Steiner symmetric extremals in Pólya–Szegö type inequalities

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1 Introduction

The expression *Pólya–Szegö principle* (or *inequality*) pertains, in a broad sense, to any statement asserting that some Dirichlet type integral depending on the gradient of real-valued functions does not increase under an appropriate rearrangement of the involved function. Typically, an operation of rearrangement maps a measurable function into an equidistributed function enjoying some additional symmetry property, and it is hence also called a symmetrization. *Schwarz spherical symmetrization* about a point and *Steiner symmetrization* about an hyperplane are probably the most popular symmetrizations in the literature. Pólya–Szegö inequalities for these symmetrizations have been known for a long time, and have seen noticeable applications in various branches of analysis, including eigenvalue problems in mathematical physics, potential theory, theory of Sobolev spaces, partial differential equations.

The analysis of functions attaining equality in Pólya–Szegö inequalities, which will be referred to as *PS-extremals*, has a much more recent history. A part from its own interest, the characterization of these extremals is relevant, for instance, in view of applications to symmetry properties of solutions to variational problems. An impulse to the study of this delicate issue was given by the paper [K2] (see also [K1]), where the symmetry of PS-extremals for Schwarz and Steiner symmetrizations is established, by classical techniques, in special classes of functions and ground domains. After a few subsequent contributions, the problem for Schwarz symmetrization, in the class of Sobolev functions, was finally solved in [BZ] (see also [FV] for an interesting alternate proof). Indeed, in [BZ] a minimal assumption is exhibited for any PS-extremal to be necessarily Schwartz symmetric. Such an assumption amounts to requiring that the set of critical points of the symmetrized extremal has Lebesgue measure zero. A version of this result in the framework of functions of bounded variations can be found in [CF].

On the other hand, in spite of various recent developments concerning Steiner symmetrization (see e.g. [ADLT], [Bae], [Br], [Bur], [BLM]), the status of the art on the question of symmetry of PS-extremals seems to be still that set in [K1]. The aim of the present paper is to settle this question in full generality. Actually, our results yield the Steiner symmetry of PS-extremals in the class of Sobolev, and more generally, BV functions, for a large class of functionals. The crucial assumption is that the derivative of the relevant extremal, in the direction orthogonal to the hyperplane of symmetrization, vanishes at most in a set of Lebesgue measure zero. This hypothesis is the analog, in the setting at hand, of the hypothesis of [BZ] mentioned above. However, unlike the case of Schwartz symmetrization, additional undispensable assumptions on the ground domain have to be imposed. Besides certain connectedness and boundedness natural conditions, the main assumption requires that, loosely speaking, no essential part of the reduced boundary of the domain lies orthogonally to the hyperplane of symmetrization.

Let us emphasize that our approach differs substantially from those of [BZ] and [FV]. In particular, whereas the description of PS-extremals for Schwarz symmetrization heavily relies upon the characterization of balls as the unique extremals in the isoperimetric inequality in $\mathbb{R}^n$, the corresponding discussion of the equality cases in the perimeter inequality for Steiner symmetrization, recently provided in [CCF], seems to be of no direct use for our purposes. Instead, a successful underlying idea in attacking the problem is to derive the symmetry of extremals from the symmetry of their subgraphs. This is most apparent when functions of bounded variation are taken into account. Indeed, in this case, our strategy is to turn the Dirichlet type functionals under consideration into geometric functionals, depending on the generalized inner normal to the reduced boundary of the subgraph of functions. This leads us to a preliminary study, of independent interest, of Pólya–Szegö type inequalities for this kind of functionals, which can be regarded as the "parametric" counterpart of the original "non-parametric" functionals.

2 Main results

We begin with some definitions and elementary facts about Steiner symmetrization of sets and functions.

A point x in the Euclidean space $\mathbb{R}^n$, $n \geq 2$, will be usually labeled by (x', y), where $x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$ and $y \in \mathbb{R}$; similarly, when $x \in \mathbb{R}^{n+1}$, we shall write x as (x', y, t). To emphasize the different roles of the variables y and t, we shall also write $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}_y$ and $\mathbb{R}^{n+1} = \mathbb{R}^{n-1} \times \mathbb{R}_y \times \mathbb{R}_t$. Consistent notations will be used for subsets of $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$.
Given any measurable subset E of $\mathbb{R}^n$, define, for $x' \in \mathbb{R}^{n-1}$,

$$E_{x'} = \{y \in \mathbb{R} : (x', y) \in E\}, \tag{2.1}$$

and

$$\ell_E(x') = \mathcal{L}^1(E_{x'}). \tag{2.2}$$

Hereafter, $\mathcal{L}^m$ denotes the outer Lebesgue measure in $\mathbb{R}^m$. Then, we define the *Steiner symmetral* E^s of E about the hyperplane $\{y = 0\}$ as

$$E^s = \{(x', y) \in \mathbb{R}^n : |y| < \ell_E(x')/2\}.$$

When $E \subset \mathbb{R}^{n-1} \times \mathbb{R}_y \times \mathbb{R}_t$, its Steiner symmetral E^s about $\{y = 0\}$ is defined analogously, after replacing (2.1)-(2.2) by parallel definitions of $E_{x',t}$ and $\ell_E(x', t)$.
Now, let Ω be a measurable subset of $\mathbb{R}^n$ and let u be a nonnegative measurable function in Ω such that, for $\mathcal{L}^{n-1}$-a.e. $x' \in \pi_{n-1}(\Omega)$,

$$\mathcal{L}^1\left(\{y \in \Omega_{x'} : u(x', y) > t\}\right) < \infty \qquad \text{for every } t > 0. \tag{2.3}$$

Here, π_{n-1} denotes orthogonal projection onto $\mathbb{R}^{n-1}$. The *Steiner rearrangement* u^s of u is the function from $\mathbb{R}^n$ into $[0, +\infty]$ given by

$$u^s(x', y) = \inf\{t > 0 : \mu_u(x', t) \leq 2|y|\} \qquad \text{for } (x', y) \in \mathbb{R}^{n-1} \times \mathbb{R}_y,$$

where

$$\mu_u(x', t) = \mathcal{L}^1\left(\{y \in \mathbb{R} : u_0(x', y) > t\}\right),$$

the *distribution function* of $u(x', \cdot)$, and u_0 denotes the continuation of u to $\mathbb{R}^n$ which vanishes outside Ω. Note that $u^s = 0$ $\mathcal{L}^n$-a.e. in $\mathbb{R}^n \setminus \Omega^s$.

The notions of Steiner symmetral of a set and of Steiner rearrangement of a function are clearly related. Actually, if $u : \Omega \to [0, +\infty)$ is as above, and

$$\mathcal{S}_u = \{(x', y, t) \in \mathbb{R}^{n+1} : (x', y) \in \Omega,\, 0 < t < u(x', y)\} ,$$

the subgraph of u, then

$$(\mathcal{S}_u)^s \text{ is equivalent to } \mathcal{S}_{u^s} \tag{2.4}$$

and

$$\{(x', y) : u(x', y) > t\}^s \text{ is equivalent to } \{(x', y) : u^s(x', y) > t\} \qquad \text{for every } t > 0 . \tag{2.5}$$

Equations (2.4)-(2.5) are easy consequences of the fact that

$$\ell_{\mathcal{S}_u}(x', t) = \mu_u(x', t) \qquad \text{for } (x', t) \in \mathbb{R}^{n-1} \times \mathbb{R}^+_t$$

and that, for $\mathcal{L}^{n-1}$-a.e. $x' \in \mathbb{R}^{n-1}$, we have $u^s(x', y) > t$ for some $y \in \mathbb{R}_y$ and $t \in \mathbb{R}^+_t$ if and only if $\mu_u(x', t) > 2|y|$.
Equation (2.5) ensures that u and u^s are equidistributed functions; in fact $u(x', \cdot)$ and $u^s(x', \cdot)$ are equidistributed for $\mathcal{L}^{n-1}$-a.e. $x' \in \pi_{n-1}(\Omega)$, by the very definition of u^s. Hence, any rearrangement invariant norm of a function, i.e. any norm depending only on the measure of its level sets (such as Lebesgue, Lorentz or Orlicz norms), is trivially preserved under Steiner rearrangement.

A much deeper property involving the Steiner rearrangement of functions enjoying differentiability properties is provided by the Pólya–Szegö principle. Theorem 2.1 below contains a version of this principle for Sobolev functions, which applies to the class of integral functionals having the form

$$\int_\Omega f(\nabla u)\, dx ,$$

where f is any convex function from $\mathbb{R}^n$ into $[0, +\infty)$, vanishing at 0, and satisfying

$$f(\xi_1, \dots, \xi_{n-1}, \xi_n) = f(\xi_1, \dots, \xi_{n-1}, -\xi_n) \qquad \text{for every } (\xi_1, \dots, \xi_n) \in \mathbb{R}^n . \tag{2.6}$$

These functionals will be evaluated at functions which vanish on the subset of $\partial\Omega$ which lies inside the cylinder $\pi_{n-1}(\Omega) \times \mathbb{R}_y$. Precisely, functions from the class

$$W^{1,1}_{0,y}(\Omega) = \Big\{u : \Omega \to \mathbb{R} : u_0 \in W^{1,1}(\omega \times \mathbb{R}_y) \text{ for every open set } \omega \subset\subset \pi_{n-1}(\Omega)\Big\}$$

are taken into account.

Theorem 2.1 *Let f be as above. Let Ω be an open subset of $\mathbb{R}^n$ and let u be a nonnegative function from $W^{1,1}_{0,y}(\Omega)$. Then $u^s \in W^{1,1}(\omega \times \mathbb{R}_y)$ for every open set $\omega \subset\subset \pi_{n-1}(\Omega)$, and*

$$\int_{\Omega^s} f(\nabla u^s)\, dx \le \int_\Omega f(\nabla u)\, dx . \tag{2.7}$$

Let us mention that $W^{1,1}_{0,y}(\Omega)$ can be replaced in Theorem 2.1 by any space $W^{1,p}_{0,y}(\Omega)$ defined analogously with $p \geq 1$ – see Remark 4.6, Section 4.
Theorem 2.1 is well known (under more restrictive assumptions on u) when $f(\xi) = |\xi|^p$, with $p \geq 1$, and, more generally, when $f(\xi) = \Phi(|\xi|)$ and Φ is a convex function from $[0, +\infty)$ into $[0, +\infty)$ vanishing at 0 – see [Br], [Bur]. Here we present an independent proof of (2.7), which has the advantage of providing us with information about functions yielding equality. Actually, our final goal in regard to (2.7) is to find out minimal assumptions for such extremal functions to be necessarily Steiner symmetric. The assumptions in question turn out to involve both the extremal u and the domain Ω, and amount to what follows.
Consider u first, and set

$$M(x') = \inf \{t > 0 : \mu_u(x', t) = 0\} \qquad \text{for } x' \in \pi_{n-1}(\Omega) .$$

Obviously, $M(x')$ agrees with $\text{esssup}\{u(x', y) : y \in \Omega_{x'}\}$ for $\mathcal{L}^{n-1}$-a.e. $x' \in \pi_{n-1}(\Omega)$. Moreover, M is a measurable function in $\pi_{n-1}(\Omega)$, owing to (2.3). We demand that, for $\mathcal{L}^{n-1}$-a.e. $x' \in \pi_{n-1}(\Omega)$, $M(x') > 0$ and the derivative of the restriction $u(x', \cdot)$ is $\mathcal{L}^1$-a.e. different from 0 in the set where $u(x', \cdot) < M(x')$. This is equivalent to the condition

$$\text{(2.8)} \quad \mathcal{L}^n \left(\{(x', y) \in \Omega : \nabla_y u(x', y) = 0\} \cap \{(x', y) \in \Omega : \text{either } M(x') = 0 \text{ or } u(x', y) < M(x')\}\right) = 0 .$$

As far as Ω is concerned, we require that

$$\text{(2.9)} \qquad \pi_{n-1}(\Omega) \text{ is connected,}$$

$$\text{(2.10)} \qquad \Omega \text{ is bounded in the direction } y ,$$

and that, in a sense, the reduced boundary $\partial^*\Omega$ of Ω is almost nowhere parallel to the y-axis inside the open cylinder $\pi_{n-1}(\Omega) \times \mathbb{R}_y$. A precise formulation of the last condition reads

$$\text{(2.11)} \qquad \begin{array}{l} \Omega \text{ has locally finite perimeter in } \pi_{n-1}(\Omega) \times \mathbb{R}_y \text{ and} \\ \mathcal{H}^{n-1}\left(\{(x', y) \in \partial^*\Omega : \nu^{\Omega}_y(x', y) = 0\} \cap (\pi_{n-1}(\Omega) \times \mathbb{R}_y)\right) = 0 , \end{array}$$

where $\mathcal{H}^k$ stands for k-dimensional Hausdorff measure, and ν^{Ω}_y denotes the component along the y-axis of the generalized inner normal ν^{Ω} to Ω.

