{\int_{\partial E}\psi(\nu_E)d\mathcal{H}^1 : E_0\subseteq E\Big\}, \tag{1}$$

where the minimum is computed among all sets $E\subset\mathbb{R}^2$ with boundary of class C^1 and containing a fixed open set E_0. Here, ψ is a Borel function, ν_E denotes the (appropriately-oriented) tangent to E and $\mathcal{H}^1$ is the 1-dimensional (Hausdorff) surface measure. Another model problem is that of *local minimizers* of the perimeter, related to

$$\min\Big\{\int_{\partial E}\psi(\nu_E)d\mathcal{H}^1 : |E_0\,\Delta\,E|\le\delta\Big\}, \tag{2}$$

where $\delta>0$ is a fixed constant.

Problems the type above, or some of their perturbations for which the solution is not as much at hand, can be attacked following the so-called direct methods of the calculus of variations. First, problems (1)

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and (2) can be ‘relaxed’ by admitting as competing sets all sets with finite perimeter (see [18], [5]). Then, if ψ is larger than a fixed constant and if its homogeneous positive extension of degree one is a *convex* function, classical results imply that the surface integral in (1) and (2) is lower semicontinuous and coercive in the appropriate topology of the L^1-convergence of characteristic functions of sets. The application of the direct methods of the calculus of variations thus yields existence of minimizing sets of finite perimeter, and, if ψ^2 is smooth and strictly convex, regularity results for minimal surfaces assure that such minimizers are regular. On the other hand if ψ^2 is *non convex* then the minimum problems (1) or (2) may not possess solutions. It can be seen (see e.g. [21]) that the application of the direct method of the calculus of variations gives minimizing sequences with increasingly wiggly boundaries (even though with equi-bounded total surface). Their limits can be described (see [4]) as minimizers of a ‘relaxed’ problem of the same type: in the case of (1) for example,

$$\min\Big\{\int_{\partial E}\overline{\psi}(\nu_E)d\mathcal{H}^1 : E_0\subseteq E\Big\}, \tag{3}$$

where the new length energy density $\overline{\psi}$ is simply the convex envelope of the one-homogeneous extension of ψ to $\mathbb{R}^n$. This process may lead to non-strictly convex integrands, which in turn may yield non-uniqueness and non-regularity of solutions. In this case it may be necessary to consider higher-order terms in the surface energy to explain solutions with sharp corners and facets (see also [30]; a similar phenomenon is studied in [19]). Note that so far the problem can be framed in an n-dimensional framework, upon replacing curves by hypersurfaces.

In this paper we study, in a two-dimensional setting, the case when we add a singular perturbation by a curvature term in (1) (or, analogously, in (2)), obtaining a minimum problem of the form

$$\min\Big\{\int_{\partial E}\Big(\psi(\nu_E)+\varepsilon^2\kappa^2\Big)d\mathcal{H}^1 : E_0\subseteq E\Big\}, \tag{4}$$

where now the minimum is taken among sets with C^2 boundary and $\kappa(x)$ denotes the curvature of ∂E at x. In this way, oscillating boundaries are penalized when introducing large curvatures.

In a way similar to [25], [24], [22], in order to understand the behaviour of minimizers for (4) we may study the (equivalent) scaled minimum problems

$$\min\Big\{\int_{\partial E}\Big(\frac{\psi(\nu_E)-\overline{\psi}(\nu_E)}{\varepsilon}+\varepsilon\kappa^2\Big)d\mathcal{H}^1 : E_0\subseteq E\Big\}. \tag{5}$$

We assume for simplicity that $\psi(\nu_E)=\overline{\psi}(\nu_E)$ precisely on a finite number of directions $\nu_1\dots,\nu_N$. One can easily check that under this assumption ψ must satisfy

$$\psi(\nu)>\frac{\sin(\nu_{i+1}-\nu)}{\sin(\nu_{i+1}-\nu_i)}\,\psi(\nu_i)+\frac{\sin(\nu-\nu_i)}{\sin(\nu_{i+1}-\nu_i)}\,\psi(\nu_{i+1}),\qquad \forall\,\nu\in(\nu_i,\nu_{i+1}),\forall\,i=1,\dots,N.$$

Note that this condition rules out a smooth behaviour near $\nu_1,\dots,\nu_N$ as in the energies considered in [19]. The problem can be then rewritten as

$$\min\Big\{\int_{\partial E}\Big(\frac{\varphi(\nu_E)}{\varepsilon}+\varepsilon\kappa^2\Big)d\mathcal{H}^1 : E_0\subseteq E\Big\}, \tag{6}$$

where $\varphi : S^1\to[0,+\infty)$ vanishes precisely on those preferred directions.

Our main result is to describe the asymptotic behaviour as $\varepsilon\to 0$ of the problems in (6), showing that minimizers E_ε, up to translations, tend to sets E which in turn minimize a limit energy. This limit energy can be computed by using the techniques of Γ-convergence (see [13], [11], [10]). We define the functionals F_ε on sets of finite perimeter as

$$F_\varepsilon(E) = \begin{cases} \displaystyle\int_{\partial E}\Big(\frac{\varphi(\nu_E)}{\varepsilon} + \varepsilon\kappa^2\Big)d\mathcal{H}^1 & \text{if } E \text{ is of class } C^2 \\ \\ +\infty & \text{otherwise,} \end{cases} \tag{7}$$

and we compute their Γ-limit G with respect to the L^1 and $L^1_{\rm loc}$-convergence of characteristic functions of sets. As an example, in the simplest case when φ is *symmetric* and the preferred directions coincide with the coordinate directions, the domain of the limit G is simply the set of the coordinate polyrectangles and $G(E) = c\,\#(V(E))$, where $V(E)$ is the set of vertices of the polyrectangle E. The constant c can be computed as

$$c = 2\int_S \sqrt{\varphi(s)}d\mathcal{H}^1(s), \tag{8}$$

where S is the minimal arc in S^1 connecting $(1,0)$ and $(0,1)$. Hence, the limit problem is trivially

$$\min\Big\{c\,\#(V(E)) : E \text{ coordinate polyrectangle, } E_0 \subseteq E\Big\} \tag{9}$$

and the minimizers E of the limit problem are simply all coordinate rectangles containing E_0. Note that the limits E of minimizers E_ε of (4) minimize both (3) and (9), so that they are coordinate rectangles containing E_0 of minimal perimeter.

In the general case, we show that the domain of the limit energy consists of those polyhedra whose tangents point in the preferred directions $\nu_1, \dots, \nu_N$, and that the limit energy is much more complex. If E contains only simple vertices (or, equivalently, if ∂E is locally Lipschitz) we define

$$F(E) = \sum\Big\{g(\nu^-(v), \nu^+(v)) :\ v \in V(E)\Big\}, \tag{10}$$

where g is given by

$$g(\nu_1, \nu_2) = 2\int_{A(\nu_1,\nu_2)} \sqrt{\varphi(s)}d\mathcal{H}^1(s) \tag{11}$$

($A(\nu_1, \nu_2)$ is the minimal arc connecting ν_1 and ν_2 in S^1) and $\nu^\pm(v)$ are the two tangents at v. If, loosely speaking, E is such that approximating sequences E_ε may be chosen 'close' to E then we prove that $G(E) = F(E)$. In the general case, the value $G(E)$ is obtained as

$$G(E) = \inf\Big\{\liminf_j F(E_j) :\ E_j \to E,\ E_j \text{ with simple vertices}\Big\}. \tag{12}$$

This formula hides two types of degenerate behaviours. First of all, we have to take into account that when two or more vertices meet at a point the set E may be approximated in many different ways and the approximation of minimal energy must be chosen. In addition, the energy G may be *non-local*: in a sense, a polyhedron may be completed by adding segments pointing in some of the preferred directions, which must be considered as degenerate parts of E; the energy $G(E)$ takes into account the 'minimal' of such completions. This effect is analogous to that highlighted in [6] for functionals depending on the square of the curvature. As a consequence of formula (12) we get that the study of minimizers of problems involving G corresponds to the analysis of minimizing sequences of corresponding problems involving F. In particular, we deduce that the limit problem of (6) admits as solutions all the convex polyhedra with tangents in the preferred directions.

Once the form of the Γ-limit is computed, we may apply our results also to other problems for which the solution is less immediate, such as

$$\min\Big\{\int_{\partial E}\Big(\frac{\varphi(\nu_E)}{\varepsilon}+\varepsilon\kappa^2\Big)d\mathcal{H}^{n-1} : |E_0\,\Delta\,E|\le\delta\Big\}, \tag{13}$$

or

$$\min\Big\{\int_{\partial E}\Big(\frac{\varphi(\nu_E)}{\varepsilon}+\varepsilon\kappa^2\Big)d\mathcal{H}^{n-1} + |E_0\,\Delta\,E|\Big\}, \tag{14}$$

where E_0 is some fixed set. The latter problem is also of interest in some models in Image Processing where energies depending on curvatures and on (the number of) vertices are considered (see [27], [23], [12]). Note that the solution to problem (14) may not be given by a set where $G(E) = F(E)$ (see the example in Section 6.2).

Finally, we note that, since the solutions of the limit problem are polyhedra with fixed orientations, it is very tempting to link this approximation result to the theory of crystalline growth as recently developed (see [28], [29], [17], [7]), where non-striclty convex ψ are considered.

The paper is organized as follows. Section 2 contains the statement of the main results in terms of Γ-convergence and the necessary notation. In Sections 3 and 4 we prove the lower and upper bounds for the limit energy. In Section 5 some cases are dealt with when the limit energy can be proven to be *local*; i.e., it can be written as a sum of energies concentrated on vertices. Finally, in Section 6 we consider the pathological case when we do not have a boundedness condition on the perimeters, giving a qualitative description of the shape of sequences with equi-bounded energy and an example when Γ-limits computed in the L^1 and $L^1_{\rm loc}$ topology differ.

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2 Main results

2.1 Statement of the main results

For every $E \subseteq \mathbb{R}^2$ of class C^2 and every $\varepsilon > 0$, we define the energy

$$F_\varepsilon(E) = \int_{\partial E}\left(\frac{1}{\varepsilon}\varphi(\nu)+\varepsilon k^2\right)d\mathcal{H}^1 \tag{15}$$

where $\nu = \nu(x)$ is the *tangent direction* to ∂E in x, defined in such a way that $(\nu_2, -\nu_1)$ coincides with the outer unit normal to ∂E in x. With $\mathcal{H}^1$ we denote the 1-dimensional Hausdorff measure, which will coincide with the line measure throughout the paper. The quantity $\kappa = \kappa(x)$ denotes the *curvature of* ∂E in x, and $\varphi : S^1 \to [0, +\infty)$ (we identify S^1 with $\mathbb{R}$ mod 2π) is a continuous function with the following property

$$\exists\,\nu_1,\dots\nu_N \in S^1, \quad \nu_1<\nu_2\cdots<\nu_N<\nu_{N+1}=\nu_1+2\pi \quad \text{such that } \varphi(\nu)=0=\{\nu_1,\dots\nu_N\}.$$

We will always assume that

$$|\nu_i-\nu_{i+1}|<\pi, \quad i=1,\dots,N.$$

We will identify sets E with their *characteristic function* χ_E, and then, with a slight abuse of notation the functional given by formula (15) will be identified with the functional $F_\varepsilon : L^1(\mathbb{R}^2) \to [0,+\infty]$ given by

$$F_\varepsilon(u) = \begin{cases} \displaystyle\int_{\partial E} \left(\frac{1}{\varepsilon}\varphi(\nu) + \varepsilon\kappa^2\right) d\mathcal{H}^1 & \text{if} \quad u = \chi_E \text{ and } E \text{ is of class } C^2 \\ +\infty & \text{otherwise.} \end{cases} \tag{16}$$

With an additional slight abuse of notation, we say that a sequence of sets $(E_n)_n \subseteq \mathbb{R}^2$ converges to $E \subseteq \mathbb{R}^2$ in $L^1(\mathbb{R}^2)$ if $\chi_{E_n} \to \chi_E$ in $L^1(\mathbb{R}^2)$.

