DISCRETE-TO-CONTINUUM LIMITS FOR STRAIN-ALIGNMENT-COUPLED SYSTEMS: MAGNETOSTRICTIVE SOLIDS, FERROELECTRIC CRYSTALS AND NEMATIC ELASTOMERS

MARCO CICALESE, ANTONIO DESIMONE AND CATERINA IDA ZEPPIERI

Abstract. In the framework of linear elasticity, we study the limit of a class of discrete free energies modeling strain-alignment-coupled systems by a rigorous coarse-graining procedure, as the number of molecules diverges. We focus on three paradigmatic examples: magnetostrictive solids, ferroelectric crystals and nematic elastomers, obtaining in the limit three continuum models consistent with those commonly employed in the current literature. We also derive the correspondent macroscopic energies in the presence of displacement boundary conditions and of various kinds of applied external fields.

Keywords: Γ-convergence, discrete systems, linear elasticity, liquid crystals, magnetostrictive solids, ferroelectric crystals, nematic elastomers.

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Contents

1. Introduction

Many physical systems show a nontrivial mechanical response to applied electric or magnetic fields. Three noticeable examples, which have received considerable attention in the recent mathematical and physical literature, are magnetostrictive solids (*i.e.*, deformable ferromagnets), ferroelectric crystals, and liquid crystal elastomers (in particular, nematic elastomers).

Some features of the microscopic origin of the coupling between mechanical and electro-magnetic effects are common in the three cases, and can be described as follows. On the one hand, a symmetry-breaking phase transformation introduces some anisotropy in the system, accompanied by spontaneous deformations. This anisotropy consists in a distinguished direction of alignment of the nematic mesogens in the case of liquid crystals, or of the electric or magnetic dipoles in the other two cases. An applied electro-magnetic field can act on the alignment direction by turning it, and hence affecting the state of deformation of the system in the absence of applied mechanical forces. This also explains why an applied electro-magnetic field may interfere with the the mechanical response of the system to applied loads. We refer to all these phenomena as *strain-alignment coupling*.

The study of strain-alignment coupling has followed two main paths. On the one hand, the continuum mechanics literature has focussed on phenomenological theories based on classical elasticity, supplemented by the introduction of additional field variables to describe the presence of nematic order, magnetization, or electric polarization. On the other hand, statistical physicists have studied these materials at a more microscopic scale, typically leading to complex lattice models which are not explicitly solvable, and have been studied by means of Monte Carlo techniques. In this paper we aim at linking these two approaches by performing a rigorous *discrete-to-continuum* limit.

Roughly speaking, the discrete-to-continuum analysis of a physical system can be explained as follows (see [11]). One considers a set of interacting material points contained in a box. Denoting by ε a positive parameter proportional to the mean distance of the material points in an equilibrium configuration, one may consider the free energy E_ε of the system. By taking the limit as ε tends to zero, the discrete structure of the physical system naturally gives rise to a continuum picture since its points start "filling" the box. In particular, one may be interested in analyzing the asymptotic properties of E_ε, when computed on equilibrium configurations. It is by now well known that, at least for the case in which equilibria are minimizers of the free energy, a single mathematical object can be used for this purpose, namely, the Γ-limit of E_ε (see [9, 2] and references therein). This gives a way of establishing a correspondence between ground states of the discrete system with those of the continuum limit: the second are "generated" as macroscopic limits of the first, the first are discrete descriptions or "approximations" of the second.

The type of asymptotic analysis described above has a double value. On the one hand, deriving a macroscopic picture of a system from the mechanisms governing its microscopic structure may help to gain a better understanding of the phenomenological models currently employed at the macroscopic scale. In this perspective, a discrete-to-continuum analysis may provide a tool to justify the macroscopic models used by practitioners, or to choose among different conflicting proposals, as it settles them on more fundamental basis. On the other hand, the same analysis can be used backwards; *i.e.,* given a continuum model, it may provide a natural discretization scheme and a discrete energy which is ready to be used for computer simulations. Among all possible discretizations of the continuum models, the ones we consider have the advantage of yielding the convergence of the ground states of the discrete energy to the ground states of the continuum one, by the properties of Γ-convergence.

Many discrete models have been studied in recent years from the standpoint of Γ-convergence. Among them we mention some of those concerning *elasticity* (and their generalization to the fracture mechanics setting) as [4, 5, 7, 10, 12] and those about *ordered systems* such as [1, 3, 6]. In these two kinds of models the relevant variables describing the physics of the system are the *strain* and the *order variable* (*e.g.*, the local magnetic moment), respectively. Our paper borrows from both of these two lines of research since the phenomenon of strain-alignment-coupling depends on the interplay between deformation and order.

Our analysis focusses on three models which have been studied in the continuum setting by many authors (see, *e.g.*, [8, 15, 17, 18], [20]-[26], [28, 38]). In this framework the energy of a strain-alignment coupled system can be described as follows. We assume that the reference configuration of our sample is an open bounded set $\Omega \subset \mathbb{R}^2$. As suggested by the physical picture discussed so far, the energy of the system depends on two variables: the *deformation field* $u : \Omega \to \mathbb{R}^2$, which describes the elastic deformations of the sample and the *ordering field* $\nu : \Omega \to V$ (the space V depends on the particular model we are considering and will be further specified in the sequel) which takes into account, as an internal or microscopic variable, the ordering properties of the sample.

Then, the free energy of the system can be written as

$$E(u,\nu) = \frac{1}{2}\int_\Omega \mathbb{C}(\mathbf{E}u - \mathbf{E}_0(\nu)) : (\mathbf{E}u - \mathbf{E}_0(\nu))\ dx + E^I(\nu) + (a.t.), \tag{1.1}$$

where $\mathbb{C}$ is the *tensor of elastic moduli*, $\mathbf{E}(u) = \frac{1}{2}(\nabla u + \nabla u^T)$ is the strain, $\mathbf{E}_0(\nu)$ is the *stress-free strain*, $E^I(\nu)$ is the energy stored by the internal variable ν and $(a.t.)$ denotes some *additional terms* such as those due to the presence of external forces and applied electric or magnetic fields or local terms in ν such as crystalline anisotropy (see Section 8 for more details). Although such terms are very relevant from the physical point of view, they do not play a crucial role in the mathematical analysis. Notice that, as highlighted in the notation, the stress-free strain turns out to depend on the internal variable ν.

If we single out the the elastic term

$$E^{EL}(u) = \frac{1}{2}\int_\Omega \mathbb{C}\mathbf{E}u : \mathbf{E}u\ dx,$$

the energy (1.1) can be rewritten as

$$E(u,\nu) = E^{EL}(u) + E^{SA}(u,\nu) + E^I(\nu)$$

where we have denoted by E^{SA} the *strain-alignment* energy; *i.e.*,

$$E^{SA}(u,\nu) = -\int_\Omega \mathbb{C}\mathbf{E}_0(\nu) : \mathbf{E}u\ dx + \frac{1}{2}\int_\Omega \mathbb{C}\mathbf{E}_0(\nu) : \mathbf{E}_0(\nu)\ dx. \tag{1.2}$$

Throughout this paper we work under the assumption of isotropic linear elasticity

$$\mathbb{C}\mathbf{A} = 2\mu\mathbf{A} + \lambda(\mathrm{tr}\mathbf{A})\mathbf{Id},$$

and, moreover, we normalize the Lamé coefficients as $\mu = \lambda = \frac{1}{2}$; as a consequence we find that in particular

$$E^{EL}(u) = \frac{1}{2}\int_{\Omega} |\mathbf{E}u|^2\, dx + \frac{1}{4}\int_{\Omega} |\mathrm{div}\, u|^2\, dx. \tag{1.3}$$

In the above stated setting, we now come to describe the three relevant continuum models we are interested in, only focusing on their distinctive features.

Magnetostrictive solids. For this class of materials the relevant internal variable is the *local magnetization* $m : \mathbb{R}^2 \to \mathbb{R}^2$, with $|m| = 1$; *i.e.,* $V = S^1$. The energy stored in Ω by the internal variable has the form

$$E^{M,I}(m) = \frac{1}{2}\int_{\Omega} |\nabla m|^2\ dx - \frac{1}{2}\int_{\mathbb{R}^2} \langle m, \mathbf{K} * m\rangle\ dx, \tag{1.4}$$

where $\mathbf{K}$ is the two-dimensional *Helmholtz kernel*; *i.e.*, the (2×2)-matrix field defined for every $z \in \mathbb{R}^2 \setminus \{0\}$ as

$$\mathbf{K}(z) = \frac{1}{2\pi|z|^2}\Big(2\frac{z}{|z|} \otimes \frac{z}{|z|} - \mathbf{Id}\Big).$$

In the theory of micromagnetics (see [13]), the second term in (1.4) is known as the magnetostatic energy.

The stress-free strain can be written as

$$\mathbf{E}_0(m) = \gamma\mathbf{Q}(m),$$

where $\gamma > 0$ and where we have denoted by $\mathbf{Q}(m)$ the tensor order parameter of the de Gennes' theory (see [19]) defined as

$$\mathbf{Q}(m) := m \otimes m - \frac{1}{2}\mathbf{Id}.$$

Setting $\gamma = 1$ and dropping the term $\frac{1}{2}\int_{\Omega} \mathbb{C}\mathbf{E}_0(v) : \mathbf{E}_0(v)\ dx$, which is constant because of the constraint $|m| = 1$, the strain-alignment-coupling energy (1.2) reads as

$$E^{M,SA}(u, m) = -\int_{\Omega} \mathbf{E}u : \mathbf{Q}(m)\ dx.$$

Then, up to additive constants, taking into account (1.3), the total free energy of a magnetostrictive solid turns out to be

$$\begin{aligned} E^{M}(u, m) \quad &= \quad \frac{1}{2}\int_{\Omega} |\mathbf{E}u|^2\ dx + \frac{1}{4}\int_{\Omega} |\mathrm{div} u|^2\ dx - \int_{\Omega} \mathbf{E}u : \mathbf{Q}(m)\ dx \\ &+ \quad \frac{1}{2}\int_{\Omega} |\nabla m|^2\ dx - \frac{1}{2}\int_{\mathbb{R}^2} \langle m, \mathbf{K} * m\rangle\ dx. \end{aligned} \tag{1.5}$$

Ferroelectric crystals. From the point of view of our analysis this class of materials can be considered as a minor variant of the previous one. The main difference with the magnetostrictive case being that the relevant microscopic variable now is the *local polarization* $p : \mathbb{R}^2 \to \mathbb{R}^2$, which does not share with m the constraint $|m| = 1$; *i.e.,* in this case $V = \mathbb{R}^2$.

The energy stored in Ω by the internal variable has the form

$$E^{F,I}(p) = \frac{1}{2}\int_{\Omega} |\nabla p|^2\ dx + \frac{1}{2}\int_{\Omega} |p|^2\ dx - \frac{1}{2}\int_{\mathbb{R}^2} \langle p, \mathbf{K} * p\rangle\ dx,$$

while the stress-free strain is now given by

$$\mathbf{E}_0(p) = \gamma|p|^2\mathbf{Q}\left(\frac{p}{|p|}\right),$$

and the strain-alignment coupling energy for this model is

$$E^{F,SA}(u,p) = -2\mu\gamma\int_\Omega |p|^2\mathbf{E}u : \mathbf{Q}\left(\frac{p}{|p|}\right)\, dx + \frac{\mu\gamma^2}{2}\int_\Omega |p|^4\, dx.$$

Thus, normalizing the constants μ, λ, γ as before, the total free energy is

$$\begin{aligned} E^F(u,p) &= \frac{1}{2}\int_\Omega |\mathbf{E}u|^2\, dx + \frac{1}{4}\int_\Omega |\mathrm{div} u|^2\, dx \\ &- \int_\Omega |p|^2\mathbf{E}u : \mathbf{Q}\left(\frac{p}{|p|}\right)\, dx + \frac{1}{4}\int_\Omega |p|^4\, dx \\ &+ \frac{1}{2}\int_\Omega |\nabla p|^2\, dx + \frac{1}{4}\int_\Omega |p|^2\, dx - \frac{1}{2}\int_{\mathbb{R}^2} \langle p, \mathbf{K} * p\rangle\, dx. \end{aligned} \tag{1.6}$$

Nematic elastomers. In this model the relevant microscopic variable is a symmetric, (2×2)-matrix $\mathbf{Q}$ with $\mathrm{tr}\mathbf{Q} = 0$ and $|\mathbf{Q}| = \frac{1}{\sqrt{2}}$; *i.e.*, $V = \{\mathbf{M} \in \mathcal{M}^{2\times 2}_{sym} : \mathrm{tr}\,\mathbf{M} = 0,\ |\mathbf{M}| = \frac{1}{\sqrt{2}}\}$. The internal energy associated with this internal variable is

$$E^{N,I}(\mathbf{Q}) = \frac{1}{2}\int_\Omega |\nabla\mathbf{Q}|^2\, dx,$$

and the stress-free strain has the form

$$\mathbf{E}_0(\mathbf{Q}) = \gamma\mathbf{Q},$$

which implies that the strain-alignment coupling energy in (1.2) is given by

$$E^{N,SA}(u,\mathbf{Q}) = -2\mu\gamma\int_\Omega \mathbf{E}u : \mathbf{Q}\, dx.$$

Thus, again setting $\mu = \lambda = \frac{1}{2}$, $\gamma = 1$, up to additive constants, the total free energy functional of a nematic elastomer reads as

$$\begin{aligned} E^N(u,\mathbf{Q}) &= \frac{1}{2}\int_\Omega |\mathbf{E}u|^2\, dx + \frac{1}{4}\int_\Omega |\mathrm{div} u|^2\, dx \\ &- \int_\Omega \mathbf{E}u : \mathbf{Q}\, dx + \frac{1}{2}\int_\Omega |\nabla\mathbf{Q}|^2\, dx. \end{aligned} \tag{1.7}$$

At a microscopic scale, we describe strain-alignment interactions by introducing two separate length scales. At the scale δ, we define a map $u : \delta\mathbb{Z}^2 \cap \Omega \to \mathbb{R}^2$ giving the deformation of a set of lattice points identified by their position in the reference configuration. These can be thought of as points of the crystal lattice in the case of ferromagnetic or ferroelectric crystals, or as the position of the crosslinkers in the case of nematic elastomers. At the scale $\varepsilon \ll \delta$ we define a sublattice $\varepsilon\mathbb{Z}^2 \cap \Omega$, where the orientational properties of the system are described by means of a map $\nu : \varepsilon\mathbb{Z}^2 \to V$ (as before, V depends on the physics of the material and will be further specified later).

As in the continuum picture, neglecting the energy contribution due to the interaction of the system with external fields or to crystalline anisotropy, we may suppose that the total energy consists of three terms: an elastic part E^{EL}_δ accounting for the energy stored by deforming the δ-lattice, an ordering part E^I_ε related to the *short* and *long-range* interactions between the mesogenic units, and a strain-alignment term $E^{SA}_{\varepsilon,\delta}$ coupling the deformation of the δ-lattice with the order of the mesogenic units. As also highlighted with the notation, the first two terms in the total energy act on the two different scales δ and ε, while the third term depends on the properties of the material on both scales. Under the hypotheses

$$\delta = \delta(\varepsilon), \quad \delta(\varepsilon) \to 0, \quad \delta(\varepsilon)/\varepsilon \to +\infty, \text{ as } \varepsilon \to 0$$

the asymptotic analysis we are going to perform relies on the computation of the Γ-limit of the free energy functional

$$E_\varepsilon(u,\nu) = E^{EL}_\varepsilon(u) + E^I_\varepsilon(\nu) + E^{SA}_\varepsilon(u,\nu),$$

where to shorten notation we have set $E^{SA}_\varepsilon := E^{SA}_{\varepsilon,\delta(\varepsilon)}$.

As we are in the setting of isotropic linear elasticity, the discrete elastic energy E^{EL}_ε can be modeled by

$$E^{EL}_\varepsilon(u) := \sum_{\xi\in\{e_1,e_2,e_1\pm e_2\}} \sum_{\alpha,\alpha+\varepsilon\xi\in\delta\mathbb{Z}^2\cap\Omega} \delta^2 \Big|\langle \frac{u(\alpha+\delta\xi)-u(\alpha)}{\delta|\xi|}, \frac{\xi}{|\xi|}\rangle\Big|^2.$$

(see Section 4 for further details).

The three different ordering parts are as follows.

In the case of **magnetostrictive solids** order is described by a discrete vector field $m : \varepsilon\mathbb{Z}^2 \cap \Omega \to \mathbb{R}^2$ with $|m(\alpha)| = 1$ for every $\alpha \in \varepsilon\mathbb{Z}^2\cap\Omega$, which represents the local magnetization of the mesogenic units. Our ordering energy $E^{I,M}_\varepsilon$ is the sum of two contributions

$$E^{M,I}_\varepsilon(m) = E^{M,SR}_\varepsilon(m) + E^{M,LR}_\varepsilon(m).$$

The first term represents the short-range interaction energy between the mesogens and is given by

$$E^{M,SR}_\varepsilon(m) := \sum_{\xi\in\{e_1,e_2\}} \sum_{\alpha,\alpha+\varepsilon\xi\in\varepsilon\mathbb{Z}^2\cap\Omega} (1 - \langle m(\alpha), m(\alpha+\varepsilon\xi)\rangle),$$

while the second term takes into account the long-range interactions between the magnetization of the mesogens and has a dipolar origin (see [29, 30]). It is defined as

$$E^{M,LR}_\varepsilon(m) := -\frac{1}{2} \sum_{\alpha,\beta\in\varepsilon\mathbb{Z}^2\cap\Omega,\ \alpha\neq\beta} \varepsilon^4 \langle \mathbf{K}(\alpha-\beta)m(\alpha), m(\beta)\rangle, \tag{1.8}$$

where $\mathbf{K}$ is the two-dimensional Helmholtz kernel.

The strain-alignment coupling energy is modeled by coupling the local ordering tensor for mesogenic units $\mathbf{Q}(m)$, with the discrete strain of the system in each direction $\xi \in \{e_1, e_2, e_1 \pm e_2\}$. Since these two objects are defined on two different lattices, the correct coupling is achieved by first averaging the tensor order parameter $\mathbf{Q}(m(\beta))$ when β ranges over the points of the ε-lattice contained in each cell $\alpha + [0,\delta]^2$ of the bigger lattice $\delta\mathbb{Z}^2\cap\Omega$, and then coupling it with the strain. Hence, the coupling term is given by

$$E^{M,SA}_\varepsilon(u,m) := -\sum_{\xi\in\{e_1,e_2,e_1\pm e_2\}} \sum_{\alpha,\alpha+\delta\xi\in\delta\mathbb{Z}^2\cap\Omega} \delta^2 \langle \frac{u(\alpha+\delta\xi)-u(\alpha)}{\delta|\xi|}, \frac{\xi}{|\xi|}\rangle \langle \overline{\mathbf{Q}}(\alpha)\frac{\xi}{|\xi|}, \frac{\xi}{|\xi|}\rangle,$$

where

$$\overline{\mathbf{Q}}(\alpha) := \sum_{\beta\in\varepsilon\mathbb{Z}^2\cap(\alpha+[0,\delta]^2)} \frac{\varepsilon^2}{\delta^2}\,\mathbf{Q}(m(\beta)).$$

For **ferroelectric crystals** the order variable is $p : \varepsilon\mathbb{Z}^2\cap\Omega \to \mathbb{R}^2$ and the ordering term is again the sum of two contributions. The long range term is given by (1.8) with p in place of m while the short range contribution now reads as

$$E_\varepsilon^{F,SR}(p) := \frac{1}{2}\sum_{\xi\in\{e_1,e_2\}}\sum_{\alpha,\alpha+\varepsilon\xi\in\varepsilon\mathbb{Z}^2\cap\Omega} |p(\alpha+\varepsilon\xi)-p(\alpha)|^2 + \frac{1}{2}\sum_{\alpha\in\varepsilon\mathbb{Z}^2\cap\Omega}\varepsilon^2|p(\alpha)|^2.$$

Moreover, the strain-alignment-coupling energy for this model is given by

$$\begin{aligned} E_\varepsilon^{F,SA}(u,p) \quad &= \quad -\sum_{\xi\in\{e_1,e_2,e_1\pm e_2\}}\sum_{\alpha,\alpha+\delta\xi\in\delta\mathbb{Z}^2\cap\Omega}\delta^2\langle\frac{u(\alpha+\delta\xi)-u(a)}{\delta|\xi|},\frac{\xi}{|\xi|}\rangle\langle\widetilde{\mathbf{Q}}(\alpha)\frac{\xi}{|\xi|},\frac{\xi}{|\xi|}\rangle \\ &\quad +\frac{1}{2}\sum_{\xi\in\{e_1,e_2,e_1\pm e_2\}}\sum_{\alpha,\alpha+\delta\xi\in\delta\mathbb{Z}^2\cap\Omega}\delta^2\Big\langle\widetilde{\mathbf{Q}}(\alpha)\frac{\xi}{|\xi|},\frac{\xi}{|\xi|}\Big\rangle^2, \end{aligned}$$

with

$$\widetilde{\mathbf{Q}}(\alpha) = \sum_{\beta\in\varepsilon\mathbb{Z}^2\cap(\alpha+[0,\delta]^2)}\frac{\varepsilon^2}{\delta^2}\,|p(\beta)|^2\,\mathbf{Q}\left(\frac{p(\beta)}{|p(\beta)|}\right).$$

Notice that for $|p(\alpha)| = 1$, $\widetilde{\mathbf{Q}}$ reduces to $\overline{\mathbf{Q}}$.

In the case of **nematic elastomers**, following Lebwhol-Lasher [31], the order variable is $n : \varepsilon\mathbb{Z}^2\cap\Omega \to \mathbb{R}^2$, again satisfying the constraint $|n(\alpha)| = 1$ for every $\alpha\in\varepsilon\mathbb{Z}^2$ and the ordering energy has only short-range contribution; it can be written as

$$E_\varepsilon^{N,I}(n) := \sum_{\xi\in\{e_1,e_2\}}\sum_{\alpha,\alpha+\varepsilon\xi\in\delta\mathbb{Z}^2\cap\Omega} 2(1-\langle n(\alpha), n(\alpha+\varepsilon\xi)\rangle^2).$$

Notice that the above energy is invariant if, at some $\alpha\in\varepsilon\mathbb{Z}^2\cap\Omega$ we replace $n(\alpha)$ by $-n(\alpha)$, thus describing the mesogenic units as *directions* and not as *spins*. In this case the strain-alignment energy is exactly as in the magnetostrictive case now with n in place of m.