We are now ready to state our result about the equality case in (2.7).

Theorem 2.2 *Let $f : \mathbb{R}^n \to [0, +\infty)$ be a strictly convex function vanishing at 0 and satisfying (2.6). Let Ω be an open subset of $\mathbb{R}^n$ fulfilling (2.9)-(2.11). Let u be a nonnegative function from $W^{1,1}_{0,y}(\Omega)$ satisfying (2.8). If*

$$\text{(2.12)} \qquad \int_{\Omega^s} f(\nabla u^s)\, dx = \int_{\Omega} f(\nabla u)\, dx < \infty ,$$

then u^s is equivalent to u (up to translations along the y-axis).

Let us emphasize that assumptions (2.8)-(2.11) in Theorem 2.2 are essentially sharp, in that they cannot be removed without effecting the conclusion. Indeed, if (2.8) is dropped, then functions whose graph is shaped like that represented in Figure 1 immediately prove that (2.12) may hold even if u does not agree with any translate of u^s.
As for the domain Ω, assumption (2.9) is easily seen to be indispensable. The example in Figure 2 demonstrates the necessity of condition (2.11); an easy modification of that example - see Figure 3 -

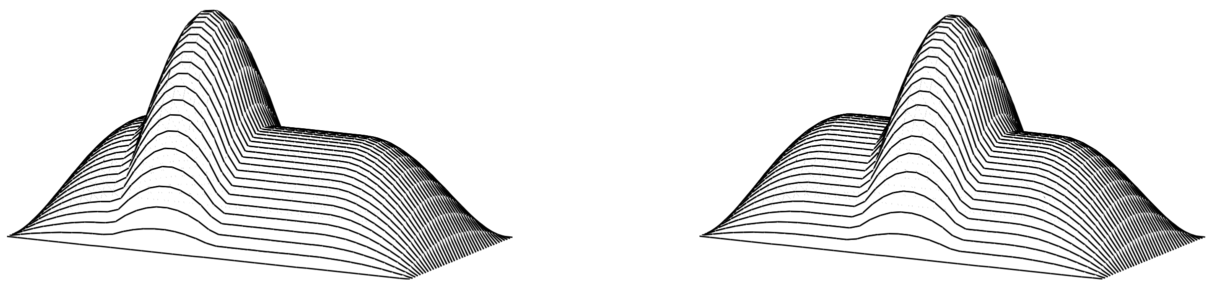

Figure 1: The graph of u is on the left; on the right, the graph of u^s

Figure 2: The domain of u is the union of two rectangles

shows that unbounded domains in the direction y cannot be allowed, even in the case where $\Omega_{x'}$ is bounded for $\mathcal{L}^{n-1}$-a.e. $x' \in \pi_{n-1}(\Omega)$.
Notice that strict convexity of f is also essential in Theorem 2.2. Actually, suppose, for instance, that $f(\xi) = |\xi|$ for $\xi \in \mathbb{R}^n$ and let u be any compactly supported function from $W^{1,1}(\mathbb{R}^n)$, whose level sets are Steiner symmetric about non coincident hyperplanes parallel to $\{y = 0\}$. Then u is not equivalent to any translate of u^s, but, as a consequence of the coarea formula for Sobolev functions,

$$\int_{\mathbb{R}^n} f(\nabla u)\, dx = \int_0^\infty P(\{u > t\})\, dt = \int_0^\infty P(\{u > t\}^s)\, dt = \int_0^\infty P(\{u^s > t\})\, dt = \int_{\mathbb{R}^n} f(\nabla u^s)\, dx\,.$$

Here, $P(E)$ denotes the perimeter of a set E. Observe that the second equality holds since $P(E) = P(E^s)$ whenever E is equivalent to a translate of E^s, and that the third equality holds because of (2.5). Similar counterexamples can be exhibited also in the case where there exist $a \geq 0$ and $b \in \mathbb{R}$ such that $f(\xi) = a|\xi| + b$ just for ξ satisfying $t_1 \leq |\xi| \leq t_2$ for some $0 \leq t_1 < t_2$.

To conclude with our comments about Theorem 2.2, let us go back to hypothesis (2.8). The counterpart of this hypothesis in the case of Schwarz symmetrization, as appears in [BZ], is that

$$\mathcal{L}^n\left(\{\nabla u^\star = 0\} \cap \{0 < u^\star < \operatorname{esssup} u\}\right) = 0\,, \tag{2.13}$$

where $u^\star$ denotes the Schwarz rearrangement of u. Condition (2.13) is weaker, in general, than the same condition imposed on u. Thus, one might ask whether (2.8) could be relaxed on replacing u by u^s. However, such a replacement is immaterial, as a consequence of the following result.

Proposition 2.3 *Let u be a nonnegative function from $W^{1,1}_{0,y}(\Omega)$. Then, for $\mathcal{L}^{n-1}$-a.e. $x' \in \pi_{n-1}(\Omega)$,*

$$\mathcal{L}^1(\{y : \nabla_y u(x', y) = 0, t < u(x', y) < M(x')\}) = \mathcal{L}^1(\{y : \nabla_y u^s(x', y) = 0, t < u^s(x', y) < M(x')\})$$

for every $t \in (0, M(x'))$.

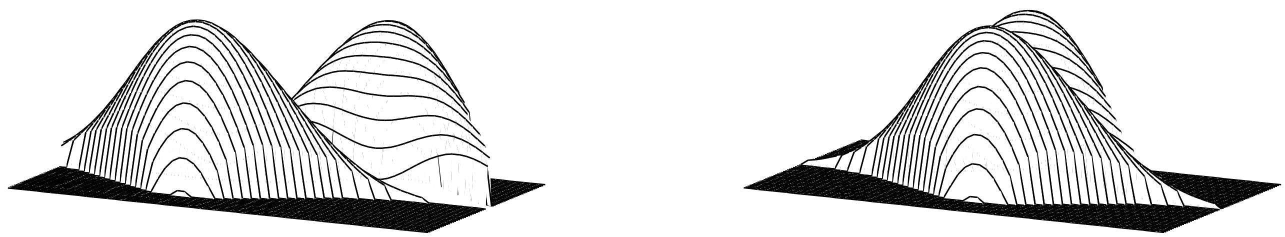

Figure 3: The domain of the function u is unbounded

We now shift to the more general framework of functions of bounded variation. Here, a version of the Pólya–Szegö inequality can still be shown to hold, provided that the involved functional is properly defined. Consider any nonnegative convex function f in $\mathbb{R}^n$ growing linearly at infinity, i.e., satisfying

$$0 \le f(\xi) \le C(1 + |\xi|) \tag{2.14}$$

for some positive constant C and for every $\xi \in \mathbb{R}^n$, and define the recession function f_∞ of f as

$$f_\infty(\xi) = \lim_{t \to +\infty} \frac{f(t\xi)}{t}.$$

Then a standard extension of the functional $\int_\Omega f(\nabla u)dx$ to the space $BV_{\rm loc}(\Omega)$ of functions of locally bounded variation in an open set $\Omega \subset \mathbb{R}^n$ is defined as

$$J_f(u;\Omega) = \int_\Omega f(\nabla u)\, dx + \int_\Omega f_\infty\left(\frac{D^s u}{|D^s u|}\right) d|D^s u| \tag{2.15}$$

at $u \in BV_{\rm loc}(\Omega)$. Here, ∇u denotes the approximate gradient of u, which agrees with the density of the absolutely continuous part, with respect to $\mathcal{L}^n$, of the measure Du, the distributional derivative of u; moreover, $D^s u$ stands for the singular part of Du with respect to $\mathcal{L}^n$, and $|D^s u|$ is its total variation. Indeed, $J_f(u;\Omega)$ is the *relaxed functional* of $\int_\Omega f(\nabla u)dx$ in $BV(\Omega)$ with respect to convergence in $L^1_{\rm loc}(\Omega)$ (see Theorem F, Section 6). A Pólya–Szegö inequality for functionals of the form (2.15) holds in the space of functions from $BV_{\rm loc}(\Omega)$ which vanish on $\partial\Omega \cap (\pi_{n-1}(\Omega) \times \mathbb{R}_y)$, in the sense that they belong to the space

$$BV_{0,y}(\Omega) = \{u : \Omega \to \mathbb{R}^n : \ u_0 \in BV(\omega \times \mathbb{R}_y) \text{ and } |Du_0|(\omega \times \mathbb{R}_y) = |Du|(\Omega \cap (\omega \times \mathbb{R}_y)) \\ \text{for every open set } \omega \subset\subset \pi_{n-1}(\Omega)\}.$$

Theorem 2.4 *Let $f : \mathbb{R}^n \to [0, +\infty)$ be a convex function vanishing at* 0 *and satisfying (2.6). Let Ω be an open subset of $\mathbb{R}^n$ and let u be a nonnegative function from $BV_{0,y}(\Omega)$. Then $u^s \in BV(\omega \times \mathbb{R}_y)$ for every open set $\omega \subset\subset \pi_{n-1}(\Omega)$, and*

$$J_f(u^s;\Omega^s) \le J_f(u;\Omega). \tag{2.16}$$

The identification of the extremals in equality (2.16) as Steiner symmetric functions is the content of the next theorem. The assumptions on u and Ω in this theorem are the same as in Theorem 2.2, provided that $\nabla_y u$ is interpreted in (2.8) as the y-component of the absolutely continuous part of Du, or, equivalently, as the y-component of the approximate gradient of u. Due to the presence of the term depending on $D^s u$

in the functional J_f, we need here additional assumptions on the recession function of f. Actually, we have to require that

$$f_\infty(\xi_1, \dots, \xi_{n-1}, \cdot) \text{ is strictly increasing on } [0, +\infty) \text{ for every } (\xi_1, \dots, \xi_{n-1}) \in \mathbb{R}^{n-1}, \tag{2.17}$$

and that

$$\text{the function } \mathbb{R}^{n-1} \ni (\xi_1, \dots, \xi_{n-1}) \mapsto f_\infty(\xi_1, \dots, \xi_{n-1}, 1) \text{ is strictly convex.} \tag{2.18}$$

Theorem 2.5 *Let $f : \mathbb{R}^n \to [0, +\infty)$ be a strictly convex function vanishing at 0 and satisfying (2.6), (2.14), (2.17) and (2.18). Let Ω be an open subset of $\mathbb{R}^n$ fulfilling (2.9)–(2.11). Let u be a nonnegative function from $BV_{0,y}(\Omega)$ satisfying (2.8). If*

$$J_f(u^s; \Omega^s) = J_f(u; \Omega) < \infty, \tag{2.19}$$

then u is equivalent to u^s (up to translations along the y-axis).

Theorems 2.2 and 2.5 have completely parallel statements and, in fact, a unified treatment of the two results could be given. However, since Theorem 2.2 admits an alternate, more direct, approach, we prefer to present separate proofs which overlap only partially. To be specific, both proofs follow the same scheme which consists of two steps: first, showing that the restrictions $u(x', \cdot)$ are symmetric about some point $b(x')$ for $\mathcal{L}^{n-1}$-a.e. $x' \in \pi_{n-1}(\Omega)$; second, proving that $b(x')$ is, in fact, constant. The proof of the latter step is essentially the same (save just a single point), and it is at this stage that assumptions (2.9)-(2.11) on Ω come into play. Assumption (2.8) on u has a role in the first step, whose proof in the case of Theorem 2.5 is more delicate. Actually, as already mentioned in Section 1, when dealing with BV functions we need to work, from the very beginning, with functionals defined on sets of locally finite perimeter and having the form

$$\int_{\partial^* E} F(\nu^E)\, d\mathcal{H}^n. \tag{2.20}$$

Here, $F : \mathbb{R}^{n+1} \to [0, +\infty)$ is a convex function such that

$$F(\lambda\xi_1, \dots, \lambda\xi_{n+1}) = \lambda F(\xi_1, \dots, \xi_{n+1}) \text{ for every } (\xi_1, \dots, \xi_{n+1}) \in \mathbb{R}^{n+1} \text{ and every } \lambda > 0. \tag{2.21}$$

Note that, if $F(\xi) = |\xi|$, then $\int_{\partial^* E} F(\nu^E) d\mathcal{H}^n$ agrees with $P(E)$.
The functional J_f is linked to the functional defined as in (2.20), with integrand given by

$$F_f(\xi_1, \dots, \xi_{n+1}) = \begin{cases} f\Big(\dfrac{(\xi_1, \dots, \xi_n)}{-\xi_{n+1}}\Big)(-\xi_{n+1}) & \text{if } \xi_{n+1} < 0 \\ f_\infty(\xi_1, \dots, \xi_n) & \text{if } \xi_{n+1} \geq 0, \end{cases} \tag{2.22}$$

by the following result.

