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A NOTE ON AN EXPONENTIAL SEMILINEAR EQUATION OF THE FOURTH ORDER

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Abstract. In this note we prove some existence and nonexistence results for a semilinear equation of the fourth order involving an exponential function via variational techniques. Such equations may occur while studying traveling waves in suspension bridges.

1. Introduction

Recently Chen and McKenna ([2]) have suggested investigating the following equation:

$$u_{xxxx} + cu_{xx} = e^u - 1 \tag{1.1}$$

as a generalization of the model for traveling waves in suspension bridges proposed by Lazer and McKenna in [4],

$$(LM) \qquad \begin{cases} \Delta^2 u + c\Delta u - b[u + (u+1)^-] = 0 & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

where b is a real parameter, Δ^2 stands for the biharmonic operator and $v^- = \max\{0, -v\}$. Moreover, in [4] the authors suggested studying problems similar to (LM), where the nonlinearity $h(u) = u + (u+1)^-$ is replaced by some function which behaves like $h(u)$ (see Problem 11 in [4]).

For example, the functions $h(u) = u + (u+1)^- = (u+1)^+ - 1$ and $g_2(u) = e^u - 1$ have similar behavior, at least if u is bounded from above.

Moreover, one can also consider nonlinear problems of the form

$$(P_k) \qquad \begin{cases} \Delta^2 u + c\Delta u - bu + a((u+1)^-)^{k-1} = 0 & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $a, b \in \mathbb{R}$ and k is greater than 2 and subcritical (see [10]).

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In this paper we are interested in a problem related to (1.1). More precisely we will consider the problem

$$(P) \qquad \begin{cases} \Delta^2 u + c\Delta u - b(e^u - 1) = 0 & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded and smooth domain of $\mathbb{R}^N$ and $b \in \mathbb{R}$. By a (weak) solution we mean a function u in $H_\Omega = H_0^1(\Omega) \cap H^2(\Omega)$ such that

$$\int_\Omega \Delta u \Delta v \, dx - c \int_\Omega Du \cdot Dv \, dx - b \int_\Omega (e^u v - v) \, dx = 0 \tag{1.2}$$

for every $v \in H_\Omega$. Let us remark that all integrals appearing in (1.2) are well defined when $N \leq 3$ and Ω is bounded.

We remark that $u \equiv 0$ is a solution of all the problems considered above. Moreover, one could expect that solutions of problems (P), (LM), and (P_k) behave similarly, since the nonlinearities are similar, but it is not the case, as we will see. In fact, it is not difficult to prove that, for example, if Ω is a bounded and smooth domain, $c < \lambda_1$ (the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$), $b = 0$, and $a \leq 0$, then *any* solution u of (P_k) is less than or equal to 0 (and on the other hand, there are nontrivial solutions of such a problem). In fact, a maximum principle for equations of the fourth order show that, if $u \not\equiv 0$, then $u < 0$ in Ω (see Section 2). But in [11] it is proved that, if $c < \lambda_1$, then (LM) has a nontrivial solution if and only if $b \geq \lambda_1(\lambda_1 - c)$ and that in such a case there exists a solution $u < 0$ in Ω. (Concerning existence and multiplicity results for problem (LM) there is a large literature. For example, see [3], [5], [6], [7], [9], and the references therein.)

Moreover, in the same spirit of [11], we are going to prove in a very simple way that, if $b \leq 0$ and $c < \lambda_1$, then problem (P) does *not* have nontrivial solutions. In particular we will prove the following results.

Proposition 1. *Suppose $b \leq 0$, Ω is a bounded and smooth domain of $\mathbb{R}^N$, and $c < \lambda_1$. Then (P) admits only the trivial solution.*

On the other hand, we will show a result for (P) which is different in nature from the one found in [11] for (LM). In fact, we will prove the existence of nontrivial solutions for positive values of b which are below $\lambda_1(\lambda_1 - c)$. More precisely, the following theorem holds.