Upon identifying the relevant discrete variables with suitably chosen continuous counterparts (see Section 2), all the above discrete energies can be viewed as being defined on the functional space $L^2(\Omega;\mathbb{R}^2)\times L^2(\Omega;V)$ and in this framework they can be described by a Γ-limit procedure.

The main result of this paper is that the discrete energies E_ε^M, E_ε^F, E_ε^N Γ-converge to the corresponding continuum energies E^M, E^F, E^N defined respectively as in (1.5) (1.6) and (1.7) (Sections 5, 6 and 7) under Dirichlet boundary conditions on the displacement (Section 5.1) and when some "additional terms" are taken into account (Section 8).

We finally remark that the above results can be suitably extended in higher dimensions but, for expositional simplicity, we focus on the two-dimensional case as it already contains the main features of the problem.

2. Notation

In this section we set some basic notation employed in the rest of the paper.

Vectors and matrices. We denote by $\{e_1, e_2\}$ the canonical basis of $\mathbb{R}^2$. Given $\xi, \eta \in \mathbb{R}^2$, we denote by $\langle \xi, \eta \rangle$ their scalar product. For any $\xi \in \mathbb{R}^2 \setminus \{0\}$ we set $\hat{\xi} := \xi/|\xi|$.

We denote by $\mathcal{M}^{2\times 2}$ the space of (2×2)-real matrices. The trace of $\mathbf{A} = (a_{il}) \in \mathcal{M}^{2\times 2}$ is denoted by $\mathrm{tr}\mathbf{A} := \sum_{i=1}^2 a_{ii}$. The scalar product of any given pair $\mathbf{A}, \mathbf{B} \in \mathcal{M}^{2\times 2}$ is defined as

$$\mathbf{A} : \mathbf{B} := \mathrm{tr}(\mathbf{A}\mathbf{B}^T) = \sum_{i,l=1}^{2} a_{il} b_{il},$$

where $\mathbf{B}^T$ denotes the transpose of $\mathbf{B}$. The norm of $\mathbf{A}$ induced by this scalar product is denoted by $|\mathbf{A}|$. We denote by $\mathcal{M}^{2\times 2}_{sym}$ the subspace of $\mathcal{M}^{2\times 2}$ of symmetric real matrices.

The symmetrized gradient of a function $u : \mathbb{R}^2 \to \mathbb{R}^2$ is defined as

$$\mathbf{E}u := \frac{\nabla u + \nabla u^T}{2},$$

where $\nabla u \in \mathcal{M}^{2\times 2}$ denotes the gradient of u, $\nabla u := (\frac{\partial u_i}{\partial x_l})$ for $i, l = 1, 2$.

We denote by S^1 the set of unitary vectors in $\mathbb{R}^2$. Given a vector $n \in S^1$, we define the traceless matrix $\mathbf{Q}(n) \in \mathcal{M}^{2\times 2}_{sym}$ as follows

$$\mathbf{Q}(n) := n \otimes n - \frac{1}{2}\mathbf{Id}. \tag{2.1}$$

where $n \otimes n := (n_i\, n_l)$ and $\mathbf{Id}$ denotes the identity matrix in $\mathcal{M}^{2\times 2}$. We also introduce the following notation

$$\mathcal{N} := \big\{\mathbf{M} \in \mathcal{M}^{2\times 2}_{sym} :\ \mathrm{tr}\,\mathbf{M} = 0,\ |\mathbf{M}| = \frac{1}{\sqrt{2}}\big\}. \tag{2.2}$$

Notice that $\mathbf{Q}(n) \in \mathcal{N}$.

Lattices. Given $\varepsilon > 0$, and $\Omega \subset \mathbb{R}^2$ a bounded open set with Lipschitz boundary, we denote by $\Omega_\varepsilon := \varepsilon\mathbb{Z}^2 \cap \Omega$ the portion of the lattice $\varepsilon\mathbb{Z}^2$ in Ω. Set $Q := [0,1)^2$, for a given $\alpha \in \varepsilon\mathbb{Z}^2$, the set $Q_\varepsilon(\alpha) := \alpha + \varepsilon\, Q$ is called a cell of the lattice $\varepsilon\mathbb{Z}^2$ with vertex in α. It is useful to divide each cell of the lattice into triangles; to this end we introduce the notation

$$T^\pm := \{x \in [0,1)^2 :\ \pm\langle x, e_2 - e_1\rangle > 0\}.$$

According to the previous definition we have that, for any $\alpha \in \varepsilon\mathbb{Z}^2$, the triangles $T^\pm_\varepsilon(\alpha) := \alpha + \varepsilon T^\pm$ partition the cell $Q_\varepsilon(\alpha)$, that is $\overline{Q_\varepsilon(\alpha)} := \overline{T^+_\varepsilon(\alpha) \cup T^-_\varepsilon(\alpha)}$, where the over bar stays for the closure of the corresponding set. We also introduce the sets

$$P^{e_1}_\varepsilon(\alpha) := \overline{T^-_\varepsilon(\alpha) \cup T^+_\varepsilon(\alpha - \varepsilon e_2)}, \qquad P^{e_2}_\varepsilon(\alpha) := \overline{T^+_\varepsilon(\alpha) \cup T^-_\varepsilon(\alpha - \varepsilon e_1)}.$$

To shorten notation, we sometimes write

$$P^{e_1+e_2}_\varepsilon(\alpha) := \overline{Q_\varepsilon(\alpha)} \quad \text{and} \quad P^{e_1-e_2}_\varepsilon(\alpha) := \overline{Q_\varepsilon(\alpha - \varepsilon e_2)}.$$

We denote by $\#(A_\varepsilon)$ the cardinality of $A_\varepsilon \subseteq \Omega_\varepsilon$.

Difference quotients. Given a function $u : \varepsilon\mathbb{Z}^2 \to \mathbb{R}^2$, a point $\alpha \in \varepsilon\mathbb{Z}^2$ and a vector $\xi \in \mathbb{Z}^2$, we denote by $D^\xi_\varepsilon u(\alpha)$ the difference quotient of u in α along the direction ξ, that is

$$D^\xi_\varepsilon u(\alpha) := \frac{u(\alpha+\varepsilon\xi) - u(\alpha)}{\varepsilon|\xi|}.$$

Interpolations. Let

$$\mathcal{C}_\varepsilon(\Omega) := \{u : \mathbb{R}^2 \to \mathbb{R}^2 : \ u(x) = u(\alpha) \quad \forall x \in Q_\varepsilon(\alpha) \cap \Omega, \ \alpha \in \varepsilon\mathbb{Z}^2\}. \tag{2.3}$$

For any $u \in \mathcal{C}_\varepsilon(\Omega)$, we define $v := A(u) : \mathbb{R}^2 \to \mathbb{R}^2$ a piecewise-affine interpolation of u on the cells of the lattice $\varepsilon\mathbb{Z}^2$ as follows: for every $x \in Q_\varepsilon(\alpha)$,

$$v(x) := \begin{cases} \langle x-\alpha, e_1\rangle D^{e_1}_\varepsilon u(\alpha) + \langle x-\alpha, e_2\rangle D^{e_2}_\varepsilon u(\alpha+\varepsilon e_1) + u(\alpha), & x \in T^-_\varepsilon(\alpha) \\ \\ \langle x-\alpha, e_1\rangle D^{e_1}_\varepsilon u(\alpha+\varepsilon e_2) + \langle x-\alpha, e_2\rangle D^{e_2}_\varepsilon u(\alpha) + u(\alpha), & x \in T^+_\varepsilon(\alpha). \end{cases} \tag{2.4}$$

The space of such piecewise-affine interpolations is denoted by $\mathcal{A}_\varepsilon(\Omega)$; *i.e.*,

$$\mathcal{A}_\varepsilon(\Omega) := \{v : \mathbb{R}^2 \to \mathbb{R}^2 \ v = A(u), \text{ for some } u \in \mathcal{C}_\varepsilon(\Omega)\}.$$

Notice that if $v \in \mathcal{A}_\varepsilon(\Omega)$, then

$$\frac{\partial v}{\partial x_i}(x) = D^{e_i}_\varepsilon u(\alpha), \quad \text{for } x \in P^{e_i}_\varepsilon(\alpha), \quad i = 1, 2.$$

We also set

$$\mathcal{B}_\varepsilon(\Omega) := \{v \in \mathcal{A}_\varepsilon(\Omega) : \ v(\alpha) \in S^1 \quad \forall \alpha \in \varepsilon\mathbb{Z}^2 : Q_\varepsilon(\alpha) \cap \Omega \neq \emptyset\}.$$

We denote by $\mathcal{A}_\varepsilon(\Omega; \mathcal{M}^{2\times 2})$ the space of all matrix-valued functions $\mathbf{M}$, whose columns belong to $\mathcal{A}_\varepsilon(\Omega)$ and we define

$$\mathcal{M}_\varepsilon(\Omega) := \{\mathbf{M} \in \mathcal{A}_\varepsilon(\Omega; \mathcal{M}^{2\times 2}) : \ \mathbf{M}(\alpha) \in \mathcal{N} \quad \forall \alpha \in \varepsilon\mathbb{Z}^2 : Q_\varepsilon(\alpha) \cap \Omega \neq \emptyset\}.$$

In all that follows the letter C stands for a generic positive constant which may vary from line to line and expression to expression within the same formula.

3. Preliminaries

3.1. Γ-convergence.

In this subsection we recall the definition and the main properties of Γ-convergence. For our purposes we limit our attention to the L^2 setting while we refer the reader to [9] for a comprehensive introduction to the subject.

In all that follows (F_j) is a sequence of functionals with $F_j : L^2(\Omega) \longrightarrow \mathbb{R} \cup \{\pm\infty\}$, for any $j \in \mathbb{N}$.

Definition 3.1 (Γ-convergence)**.** *A sequence of functionals (F_j), Γ-converges with respect to the $L^2(\Omega)$ convergence to the functional $F : L^2(\Omega) \longrightarrow \mathbb{R} \cup \{\pm\infty\}$ if for all $u \in L^2(\Omega)$ we have*

(i) (lower bound) *for every sequence $(u_j) \subset L^2(\Omega)$, $u_j \to u$ in $L^2(\Omega)$*

$$F(u) \leq \liminf_{j \to +\infty} F_j(u_j);$$

(ii) (upper bound) *there exists a sequence $(\bar{u}_j) \subset L^2(\Omega)$, $\bar{u}_j \to u$ in $L^2(\Omega)$ such that*

$$F(u) \geq \limsup_{j \to +\infty} F_j(\bar{u}_j).$$

The functional F is called the Γ-limit of (F_j) with respect to the $L^2(\Omega)$ convergence.

If we define the *lower* and *upper* Γ*-limits* (which always exist)

$$\Gamma\text{-}\liminf_{j\to+\infty} F_j(u) := \inf\{\liminf_{j\to+\infty} F_j(u_j) :\ u_j \to u \ \text{ in } \ L^2(\Omega)\}$$

and

$$\Gamma\text{-}\limsup_{j\to+\infty} F_j(u) := \inf\{\limsup_{j\to+\infty} F_j(u_j) :\ u_j \to u \ \text{ in } \ L^2(\Omega)\},$$

then the sequence (F_j) Γ-converges to F if and only if

$$\Gamma\text{-}\liminf_{j\to+\infty} F_j(u) = \Gamma\text{-}\limsup_{j\to+\infty} F_j(u) = F(u),$$

for every $u \in L^2(\Omega)$.

We note that upper and lower Γ-limits are $L^2(\Omega)$ lower semicontinuos functionals (see [9], Proposition 1.28).

Definition 3.2 (Equi-coercivity). *The sequence of functionals (F_j) is $L^2(\Omega)$ equi-coercive if for any given $(u_j) \subset L^2(\Omega)$ with $\sup_j F_j(u_j) < +\infty$, up to subsequences, we have that $u_j \to u$ in $L^2(\Omega)$, for some $u \in L^2(\Omega)$.*

The following theorem states the fundamental property of Γ-convergence.

Theorem 3.3 (Convergence of minimum problems). *Let (F_j) be a $L^2(\Omega)$ equi-coercive sequence of functionals Γ-converging to F with respect to the $L^2(\Omega)$ convergence, then*

(1) (convergence of minima) *there exists* $\min_{L^2(\Omega)} F$ *and*

$$\lim_{j\to+\infty} \inf_{L^2(\Omega)} F_j = \min_{L^2(\Omega)} F$$

(2) (convergence of minimum points) *if $(u_j) \subset L^2(\Omega)$ is a* minimizing sequence *for (F_j) (i.e., $\lim_j F_j(u_j) = \lim_j \inf F_j$) then, up to subsequences, $u_j \to \bar{u}$ in $L^2(\Omega)$ and*

$$F(\bar{u}) = \min_{L^2(\Omega)} F.$$

3.2. **Some preliminary lemmas.** For the reader's convenience, in this subsection we collect two lemmas that will be used in the following.

We start setting some notation. For any $y \in \overline{Q}$ and $\varepsilon_j > 0$ such that $\varepsilon_j \to 0$ as $j \to +\infty$ we denote by $T_y^{\varepsilon_j}$ the operator which maps $u :\ \mathbb{R}^2 \to \mathbb{R}^2$ in $T_y^{\varepsilon_j} u :\ \mathbb{R}^2 \to \mathbb{R}^2$ defined as

$$T_y^{\varepsilon_j} u(x) := u\Big(\varepsilon_j y + \varepsilon_j \left[\frac{x}{\varepsilon_j}\right]\Big),$$

where, for all $z = (z_1, z_2) \in \mathbb{R}^2$, $[z] := ([z_1], [z_2])$, $[z_i]$ being the integer part of z_i, $i = 1, 2$. Note that $T_y^{\varepsilon_j} u$ is constant on each cell of the lattice $\varepsilon_j \mathbb{Z}^2$ and thus can be identified with a discrete function mapping $\varepsilon_j \mathbb{Z}^2$ into $\mathbb{R}^2$.

The following approximation result is a straightforward generalization of the result stated in [3], Lemma 4.

Lemma 3.4. *Let $u_j \to u$ in $W^{1,2}_{loc}(\mathbb{R}^2, \mathbb{R}^2)$. Then, for any open set $B \subset \mathbb{R}^2$ such that $\overline{B} \subset \mathbb{R}^2$,*

$$\lim_{j\to+\infty} \int_Q \|A(T_y^{\varepsilon_j} u_j) - u\|_{W^{1,2}(B;\mathbb{R}^2)}\, dy = 0,$$

where $A(T_y^{\varepsilon_j} u_j)$ is the piecewise-affine interpolation of $T_y^{\varepsilon_j} u_j$ defined as in (2.4).

Remark 3.5. Since trivially

$$\min_{y\in\overline{Q}} \|A(T_y^{\varepsilon_j} u_j) - u\|_{W^{1,2}(B;\mathbb{R}^2)} \le \int_Q \|A(T_y^{\varepsilon_j} u_j) - u\|_{W^{1,2}(B;\mathbb{R}^2)}\, dy,$$

Lemma 3.4 in particular yields the existence of a sequence $(y_j) \subset \overline{Q}$ such that

$$\lim_{j\to+\infty} \|A(T_{y_j}^{\varepsilon_j} u_j) - u\|_{W^{1,2}(B;\mathbb{R}^2)} = 0,$$

for any open set $B \subset \mathbb{R}^2$ such that $\overline{B} \subset \mathbb{R}^2$.

The following algebraic lemma gives a characterization of traceless matrices in $\mathcal{M}^{2\times 2}_{sym}$, with fixed norm.

Lemma 3.6. *Let* $\mathbf{A} \in \mathcal{N}$*, with* $\mathcal{N}$ *as in* (2.2)*, then*

$$\mathbf{A} = \nu \otimes \nu - \frac{1}{2}\,\mathbf{Id}, \quad \textit{for some } \nu \in S^1.$$

Proof. By hypothesis

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & -a_{11} \end{pmatrix}$$

and

$$|\mathbf{A}|^2 = 2a_{11}^2 + 2a_{12}^2 = \frac{1}{2}. \tag{3.1}$$

If we consider the matrix $\mathbf{A} + \frac{1}{2}\,\mathbf{Id}$, in view of (3.1) we get that

$$\det\big(\mathbf{A} + \frac{1}{2}\,\mathbf{Id}\big) = \frac{1}{4} - a_{11}^2 - a_{12}^2 = 0,$$

hence $\mathbf{A} + \frac{1}{2}\,\mathbf{Id}$ has rank 1. As a consequence

$$\mathbf{A} + \frac{1}{2}\,\mathbf{Id} = \nu \otimes \nu'$$

for some $\nu \in S^1$ and $\nu' \in \mathbb{R}^2$, $\nu' \neq 0$. By the symmetry of $\mathbf{A}$ we deduce

$$\nu_1\nu_2' = \nu_2\nu_1', \tag{3.2}$$

while $\mathrm{tr}\mathbf{A} = 0$ gives

$$\nu_1\nu_1' + \nu_2\nu_2' = 1. \tag{3.3}$$

Then if $\nu_1 \neq 0$, by combining (3.2) and (3.3) we find

$$\nu_1\nu_1' + \frac{\nu_2^2}{\nu_1}\,\nu_1' = 1 \iff \nu_1'\left(\frac{\nu_1^2+\nu_2^2}{\nu_1}\right) = 1 \iff \nu_1' = \nu_1,$$

thus by (3.2) the thesis. If $\nu_1 = 0$ then $|\nu_2| = 1$, hence again (3.3) and (3.2) yield the thesis. □

3.3. **Korn's Inequalities.** In this subsection we recall two variants of the Korn Inequality that we use in the proof of Proposition 5.2 and Proposition 5.14. We specialize them to our setting while we refer the reader to [16, 34] (and references therein) for more general statements.

Proposition 3.7. *Let* Ω *be a bounded open Lipschitz subset of* $\mathbb{R}^2$*, then there exists a positive constant* C *such that*

(i) (Korn's Inequality)

$$\|\nabla u\|_{L^2(\Omega)} \le C\big(\|\mathbf{E}(u)\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\big), \quad \forall u \in W^{1,2}(\Omega);$$

(ii) (Korn's Inequality with boundary data)

$$\|\nabla u\|_{L^2(\Omega)} \leq C\|\mathbf{E}(u)\|_{L^2(\Omega)}, \quad \forall u \in W_0^{1,2}(\Omega).$$

4. The models

In this section we give a detailed description of the discrete models we deal with in this paper. We assume that our sample is contained in a bounded open set $\Omega \subset \mathbb{R}^2$ on which we construct two lattices Ω_δ and Ω_ε, with $\varepsilon \ll \delta \ll 1$. In all that follows we assume $\delta = \delta(\varepsilon)$ to be chosen in a way such that the above inequalities holds true (see Figure 1).

We consider three different types of materials which exhibit a strain-alignment-coupling phenomenon: magnetostrictive solids, ferroelectric crystals and nematic elastomers. In all these models the total free energy E is given by the sum of three contributions: an elastic energy E^{EL}, an ordering energy E^I and a strain-alignment-coupling energy E^{SA}.

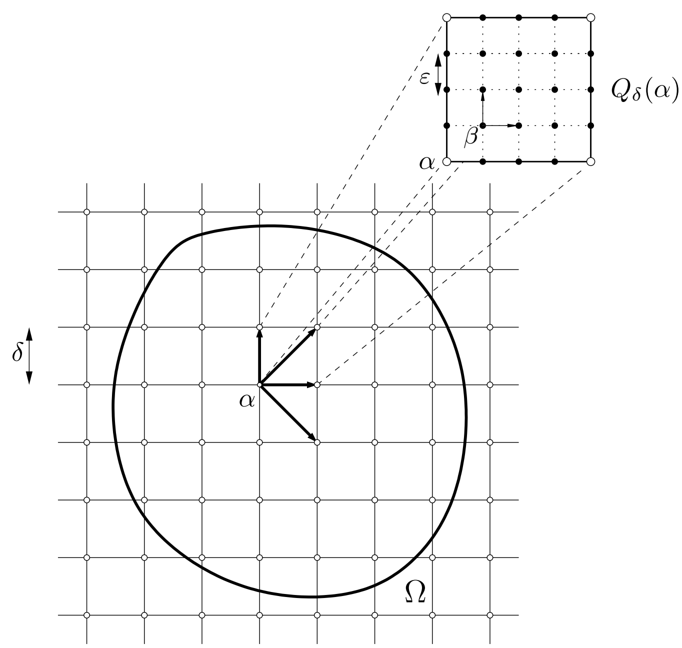

Figure 1. The discrete system in the reference configuration.

In what follows we specialize these energy terms depending on the three different models we are going to consider.

4.1. **Elastic energy.** We work in the small deformations regime. The deformation of each point of the lattice is described by a map $u : \Omega_\delta \to \mathbb{R}^2$. We suppose that the elastic energy associated with the displacement of a point in the reference configuration is determined by the pairwise interactions of this point with "few" *neighbors*. Specifically, we consider only the interactions between a point $\alpha \in \Omega_\delta$ and those points $\alpha + \delta\xi \in \Omega_\delta$, with $\xi \in \{e_1, e_2, e_1 \pm e_2\}$. This choice leads, in the continuum limit, to the elastic energy of linear elasticity corresponding to a particular choice of the Lamé coefficients. Nevertheless, we remark that following [7], more general interactions may be taken into account thus yielding a continuum model for all the possible Lamé coefficients. Moreover, the choice of considering only the interactions between few neighboring points of the lattice does not play a fundamental role. Indeed, a generalization to a model taking into account interactions between all the points of the lattice is possible (see [7]), but since this would lead to much less readable formulas without giving a better insight into the physics of the problem, we work in the simplified setting discussed above. Then, for any $u : \Omega_\delta \to \mathbb{R}^2$, the elastic energy E^{EL}_ε has the form

$$E^{EL}_\varepsilon(u) := \sum_{\xi\in X} \sum_{\alpha\in\mathcal{R}^\xi_\delta(\Omega)} \delta^2 |\langle D^\xi_\delta u(\alpha), \hat{\xi}\rangle|^2, \tag{4.1}$$

where $X := \{e_1, e_2, e_1 \pm e_2\}$, and for any $\xi \in X$, $\mathcal{R}^\xi_\delta(\Omega) := \{\alpha \in \Omega_\delta : \ \alpha + \delta\xi \in \Omega_\delta\}$.

4.2. **Energies stored by the internal variable.** As already pointed out, the internal variable ν defined on the points of the lattice Ω_ε contributes to the total free energy with a term promoting local alignment. Since ν has a different physical meaning depending on the model we are considering, the three corresponding internal energy terms are also different.

4.2.1. **Magnetostrictive solids.** For a magnetostrictive solid the internal variable gives the orientation of the magnetic moments and it is described, for every $\alpha \in \Omega_\varepsilon$, by a vector $m(\alpha) \in S^1$.