Proposition 2.6 *Let $f : \mathbb{R}^n \to [0, +\infty)$ be a convex function vanishing at 0 and satisfying (2.14) and let F_f be the function associated with f as in (2.22). Then F_f is a convex function satisfying (2.21). Moreover, if Ω is an open subset of $\mathbb{R}^n$, then*

$$J_f(u; B) = \int_{\partial^* \mathcal{S}_u \cap (B \times \mathbb{R}_t)} F_f(\nu^{\mathcal{S}_u})\, d\mathcal{H}^n \tag{2.23}$$

for every nonnegative function $u \in BV_{\text{loc}}(\Omega)$ and for every Borel set $B \subset \Omega$.

Thanks to Proposition 2.6, the proof of Theorem 2.4 can be reduced to the proof of a Pólya–Szegö type inequality for functionals as in (2.20); moreover, information about extremal sets in such inequality yields information about extremal functions in (2.19). The relevant Pólya–Szegö inequality for the functionals $\int_{\partial^* E} F(\nu^E) d\mathcal{H}^n$ is contained in the next theorem. Besides (2.21), the integrand F will be assumed to satisfy

$$F(\xi_1, \dots, \xi_{n-1}, \xi_n, \xi_{n+1}) = F(\xi_1, \dots, \xi_{n-1}, -\xi_n, \xi_{n+1}) \qquad \text{for every } (\xi_1, \dots, \xi_{n+1}) \in \mathbb{R}^{n+1}. \tag{2.24}$$

Theorem 2.7 *Let $F : \mathbb{R}^{n+1} \to [0, +\infty)$ be a convex function satisfying (2.21) and (2.24). Let U be an open subset of $\mathbb{R}^{n-1} \times \mathbb{R}_t$ and let E be a set of finite perimeter in $U \times \mathbb{R}_y$ such that $\ell_E < +\infty$ $\mathcal{L}^n$-a.e. in U. Then*

$$\int_{\partial^* E^s \cap (A \times \mathbb{R}_y)} F(\nu^{E^s})\, d\mathcal{H}^n \le \int_{\partial^* E \cap (A \times \mathbb{R}_y)} F(\nu^E)\, d\mathcal{H}^n \tag{2.25}$$

for every Borel set $A \subset U$. Moreover, if E is a set of finite perimeter in $\mathbb{R}^{n+1}$, then

$$\int_{\partial^* E^s} F(\nu^{E^s})\, d\mathcal{H}^n \le \int_{\partial^* E} F(\nu^E)\, d\mathcal{H}^n\,. \tag{2.26}$$

A first set of conclusions about the equality cases in (2.25)$-$(2.26) is provided by the following result. Here, the assumptions on F have to be reinforced by requiring that

$$F(\xi_1, \dots, \xi_{n-1}, \cdot, \xi_{n+1}) \text{ is strictly increasing in } [0, +\infty) \quad \text{for every } (\xi_1, \dots, \xi_{n-1}, \xi_{n+1}) \in \mathbb{R}^n\,, \tag{2.27}$$

and that there exists a convex set $K \subset \mathbb{R}^{n-1} \times \mathbb{R}_t$ such that the function

$$K \ni (\xi_1, \dots, \xi_{n-1}, \xi_{n+1}) \mapsto F(\xi_1, \dots, \xi_{n-1}, 1, \xi_{n+1}) \quad \text{is strictly convex.} \tag{2.28}$$

In what follows, the essential projection of a set $E \subset \mathbb{R}^{n+1}$ onto $\mathbb{R}^{n-1} \times \mathbb{R}_t$ is defined as

$$\pi_{n-1,n+1}(E)^+ = \{(x', t) \in \mathbb{R}^{n-1} \times \mathbb{R}_t : \ell_E(x', t) > 0\}\,.$$

The essential projection $\pi_{n-1}(E)^+$ onto $\mathbb{R}^{n-1}$ is defined similarly.

Theorem 2.8 *Let $F : \mathbb{R}^{n+1} \to [0, +\infty)$ be a convex function satisfying (2.21), (2.24), (2.27) and (2.28) for some convex set $K \subset \mathbb{R}^{n-1} \times \mathbb{R}_t$. Let U be an open subset of $\mathbb{R}^{n-1} \times \mathbb{R}_t$. Let E be a set of finite perimeter in $U \times \mathbb{R}_y$ satisfying $\ell_E < +\infty$ $\mathcal{L}^n$-a.e. in U, and let A be a Borel subset of U such that*

$$\left(\frac{\nu_1^E}{\nu_y^E}, \dots, \frac{\nu_{n-1}^E}{\nu_y^E}, \frac{\nu_t^E}{\nu_y^E}\right) \in K \qquad \mathcal{H}^n\text{-a.e. on } \partial^* E \cap (A \times \mathbb{R}_y)\,, \tag{2.29}$$

where $\nu^E = (\nu_1^E, \dots, \nu_{n-1}^E, \nu_y^E, \nu_t^E)$. If

$$\int_{\partial^* E^s \cap (A \times \mathbb{R}_y)} F(\nu^{E^s})\, d\mathcal{H}^n = \int_{\partial^* E \cap (A \times \mathbb{R}_y)} F(\nu^E)\, d\mathcal{H}^n < \infty\,, \tag{2.30}$$

then there exist two functions $y_1, y_2 : \pi_{n-1,n+1}(E)^+ \cap A \to \mathbb{R}$ such that, for $\mathcal{L}^n$-a.e. $(x', t) \in \pi_{n-1,n+1}(E)^+ \cap A$,

$$E_{x',t} \text{ is equivalent to } (y_1(x', t), y_2(x', t))\,, \tag{2.31}$$

$$\nu_i^E(x', y_1(x', t), t) = \nu_i^E(x', y_2(x', t), t) \qquad \text{for } i = 1, \dots, n-1, t\,, \tag{2.32}$$

$$\nu_y^E(x', y_1(x', t), t) = -\nu_y^E(x', y_2(x', t), t)\,. \tag{2.33}$$

Theorems 2.7 and 2.8 contain all the material concerning the functionals (2.20) needed in our proof of Theorem 2.5. Notice, however, that Theorem 2.8 leaves open the problem of whether any set satisfying (2.30) is necessarily Steiner symmetric. Simple examples show that this is not the case, even when $F(\xi) = |\xi|$ (see e.g. [CCF]). For completeness, we conclude with a result, in the spirit of Theorems 2.2 and 2.5, which provides minimal additional assumptions ensuring Steiner symmetry of extremal sets in (2.26).

Theorem 2.9 *Let F be as in Theorem 2.8. Let U be an open subset of $\mathbb{R}^{n-1} \times \mathbb{R}_t$. Let E be a set of finite perimeter in $U \times \mathbb{R}_y$ and let U be an open subset of $\mathbb{R}^{n-1} \times \mathbb{R}_t$ satisfying $\ell_E < +\infty$ $\mathcal{L}^n$-a.e. in U and such that (2.30) is fulfilled with $A = U$. Assume either that*

$$F(\xi_1, \ldots, \xi_{n-1}, 0, \xi_{n+1}) > 0 \qquad \textit{for every} \quad (\xi_1, \ldots, \xi_{n-1}, \xi_{n+1}) \in \mathbb{R}^n \setminus \{0\} \tag{2.34}$$

and

$$\mathcal{H}^n \Big(\{x \in \partial^* E^s : \nu_y^{E^s}(x) = 0\} \cap (U \times \mathbb{R}) \Big) = 0 \,, \tag{2.35}$$

or that

$$\mathcal{H}^n \Big(\{x \in \partial^* E : \nu_y^{E}(x) = 0\} \cap (U \times \mathbb{R}) \Big) = 0 \,. \tag{2.36}$$

Assume also that the precise representative ℓ_E^* of ℓ_E satisfies

$$\ell_E^*(x', t) > 0 \qquad \textit{for } \mathcal{H}^{n-1}\textit{-a.e. } (x', t) \in U \,. \tag{2.37}$$

Then $E \cap (U_\alpha \times \mathbb{R}_y)$ is equivalent to a translate of $E^s \cap (U_\alpha \times \mathbb{R}_y)$ along the y-axis for each connected component U_α of U. In particular, if U is connected and $\mathcal{L}^n(\pi_{n-1,n+1}(E)^+ \setminus U) = 0$, then E is equivalent to E^s (up to translations along the y-axis).

Observe that assumption (2.36) implies (2.35) (see Lemma 5.7); the reverse implication holds if E satisfies both (2.30) and (2.34) (see Lemma 5.7 again). If any of the last two conditions is dropped, then (2.35) may hold without (2.36) being fulfilled. Counterexamples in this connection are easy. An example in the case where (2.30) is not in force and ($F(\xi) = |\xi|$) is given in [CCF]. On the other hand, if F is any function as in Theorems 2.8 – 2.9 (with $n = 2$) such that $F(\xi_1, 0, \xi_3) = 0$ for $(\xi_1, \xi_3) \in \mathbb{R}^2$, and E is the set depicted in figure 4, then (2.30) holds and (2.35) is satisfied with $\Omega = \mathbb{R}^2$, but (2.36) is not.

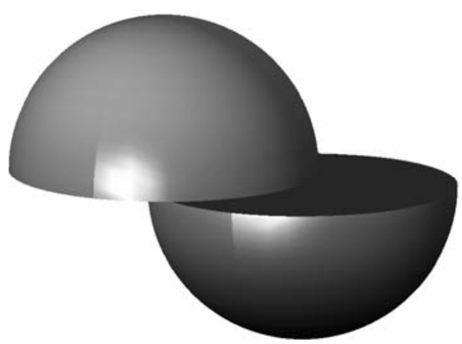

Figure 4: The Steiner simmetral of the set E displayed in the picture is a ball

Owing to Proposition 2.6, one might hope that Theorem 2.5 could be derived from Theorem 2.9; this seems to be possible, however, only under extra regularity assumptions on u.

In view of the above discussion, it should be clear that our approach requires, besides symmetrization techniques, a considerable use of tools from geometric measure theory and from the related theory of BV functions. The necessary material on these topics is collected in the next section. Section 4 is devoted to functionals defined on Sobolev functions, i.e. to the proofs of Theorems 2.1−2.2. The functionals of the generalized normal are the object of Section 5, where Theorems 2.7−2.9 are established. Finally, proofs of Theorems 2.4−2.5, dealing with BV functions, are presented in Section 6.

3 Background

Let Ω be an open subset of $\mathbb{R}^n$. The Sobolev space $W^{1,p}(\Omega)$ is the collection of all functions from $L^p(\Omega)$ which are weakly differentiable in Ω and whose gradient belongs to $L^1(\Omega)$. By $W^{1,p}_{\rm loc}(\Omega)$ we denote the space of those functions which belong to $W^{1,p}(\Omega')$ for every open set $\Omega' \subset\subset \Omega$.

The space of functions of bounded variation in Ω is denoted by $BV(\Omega)$. Recall that a function $u \in L^1(\Omega)$ is said to be of bounded variation in Ω if its distributional gradient Du is a vector-valued Radon measure in Ω whose total variation $|Du|$ is finite in Ω. The space $BV_{\rm loc}(\Omega)$ is defined accordingly.

Pointwise properties of Sobolev and BV functions are suitably described in terms of approximate continuity and differentiability. Given a measurable set E in $\mathbb{R}^n$ and a point $x \in \mathbb{R}^n$, the *density* of E at x is defined by

$$D(E,x) = \lim_{r\to 0} \frac{\mathcal{L}^n(E \cap B_r(x))}{\mathcal{L}^n(B_r(x))}\,,$$

provided that the limit on the right-hand side exists. Here, $B_r(x)$ denotes the ball, centered at x, having radius r. The *essential boundary* of E is the Borel set

$$\partial^M E = \mathbb{R}^n \setminus \{x \in \mathbb{R}^n : \text{either}\, D(E,x) = 0 \ \text{ or } \ D(E,x) = 1\}\,.$$

One has

$$\partial^M(E' \cup E'') \cup \partial^M(E' \cap E'') \subset \partial^M E' \cup \partial^M E'' \tag{3.1}$$

for any measurable sets E' and E'' in $\mathbb{R}^n$.