For $\theta_1, \theta_2 \in S^1$, $\theta_1 \neq \theta_2 + \pi$, let $A_{(\theta_1,\theta_2)}$ denote the shorter of the two arcs in S^1 connecting θ_1 and θ_2. We assume that $A_{(\theta_1,\theta_2)}$ is oriented in the direction going from θ_1 to θ_2. We define $g : S^1 \times S^1 \to [0,+\infty)$ in the following way

$$g(\theta_1,\theta_2) = \begin{cases} \displaystyle 2\int_{A_{(\theta_1,\theta_2)}} \sqrt{\varphi(\nu)}\, d\mathcal{H}^1(\nu) & \text{if} \quad \theta_i \in \{\nu_1, \dots \nu_N\} \quad i = 1,2 \\ +\infty & \text{otherwise.} \end{cases} \tag{17}$$

Note that $W(\theta_2,\theta_1) = W(\theta_1,\theta_2)$.

An *admissible polyhedron* is a set $P \subseteq \mathbb{R}^2$ whose boundary is a polygonal composed of segments whose directions lie in the set $\{\nu_1, \dots \nu_N\}$. We set

$$\mathcal{P} = \{P \,:\, P \text{ is an admissible polyhedron}\}.$$

We also define the class

$$\mathcal{R} = \left\{P \in \mathcal{P} \,:\, \partial P \text{ is piecewise } C^1\right\},$$

and we call *regular admissible polyhedra* the elements of $\mathcal{R}$. The difference between an admissible and a regular admissible polyhedron is that each vertex of a polyhedron of the second type is the endpoint of precisely two sides.

Given a polyhedron P in $\mathbb{R}^2$, we define the set $V(P) \subseteq \mathbb{R}^2$ of the *vertices of* P to be

$$V(P) = \{x \in \partial P \,:\, \partial P \text{ is not } C^1 \text{ at } x\}.$$

We define also the functional $F_\mathcal{R} : \mathcal{P} \to \mathbb{R}$ in the following way

$$F_\mathcal{R}(E) = \begin{cases} \displaystyle\sum_{v \in V(E)} W(\nu^-(v), \nu^+(v)), & \text{if } E \in \mathcal{R}; \\ +\infty, & \text{if } E \notin \mathcal{R}. \end{cases}$$

Here, $\nu^-(v), \nu^+(v)$ denote the directions of the two sides intersecting in $v \in V(E)$. This functional will be identified with a functional $F_\mathcal{R} : L^1(\mathbb{R}^2) \to [0,+\infty]$ in the same spirit of (16).

We also set

$$G = sc^-(F_\mathcal{R}),$$

where sc^- denotes the sequential lower semi-continuous envelope, understood in the sense of the L^1-topology with uniform bounds on the perimeters, namely

$$sc^-(F_\mathcal{R})(E) = \inf\left\{\liminf_n F_\mathcal{R}(E_n) \,:\, E_n \to E \text{ in } L^1(\mathbb{R}^2),\ \sup_n \mathcal{H}^1(\partial E_n) < +\infty\right\}.$$

Remark 2.1 It is easy to check that G is finite only on (characteristic functions of) admissible polyhedra. Moreover, given an admissible polyhedron P, there always exists a sequence $(P_n)_n$ of regular polyhedra which converge to P in $L^1(\mathbb{R}^2)$, and for which $\sup_n \mathcal{H}^1(\partial P_n) < +\infty$ and $\sup_n F_{\mathcal{R}}(P_n) < +\infty$. In fact, it is sufficient to take

$$P_n = \left\{ x \in P \,:\, \mathrm{dist}(x, P) \leq \frac{1}{n} \right\}. \tag{18}$$

We remark that in general the sequence given by formula (18) does not recover the infimum in the definition of $G(E)$.

Remark 2.2 Given an admissible polyhedron P, there always exists a sequence $(P_n)_n$ of regular polyhedra which converge to P in $L^1(\mathbb{R}^2)$, and for which $G(P) = F_{\mathcal{R}}(P_n)$ for sufficiently large n. In fact, whenever the quantities $F_{\mathcal{R}}(P_n)$ remain bounded, they have range in a finite set of numbers, and the infimum is always attained.

Our main result is the following Γ-convergence theorem (for a general introduction to the subject we refer to [13], [11]).

Theorem 2.1 *For $\varepsilon > 0$, let $F_\varepsilon : L^1(\mathbb{R}^2) \to [0, +\infty]$ be the functional given by formula* (16)*. Then there holds*

$$\Gamma\text{-}\lim_{\varepsilon \to 0} F_\varepsilon = G \tag{19}$$

with respect to the convergence in $L^1(\mathbb{R}^2)$ with uniform bounds of the lengths of the perimeters. More precisely, by (19) *we mean:*
(i) (closure) *if $\sup_\varepsilon \mathcal{H}^1(E_\varepsilon) < +\infty$, $\sup_\varepsilon F_\varepsilon(E_\varepsilon) < +\infty$ and $E_\varepsilon \to u$ in $L^1(\mathbb{R}^2)$ then there exists $P \in \mathcal{P}$ such that $u = \mathcal{P}$;*
(ii) (Γ-liminf inequality) *for all $P \in \mathcal{P}$ and for all $E_\varepsilon \to P$ in $L^1(\mathbb{R}^2)$ with $\sup_\varepsilon \mathcal{H}^1(E_\varepsilon) < +\infty$, we have $G(P) \leq \liminf_\varepsilon F_\varepsilon(E_\varepsilon)$;*
(iii) (Γ-limsup inequality) *for all $P \in \mathcal{P}$ there exists $E_\varepsilon \to P$ in $L^1(\mathbb{R}^2)$ with $\sup_\varepsilon \mathcal{H}^1(E_\varepsilon) < +\infty$ such that $G(P) = \lim_\varepsilon F_\varepsilon(E_\varepsilon)$.*

Remark 2.3 (Convergence of minimum problems) From Theorem 2.1 we obtain the convergence of the minimum values of problems (13) and (14) to the minimum values

$$\min \Big\{ G(P) : P \in \mathcal{P},\ |E_0 \,\Delta\, P| \leq \delta \Big\} = \inf \Big\{ \sum_{v \in V(P)} g(\nu^-(v), \nu^+(v)) : P \in \mathcal{R},\ |E_0 \,\Delta\, P| \leq \delta \Big\},$$

and

$$\min \Big\{ G(P) + |E_0 \,\Delta\, P| : P \in \mathcal{P} \Big\} = \inf \Big\{ \sum_{v \in V(P)} g(\nu^-(v), \nu^+(v)) + |E_0 \,\Delta\, P| : P \in \mathcal{R} \Big\},$$

respectively, provided that we may find a sequence of minimizers with equibounded perimeter. This property is a well-known result of Γ-convergence, once we notice that the equi-boundedness of the perimeters ensures compactness of the minimizing sequence (upon, possibly, a translation), and that the constraints or the additional terms are 'compatible' with Γ-convergence. To check this for problem (13), it is sufficient to notice that a slight modification of the argument in the proof of Theorem 2.1(iii) allows to obtain that we may suppose $|E_0 \,\Delta\, E| \leq \delta$, while it is clear that the addition of the perturbation in (14) is compatible since it is continuous with respect to the L^1-convergence.

Remark 2.4 The results of Theorem 2.1 remain valid if F_ε has the form

$$F_\varepsilon(E) = \int_{\partial E} \left(\frac{1}{\varepsilon} \varphi(\nu) + \varepsilon \kappa^2 \right) d\mathcal{H}^1 + c\mathcal{H}^1(\partial E) \tag{20}$$

with $c > 0$; i.e., if we add a term proportional to the length of ∂E. In this case, we similarly modify $F_{\mathcal{R}}(E)$ by setting

$$F_{\mathcal{R}}(E) = \sum_{v \in V(E)} g(\nu^-(v), \nu^+(v)) + c\mathcal{H}^1(\partial E)$$

on $\mathcal{R}$. Note that in this case the equi-boundedness condition on the perimeters is redundant.

For the case when we drop the equi-boundedness condition on the perimeters and we consider the $L^1_{\rm loc}$ convergence we refer to Section 6.

We conclude this section by deducing a convergence result for the minimum problems in (1) as an example of application of Theorem 2.1.

Corollary 2.1 *Let ψ and $\overline{\psi}$ be as in the Introduction. Let E_0 be a bounded connected open set and let E_ε be minimizers for the problems*

$$m_\varepsilon = \min\Big\{ \int_{\partial E} \Big(\psi(\nu_E) + \varepsilon^2\kappa^2\Big) d\mathcal{H}^1 : E_0 \subseteq E\Big\}.$$

Then, upon translations and passage to a subsequence, E_ε converge to a polyhedron P which minimizes both

$$m = \min\Big\{ \int_{\partial E} \overline{\psi}(\nu_E) d\mathcal{H}^1 : E_0 \subseteq E\Big\} \tag{21}$$

and

$$m^{(1)} = \min\Big\{ \sum_{v \in V(E)} g(\nu^-(v), \nu^+(v)) : E_0 \subseteq E, E \in \mathcal{R}\Big\}. \tag{22}$$

Proof. We just sketch the proof, including details only for the passages involving Γ-convergence.

By a relaxation argument (see [4]) and the density of sets with regular boundary we may suppose that E_ε converges to a minimizer $\overline{E}$ of (21), which is connected since E_0 is. On the other hand, E_ε is also a minimizer of

$$m^{(1)}_\varepsilon = \min\Big\{ \int_{\partial E} \Big(\frac{\psi(\nu_E)}{\varepsilon} + \varepsilon\kappa^2\Big) d\mathcal{H}^1 - \frac{m}{\varepsilon} : E_0 \subseteq E\Big\}.$$

Define $\varphi = \psi - \overline{\psi}$. By using Lemma 3.1 and the construction of Section 4, one can check that

$$m \le \int_{\partial E_\varepsilon} \overline{\psi}(\nu_{E_\varepsilon}) d\mathcal{H}^1 \le m + o(\varepsilon),$$

and that E_ε is an $o(1)$-minimizer of

$$\begin{aligned}\tilde{m}^{(1)}_\varepsilon &= \min\Big\{ \int_{\partial E} \Big(\frac{\psi(\nu_E) - \overline{\psi}(\nu_E)}{\varepsilon} + \varepsilon\kappa^2\Big) d\mathcal{H}^1 : E_0 \subseteq E\Big\} \\ &= \min\Big\{ \int_{\partial E} \Big(\frac{\varphi(\nu_E)}{\varepsilon} + \varepsilon\kappa^2\Big) d\mathcal{H}^1 : E_0 \subseteq E\Big\}.\end{aligned}$$

We may apply Theorem 2.1 and Remark 2.4 as the perimeters of E_ε are equibounded since $\psi \ge c$. We then obtain that $\overline{E}$ is a (convex) polyhedron which minimizes (21) and also (22). ■

2.2 Notation

We introduce some preliminary notation and definitions.