In this case the internal energy $E^{M,I}_\varepsilon$ is given by the sum of two contributions taking into account short-range and long-range interactions, respectively; *i.e.*,

$$E^{M,I}_\varepsilon(m) = E^{M,SR}_\varepsilon(m) + E^{M,LR}_\varepsilon(m).$$

4.2.2. *Short-range interactions: the XY model.* For any given $m : \Omega_\varepsilon \to S^1$, the short-range energy $E^{M,SR}_\varepsilon$ is defined as

$$E^{M,SR}_\varepsilon(m) := \sum_{\xi\in Y} \sum_{\alpha\in\mathcal{R}^\xi_\varepsilon(\Omega)} (1 - \langle m(\alpha), m(\alpha + \varepsilon\xi)\rangle), \tag{4.2}$$

where $Y := \{e_1, e_2\}$. It represents the so called exchange-interaction energy between the magnetic dipoles and it favors their parallel alignment.

Notice that (4.2) is obtained scaling by ε^2 the energy of a so called XY-spin system. In formula

$$E^{M,SR}_\varepsilon = \frac{E^{M,SR}_{\varepsilon,0} - \min E^{M,SR}_{\varepsilon,0}}{\varepsilon^2},$$

where

$$E^{M,SR}_{\varepsilon,0}(m) := -\sum_{\xi\in Y} \sum_{\alpha\in\mathcal{R}^\xi_\varepsilon(\Omega)} \varepsilon^2 \langle m(\alpha), m(\alpha + \varepsilon\xi)\rangle$$

is the energy associated to an XY-spin model (see [3] for lower-order scalings). Moreover, since $|m(\alpha+\varepsilon\xi)-m(\alpha)|^2 = 2-2\langle m(\alpha), m(\alpha+\varepsilon\xi)\rangle$, we may equivalently write

$$E^{M,SR}_\varepsilon(m) = \frac{1}{2}\sum_{\xi\in Y} \sum_{\alpha\in\mathcal{R}^\xi_\varepsilon(\Omega)} |m(\alpha + \varepsilon\xi) - m(\alpha)|^2 = \frac{1}{2}\sum_{\xi\in Y} \sum_{\alpha\in\mathcal{R}^\xi_\varepsilon(\Omega)} \varepsilon^2 |D^\xi_\varepsilon m(\alpha)|^2. \tag{4.3}$$

4.2.3. *Long-range interactions: the dipolar energy.* The $E_\varepsilon^{M,LR}$ term in the internal energy is due to the long-range interactions between the magnetic moments. It has dipolar origin (see [29], or [30] for its analog in 3-dimensions). It has the form

$$E_\varepsilon^{M,LR}(m) = -\frac{1}{2} \sum_{\alpha,\beta\in\Omega_\varepsilon,\ \alpha\neq\beta} \varepsilon^4 \langle \mathbf{K}(\alpha-\beta)m(\alpha), m(\beta)\rangle \tag{4.4}$$

where $\mathbf{K} : \mathbb{R}^2 \setminus \{0\} \to \mathcal{M}^{2\times 2}_{sym}$ is the two-dimensional Helmholtz kernel

$$\mathbf{K}(z) = \frac{1}{2\pi|z|^2}\Big(2\hat{z}\otimes\hat{z} - \mathbf{Id}\Big). \tag{4.5}$$

4.2.4. **Ferroelectric crystals.** The previous description can be suitably adapted to the case of ferroelectric crystals. Here the polarization field of each molecule in $\alpha \in \Omega_\varepsilon$ is described by a vector $p(\alpha) \in \mathbb{R}^2$. Since we consider possibly unbounded polarization fields, the energetic description of this model is slightly different from that of the magnetostrictive case.

As before, the energy stored by the internal variable $E_\varepsilon^{F,I}$ is the sum of two contributions taking into account short-range and long-range interactions, respectively. We have

$$E_\varepsilon^{F,I}(p) := E_\varepsilon^{F,SR}(p) + E_\varepsilon^{F,LR}(p).$$

4.2.5. *Short-range interactions.* For the case of a ferroelectric crystal we consider a short-range energy given by

$$E_\varepsilon^{F,SR}(p) = \frac{1}{2}\sum_{\xi\in Y}\sum_{\alpha\in\mathcal{R}_\varepsilon^\xi(\Omega)} \varepsilon^2|D_\varepsilon^\xi p(\beta)|^2 + \frac{1}{2}\sum_{\alpha\in\Omega_\varepsilon}\varepsilon^2|p(\beta)|^2, \tag{4.6}$$

where the second term (4.6) has no counterpart in (4.3) and has been added in order to extend the analysis to unbounded polarization fields.

4.2.6. *Long-range interactions: the dipolar energy.* The $E_\varepsilon^{F,LR}$ term has the same origin and the same form of the corresponding term of the magnetostrictive case, being

$$E_\varepsilon^{F,LR}(p) = -\frac{1}{2} \sum_{\alpha,\beta\in\Omega_\varepsilon,\ \alpha\neq\beta} \varepsilon^4 \langle \mathbf{K}(\alpha-\beta)p(\alpha), p(\beta)\rangle. \tag{4.7}$$

4.2.7. **Nematic elastomers: the Lebwhol-Lasher model.** For a nematic elastomer the interaction between mesogenic units has mainly steric origin. As a consequence, the internal energy can be assumed to be only short-range. Regarding the alignment properties of these systems, no distinction is possible between "heads" and "tails" of each mesogenic unit. This suggests to describe each unit by means of a *director* field; *i.e.,* a field with values in the projective plane and to employ the energy (4.3) to promote their alignment.

An alternative description is given by the Lebwhol-Lasher model [31]. In this model the orientation of the nematic is described, as in the magnetic case, by a vector field $n : \Omega_\varepsilon \to S^1$, while the identification between heads and tails of the molecules (n and $-n$) is realized at the energetic level. More precisely, the ordering energy (4.3) is replaced by a new energy E^{LL} which is invariant if $n(\alpha)$ is replaced by $-n(\alpha)$ for some $\alpha \in \Omega_\varepsilon$. This is also our choice.

For a given vector field $n : \Omega_\varepsilon \to S^1$, the ordering energy in the Lebwhol-Lasher model is defined as

$$E_\varepsilon^{LL}(n) := \sum_{\xi \in Y} \sum_{\alpha \in \mathcal{R}_\varepsilon^\xi(\Omega)} 2\big(1 - \langle n(\alpha), n(\alpha + \varepsilon\xi)\rangle^2\big). \tag{4.8}$$

Then, a straightforward calculation gives

$$\begin{aligned} E_\varepsilon^{LL}(n) &= \sum_{\xi \in Y} \sum_{\alpha \in \mathcal{R}_\varepsilon^\xi(\Omega)} \big(1 - \mathbf{Q}(n(\alpha)) : \mathbf{Q}(n(\alpha + \varepsilon\xi))\big) \\ &= \frac{1}{2} \sum_{\xi \in Y} \sum_{\alpha \in \mathcal{R}_\varepsilon^\xi(\Omega)} \varepsilon^2 |D_\varepsilon^\xi \mathbf{Q}(n(\alpha))|^2, \end{aligned} \tag{4.9}$$

where $\mathbf{Q}(n)$, defined as in (2.1), represents the tensor-order parameter of the de Gennes' theory and plays, in this case, the role of the meaningful internal variable (see Remark 7.2).

4.3. **Strain-alignment energies.** The systems under investigation deform macroscopically upon a change in the orientational ordering. This is modeled by an energy measuring the coupling between order at scale ε and deformation at scale δ. The character of the coupling strongly depends on the detailed structure of the molecules forming the system, even if some properties can be captured by a variant of the model proposed by Uchida and Onuki in [35] for nematic elastomers (see also [19] and [37]) in the small-deformation regime.

4.3.1. **Magnetostrictive solids.** In this case the strain-alignment energy models the coupling of the local ordering tensor for the magnetic moments $\mathbf{Q}(m)$, with the discrete strain of the system in each direction $\xi \in X$. These two objects are defined on two different lattices, thus the correct coupling can be achieved by first averaging the tensor order parameter $\mathbf{Q}(m(\beta))$ when β ranges over the points pf the ε-lattice contained in each cell $Q_\delta(\alpha)$ of the lattice Ω_δ and then coupling it with the strain. Then, for given $u : \Omega_\delta \to \mathbb{R}^2$ and $m : \Omega_\varepsilon \to S^1$ the coupling term is

$$E_\varepsilon^{M,SA}(u, m) := -\sum_{\xi \in X} \sum_{\alpha \in \mathcal{R}_\delta^\xi(\Omega)} \delta^2 \langle D_\delta^\xi u(\alpha), \hat{\xi}\rangle \sum_{\beta \in Q_\delta(\alpha) \cap \Omega_\varepsilon} \frac{\varepsilon^2}{\delta^2} \big\langle \mathbf{Q}(m(\beta))\, \hat{\xi}, \hat{\xi}\big\rangle. \tag{4.10}$$

In what follows, for every fixed $\alpha \in \Omega_\delta$ and for each $\xi \in X$, it is useful to choose to average the local ordering parameter on the corresponding "cell" $P_\delta^\xi(\alpha)$ instead of considering the same cell $Q_\delta(\alpha)$ for any direction ξ. With this choice the strain-alignment energy $E_\varepsilon^{M,SA}$ can be rephrased as

$$E_\varepsilon^{M,SA}(u, m) = -\sum_{\xi \in X} \sum_{\alpha \in \mathcal{R}_\delta^\xi(\Omega)} \delta^2 \langle D_\delta^\xi u(\alpha), \hat{\xi}\rangle \sum_{\beta \in P_\delta^\xi(\alpha) \cap \Omega_\varepsilon} \frac{\varepsilon^2}{\delta^2} \big\langle \mathbf{Q}(m(\beta))\, \hat{\xi}, \hat{\xi}\big\rangle. \tag{4.11}$$

It will be clear in what follows that the energy defined as above has the same asymptotic behavior, as $\varepsilon \to 0$, of (4.10).

Clearly, other choices are possible. For instance, in [32] (see also [33]) Pasini, Skačej and Zannoni proposed a model in which the coupling term is obtained by averaging the variable m (instead of $\mathbf{Q}(m)$) on the cells of the bigger lattice $\delta\mathbb{Z}^2$. In Remark 5.6 we show that, in the limit as $\varepsilon \to 0$, this model is "asymptotically equivalent" to the one we consider.

4.3.2. **Ferroelectric crystals.** For this class of materials the strain-alignment-coupling energy, for a given configuration $u : \Omega_\delta \to \mathbb{R}^2$ and $p : \Omega_\varepsilon \to \mathbb{R}^2$, is

$$\begin{aligned} E^{F,SA}_\varepsilon(u,p) &= -\sum_{\xi\in X}\sum_{\alpha\in\mathcal{R}^\xi_\delta(\Omega)} \delta^2\langle D^\xi_\delta u(\alpha),\hat{\xi}\rangle \sum_{\beta\in P^\xi_\delta(\alpha)\cap\Omega_\varepsilon} \frac{\varepsilon^2}{\delta^2}\,|p(\beta)|^2\Big\langle \mathbf{Q}(\hat{p}(\beta))\,\hat{\xi},\hat{\xi}\Big\rangle \\ &\quad +\frac{1}{2}\sum_{\xi\in X}\sum_{\alpha\in\mathcal{R}^\xi_\delta(\Omega)} \delta^2\Big\langle \sum_{\beta\in P^\xi_\delta(\alpha)\cap\Omega_\varepsilon} \frac{\varepsilon^2}{\delta^2}\,|p(\beta)|^2\mathbf{Q}(\hat{p}(\beta))\,\hat{\xi},\hat{\xi}\Big\rangle^2. \end{aligned} \tag{4.12}$$

4.3.3. **Nematic elastomers.** In this case the strain-alignment-coupling energy is the same as the one considered in the magnetostrictive case; *i.e.,* for a given configuration $u : \Omega_\delta \to \mathbb{R}^2$ and $n : \Omega_\varepsilon \to S^1$,

$$E^{N,SA}_\varepsilon(u,n) = -\sum_{\xi\in X}\sum_{\alpha\in\mathcal{R}^\xi_\delta(\Omega)} \delta^2\langle D^\xi_\delta u(\alpha),\hat{\xi}\rangle \sum_{\beta\in P^\xi_\delta(\alpha)\cap\Omega_\varepsilon} \frac{\varepsilon^2}{\delta^2}\,\big\langle \mathbf{Q}(n(\beta))\,\hat{\xi},\hat{\xi}\big\rangle.$$

5. Magnetostrictive solids

At the microscopic scale the free energy E^M_ε associated with a magnetostrictive solid whose configuration is described by the fields u and m, is given by

$$E^M_\varepsilon(u,m) := E^{EL}_\varepsilon(u) + E^{M,SR}_\varepsilon(m) + E^{M,LR}_\varepsilon(m) + E^{M,SA}_\varepsilon(u,m),$$

with $E^{EL}_\varepsilon, E^{M,SR}_\varepsilon$, $E^{M,LR}_\varepsilon$ and $E^{M,SA}_\varepsilon$ as in (4.1), (4.3), (4.4) and (4.11), respectively.

In this section we study the asymptotic behavior of the energies E^M_ε, as $\varepsilon \to 0$, by means of Γ-convergence. To this end, we let (ε_j) be a sequence of positive real numbers, converging to zero as $j \to +\infty$ and set $E^M_j := E^M_{\varepsilon_j}$. With this definition, each energy E^M_j is defined on a different space. In order to study the Γ-convergence of the functionals E^M_j as $j \to +\infty$, we need to identify each pair $(u,m) : \Omega_{\delta_j} \times \Omega_{\varepsilon_j} \to \mathbb{R}^2 \times S^1$, with a pair of functions belonging to a subspace of a common functional space on Ω. To this end we identify each (u,m) with $(A(u),A(m)) \in \mathcal{A}_{\delta_j}(\Omega)\times\mathcal{B}_{\varepsilon_j}(\Omega) \subset L^2(\Omega;\mathbb{R}^2)\times L^2(\Omega;\mathbb{R}^2)$ as in Section 2. To not overburden notation, when no confusion is possible, we write u,m in place of $A(u),A(m)$.

With the above identification in mind, we may extend the energies $E^{EL}_j := E^{EL}_{\varepsilon_j}$, $E^{M,SR}_j := E^{M,SR}_{\varepsilon_j}$, $E^{M,LR}_j := E^{M,LR}_{\varepsilon_j}$, $E^{M,SA}_j := E^{M,SA}_{\varepsilon_j}$ by identifying them with the functionals (not relabeled) $E^{EL}_j, E^{M,SR}_j : L^2(\Omega;\mathbb{R}^2) \longrightarrow [0,+\infty]$ defined respectively as

$$E^{EL}_j(u) := \begin{cases} \displaystyle\sum_{\xi\in X}\sum_{\alpha\in\mathcal{R}^\xi_{\delta_j}(\Omega)} \delta_j^2|\langle D^\xi_{\delta_j}u(\alpha),\hat{\xi}\rangle|^2 & \text{if } u\in\mathcal{A}_{\delta_j}(\Omega)\\ +\infty & \text{otherwise,}\end{cases}$$

and

$$E^{M,SR}_j(m) := \begin{cases} \displaystyle\frac{1}{2}\sum_{\xi\in Y}\sum_{\alpha\in\mathcal{R}^\xi_{\varepsilon_j}(\Omega)} {\varepsilon_j}^2|D^\xi_{\varepsilon_j}m(\alpha)|^2 & \text{if } m\in\mathcal{B}_{\varepsilon_j}(\Omega)\\ +\infty & \text{otherwise,}\end{cases}$$

while $E^{M,LR}_j : L^2(\Omega;\mathbb{R}^2) \longrightarrow \mathbb{R}\cup\{+\infty\}$ and $E^{M,SA}_j : L^2(\Omega;\mathbb{R}^2)\times L^2(\Omega;\mathbb{R}^2) \longrightarrow \mathbb{R}\cup\{+\infty\}$ are defined respectively as

$$E_j^{M,LR}(m) := \begin{cases} -\dfrac{1}{2} \displaystyle\sum_{\alpha,\beta\in\Omega_{\varepsilon_j},\ \alpha\neq\beta} \varepsilon_j^4 \langle \mathbf{K}(\alpha-\beta)m(\alpha), m(\beta)\rangle & \text{if } m \in \mathcal{B}_{\varepsilon_j}(\Omega) \\ +\infty & \text{otherwise,} \end{cases} \tag{5.1}$$

and

$$E_j^{M,SA}(u,m) := \begin{cases} -\displaystyle\sum_{\xi\in X} \sum_{\alpha\in\mathcal{R}^{\xi}_{\delta_j}(\Omega)} \delta_j^2 \langle D^{\xi}_{\delta_j} u(\alpha), \hat{\xi}\rangle \sum_{\beta\in P^{\xi}_{\delta_j}(\alpha)\cap\Omega_{\varepsilon_j}} \frac{\varepsilon_j^2}{\delta_j^2} \left\langle \mathbf{Q}(m(\beta))\,\hat{\xi}, \hat{\xi}\right\rangle \\ \qquad\qquad\qquad\qquad \text{if } (u,m) \in \mathcal{A}_{\delta_j}(\Omega)\times\mathcal{B}_{\varepsilon_j}(\Omega) \\ +\infty \qquad\qquad\qquad\quad \text{otherwise.} \end{cases}$$

Thus we finally have $E_j^M : L^2(\Omega;\mathbb{R}^2)\times L^2(\Omega;\mathbb{R}^2) \longrightarrow \mathbb{R}\cup\{+\infty\}$ with

$$E_j^M(u,m) = E_j^{EL}(u) + E_j^{M,SR}(m) + E_j^{M,LR}(m) + E_j^{M,SA}(u,m). \tag{5.2}$$

Remark 5.1. The choice of identifying the discrete functions u and m with their piecewise-affine interpolations is mainly suggested by the form of the energies which involves “discrete gradients”. Nevertheless in all that follows piecewise-constant interpolations of u and m may be equivalently considered (see *e.g.*, [7], Proposition A.1 and Remark A.2).

The procedure leading to the computation of the Γ-limit of E_j^M is divided into two main steps. In the first one we limit our analysis to the asymptotic behavior of the functionals

$$\mathcal{E}_j^M(u,m) := E_j^{EL}(u) + E_j^{M,SR}(m) + E_j^{M,SA}(u,m), \tag{5.3}$$

finally proving Theorem 5.4. In the second step, appealing to the works by Firoozye [27] and James and Müller [30], we show that $E_j^{M,LR}$ satisfies a “continuity” property (Proposition 5.8) which in turn provides some kind of “stability” of the Γ-convergence of $\mathcal{E}_j^M$ under the addition of $E_j^{M,LR}$, thus easily yielding a Γ-convergence result for the whole free energy E_j^M (Theorem 5.13).

Step one: a Γ-convergence result for $\mathcal{E}_j^M$.
With the following proposition, we prove a compactness result for sequences (u_j, m_j) converging in $L^2(\Omega;\mathbb{R}^2)\times L^2(\Omega;\mathbb{R}^2)$ and with equi-bounded energy $\mathcal{E}_j^M$. Hence, as a by-product we identify the domain of the Γ-limit of $\mathcal{E}_j^M$.

Proposition 5.2. *Let* $(\mathcal{E}_j^M)$ *be the sequence of functionals defined in* (5.3) *and let* $(u_j, m_j) \to (u,m)$ *in* $L^2(\Omega;\mathbb{R}^2)\times L^2(\Omega;\mathbb{R}^2)$ *be such that*

$$\sup_j \mathcal{E}_j^M(u_j, m_j) < +\infty. \tag{5.4}$$

Then,

$$(u_j, m_j) \rightharpoonup (u,m) \quad \text{in} \quad W^{1,2}(\Omega;\mathbb{R}^2)\times W^{1,2}(\Omega;\mathbb{R}^2),$$

with $m \in S^1$ *a.e. in* Ω.