For any measurable function u in an open set $\Omega \subset \mathbb{R}^n$, the *approximate upper* and *lower limit* of u at a point x are defined as

$$u_+(x) = \inf\{t : D(\{u > t\}, x) = 0\} \quad \text{and} \quad u_-(x) = \sup\{t : D(\{u < t\}, x) = 0\}\,,$$

respectively. The function u is said to be *approximately continuous* at x if $u_-(x)$ and $u_+(x)$ are equal and finite. The common value of $u_-(x)$ and $u_+(x)$ at a point of approximate continuity x is called the *approximate limit* of u at x and is denoted by $\widetilde{u}(x)$. By C_u we denote the Borel set of all points at which u is approximately continuous. The *precise representative* u^* of u is defined as

$$u^*(x) = \begin{cases} \dfrac{u_-(x) + u_+(x)}{2} & \text{if } u_-(x) \text{ and } u_+(x) \text{ are both finite} \\[2ex] 0 & \text{otherwise.} \end{cases}$$

Clearly, $u^* \equiv \widetilde{u}$ in C_u. A locally integrable function u in Ω is said to be *approximately differentiable* at $x \in C_u$ if there exists a vector $\nabla u(x)$ in $\mathbb{R}^n$, called the approximate gradient of u at x, such that

$$\lim_{r\to 0} \frac{1}{\mathcal{L}^n(B_r(x))} \int_{B_r(x)} \frac{|u(z) - \widetilde{u}(x) - \langle \nabla u(x), z - x\rangle|}{r}\, dz = 0\,.$$

The set of all points $x \in C_u$ where u is approximately differentiable is a Borel set denoted by $\mathcal{D}_u$. The subset of $\mathcal{D}_u$ where $\nabla u \neq 0$ and the subset where $\nabla u = 0$ will be denoted by $\mathcal{D}_u^+$ and $\mathcal{D}_u^0$, respectively. If $u \in BV(\Omega)$, then $\mathcal{L}^n(\Omega \setminus \mathcal{D}_u) = 0$. Moreover, on denoting by $D^a u$ and by $D^s u$ the absolutely continuous part and the singular part, respectively, of Du with respect to $\mathcal{L}^n$, we have that ∇u agrees $\mathcal{L}^n$-a.e. with the density of $D^a u$ with respect to $\mathcal{L}^n$, and that $|D^s u|(\mathcal{D}_u) = 0$. Thus, in particular, $W^{1,1}(\Omega)$ can be identified with the subspace of $BV(\Omega)$ of those functions in $BV(\Omega)$ such that $|D^s u|(B) = 0$ for every Borel set $B \subset \Omega$ satisfying $\mathcal{L}^n(B) = 0$.

A measurable subset E of $\mathbb{R}^n$ is said to be of *finite perimeter* in an open set $\Omega \subset \mathbb{R}^n$ if $D\chi_E$ is a vector-valued Radon measure with finite total variation in Ω. The perimeter of E in a Borel subset B of Ω is defined by

$$P(E; B) = |D\chi_E|(B).$$

When $B = \mathbb{R}^n$, we shall simply write $P(E)$ instead of $P(E; \mathbb{R}^n)$. If $\chi_E \in BV_{\mathrm{loc}}(\Omega)$, then we say that E has *locally finite perimeter* in Ω.

A characterization of functions of bounded variation in terms of their subgraphs is provided by the following theorem (see [GMS, Part I: Chap.4, Sect.1.5, Theorem 1 and Chap.4, Sect.2.4, Theorem 4]). Notice that this theorem and Theorem C below are stated in terms of a modified notion of subgraph of a function $u : \Omega \to \mathbb{R}$, defined as

$$\mathcal{S}_u^- = \{(x, t) \in \mathbb{R}^{n+1} : x \in \Omega, \, t < u(x)\}.$$

Theorem A *Let Ω be a bounded open subset of $\mathbb{R}^n$ and let u be a nonnegative function from $L^1(\Omega)$. Then $\mathcal{S}_u^-$ is a set of finite perimeter in $\Omega \times \mathbb{R}_t$ if and only if $u \in BV(\Omega)$. Moreover, in this case,*

$$P(\mathcal{S}_u^-; B \times \mathbb{R}_t) = \int_B \sqrt{1 + |\nabla u|^2}\, dx + |D^s u|(B)$$

for every Borel set $B \subset \Omega$.

Let E be a set of locally finite perimeter in an open subset Ω of $\mathbb{R}^n$. We denote by ν_i^E, $i = 1, \ldots, n$, the derivative of the measure $D_i\chi_E$ with respect to $|D\chi_E|$. The *reduced boundary* $\partial^* E$ of E is the set of all points $x \in \Omega$ such that the vector $\nu^E(x) = (\nu_1^E(x), \ldots, \nu_n^E(x))$ exists and satisfies $|\nu^E(x)| = 1$. The vector $\nu^E(x)$ is called the *generalized inner normal* to E at x. We have that (see [AFP, Theorem 3.59])

$$D\chi_E = \nu^E \mathcal{H}^{n-1} \llcorner \partial^* E. \tag{3.2}$$

Equation (3.2) implies that

$$|D\chi_E| = \mathcal{H}^{n-1} \llcorner \partial^* E \tag{3.3}$$

and that

$$|D_i\chi_E| = |\nu_i^E| \mathcal{H}^{n-1} \llcorner \partial^* E, \qquad i = 1, \ldots, n. \tag{3.4}$$

Since $\partial^* E$ is a *countably* $(n-1)$*-rectifiable set* whose approximate tangent plane at any $x \in \partial^* E$ is orthogonal to $\nu^E(x)$, then by the locality of the approximate tangent plane (see [AFP, Remark 2.87]) one gets

Theorem B *Let Ω be an open subset of $\mathbb{R}^n$ and let E' and E'' be sets of locally finite perimeter in Ω. Then $\nu^{E'}(x) = \pm\nu^{E''}(x)$ for $\mathcal{H}^{n-1}$-a.e. $x \in \partial^* E' \cap \partial^* E''$.*

A result by Federer ([AFP, Theorem 3.61]) tells us that if E is a set of locally finite perimeter in Ω, then

$$\partial^* E \cap \Omega \subset \partial^M E \cap \Omega \quad \text{and} \quad \mathcal{H}^{n-1}\Big((\partial^M E \setminus \partial^* E) \cap \Omega\Big) = 0\,. \tag{3.5}$$

The reduced boundary of level sets plays a role in the *coarea formula* for Sobolev and BV functions. In the general version for BV functions, such a formula tells us that, given any $u \in BV(\Omega)$ and any Borel function $g : \Omega \to [0, +\infty]$,

$$\int_\Omega g\, d|Du| = \int_{-\infty}^{+\infty} dt \int_{\Omega\cap\partial^*\{u>t\}} g\, d\mathcal{H}^{n-1} = \int_{-\infty}^{+\infty} dt \int_{\{u_-\le t\le u_+\}} g\, d\mathcal{H}^{n-1}\,. \tag{3.6}$$

Notice that, if $u \in W^{1,1}(\Omega)$, then

$$u_-(x) = u_+(x) = u^*(x) \qquad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \mathbb{R}^n$$

(see [EG, Theorem 1, Sect.4.8]). Thus, in this case, equation (3.6) can be written as

$$\int_\Omega g|\nabla u|\, dx = \int_{-\infty}^{+\infty} dt \int_{\Omega\cap\partial^*\{u>t\}} g\, d\mathcal{H}^{n-1} = \int_{-\infty}^{+\infty} dt \int_{\{u^*=t\}} g\, d\mathcal{H}^{n-1}\,. \tag{3.7}$$

The approximate gradient of a BV function and the generalized inner normal to its subgraph are related by the next result (see [GMS, Part I: Chap.4, Sect.1.5, Theorems 4 and 5]).

Theorem C *Let Ω be an open subset of $\mathbb{R}^n$ and let $u \in BV(\Omega)$. Then*

$$\nu^{\mathcal{S}_u^-}(x,t) = \left(\frac{\nabla_1 u(x)}{\sqrt{1+|\nabla u(x)|^2}}, \dots, \frac{\nabla_n u(x)}{\sqrt{1+|\nabla u(x)|^2}}, \frac{-1}{\sqrt{1+|\nabla u(x)|^2}}\right) \tag{3.8}$$

for $\mathcal{H}^n$-a.e. $(x,t) \in \partial^*\mathcal{S}_u^- \cap (\mathcal{D}_u \times \mathbb{R}_t)$ and

$$\nu_{n+1}^{\mathcal{S}_u^-}(x,t) = 0 \qquad \text{for } \mathcal{H}^n\text{-a.e. } (x,t) \in \partial^*\mathcal{S}_u^- \cap [(\Omega \setminus \mathcal{D}_u) \times \mathbb{R}_t]\,.$$

In particular, if $u \in W^{1,1}(\Omega)$, then (3.8) holds for $\mathcal{H}^n$-a.e. $(x,t) \in \partial^*\mathcal{S}_u^- \cap (\Omega \times \mathbb{R}_t)$.

Observe that, if Ω is any open set and $u \in BV(\Omega)$, then, by Theorem A, the set $\mathcal{S}_u^-$ is of locally finite perimeter in $\Omega \times \mathbb{R}_t$. Therefore, also $\mathcal{S}_u$ is a set of locally finite perimeter in $\Omega \times \mathbb{R}_t$ and

$$\partial^*\mathcal{S}_u \cap (\Omega \times \mathbb{R}_t^+) = \partial^*\mathcal{S}_u^- \cap (\Omega \times \mathbb{R}_t^+) \qquad \text{and} \qquad \nu^{\mathcal{S}_u} \equiv \nu^{\mathcal{S}_u^-} \text{ on } \partial^*\mathcal{S}_u \cap (\Omega \times \mathbb{R}_t^+)\,. \tag{3.9}$$

A special case of the coarea formula for rectifiable sets ([AFP, Theorem 2.93 and Remark 2.94]) states that if E is a set of locally finite perimeter in Ω and g is any Borel function from Ω into $[0, +\infty]$, then

$$\int_{\partial^*E\cap\Omega} g(x)|\nu_n^E(x)|\, d\mathcal{H}^{n-1}(x) = \int_{\pi_{n-1}(\Omega)} dx' \int_{(\partial^*E\cap\Omega)_{x'}} g(x',y)\, d\mathcal{H}^0(y)\,. \tag{3.10}$$

We conclude this section with two theorems concerning one-dimensional sections of sets of finite perimeter and one-dimensional restrictions of BV functions, respectively. The first result is due to Vol'pert ([V]). In the present form, it can be easily deduced from [AFP, Theorem 3.108].

Theorem D *Let E be a set of finite perimeter in Ω. Then, for $\mathcal{L}^{n-1}$-a.e. $x' \in \pi_{n-1}(\Omega)$,*

$$E_{x'} \text{ has finite perimeter in } \Omega_{x'}\,; \tag{3.11}$$

(3.12) $$(\partial^* E \cap \Omega)_{x'} = \partial^*(E_{x'}) \cap \Omega_{x'}\,;$$

(3.13) $$\nu_n^E(x',y) \neq 0 \textit{ for every } y \textit{ such that } (x',y) \in \partial^* E \cap \Omega\,;$$

(3.14) $$\begin{cases} \lim\limits_{z\to y^+} \chi_E^*(x',z) = 1, \quad \lim\limits_{z\to y^-} \chi_E^*(x',z) = 0 & \textit{if } \nu_n^E(x',y) > 0 \\ \lim\limits_{z\to y^+} \chi_E^*(x',z) = 0, \quad \lim\limits_{z\to y^-} \chi_E^*(x',z) = 1 & \textit{if } \nu_n^E(x',y) < 0\,. \end{cases}$$

In particular, there exists a Borel set $G_E \subset \pi_{n-1}(E)^+ \cap \pi_{n-1}(\Omega)$ *satisfying* $\mathcal{L}^{n-1}(\pi_{n-1}(E)^+ \cap \pi_{n-1}(\Omega) \setminus G_E) = 0$ *and such that (3.11)–(3.14) hold for every* $x' \in G_E$*.*

Theorem E *Let* $u \in BV(\Omega)$ *and set* $v^{x'}(y) = u(x',y)$ *for every* $x' \in \pi_{n-1}(\Omega)$ *and* $y \in \Omega_{x'}$*. Then, for* $\mathcal{L}^{n-1}$*-a.e.* $x' \in \pi_{n-1}(\Omega)$*, the function* $v^{x'} \in BV(\Omega_{x'})$ *and ,*

(3.15) $$(C_u)_{x'} \subset C_{v^{x'}}\,,$$

(3.16) $$\frac{dv^{x'}}{dy}(y) = \nabla_y u(x',y) \qquad \textit{for } \mathcal{L}^1\textit{-a.e. } y \in \Omega_{x'}\,,$$