Theorem 2. *Suppose $N \leq 3$, Ω is a bounded and smooth domain of $\mathbb{R}^N$, $c < \lambda_1$ and $0 < b < \lambda_1(\lambda_1 - c)$. Then (P) has a nontrivial solution.*

We remark that such results cover the "physical case" ($N \leq 3$).

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2. Notations, a remark on (P_k) and proof of Proposition 1

Let Ω be a bounded and smooth domain of $\mathbb{R}^N$, $N \geq 1$, $b, c \in \mathbb{R}$, and $b \leq 0$. Let us consider the following problem:

$$(P) \qquad \begin{cases} \Delta^2 u + c\Delta u - b(e^u - 1) = 0 & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases}$$

Denote by λ_i the i-th eigenvalue of $-\Delta$ in $H_0^1(\Omega)$ with eigenvalue e_i. It will also be useful to introduce the couple of eigenvalues and eigenfunctions $(\Lambda_i, E_i)_i$ of the problem

$$\begin{cases} \Delta^2 E + c\Delta E = \Lambda E & \text{in } \Omega, \\ E = \Delta E = 0 & \text{on } \partial\Omega. \end{cases}$$

We remark that, if $c < \lambda_1$, it turns out that $\Lambda_i = \lambda_i^2 - c\lambda_i$ and $E_i = e_i \ \forall\, i$. Recall that one can choose $e_1 > 0$ in Ω.

We will use the following version of the maximum principle for fourth-order equations (see [8]).

Theorem 3 (Maximum Principle)**.** *Suppose $c < \lambda_1$ and u satisfies*

$$\begin{cases} \Delta^2 u + c\Delta u \leq 0 & \textit{in } \Omega, \\ u = \Delta u = 0 & \textit{on } \partial\Omega. \end{cases} \tag{2.1}$$

Then either $u \equiv 0$, or for every ball B contained in Ω, $\sup_B u < 0$.

Of course (2.1) is considered in the sense of distributions; that is, $\forall\, v$ in H_Ω, $v \geq 0$, one has

$$\int_\Omega \Delta u \Delta v \, dx - c \int_\Omega Du \cdot Dv \, dx \leq 0.$$

A straightforward application of this theorem implies the following proposition.

Proposition 4. *Suppose $b = 0$, $a \leq 0$, and $c < \lambda_1$. Then any nontrivial solution u of problem (P_k) is such that $u < 0$ in Ω.*

On the other hand, one can prove that there is *always* a nontrivial solution of problem (P_k) (see [10], where also multiplicity results are shown). On the other hand, there are also solutions of (P_k) which are positive in Ω; for example, if $b = \lambda_1^2 - c\lambda_1$, all functions of the form $u = te_1$, $t > 0$, solve (P_k).

On the contrary, we are now going to prove Proposition 1, which moves in an opposite direction.

Proof of Proposition 1. Assume for the sake of contradiction that there exists $u \not\equiv 0$ which is a solution of (1.2). Then

$$\int_\Omega |\Delta u|^2\,dx - c\int_\Omega |Du|^2\,dx = b\int_\Omega (e^u - 1)u\,dx \le 0,$$

since $(e^s - 1)s \ge 0\ \forall\, s \in \mathbb{R}$.

But the left-hand side of such an inequality is greater than or equal to

$$(\lambda_1^2 - c\lambda_1)\int_\Omega u^2\,dx,$$

which is positive, a contradiction. □

3. Proof of Theorem 2

Problem (P) has a natural variational structure. In fact, its solutions are the critical points of the functional defined on the Hilbert space $H_\Omega = H_0^1(\Omega) \cap H^2(\Omega)$ as

$$f(u) = \frac{1}{2}\int_\Omega |\Delta u|^2\,dx - \frac{c}{2}\int_\Omega |Du|^2 - b\int_\Omega (e^u - u - 1)\,dx.$$

It is clear, in fact, that f is a smooth functional on H_Ω, since $N \le 3$.

Let us remark that the space H_Ω can be endowed with the scalar product $(u, v) = \int \Delta u \Delta v$, which induces the norm $\|u\| = (\int_\Omega |\Delta u|^2)^{1/2}$, equivalent to the standard norm of H^2 (see [10]).