Proof. For any $\xi \in X$ we have

$$\begin{aligned}
&\Big| \sum_{\alpha \in \mathcal{R}^\xi_{\delta_j}(\Omega)} \delta_j^2 \langle D^\xi_{\delta_j} u(\alpha), \hat{\xi} \rangle \sum_{\beta \in P^\xi_{\delta_j}(\alpha) \cap \Omega_{\varepsilon_j}} \frac{\varepsilon_j^2}{\delta_j^2} \left\langle \mathbf{Q}(m(\beta))\, \hat{\xi}, \hat{\xi} \right\rangle \Big| \\
\leq \quad & |\mathbf{Q}(m)| \, \frac{\varepsilon_j^2}{\delta_j^2} \sup_{\alpha \in \Omega_{\delta_j}} \#(P^\xi_{\delta_j}(\alpha) \cap \Omega_{\varepsilon_j}) \sum_{\alpha \in \mathcal{R}^\xi_{\delta_j}(\Omega)} \delta_j^2 |\langle D^\xi_{\delta_j} u(\alpha), \hat{\xi} \rangle| \\
= \quad & \frac{C_j}{\sqrt{2}} \sum_{\alpha \in \mathcal{R}^\xi_{\delta_j}(\Omega)} \delta_j^2 |\langle D^\xi_{\delta_j} u(\alpha), \hat{\xi} \rangle|, \qquad (5.5)
\end{aligned}$$

where

$$C_j := \frac{\varepsilon_j^2}{\delta_j^2} \sup_{\alpha} \#(P^\xi_{\delta_j}(\alpha) \cap \Omega_{\varepsilon_j}), \qquad \lim_{j \to +\infty} C_j = 1, \qquad (5.6)$$

thus (5.3) and (5.5) give

$$\mathcal{E}^M_j(u_j, m_j) \geq E^{EL}_j(u_j) + E^{M,SR}_j(m_j) - \frac{C_j}{\sqrt{2}} \sum_{\xi \in X} \sum_{\alpha \in \mathcal{R}^\xi_{\delta_j}(\Omega)} \delta_j^2 |\langle D^\xi_{\delta_j} u_j(\alpha), \hat{\xi} \rangle|. \qquad (5.7)$$

Moreover,

$$\begin{aligned}
& E^{EL}_j(u_j) - \frac{C_j}{\sqrt{2}} \sum_{\xi \in X} \sum_{\alpha \in \mathcal{R}^\xi_{\delta_j}(\Omega)} \delta_j^2 |\langle D^\xi_{\delta_j} u_j(\alpha), \hat{\xi} \rangle| \\
= \quad & \sum_{\xi \in X} \sum_{\alpha \in \mathcal{R}^\xi_{\delta_j}(\Omega)} \delta_j^2 \Big(|\langle D^\xi_{\delta_j} u_j(\alpha), \hat{\xi} \rangle|^2 - \frac{C_j}{\sqrt{2}} |\langle D^\xi_{\delta_j} u_j(\alpha), \hat{\xi} \rangle| \Big) \\
= \quad & \sum_{\xi \in X} \sum_{\alpha \in \mathcal{R}^\xi_{\delta_j}(\Omega) : |\langle D^\xi_{\delta_j} u_j(\alpha), \hat{\xi} \rangle| \leq 1} \delta_j^2 \Big(|\langle D^\xi_{\delta_j} u_j(\alpha), \hat{\xi} \rangle|^2 - \frac{C_j}{\sqrt{2}} |\langle D^\xi_{\delta_j} u_j(\alpha), \hat{\xi} \rangle| \Big) \\
+ \quad & \sum_{\xi \in X} \sum_{\alpha \in \mathcal{R}^\xi_{\delta_j}(\Omega) : |\langle D^\xi_{\delta_j} u_j(\alpha), \hat{\xi} \rangle| > 1} \delta_j^2 \Big(|\langle D^\xi_{\delta_j} u_j(\alpha), \hat{\xi} \rangle|^2 - \frac{C_j}{\sqrt{2}} |\langle D^\xi_{\delta_j} u_j(\alpha), \hat{\xi} \rangle| \Big) \\
=: \quad & S_{1,j}(u_j) + S_{2,j}(u_j).
\end{aligned}$$

The following estimates for $S_{1,j}(u_j)$ and $S_{2,j}(u_j)$ hold true:

$$\begin{aligned}
S_{1,j}(u_j) \quad \geq \quad & \sum_{\xi \in X} \Big(\sum_{\alpha \in \mathcal{R}^\xi_{\delta_j}(\Omega) : |\langle D^\xi_{\delta_j} u_j(\alpha), \hat{\xi} \rangle| \leq 1} \delta_j^2 \, |\langle D^\xi_{\delta_j} u_j(\alpha), \hat{\xi} \rangle|^2 \\
- \quad & \delta_j^2 \, \frac{C_j}{\sqrt{2}} \#(\{\alpha \in \mathcal{R}^\xi_{\delta_j}(\Omega) : \ |\langle D^\xi_{\delta_j} u_j(\alpha), \hat{\xi} \rangle| \leq 1\}) \Big) \\
\geq \quad & \sum_{\xi \in X} \sum_{\alpha \in \mathcal{R}^\xi_{\delta_j}(\Omega) : |\langle D^\xi_{\delta_j} u_j(\alpha), \hat{\xi} \rangle| \leq 1} \delta_j^2 \, |\langle D^\xi_{\delta_j} u_j(\alpha), \hat{\xi} \rangle|^2 - C|\Omega|, \qquad (5.8)
\end{aligned}$$

$$S_{2,j}(u_j) \geq \Big(1 - \frac{C_j}{\sqrt{2}} \Big) \sum_{\xi \in X} \sum_{\alpha \in \mathcal{R}^\xi_{\delta_j}(\Omega) : |\langle D^\xi_{\delta_j} u_j(\alpha), \hat{\xi} \rangle| > 1} \delta_j^2 \, |\langle D^\xi_{\delta_j} u_j(\alpha), \hat{\xi} \rangle|^2. \qquad (5.9)$$

Then, gathering (5.7)-(5.9) we get

$$\mathcal{E}_j^M(u_j, m_j) \geq \Big(1 - \frac{C_j}{\sqrt{2}}\Big) E_j^{EL}(u_j) + E_j^{M,SR}(m_j) - C|\Omega|,$$

which by (5.4) and (5.6) permits to deduce

$$\sup_j E_j^{EL}(u_j) < +\infty \quad \text{and} \quad \sup_j E_j^{M,SR}(m_j) < +\infty. \tag{5.10}$$

We now come to prove that $u \in W^{1,2}(\Omega;\mathbb{R}^2)$. If for any $\xi \in X$ we define

$$\Omega_{\delta_j}^\xi := \{\alpha \in \Omega_{\delta_j} : \ P_{\delta_j}^\xi(\alpha) \subset \Omega\}, \tag{5.11}$$

we trivially have

$$E_j^{EL}(u_j) \geq \frac{1}{2} \sum_{\xi \in X} \sum_{\alpha \in \Omega_{\delta_j}^\xi} \delta_j^2 |\langle D_{\delta_j}^\xi u_j(\alpha), \hat{\xi}\rangle|^2.$$

Then, in view of definition (2.4), it can be easily checked that

$$\delta_j^2 |\langle D_{\delta_j}^\xi u_j(\alpha), \hat{\xi}\rangle|^2 = \int_{P_{\delta_j}^\xi(\alpha)} |\langle \nabla u_j \hat{\xi}, \hat{\xi}\rangle|^2 \, dx, \tag{5.12}$$

for any $\xi \in X$. Moreover, we have

$$\Big|\Omega \setminus \bigcup_{\alpha \in \Omega_{\delta_j}^\xi} P_{\delta_j}^\xi(\alpha)\Big| = O(\delta_j), \quad \text{as } j \to +\infty \tag{5.13}$$

for every $\xi \in X$. Hence, if for any fixed $\eta > 0$ we define $\Omega^\eta := \{x \in \Omega : \operatorname{dist}(\Omega, \partial\Omega) > \eta\}$, gathering (5.12) and (5.13) we find that, for j sufficiently large,

$$\begin{aligned} E_j^{EL}(u_j) &\geq \frac{1}{2} \sum_{\xi \in X} \int_{\bigcup_{\alpha \in \Omega_{\delta_j}^\xi} P_{\delta_j}^\xi(\alpha)} |\langle \nabla u_j \hat{\xi}, \hat{\xi}\rangle|^2 \, dx \\ &\geq \frac{1}{2} \int_{\Omega^\eta} \sum_{\xi \in X} |\langle \nabla u_j \hat{\xi}, \hat{\xi}\rangle|^2 \, dx \\ &= \frac{1}{2} \int_{\Omega^\eta} |\mathbf{E} u_j|^2 \, dx + \frac{1}{4} \int_{\Omega^\eta} |\operatorname{div} u_j|^2 \, dx, \end{aligned} \tag{5.14}$$

the last equality following by a direct calculation. Since $\sup_j E_j^{EL}(u_j) < +\infty$, from (5.14) we get that in particular

$$\sup_j \int_{\Omega^\eta} |\mathbf{E} u_j|^2 \, dx < +\infty \quad \forall \eta > 0.$$

Therefore, taking the sup on η gives

$$\sup_j ||\mathbf{E} u_j||_{L^2(\Omega;\mathcal{M}^{2\times 2})} < +\infty.$$

This, combined with $\sup_j ||u_j||_{L^2(\Omega;\mathbb{R}^2)} < +\infty$, by invoking the Korn's Inequality (*cfr.* Proposition 3.7(i)), permits to deduce that

$$\sup_j ||\nabla u_j||_{L^2(\Omega;\mathcal{M}^{2\times 2})} < +\infty,$$

thus finally $u_j \rightharpoonup u$ in $W^{1,2}(\Omega;\mathbb{R}^2)$.

We now come to show that $m \in W^{1,2}(\Omega; S^1)$. A similar analysis to the one performed for E_j^{EL} applies for $E_j^{M,SR}$ as well and it easily leads to

$$E_j^{M,SR}(m_j) \geq \frac{1}{2} \int_{\Omega^\eta} \sum_{\xi \in Y} |\nabla m_j \xi|^2 \, dx = \frac{1}{2} \int_{\Omega^\eta} |\nabla m_j|^2 \, dx, \tag{5.15}$$

for every $\eta > 0$. Then, (5.10) and (5.15) imply

$$\sup_j \int_{\Omega^\eta} |\nabla m_j|^2 \, dx < +\infty \quad \forall \eta > 0.$$

Therefore

$$\sup_j ||\nabla m_j||_{L^2(\Omega;\mathcal{M}^{2\times 2})} < +\infty,$$

which, together with $\sup_j ||m_j||_{L^\infty(\Omega;\mathbb{R}^2)} < +\infty$, yields $m_j \rightharpoonup m$ in $W^{1,2}(\Omega; \mathbb{R}^2)$. Thus, it remains to prove that $|m| = 1$ a.e. in Ω.

To this effect, with $m_0 \in S^1$ given, we define the piecewise-constant function $\tilde{m}_j \in \mathcal{C}_{\varepsilon_j}(\Omega)$ as

$$\tilde{m}_j(x) := \begin{cases} m_j(\alpha) & \text{if } \; x \in Q_{\varepsilon_j}(\alpha) \cap \Omega, \; \alpha \in \Omega_{\varepsilon_j} \\ m_0 & \text{otherwise} \end{cases} \tag{5.16}$$

then, Proposition A.1 and Remark A.2 in [7] permit to conclude that $\tilde{m}_j \to m$ in $L^2(\Omega; \mathbb{R}^2)$ and, as $|\tilde{m}_j| = 1$ a.e. in Ω, we finally deduce that $|m| = 1$ a.e. in Ω. □

The next proposition, which will be a key ingredient in the proof of the Γ-convergence result Theorem 5.4, states a continuity property for the sequence $(E_j^{M,SA})$ with respect to the weak-$W^{1,2}(\Omega; \mathbb{R}^2)\times$ strong-$L^2(\Omega; \mathbb{R}^2)$ convergence.

Proposition 5.3. *Let $(u_j, m_j) \subset \mathcal{A}_{\delta_j}(\Omega) \times \mathcal{B}_{\varepsilon_j}(\Omega)$ be such that*

$$u_j \rightharpoonup u \;\; \textit{in} \;\; W^{1,2}(\Omega; \mathbb{R}^2), \qquad m_j \to m \;\; \textit{in} \;\; L^2(\Omega; \mathbb{R}^2),$$

then

$$\lim_{j\to+\infty} E_j^{M,SA}(u_j, m_j) = -\int_\Omega \mathbf{E}u : \mathbf{Q}(m) \, dx.$$

Proof. By virtue of the Lipschitz-regularity assumption on Ω, without loss of generality, we may suppose that $\sup_j ||u_j||_{W^{1,2}(\Omega';\mathbb{R}^2)} < +\infty$ with Ω' open, bounded such that $\Omega \subset\subset \Omega'$.

For any $\xi \in X$ we set

$$E_j^{M,SA}(\xi)(u,m) := \sum_{\alpha \in \mathcal{R}^\xi_{\delta_j}(\Omega)} \delta_j^2 \langle D^\xi_{\delta_j} u(\alpha), \hat{\xi} \rangle \sum_{\beta \in P^\xi_{\delta_j}(\alpha)\cap\Omega_{\varepsilon_j}} \frac{\varepsilon_j^2}{\delta_j^2} \left\langle \mathbf{Q}(m(\beta))\, \hat{\xi}, \hat{\xi} \right\rangle,$$

thus

$$E_j^{M,SA}(u,m) = -\sum_{\xi \in X} E_j^{M,SA}(\xi)(u,m). \tag{5.17}$$

Let $u_j \rightharpoonup u$ in $W^{1,2}(\Omega; \mathbb{R}^2)$, $m_j \to m$ in $L^2(\Omega; \mathbb{R}^2)$ and let $\tilde{m}_j$ be as in (5.16); we start proving that for any $\xi \in X$

$$E_j^{M,SA}(\xi)(u_j, m_j) = \int_\Omega \Big(\langle \nabla u_j \hat{\xi}, \hat{\xi} \rangle \, \fint_{P^\xi_{\delta_j}([x]_\xi)} \big\langle \mathbf{Q}(\tilde{m}_j(y)) \hat{\xi}, \hat{\xi} \big\rangle \, dy \Big) \, dx + o(1) \tag{5.18}$$

as $j \to +\infty$, where for $x = (x_1, x_2) \in \Omega$

$$[x]_\xi := \begin{cases} [\frac{x_1}{\delta_j}]\delta_j\, e_1 + ([\frac{x_1 - x_2}{\delta_j}] + [\frac{x_1}{\delta_j}])\delta_j\, e_2 & \text{if } \xi = e_1 \\ ([\frac{x_2}{\delta_j}] - [\frac{x_2 - x_1}{\delta_j}])\delta_j\, e_1 + [\frac{x_2}{\delta_j}]\delta_j\, e_2 & \text{if } \xi = e_2 \\ [\frac{x_1}{\delta_j}]\delta_j\, e_1 + [\frac{x_2}{\delta_j}]\delta_j\, e_2 & \text{if } \xi = e_1 + e_2 \\ [\frac{x_1}{\delta_j}]\delta_j\, e_1 + ([\frac{x_2}{\delta_j}] - 1)\delta_j\, e_2 & \text{if } \xi = e_1 - e_2. \end{cases}$$

Notice that by definition

$$P^\xi_{\delta_j}([x]_\xi) = P^\xi_{\delta_j}(\alpha), \quad \text{for } x \in P^\xi_{\delta_j}(\alpha)$$

and for every $\xi \in X$.

We have

$$\begin{aligned} &\Big| E_j^{M,SA}(\xi)(u_j, m_j) - \int_\Omega \Big(\langle \nabla u_j\, \hat\xi, \hat\xi\rangle ⨍_{P^\xi_{\delta_j}([x]_\xi)} \langle \mathbf{Q}(\tilde m_j(y))\hat\xi, \hat\xi\rangle\, dy \Big)\, dx \Big| \\ \le\ & \Big| E_j^{M,SA}(\xi)(u_j, m_j) - \sum_{\alpha \in \Omega^\xi_{\delta_j}} \delta_j^2 \langle D^\xi_{\delta_j} u_j(\alpha), \hat\xi\rangle \sum_{\beta \in P^\xi_{\delta_j}(\alpha) \cap \Omega_{\varepsilon_j}} \frac{\varepsilon_j^2}{\delta_j^2} \big\langle \mathbf{Q}(m_j(\beta))\hat\xi, \hat\xi \big\rangle \Big| \\ +\ & \Big| \sum_{\alpha \in \Omega^\xi_{\delta_j}} \delta_j^2 \langle D^\xi_{\delta_j} u_j(\alpha), \hat\xi\rangle \Big(⨍_{P^\xi_{\delta_j}(\alpha)} \langle \mathbf{Q}(\tilde m_j(y))\hat\xi, \hat\xi\rangle\, dy - \sum_{\beta \in P^\xi_{\delta_j}(\alpha) \cap \Omega_{\varepsilon_j}} \frac{\varepsilon_j^2}{\delta_j^2} \big\langle \mathbf{Q}(m_j(\beta))\hat\xi, \hat\xi \big\rangle \Big) \Big| \\ +\ & \Big| \int_{\Omega \setminus \bigcup_{\alpha \in \Omega^\xi_{\delta_j}} P^\xi_{\delta_j}(\alpha)} \Big(\langle \nabla u_j\, \hat\xi, \hat\xi\rangle ⨍_{P^\xi_{\delta_j}([x]_\xi)} \langle \mathbf{Q}(\tilde m_j(y))\hat\xi, \hat\xi\rangle\, dy \Big)\, dx \Big| \\ =:\ & I^\xi_{1,j} + I^\xi_{2,j} + I^\xi_{3,j}, \end{aligned}$$

with $\Omega^\xi_{\delta_j}$ as in (5.11). We now turn to estimate $I^\xi_{i,j}$, for $i = 1, 2, 3$.

By the uniform boundedness of $(\mathbf{Q}(m_j))$ and applying Hölder's Inequality we get

$$\begin{aligned} I^\xi_{1,j} &\le \sum_{\alpha \in \mathcal{R}^\xi_{\delta_j}(\Omega) \setminus \Omega^\xi_{\delta_j}} \delta_j^2 |\langle D^\xi_{\delta_j} u_j(\alpha), \hat\xi, \rangle|\, |\mathbf{Q}(m_j(\beta))|\, \frac{\varepsilon_j^2}{\delta_j^2} \#(P^\xi_{\delta_j}(\alpha) \cap \Omega_{\varepsilon_j}) \\ &\le ||\nabla u_j||_{L^2(\Omega')} \big(\delta_j^2\, \#(\mathcal{R}^\xi_{\delta_j}(\Omega) \setminus \Omega^\xi_{\delta_j}) \big)^{\frac12} \frac{1}{\sqrt2} \frac{\varepsilon_j^2}{\delta_j^2} \sup_\alpha \#(P^\xi_{\delta_j}(\alpha) \cap \Omega_{\varepsilon_j}), \end{aligned}$$

then from

$$\#(\mathcal{R}^\xi_{\delta_j}(\Omega) \setminus \Omega^\xi_{\delta_j}) = O\Big(\frac{1}{\delta_j}\Big) \quad \text{and} \quad \sup_\alpha \#(P^\xi_{\delta_j}(\alpha) \cap \Omega_{\varepsilon_j}) = O\Big(\frac{\delta_j^2}{\varepsilon_j^2}\Big), \quad j \to +\infty,$$

we deduce that

$$I^\xi_{1,j} = O(\sqrt{\delta_j}) \quad \text{as} \quad j \to +\infty. \tag{5.19}$$

If we set $\mathcal{Q}^\xi_{\alpha,j} := \{\beta \in \Omega_{\varepsilon_j} : \ Q_{\varepsilon_j}(\beta) \subset P^\xi_{\delta_j}(\alpha)\}$, the following estimate on $I^\xi_{2,j}$ holds:

$$
\begin{aligned}
I^\xi_{2,j} \ &\leq \ \sum_{\alpha\in\Omega^\xi_{\delta_j}} \delta_j^2 |\langle D^\xi_{\delta_j} u_j(\alpha), \hat{\xi}\rangle| \, \frac{1}{\delta_j^2}\Big(\sum_{\beta\in(P^\xi_{\delta_j}(\alpha)\cap\Omega_{\varepsilon_j})\setminus\mathcal{Q}^\xi_{\alpha,j}} \varepsilon_j^2 \, |\langle \mathbf{Q}(m_j(\beta))\hat{\xi}, \hat{\xi}\rangle| \\
&+ \ \int_{P^\xi_{\delta_j}(\alpha)\setminus\bigcup_{\beta\in\mathcal{Q}^\xi_{\alpha,j}} Q_{\varepsilon_j}(\beta)} |\langle \mathbf{Q}(m_j(y))\hat{\xi}, \hat{\xi}\rangle| \, dy\Big) \\
&\leq \ \sum_{\alpha\in\Omega^\xi_{\delta_j}} \delta_j^2 |\langle D^\xi_{\delta_j} u_j(\alpha), \hat{\xi}\rangle| \, \frac{1}{\delta_j^2} \, |\mathbf{Q}(m_j(\beta))| \, \Big(\varepsilon_j^2 \, \#\big((P^\xi_{\delta_j}(\alpha)\cap\Omega_{\varepsilon_j}) \setminus \mathcal{Q}^\xi_{\alpha,j}\big) \\
&+ \ |P^\xi_{\delta_j}(\alpha) \setminus \bigcup_{\beta\in\mathcal{Q}^\xi_{\alpha,j}} Q_{\varepsilon_j}(\beta)|\Big) \\
&\leq \ \frac{1}{\sqrt{2}} ||\nabla u_j||_{L^2(\Omega')} \big(\delta_j^2 \, \#(\Omega^\xi_{\delta_j})\big)^{\frac{1}{2}} \, \frac{1}{\delta_j^2} \, \Big(\varepsilon_j^2 \, \sup_\alpha \#\big((P^\xi_{\delta_j}(\alpha)\cap\Omega_{\varepsilon_j}) \setminus \mathcal{Q}^\xi_{\alpha,j}\big) \\
&+ \ \sup_\alpha |P^\xi_{\delta_j}(\alpha) \setminus \bigcup_{\beta\in\mathcal{Q}^\xi_{\alpha,j}} Q_{\varepsilon_j}(\beta)|\Big).
\end{aligned}
$$

Since

$$
\delta_j^2 \, \#(\Omega^\xi_{\delta_j}) = O(1), \quad \sup_\alpha \#\big((P^\xi_{\delta_j}(\alpha)\cap\Omega_{\varepsilon_j}) \setminus \mathcal{Q}^\xi_{\alpha,j}\big) = O\Big(\frac{\delta_j}{\varepsilon_j}\Big),
$$

$$
\sup_\alpha |P^\xi_{\delta_j}(\alpha) \setminus \bigcup_{\beta\in\mathcal{Q}^\xi_{\alpha,j}} Q_{\varepsilon_j}(\beta)| = O(\varepsilon_j \, \delta_j), \quad \text{as} \ \ j \to +\infty
$$

we can infer that

$$
I^\xi_{2,j} = O\Big(\frac{\varepsilon_j}{\delta_j}\Big) \quad \text{as} \quad j \to +\infty. \tag{5.20}
$$

When we come to consider $I^\xi_{3,j}$. Hölder's Inequality immediately gives

$$
I^\xi_{3,j} \leq \frac{1}{\sqrt{2}} \, ||\nabla u_j||_{L^2(\Omega')} \, \big| \, \Omega \setminus \bigcup_{\alpha\in\Omega^\xi_{\delta_j}} P^\xi_{\delta_j}(\alpha)\big|^{\frac{1}{2}} = O(\sqrt{\delta}_j) \quad \text{as} \quad j \to +\infty. \tag{5.21}
$$

Finally, gathering (5.19)-(5.21) permits to deduce (5.18).

We now want to prove that

$$
\lim_{j\to+\infty} \int_\Omega \Big(\langle \nabla u_j \, \hat{\xi}, \hat{\xi}\rangle \, \fint_{P^\xi_{\delta_j}([x]_\xi)} \langle \mathbf{Q}(\tilde{m}_j(y))\hat{\xi}, \hat{\xi}\rangle \, dy\Big) \, dx = \int_\Omega \langle \nabla u \, \hat{\xi}, \hat{\xi}\rangle \, \langle \mathbf{Q}(m)\hat{\xi}, \hat{\xi}\rangle \, dx, \tag{5.22}
$$

for any $\xi \in X$. To this end, it suffices to show that

$$
\fint_{P^\xi_{\delta_j}([\cdot]_\xi)} \langle \mathbf{Q}(\tilde{m}_j(y))\hat{\xi}, \hat{\xi}\rangle \, dy \longrightarrow \langle \mathbf{Q}(m)\hat{\xi}, \hat{\xi}\rangle \quad \text{in} \quad L^1(\Omega;\mathbb{R}). \tag{5.23}
$$

Indeed, since

$$
\sup_j \Big\| \fint_{P^\xi_{\delta_j}([\cdot]_\xi)} \langle \mathbf{Q}(\tilde{m}_j(y))\hat{\xi}, \hat{\xi}\rangle \, dy \Big\|_{L^\infty(\Omega,\mathbb{R})} < +\infty,
$$

once proved, (5.23) permits to deduce that in particular

$$\fint_{P^{\xi}_{\delta_j}([\cdot]_\xi)} \langle \mathbf{Q}(\tilde m_j(y))\hat\xi, \hat\xi\rangle\, dy \longrightarrow \langle \mathbf{Q}(m)\hat\xi,\hat\xi\rangle \quad \text{in} \quad L^2(\Omega;\mathbb{R})$$

and this combined with $\nabla u_j \rightharpoonup \nabla u$ in $L^2(\Omega;\mathcal{M}^{2\times 2})$ finally implies (5.22).