Given a polyhedron P in $\mathbb{R}^2$, we define a *side of* P to be the closure of a component of $\partial P \setminus V(P)$; we also define

$$\overline{s}(P) = \inf\{|s| \,:\, s \text{ is a side of } P\}.$$

If $\gamma^i : [a_i, b_i] \to \mathbb{R}^2 \quad i = 1,2$ are two curves with $\gamma^1(b_1) = \gamma^2(a_2)$, we define $\gamma^1 * \gamma^2 : [a_1, a_2 + b_2 - b_1] \to \mathbb{R}^2$ as

$$\gamma^1 * \gamma^2(t) = \begin{cases} \gamma^1(t) & t \in [a_1, b_1] \\ \gamma^2(t - b_1 + a_1) & t \in [b_1, b_1 + b_2 - a_2]. \end{cases}$$

Similarly, we define inductively

$$\gamma^1 * \cdots * \gamma^k = \left(\gamma^1 * \cdots * \gamma^{k-1}\right) * \gamma^k.$$

Given a curve $c : [a, b] \to \mathbb{R}^2$, we denote by $\mathrm{im}(c)$ its image, and if c is of class C^2, and $t \in [a, b]$ is such that $c'(t) \neq 0$, we define $\kappa(c(t))$ to be the curvature of c at $c(t)$.

Given two sequences $(A_n)_n, (B_n)_n$ of subsets of $\mathbb{R}^2$ such that $A_n \cap B_n = \emptyset \; \forall n \in \mathbb{N}$, and given $\nu \in S^1$, we say that $(B_n)_n$ *falls into line with respect to* $(A_n)_n$ *in the direction* ν if for every $\delta > 0$ it is

$$\left| \frac{x - y}{|x - y|} - \nu \right| < \delta, \quad \forall x \in A_n, \forall y \in B_n, \qquad \text{for } n \text{ sufficiently large.}$$

We say that a family of curves $\gamma_n : (a_n, b_n) \to \mathbb{R}^2$ *falls into line in the direction* ν if for every $\eta > 0$ and for every sequence of pairs (x_n, y_n), $x_n, y_n \in \mathrm{im}(\gamma_n)$, with $|x_n - y_n| > \eta$, and such that $\gamma_n^{-1}(x_n) > \gamma_n^{-1}(y_n)$, the sequence $(x_n)_n$ falls into line with respect to $(y_n)_n$ in the direction ν.

Given a piecewise C^1 curve $\gamma : S^1 \to \mathbb{R}^2$, and given a point x which does not belong to $\mathrm{im}(\gamma)$, we define $\mathrm{ind}(x, \gamma)$ to be the winding number of γ around x, namely (in complex notation)

$$\mathrm{ind}(\gamma, x) = \frac{1}{2\pi i} \int_{S^1} \frac{\dot{\gamma}(t)}{\gamma(t) - x}\, dt.$$

Finally, we say that two segments $[x_1, x_2], [y_1, y_2] \subseteq \mathbb{R}^2$ *do not intersect transversally* if the condition below holds true

$$(NT) \qquad [x_1, x_2] \cap [y_1, y_2] \cap \{x_1 \cup x_2 \cup y_1 \cup y_2\} = \emptyset.$$

Given $\theta_1, \theta_2 \in S^1$, the sum $\theta_1 + \theta_2$ will denote, unless it is explicitly remarked, the sum as elements of the group S^1 endowed with its natural structure.

3 The Γ-lim inf inequality

This section is devoted to the proof of the Γ-lim inf inequality in Theorem 2.1.

We consider sequences $(E_n) \subseteq \mathbb{R}^2, \varepsilon_n \to 0^+$ for which

(H_1) $\quad \chi_{E_n} \to u$ in $L^1(L^1_{\mathrm{loc}})(\mathbb{R}^2)$;

(H_2) $\quad \sup_n \mathcal{H}^1(\partial E_n) < +\infty$;

(H_3) $\quad \sup_n F_{\varepsilon_n}(E_n) < +\infty$.

Our first aim is to prove that the sequence $(E_n)_n$ converges in $L^1(\mathbb{R}^2)$ to some admissible polyhedron P. In fact we have the following result.

Proposition 3.1 *Let $\varepsilon_n \to 0$ and let $(E_n)_n$ satisfy hypotheses (H_1), (H_2) and (H_3). Then there exists an admissible polyhedron $P \in \mathcal{P}$ such that $u = \chi_P$, and for which there holds*

$$G(u) \leq \liminf_n F_{\varepsilon_n}(E_n). \tag{23}$$

Before proving Proposition 3.1 we introduce some preliminary result.

Lemma 3.1 *Let $a, b, \delta \in \mathbb{R}$, $a < b$, $\delta > 0$, and let $\nu_i \in \varphi^{-1}(0)$. Then for every curve $\eta : [a,b] \to A_{(\nu_i,\nu_{i+1})}$ of class C^1 with*

$$\eta(a) = \nu_i + \delta, \qquad \eta(b) = \nu_{i+1} - \delta,$$

we have

$$\int_a^b \left(\frac{1}{\varepsilon}\, \varphi(\eta(t)) + \varepsilon\, \|\dot\eta(t)\|^2 \right) dt \geq g(\nu_i, \nu_{i+1}) + o_\delta(1), \tag{24}$$

where $o_\delta(1) \to 0$ as $\delta \to 0$.

Proof. This is a simple consequence of the Young inequality, in fact we obtain

$$\begin{aligned}\int_a^b \left(\frac{1}{\varepsilon}\, \varphi(\eta(t)) + \varepsilon\, \|\dot\eta(t)\|^2 \right) dt \quad &\geq \quad 2 \int_a^b \sqrt{\varphi(\eta(t))}\, |\dot\eta(t)|\, dt \\ &\geq \quad 2 \int_{\nu_i+\delta}^{\nu_{i+1}-\delta} \sqrt{\varphi(\eta(t))}\, dt \geq 2 \int_{\nu_i}^{\nu_{i+1}} \sqrt{\varphi(\eta(t))}\, dt + o_\delta(1),\end{aligned}$$

which is the desired inequality. ■

Now we consider a family of curves $\gamma_n : S^1 \to \mathbb{R}^2$ of class C^2 with the following properties

$$\sup_n \int_{S^1} \left(\frac{1}{\varepsilon_n}\, \varphi\left(\frac{\dot\gamma_n}{|\dot\gamma_n|} \right) + \varepsilon_n \left(\frac{d}{dt}\, \frac{\dot\gamma_n}{|\dot\gamma_n|} \right)^2 \right) dt = M < +\infty, \tag{25}$$

$$\sup_n \int_{S^1} |\dot\gamma_n|\; dt < +\infty. \tag{26}$$

We suppose also that the curves γ_n are parametrized proportionally to their arc length, namely that there holds

$$|\dot\gamma_n(t)| = \frac{1}{2\pi} \int_{S^1} |\dot\gamma_n|\; ds; \qquad \text{for all } t \in S^1 \text{ and for all } n \in \mathbb{N}.$$

We want to describe the limit shape of the curves γ_n when $n \to +\infty$. In order to do this, we set for $\delta > 0$

$$S_\delta = S^1 \setminus ([\nu_1 - \delta, \nu_1 + \delta] \cup \cdots \cup [\nu_N - \delta, \nu_N + \delta]),$$

and

$$C(\delta) = \inf_{\nu \in S_\delta} \varphi(\nu). \tag{27}$$

If $\eta : [a,b] \to S_\delta$ is a curve of class C^1, then there holds clearly

$$\int_a^b \left(\frac{1}{\varepsilon} \varphi(\eta(t)) + \varepsilon \|\dot\eta(t)\|^2 \right) dt \geq \frac{1}{\varepsilon}\, (b-a)\, C(\delta); \tag{28}$$

hence, using (25) and (28) with $\eta = \dot\gamma_n$ and $\varepsilon = \varepsilon_n$, we deduce

$$\mathcal{H}^1(\{t \in [0,T_n] : \dot\gamma_n(t) \in S_\delta\}) \le \frac{\varepsilon_n}{C(\delta)} \int_{\dot\gamma_n \in S_\delta} \left(\frac{1}{\varepsilon_n} \varphi\left(\frac{\dot\gamma_n(t)}{|\dot\gamma_n|}\right) + \varepsilon_n\, \kappa^2(\gamma_n(t)) \right) dt \le \frac{\varepsilon_n M}{C(\delta)}.$$

From this inequality we deduce the existence of a sequence $\delta_n \to 0$ such that

$$\mathcal{L}^1(I_n) \to 0 \quad \text{as } n \to +\infty, \tag{29}$$

where we have set

$$I_n = \left\{ t \in S^1 : \frac{\dot\gamma_n(t)}{|\dot\gamma_n(t)|} \in S_{\delta_n} \right\}.$$

Since S_δ is open, the components of I_n are at most countable: denote by $I_n^j = (\sigma_n^j, \theta_n^j)$, $j = 1, \dots, k_n$, those components of I_n for which $\dot\gamma_n(a_n^j) \ne \dot\gamma_n(b_n^j)$. From assumption (H_3) and from Lemma 3.1 it follows that $\sup_n k_n < +\infty$ and so, passing to a subsequence, we can assume that $k_n = \bar k$ for all n. We also set

$$J_n = S^1 \setminus \bigcup_{j=1}^{\bar k} I_n^j. \tag{30}$$

Lemma 3.2 *Let $(\theta_n^h, \sigma_n^{h+1})$ be a component of J_n such that*

$$\dot\gamma_n(\theta_n^h) = |\dot\gamma_n|\,(\nu_i \pm \delta_n), \qquad \textit{for some } \nu_i \in \{\nu_1, \dots, \nu_N\}.$$

Then $\gamma_n|_{(\theta_n^h, \sigma_n^{h+1})}$ falls into line in the direction ν_i.