(3.17) $$|D^s v^{x'}|\,((\mathcal{D}_u)_{x'}) = 0\,.$$

In particular, for $\mathcal{L}^{n-1}$*-a.e.* $x' \in \pi_{n-1}(\Omega)$*, the set* $\{y : v^{x'}(y) > t\}$ *is of finite perimeter in* $\Omega_{x'}$ *for* $\mathcal{L}^1$*-a.e.* $t \in \mathbb{R}$*, and* $\mathcal{L}^1(\{y : |v^{x'}(y)| > t\}) < \infty$ *for every* $t > 0$*.*
Moreover, if $u \in W^{1,1}(\Omega)$*, then, for* $\mathcal{L}^{n-1}$*-a.e.* $x' \in \pi_{n-1}(\Omega)$*, the function* $v^{x'} \in W^{1,1}(\Omega_{x'})$ *and*

(3.18) $$u^*(x',y) = (v^{x'})^*(y) \qquad \textit{for every } y \in \Omega_{x'}\,.$$

Proof. We prove only (3.15), (3.17) and (3.18), the other assertions being straightforward consequences of [AFP, Theorem 3.107]. Let us denote by J_u the *jump set* of u (see e.g. [AFP, Definition 3.67]). By [AFP, Proposition 3.69], $J_u \subset \Omega \setminus C_u$. Moreover, owing to [AFP, Theorem 3.108], the function $u^*(x',\cdot)$ is continuous in $\Omega_{x'} \setminus (J_u)_{x'}$ for $\mathcal{L}^{n-1}$-a.e. $x' \in \pi_{n-1}(\Omega)$. Therefore, for any such x', we have that $(C_u)_{x'} \subset \Omega_{x'} \setminus (J_u)_{x'} \subset C_{v^{x'}}$, whence (3.15) follows.
Consider now (3.17). By [AFP, equation (3.104)] we have that

(3.19) $$\int_\Omega \varphi(x',y)\, dD_y^s u(x',y) = \int_{\pi_{n-1}(\Omega)} dx' \int_{\Omega_{x'}} \varphi(x',y)\, dD^s v^{x'}(y)$$

for any bounded Borel function φ in Ω. Let us denote by $\mathcal{C}$ a countable dense subset of $C_0(\mathbb{R})$. Fixed any $\psi \in \mathcal{C}$, apply (3.19) with $\varphi(x',y) = g(x')\psi(y)\chi_{\mathcal{D}_u}(x',y)$, where g is any function from $C_0(\pi_{n-1}(\Omega))$. Owing to the arbitrariness of g, we get that there exists a measurable subset N_ψ of $\pi_{n-1}(\Omega)$ satisfying $\mathcal{L}^{n-1}(\pi_{n-1}(\Omega) \setminus N_\psi) = 0$ and such that

$$\int_{\Omega_{x'}} \psi(y)\chi_{\mathcal{D}_u}(x',y)\, dD^s v^{x'}(y) = 0 \qquad \text{for every } x' \in N_\psi\,.$$

Hence, $|D^s v^{x'}|((\mathcal{D}_u)_{x'}) = 0$ for every $x' \in \cap_{\psi\in\mathcal{C}} N_\psi$, and (3.17) holds.
Finally, recall that if $u \in W^{1,1}(\Omega)$, then $u^*(x',\cdot)$ is absolutely continuous for $\mathcal{L}^{n-1}$-a.e. $x' \in \pi_{n-1}(\Omega)$ (see e.g. (see [EG])). On the other hand, $v^{x'} \in W^{1,1}(\Omega_{x'})$ for $\mathcal{L}^{n-1}$-a.e. $x' \in \pi_{n-1}(\Omega)$, and thus $(v^{x'})^*$ is also absolutely continuous. Since, for $\mathcal{L}^{n-1}$-a.e. $x' \in \pi_{n-1}(\Omega)$, one has $v^{x'}(y) = u^*(x',y)$ for $\mathcal{L}^1$-a.e. $y \in \Omega_{x'}$, equation (3.18) immediately follows. $\square$

4 Functionals of Sobolev functions

Our proof of Theorem 2.2 consists of several steps, one of which provides a proof of Theorem 2.1. The first step is contained in the following lemma, and yields formulas for the approximate gradient of the distribution function of a Sobolev function.

Lemma 4.1 *Let Ω be an open subset of $\mathbb{R}^n$ satisfying (2.10), and let u be a nonnegative function from $W^{1,1}_{0,y}(\Omega)$. Then $\mu_u \in BV(\omega \times \mathbb{R}^+_t)$ for every open set $\omega \subset\subset \pi_{n-1}(\Omega)$, and, for $\mathcal{L}^{n-1}$-a.e. $x' \in \pi_{n-1}(\mathcal{S}_u)^+$,*

$$\nabla_t \mu_u(x',t) = -\int_{\partial^*\{y: u(x',y)>t\}} \frac{1}{|\nabla_y u|}\, d\mathcal{H}^0 , \tag{4.1}$$

$$\nabla_i \mu_u(x',t) = \int_{\partial^*\{y: u(x',y)>t\}} \frac{\nabla_i u}{|\nabla_y u|}\, d\mathcal{H}^0 \qquad i=1,\dots,n-1, \tag{4.2}$$

for $\mathcal{L}^1$-a.e. $t \in (0, M(x'))$.

Remark 4.2 The set $\partial^*\{y : u(x',y) > t\}$ can be replaced by $\{y : u^*(x',y) = t\}$ in formulas (4.1)-(4.2). Actually,

$$\begin{aligned}\partial^*\{y:\ u(x',y)>t\} &= \partial^*\{y:\ u(x',\cdot)^*(y)>t\} = \partial^M\{y: u(x',\cdot)^*(y)>t\}\\ &= \{y: (u(x',\cdot)^*)_-(y)\le t\le (u(x',\cdot)^*)_+(y)\} = \{y: u(x',\cdot)^*(y)=t\} = \{y: u^*(x',y)=t\},\end{aligned}$$

for $\mathcal{L}^1$-a.e. $t > 0$, where the second equality is due to (3.5), the third (which holds for $\mathcal{L}^1$-a.e. $t > 0$) is a consequence of (2.13) and (2.20) of [CF], the fourth holds because precise representatives of one–dimensional Sobolev functions are continuous, and the fifth follows from (3.18).

Let us note that, in the special case where u is smooth enough and satisfies (2.8), formulas (4.1)-(4.2) can be derived from [FM, Theorem 2.2].

Proof of Lemma 4.1. Let L be a positive number such that $\Omega \subset \mathbb{R}^{n-1} \times (-L, L)$ and let ω be an open set such that $\omega \subset\subset \pi_{n-1}(\Omega)$. For simplicity of notation, throughout the proof the continuation by 0 of u outside Ω will be still denoted by u. Hence, $u \in W^{1,1}(\omega \times \mathbb{R}_y)$ and $u(x',y) = 0$ if $|y| > L$. Let $\varphi \in C^1_0(\omega \times \mathbb{R}^+_t)$. Fubini's theorem and a standard rule on the differentiation of integrals ensure that

$$\begin{aligned}\int_{\omega\times\mathbb{R}^+_t} \nabla_i\varphi(x',t)\mu_u(x',t)\,dx'dt &= \int_{\omega\times\mathbb{R}_y\times\mathbb{R}^+_t} \nabla_i\varphi(x',t)\chi_{\mathcal{S}_u}(x',y,t)\,dx'dydt\\ &= \int_{\omega\times\mathbb{R}_y} dx'dy \int_0^{u(x',y)} \nabla_i\varphi(x',t)\,dt = \int_{\omega\times(-L,L)} \nabla_i\left(\int_0^{u(x',y)}\varphi(x',t)\,dt\right)dx'dy\\ &\quad -\int_{\omega\times(-L,L)} \varphi(x',u(x',y))\nabla_i u(x',y)\,dx'dy, \qquad i=1,\dots,n-1.\end{aligned} \tag{4.3}$$

The last but one integral over $\omega \times (-L, L)$ is easily seen to vanish. On the other hand,

$$\begin{aligned}&\int_{\partial^*\mathcal{S}_u\cap(\omega\times(-L,L)\times\mathbb{R}^+_t)} \varphi(x',t)\nabla_i u(x',y)|\nu_t^{\mathcal{S}_u}(x',y,t)|\,d\mathcal{H}^n\\ &= \int_{\omega\times(-L,L)} dx'dy \int_{(\partial^*\mathcal{S}_u)_{x',y}\cap\mathbb{R}^+_t} \varphi(x',t)\nabla_i u(x',y)\,d\mathcal{H}^0(t)\\ &= \int_{\omega\times(-L,L)} \varphi(x',u(x',y))\nabla_i u(x',y)\,dx'dy, \qquad i=1,\dots,n-1,\end{aligned} \tag{4.4}$$

where the first equality follows from an application of the coarea formula (3.10), and the second holds since, by Theorem D, $(\partial^*\mathcal{S}_u)_{x',y} \cap \mathbb{R}_t^+ = \partial^*(\mathcal{S}_u)_{x',y} \cap \mathbb{R}_t^+ = \partial^*(0, u(x',y)) \cap \mathbb{R}_t^+$ for $\mathcal{L}^n$-a.e. $(x',y) \in \omega \times (-L,L)$. Owing to Theorem C and to (3.9),

$$\nu^{\mathcal{S}_u}(x',y,t) = \left(\frac{\nabla_1 u(x',y)}{\sqrt{1+|\nabla u|^2}}, \ldots, \frac{\nabla_{n-1} u(x',y)}{\sqrt{1+|\nabla u|^2}}, \frac{\nabla_y u(x',y)}{\sqrt{1+|\nabla u|^2}}, \frac{-1}{\sqrt{1+|\nabla u|^2}}\right) \tag{4.5}$$

for $\mathcal{H}^n$-a.e. $(x',y,t) \in \partial^*\mathcal{S}_u \cap (\omega \times (-L,L) \times \mathbb{R}_t^+)$.
Combining (4.3)-(4.5) yields that, for $i = 1, \ldots, n-1$,

$$\int_{\omega\times\mathbb{R}_t^+} \nabla_i \varphi(x',t)\mu_u(x',t)\, dx'dt = -\int_{\partial^*\mathcal{S}_u \cap(\omega\times(-L,L)\times\mathbb{R}_t^+)} \varphi(x',t)\frac{\nabla_i u(x',y)}{\sqrt{1+|\nabla u|^2}}\, d\mathcal{H}^n\,. \tag{4.6}$$

Equation (4.6) easily implies that the distributional derivative $D_i\mu_u$ is a finite Radon measure in $\omega\times\mathbb{R}_t^+$. A similar argument shows that the same conclusion is also true for the distributional derivative $D_t\mu_u$. Moreover, since $\int_{\omega\times\mathbb{R}_t^+}\mu_u(x',t)dx'dt = \int_{\omega\times\mathbb{R}_y} u(x',y)dx'dy < \infty$, then $\mu_u \in L^1(\omega\times\mathbb{R}_t^+)$. Thus, $\mu_u \in BV(\omega\times\mathbb{R}_y)$. Equation (4.6) also entails that

$$\int_{\omega\times\mathbb{R}_t^+} \varphi(x',t)\, dD_i\mu_u = \int_{\partial^*\mathcal{S}_u\cap(\omega\times(-L,L)\times\mathbb{R}_t^+)} \varphi(x',t)\frac{\nabla_i u(x',y)}{\sqrt{1+|\nabla u|^2}}\, d\mathcal{H}^n \tag{4.7}$$

for every $\varphi \in C_0^1(\omega\times\mathbb{R}_t^+)$, and hence, by density, for every $\varphi \in C(\omega\times\mathbb{R}_t^+)$ as well. We want now to show that (4.7) continues to hold for every bounded Borel function in $\omega\times\mathbb{R}_t^+$. To this purpose, let us define the Borel measure m as

$$m(B) = |D_i\mu_u|(B) + \mathcal{H}^n(\partial^*\mathcal{S}_u \cap (B\times\mathbb{R}_y))$$