We prove (5.23) only for $\xi = e_1$, the proof of the other cases being analogous.

A change of variable and Fubini's Theorem yield

$$\begin{aligned}
&\Big\| \fint_{P^{e_1}_{\delta_j}([x]_{e_1})} \langle \mathbf{Q}(\tilde m_j(y))e_1, e_1\rangle\, dy - \langle \mathbf{Q}(m)e_1,e_1\rangle \Big\|_{L^1(\Omega;\mathbb{R})} \\
\le\ & \int_\Omega \frac{1}{\delta_j^2}\int_{x-\delta_j\, e_1+\delta_j P^{e_1}_2(0)} |\langle \mathbf{Q}(\tilde m_j(y))e_1,e_1\rangle\, dy - \langle \mathbf{Q}(m(x))e_1,e_1\rangle|\, dx \\
=\ & \int_\Omega \int_{P^{e_1}_2(0)} |\langle \mathbf{Q}(\tilde m_j(x-\delta_j\, e_1+\delta_j z))e_1,e_1\rangle\, dz - \langle \mathbf{Q}(m(x))e_1,e_1\rangle|\, dx \\
\le\ & \int_{P^{e_1}_2(0)} \int_\Omega |\langle \mathbf{Q}(\tilde m_j(x-\delta_j\, e_1+\delta_j z))e_1,e_1\rangle - \langle \mathbf{Q}(\tilde m_j(x))e_1,e_1\rangle|\, dx\, dz \\
+\ & 4\|\langle(\mathbf{Q}(\tilde m_j) - \mathbf{Q}(m))\, e_1, e_1\rangle\|_{L^1(\Omega;\mathbb{R})}.
\end{aligned}$$

As we know that $m_j \to m$ in $L^2(\Omega;\mathbb{R}^2)$, we have that $\tilde m_j \to \tilde m$ in $L^2_{loc}(\mathbb{R}^2;\mathbb{R}^2)$, with

$$\tilde m = \begin{cases} m & \text{in } \Omega \\ m_0 & \text{otherwise}\end{cases}$$

and $|\tilde m_j| = |\tilde m| = 1$ a.e. in $\mathbb{R}^2$. We deduce that

$$\mathbf{Q}(\tilde m_j) \longrightarrow \mathbf{Q}(\tilde m) \quad \text{in} \quad L^1_{loc}(\mathbb{R}^2;\mathcal{M}^{2\times 2}).$$

Then, by the uniform continuity of the translation operator for strongly converging sequences in $L^1_{loc}(\mathbb{R}^2)$, we get (5.23). Gathering (5.18) and (5.22) finally yields

$$\lim_{j\to+\infty} E^{M,SA}_j(u_j,n_j) = -\int_\Omega \sum_{\xi\in X} \langle \nabla u\,\hat\xi,\hat\xi\rangle\, \langle \mathbf{Q}(m)\hat\xi,\hat\xi\rangle\, dx = -\int_\Omega \mathbf{E}u : \mathbf{Q}(m)\, dx,$$

with the second equality following by a direct calculation. □

We are now ready to prove a Γ-convergence result for the sequence $(\mathcal{E}^M_j)$.

In what follows we additionally assume that Ω is a simply connected set (see Remark 5.5).

Theorem 5.4 (Γ-convergence of $\mathcal{E}^M_j$)**.** *The sequence of functionals $(\mathcal{E}^M_j)$ defined in* (5.3) *Γ-converges with respect to the $L^2(\Omega;\mathbb{R}^2)\times L^2(\Omega;\mathbb{R}^2)$ convergence, to the functional $\mathcal{E}^M : L^2(\Omega;\mathbb{R}^2)\times L^2(\Omega;\mathbb{R}^2) \longrightarrow \mathbb{R}\cup\{+\infty\}$ given by*

$$\mathcal{E}^M(u,m) = \begin{cases} \dfrac12\displaystyle\int_\Omega |\mathbf{E}u|^2\,dx + \dfrac14\int_\Omega |\operatorname{div} u|^2\,dx + \dfrac12\int_\Omega |\nabla m|^2\,dx - \int_\Omega \mathbf{E}u:\mathbf{Q}(m)\,dx \\ \qquad\qquad\qquad\qquad \textit{if } (u,m)\in W^{1,2}(\Omega;\mathbb{R}^2)\times W^{1,2}(\Omega;S^1) \\ +\infty \qquad\qquad\qquad\quad \textit{otherwise.}\end{cases}$$

Proof. Lower bound. Let $(u_j,m_j)\subset \mathcal{A}_{\delta_j}(\Omega)\times\mathcal{B}_{\varepsilon_j}(\Omega)$ be a sequence of functions converging to (u,m) in $L^2(\Omega;\mathbb{R}^2)\times L^2(\Omega;\mathbb{R}^2)$. We have to prove that

$$\liminf_{j\to+\infty} \mathcal{E}^M_j(u_j,m_j) \ge \mathcal{E}^M(u,m).$$

Without loss of generality, we may assume that

$$\liminf_{j\to+\infty} \mathcal{E}_j^M(u_j, m_j) < +\infty,$$

then, by virtue of Proposition 5.2

$$(u_j, m_j) \rightharpoonup (u, m) \quad \text{in} \quad W^{1,2}(\Omega;\mathbb{R}^2) \times W^{1,2}(\Omega;\mathbb{R}^2),$$

with $|m| = 1$ a.e. in Ω. Moreover, in view of (5.14) and (5.15) we have

$$\begin{aligned} \liminf_{j\to+\infty} \left(E_j^{EL}(u_j) + E^M(m_j)\right) &\geq \liminf_{j\to+\infty} \left(\frac{1}{2}\int_{\Omega^\eta} |\mathbf{E}u_j|^2\,dx + \frac{1}{4}\int_{\Omega^\eta} |\operatorname{div} u_j|^2\,dx\right) \\ &+ \frac{1}{2}\liminf_{j\to+\infty}\int_{\Omega^\eta} |\nabla m_j|^2\,dx, \end{aligned}$$

for $\eta > 0$ and for j sufficiently large.

Since

$$|\mathbf{E}v|^2 + \frac{1}{2}|\operatorname{div} v|^2 = \frac{1}{2}\Big(\frac{1}{2}|\nabla v + \nabla v^T|^2 + |\operatorname{tr}\nabla v|^2\Big) =: f(\nabla v)$$

and f is convex, by sequential weak-$W^{1,2}(\Omega;\mathbb{R}^2)\times W^{1,2}(\Omega;\mathbb{R}^2)$ lower semicontinuity, we get

$$\liminf_{j\to+\infty} \left(E_j^{EL}(u_j) + E^M(m_j)\right) \geq \frac{1}{2}\int_{\Omega^\eta} |\mathbf{E}u|^2\,dx + \frac{1}{4}\int_{\Omega^\eta} |\operatorname{div} u|^2\,dx + \frac{1}{2}\int_{\Omega^\eta} |\nabla m|^2\,dx,$$

for every $\eta > 0$. Invoking Proposition 5.3 yields

$$\begin{aligned} \liminf_{j\to+\infty} \mathcal{E}_j^M(u_j, m_j) &= \liminf_{j\to+\infty} \left(E_j^{EL}(u_j) + E^M(m_j)\right) + \lim_{j\to+\infty} E_j^{M,SA}(u_j, m_j) \\ &\geq \frac{1}{2}\int_{\Omega^\eta} |\mathbf{E}u|^2\,dx + \frac{1}{4}\int_{\Omega^\eta} |\operatorname{div} u|^2\,dx + \frac{1}{2}\int_{\Omega^\eta} |\nabla m|^2\,dx \\ &- \int_\Omega \mathbf{E}(u) : \mathbf{Q}(m)\,dx, \end{aligned}$$

for every $\eta > 0$. Finally, taking the sup on η gives the lower bound.

Upper bound. We have to prove that, for any $(u, m) \in W^{1,2}(\Omega;\mathbb{R}^2)\times W^{1,2}(\Omega;S^1)$, there exists a sequence $(u_j, m_j) \subset \mathcal{A}_{\delta_j}(\Omega)\times\mathcal{B}_{\varepsilon_j}(\Omega)$ converging to (u, m) in $L^2(\Omega;\mathbb{R}^2)\times L^2(\Omega;\mathbb{R}^2)$ and such that

$$\limsup_{j\to+\infty} \mathcal{E}_j^M(u_j, m_j) \leq \mathcal{E}^M(u, m).$$

By the assumption on Ω, without loss of generality, we may suppose that $(u, m) \in W^{1,2}(\Omega', \mathbb{R}^2) \times W^{1,2}(\Omega', S^1)$, with $\Omega \subset\subset \Omega'$.

By Remark 3.5 there exist two sequences $(y_j), (z_j) \subset \overline{Q}$ such that, setting $v_j := T_{y_j}^{\delta_j} u$ and $w_j := T_{z_j}^{\varepsilon_j} m$, the sequence $(u_j, m_j) := (A(v_j), A(w_j))$ is such that $(u_j, m_j) \subset \mathcal{A}_{\delta_j}(\Omega) \times \mathcal{B}_{\varepsilon_j}(\Omega)$ and

$$(u_j, m_j) \to (u, m) \quad \text{in} \quad W^{1,2}(\Omega', \mathbb{R}^2) \times W^{1,2}(\Omega', \mathbb{R}^2). \tag{5.24}$$

Moreover, a direct calculation gives

$$\begin{aligned} E_j^{EL}(u_j) + E_j^{M,SR}(m_j) &\leq \frac{1}{2}\int_{\Omega+B_\eta(0)} |\mathbf{E}u_j|^2\,dx + \frac{1}{4}\int_{\Omega+B_\eta(0)} |\operatorname{div} u_j|^2\,dx \\ &+ \frac{1}{2}\int_{\Omega+B_\eta(0)} |\nabla m_j|^2\,dx, \end{aligned} \tag{5.25}$$

being $B_\eta(0)$ the ball centered at 0 and with radius η, such that $\sqrt{2}\delta_j < \eta < \operatorname{dist}(\partial\Omega, \mathbb{R}^2 \setminus \Omega')$, for j large enough. In view of (5.24), (5.25) and appealing to Proposition 5.3 we find

$$\begin{aligned}\limsup_{j\to+\infty} \mathcal{E}_j^M(u_j, m_j) &= \limsup_{j\to+\infty} \left(E_j^{EL}(u_j) + E_j^{M,SR}(m_j)\right) + \lim_{j\to+\infty} E_j^{M,SA}(u_j, m_j) \\ &\leq \frac{1}{2}\int_{\Omega+B_\eta(0)} |\mathbf{E}u|^2\,dx + \frac{1}{4}\int_{\Omega+B_\eta(0)} |\operatorname{div} u|^2\,dx \\ &+ \frac{1}{2}\int_{\Omega+B_\eta(0)} |\nabla m|^2\,dx - \int_\Omega \mathbf{E}(u) : \mathbf{Q}(m).\end{aligned}$$

Finally, taking the limit as $\eta \to 0$, we get the upper bound. □

Remark 5.5. The simply connectedness assumption on Ω plays a crucial role in the proof of the upper bound inequality in Theorem 5.4 since we need it to extend a function $m \in W^{1,2}(\Omega; S^1)$ to a function belonging to $W^{1,2}(\mathbb{R}^2; S^1)$. Nevertheless, it is worth pointing out that this hypothesis can be removed if in the energy $E_\varepsilon^{M,SR}$ we decide to take into account only the interactions between those points $\alpha \in \Omega_\varepsilon$ such that $Q_\varepsilon(\alpha) \subset\subset \Omega$. Clearly, this possible modification would not affect the Γ-limit $\mathcal{E}^M$.

Remark 5.6 (The model by Pasini, Skačej and Zannoni)**.** As already pointed out in Section 4, other functionals have been considered to describe the strain-alignment energy. Here we discuss the one Pasini, Skačej and Zannoni proposed in [32] (see also [33]). The main difference with our model is in the choice of averaging the field m rather than the tensor order parameter $\mathbf{Q}(m)$ on the cells of the lattice Ω_δ. In our notation, this amounts to consider a coupling term of the form

$$F_j^{M,SA}(u,m) := -\sum_{\xi\in X} \sum_{\alpha\in\mathcal{R}^\xi_{\delta_j}(\Omega)} \delta_j^2 \langle D^\xi_{\delta_j} u(\alpha), \hat{\xi}\rangle \left\langle \mathbf{Q}(\overline{m}(\alpha))\,\hat{\xi}, \hat{\xi}\right\rangle,$$

with

$$\overline{m}(\alpha) := \frac{1}{\delta_j^2} \sum_{\beta\in P^\xi_{\delta_j}(\alpha)\cap\Omega_{\varepsilon_j}} \varepsilon_j^2\, m(\beta)$$

and where, abusing notation, $\mathbf{Q}(\overline{m}(\alpha))$ still denotes the tensor order parameter, defined as in (2.1), even if now $\overline{m}(\alpha) \notin S^1$.

In this remark we want to show that the two models involving the two different strain-alignment terms $E_j^{M,SA}$ and $F_j^{M,SA}$ are "asymptotically equivalent"; *i.e.*, they have the same Γ-limit $\mathcal{E}^M$. To this end, it suffices to show that

$$\lim_{j\to+\infty} F_j^{M,SA}(u_j, m_j) = -\int_\Omega \mathbf{E}(u) : \mathbf{Q}(m)\,dx, \tag{5.26}$$

for every $(u_j, m_j) \subset \mathcal{A}_{\delta_j}(\Omega) \times \mathcal{B}_{\varepsilon_j}(\Omega)$ such that $u_j \rightharpoonup u$ in $W^{1,2}(\Omega;\mathbb{R}^2)$ and $m_j \to m$ in $L^2(\Omega;\mathbb{R}^2)$. Following the line of the proof of Proposition 5.3, it can be easily proved that, setting

$$F_j^{M,SA}(\xi)(u,m) := \sum_{\alpha\in\mathcal{R}^\xi_{\delta_j}(\Omega)} \delta_j^2 \langle D^\xi_{\delta_j} u(\alpha), \hat{\xi}\rangle \left\langle \mathbf{Q}(\overline{m}(\alpha))\,\hat{\xi}, \hat{\xi}\right\rangle,$$

for any $\xi \in X$, we get

$$F_j^{M,SA}(\xi)(u_j, m_j) = \int_\Omega \Big(\langle \nabla u_j \hat{\xi}, \hat{\xi} \rangle \, \Big\langle \mathbf{Q}\Big(\fint_{P^\xi_{\delta_j}([x]_\xi)} \tilde{m}_j(y)\, dy \Big) \hat{\xi}, \hat{\xi} \Big\rangle \Big)\, dx + o(1),$$

as $j \to +\infty$, with $\tilde{m}_j$ as in (5.16). Then, since $\nabla u_j \rightharpoonup \nabla u$ in $L^2(\Omega; \mathcal{M}^{2\times 2})$, the equivalence reduces to prove that

$$\mathbf{Q}\Big(\fint_{P^\xi_{\delta_j}([\cdot]_\xi)} \tilde{m}_j(y)\, dy \Big) \longrightarrow \mathbf{Q}(m) \quad \text{in} \quad L^2(\Omega; \mathcal{M}^{2\times 2}) \tag{5.27}$$

as $j \to +\infty$.

By definition

$$\begin{aligned} \mathbf{Q}\Big(\fint_{P^\xi_{\delta_j}([x]_\xi)} \tilde{m}_j(y)\, dy \Big) - \mathbf{Q}(m(x)) &= \Big(\fint_{P^\xi_{\delta_j}([x]_\xi)} \tilde{m}_j(y)\, dy - m(x) \Big) \otimes \fint_{P^\xi_{\delta_j}([x]_\xi)} \tilde{m}_j(y)\, dy \\ &+ m(x) \otimes \Big(\fint_{P^\xi_{\delta_j}([x]_\xi)} \tilde{m}_j(y)\, dy - m(x) \Big). \end{aligned}$$

Thus, in view of

$$\sup_j \Big\| \fint_{P^\xi_{\delta_j}([x]_\xi)} \tilde{m}_j(y)\, dy - m(x) \Big\|_{L^\infty(\Omega;\mathbb{R}^2)} < +\infty,$$

it is enough to show that

$$\fint_{P^\xi_{\delta_j}([x]_\xi)} \tilde{m}_j(y)\, dy - m(x) \longrightarrow 0 \quad \text{a.e. in } \ \Omega$$

as $j \to +\infty$. Since the same argument performed in the proof of Proposition 5.3 easily gives the above convergence, we deduce (5.27), hence (5.26), thus finally the desired equivalence.

Step two: asymptotic behavior of the long range internal energy $E_j^{M,LR}$ and a Γ-convergence result for E_j^M.

In order to prove a Γ-convergence result for the total free energy E_j^M we need to perform a preliminary asymptotic analysis for $E_j^{M,LR}$. Such analysis relies on a result stated in [27], Theorem 3 (see also Section 8 for its application to magnetostatics). In order to proceed we start by recalling some notation, definitions, and results contained in [27]. We refer to [27, 30, 36] for the details.

Let $f : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$, $f' : S^1 \to \mathbb{R}$ such that $f(x) := |x|^{-2} f'(\hat{x})$. We say that f is an *admissible function* if

(i) $f' \in C^\infty(S^1; \mathbb{R})$,
(ii) $\int_{S^1} f' = 0$.

Let $\tilde{f} \in C^\infty(S^1; \mathbb{R})$ be defined as

$$\tilde{f}(s) = \sum_{\ell=0}^{\infty} (-1)^\ell \, a_\ell \, \frac{\Gamma(\frac{\ell}{2})}{\Gamma(\frac{1}{2}(\ell+2))} Y_\ell(s),$$

where a_ℓ denote the coefficients of the development of f' in spherical harmonics Y_ℓ and Γ is the Euler function; *i.e.*, $\Gamma(t) := \int_0^{+\infty} x^{t-1} e^{-x}\, dx$, for $t > 0$.

Let

$$S^f := \lim_{t\to 0} \sum_{z\in\mathbb{Z}^2\setminus\{0\}} e^{-t|z|}|z|^{-2} f'(\hat{z}).$$

It can be proved that if f' is an admissible function, then $S^f \in \mathbb{R}$.

Let $g^f \in C^\infty(S^1;\mathbb{R})$ be defined as

$$g^f(s) := \tilde{f}(s) + S^f.$$

We say that $\mathbf{F} : \mathbb{R}^2 \to \mathcal{M}^{2\times 2}$, $\mathbf{F} := (f_{il})$ is an *admissible kernel* if its entries f_{il} are admissible functions, for $i, l = 1, 2$. Moreover we set $\mathbf{G_F} : S^1 \to \mathcal{M}^{2\times 2}$ as $\mathbf{G_F} := (g^{f_{il}})$.

For the reader's convenience in what follows we present a simplified version, useful for our purposes, of two results by Firoozye. We remark that, in its simplified version, the first result we are going to state is a straightforward consequence of Wainger's Theorems 6 and 7 in [36].

Theorem 5.7 (Theorem 3, [27])**.** *Let* $\mathbf{F}$ *be an admissible kernel and let* $(m_j) \subset L^2(\mathbb{R}^2;\mathbb{R}^2)$ *be such that* $m_j \to m$ *in* $L^2(\mathbb{R}^2;\mathbb{R}^2)$*. Then*

$$\lim_{j\to+\infty} \sum_{\alpha,\beta\in\varepsilon_j\mathbb{Z}^2,\ \alpha\neq\beta} \varepsilon_j^4 \langle \mathbf{F}(\alpha-\beta)m_j(\alpha), m_j(\beta)\rangle = \int_{\mathbb{R}^2} \mathbf{G_F}(\hat{y}) : \big(\mathcal{F}(m)(y)\otimes\overline{\mathcal{F}(m)}(y)\big)\, dy,$$

where $\mathcal{F}(m)$ *is the Fourier transform of* m *and* $\overline{\mathcal{F}(m)}$ *is its conjugate.*

We are now ready to analyze the asymptotics of the sequence of energies $(E_j^{M,LR})$.

Proposition 5.8. *Let* $(m_j) \subset \mathcal{B}_{\varepsilon_j}(\Omega)$ *be such that* $m_j \to m$ *in* $L^2(\Omega;\mathbb{R}^2)$*. Then,*

$$\lim_{j\to+\infty} E_j^{M,LR}(m_j) = -\frac{1}{2}\int_{\mathbb{R}^2} \langle \tilde{m}, \mathbf{K} * \tilde{m}\rangle\ dx - \frac{1}{4}|\Omega|,$$

where $\tilde{m} := m\chi_\Omega$ *and* $\mathbf{K}$ *is as in* (4.5)*.*

Proof. A direct computation yields that the Helmholtz kernel $\mathbf{K} = (k_{il})$ is an admissible kernel. Moreover, thanks to the symmetry of the square lattice it holds that $S^{k_{il}} = 0$ for any k_{il}, $i, l = 1, 2$ (see Remark 6.4 in [30]).