Proof. Let $\eta > 0$, and let $\alpha_n, \beta_n \in (\theta_n^h, \sigma_n^{h+1})$ be such that $|\gamma_n(\alpha_n) - \gamma_n(\beta_n)| > \eta$. Then, since γ_n is parametrized proportionally to the arc lenght, there holds

$$\eta < |\gamma_n(\alpha_n) - \gamma_n(\beta_n)| \le \int_{\alpha_n}^{\beta_n} |\dot\gamma_n(t)|\, dt = |\dot\gamma_n|\; \mathcal{L}_1((\alpha_n, \beta_n)), \tag{31}$$

so in particular we have

$$\frac{\eta}{\sup_j |\dot\gamma_j|} \le \frac{\eta}{|\dot\gamma_n|} < \mathcal{L}_1((\alpha_n, \beta_n)) < 2\pi.$$

Hence by equation (26) the quantities $\mathcal{L}_1((\alpha_n, \beta_n))$ are uniformly bounded from above and from below. Set

$$\rho_n = \int_{(\alpha_n,\beta_n)\setminus I_n} \dot\gamma_n(t) dt; \qquad \tau_n = \int_{(\alpha_n,\beta_n)\cap I_n} \dot\gamma_n(t) dt.$$

Equations (26) and (29) imply that $\tau_n \to 0$ as $n \to +\infty$. We also have

$$\int_{(\alpha_n,\beta_n)\setminus I_n} \dot\gamma_n(t) dt = |\dot\gamma_n|\; \mathcal{L}_1((\alpha_n, \beta_n) \setminus I_n)\, \nu_i + \int_{(\alpha_n,\beta_n)\setminus I_n} (\dot\gamma_n(t) - |\dot\gamma_n|\; \nu_i)\; dt,$$

so from (31) and the definition of I_n we deduce

$$\rho_n = |\dot\gamma_n|\; \mathcal{L}_1((\alpha_n, \beta_n))\, \nu_i + o(1). \tag{32}$$

From this expression and from the fact that $\rho_n \to 0$ it follows that

$$\frac{\gamma_n(\beta_n) - \gamma_n(\alpha_n)}{|\gamma_n(\beta_n) - \gamma_n(\alpha_n)|} - \nu_i = \frac{\rho_n + \tau_n}{|\rho_n + \tau_n|} - \nu_i = \frac{\rho_n}{|\rho_n|} - \nu_i + o(1) = o(1).$$

This concludes the proof. ■

The next lemma shows that γ_n, restriced to a component of J_n, converges uniformly to a segment in direction ν_i parametrized by arc lenght.

Lemma 3.3 *Let $(\theta_n^h, \sigma_n^{h+1})$ be a component of J_n as in Lemma 3.2. Then, given any $\rho > 0$, there exists $n_\rho \in \mathbb{N}$ such that*

$$\|\gamma_n(\beta_n) - \gamma_n(\alpha_n) - |\dot{\gamma}_n|\,(\beta_n - \alpha_n)\,\nu_i\| < \rho, \qquad \forall \alpha_n, \beta_n \in (\theta_n^h, \sigma_n^{h+1}),\ \forall n \geq n_\rho. \tag{33}$$

PROOF. It follows easily from $\gamma_n(\beta_n) - \gamma_n(\alpha_n) = \rho_n + \tau_n$, equation (32), and the fact that $\tau_n \to 0$ as $n \to +\infty$. ■

Let us now introduce some additional notation. We define the class

$$\mathcal{C} = \left\{ \{\gamma^1, \ldots, \gamma^k\} \mid \gamma^i : S^1 \to \mathbb{R}^2 \text{ is piecewise } C^1, \frac{\dot{\gamma}^i}{|\dot{\gamma}^i|} \in \{\nu_1, \ldots, \nu_N\} \text{ a.e. in } S^1, i = 1, \ldots, k \right\}.$$

Let $\gamma = \{\gamma^1, \ldots, \gamma^k\} \in \mathcal{C}$. Then for all i $\operatorname{im}(\gamma^i)$ is composed by a finite number of segments with directions $\nu_{j_1}, \ldots, \nu_{j_{l_i}}$. We define $\tilde{F} : \mathcal{C} \to \mathbb{R}$ in the following way

$$\tilde{F}(\gamma) = \sum_{i=1}^{k} \sum_{h=1}^{j_i} g(\nu_{j_h}, \nu_{j_{h+1}}).$$

Proposition 3.2 *Let ε_n and let $(E_n)_n$ satisfy hypotheses (H_1), (H_2) and (H_3) above. Let γ_n^j, $j = 1, \ldots, l$ (passing to a subsequence we can suppose that the number l is independent of n) be parametrizations of the components of ∂E_n. Then there exist a polyhedron $P \in \mathcal{P}$ such that $u = \chi_P$, there exist integers h, k, $k \leq h \leq l$, and there exists $\gamma = \{\gamma^1, \ldots, \gamma^k\} \in \mathcal{C}$ with the following properties*

- (Γ_1) *$\gamma_n^j \to \gamma^j$, $j = 1, \ldots, k$, uniformly on S^1, and $\gamma_n^j \to x^j \in \mathbb{R}^2$, $j = k+1, \ldots, h$, uniformly on S^1.*
- (Γ_2) *the segments of im(γ) do not intersect transversally;*
- (Γ_3) *for a.e. $x \in \mathbb{R}^2$, it is $\sum_{i=1}^k \mathrm{ind}(\gamma^i, x) \in \{0, 1\}$, and $\chi_P(x) = \sum_{i=1}^k \operatorname{ind}(\gamma^i, x)$;*
- (Γ_4) *$\tilde{F}(\gamma) \leq \liminf_n F_{\varepsilon_n}(E_n)$;*

PROOF. Let $i \in \{1, \ldots, h\}$, and consider the sequence of curves γ_n^i which parametrize the i-th component of ∂E_n. This sequence satisfies conditions (25) and (26), hence we can repeat for them the constructions above. Let J_n^i be the counterpart of the set J_n for the curve γ_n^i. We can also suppose that the number of components of J_n^i is a constant k^i independent of n. From Lemma 3.3 it follows that,

$$\text{up to translation, } \gamma_n^i \to \gamma^i \text{ uniformly on } S^1, \qquad \text{for some curve } \gamma^i \in \mathcal{C}, \tag{34}$$

or

$$\text{up to translation, } \gamma_n^i \to x^i \text{ uniformly on } S^1, \qquad \text{for some point } x^i \in \mathbb{R}^2. \tag{35}$$

Up to a permutation of the indices, there exist $h, k \in \mathbb{N}$, $0 \leq k \leq h \leq l$ such that $(\gamma_n^1)_n, \ldots, (\gamma_n^k)_n$ converge uniformly in S^1 to some $\gamma^1, \ldots, \gamma^k \in \mathcal{C}$, and that $(\gamma_n^{k+1})_n, \ldots, (\gamma_n^h)_n$ converge uniformly in S^1 to some points $x^{k+1}, \ldots, x^h \in \mathbb{R}^2$. Define γ to be $\gamma = \{\gamma^1, \ldots, \gamma^k\}$, so that also $\gamma \in \mathcal{C}$. Condition (Γ_1) is automatically satisfied. Condition (Γ_2) follows easily from the fact that the sets E_n are of class C^2.

From equations (34) and (35) we deduce

$$\mathcal{H}^2(B_\gamma) = 0, \quad \text{where} \quad B_\gamma = \Big(\bigcup_{i=1}^k \operatorname{im}(\gamma^i)\Big) \cup \Big(\bigcup_{i=k+1}^h x^i\Big). \tag{36}$$

By the continuity of the winding number with respect to the uniform convergence we have

$$\lim_n \sum_{i=1}^h \mathrm{ind}\,(\gamma_n^i, x) = \sum_{i=1}^h \mathrm{ind}\,(\gamma^i, x), \qquad \text{for all } x \in \mathbb{R}^2 \setminus B_\gamma,$$

hence, since the index is integer-valued there holds

$$\sum_{i=1}^h \mathrm{ind}\,(\gamma_n^i, x) = \sum_{i=1}^k \mathrm{ind}\,(\gamma^i, x), \qquad \text{for } n \text{ large and for all } x \in \mathbb{R}^2 \setminus B_\gamma.$$

From this we can deduce that, setting

$$P = \Big\{ x \in \mathbb{R}^2 \setminus B_\gamma \ : \ \lim_n \sum_{i=1}^h \mathrm{ind}\,(\gamma_n^i, x) = 1 \Big\},$$

we have

$$\begin{cases} x \in P \quad \Rightarrow \quad x \in E_n & \text{for } n \text{ large;} \\ x \notin P \quad \Rightarrow \quad x \notin E_n & \text{for } n \text{ large.} \end{cases}$$

This implies that

$$\chi_{E_n} \to \chi_P \quad \text{as } n \to +\infty, \qquad \text{a.e. in } \mathbb{R}^2,$$

and proves condition (Γ_3). Property (Γ_4) follows from Lemma 3.1. ■

Lemma 3.4 *Suppose that $\gamma \in \mathcal{C}$ satisfies conditions (Γ_1) and (Γ_2) in Proposition 3.2. Then there exists a sequence of regular polyhedra $(P_n)_n \subseteq \mathcal{R}$ such that*

$$\chi_{P_n} \to \chi_P \quad \text{in } L^1(\mathbb{R}^2); \qquad F_{\mathcal{R}}(P_n) \le \tilde{F}(\gamma). \tag{37}$$

Proof. For the proof of this Lemma we refer to [14]. ■

Finally, we are in position to prove Proposition 3.1.

Proof of Proposition 3.1. Let P be the polyhedron given by Proposition 3.2, and let $(P_n)_n \subseteq \mathcal{R}$ be the sequence of regular polyhedra given by Lemma 3.4. Then, by equation (37) and by property (Γ_4) there holds

$$F_{\mathcal{R}}(P_n) \le \tilde{F}(\gamma) \le \liminf_n F_{\varepsilon_n}(E_n).$$

Finally, by the definition of G we have

$$G(P) \le \liminf_n F_{\mathcal{R}}(P_n) \le \liminf_n F_{\varepsilon_n}(E_n).$$

This concludes the proof. ■

4 The Γ-lim sup inequality

The goal of this section is to prove the Γ-lim sup inequality in Theorem 2.1. Starting with a regular admissible polyhedron P, we modify it near its vertices and we obtain a sequence of sets E_n of class C^2 which converge to P and such that $F_{\varepsilon_n}(E_n)$ is as small as possible. Then we treat the general case of an admissible polyhedron by approximating it with regular polyhedra.

Proposition 4.1 *Let $P \in \mathcal{R}$ be an admissible regular polyhedron. Then, given any sequence $\varepsilon_n \to 0^+$, there exists a sequence of sets $(E_n)_n$ of class C^2 such that*

$$\chi_{E_n} \to \chi_P \ in \ L^1(\mathbb{R}^2); \qquad \limsup_n F_{\varepsilon_n}(E_n) \le F_{\mathcal{R}}(P).$$

Proof. Let v be a vertex of P: since P is regular, there are exactly two sides of P intersecting v. Without loss of generality, we can suppose that the directions of these sides, which we denote by l_1 and l_2, are ν_1 and ν_2 respectively. Let $\lambda : \left(-\frac{1}{2}|l_1|, \frac{1}{2}|l_2|\right) \to \mathbb{R}^2$ be defined by

$$\lambda(t) = \begin{cases} v - t\,\nu_1, & t \in \left[-\frac{1}{2}l_1, 0\right]; \\ v + t\,\nu_2, & t \in \left[0, \frac{1}{2}l_2\right]. \end{cases} \tag{38}$$

The curve λ defined in this way parametrizes part of l_1 for $t < 0$ and part of l_2 for $t > 0$. Our aim is to find a sequence of regular curves $\lambda_n : \left[-\frac{1}{2}|l_1|, \frac{1}{2}|l_2|\right] \to \mathbb{R}^2$ with the following properties:

$$\lambda_n \to \lambda \qquad \text{uniformly on } \left[-\frac{1}{2}|l_1|, \frac{1}{2}|l_2|\right]; \tag{39}$$

$$\lim_n \int_{\left(-\frac{1}{2}|l_1|, \frac{1}{2}|l_2|\right)} \left(\frac{1}{\varepsilon_n} \varphi\left(\frac{\dot\lambda_n}{|\dot\lambda_n|}\right) + \varepsilon_n\, \kappa^2(\lambda_n) \right) dt = g(\nu_1, \nu_2). \tag{40}$$

Since φ is assumed to be of class C^1 in $S^1 \setminus \{\nu_1, \dots, \nu_N\}$, the following Cauchy problem

$$\begin{cases} y'(t) = \sqrt{\varphi(y(t))} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} y \\ y(0) = \dfrac{\nu_1 + \nu_2}{2}. \end{cases} \tag{41}$$

admits a unique maximal solution $u : (a, b) \to S^1$, with $-\infty \le a < 0, 0 < b \le +\infty$. It is immediate to check that that u is a C^1 increasing function which tends to ν_1 (respectively, ν_2) as $t \to a$ (respectively, $t \to b$).