at any Borel subset B of $\omega\times\mathbb{R}_t^+$. Let φ be any bounded Borel function in $\omega\times\mathbb{R}_t^+$. Then, by Lusin's theorem, for any $\varepsilon > 0$ there exists $\varphi_\varepsilon \in C(\omega\times\mathbb{R}_t^+)$ satisfying $\|\varphi_\varepsilon\|_{L^\infty} \le \|\varphi\|_{L^\infty}$ and $m(\{(x',t) : \varphi_\varepsilon(x',t) \ne \varphi(x',t)\}) < \varepsilon$. Since (4.7) holds for φ_ε, it is easily seen that the absolute value of the difference of the left-hand and right-hand side of (4.7) for such a φ does not exceed $4\varepsilon\|\varphi\|_{L^\infty}$. Thus, (4.7) holds also for φ, thanks to the arbitrariness of ε.
Now, let $G_{\mathcal{S}_u}$ be the Borel set in $\pi_{n-1,n+1}(\mathcal{S}_u)^+ \cap (\omega\times\mathbb{R}_t^+)$, given by Theorem D, for which (3.11)-(3.14) hold and let $g \in C_0(\omega\times\mathbb{R}_t^+)$. On applying (4.7) with $\varphi = \chi_{G_{\mathcal{S}_u}} g$, making use of (4.5), and applying the coarea formula (3.10) we infer that

$$\begin{aligned}
\int_{\omega\times\mathbb{R}_t^+} g(x',t)\chi_{G_{\mathcal{S}_u}}(x',t)\, dD_i\mu_u &= \int_{\partial^*\mathcal{S}_u\cap(\omega\times\mathbb{R}_y\times\mathbb{R}_t^+)} g(x',t)\chi_{G_{\mathcal{S}_u}}(x',t)\frac{\nabla_i u(x',y)}{\sqrt{1+|\nabla u|^2}}\, d\mathcal{H}^n \\
&= \int_{\partial^*\mathcal{S}_u\cap(\omega\times\mathbb{R}_y\times\mathbb{R}_t^+)} g(x',t)\chi_{G_{\mathcal{S}_u}}(x',t)\frac{\nabla_i u(x',y)}{|\nabla_y u(x',y)|}|\nu_y^{\mathcal{S}_u}(x',y,t)|\, d\mathcal{H}^n \\
&= \int_{\omega\times\mathbb{R}_t^+} g(x',t)\chi_{G_{\mathcal{S}_u}}(x',t)\, dx'dt \int_{(\partial^*\mathcal{S}_u)_{x',t}} \frac{\nabla_i u(x',y)}{|\nabla_y u(x',y)|}\, d\mathcal{H}^0(y)\,.
\end{aligned}$$

Owing to the arbitrariness of g, we may conclude that $D_i\mu_u \llcorner G_{\mathcal{S}_u}$ is absolutely continuous with respect to $\mathcal{L}^n$ and agrees with $\chi_{G_{\mathcal{S}_u}}\left(\int_{(\partial^*\mathcal{S}_u)_{x',t}} \frac{\nabla_i u}{|\nabla_y u|}\, d\mathcal{H}^0\right)\mathcal{L}^n$. Consequently, by (3.12), equation (4.2) holds for $\mathcal{L}^n$-a.e. $(x',t) \in \pi_{n-1,n+1}(\mathcal{S}_u)^+ \cap (\omega\times\mathbb{R}_t^+)$. It is easy to verify that

$$\pi_{n-1,n+1}(\mathcal{S}_u)^+ \quad \text{is equivalent to} \quad \bigcup_{x'\in\pi_{n-1}(\mathcal{S}_u)^+} \{x'\}\times(0, M(x'))\,. \tag{4.8}$$

Hence, for $\mathcal{L}^{n-1}$-a.e. $x' \in \pi_{n-1}(\mathcal{S}_u)^+$, equation (4.2) holds for $\mathcal{L}^1$-a.e. $t \in (0, M(x'))$. The proof of (4.1) is analogous. □

Lemma 4.3 *Let Ω be an open subset of $\mathbb{R}^n$ and let $p \geq 1$. If $u \in W^{1,p}_{0,y}(\Omega)$, then $u^s \in W^{1,p}_{0,y}(\Omega^s)$.*

Proof. Fixed any open set $\omega \subset\subset \pi_{n-1}(\Omega)$, let φ be any function from $C^1(\mathbb{R}^{n-1})$ which is compactly supported in $\pi_{n-1}(\Omega)$ and satisfies $\varphi = 1$ in ω. Since the function $v = \varphi u_0$ belongs to $W^{1,p}(\mathbb{R}^n)$, then $v^s \in W^{1,p}(\mathbb{R}^n)$ (see [Bur]). However, $v^s(x', y) = u^s(x', y)$ for every $x' \in \omega$ and $y \in \mathbb{R}$. Hence, $u^s \in W^{1,p}(\omega \times \mathbb{R}_y)$, and the conclusion follows. □

Remark 4.4 Assume that Ω and u are as in Lemma 4.1. Then, by Lemma 4.3, the function $u^s \in W^{1,1}_{0,y}(\Omega)$. Moreover, by (2.4), $\pi_{n-1}(\mathcal{S}_u)^+$ is equivalent to $\pi_{n-1}(\mathcal{S}_{u^s})^+$. Since $\mu_{u^s} = \mu_u$, then an application of Lemma 4.1 to u^s yields, for $\mathcal{L}^{n-1}$-a.e. $x' \in \pi_{n-1}(\mathcal{S}_u)^+$,

$$\nabla_t \mu_u(x', t) = -\frac{2}{|\nabla_y u^s|}_{|\partial^*\{y: u^s(x',y) > t\}} \tag{4.9}$$

and

$$\nabla_i \mu_u(x', t) = 2\frac{\nabla_i u^s}{|\nabla_y u^s|}_{|\partial^*\{y: u^s(x',y) > t\}} \tag{4.10}$$

for $\mathcal{L}^1$-a.e. $t \in (0, M(x'))$.

Formulas (4.1)-(4.2) are a key tool in our proof of Theorem 2.1. In particular, unlike other available proofs of Pólya–Szegö type inequalities for Steiner symmetrization, they enable us to avoid approximation arguments, at least when (2.8) and (2.10) are in force, and thus to derive information about the equality case in (2.7).

Before we go into the proof of Theorem 2.1, let us establish Proposition 2.3.

Proof of Proposition 2.3. We have, for $\mathcal{L}^{n-1}$-a.e. $x' \in \mathbb{R}^{n-1}$,

$$\begin{aligned} \mu_u(x', t) \;=\; & \mathcal{L}^1\left(\{y : u(x', y) = M(x')\}\right) + \mathcal{L}^1\left(\{y : \nabla_y u(x', y) = 0,\, t < u(x', y) < M(x')\}\right) \\ & + \mathcal{L}^1\left(\{y : \nabla_y u(x', y) \neq 0,\, t < u(x', y) < M(x')\}\right) \end{aligned} \tag{4.11}$$

for every $t \in (0, M(x'))$. Now, if $v^{x'}$ is defined as in Theorem E, then, for $\mathcal{L}^{n-1}$-a.e. $x' \in \mathbb{R}^{n-1}$,

$$\begin{aligned} & \mathcal{L}^1\left(\{y : \nabla_y u(x', y) \neq 0,\, t < u(x', y) < M(x')\}\right) = \int_{\Omega_{x'}} \chi_{\{(v^{x'})^* > t\}} \chi_{\{\nabla_y u(x', \cdot) \neq 0\}} \frac{\left|\frac{d}{dy} v^{x'}(y)\right|}{|\nabla_y u(x', y)|} dy \\ & = \int_t^{M(x')} d\tau \int_{\{(v^{x'})^* = \tau\}} \frac{1}{|\nabla_y u(x', y)|} d\mathcal{H}^0(y) = \int_t^{M(x')} d\tau \int_{\partial^*\{(v^{x'})^* > \tau\}} \frac{1}{|\nabla_y u(x', y)|} d\mathcal{H}^0(y) \\ & = \int_t^{M(x')} d\tau \int_{\partial^*\{u(x', \cdot) > \tau\}} \frac{1}{|\nabla_y u(x', y)|} d\mathcal{H}^0(y) \end{aligned} \tag{4.12}$$

for every $t \in (0, M(x'))$, where the first equality holds thanks to (3.16), and the second and third follow from an application of the coarea formula (3.7). Hence, for $\mathcal{L}^{n-1}$-a.e. $x' \in \mathbb{R}^{n-1}$,

$$\begin{aligned} \mu_u(x', t) \;=\; & \mathcal{L}^1\left(\{y : u(x', y) = M(x')\}\right) + \mathcal{L}^1\left(\{y : \nabla_y u(x', y) = 0,\, t < u(x', y) < M(x')\}\right) \\ & + \int_t^{M(x')} d\tau \int_{\partial^*\{u(x', \cdot) > \tau\}} \frac{1}{|\nabla_y u(x', y)|} d\mathcal{H}^0(y) \end{aligned} \tag{4.13}$$

for every $t \in (0, M(x'))$. Since $\mu_u = \mu_{u^s}$, then, analogously, for $\mathcal{L}^{n-1}$-a.e. $x' \in \mathbb{R}^{n-1}$,

$$\mu_u(x',t) = \mathcal{L}^1(\{y:\ u^s(x',y) = M(x')\}) + \mathcal{L}^1(\{y:\ \nabla_y u^s(x',y) = 0,\, t < u^s(x',y) < M(x')\}) + \int_t^{M(x')} d\tau \int_{\partial^*\{u^s(x',\cdot)>\tau\}} \frac{1}{|\nabla_y u^s(x',y)|} d\mathcal{H}^0(y) \tag{4.14}$$

for every $t \in (0, M(x'))$. By (3.16) and Lemma 2.4 of [CF], for $\mathcal{L}^{n-1}$-a.e. $x' \in \mathbb{R}^{n-1}$,

$$\frac{d}{dt}\mathcal{L}^1(\{y:\ \nabla_y u(x',y) = 0,\ u(x',y) > t\}) = \frac{d}{dt}\mathcal{L}^1(\{y:\ \nabla_y u^s(x',y) = 0,\ u^s(x',y) > t\})$$

for $\mathcal{L}^1$-a.e. $t > 0$. Consequently,

$$\int_t^{M(x')} d\tau \int_{\partial^*\{y:u(x',y)>\tau\}} \frac{1}{|\nabla_y u(x',y)|} d\mathcal{H}^0(y) = \int_t^{M(x')} d\tau \int_{\partial^*\{y:u^s(x',y)>\tau\}} \frac{1}{|\nabla_y u^s(x',y)|} d\mathcal{H}^0(y) \tag{4.15}$$

for $\mathcal{L}^{n-1}$-a.e. $x' \in \mathbb{R}^{n-1}$ and for $\mathcal{L}^1$-a.e. $t \in (0, M(x'))$. Since $\mathcal{L}^1(\{y:\ u(x',y) = M(x')\}) = \mathcal{L}^1(\{y:\ u^s(x',y) = M(x')\})$, equations (4.13)-(4.14) yield the conclusion. □

Proof of Theorem 2.1. We shall prove a stronger inequality than (2.7), namely that

$$\int_{B\times\mathbb{R}_y} f(\nabla u^s)\, dx \le \int_{B\times\mathbb{R}_y} f(\nabla u)\, dx \tag{4.16}$$

for every Borel set $B \subset \pi_{n-1}(\Omega)$. Here, and throughout the proof, we denote the extension u_0 simply by u.