We first prove the usual compactness condition, known as the Palais-Smale condition at level a, $(PS)_a$ for short: if $(u_n)_n$ in H_Ω is such that $f(u_n) \to a$ and $f'(u_n) \to 0$, then there exists a subsequence $(u_{n_k})_k$ which converges.

Proposition 5 (Palais-Smale)**.** *Assume $N \le 3$, $c \ne \lambda_i\ \forall\, i$, and $b \in \mathbb{R}$. Then f satisfies $(PS)_a\ \forall\, a$ in $\mathbb{R}$.*

Proof. Let $(u_n)_n$ be such that $f(u_n) \to a$ and $f'(u_n) \to 0$. In order to prove $(PS)_a$ it is enough to prove that $(u_n)_n$ is bounded. Indeed, one has

$$\operatorname{grad} f(u) = u + i^*(c\Delta u - be^u - 1),$$

where i^* is the adjoint (compact) operator to the immersion of H_Ω into $L^2(\Omega)$.

First of all observe that

$$\frac{f'(u_n)(u_n)}{\|u_n\|} = \frac{1}{\|u_n\|}\left\{\int_\Omega |\Delta u_n|^2\,dx - c\int_\Omega |Du_n|^2\,dx - b\int_\Omega [e^{u_n}u_n - u_n]\,dx\right\}$$
$$= \frac{1}{\|u_n\|}\left\{2f(u_n) + b\int_\Omega [2e^{u_n} - e^{u_n}u_n - u_n - 2]\,dx\right\}$$

so that

$$\frac{\int_\Omega [2e^{u_n} - e^{u_n}u_n]\,dx}{\|u_n\|} \to \int_\Omega u\,dx. \tag{3.1}$$

Moreover, $\frac{f(u_n)}{\|u_n\|} \to 0$, which implies

$$\lim_{n\to\infty}\left\{\|u_n\| - \frac{c\int_\Omega |Du_n|^2\,dx}{\|u_n\|} - \frac{b\int_\Omega e^{u_n}\,dx}{\|u_n\|}\right\} = -b\int_\Omega u\,dx, \tag{3.2}$$

and $\frac{f'(u_n)(u_n)}{\|u_n\|} \to 0$, so that

$$\lim_{n\to\infty}\left\{\|u_n\| - \frac{c\int_\Omega |Du_n|^2\,dx}{\|u_n\|} - \frac{b\int_\Omega e^{u_n}\,dx}{\|u_n\|}\right\} \tag{3.3}$$
$$= \lim_{n\to\infty} b\frac{\int_\Omega [e^{u_n}u_n - e^{u_n} - u_n]\,dx}{\|u_n\|}.$$

Combining (3.2) and (3.3) we get

$$\lim_{n\to\infty}\frac{\int_\Omega [e^{u_n}u_n - e^{u_n}]\,dx}{\|u_n\|} = 0. \tag{3.4}$$

In this way, (3.1) and (3.4) imply

$$\lim_{n\to\infty}\frac{\int_\Omega e^{u_n}\,dx}{\|u_n\|} = 0 \tag{3.5}$$

and

$$\lim_{n\to\infty}\frac{\int_\Omega e^{u_n}u_n\,dx}{\|u_n\|} = 0.$$

Now compute $f'(u_n)(E_i)$, which goes to 0 for every i; in this way, we get

$$\Lambda_i\frac{\int_\Omega u_nE_i\,dx}{\|u_n\|} - \frac{b\int_\Omega [e^{u_n}E_i - E_i]\,dx}{\|u_n\|} \to 0.$$

But the second integral goes to 0 as n goes to ∞ by (3.5), since E_i is bounded for every i, so that

$$\Lambda_i\int_\Omega uE_i = 0 \qquad \forall\, i;$$

that is, $u \equiv 0$, since $\Lambda_i \neq 0$ (for $c \neq \lambda_i \ \forall\, i$).

On the other hand, computing $\frac{f(u_n)}{\|u_n\|^2}$ and passing to the limit, we get

$$1 - c\int_\Omega |Du|^2\,dx = 0;$$

that is, $u \not\equiv 0$, and a contradiction arises.