For every $c, d \in (a, b)$, with $c < 0 < d$ (c and d will be taken sufficiently close to a and b), define $e = c - (u(c) - \nu_1)$ and $f = d + (\nu_2 - u(d))$; note that $e < c < d < f$. We can find a nondecreasing function $\eta : [e, f] \to A_{(\nu_1, \nu_2)}$ of class C^1, such that

$$\eta\,(e) = \nu_1; \quad \dot\eta\,(e) = 0; \tag{42}$$

$$\eta\,(f) = \nu_1; \quad \dot\eta\,(f) = 0; \tag{43}$$

$$\begin{cases} \eta(t) = u(t), & t \in (c, d); \\ |\dot\eta(t)| < 2\,|u(c) - \nu_1|, & t \in (e, c); \\ |\dot\eta(t)| < 2\,|\nu_2 - u(d)|, & t \in (d, f). \end{cases} \tag{44}$$

For $\varepsilon > 0$, let η_ε denote the unique continuous extension of η to the interval $\left[-\frac{1}{\varepsilon}\frac{1}{2}|l_1|, \frac{1}{\varepsilon}\frac{1}{2}|l_2|\right]$ for which

$$\dot\eta_\varepsilon(t) = \begin{cases} \nu_1, & t \in \left[-\frac{1}{\varepsilon}\frac{1}{2}|l_1|, e\right]; \\ \nu_2, & t \in \left[f, \frac{1}{\varepsilon}\frac{1}{2}|l_2|\right]. \end{cases} \tag{45}$$

Finally, for $\varepsilon_n \to 0^+$, define $\lambda_n : \left[-\frac{1}{2}|l_1|, \frac{1}{2}|l_2|\right] \to \mathbb{R}^2$ to be

$$\lambda_n(t) = v + \int_0^t \eta_{\varepsilon_n}\left(\frac{s}{\varepsilon_n}\right) ds, \qquad t \in \left[-\frac{1}{2}|l_1|, \frac{1}{2}|l_2|\right].$$

Since η_{ε_n} is an S^1-valued curve of class C^1, it follows that λ_n is of class C^2 and is parametrized by arc lenght. For $t < 0$ it turns out that

$$\begin{aligned}\lambda_n(t) - \lambda(t) &= v + \int_0^t \eta_{\varepsilon_n}\left(\frac{s}{\varepsilon_n}\right) ds - v - t\,\nu_1 \\ &= \int_0^{\varepsilon_n e} \eta_{\varepsilon_n}\left(\frac{s}{\varepsilon_n}\right) ds + \int_{\varepsilon_n e}^t \eta_{\varepsilon_n}\left(\frac{s}{\varepsilon_n}\right) ds - t\,\nu_1.\end{aligned}$$

Since $|\eta_{\varepsilon_n}| = 1$, and since $\dot\eta_{\varepsilon_n}(t) = \nu_1$ for $t < e$, it follows that

$$\lambda_n(t) - \lambda(t) \to 0, \qquad \text{uniformly for } t \in \left[-\frac{1}{2}|l_1|, 0\right].$$

In the same way one can show that

$$\lambda_n(t) - \lambda(t) \to 0, \qquad \text{uniformly for } t \in \left[0, \frac{1}{2}|l_1|,\right],$$

so we have proved (39).

Using the definition of λ_n and the change of variable $\frac{s}{\varepsilon_n} = y$, we find

$$\int_{-\frac{1}{2}|l_1|}^{\frac{1}{2}|l_2|} \left(\frac{1}{\varepsilon_n}\varphi\Big(\frac{\dot\lambda_n}{|\dot\lambda_n|}\Big) + \varepsilon_n\,\kappa^2(\lambda_n)\right) ds = \int_{-\frac{1}{2}\frac{1}{\varepsilon_n}|l_1|}^{\frac{1}{2}\frac{1}{\varepsilon_n}|l_2|} \left(\frac{1}{\varepsilon_n}\varphi\left(\eta_{\varepsilon_n}\right) + \varepsilon_n\left(\dot\eta_{\varepsilon_n}\right)^2\right) dy;$$

then, taking into account equation (45), one has

$$\int_{-\frac{1}{2}\frac{1}{\varepsilon_n}|l_1|}^{\frac{1}{2}\frac{1}{\varepsilon_n}|l_2|} \left(\frac{1}{\varepsilon_n}\varphi\left(\eta_{\varepsilon_n}\right) + \varepsilon_n\left(\dot\eta_{\varepsilon_n}\right)^2\right) dt = \int_e^f \left(\varphi\left(\eta_{\varepsilon_n}\right) + \left(\dot\eta_{\varepsilon_n}\right)^2\right) dt.$$

Dividing the interval (e, f) into (e, c), (c, d) and (d, f), by equation (41) we get

$$\begin{aligned}\int_e^f \left(\varphi\left(\eta_{\varepsilon_n}\right) + \left(\dot\eta_{\varepsilon_n}\right)^2\right) dt \quad &\leq \quad |c - e|\left(\sup_{(e,c)}\varphi + \sup_{(e,c)}\dot\eta^2_{\varepsilon_n}\right) \\ &\quad + g(\nu_1, \nu_2) + |f - d|\left(\sup_{(d,f)}\varphi + \sup_{(d,f)}\dot\eta^2_{\varepsilon_n}\right).\end{aligned}$$

Using the expression of e, f, and taking into account (44), we deduce

$$\begin{aligned}\int_e^f \left(\varphi\left(\eta_{\varepsilon_n}\right) + \left(\dot\eta_{\varepsilon_n}\right)^2\right) dt \quad &\leq \quad g(\nu_1, \nu_2) + |u(c) - \nu_1|\left(\sup_{(e,c)}\varphi + 4\,|u(c) - \nu_1|^2\right) \\ &\quad + |\nu_2 - u(d)|\left(\sup_{(e,c)}\varphi + 4\,|\nu_2 - u(d)|^2\right).\end{aligned}$$

Hence, choosing $c = c(n)$ and $d = d(n)$ depending on n and such that

$$|u(c) - \nu_1| + |\nu_2 - u(d)| \to 0 \qquad \text{as } n \to +\infty,$$

also (40) follows.

Now consider a component Θ of ∂P. Let $v_1, \dots, v_{i_\Theta}$ denote an ordering of the vertices of Θ along the parametrization of λ, and let λ_j be the curve defined above corresponding to the vertex $v_j, j = 1, \dots, i_\Theta$. Then we can choose as parametrization for Θ the piecewise-C^2 curve λ_Θ given by

$$\lambda_\Theta = \lambda_1 * \cdots * \lambda_{i_\Theta}.$$

For $j \in \{1, \dots, i_\Theta\}$, let $\lambda_{j,n}$ be a sequence of curves which satisfy (39) and (40) with $\lambda = \lambda_j$ and $\nu^-(v_j), \nu^+(v_j)$ instead of ν_1 and ν_2. If we consider the sequence of curves

$$\lambda_{p,n} = \lambda_{1,n} * \cdots * \lambda_{i_\Theta,n}, \qquad n \in \mathbb{N},$$

they will converge uniformly to λ_Θ on their domain (a_Θ, b_Θ). In general the curve $\lambda_{\Theta,n}$ is not closed, but since λ_Θ is closed there holds

$$\lambda_{\Theta,n}(a_\Theta) - \lambda_{\Theta,n}(b_\Theta) \to 0, \qquad \text{as } n \to +\infty.$$

Consider the curve $\lambda_{1,n}$. Since the directions of its two rectilinear parts are linearly independent, it is sufficient to modify slightly the lenght of these parts in such a way that $\lambda_{\Theta,n}$ transforms into a closed curve $\overline{\lambda}_{\Theta,n}$.

Repeating this procedure for all the components of ∂P we obtain a set E_n whose boundary is parametrized by the union of the curves $(\overline{\lambda}_{\Theta,n})_\Theta$. The sequence E_n will satisfiy the required properties in the proposition. ■

Remark 4.1 From the proof of Proposition 4.1 it follows that we can choose λ_n satisfying (40) and

(46)

$$\lambda_n \text{ coincides with } \lambda \text{ in a neighbourhood of } \left\{-\frac{1}{2}|l_1|, \frac{1}{2}|l_2|\right\}; \qquad \|\lambda_n - \lambda\|_\infty \le 2(|e(n)| + |f(n)|)\,\varepsilon_n,$$

where $e(n) = c(n) - u(c(n)) + \nu_1$ and $f(n) = d(n) + u(c(n)) - \nu_2$.

As an immediate consequence of Proposition 4.1 we have the following corollary.

Corollary 4.1 (Γ-limsup inequality) *Let $P \in \mathcal{P}$ be an admissible polyhedron. Then, for every $(\varepsilon_n)_n$ with $\varepsilon_n \to 0^+$ there exists a sequence of sets E_n of class C^2 such that*

$$E_n \to P \text{ in } L^1(\mathbb{R}^2) \qquad \text{and} \qquad \limsup_n F_{\varepsilon_n}(E_n) \le G(P).$$

Proof. By Remark 2.2, there exists a sequence $(P_k)_k \subseteq \mathcal{R}$ of regular polyhedra such that

$$\chi_{P_k} \to \chi_P \text{ in } L^1(\mathbb{R}^2); \qquad \limsup_k F_{\mathcal{R}}(P_k) = G(P); \qquad \sup_k \mathcal{H}^1(\partial P_k) < +\infty.$$

Then, by Proposition 4.1, for every $k \in \mathbb{N}$ there exists a sequence $(E_n^k)_n$ of sets of class C^2 such that

$$E_n^k \to P_k; \qquad F_{\varepsilon_n}(E_n^k) \to F_{\mathcal{R}}(P_k), \qquad \text{as } n \to +\infty.$$

Hence we can choose a sequence of natural numbers $n(k)$ with $n(k_2) > n(k_1)$ if $k_2 > k_1$ such that

$$\|\chi_{E_n^k} - \chi_{P_k}\|_{L^1(\mathbb{R}^2)} \le \frac{1}{k}, \qquad F_{\varepsilon_n}(E_n^k) \le F_{\mathcal{R}}(P_k) + \frac{1}{k}.$$

So, if we choose

$$E_n = E_n^k, \qquad \text{for } n(k) \le n < n(k+1),$$

the sequence $(E_n)_n$ satisfies the desired properties. ■

5 Some local cases

In this section we study some specific cases for which the Γ-limit G has a local expression, namely it is the sum over the vertices of a quantity depending only on the single vertices.