Since for every $i, l = 1, 2$, k_{il} is a spherical harmonics of order 2, we have that

$$\mathbf{G_K}(s) = \Big(\frac{1}{2}\mathbf{Id} - s\otimes s\Big).$$

Let us consider the sequence $(\tilde{m}_j) \subset \mathcal{C}_{\varepsilon_j}(\Omega)$ of piecewise constant functions defined as

$$\tilde{m}_j(x) = \begin{cases} m_j(\alpha) & \text{if } x \in Q_{\varepsilon_j}(\alpha),\ Q_{\varepsilon_j}(\alpha)\cap\Omega\neq\emptyset,\ \alpha\in\Omega_{\varepsilon_j} \\ 0 & \text{otherwise.}\end{cases}$$

We have that $\tilde{m}_j \to \tilde{m}$ in $L^2(\mathbb{R}^2,\mathbb{R}^2)$ and $E_j^{M,LR}(m_j) = E_j^{M,LR}(\tilde{m}_j)$. Hence by applying Theorem 5.7 we get that

$$\lim_{j\to+\infty} E_j^{M,LR}(m_j) = -\frac{1}{2}\int_{\mathbb{R}^2} \mathbf{G_K}(\hat{y}) : \big(\mathcal{F}(\tilde{m})(y)\otimes\overline{\mathcal{F}(\tilde{m})}(y)\big)\, dy.$$

Then, the conclusion follows passing to the real space variable. Indeed, since

$$\mathcal{F}(\mathbf{K})(y) = -\hat{y}\otimes\hat{y},$$

appealing to the properties of the Fourier transform, we deduce that

$$\lim_{j\to+\infty} E_j^{M,LR}(m_j) = -\frac{1}{2}\int_{\mathbb{R}^2} \langle \tilde{m}, \mathbf{K} * \tilde{m}\rangle \, dx - \frac{1}{2}\int_{\Omega} \frac{1}{2}|m|^2 \, dx. \tag{5.28}$$

Finally, the thesis follows observing that $|m| = 1$ a.e. in Ω. □

Remark 5.9. Note that in the continuum micromagnetic theory the first term in the right hand side of (5.28) is called the magnetostatic energy. Indeed it can be easily shown (see [13]) that, since

$$k_{il} = \frac{\partial^2}{\partial x_i x_l}\Delta^{-1}(\delta_0)$$

in the sense of distributions, δ_0 being the Dirac mass centered at 0, the term $\mathbf{K}*\, m$ is the distributional solution h to the magnetostatic equation

$$\begin{cases} \operatorname{div}(\nabla\psi + m\chi_\Omega) = 0 & \text{in } \mathbb{R}^2 \\ h = -\nabla\psi. \end{cases} \tag{5.29}$$

We recall now a second result from [27, Section 8].

Theorem 5.10. *Let $\mathbf{K} : \mathbb{R}^2 \to \mathcal{M}^{2\times 2}$ be the two-dimensional Helmholtz kernel* (4.5) *and let $m : \mathbb{R}^2 \to \mathbb{R}^2$. Then*

$$\liminf_{j\to+\infty}\Big(-\frac{1}{2}\sum_{\alpha,\beta\in\varepsilon_j\mathbb{Z}^2,\ \alpha\neq\beta} \varepsilon_j^4 \langle \mathbf{K}(\alpha-\beta)m(\alpha), m(\beta)\rangle\Big) \geq \limsup_{j\to+\infty}\Big(-\frac{1}{4}\sum_{\alpha\in\varepsilon_j\mathbb{Z}^2} \varepsilon_j^2 |m(\alpha)|^2\Big).$$

Remark 5.11. Notice that applying Theorem 5.10 with $\tilde{m} = m\chi_\Omega$, $m : \Omega_\varepsilon \to S^1$, yields

$$\liminf_{j\to+\infty} E_j^{M,LR}(m) \geq -\frac{1}{4}|\Omega|. \tag{5.30}$$

We now come to study the asymptotic behavior of the total free energies E_j^M, starting with a compactness result.

Proposition 5.12. *Let (E_j^M) be the sequence of functionals defined in* (5.2) *and let $(u_j, m_j) \to (u, m)$ in $L^2(\Omega;\mathbb{R}^2)\times L^2(\Omega;\mathbb{R}^2)$ be such that*

$$\sup_j E_j^M(u_j, m_j) < +\infty.$$

Then,

$$(u_j, m_j) \rightharpoonup (u, m) \quad \text{in} \quad W^{1,2}(\Omega;\mathbb{R}^2)\times W^{1,2}(\Omega;\mathbb{R}^2),$$

with $m \in S^1$ a.e. in Ω.

Proof. The proof is straightforward from (5.30) and Proposition 5.2. □

The following theorem states the desired Γ-convergence result for E_j^M.

Theorem 5.13 (Γ-convergence of E_j^M)**.** *The sequence of functionals (E_j^M) defined in* (5.2) *Γ-converges with respect to the $L^2(\Omega;\mathbb{R}^2)\times L^2(\Omega;\mathbb{R}^2)$ convergence, to the functional $E^M : L^2(\Omega;\mathbb{R}^2)\times L^2(\Omega;\mathbb{R}^2) \longrightarrow \mathbb{R}\cup\{+\infty\}$ given by*

$$E^M(u, m) = \mathcal{E}^M(u, m) - \frac{1}{2}\int_{\mathbb{R}^2} \langle \tilde{m}, \mathbf{K} * \tilde{m}\rangle \, dx - \frac{1}{4}|\Omega|,$$

with $\mathcal{E}^M$ as in Theorem 5.4 and $\tilde{m} = m\chi_\Omega$.

Proof. Lower bound. Let $(u_j, m_j) \subset \mathcal{A}_{\delta_j}(\Omega) \times \mathcal{B}_{\varepsilon_j}(\Omega)$ be a sequence of functions converging to (u, m) in $L^2(\Omega;\mathbb{R}^2) \times L^2(\Omega;\mathbb{R}^2)$. We have to prove that

$$\liminf_{j\to+\infty} E_j^M(u_j, m_j) \geq E^M(u, m).$$

Without loss of generality, we may assume that

$$\liminf_{j\to+\infty} E_j^M(u_j, m_j) < +\infty,$$

which, by virtue of Proposition 5.12 gives

$$(u_j, m_j) \rightharpoonup (u, m) \quad \text{in} \quad W^{1,2}(\Omega;\mathbb{R}^2) \times W^{1,2}(\Omega;\mathbb{R}^2),$$

with $|m| = 1$ a.e. in Ω. Finally, combining Theorem 5.4 and Proposition 5.8 immediately yields

$$\begin{aligned}\liminf_{j\to+\infty} E_j^M(u_j, m_j) &= \liminf_{j\to+\infty} \mathcal{E}_j^M(u_j, m_j) + \lim_{j\to+\infty} E^{M,LR}(m_j) \\ &\geq \mathcal{E}^M(u, m) - \frac{1}{2}\int_{\mathbb{R}^2} \langle \tilde{m}, \mathbf{K} * \tilde{m}\rangle \, dx - \frac{1}{4}|\Omega|,\end{aligned}$$

hence, the lower bound.

Upper bound. Let $(u_j, m_j) \subset \mathcal{A}_{\delta_j}(\Omega) \times \mathcal{B}_{\varepsilon_j}(\Omega)$ be as in the proof of the upper bound inequality of Theorem 5.4. Then, again by Theorem 5.4 and Proposition 5.8 we have

$$\begin{aligned}\limsup_{j\to+\infty} E_j^M(u_j, m_j) &= \limsup_{j\to+\infty} \mathcal{E}_j^M(u_j, m_j) + \lim_{j\to+\infty} E^{M,LR}(m_j) \\ &\leq \mathcal{E}^M(u, m) - \frac{1}{2}\int_{\mathbb{R}^2} \langle \tilde{m}, \mathbf{K} * \tilde{m}\rangle \, dx - \frac{1}{4}|\Omega|\end{aligned}$$

and thus the thesis. □

5.1. **Boundary value problems.** In this section we discuss the asymptotic behavior of (E_j^M) in the presence of boundary constraints on the displacement. To this end we need to properly define a new sequence of energies. Given $\varphi \in W^{1,\infty}_{loc}(\mathbb{R}^2;\mathbb{R}^2)$, for any $\delta > 0$, we set

$$\mathcal{A}_\delta^\varphi(\Omega) := \{u \in \mathcal{A}_\delta(\Omega) \,:\, u(\alpha) = \varphi(\alpha) \quad \text{if} \quad Q_{2\delta}(\alpha) \cap \Omega^c \neq \emptyset\}, \tag{5.31}$$

where $\Omega^c := \mathbb{R}^2 \setminus \Omega$. We define $E_j^{M,\varphi} : L^2(\Omega;\mathbb{R}^2) \times L^2(\Omega;\mathbb{R}^2) \longrightarrow \mathbb{R} \cup \{+\infty\}$ as

$$E_j^{M,\varphi}(u, m) = \begin{cases} E_j^M(u, m) & \text{if } u \in \mathcal{A}_{\delta_j}^\varphi(\Omega) \\ +\infty & \text{otherwise,}\end{cases} \tag{5.32}$$

with E_j^M as in (5.2).

Thanks to the boundary constraint on u, we are now able to prove a compactness result for a sequence $(u_j, m_j) \subset L^2(\Omega;\mathbb{R}^2) \times L^2(\Omega;\mathbb{R}^2)$ with equi-bounded energy $E_j^{M,\varphi}$, without any *a priori* convergence requirement on this sequence (*cfr.* Proposition 5.12). This yields in particular the $L^2(\Omega;\mathbb{R}^2) \times L^2(\Omega;\mathbb{R}^2)$ equi-coercivity of $(E_j^{M,\varphi})$.

Proposition 5.14 (Equi-coercivity)**.** *Let (u_j, m_j) be a sequence in $L^2(\Omega;\mathbb{R}^2) \times L^2(\Omega;\mathbb{R}^2)$ such that*

$$\sup_j E_j^{M,\varphi}(u_j, m_j) < +\infty. \tag{5.33}$$

Then, there exists a pair $(u,m) \in W^{1,2}(\Omega;\mathbb{R}^2)\times W^{1,2}(\Omega;S^1)$ *with* $u-\varphi \in W_0^{1,2}(\Omega;\mathbb{R}^2)$ *such that, up to subsequences,*

$$(u_j,m_j) \rightharpoonup (u,m) \quad in \quad W^{1,2}(\Omega;\mathbb{R}^2)\times W^{1,2}(\Omega;\mathbb{R}^2).$$

Proof. In view of (5.30), following the proof of Proposition 5.2 we immediately deduce that (5.33) implies

$$\sup_j E_j^{EL}(u_j) < +\infty \quad \text{and} \quad \sup_j E_j^{M,SR}(m_j) < +\infty.$$

Since the compactness of (m_j) is as in the proof of Proposition 5.2, here we only address the proof of the compactness of (u_j). By (5.33) we get that in particular $(u_j) \subset \mathcal{A}_{\delta_j}^{\varphi}(\Omega)$, hence $u_j - \varphi \in W_0^{1,2}(\Omega;\mathbb{R}^2)$ for any fixed j. Then, the variant of the Korn Inequality stated in Proposition 3.7(ii) gives

$$||\nabla(u_j-\varphi)||_{L^2(\Omega;\mathcal{M}^{2\times 2})} \le C||\mathbf{E}(u_j-\varphi)||_{L^2(\Omega;\mathcal{M}^{2\times 2})}. \tag{5.34}$$

Moreover, the same construction performed in the proof of Proposition 5.2 together with $\sup_j E_j^{EL}(u_j) < +\infty$, by virtue of (5.34) now yields $\sup_j ||\nabla u_j||_{L^2(\Omega;\mathcal{M}^{2\times 2})} < +\infty$. This, combined with the Poincaré Inequality, implies

$$\sup_j ||u_j||_{W^{1,2}(\Omega;\mathbb{R}^2)} < +\infty,$$

and finally

$$u_j \rightharpoonup u \quad \text{in} \quad W^{1,2}(\Omega;\mathbb{R}^2), \quad u-\varphi \in W_0^{1,2}(\Omega;\mathbb{R}^2).$$

□

We are ready to prove a Γ-convergence result for $E_j^{M,\varphi}$.

Theorem 5.15 (Γ-convergence of $E_j^{M,\varphi}$)**.** *The sequence of functionals* $(E_j^{M,\varphi})$ *defined in* (5.32) Γ*-converges with respect to the* $L^2(\Omega;\mathbb{R}^2)\times L^2(\Omega;\mathbb{R}^2)$ *convergence to the functional* $E^{M,\varphi} : L^2(\Omega;\mathbb{R}^2)\times L^2(\Omega;\mathbb{R}^2) \longrightarrow \mathbb{R}\cup\{+\infty\}$ *given by*

$$E^{M,\varphi}(u,m) = \begin{cases} E^M(u,m) & \text{if } (u-\varphi,m) \in W_0^{1,2}(\Omega;\mathbb{R}^2)\times W^{1,2}(\Omega;S^1) \\ +\infty & \text{otherwise,} \end{cases}$$

with E^M *as in Theorem 5.13.*

Proof. Lower bound. Let $(u_j,m_j) \subset \mathcal{A}_{\delta_j}^{\varphi}(\Omega)\times\mathcal{B}_{\varepsilon_j}(\Omega)$ be a sequence of functions converging to (u,m) in $L^2(\Omega;\mathbb{R}^2)\times L^2(\Omega;\mathbb{R}^2)$ and such that $\sup_j E_j^{M,\varphi}(u_j,m_j) < +\infty$. Then, Proposition 5.14 gives

$$(u-\varphi,m) \in W_0^{1,2}(\Omega;\mathbb{R}^2)\times W^{1,2}(\Omega;S^1),$$

while the lower bound in Theorem 5.13 yields

$$\liminf_{j\to+\infty} E_j^M(u_j,m_j) \ge E^M(u,m),$$

therefore

$$\liminf_{j\to+\infty} E_j^{M,\varphi}(u_j,m_j) \ge E^{M,\varphi}(u,m).$$

Upper bound. As the imposed boundary constraint on the displacement only modify the elastic term in the energy, we may prove the upper bound inequality limiting our attention to this term, the upper bound for the whole $E_j^{M,\varphi}$ easily following as in the proof of Theorem 5.13 (see also Theorem 5.4).

Setting

$$E^{EL}(u) := \frac{1}{2}\int_{\Omega} |\mathbf{E}u|^2\,dx + \frac{1}{4}\int_{\Omega} |\operatorname{div} u|^2\,dx,$$

we have to show that

$$\Gamma\text{-}\limsup_{j\to+\infty} E_j^{EL}(u) \le E^{EL}(u), \tag{5.35}$$

for every u such that $u-\varphi \in W_0^{1,2}(\Omega;\mathbb{R}^2)$.

We prove (5.35) by density. To this end let $u \in W^{1,2}(\Omega;\mathbb{R}^2)$ be such that $\Omega^0 := \operatorname{spt}(u-\varphi) \subset\subset \Omega$. Given $\eta > 0$, we let A'' be an open subset of Ω satisfying $\Omega^0 \subset\subset A'' \subset\subset \Omega$ and

$$\int_{\Omega\setminus A''} |\nabla u|^2\; dx = \int_{\Omega\setminus A''} |\nabla \varphi|^2\; dx \le \eta. \tag{5.36}$$

Now let A' be an open subset of Ω such that $A'' \subset\subset A' \subset\subset \Omega$. Setting $d := \operatorname{dist}(A'', A'^c)$, for any $k \in \{1,\dots,N\}$ with $N \in \mathbb{N}$, we define

$$A_k := \Big\{x \in A' :\; \operatorname{dist}(x, A'') < k\frac{d}{N}\Big\}.$$

Notice that $A'' \subset A_k \subset A_{k+1}$, for every $k \in \{1,\dots,N-1\}$ and $A_N = A'$.

We recall that Theorem 5.13 ensures the existence of a sequence $(u_j) \subset \mathcal{A}_{\delta_j}(\Omega)$ such that $u_j \rightharpoonup u$ in $W^{1,2}(\Omega;\mathbb{R}^2)$, for which

$$\lim_{j\to+\infty} E_j^{EL}(u_j) = E^{EL}(u). \tag{5.37}$$

We want to show that we can construct a recovery sequence $(\bar u_j) \subset \mathcal{A}^{\varphi}_{\delta_j}(\Omega)$ for $E_j^{M,\varphi}$, suitably modifying (u_j) "near" the boundary of Ω.

We start by considering the cut-off functions $\phi_k \in C^\infty(\Omega;[0,1])$ defined as

$$\phi_k = \begin{cases} 1 & \text{in } A_k \\ 0 & \text{in } \Omega\setminus A_{k+1} \end{cases}$$

and satisfying $\|\nabla\phi_k\|_{L^\infty(\Omega;\mathbb{R})} \le \frac{N}{d}$, for every $k \in \{1,2,\dots,N\}$. For any $\alpha \in \Omega_{\delta_j}$ we set

$$u_j^k(\alpha) := \phi_k(\alpha)u_j(\alpha) + (1-\phi_k(\alpha))\varphi(\alpha)$$

and we consider its piecewise-affine interpolation $A(u_j^k)$, defined as in (2.4). From $u_j \to u$ in $L^2(\Omega;\mathbb{R}^2)$ we get that $A(u_j^k) \to u$ in $L^2(\Omega;\mathbb{R}^2)$, and moreover $A(u_j^k) \in \mathcal{A}^{\varphi}_{\delta_j}(\Omega)$, for j large enough.

Note that

$$\begin{aligned} D^{\xi}_{\delta_j} u_j^k(\alpha) = \phi_k(\alpha+\varepsilon\xi) D^{\xi}_{\delta_j} u_j(\alpha) + (1-\phi_k(\alpha+\varepsilon\xi))\, D^{\xi}_{\delta_j}\varphi(\alpha) \\ + (u_j(\alpha)-\varphi(\alpha))\, D^{\xi}_{\delta_j}\phi_k(\alpha), \end{aligned}$$

for any $\xi \in X$, thus we easily deduce

$$|D^{\xi}_{\delta_j} u_j^k(\alpha)|^2 \le C\left(|D^{\xi}_{\delta_j} u_j(\alpha)|^2 + |D^{\xi}_{\delta_j}\varphi(\alpha)|^2 + \frac{N^2}{d^2}|u_j(\alpha)-\varphi(\alpha)|^2\right). \tag{5.38}$$

Setting, for every $\xi \in X$,

$$S_j^{k,\xi} := \{x = y + t\xi,\; |t| \le \delta_j,\; y \in \overline{A}_{k+1}\setminus A_k\},$$

we get

$$R^{\xi}_{\delta_j}(\Omega) \subseteq R^{\xi}_{\delta_j}(A_k) \cup R^{\xi}_{\delta_j}(\Omega \setminus \overline{A}_{k+1}) \cup R^{\xi}_{\delta_j}\left(S^{k,\xi}_j\right). \tag{5.39}$$

Then, by collecting all the interactions in the energy according to (5.39) we have

$$E^{EL}_j(u^k_j) \leq E^{EL}_j(u_j) + \sum_{\xi \in X} \sum_{\alpha \in \mathcal{R}^{\xi}_{\delta_j}(\Omega \setminus A'')} \delta_j^2 |D^{\xi}_{\delta_j}\varphi(\alpha)|^2 + I_{j,k}(u_j) \tag{5.40}$$

where by virtue of (5.38), we have set

$$I_{j,k}(u_j) := C \sum_{\xi \in X} \sum_{\alpha \in \mathcal{R}^{\xi}_{\delta_j}(S^{k,\xi}_j)} \delta_j^2 \left(|D^{\xi}_{\delta_j} u_j(\alpha)|^2 + |D^{\xi}_{\delta_j}\varphi(\alpha)|^2 + \frac{N^2}{d^2}|u_j(\alpha) - \varphi(\alpha)|^2 \right).$$

Since in view of (5.40) we get

$$\sum_{k=1}^N E^{EL}_j(u^k_j) \leq N E^{EL}_j(u_j) + N \sum_{\xi \in X} \sum_{\alpha \in \mathcal{R}^{\xi}_{\delta_j}(\Omega \setminus A'')} \delta_j^2 |D^{\xi}_{\delta_j}\varphi(\alpha)|^2 + \sum_{k=1}^N I_{j,k}(u_j),$$

we may infer the existence of an integer $k(j) \in \{1, 2, \dots, N\}$ such that setting $\bar{u}_j := A(u^{k(j)}_j)$, we have

$$E^{EL}_j(\bar{u}_j) \leq E^{EL}_j(u_j) + \sum_{\xi \in X} \sum_{\alpha \in \mathcal{R}^{\xi}_{\delta_j}(\Omega \setminus A'')} \delta_j^2 |D^{\xi}_{\delta_j}\varphi(\alpha)|^2 + \frac{1}{N}\sum_{k=1}^N I_{j,k}(u_j).$$

Moreover, setting $L_\varphi := \|\nabla \varphi\|_{L^\infty(\Omega;\mathbb{R}^2)}$ and denoting by w_j and φ_j the piecewise-constant interpolations on the cells of the lattice $\delta_j\mathbb{Z}^2$ of u_j and of φ respectively, we have

$$\frac{1}{N}\sum_{k=1}^N I_{j,k}(u_j) \leq \frac{C}{N}\Big(\int_{\Omega \setminus A''} |\nabla u_j|^2 + L^2_\varphi |\Omega \setminus A''| \Big) + N \int_{\Omega \setminus A''} |w_j(x) - \varphi_j(x)|^2 \, dx, \tag{5.41}$$

and

$$\sum_{\xi \in X} \sum_{\alpha \in \mathcal{R}^{\xi}_{\delta_j}(\Omega \setminus A'')} \delta_j^2 |D^{\xi}_{\delta_j}\varphi(\alpha)|^2 \leq C \int_{\Omega \setminus A''} |\nabla \varphi|^2 \, dx. \tag{5.42}$$

Since by construction

$$\lim_{j \to +\infty} \|w_j(x) - \varphi_j(x)\|_{L^2(\Omega \setminus A'';\mathbb{R}^2)} = 0,$$

gathering (5.40), (5.41) and (5.42) we finally get

$$\begin{aligned}
\limsup_{j \to +\infty} E^{EL}_j(\bar{u}_j) &\leq \limsup_{j \to +\infty} E^{EL}_j(u_j) + \int_{\Omega \setminus A''} |\nabla \varphi|^2 \, dx \\
&+ \frac{C}{N}\Big(\int_{\Omega \setminus A''} |\nabla \varphi|^2 + L^2_\varphi |\Omega \setminus A''| \Big) \\
&\leq E^{EL}(u) + \eta + \frac{C}{N}(\eta + 1),
\end{aligned}$$

where the last inequality follows by (5.37) and (5.36). Then, (5.35) follows by the arbitrariness of N, η and by the definition of Γ-limsup.

We finally remark that as the set of all functions $u \in W^{1,p}(\Omega;\mathbb{R}^2)$ such that $\mathrm{spt}(u-\varphi) \subset\subset \Omega$ is dense in $W_0^{1,p}(\Omega;\mathbb{R}^2)+\varphi$ with respect to the $W^{1,p}(\Omega;\mathbb{R}^2)$ convergence and E^{EL} is continuous with respect to the same convergence, the general case $u-\varphi \in W_0^{1,2}(\Omega;\mathbb{R}^2)$ can be recovered by a standard diagonalization argument, which relies also on the lower-semicontinuity of the Γ-limsup (see *e.g.*, [9], Remark 1.29). □

In view of Theorem 5.15 and by the fundamental property of Γ-convergence Theorem 3.3, we derive the following result about the convergence of minimum problems with Dirichlet boundary data.