In this way, $(u_n)_n$ is bounded and the $(PS)_a$ follows. □

We now want to prove that, if $0 < b < \lambda_1(\lambda_1 - c)$, f satisfies the geometrical assumptions to exhibit a mountain-pass structure (see [1]). To this purpose it is enough to show that if $|u| \leq \rho$, there exists a constant M_ρ, going to 1 as $\rho \to 0$, such that

$$f(u) \geq \frac{1}{2}\int_\Omega \left\{\Delta u|^2 - c|Du|^2 - bM_\rho u^2\right\} dx.$$

In fact, write $u = \sum \alpha_i e_i$, so that

$$f(u) \geq \frac{1}{2}\sum \alpha_i^2(\lambda_i^2 - c\lambda_i - bM_\rho) = \frac{1}{2}\sum \lambda_i^2\alpha_i^2\frac{\lambda_i^2 - c\lambda_i - bM_\rho}{\lambda_i^2}$$

$$\geq \frac{1}{2}\|u\|^2 \inf_i \frac{\lambda_i^2 - c\lambda_i - bM_\rho}{\lambda_i^2} = \frac{1}{2}\|u\|^2 m_\rho,$$

where

$$m_\rho = \min\left\{1, \frac{\lambda_1^2 - c\lambda_1 - bM_\rho}{\lambda_1^2}\right\} > 0.$$

In particular, $m_\rho = \frac{\lambda_1^2 - c\lambda_1 - bM_\rho}{\lambda_1^2}$ if $c \geq 0$ or $c < 0$ and $b \geq -c\lambda_1$, and $m_\rho = 1$ if $c < 0$ and $b < -c\lambda_1$.

In this way, if ρ is small enough, $\lambda_1^2 - c\lambda_1 - bM_\rho > 0$ and $u = 0$ is a strict local minimum for f.

Now take $t > 0$ and consider

$$f(te_1) = \frac{t^2}{2}(\lambda_1^2 - c\lambda_1) - a\int_\Omega \left[e^{te_1} - te_1 - 1\right] dx.$$

If $t \to +\infty$, it is clear that $f(te_1) \to -\infty$.

We have thus proved that f satisfies all the assumptions of the mountain-pass theorem. Therefore f has a nontrivial critical point; that is, (P) has a nontrivial solution.

So, we only have to prove that, if $|u| \leq \rho$, there exists M_ρ such that $e^u - u - 1 \leq M_\rho u^2/2$. Let us define

$$M_\rho = \sup_{|u|\leq\rho} 2\frac{e^u - u - 1}{u^2}.$$

We want to show that $M_\rho = 2(e^\rho - \rho - 1)/\rho^2$.

Assume first $0 \leq u \leq \rho$, and consider the function $g(u) = \frac{e^u - u - 1}{u^2}$ (where $g(0) := \lim_{u\to 0} g(u) = 1/2$). Then $g'(u) = \frac{(u-2)e^u + u + 2}{u^3}$. Setting $m(u) = (u-2)e^u + u + 2$, we get $m''(u) = ue^u$, so that m is convex on $[0, \rho]$. So m' is increasing. But $m'(0) = 0$, so m is increasing for u positive; that is, $m(u) > 0$, since $m(0) = 0$. This means $g'(u) > 0$ for $u > 0$, and so $g(u) \leq \frac{e^\rho - \rho - 1}{\rho^2}$. Note that $M_\rho \downarrow 1$ as $\rho \to 0$.

Now assume $u < 0$. Set $v = -u$, and consider the function $h(v) = \frac{e^{-v} + v - 1}{v^2}$. Reasoning as in the previous case, it is easy to show that $h(v) \leq h(0^-) = 1/2\ \forall\, v > 0$. We have thus proved that

$$\sup_{|u|\leq\rho} \frac{e^u - u - 1}{u^2} = \frac{e^\rho - \rho - 1}{\rho^2}.$$

Theorem 2 is thus completely proved.

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