5.1 A non-symmetric case

In this section we treat the following particular case. We assume that the function φ satisfies the following conditions

(i) $\varphi \in C^1(S^1)$;

(ii) $\varphi^{-1}(0) = \{\nu_1, \dots, \nu_N\}$, and $\nu_i \in \varphi^{-1}(0) \Rightarrow -\nu_i \notin \varphi^{-1}(0)$.

Under these hypotheses we will prove that $\Gamma\text{-}\lim_{\varepsilon\to 0} F_\varepsilon$ has a local expression. Namely, to every vertex of an admissible polyhedron P is associated a quantity $E(v)$, and $\Gamma\text{-}\lim_{\varepsilon\to 0} F_\varepsilon(P)$ is the sum of $E(v)$ over the vertices v of P, see Proposition 5.4. In order to state this result precisely we introduce some additional notation.

Let P be an admissible polyhedron, and let v be a vertex of P. Let $l_1, \dots, l_{2k}$ be the sides of P which intersect at v. If condition (i) above is satisfied, then for each of these segments l_j, $j = 1, \dots, 2k$, is uniquely determined a tangent direction $\nu(l_j) = \nu_{i_j} \in \varphi^{-1}(0)$.

To each l_j we can associate an orientation $\sigma_v(l_j)$ respect to v, namely we set

$$\begin{cases} \sigma_v(l_j) = -1, & \text{if } l_j \text{ is oriented toward } v; \\ \sigma_v(l_j) = 1, & \text{if } -l_j \text{ is oriented toward } v, \end{cases} \qquad j = 1, \dots, 2k.$$

If the segments $l_1, \dots, l_{2k}$, are ordered in such a way that $\nu_{i_1} < \nu_{i_2} < \dots < \nu_{i_{2k}}$, then clearly it must be

$$\sigma_v(l_j) \cdot \sigma_v(l_{j+1}) = -1, \qquad j = 1, \dots, 2k-1, \qquad \text{and} \qquad \sigma_v(l_{2k}) \cdot \sigma_v(l_1) = -1.$$

Definition 5.1 *An* admissible decomposition ω *of* v *is a partition of* $l_1, \dots, l_{2k}$ *in pairs* $\left(l_i^-, l_i^+\right)$, $i = 1, \dots, k$, *such that*

$$(AD_1) \qquad \sigma_v(l_i^-) = -1, \quad \sigma_v(l_i^+) = 1; \qquad i = 1, \dots, k,$$

and

$$(AD_2) \qquad \nu(l_i^-) < \min\left\{\nu(l_j^-), \nu(l_j^+)\right\} < \max\left\{\nu(l_j^-), \nu(l_j^+)\right\} < \nu(l_i^+), \qquad i, j = 1, \dots, k, \quad i \neq j.$$

We set also

$$\Omega_v = \{\omega \mid \omega \textit{ is an admissible decomposition for } v\}.$$

Remark 5.2 Every vertex $v \in V(P)$ admits an admissible decomposition. In fact, if the versors $\nu_{i_1}, \dots, \nu_{i_{2k}}$, are ordered in such a way that $\nu_{i_1} < \nu_{i_2} < \dots < \nu_{i_{2k}}$, then one can take

$$l_i^- = l_{2i-1}, \quad l_i^+ = l_{2i} \qquad i = 1, \dots, k.$$

To each admissible decomposition $\omega = \{(l_i^-, l_i^+)\}_i$ of a vertex v, we associate the energy $\psi(\omega)$ defined by

$$(47) \qquad \psi(\omega) = \sum_{i=1}^{k} g\left(\nu(l_i^-), \nu(l_i^+)\right),$$

and we define

$$(48) \qquad E(v) = \min\left\{\psi(\omega) \mid \omega \in \Omega_v\right\}.$$

Lemma 5.3 *Let $\gamma \in \mathcal{C}$ satisfy conditions (Γ_2) and (Γ_3) in Proposition 3.2, and let P be the polyhedron associated to γ from (Γ_3). Let $v \in V(P)$ and let $l_1, \ldots, l_{2k}$ be the segments of γ which intersect v. Let $l_1^-, \ldots, l_k^-$ be the segments of $\{l_1, \ldots, l_{2k}\}$ which are oriented toward v, and let $l_1^+, \ldots, l_k^+$ be the elements of $\{l_1, \ldots, l_{2k}\}$ which, following the parametrization of γ, are after $l_1^-, \ldots, l_k^-$ respectively. Then $\omega_v^\gamma = (l_j^-, l_j^+)$, $j = 1, \ldots, k$, is an admissible decomposition of v.*

PROOF. Property (AD_1) is immediate to verify. Condition (AD_2) is equivalent to the fact that adjacent sides must have opposite orientations. ■

Proposition 5.4 *Suppose φ satisfies conditions* (i) *and* (ii) *above and let P be an admissible polyhedron. Then there holds*

$$G(\chi_P) = \sum_{v \in V(P)} E(v). \tag{49}$$

PROOF. Let us prove first the Γ-lim inf inequality. Let $\varepsilon_n \to 0$, let $(E_n)_n$ satisfy hypotheses $(H_1) - (H_3)$, and let $u = \chi_P$. Let $\gamma \in \mathcal{C}$ be given by Proposition 3.2. Then, if ω_v^γ is given by Lemma 5.3, there holds

$$\tilde{F}(\gamma) = \sum_{v \in V(P)} E(\omega_v^\gamma).$$

Finally, using equation (48) and property (Γ_4) in Proposition 3.2 we get

$$\sum_{v \in V(P)} E(v) \le \sum_{v \in V(P)} E(\omega_\gamma^v) = \tilde{F}(\gamma) \le \liminf_n F_{\varepsilon_n}(E_n).$$

This proves the Γ-lim inf inequality; let us now turn to the Γ-lim sup inequality.

Let $v \in V(P)$ and let $\overline{\omega}_v$ be an admissible decomposition of v which realizes the minimum energy, namely for which

$$\psi(\overline{\omega}_v) = E(v).$$

The set of the admissible decompositions $\overline{\omega}_v$, when v varies over $V(P)$, determines an element $\gamma \in \mathcal{C}$ in the following way.

Given a side l^1 of P, are uniquely determined two vertices v_1 and v_2 and two indices i_1 and i_2 for which, if we set $\overline{\omega}_1 = \left\{(l_{i,1}^+, l_{i,1}^-)\right\}_i$ and $\overline{\omega}_2 = \left\{(l_{i,2}^+, l_{i,2}^-)\right\}_i$, we have

$$l^1 = l_{i_1,1}^+ = l_{i_2,2}^-.$$

Let $l^2 = l_{i_2,2}^-$; reasoning as above, there exist an unique vertex v_3 an an unique index i_3 for which, if we set $\overline{\omega}_3 = \left\{(l_{i,3}^+, l_{i,3}^-)\right\}_i$, there holds

$$l^2 = l_{i_2,2}^+ = l_{i_3,3}^-.$$

Continuing in this way, we obtain a first segment l^{j_1} for which $l^{j_1} = l_{i_1,1}^-$. Let $c^i : [\alpha^i, \beta^i] \to \mathbb{R}^2$, $i = 1, \ldots, j$ be parametrizations of the sides l^i, and consider the closed curve γ^1 defined by

$$\gamma^1 = c^1 * \cdots * c^j.$$

Up to reparametrizations, we can suppose that γ^1 is defined on S^1. In the same way, we define the curves $\gamma^2, \ldots, \gamma^k : S^1 \to \mathbb{R}^2$ until all the remaining sides of P are considered.

Now we fix a number $M > 0$, a sequence of positive numbers δ_n converging to zero, and we consider the set

$$A_n = \left\{\cup \overline{B}_{M\delta_n}(v) \,|\, v \in V(P)\right\}.$$

Let γ^1 be the curve defined above, and let $\xi_n^1 = \{t \in S^1 : \gamma^1(t) \in A_n\}$. The set ξ_n^1 is a finite union of closed intervals $[\alpha_n^{1,i}, \beta_n^{1,i}]$, $i = 1, \dots, j_1$, and we denote by $(\sigma_n^{1,i}, \tau_n^{1,i})$, $i = 1, \dots, j_1$, the components of $S^1 \setminus \xi_n^1$, where we have taken $\sigma_n^{1,i} = \beta_n^{1,i}$. Setting $\overline{c}_n^{1,i} = \gamma^1|_{[\alpha_n^{1,i}, \beta_n^{1,i}]}$, and $\hat{c}_n^{1,i} = \gamma^1|_{[\sigma_n^{1,i}, \tau_n^{1,i}]}$, it is clear that

$$\gamma^1 = \overline{c}_n^{1,1} * \hat{c}_n^{1,1} * \overline{c}_n^{1,2} * \dots * \overline{c}_n^{1,j_1} * \hat{c}_n^{1,j_1}.$$

Of course, we can write a similar expression for $\gamma^2, \dots, \gamma^k$.

We observe that the maps $\overline{c}_n^{i,l}$, $i = 1, \dots, k$, $l = 1, \dots, j_i$, are union of two rectilinear curves with directions $\nu_-^{i,l}$ and $\nu_+^{i,l}$ (following the order of the parametrization), while the curves $\hat{c}_n^{i,l}$ are rectilinear with direction $\nu_+^{i,l}$.

We define also the curves

$$\tilde{c}_n^{i,l}(t) = \overline{c}_n^{i,l}(t) + \delta_n \left(\nu_-^{i,l} + \nu_+^{i,l}\right), \qquad t \in [\alpha_n^{i,l}, \beta_n^{i,l}];$$

where the above sum $\nu_-^{i,l} + \nu_+^{i,l}$is now a sum of elements in $\mathbb{R}^2$. It follows from property (AD_2) that the images of the curves $\tilde{c}_n^{i,l}$ are all disjoint when i varies from 1 to k, and l varies from 1 to j_i. We have also

$$\frac{(\tilde{c}_n^{i,l})'(\beta_n^{i,l})}{\left|(\tilde{c}_n^{i,l})'(\beta_n^{i,l})\right|} = \nu_+^{i,l} = \nu_-^{i,l+1} = \frac{(\tilde{c}_n^{i,l+1})'(\alpha_n^{i,l+1})}{\left|(\tilde{c}_n^{i,l+1})'(\alpha_n^{i,l+1})\right|}, \qquad \text{for all } i = 1, \dots, k,\ l = 1, \dots, j_i. \tag{50}$$

Now we choose a function $\eta : [0,1] \to [0,1]$ of class C^∞ and which satisfies the following properties

$$\begin{cases} \eta \equiv 0 \text{ in a neighbourhood of } 0; \\ \eta \equiv 1 \text{ in a neighbourhood of } 1; \\ \eta' \geq 0; \quad |\eta'| \leq 2; \quad |\eta''| \leq 4, \end{cases} \tag{51}$$

and for $a, b > 0$, let $\eta_{a,b} : [0,1] \to \mathbb{R}^2$ be defined by

$$\eta_{a,b}(t) = \begin{pmatrix} a\,t \\ b\,\eta(t) \end{pmatrix}; \qquad t \in [0,1].$$

Using simple computations, one can check that

$$|\kappa\left(\eta_{a,b}(t)\right)| \leq 4\,\frac{b}{a}, \qquad \text{for all } t \in [0,1]. \tag{52}$$

We recall $\kappa\left(\eta_{a,b}(t)\right)$ denotes the curvature of $\eta_{a,b}$ at $\eta_{a,b}(t)$.