Step 1. We assume here that Ω satisfies (2.10) and that u is a nonnegative function from $W^{1,1}_{0,y}(\Omega)$ such that

$$\mathcal{L}^1(\{y:\ \nabla_y u(x',y) = 0\} \cap \{y:\ 0 < u(x',y) < M(x')\}) = 0 \tag{4.17}$$

for $\mathcal{L}^{n-1}$-a.e. $x' \in \pi_{n-1}(\Omega)$ such that $M(x') > 0$. By Proposition 2.3, equation (4.17) is fulfilled with u replaced by u^s as well. Hence, by Theorem E and the coarea formula (3.7), we have that

$$\int_{\{y:u^s(x',y)>0\}} f(\nabla u^s(x',y))\, dy = \int_0^{M(x')} dt \int_{\partial^*\{y:u^s(x',y)>t\}} \frac{1}{|\nabla_y u^s|} f(\nabla u^s)\, d\mathcal{H}^0 \tag{4.18}$$

for $\mathcal{L}^{n-1}$-a.e. $x' \in \pi_{n-1}(\Omega)$ such that $M(x') > 0$. Thus, assumption (2.6) and formulas (4.9)-(4.10) ensure that, for any such x',

$$\begin{aligned}
&\int_{\partial^*\{y:u^s(x',y)>t\}} \frac{1}{|\nabla_y u^s|} f(\nabla_1 u^s, \ldots, \nabla_{n-1} u^s, \nabla_y u^s)\, d\mathcal{H}^0 \\
&= \int_{\partial^*\{y:u^s(x',y)>t\}} \frac{1}{|\nabla_y u^s|} f(\nabla_1 u^s, \ldots, \nabla_{n-1} u^s, |\nabla_y u^s|)\, d\mathcal{H}^0 \\
&= -\nabla_t \mu_u(x',t) f\Big(\frac{\nabla_1 \mu_u(x',t)}{-\nabla_t \mu_u(x',t)}, \ldots, \frac{\nabla_{n-1}\mu_u(x',t)}{-\nabla_t \mu_u(x',t)}, \frac{2}{-\nabla_t \mu_u(x',t)}\Big)
\end{aligned} \tag{4.19}$$

for $\mathcal{L}^1$-a.e. $t \in (0, M(x'))$. For $\mathcal{L}^{n-1}$-a.e. $x' \in \pi_{n-1}(\Omega)$, the set $\{y : u(x',y) > t\}$ is of finite perimeter in $\mathbb{R}$ for $\mathcal{L}^1$-a.e. $t > 0$, and $\mathcal{L}^1(\{y : u(x',y) > t\}) < \infty$ for $t > 0$ (see Theorem E). Then, by the isoperimetric inequality in $\mathbb{R}$,

$$2 \le \mathcal{H}^0(\partial^*\{y:\ u(x',y) > t\}) = \int_{\partial^*\{y:u(x',y)>t\}} d\mathcal{H}^0 \tag{4.20}$$

for $\mathcal{L}^{n-1}$-a.e. $x' \in \pi_{n-1}(\Omega)$ such that $M(x') > 0$ and for $\mathcal{L}^1$-a.e. $t \in (0, M(x'))$. Assumption (2.6) implies that $f(\xi_1, \ldots, \xi_{n-1}, \cdot)$ is non decreasing in $[0, +\infty)$ for every $(\xi_1, \ldots, \xi_{n-1}) \in \mathbb{R}^{n-1}$. Consequently, by (4.20) and Lemma 4.1, we have that, for $\mathcal{L}^{n-1}$-a.e. $x' \in \pi_{n-1}(\Omega)$ such that $M(x') > 0$,

$$
\begin{aligned}
(4.21)\qquad & -\nabla_t \mu_u(x',t) f\Big(\frac{\nabla_1 \mu_u(x',t)}{-\nabla_t \mu_u(x',t)}, \ldots, \frac{\nabla_{n-1} \mu_u(x',t)}{-\nabla_t \mu_u(x',t)}, \frac{2}{-\nabla_t \mu_u(x',t)}\Big) \\
& \le f\left(\frac{\displaystyle\int_{\partial^*\{\ldots\}} \frac{\nabla_1 u}{|\nabla u_y|} d\mathcal{H}^0}{\displaystyle\int_{\partial^*\{\ldots\}} \frac{d\mathcal{H}^0}{|\nabla u_y|}}, \ldots, \frac{\displaystyle\int_{\partial^*\{\ldots\}} \frac{\nabla_{n-1} u}{|\nabla u_y|} d\mathcal{H}^0}{\displaystyle\int_{\partial^*\{\ldots\}} \frac{d\mathcal{H}^0}{|\nabla u_y|}}, \frac{\displaystyle\int_{\partial^*\{\ldots\}} d\mathcal{H}^0}{\displaystyle\int_{\partial^*\{\ldots\}} \frac{d\mathcal{H}^0}{|\nabla u_y|}} \right) \cdot \int_{\partial^*\{\ldots\}} \frac{d\mathcal{H}^0}{|\nabla u_y|}
\end{aligned}
$$

for $\mathcal{L}^1$-a.e. $t \in (0, M(x'))$. Here, $\partial^*\{\ldots\}$ is a shorthand for $\partial^*\{y : u(x', y) > t\}$. Since f is a convex function, then Jensen's inequality and assumption (2.6) ensure that the last expression

$$
(4.22)\qquad \le \int_{\partial^*\{y:u(x',y)>t\}} \frac{1}{|\nabla_y u|} f(\nabla_1 u, \ldots, \nabla_{n-1} u, \nabla_y u)\, d\mathcal{H}^0 .
$$

Combining (4.19), (4.21) and (4.22) leads to

$$
\begin{aligned}
(4.23)\qquad & \int_{\partial^*\{y:u^s(x',y)>t\}} \frac{1}{|\nabla_y u^s|} f(\nabla_1 u^s, \ldots, \nabla_{n-1} u^s, \nabla_y u^s)\, d\mathcal{H}^0 \\
& \le \int_{\partial^*\{y:u(x',y)>t\}} \frac{1}{|\nabla_y u|} f(\nabla_1 u, \ldots, \nabla_{n-1} u, \nabla_y u)\, d\mathcal{H}^0
\end{aligned}
$$

for $\mathcal{L}^{n-1}$-a.e. $x' \in \pi_{n-1}(\Omega)$ such that $M(x') > 0$ and for $\mathcal{L}^1$-a.e. $t \in (0, M(x'))$. Now, given any Borel subset B of $\pi_{n-1}(\Omega)$, set $\widetilde{B} = B \cap \{x' : M(x') > 0\}$. Since $\nabla u^s = 0 = \nabla u$ $\mathcal{L}^{n-1}$-a.e. in $(B \setminus \widetilde{B}) \times \mathbb{R}_y$, on integrating (4.23) first with respect to t over $(0, M(x'))$, and then with respect to x' over $\widetilde{B}$, one gets

$$
\begin{aligned}
(4.24)\qquad \int_{B\times\mathbb{R}_y} f(\nabla u^s)\, dx'dy &= \int_{\widetilde{B}} dx' \int_{\{y:u^s(x',y)>0\}} f(\nabla u^s)\, dy \\
&= \int_{\widetilde{B}} dx' \int_0^{M(x')} dt \int_{\partial^*\{y:u^s(x',y)>t\}} \frac{1}{|\nabla_y u^s|} f(\nabla u^s) d\mathcal{H}^0 \\
&\le \int_{\widetilde{B}} dx' \int_0^{M(x')} dt \int_{\partial^*\{y:u(x',y)>t\}} \frac{1}{|\nabla_y u|} f(\nabla u) d\mathcal{H}^0 = \int_{B\times\mathbb{R}_y} f(\nabla u) dx'dy.
\end{aligned}
$$

Notice that here we have made use of (4.18) and of an analogous equality for u.

Step 2. Let u be any nonnegative function from $W^{1,1}_{0,y}(\Omega)$ and let ω be any open set such that $\omega \subset\subset \pi_{n-1}(\Omega)$. Lemma 4.5 below ensures that there exists a sequence $\{u_h\}$ of nonnegative Lipschitz functions, with compact support in $\mathbb{R}^n$, satisfying (4.17) and converging strongly to u in $W^{1,1}(\omega \times \mathbb{R}_y)$. Assume, for a moment, that f satisfies (2.14). Then f is globally Lipschitz continuous, and hence $f(\nabla u_h)$ converges to $f(\nabla u)$ in $L^1(\omega \times \mathbb{R}_y)$. On the other hand, since Steiner rearrangement is continuous in L^1 (see e.g. [K1, pag. 23]), then u_h^s converges to u^s in $L^1(\omega \times \mathbb{R}_y)$. Thus, by lower semicontinuity (see [But, Theorem 4.2.8]) and by (4.24), we get

$$
\int_{\omega\times\mathbb{R}_y} f(\nabla u^s)\, dx \le \liminf_{h\to\infty} \int_{\omega\times\mathbb{R}_y} f(\nabla u_h^s)\, dx \le \liminf_{h\to\infty} \int_{\omega\times\mathbb{R}_y} f(\nabla u_h)\, dx = \int_{\omega\times\mathbb{R}_y} f(\nabla u)\, dx .
$$

Hence (4.16) follows. Let us now remove assumption (2.14). Since f is nonnegative and convex, there exist sequences $\{a_j\}$ of vectors in $a_j \in \mathbb{R}^n$ and $\{b_j\}$ of real numbers b_j such that

$$
(4.25)\qquad f(\xi) = \sup_{j\in\mathbb{N}} \{\langle a_j, \xi\rangle + b_j\} = \sup_{j\in\mathbb{N}} \{(\langle a_j, \xi\rangle + b_j)^+\} \qquad \text{for every } \xi \in \mathbb{R}^n .
$$

Moreover, since f satisfies (2.6), then

(4.26) $$f(\xi) = \sup_{j\in\mathbb{N}}\{(\langle a_j,\xi\rangle + b_j)^+, (\langle \overline{a}_j,\xi\rangle + b_j)^+\} \qquad \text{for every } \xi \in \mathbb{R}^n\,,$$

where $\overline{a}_j = ((a_j)_1, \dots, (a_j)_{n-1}, -(a_j)_n)$. Set, for $N \in \mathbb{N}$,

(4.27) $$f_N(\xi) = \sup_{1\leq j\leq N}\{(\langle a_j,\xi\rangle + b_j)^+, (\langle \overline{a}_j,\xi\rangle + b_j)^+\} \qquad \text{for } \xi \in \mathbb{R}^n\,.$$

Obviously, $f_N(\xi)$ converges monotonically to $f(\xi)$ for every $\xi \in \mathbb{R}^n$. Since f_N satisfies (2.6) and (2.14), then (4.16) holds with f replaced by f_N. Inequality (4.16) then follows by monotone convergence. □

Lemma 4.5 *Let ω be an open subset of $\mathbb{R}^{n-1}$, and let u be a nonnegative function from $W^{1,p}(\omega\times\mathbb{R}_y)$. Then for every open set $\omega' \subset\subset \omega$ and for every $\varepsilon > 0$ there exists a nonnegative Lipschitz continuous function w, with compact support in $\mathbb{R}^n$, such that*

(4.28) $$\mathcal{L}^n\left(\{x\in\mathbb{R}^n : w(x) > 0,\, \nabla_y w(x) = 0\}\right) = 0$$

and

(4.29) $$\|u - w\|_{W^{1,p}(\omega'\times\mathbb{R}_y)} < \varepsilon\,.$$

Proof. On multiplying u by a function $\varphi \in C^1(\mathbb{R}^{n-1})$ with compact support in ω and such that $\varphi \equiv 1$ in ω', we may assume, without loss of generality, that $u \in W^{1,p}(\mathbb{R}^n)$. Thus, given any $\varepsilon > 0$, there exists a function $u_\varepsilon \in C^1_0(\mathbb{R}^n)$ such that $u_\varepsilon \geq 0$ and $\|u - u_\varepsilon\|_{W^{1,p}(\mathbb{R}^n)} < \varepsilon$. Let r be any real number greater than 1 and such that the support of u_ε is contained in $B_r(0)$. Standard approximation results ensure that a polynomial p_ε exists such that $\|u_\varepsilon - p_\varepsilon\|_{C^1(\overline{B}_{2r}(0))} < \varepsilon$. On replacing, if necessary, p_ε by $p_\varepsilon + \varepsilon$ we may assume that $p_\varepsilon(x) > 0$ for every $x \in B_{2r}(0)$. Similarly, on replacing, if necessary, p_ε by $p_\varepsilon + \delta y^2$, where δ is a sufficiently small positive number, we may also assume that $\nabla_y p_\varepsilon$ does not vanish identically. Let η_r denote the function from $\mathbb{R}^n$ into $\mathbb{R}$ defined as $\eta_r(x) = 1$ if $|x| \leq r$, $\eta_r(x) = (4r^2 - |x|^2)/3r^2$ if $r < |x| \leq 2r$ and $\eta_r(x) = 0$ if $|x| \geq 2r$. Set $w = p_\varepsilon\eta_r$. Then it is easily seen that a constant c, depending only on n, exists such that

$$\|u - w\|_{W^{1,p}(\mathbb{R}^n)} < \varepsilon + c\varepsilon r^n\,,$$

whence (4.29) follows. Finally, (4.28) is a consequence of the fact that $w(x) > 0$ if and only if $x \in B_{2r}(0)$, and that w agrees will the polynomial p_ε in $B_r(0)$ and with the polynomial $p_\varepsilon\eta_r$ in $B_{2r}(0)\setminus\overline{B}_r(0)$. □

Remark 4.6 Inequality (4.16) continues to hold even if, instead of assuming $u \in W^{1,1}_{0,y}(\Omega)$, we take $u \in W^{1,p}_{0,y}(\Omega)$ for some $p > 1$. To verify this assertion, set, for $\varepsilon > 0$, $u_\varepsilon = \max\{u - \varepsilon, 0\}$ and observe that, since the support of u_ε has finite measure in $\omega\times\mathbb{R}_y$ for any open set $\omega \subset\subset \pi_{n-1}(\Omega)$, then $u_\varepsilon \in W^{1,1}_{0,y}(\Omega)$. Since $(u_\varepsilon)^s = (u^s)_\varepsilon$ and since $\nabla u_\varepsilon = \nabla u\chi_{\{u>\varepsilon\}}$ $\mathcal{L}^n$-a.e. in $\mathbb{R}^n$, then by (4.16) applied with u replaced by u_ε, and by monotone convergence,

$$\begin{aligned}\int_{B\times\mathbb{R}_y} f(\nabla u^s)\,dx &= \lim_{\varepsilon\to0^+}\int_{B\times\mathbb{R}_y} f(\nabla(u^s)_\varepsilon)\,dx = \lim_{\varepsilon\to0^+}\int_{B\times\mathbb{R}_y} f(\nabla(u_\varepsilon)^s)\,dx \\ &\leq \lim_{\varepsilon\to0^+}\int_{B\times\mathbb{R}_y} f(\nabla u_\varepsilon)\,dx = \int_{B\times\mathbb{R}_y} f(\nabla u)\,dx\,.\end{aligned}$$

By the next lemma, we begin our characterization of extremals in (2.7).