Corollary 5.16 (Convergence of minimum problems for E_j^M)**.** *For any* $\varphi \in W^{1,\infty}_{loc}(\mathbb{R}^2;\mathbb{R}^2)$

$$\lim_{j\to+\infty} \inf\{E_j^M(u,m) : (u,m) \in \mathcal{A}^{\varphi}_{\delta_j}(\Omega)\times\mathcal{B}_{\varepsilon_j}(\Omega)\}$$
$$= \min\{E^M(u,m) : \ (u-\varphi,m) \in W_0^{1,2}(\Omega;\mathbb{R}^2)\times W^{1,2}(\Omega;S^1)\}.$$

Moreover, if $(u_j,m_j) \subset \mathcal{A}^{\varphi}_{\delta_j}(\Omega)\times\mathcal{B}_{\varepsilon_j}(\Omega)$ *is a minimizing sequence for* (E_j^M) *then, up to subsequences,* $(u_j,m_j)\to(\bar u,\bar m)$ *in* $L^2(\Omega;\mathbb{R}^2)\times L^2(\Omega;\mathbb{R}^2)$ *and*

$$E^M(\bar u,\bar m) = \min\{E^M(u,m) : (u-\varphi,m) \in W_0^{1,2}(\Omega;\mathbb{R}^2)\times W^{1,2}(\Omega;S^1)\}.$$

Proof. Proposition 5.14 and Theorem 5.15 ensure that we are in the hypotheses of Theorem 3.3 which immediately yields the thesis. □

6. Ferroelectric crystals

In this section we study the asymptotic behavior of the total free energy associated to a given configuration (u,p) of a ferroelectric crystal

$$E_\varepsilon^F(u,p) := E_\varepsilon^{EL}(u) + E_\varepsilon^{F,SR}(p) + E_\varepsilon^{F,LR}(p) + E_\varepsilon^{F,SA}(u,p),$$

with E_ε^{EL} as in (4.1), $E_\varepsilon^{F,SR}$, $E_\varepsilon^{F,LR}$, $E_\varepsilon^{F,SA}$ as in (4.6), (4.7), (4.12), respectively, under Dirichelet boundary constraint on u.

As the asymptotic analysis for E_ε^F can be inferred, up to minor changes, from the one for E_ε^M, in what follows we only discuss some points which reduce the study of ferroelectric crystals to that of magnetostrictive solids treated in Section 5.

We set $E_j^{F,SR} := E_{\varepsilon_j}^{F,SR}$, $E_j^{F,LR} := E_{\varepsilon_j}^{F,LR}$, $E_j^{F,SA} := E_{\varepsilon_j}^{F,SA}$ and we perform the usual identification defining $E_j^{F,SR} : L^2(\Omega;\mathbb{R}^2) \longrightarrow [0,+\infty]$, $E_j^{F,LR} : L^2(\Omega;\mathbb{R}^2) \longrightarrow \mathbb{R}\cup\{+\infty\}$, $E_j^{F,SA} : L^2(\Omega;\mathbb{R}^2)\times L^2(\Omega;\mathbb{R}^2) \longrightarrow \mathbb{R}\cup\{+\infty\}$ respectively as

$$E_j^{F,SR}(p) := \begin{cases} \dfrac{1}{2}\displaystyle\sum_{\xi\in Y}\sum_{\alpha\in\mathcal{R}^{\xi}_{\varepsilon_j}(\Omega)} {\varepsilon_j}^2|D^{\xi}_{\varepsilon_j}p(\beta)|^2 + \dfrac{1}{2}\sum_{\alpha\in\Omega_{\varepsilon_j}} {\varepsilon_j}^2|p(\beta)|^2 & \text{if } p\in\mathcal{A}_{\varepsilon_j}(\Omega) \\ +\infty & \text{otherwise,} \end{cases}$$

$$E_j^{F,LR}(m) := \begin{cases} -\dfrac{1}{2}\displaystyle\sum_{\alpha,\beta\in\Omega_{\varepsilon_j},\ \alpha\neq\beta} \varepsilon_j^4\langle\mathbf{K}(\alpha-\beta)p(\alpha),p(\beta)\rangle & \text{if } p\in\mathcal{A}_{\varepsilon_j}(\Omega) \\ +\infty & \text{otherwise,} \end{cases} \tag{6.1}$$

and

$$E_j^{F,SA}(u,p) := \begin{cases} -\sum_{\xi\in X}\sum_{\alpha\in\mathcal{R}^\xi_{\delta_j}(\Omega)} \delta_j^2\langle D^\xi_{\delta_j}u(\alpha),\hat{\xi}\rangle \sum_{\beta\in P^\xi_{\delta_j}(\alpha)\cap\Omega_{\varepsilon_j}} \frac{\varepsilon_j^2}{\delta_j^2}\,|p(\beta)|^2\big\langle \mathbf{Q}(\hat{p}(\beta))\,\hat{\xi},\hat{\xi}\big\rangle \\ \qquad +\frac{1}{2}\sum_{\xi\in X}\sum_{\alpha\in\mathcal{R}^\xi_{\delta_j}(\Omega)} \delta_j^2\Big\langle \sum_{\beta\in P^\xi_{\delta_j}(\alpha)\cap\Omega_{\varepsilon_j}} \frac{\varepsilon_j^2}{\delta_j^2}\,|p(\beta)|^2\mathbf{Q}(\hat{p}(\beta))\,\hat{\xi},\hat{\xi}\Big\rangle^2 \\ \qquad\qquad \text{if } (u,p)\in\mathcal{A}_{\delta_j}(\Omega)\times\mathcal{A}_{\varepsilon_j}(\Omega) \\ +\infty \qquad \text{otherwise.} \end{cases}$$

Thus $E_j^F : L^2(\Omega;\mathbb{R}^2) \longrightarrow \mathbb{R}\cup\{+\infty\}$ is given by

$$E_j^F(u,p) := E_j^{EL}(u) + E_j^{F,SR}(p) + E^{F,LR}(p) + E^{F,SA}(u,p) \tag{6.2}$$

and finally, for any $\varphi\in W^{1,\infty}_{loc}(\mathbb{R}^2;\mathbb{R}^2)$, we define $E_j^{F,\varphi} : L^2(\Omega;\mathbb{R}^2)\times L^2(\Omega;\mathbb{R}^2) \longrightarrow \mathbb{R}\cup\{+\infty\}$ as

$$E_j^{F,\varphi}(u,p) = \begin{cases} E_j^F(u,p) & \text{if } u\in\mathcal{A}^\varphi_{\delta_j}(\Omega) \\ +\infty & \text{otherwise,} \end{cases} \tag{6.3}$$

with E_j^F as in (6.2) and $\mathcal{A}^\varphi_{\delta_j}(\Omega)$ as in (5.31).

We start proving an equi-coercivity result for $(E_j^{F,\varphi})$.

Proposition 6.1 (Equi-coercivity)**.** *Let (u_j,p_j) be a sequence in $L^2(\Omega;\mathbb{R}^2)\times L^2(\Omega;\mathbb{R}^2)$ such that*

$$\sup_j E_j^{F,\varphi}(u_j,p_j) < +\infty. \tag{6.4}$$

Then, there exists a pair $(u,p)\in W^{1,2}(\Omega;\mathbb{R}^2)\times W^{1,2}(\Omega;\mathbb{R}^2)$ with $u-\varphi\in W^{1,2}_0(\Omega;\mathbb{R}^2)$ such that, up to subsequences,

$$(u_j,p_j) \rightharpoonup (u,p) \quad \textit{in} \quad W^{1,2}(\Omega;\mathbb{R}^2)\times W^{1,2}(\Omega;\mathbb{R}^2).$$

Proof. We observe that

$$E_j^{EL}(u) + E_j^{F,SA}(u,p) = \frac{1}{2}\,E_j^{EL}(u)$$
$$+\frac{1}{2}\sum_{\xi\in X}\sum_{\alpha\in\mathcal{R}^\xi_{\delta_j}(\Omega)} \delta_j^2\Big(\langle D^\xi_{\delta_j}u(\alpha),\hat{\xi}\rangle - \sum_{\beta\in P^\xi_{\delta_j}(\alpha)\cap\Omega_{\varepsilon_j}} \frac{\varepsilon_j^2}{\delta_j^2}\,|p(\beta)|^2\big\langle\mathbf{Q}(\hat{p}(\beta))\,\hat{\xi},\hat{\xi}\big\rangle\Big)^2. \tag{6.5}$$

Moreover, in view of Theorem 5.10 we have that

$$E_j^{F,SR}(p) + E_j^{F,LR}(p) \geq \frac{1}{2}\,E_j^{F,SR}(p), \tag{6.6}$$

for j sufficiently large. Hence gathering (6.5) and (6.6), (6.4) permits to deduce that

$$\sup_j E_j^{EL}(u_j) < +\infty \quad \text{and} \quad \sup_j E_j^{F,SR}(p_j) < +\infty.$$

Then, arguing as in the proof of Proposition 5.14 and taking into account the definition of $E_j^{F,SR}$ we immediately get the thesis. □

We establish the following Γ-convergence result for the energies $E_j^{F,\varphi}$.

Theorem 6.2 (Γ-convergence of $E_j^{F,\varphi}$)**.** *The sequence of functionals* $(E_j^{F,\varphi})$ *defined in* (6.3) Γ*-converges with respect to the* $L^2(\Omega;\mathbb{R}^2)\times L^2(\Omega;\mathbb{R}^2)$ *convergence, to the functional* $E^{F,\varphi}: L^2(\Omega;\mathbb{R}^2)\times L^2(\Omega;\mathbb{R}^2)\longrightarrow \mathbb{R}\cup\{+\infty\}$ *given by*

$$E^{F,\varphi}(u,p)=\begin{cases}E^F(u,p) & \text{if } (u-\varphi,p)\in W_0^{1,2}(\Omega;\mathbb{R}^2)\times W^{1,2}(\Omega;\mathbb{R}^2)\\ +\infty & \text{otherwise,}\end{cases}$$

with E^F *as in* (1.6)*.*

Proof. Upon noticing that Proposition 6.1 ensures that a seqence $(u_j,p_j)\subset L^2(\Omega;\mathbb{R}^2)\times L^2(\Omega;\mathbb{R}^2)$ with $\sup_j E_j^F(u_j,p_j)<+\infty$, in particular satisfies

$$p_j\to p \quad \text{in} \quad L^4(\Omega;\mathbb{R}^2),$$

arguing as in the proof of Proposition 5.3 it is easy to show that

$$\lim_{j\to+\infty} E_j^{F,SA}(u_j,p_j)=-\int_\Omega |p|^2\mathbf{E}u:\mathbf{Q}(\hat{p})\,dx+\frac{1}{4}\int_\Omega |p|^4\,dx.$$

Then, the the Γ-convergence result for $E_j^{F,\varphi}$ follows by appling the same arguments employed in the proof of Theorem 5.15 (see also Theorem 5.4 and Theorem 5.13). □

As in Section 5, we conclude the analysis for $E_j^{F,\varphi}$ observing that, as a consequence of Proposition 6.4 and Theorem 6.2, we may derive the usual result about the convergence of minimum problems.

Corollary 6.3 (Convergence of minimum problems for E_j^F)**.** *For any* $\varphi\in W^{1,\infty}_{loc}(\mathbb{R}^2;\mathbb{R}^2)$,

$$\lim_{j\to+\infty}\inf\{E_j^F(u,p):(u,p)\in\mathcal{A}^{\varphi}_{\delta_j}(\Omega)\times\mathcal{A}_{\varepsilon_j}(\Omega)\}$$
$$=\min\{E^F(u,p):\ (u-\varphi,p)\in W_0^{1,2}(\Omega;\mathbb{R}^2)\times W^{1,2}(\Omega;\mathbb{R}^2)\}.$$

Moreover, if $(u_j,p_j)\subset\mathcal{A}^{\varphi}_{\delta_j}(\Omega)\times\mathcal{A}_{\varepsilon_j}(\Omega)$ *is a minimizing sequence for* (E_j^F) *then, up to subsequences,* $(u_j,p_j)\to(\bar{u},\bar{p})$ *in* $L^2(\Omega;\mathbb{R}^2)\times L^2(\Omega;\mathbb{R}^2)$ *and*

$$E^F(\bar{u},\bar{p})=\min\{E^F(u,p):(u-\varphi,p)\in W_0^{1,2}(\Omega;\mathbb{R}^2)\times W^{1,2}(\Omega;\mathbb{R}^2)\}.$$

7. Nematic elastomers

In this section we analyze the asymptotic behavior of the free energy

$$E_\varepsilon^N(u,n)=E_\varepsilon^{EL}(u)+E_\varepsilon^{LL}(n)+E_\varepsilon^{M,SA}(u,n)$$

of a nematic elastomer whose ordering term E_ε^{LL} follows by the Lebwhol-Lasher theory and, for every $n:\Omega_\varepsilon\to S^1$, is given by (4.8). According to (4.9), this can be written as

$$E_\varepsilon^{LL}(n)=\frac{1}{2}\sum_{\xi\in Y}\sum_{\alpha\in\mathcal{R}_\varepsilon^\xi(\Omega)}\varepsilon^2|D_\varepsilon^\xi\mathbf{Q}(n(\alpha))|^2.$$

As already observed, this suggests that the meaningful variable now is $\mathbf{Q}(n)$. Then, since the dependence on n of the strain-alignment energy is given in terms of $\mathbf{Q}(n)$ too, with a slight abuse of notation, we prefer to write the free energy as $E_\varepsilon^N(u,\mathbf{Q}(n))$ and we also set

$$E_\varepsilon^{N,SR}(\mathbf{Q}(n)):=E^{LL}(n)\quad E^{N,SA}(u,\mathbf{Q}(n)):=E^{M,SA}(u,n).$$

As for the cases of magnetostrictive solids and ferroelectric crystals, we may identify the energies E_ε^N with their continuous counterparts now defined on $L^2(\Omega;\mathbb{R}^2)\times L^2(\Omega;\mathcal{M}^{2\times 2})$. To this end we proceed as in Section 5 identifying each function $\mathbf{Q}(n):\Omega_\varepsilon\to\mathcal{N}$ with its piecewise affine interpolation as in (2.4). Then, setting $E_j^{N,SR}:=E_{\varepsilon_j}^{N,SR}$ and $E_j^{N,SA}:=E_{\varepsilon_j}^{N,SA}$, we are led to consider the functionals (not relabeled) $E_j^{N,SR}:L^2(\Omega;\mathcal{M}^{2\times 2})\longrightarrow[0,+\infty]$ and $E_j^{N,SA}:L^2(\Omega;\mathbb{R}^2)\times L^2(\Omega;\mathcal{M}^{2\times 2})\longrightarrow\mathbb{R}\cup\{+\infty\}$ defined respectively as

$$E^{N,SR}(\mathbf{Q}(n)):=\begin{cases}\dfrac{1}{2}\displaystyle\sum_{\xi\in Y}\sum_{\alpha\in\mathcal{R}_\varepsilon^\xi(\Omega)}\varepsilon^2|D_\varepsilon^\xi\mathbf{Q}(n(\alpha))|^2 & \text{if } \mathbf{Q}(n)\in\mathcal{M}_{\varepsilon_j}(\Omega)\\ +\infty & \text{otherwise}\end{cases}$$

and

$$E^{N,SA}(u,\mathbf{Q}(n)):=\begin{cases}-\displaystyle\sum_{\xi\in X}\sum_{\alpha\in\mathcal{R}_{\delta_j}^\xi(\Omega)}\delta_j^2\langle D_{\delta_j}^\xi u(\alpha),\hat\xi\rangle\sum_{\beta\in P_{\delta_j}^\xi(\alpha)\cap\Omega_{\varepsilon_j}}\frac{\varepsilon_j^2}{\delta_j^2}\left\langle\mathbf{Q}(n(\beta))\,\hat\xi,\hat\xi\right\rangle\\ \qquad\qquad\qquad\qquad\text{if } (u,\mathbf{Q}(n))\in\mathcal{A}_{\delta_j}(\Omega)\times\mathcal{M}_{\varepsilon_j}(\Omega)\\ +\infty\qquad\qquad\qquad\text{otherwise.}\end{cases}$$

Thus, finally $E_j^N:L^2(\Omega;\mathbb{R}^2)\times L^2(\Omega;\mathcal{M}^{2\times 2})\longrightarrow\mathbb{R}\cup\{+\infty\}$ is given by

$$E_j^N(u,\mathbf{Q}(n)):=E_j^{EL}(u)+E_j^{N,SR}(\mathbf{Q}(n))+E_j^{N,SA}(u,\mathbf{Q}(n)). \tag{7.1}$$

If we now let $\varphi\in W^{1,\infty}(\mathbb{R}^2;\mathbb{R}^2)$, we may define $E_j^{N,\varphi}:L^2(\Omega;\mathbb{R}^2)\times L^2(\Omega;\mathcal{M}^{2\times 2})\longrightarrow\mathbb{R}\cup\{+\infty\}$ as

$$E_j^{N,\varphi}(u,\mathbf{Q}(n)):=\begin{cases}E_j^N(u,\mathbf{Q}(n)) & \text{if } u\in\mathcal{A}_{\delta_j}^\varphi(\Omega)\\ +\infty & \text{otherwise,}\end{cases} \tag{7.2}$$

with $\mathcal{A}_{\delta_j}^\varphi(\Omega)$ as in (5.31).

In what follows we prove the analogue of the compactness result stated in Proposition 5.14 for the nematic elastomer energies $E_j^{N,\varphi}$.

Proposition 7.1 (Equi-coercivity)**.** *Let $(u_j,\mathbf{Q}(n)_j)$ be a sequence in $L^2(\Omega;\mathbb{R}^2)\times L^2(\Omega;\mathcal{M}^{2\times 2})$ such that*

$$\sup_j E_j^{N,\varphi}(u_j,\mathbf{Q}(n)_j)<+\infty.$$

Then, there exists a pair $(u,\mathbf{M})\in W^{1,2}(\Omega;\mathbb{R}^2)\times W^{1,2}(\Omega;\mathcal{M}^{2\times 2})$ with $u-\varphi\in W_0^{1,2}(\Omega;\mathbb{R}^2)$ such that, up to subsequences,

$$(u_j,\mathbf{Q}(n)_j)\rightharpoonup(u,\mathbf{M})\quad\text{in}\quad W^{1,2}(\Omega;\mathbb{R}^2)\times W^{1,2}(\Omega;\mathcal{M}^{2\times 2}).$$

Moreover, $\mathbf{M}=\mathbf{Q}(\nu)$ for some $\nu:\Omega\to S^1$.

Proof. Arguing as in the proof of Proposition 5.14 (see also Proposition 5.2) directly yields $u_j\rightharpoonup u$ in $W^{1,2}(\Omega;\mathbb{R}^2)$, $u-\varphi\in W_0^{1,2}(\Omega;\mathbb{R}^2)$ and $\mathbf{Q}(n)_j\rightharpoonup\mathbf{M}$ in $W^{1,2}(\Omega;\mathcal{M}^{2\times 2})$.

We now consider the sequence $(\mathbf{M}_j)$ of piecewise-constant functions defined as

$$\mathbf{M}_j(x):=\begin{cases}\mathbf{Q}(n(\alpha))_j & \text{if } x\in Q_{\varepsilon_j}(\alpha)\cap\Omega,\ \alpha\in\Omega_{\varepsilon_j}\\ \mathbf{Q}(n_0) & \text{if } x\in Q_{\varepsilon_j}(\alpha)\cap\Omega,\ \alpha\notin\Omega_{\varepsilon_j},\end{cases} \tag{7.3}$$

with $n_0 \in S^1$. Then, by virtue of the definition of the tensor order parameter $\mathbf{Q}(n)$ we get $\mathbf{M}_j \in \mathcal{N}$ a.e. in Ω. Moreover, appealing to Proposition A.1 and Remark A.2 in [7] we have

$$\mathbf{M}_j \to \mathbf{M} \quad \text{in} \quad L^2(\Omega; \mathcal{M}^{2\times 2}),$$

which in particular implies that, up to subsequences,

$$\mathbf{M}_j(x) \to \mathbf{M}(x) \quad \text{a.e. in } \Omega, \tag{7.4}$$

and this permits to deduce that the limit function $\mathbf{M}$ belongs to $\mathcal{N}$ for a.e. $x \in \Omega$. Then, by Lemma 3.6 we can infer the existence of a function $\nu : \Omega \to S^1$, such that $\mathbf{M}(x) = \mathbf{Q}(\nu(x))$ a.e. in Ω and this concludes the proof. □

Remark 7.2. In the model we have considered the meaningful variable which describes the nematic order is the tensor order parameter $\mathbf{Q}(n)$. Since the (2×2)-symmetric matrix $\mathbf{Q}(n)$ determines through its entries $(n_1^2, n_2^2, n_1 n_2)$ the direction of the corresponding vector $n = (n_1, n_2)$, it is not surprising that Proposition 7.1 asserts, among other things, that a sequence $(\mathbf{Q}(n)_j)$ with equi-bounded nematic energy, converges to a "direction"; *i.e.,* to a matrix-valued function $\mathbf{Q}(\nu)$, for some $\nu \in S^1$.

On the other hand, this compactness result gives no information on the asymptotic behavior of sequences $(n_j) = ((n_1)_j, (n_2)_j)$ with $E_j^{LL}(n_j)$ equi-bounded. Indeed, by Proposition 7.1 we can deduce that

$$(n_1^2)_j \rightharpoonup \nu_1^2, \quad (n_2^2)_j \rightharpoonup \nu_2^2, \quad (n_1 n_2)_j \rightharpoonup \nu_1\nu_2 \ \text{ in } \ W^{1,2}(\Omega;\mathbb{R}),$$

where ν is in general not determined by the weak-$W^{1,2}$ limit of (n_j) which may even not exist. In fact, if we consider the discrete sequence of vectors given by

$$n_j(\alpha_1,\alpha_2) := \Big(\sqrt{\varepsilon_j}\cos\frac{\pi\alpha_1}{\varepsilon_j}, \sqrt{1-\varepsilon_j}\Big),$$

we clearly have

$$|n_j| = 1, \quad E_j^{LL}(n_j) = 0, \quad n_j \to e_2 \quad \text{in} \quad L^2(\Omega;\mathbb{R}^2)$$

while

$$|\nabla n_j| = O\Big(\frac{1}{\sqrt{\varepsilon_j}}\Big), \quad \text{as } \ j \to +\infty.$$

We establish the following Γ-converge result for the sequence of nematic elastomers energy with Dirichelet boundary conditions on the displacement.