Fix $i \in \{1, \dots, k\}$, $l \in \{1, \dots, j_i\}$, and consider the points $\tilde{c}_n^{i,l}(\beta_n^{i,l})$ and $\tilde{c}_n^{i,l+1}(\alpha_n^{i,l+1})$; then by equation (50) there exist unique numbers $a, b > 0$, and an unique affine isometry T of $\mathbb{R}^2$ for which the curve $T \circ \eta_{a,b}$ possesses the following properties (we omit the dependence of a, b, T on the indices i, l and n):

$$\begin{cases} T \circ \eta_{a,b}(0) = \tilde{c}_n^{i,l}(\beta_n^{i,l}); \quad T \circ \eta_{a,b}(1) = \tilde{c}_n^{i,l+1}(\alpha_n^{i,l+1}); \\ \left(T \circ \eta_{a,b}\right)'(0) = \nu_+^{i,l}; \quad \left(T \circ \eta_{a,b}\right)'(1) = \nu_+^{i,l}. \end{cases}$$

One can easily check that

$$|b| \leq 2\,\delta_n, \quad a \geq \frac{1}{2}\,\overline{s}; \qquad \text{for } n \text{ large};$$

see the Notation for the definition of $\overline{s}$. From these equations and from (52), it follows that

$$\left|\frac{(T \circ \eta_{a,b})'}{|(T \circ \eta_{a,b})'|} - \nu_i\right| \leq 8\,\frac{\delta_n}{\overline{s}(P)}; \qquad |\kappa(T \circ \eta_{a,b})| \leq 16\,\frac{\delta_n}{\overline{s}(P)}. \tag{53}$$

Denote by $\tilde{C}_n^{i,l}$ the curve $\eta_{a,b}$, where a, b are chosen as above depending on i, l, n, and consider

$$\tilde{\gamma}_n^i = \overline{c}_n^{1,1} * \tilde{C}_n^{1,1} * \overline{c}_n^{1,2} * \cdots * \overline{c}_n^{1,j_i} * \tilde{C}_n^{1,j_i}.$$

It follows from the first equation in (53) that if M is sufficiently large, then the curves $\tilde{\gamma}_n^i$, $i = 1, \ldots, k$ are simple, mutually disjoint, and the union of their images is the boundary of a piecewise C^2 set $\tilde{E}_n \subseteq \mathbb{R}^2$. It is clear that $\tilde{E}_n \to P$ in $L^1(\mathbb{R}^2)$.

Let $\varepsilon_n \to 0$: for every $i \in \{1, \ldots, k\}$ and every $l \in \{1, \ldots, j_i\}$, let $a^{i,l}, b^{i,l}$, etc., be the analogous of a, b, c, d in the proof of Proposition 4.1 when we consider $v^{i,l}$, $\nu_-^{i,l}$ and $\nu_+^{i,l}$. Since φ is assumed to be of class C^1, we can choose $\delta_n \to 0$ and $e^{i,l}(n), f^{i,l}(n)$ with the following properties:

(i) $\displaystyle\lim_n \frac{\delta_n}{\varepsilon_n \left(|e^{i,l}(n)| + |f^{i,l}(n)|\right)} = +\infty$ for all $i \in \{1, \ldots, k\}$ and every $l \in \{1, \ldots, j_i\}$;

(ii) $\displaystyle\lim_n \frac{1}{\varepsilon_n} C\left(\frac{8}{\overline{s}}\delta_n\right) = 0$;

see (27) for the definition of $C(\delta)$.

We have

$$\int_{[0,1]} \frac{1}{\varepsilon_n} \varphi\Big(\frac{(\tilde{C}_n^{i,l})'}{|(\tilde{C}_n^{i,l})'|}\Big) dt + \varepsilon_n \int_{[0,1]} \kappa^2(\tilde{C}_n^{i,l})\, dt \le \frac{1}{\varepsilon_n} C\left(\frac{8}{\overline{s}}\delta_n\right) + \varepsilon_n \left(\frac{16}{\overline{s}}\right)^2 \delta_n^2.$$

From property (ii) above and from (53), it follows that

$$\lim_n \left(\int_{[0,1]} \frac{1}{\varepsilon_n} \varphi\Big(\frac{(\tilde{C}_n^{i,l})'}{|(\tilde{C}_n^{i,l})'|}\Big) dt + \varepsilon_n \int_{[0,1]} \kappa^2(\tilde{C}_n^{i,l})\, dt \right) = 0. \tag{54}$$

By Remark 4.1, for every $i \in \{1, \ldots, k\}$, every $l \in \{1, \ldots, j_i\}$ and every n sufficiently large it is possible to choose a curve $\overline{C}_n^{i,l} : [\alpha_n^{i,l}, \beta_n^{i,l}] \to \mathbb{R}^2$ such that

$$\left| \overline{C}_n^{i,l}(t) - \overline{c}_n^{i,l}(t) \right| \le 2\varepsilon_n \left(|e^{i,l}(n)| + |f^{i,l}(n)|\right); \tag{55}$$

$$\overline{C}_n^{i,l} \text{ coincides with } \overline{c}_n^{i,l} \text{ in a neighbourhood of } \{\alpha_n^{i,l}, \beta_n^{i,l}\}; \tag{56}$$

$$\int_{[\alpha_n^{i,l}, \beta_n^{i,l}]} \frac{1}{\varepsilon_n} \varphi\Big(\frac{(\overline{C}_n^{i,l})'}{|(\overline{C}_n^{i,l})'|}\Big) dt + \int_{[\alpha_n^{i,l}, \beta_n^{i,l}]} \varepsilon_n\, \kappa^2(\overline{C}_n^{i,l})\, dt \to g(\nu_-^{i,l}, \nu_+^{i,l}). \tag{57}$$

Let $\overline{\gamma}^i$ be the curve defined by

$$\tilde{\gamma}_n^i = \overline{C}_n^{i,1} * \tilde{C}_n^{i,1} * \overline{C}_n^{i,2} * \cdots * \overline{C}_n^{i,j_i} * \tilde{C}_n^{i,j_i}.$$

From (56) it follows that the curve $\tilde{\gamma}_n^i$, $i = 1, \ldots, k$, are curves of class C^2, while (55) implies that they are simple, mutually disjoint, and the union of their images is the boundary of a C^2 set $\overline{E}_n \subseteq \mathbb{R}^2$. Again, $\tilde{E}_n \to P$ in $L^1(\mathbb{R}^2)$. Moreover from (57) one can deduce that

$$\limsup_n F_{\varepsilon_n}(\overline{E}_n) \le \sum_{v \in V(P)} E(v).$$

This concludes the proof. ■

5.2 A symmetric case

In this section we treat the case in which the admissile polyhedra are polyrectangles, and the function φ is symmetric with respect to the axes x and y. A direct proof of Theorem 5.1 is also presented in [10], Appendix B.

Theorem 5.1 *Let $\mathbf{e}_1, \mathbf{e}_2$ be the canonical basis of $\mathbb{R}^2$, and suppose that φ satisfies the conditions*

$$(58) \qquad \varphi^{-1}(0) = \{\nu_1, \dots, \nu_4\}, \quad \text{where} \quad \nu_1 = \mathbf{e}_1, \quad \nu_2 = \mathbf{e}_2, \quad \nu_3 = -\mathbf{e}_1, \quad \nu_4 = -\mathbf{e}_2,$$

and

$$(59) \qquad g_0 := g(\nu_i, \nu_{i+1}) \quad \text{is independent of } i = 1, \dots, 4.$$

Then the admissible polyhedra are polyrectangles and for every $P \in \mathcal{P}$ there holds

$$G(P) = g_0 \times \sharp\{\text{vertices of } P\}.$$

Proof. Let us prove first the Γ-lim inf inequality. We note that if $P \in \mathcal{R}$, then one has

$$(60) \qquad F_{\mathcal{R}}(P) = \sharp\{\text{vertices of } P\} = \sharp\{\text{sides of } P\}.$$

Let $E \in \mathcal{P}$, and let $E_k \in \mathcal{R}$, $E_k \to E$ in $L^1(\mathbb{R}^2)$. Then, since it must be $\sharp\{\text{sides of } E_k\} \geq \sharp\{\text{sides of } E\}$ for k large, it follows from (60) that

$$F_{\mathcal{R}}(E_k) \geq \sharp\{\text{sides of } E\} \geq \sharp\{\text{vertices of } E\}, \qquad \text{for } k \text{ large}.$$

Hence we have also

$$G(E) = sc^-(F_{\mathcal{R}})(E) \geq \sharp\{\text{vertices of } E\},$$

which is the Γ-lim inf inequality. Let us prove now the Γ-lim sup inequality. Given a polyrectangle E, and given a number $\sigma > 0$, consider the set E_σ defined by

$$E_\sigma = \{x \in E \,:\, \text{dist}(x, \partial E) \leq \sigma\}.$$

Then, if σ is sufficiently small, $E_\sigma \in \mathcal{R}$, and $\sharp\{\text{sides of } E_\sigma\} \leq \sharp\{\text{sides of } E\}$. This concludes the proof. ■

6 Pathological cases

In this section we consider the case in which it is not required the uniform boundedness of the perimeters in the definition of convergence. In this situation, it is possible to have the convergence in the $L^1_{\text{loc}}(\mathbb{R}^2)$ sense without having convergence in $L^1(\mathbb{R}^2)$, so we are led to consider the quantity

$$\overline{G}(E) = \inf\{\liminf_n F_{\varepsilon_n}(E_n) \,:\, E_n \to E \text{ in } L^1_{\text{loc}}(\mathbb{R}^2)\}.$$

We recall that, by Theorem 2.1, $G(E) = \inf\{\liminf_n F_{\varepsilon_n}(E_n) \,:\, E_n \to E \text{ in } L^1(\mathbb{R}^2), \sup_n \mathcal{H}_1(\partial E_n) < +\infty\}$, so it is clearly $\overline{G}(E) \leq G(E)$. In Section 6.1 we describe the asymptotic shape of the subsequences $(E_n)_n$ for which $\sup_n F_{\varepsilon_n}(E_n) < +\infty$, highlighting similarities with Section 3. However, in general $\overline{G} < G$, and in Section 6.2 we exhibit an example of a function φ and of a polyhedron P for which $\overline{G}(P)$ is stricly less than $G(P)$.

6.1 Asymptotic shape of minimizers

In this subsection we describe the limit shape of a sequence of sets $(E_n)_n$ for which just condition (H_3) holds, thus without assuming that the perimeters ∂E_n are uniformly bounded.