Theorem 7.3 (Γ-convergence of $E_j^{N,\varphi}$)**.** *The sequence of functionals $(E_j^{N,\varphi})$ defined in (7.2) Γ-converges with respect to the $L^2(\Omega;\mathbb{R}^2)\times L^2(\Omega;\mathcal{M}^{2\times 2})$ convergence to the functional $E^{N,\varphi} : L^2(\Omega;\mathbb{R}^2)\times L^2(\Omega;\mathcal{M}^{2\times 2}) \longrightarrow \mathbb{R}\cup\{+\infty\}$ given by*

$$E^{N,\varphi}(u,\mathbf{Q}(\nu)) = \begin{cases} \dfrac12\displaystyle\int_\Omega |\mathbf{E}u|^2\,dx + \frac14\int_\Omega |\operatorname{div} u|^2\,dx + \frac12\int_\Omega |\nabla\mathbf{Q}(\nu)|^2\,dx \\ \qquad\qquad\qquad\qquad\qquad -\displaystyle\int_\Omega \mathbf{E}u : \mathbf{Q}(\nu)\,dx \\ \quad \textit{if } \ (u-\varphi,\mathbf{Q}(\nu)) \in W_0^{1,2}(\Omega;\mathbb{R}^2)\times W^{1,2}(\Omega;\mathcal{M}^{2\times 2}) \\ \qquad \textit{with } \ \nu : \Omega \to S^1 \\ +\infty \qquad \textit{otherwise.} \end{cases}$$

Proof. Lower bound. Appealing to Proposition 7.1 and noticing that now

$$\lim_{j\to+\infty} E_j^{N,SA}(u_j,\mathbf{Q}(n)_j) = -\int_\Omega \mathbf{E}u:\mathbf{Q}(\nu)\,dx.$$

for every $(u_j,\mathbf{Q}(n)_j)\subset\mathcal{A}_{\delta_j}(\Omega)\times\mathcal{M}_{\varepsilon_j}(\Omega)$ such that

$$u_j\rightharpoonup u \ \text{ in } \ W^{1,2}(\Omega;\mathbb{R}^2), \qquad \mathbf{Q}(n)_j\to\mathbf{Q}(\nu) \ \text{ in } \ L^2(\Omega;\mathcal{M}^{2\times 2}),$$

the proof of the lower bound exactly follows that of Theorem 5.4.

Upper bound. We have to prove that, for any $(u,\mathbf{M})\in W^{1,2}(\Omega;\mathbb{R}^2)\times W^{1,2}(\Omega;\mathcal{M}^{2\times 2})$ with $u-\varphi\in W_0^{1,2}(\Omega;\mathbb{R}^2)$ and $\mathbf{M}=\mathbf{Q}(\nu)$ a.e. in Ω, for some $\nu:\Omega\to S^1$, there exists a sequence $(\bar u_j,\mathbf{Q}(n)_j)\subset\mathcal{A}_{\delta_j}^{\varphi}(\Omega)\times\mathcal{M}_{\varepsilon_j}(\Omega)$ converging to $(u,\mathbf{M})$ in $L^2(\Omega;\mathbb{R}^2)\times L^2(\Omega;\mathbb{R}^2)$ and such that

$$\limsup_{j\to+\infty} E_j^{N,\varphi}(u_j,\mathbf{Q}(n)_j)\le E^{N,\varphi}(u,\mathbf{M}).$$

Since $\bar u_j$ can be taken exactly as in the proof of the upper bound of Theorem 5.15, we only focus on the construction of a recovery sequence for the nematic variable $\mathbf{M}=\mathbf{Q}(\nu)$. By the regularity assumption on Ω, without loss of generality, we may suppose that $\mathbf{M}\in W^{1,2}(\Omega',\mathcal{M}^{2\times 2})$ with $\mathbf{M}=\mathbf{Q}(\nu)$ a.e. in Ω' and $\Omega\subset\subset\Omega'$. Indeed, by virtue of the characterization of $\mathbf{Q}(\nu)$, we have that

$$\mathbf{M}=\begin{pmatrix} a_1 & a_2\\ a_2 & -a_1\end{pmatrix} \qquad \text{a.e. in } \ \Omega$$

being $a_1,a_2:\Omega\to\mathbb{R}$ such that setting $a:=(2a_1,2a_2)$, we have $a\in W^{1,2}(\Omega;S^1)$. Hence the function a can be extended to a function (non relabeled) $a\in W^{1,2}(\Omega';S^1)$. As a consequence, $\mathbf{M}$ can be extended to a function belonging to $W^{1,2}(\Omega',\mathcal{M}^{2\times 2})$ and preserving the constraint $\mathbf{M}=\mathbf{Q}(\nu)$ a.e. in Ω', for some $\nu:\Omega'\to S^1$.

Moreover, Remark 3.5 ensures the existence of a sequence $(z_j)\subset\overline{Q}$ such that, setting $\mathbf{W}_j:=T_{z_j}^{\varepsilon_j}\mathbf{M}$, then $\mathbf{M}_j:=A(\mathbf{W}_j)$ is such that $(\mathbf{M}_j)\subset\mathcal{M}_{\varepsilon_j}(\Omega)$ and

$$\mathbf{M}_j\to\mathbf{M} \quad\text{in}\quad W^{1,2}(\Omega',\mathcal{M}^{2\times 2}).$$

Therefore, the thesis immediately follows taking as a recovery sequence the pair $(\bar u_j,\mathbf{M}_j)$ (see the proofs of Theorem 5.4 and Theorem 5.15). □

Proposition 7.1 and Theorem 7.3 permit to deduce the following result on the convergence of associated minimum problems.

Corollary 7.4 (Convergence of minimum problems for E_j^N)**.** *For any $\varphi\in W_{loc}^{1,\infty}(\mathbb{R}^2;\mathbb{R}^2)$,*

$$\lim_{j\to+\infty}\inf\{E_j^N(u,\mathbf{Q}(\nu)):(u,\mathbf{Q}(\nu))\in\mathcal{A}_{\delta_j}^{\varphi}(\Omega)\times\mathcal{M}_{\varepsilon_j}(\Omega)\}$$
$$=\min\{E^N(u,\mathbf{Q}(\nu)):\ (u-\varphi,\mathbf{Q}(\nu))\in W_0^{1,2}(\Omega;\mathbb{R}^2)\times W^{1,2}(\Omega;\mathcal{N})\}.$$

Moreover, if $(u_j,\mathbf{Q}(n)_j)\subset\mathcal{A}_{\delta_j}^{\varphi}(\Omega)\times\mathcal{M}_{\varepsilon_j}(\Omega)$ is a minimizing sequence for (E_j^N) then, up to subsequences, $(u_j,\mathbf{Q}(n)_j)\to(\bar u,\overline{\mathbf{Q}}(\nu))$ in $L^2(\Omega;\mathbb{R}^2)\times L^2(\Omega;\mathcal{M}^{2\times 2})$ and

$$E^N(\bar u,\overline{\mathbf{Q}}(\nu))=\min\{E^N(u,\mathbf{Q}(\nu)):(u-\varphi,\mathbf{Q}(\nu))\in W_0^{1,2}(\Omega;\mathbb{R}^2)\times W^{1,2}(\Omega;\mathcal{N})\}.$$

8. Additional terms in the energy

In this last section we extend the previous results to the case when the considered discrete system is subject to an external force, to an applied electric or magnetic field, or when crystalline anisotropy is taken into account. In what follows, we first introduce the discrete energies in all the cases of interest and then prove that these functionals are "continuous perturbations" with respect to the L^2 convergence. At the end of the section, we simply state the compactness and Γ-convergence results for the free energies with these additional terms without detailing their proofs. These are straightforward and can be easily obtained by reasoning as in Section 5.

8.1. **External force.** Let $f \in L^2_{loc}(\mathbb{R}^2, \mathbb{R}^2)$ denote an applied force field. The work done to deform the discrete system in presence of this field adds a new term in the free energy. For a given deformation $u : \Omega_\delta \to \mathbb{R}^2$ it is denoted by $E^{EL}_{ext,\varepsilon}(u)$ and is given by

$$E^{EL}_{ext,\varepsilon}(u) = -\sum_{\alpha\in\Omega_\delta} \delta^2 \langle f(\alpha), u(\alpha)\rangle.$$

We may identify this energy with the functional (not relabelled) $E^{EL}_{ext,\varepsilon} : L^2(\Omega, \mathbb{R}^2) \longrightarrow \mathbb{R} \cup \{+\infty\}$ defined as

$$E^{EL}_{ext,\varepsilon}(u) := \begin{cases} -\displaystyle\sum_{\alpha\in\Omega_\delta} \delta^2 \langle f(\alpha), u(\alpha)\rangle & \text{if } \ u \in \mathcal{A}_\delta(\Omega) \\ +\infty & \text{otherwise.} \end{cases} \tag{8.1}$$

Let $E^{EL}_{ext,j}(u) := E^{EL}_{ext,\varepsilon_j}(u)$; the following proposition states a continuity result for the coupling with the external force field.

Proposition 8.1. *Let $(u_j) \subset \mathcal{A}_{\delta_j}(\Omega)$ be such that $u_j \to u$ in $L^2(\Omega;\mathbb{R}^2)$. Then,*

$$\lim_{j\to+\infty} E^{EL}_{ext,j}(u_j) = E^{EL}_{ext}(u) := -\int_\Omega \langle f, u\rangle\, dx.$$

Proof. Let $(u_j) \subset \mathcal{A}_{\delta_j}(\Omega)$ be such that $u_j \to u$ in $L^2(\Omega, \mathbb{R}^2)$. Then, the sequences of functions (v_j) and (f_j) defined respectively as

$$v_j(x) := u_j(\alpha), \quad f_j(x) := f(\alpha) \quad \text{for } x \in Q_{\delta_j}(\alpha), \quad \alpha \in \delta_j\mathbb{Z}^2$$

are such that $v_j \to u$ and $f_j \to f$ in $L^2(\Omega, \mathbb{R}^2)$. Moreover, we have that

$$E^{EL}_{ext,j}(u_j) = -\int_\Omega \langle f_j, v_j\rangle\, dx + o(1),$$

hence the claim follows passing to the limit as $j \to +\infty$. □

8.2. **Magnetic and electric applied fields.** The internal variable describing orientation may couple with an applied magnetic or electric field. With the same arguments employed in the proof of Proposition 8.1 it is possible to show that the corresponding terms in the energy are continuous with respect to the strong L^2 convergence. We distinguish three cases.

8.2.1. **Magnetostrictive solids in a magnetic field.** Let $h \in L^2_{loc}(\mathbb{R}^2,\mathbb{R}^2)$ be the external magnetic field. Then, for any configuration $m : \Omega_\varepsilon \to S^1$, the free energy due to the interactions of the magnetic moments with the applied field is given by

$$E^M_{ext,\varepsilon}(m) = -\sum_{\alpha\in\Omega_\varepsilon} \varepsilon^2\langle h(\alpha), m(\alpha)\rangle.$$

As before, we set $E^M_{ext,j} := E^M_{ext,\varepsilon_j}$ and we identify it with the functional (not relabeled) $E^M_{ext,j} : L^2(\Omega,\mathbb{R}^2) \longrightarrow \mathbb{R}\cup\{+\infty\}$ given by

$$E^M_{ext,j}(m) := \begin{cases} -\displaystyle\sum_{\alpha\in\Omega_{\varepsilon_j}} \varepsilon_j^2\langle h(\alpha), m(\alpha)\rangle & \text{if } m \in \mathcal{B}_{\varepsilon_j}(\Omega) \\ +\infty & \text{otherwise.} \end{cases} \tag{8.2}$$

The following result states that the analog of Proposition 8.1 holds true also for the coupling with an applied magnetic field.

Proposition 8.2. *Let $(m_j) \subset \mathcal{B}_{\varepsilon_j}(\Omega)$ be such that $m_j \to m$ in $L^2(\Omega;\mathbb{R}^2)$. Then,*

$$\lim_{j\to+\infty} E^M_{ext,j}(m_j) = E^M_{ext}(m) := -\int_\Omega \langle h, m\rangle\,dx.$$

8.2.2. **Ferroelectric crystals in an electric field.** If we denote by $e \in L^2_{loc}(\mathbb{R}^2,\mathbb{R}^2)$ the applied electric field, then, for any configuration $p : \Omega_\varepsilon \to \mathbb{R}^2$, the free energy due to the interaction of the polarization with the external field is given by

$$E^F_{ext,\varepsilon}(p) = -\sum_{\alpha\in\Omega_\varepsilon} \varepsilon^2\langle e(\alpha), p(\alpha)\rangle.$$

As before, we set $E^F_{ext,j} := E^F_{ext,\varepsilon_j}$ and we identify it with the functional (not relabeled) $E^F_{ext,j} : L^2(\Omega,\mathbb{R}^2) \longrightarrow \mathbb{R}\cup\{+\infty\}$ given by

$$E^F_{ext,j}(m) := \begin{cases} -\displaystyle\sum_{\alpha\in\Omega_{\varepsilon_j}} \varepsilon_j^2\langle e(\alpha), p(\alpha)\rangle & \text{if } p \in \mathcal{A}_{\varepsilon_j}(\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

The following result holds.

Proposition 8.3. *Let $(p_j) \subset \mathcal{A}_{\varepsilon_j}(\Omega)$ be such that $p_j \to p$ in $L^2(\Omega;\mathbb{R}^2)$. Then,*

$$\lim_{j\to+\infty} E^F_{ext,j}(p_j) = E^F_{ext}(p) := -\int_\Omega \langle e, p\rangle\,dx.$$

8.2.3. **Nematic elastomers in a magnetic field.** When a nematic elastomer is subject to an applied magnetic field $h \in L^2_{loc}(\mathbb{R}^2,\mathbb{R}^2)$, an additional term in the energy has to be considered. For any configuration $n : \Omega_\varepsilon \to S^1$, the free energy due to the interactions of the nematic mesogens with the external field is given by

$$E^N_{ext,\varepsilon}(\mathbf{Q}(n)) = -\sum_{\alpha\in\Omega_\varepsilon} \varepsilon^2\langle \mathbf{Q}(n(\alpha))h(\alpha), h(\alpha)\rangle.$$

As before, we set $E^N_{ext,j} := E^N_{ext,\varepsilon_j}$ and we identify it with the functional (not relabeled) $E^N_{ext,j} : L^2(\Omega,\mathcal{M}^{2\times 2}) \longrightarrow \mathbb{R}\cup\{+\infty\}$ given by

$$E^N_{ext,j}(\mathbf{Q}(n)) := \begin{cases} -\displaystyle\sum_{\alpha\in\Omega_{\varepsilon_j}} \varepsilon_j^2\langle \mathbf{Q}(n(\alpha))h(\alpha), h(\alpha)\rangle & \text{if } \mathbf{Q}(n) \in \mathcal{M}_{\varepsilon_j}(\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

The following result holds.

Proposition 8.4. *Let* $(\mathbf{Q}(n)_j) \subset \mathcal{M}_{\varepsilon_j}(\Omega)$ *be such that* $\mathbf{Q}(n)_j \to \mathbf{Q}(\nu)$ *in* $L^2(\Omega, \mathcal{M}^{2\times 2})$. *Then,*

$$\lim_{j\to+\infty} E^N_{ext,j}(\mathbf{Q}(n)_j) = E^N_{ext}(\mathbf{Q}(\nu)) := -\int_\Omega \langle \mathbf{Q}(\nu)h, h\rangle \, dx.$$

8.3. **Anisotropy terms.** The interaction with an underlying lattice structure often results in the existence of *easy axes* of alignment; *i.e.,* energetically favored alignment directions for the internal variable. In the ferromagnetic and ferroelectric case, this effect is known as *crystalline anisotropy.* In the nematic elastomers case, memory of the orientation of the nematic mesogens during the cross-linking reactions may induce anisotropy. We discuss here a concrete example only for the magnetostrictive case, since completely analogous results hold in the ferroelectric and in the nematic case.

For any configuration $m : \Omega_\varepsilon \to S^1$, a cubic crystalline anisotropy, with easy axes e_1 and e_2 is described by the energy term

$$E^M_{an,\varepsilon}(m) = -\sum_{\alpha\in\Omega_\varepsilon} \varepsilon^2 \langle m(\alpha), e_1\rangle^2 \langle m(\alpha), e_2\rangle^2 \,,$$

see, e.g., [20]. As before, we set $E^M_{an,j} := E^M_{an,\varepsilon_j}$ and we identify it with the functional (not relabeled) $E^M_{an,j} : L^2(\Omega, \mathbb{R}^2) \longrightarrow \mathbb{R} \cup \{+\infty\}$ given by

$$E^M_{an,j}(m) := \begin{cases} -\displaystyle\sum_{\alpha\in\Omega_\varepsilon} \varepsilon_j^2 \langle m(\alpha), e_1\rangle^2 \langle m(\alpha), e_2\rangle^2 & \text{if } m \in \mathcal{B}_{\varepsilon_j}(\Omega) \\ +\infty & \text{otherwise.} \end{cases} \tag{8.3}$$

The following result holds true.

Proposition 8.5. *Let* $(m_j) \subset \mathcal{B}_{\varepsilon_j}(\Omega)$ *be such that* $m_j \to m$ *in* $L^2(\Omega; \mathbb{R}^2)$. *Then,*

$$\lim_{j\to+\infty} E^M_{an,j}(m_j) = E^M_{an}(m) := -\int_\Omega \langle m, e_1\rangle^2 \langle m, e_2\rangle^2 \; dx.$$

8.4. **Additional convergence results.** We end this section stating the equi-coercivity and the Γ-convergence results when the additional terms in the energy are taken into account. Since the claims in these propositions are similar in the three cases, we state them only for magnetostrictive solids.

With $\varphi \in W^{1,\infty}_{loc}(\mathbb{R}^2; \mathbb{R}^2)$ we define the total energy of the discrete magnetostrictive system, when all the additional terms discussed in this section are considered, as $E^{M,\varphi}_{tot,j} : L^2(\Omega; \mathbb{R}^2) \times L^2(\Omega; \mathbb{R}^2) \longrightarrow \mathbb{R} \cup \{+\infty\}$ given by

$$E^{M,\varphi}_{tot,j}(u, m) = \begin{cases} E^M_j(u, m) + E^{EL}_{ext,j}(u) + E^M_{ext,j}(m) + E^M_{an,j}(m) & \text{if } u \in \mathcal{A}^\varphi_{\delta_j}(\Omega) \\ +\infty & \text{otherwise,} \end{cases} \tag{8.4}$$

with E^M_j, $E^{EL}_{ext,j}$, $E^M_{ext,j}$, $E^M_{an,j}$ as in (5.2), (8.1), (8.2) and (8.3), respectively.

The following results hold true.

Proposition 8.6 (Equi-coercivity of the total energy)**.** *Let* (u_j, m_j) *be a sequence in* $L^2(\Omega; \mathbb{R}^2) \times L^2(\Omega; \mathbb{R}^2)$ *such that*

$$\sup_j E^{M,\varphi}_{tot,j}(u_j, m_j) < +\infty. \tag{8.5}$$

Then, there exists a pair $(u,m) \in W^{1,2}(\Omega;\mathbb{R}^2)\times W^{1,2}(\Omega;S^1)$ *with* $u-\varphi \in W_0^{1,2}(\Omega;\mathbb{R}^2)$ *such that, up to subsequences,*

$$(u_j, m_j) \rightharpoonup (u,m) \quad in \quad W^{1,2}(\Omega;\mathbb{R}^2)\times W^{1,2}(\Omega;\mathbb{R}^2).$$

Theorem 8.7 (Γ-convergence of the total energy $E^{M,\varphi}_{tot,j}$)**.** *The sequence of functionals* $(E^{M,\varphi}_{tot,j})$ *defined in* (8.4) Γ*-converges with respect to the* $L^2(\Omega;\mathbb{R}^2)\times L^2(\Omega;\mathbb{R}^2)$ *convergence to the functional* $E^{M,\varphi}_{tot} : L^2(\Omega;\mathbb{R}^2)\times L^2(\Omega;\mathbb{R}^2) \longrightarrow \mathbb{R}\cup\{+\infty\}$ *given by*

$$E^{M,\varphi}_{tot}(u,m) = \begin{cases} E^M(u,m) + E^{EL}_{ext}(u) + E^M_{ext}(m) + E^M_{an}(m) \\ \qquad\qquad if \ \ (u-\varphi, m) \in W_0^{1,2}(\Omega;\mathbb{R}^2)\times W^{1,2}(\Omega;S^1) \\ +\infty \qquad\quad otherwise, \end{cases}$$

with E^M *as in Theorem 5.13 and* E^{EL}_{ext}*,* E^M_{ext}*,* E^M_{an} *as in Propositions 8.1, 8.2 and 8.5, respectively.*

By the two previous results we derive the following corollary.

Corollary 8.8 (Convergence of minimum problems for $E^M_{tot,j}$)**.** *For any* $\varphi \in W^{1,\infty}_{loc}(\mathbb{R}^2;\mathbb{R}^2)$*,*

$$\lim_{j\to+\infty} \inf\{E^M_{tot,j}(u,m) : (u,m) \in \mathcal{A}^{\varphi}_{\delta_j}(\Omega)\times\mathcal{B}_{\varepsilon_j}(\Omega)\}$$
$$= \min\{E^M_{tot}(u,m) : \ (u-\varphi,m) \in W_0^{1,2}(\Omega;\mathbb{R}^2)\times W^{1,2}(\Omega;S^1)\}.$$

Moreover, if $(u_j,m_j) \subset \mathcal{A}^{\varphi}_{\delta_j}(\Omega)\times\mathcal{B}_{\varepsilon_j}(\Omega)$ *is a minimizing sequence for* $(E^M_{tot,j})$ *then, up to subsequences,* $(u_j,m_j)\to(\bar u,\bar m)$ *in* $L^2(\Omega;\mathbb{R}^2)\times L^2(\Omega;\mathbb{R}^2)$ *and*

$$E^M_{tot}(\bar u,\bar m) = \min\{E^M_{tot}(u,m) : (u-\varphi,m) \in W_0^{1,2}(\Omega;\mathbb{R}^2)\times W^{1,2}(\Omega;S^1)\}.$$

The analogue of the previous statements hold in the case of ferroelectric crystals or nematic elastomers.

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(Marco Cicalese) Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Università di Napoli Federico II, Via Cintia, 80126 Napoli, Italy
E-mail address, Marco Cicalese: cicalese@unina.it

(Antonio DeSimone, Caterina Ida Zeppieri) SISSA, Via Beirut 2-4, 34014 Trieste, Italy
E-mail address, Antonio DeSimone: desimone@sissa.it
E-mail address, Caterina Ida Zeppieri: zeppieri@sissa.it