We suppose that ∂E_n possesses just one component; the general case requires only simple modifications. Let γ_n be a parametrization of ∂E_n proportional to the arc lenght. First, we note that Lemmas 3.1 and 3.2 remain unchanged, so we can define as in Section 3 the quantities $\delta_n \to 0$, I_n and J_n with $|I_n| \to 0$. In general, we will not have uniform convergence on the components of J_n as in Lemma 3.3, but we recover it under some suitable rescaling.

Lemma 6.1 *Let J_n be defined as in (30), and let $(\theta_n^h, \sigma_n^{h+1})$ be a component of J_n such that*

$$\dot\gamma_n(\theta_n^h) = |\dot\gamma_n|\,(\nu_i \pm \delta_n) \qquad \textit{for some } \nu_i \in \{\nu_1, \dots, \nu_N\},$$

and such that $|\gamma_n(\theta_n^h) - \gamma_n(\sigma_n^{h+1})| \to +\infty$ as $n \to +\infty$. Let $\tilde\gamma_n : (\theta_n^h, \sigma_n^{h+1})$ be defined by

$$\tilde\gamma_n(t) = \frac{1}{|\gamma_n(\theta_n^h) - \gamma_n(\sigma_n^{h+1})|}\,(\gamma_n(t) - \gamma_n(\theta_n^h)).$$

Then we have

$$\sup_{t \in (\theta_n^h, \sigma_n^{h+1})} |\tilde\gamma_n(t) - \nu_i\, t| \to 0, \qquad \textit{as } n \to +\infty.$$

Proof. We have $|\tilde\gamma_n(t)| \le C$ on $(\theta_n^h, \sigma^{h+1})$, and moreover

$$\int_{(\theta_n^h, \sigma_n^{h+1})} \frac{1}{\varepsilon_n}\,\varphi\left(\frac{\dot{\tilde\gamma}_n}{|\dot{\tilde\gamma}_n|}\right) dt \le \int_{(\theta_n^h, \sigma_n^{h+1})} \frac{1}{\varepsilon_n}\,\varphi\left(\frac{\dot\gamma_n}{|\dot\gamma_n|}\right) dt.$$

Hence, considering the curve $\tilde\gamma_n$, we are in the same situation of Lemma 3.3, so our statement follows. ■

Passing to a subsequence, we find an integer k, and k sequences of points $(x_n^1)_n, \dots, (x_n^k)_n$ such that

$$\mathrm{dist}(\gamma_n(I_n), \{x_n^1, \dots, x_n^k\}) \to 0, \qquad \text{as } n \to +\infty.$$

In this case, the mutual distances of the points x_n^i can go to infinity. However it turns out that the sequences of points $\{x_n^1, \dots, x_n^k\}$ arrange in "clusters", and the limit shape of some rescaled portion of E_n is still polyhedral.

In fact, let

$$d_n^1 = \sup\{|x_n^i - x_n^j| \,:\, i, j \in \{1, \dots, k\}, i \ne j\},$$

and consider the sequence of sets

$$E_n^1 = (d_n^1)^{-1}\,(E_n - x_n^1).$$

Let γ_n^1 be a parametrization of ∂E_n^1. Then, there exists a number $k^1 \le k$ and k_1 sequences of points $(x_n^{1,1})_n, \dots, (x_n^{1,k_1})_n$ such that

$$\mathrm{dist}(\gamma_n^1(I_n), \{x_n^{1,1}, \dots, x_n^{1,k_1}\}) \to 0, \qquad \text{as } n \to +\infty.$$

From Lemma 6.1, it is easy to see that the sequence E_n^1 converges in $L^1(\mathbb{R}^2)$ to some admissible polyhedron $P^1 \in \mathcal{P}$.

If we choose a different rescaling for the set E_n, we can obtain some "finer" structures of these sets. In fact, consider the set of indices $\{i_1, \dots, i_j\} \subseteq \{1, \dots, k\}$, for which

$$\lim_n\,(d_n^1)^{-1}\,|x_n^{i_l} - x_n^1| \to 0, \qquad l = 1, \dots, j.$$

and define d_n^2 to be

$$d_n^2 = \sup\{|x_n^{i_l} - x_n^{i_h}| \ : \ l, h \in \{i_1, \ldots, i_j\}, l \neq h\};$$

it is clear that $(d_n^1)^{-1} d_n^2 \to 0$. Consider the sequence of sets E_n^2

$$E_n^2 = (d_n^2)^{-1} (E_n - x_n^1).$$

Then, using the arguments above, one can check that $E_n^2 \to P^2$ in $L^1_{\rm loc}(\mathbb{R}^2)$, where $P^2 \subseteq \mathbb{R}^2$ is a set which boundary is composed by segments, half-lines or lines oriented in the directions $\{\nu_1, \ldots, \nu_N\}$. In some sense P^2 could be considered as a polyhedron with some sides of infinite lenght.

Of course, the same result is true if one considers suitable rescalings at the points x_n^i for $i \neq 1$.

6.2 An example in which $\overline{\mathrm{G}} \neq \mathrm{G}$

In this subsection we consider the following particular case, namely $\varphi^{-1}(0) = \{\nu_1, \ldots, \nu_5\}$ with

$$\nu_1 = (1,0); \quad \nu_2 = (0,1); \quad \nu_3 = \frac{\sqrt{2}}{2}(-1,1); \quad \nu_4 = \frac{\sqrt{2}}{2}(-1,-1); \quad \nu_5 = (0,-1), \tag{61}$$

and

$$g(\nu_1, \nu_2) = g(\nu_2, \nu_3) = g(\nu_4, \nu_5) = g(\nu_1, \nu_2) = 1; \qquad g(\nu_5, \nu_1) = 5. \tag{62}$$

Let $p_i, q_i \in \mathbb{R}^2$, $i = 1, \ldots, 3$, be given by

$$p_1 = (0,0), \quad p_2 = (1,0), \quad p_3 = (1,1); \qquad q_1 = (2,0), \quad q_2 = (3,0), \quad q_3 = (2,1),$$

and let P be the polyhedron defined as follows (see figure (a))

$$P = \left\{ \sum_{i=1}^3 t_i \, p_i \, | \, t_i \geq 0, \sum_{i=1}^3 t_i = 1 \right\} \cup \left\{ \sum_{i=1}^3 t_i \, q_i \, | \, t_i \geq 0, \sum_{i=1}^3 t_i = 1 \right\}.$$

It is clear from (61) that $P \in \mathcal{P}$. We show that in this case $\overline{G}(P)$ is strictly less than $G(P)$.

In fact, let $(E_n)_n \subseteq \mathcal{R}$ be a sequence of sets of class C^2 as in figure (b). It is clear that the boundary of E_n has just one component and from (62) one can check that $F_{\mathcal{R}}(P_n) = 17 + o(1)$, where $o(1) \to 0$ as $n \to +\infty$.

Now, suppose by contradiction that $G(P) = \overline{G}(P) \leq 17$, namely that there exists $(E_n)_n \subseteq \mathbb{R}^2$ with

$$E_n \to E \text{ in } L^1(\mathbb{R}^2), \qquad \sup_n \mathcal{H}_1(\partial E_n) < +\infty, \qquad \lim_n F_{\varepsilon_n}(E_n) \leq 17.$$

Passing to a subsequence, we can assume that the number of the components of ∂E_n is a fixed number k independent of n. By Lemma 3.1, it turns out that $F_{\varepsilon_n}(E_n) \geq 9k + o(1)$, so, since we are assuming that $F_{\varepsilon_n}(E_n) \leq 17 + o(1)$, it follows that $k = 1$.

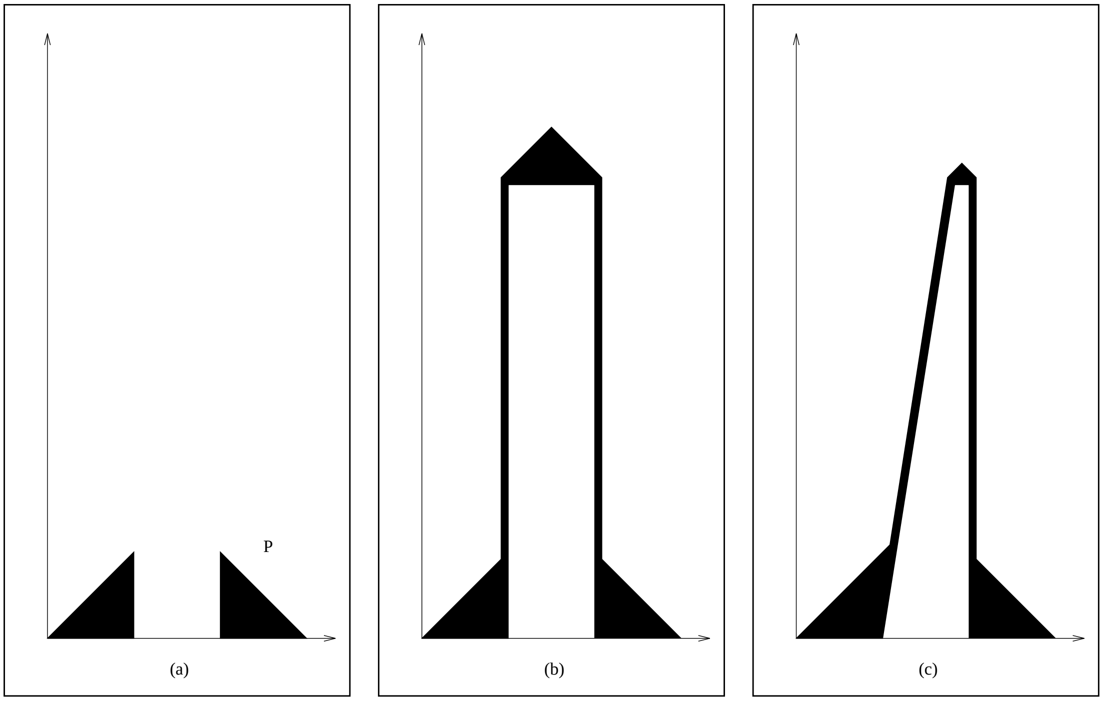
Let $\gamma_n : S^1 \to \mathbb{R}^2$ be a parametrization of ∂E_n proportional to the arc lenght. Then we can apply Proposition 3.2, and we find a curve $\gamma : S^1 \to \mathbb{R}^2$, $\gamma \in \mathcal{C}$, for which $\gamma_n \to \gamma$ uniformly on S^1, and for which $P = \{x \in \mathbb{R}^2 \ : \ \mathrm{ind}(\gamma, x) = 1\}$.

Consider the set

$$A = \{t \in S^1 \ : \ 1 < (\gamma)_x(t) < 2, \dot{\gamma}(t) \in \{\nu_3, \nu_4\}\}.$$

Since γ has just one component, it must be $A \neq \emptyset$, and since $-\nu_3$ and $-\nu_4$ do not belong to $\varphi^{-1}(0)$, it should be $\gamma(A) \subseteq \partial P$, which is a contradiction.

Remark 6.1 It is possible to have $\overline{G}(P) < G(P)$ also if we require strong L^1 convergence in the definition of $\overline{G}$. In fact, if φ is of class C^1, one could choose a sequence of approximating sets $(E_n)_n$ as in figure (c). Reasoning as in Section 5, one can prove that $F_{\varepsilon_n}(E_n) = 17 + o(1)$.



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