Lemma 4.7 *Let f be as in Theorem 2.2. Let Ω be an open subset of $\mathbb{R}^n$ satisfying (2.10) and let u be a nonnegative function from $W^{1,1}_{0,y}(\Omega)$ fulfilling (2.8) and (2.12). Then, for $\mathcal{L}^n$-a.e. $(x',t) \in \pi_{n-1,n+1}(\mathcal{S}_u)^+$, there exist $y_1(x',t), y_2(x',t) \in \mathbb{R}$ such that $y_1(x',t) < y_2(x',t)$ and that*

$$\{y : u(x',y) > t\} \text{ is equivalent to } (y_1(x',t), y_2(x',t)), \tag{4.30}$$

$$\nabla_i u(x', y_1(x',t)) = \nabla_i u(x', y_2(x',t)) \qquad i = 1, \dots, n-1, \tag{4.31}$$

$$\nabla_y u(x', y_1(x',t)) = -\nabla_y u(x', y_2(x',t)). \tag{4.32}$$

Proof. As in the proof of Theorem 2.1, we denote the extension u_0 by u. Assumption (2.12) ensures that equality necessarily holds in (4.24), with B replaced by $\pi_{n-1}(\Omega)$. Thus, since, by (2.8), $u > 0$ $\mathcal{L}^n$-a.e. in Ω, equality holds in (4.23) for $\mathcal{L}^{n-1}$-a.e. $x' \in \pi_{n-1}(\Omega)$ and $\mathcal{L}^1$-a.e. $t \in (0, M(x'))$, and hence equality holds both in (4.21) and (4.22) for the same x' and t. Inasmuch as $f(\xi_1, \dots, \xi_{n-1}, \cdot)$ is strictly increasing in $[0, +\infty)$, equality holds in (4.20) whenever it holds in (4.21). Thus, by the isoperimetric theorem in $\mathbb{R}$, the set $\{y : u(x',y) > t\}$ is equivalent to some interval, say $(y_1(x',t), y_2(x',t))$ for $\mathcal{L}^{n-1}$-a.e. $x' \in \pi_{n-1}(\Omega)$ and for $\mathcal{L}^1$-a.e. $t \in (0, M(x'))$. Since $u > 0$ $\mathcal{L}^n$-a.e. in Ω, then $\pi_{n-1}(\mathcal{S}_u)^+$ is equivalent to $\pi_{n-1}(\Omega)$. Thus, (4.8) implies that

$$\pi_{n-1,n+1}(\mathcal{S}_u)^+ \quad \text{is equivalent to} \quad \bigcup_{x' \in \pi_{n-1}(\Omega)} \{x'\} \times (0, M(x')). \tag{4.33}$$

Hence (4.30) follows.

Inequality (4.22) is derived via Jensen's inequality. Thus, since f is strictly convex, if equality holds in (4.22) for $\mathcal{L}^{n-1}$-a.e. $x' \in \pi_{n-1}(\Omega)$ and for $\mathcal{L}^1$-a.e. $t \in (0, M(x'))$, then for the same values of x' and t

$$\nabla_i u_{|\partial^*\{y: u(x',y) > t\}} = c_i \qquad i = 1, \dots, n-1 \tag{4.34}$$

and

$$|\nabla_y u|_{|\partial^*\{y: u(x',y) > t\}} = c_y \tag{4.35}$$

for some constants $c_1, \dots, c_{n-1}, c_y$. Since $\partial^*\{y : u(x',y) > t\} = \{y_1(x',t), y_2(x',t)\}$ for $\mathcal{L}^{n-1}$-a.e. $x' \in \pi_{n-1}(\Omega)$ and for $\mathcal{L}^1$-a.e. $t \in (0, M(x'))$, then (4.33), (4.34) and (4.35) tell us that, for $\mathcal{L}^n$-a.e. $(x',t) \in \pi_{n-1,n+1}(\mathcal{S}_u)^+$, equation (4.31) holds and

$$|\nabla_y u(x', y_1(x',t))| = |\nabla_y u(x', y_2(x',t))|. \tag{4.36}$$

Finally, it is easily seen from (4.30) that for $\mathcal{L}^n$-a.e. $x' \in \pi_{n-1}(\Omega)$ the function $y_1(x', \cdot)$ agrees $\mathcal{L}^1$-a.e. in $(0, M(x'))$ with a nondecreasing function, whereas $y_2(x', \cdot)$ agrees $\mathcal{L}^1$-a.e. with a nonincreasing function in the same interval. Therefore (see e.g. [CF, Lemma 4.1]), $u(x', \cdot)$ is equivalent to a function (whose level sets are open intervals) which is nondecreasing in $(-\infty, \beta_1(x'))$ and nonincreasing in $(\beta_2(x'), +\infty)$, where

$$\beta_1(x') = \operatorname*{ess\,sup}_{t < M(x')} y_1(x',t) \qquad \beta_2(x') = \operatorname*{ess\,inf}_{t < M(x')} y_2(x',t). \tag{4.37}$$

Hence, equation (4.36) implies, in fact, (4.32). □

Lemma 4.8 *Let f be as in Theorem 2.2. Let Ω be an open subset of $\mathbb{R}^n$ satisfying (2.10) and let u be a nonnegative function from $W^{1,1}_{0,y}(\Omega)$ fulfilling (2.8) and (2.12). Let $y_1(x',t)$ and $y_2(x',t)$ be defined as in Lemma 4.7. Then, there exists a function $b:\pi_{n-1}(\Omega)\to\mathbb{R}$ such that, for $\mathcal{L}^{n-1}$-a.e. $x'\in\pi_{n-1}(\Omega)$,*

$$\frac{1}{2}(y_1(x',t)+y_2(x',t))=b(x') \qquad \text{for } \mathcal{L}^1\text{-a.e. } t\in(0,M(x'))\,. \tag{4.38}$$

Proof. By (2.8), $M(x')>0$ for $\mathcal{L}^{n-1}$-a.e. $x'\in\pi_{n-1}(\Omega)$. Let N be the subset of $\pi_{n-1}(\Omega)$, satisfying $\mathcal{L}^{n-1}(\pi_{n-1}(\Omega)\backslash N)=0$, such that, for $x'\in N$, $M(x')>0$ and (4.30)-(4.32) hold for $\mathcal{L}^1$-a.e. $t\in(0,M(x'))$. Thus, for every $x'\in N$, the set $\{y:\,u(x',y)=M(x')\}$ is equivalent to $[\beta_1(x'),\beta_2(x')]$, where $\beta_i(x')$, $i=1,2$, are given by (4.37). Define now $I_1=(-\infty,\beta_1(x'))$ and $I_2=(\beta_2(x'),+\infty)$, and set $v_i^{x'}=u(x',\cdot)_{|I_i}$, $i=1,2$. By (2.8) and by Fubini's theorem,

$$\mathcal{L}^1\Big(\{y:\,\nabla_y u(x',y)=0\}\cap\{y:\,0<v_i^{x'}(y)<M(x')\}\Big)=0 \qquad i=1,2\,, \tag{4.39}$$

for $\mathcal{L}^{n-1}$-a.e. $x'\in N$. We have, for $\mathcal{L}^{n-1}$-a.e. $x'\in N$ and for every $t\in(0,M(x'))$,

$$\begin{aligned} \mu_{v_i^{x'}}(t) &= \mathcal{L}^1\Big(\{y:\,\nabla_y u(x',y)\neq 0\}\cap\{y:\,t<v_i^{x'}(y)<M(x')\}\Big) \\ &= \int_t^{M(x')} d\tau\int_{I_i\cap\partial^*\{u(x',\cdot)>\tau\}}\frac{1}{|\nabla_y u(x',y)|}\,d\mathcal{H}^0(y)=\int_t^{M(x')}\frac{1}{|\nabla_y u(x',y_i(x',\tau))|}\,d\tau \qquad i=1,2\,. \end{aligned} \tag{4.40}$$

Notice that the first equality in (4.40) is a consequence of (4.39), the second holds analogously as in (4.12), and the third is due to (4.30). From (4.40) and (4.32) we deduce that, for $\mathcal{L}^{n-1}$-a.e. $x'\in\pi_{n-1}(\Omega)$,

$$\mu_{v_1^{x'}}(t)=\mu_{v_2^{x'}}(t) \qquad \text{for every } t\in(0,M(x'))\,.$$

Since

$$y_i(x',t)=\beta_i(x')+(-1)^i\mu_{v_i^{x'}}(t) \qquad i=1,2\,,$$

equation (4.38) follows with $b(x')=\dfrac{\beta_1(x')+\beta_2(x')}{2}$. □

Lemma 4.9 *Let f be as in Theorem 2.2. Let Ω be an open subset of $\mathbb{R}^n$ satisfying (2.10). Let u be a nonnegative function from $W^{1,1}_{0,y}(\Omega)$ fulfilling (2.8) and (2.12). If $b:\pi_{n-1}(\Omega)\to\mathbb{R}$ is the function defined in Lemma 4.8, then*

$$b(x')=\frac{\displaystyle\int_{\Omega_{x'}} y\,dy}{\mu_u(x',0)} \qquad \text{for } \mathcal{L}^{n-1}\text{-a.e. } x'\in\pi_{n-1}(\Omega)\,. \tag{4.41}$$

Proof. By (4.30) and (4.33), for $\mathcal{L}^{n-1}$-a.e. $x'\in\pi_{n-1}(\Omega)$,

$$\mathcal{L}^1\left(\{y:\,u(x',y)>t\}\right)=y_2(x',t)-y_1(x',t) \tag{4.42}$$

and

$$\int_{\{y:u(x',y)>t\}} y\,dy=\frac{1}{2}(y_2^2(x',t)-y_1^2(x',t))$$

for $\mathcal{L}^1$-a.e. $t \in (0, M(x'))$. Hence, by (4.38), for $\mathcal{L}^{n-1}$-a.e. $x' \in \pi_{n-1}(\Omega)$,

$$b(x') = \frac{\int_{\{y: u(x',y)>t\}} y dy}{\mathcal{L}^1\left(\{y:\ u(x',y) > t\}\right)} \qquad \text{for } \mathcal{L}^1\text{-a.e.}\, t \in (0, M(x'))\,. \tag{4.43}$$

Thus, for $\mathcal{L}^{n-1}$-a.e. $x' \in \pi_{n-1}(\Omega)$, we have that $M(x') > 0$, that (4.43) holds for $\mathcal{L}^1$-a.e. $t \in (0, M(x'))$, and that the function $u(x', \cdot)$ is measurable. Fix any such x', and choose any decreasing sequence $\{t_j\}$ satisfying $\lim_{j\to\infty} t_j = 0$ and such that (4.43) is fulfilled with $t = t_j$. The sequence of sets $\{y : u(x', y) > t_j\}$ is obviously nondecreasing, and $\{y : u(x', y) > 0\} = \bigcup_{j\in\mathbb{N}} \{y : u(x', y) > t_j\}$. Moreover,

$$\{y : u(x', y) > 0\} \text{ is equivalent to } \Omega_{x'} \text{ for } \mathcal{L}^{n-1}\text{-a.e. } x' \in \pi_{n-1}(\Omega)\,, \tag{4.44}$$

owing to (2.8). Thus, since $\Omega_{x'}$ is a bounded subset of $\mathbb{R}$,

$$\lim_{j\to\infty} \int_{\{y:u(x',y)>t_j\}} y\, dy = \int_{\{y:u(x',y)>0\}} y\, dy = \int_{\Omega_{x'}} y\, dy$$

and

$$\lim_{j\to\infty} \mathcal{L}^1\left(\{y:\ u(x', y) > t_j\}\right) = \mathcal{L}^1\left(\{y:\ u(x', y) > 0\}\right) = \mu_u(x', 0)$$

for $\mathcal{L}^{n-1}$-a.e. $x' \in \pi_{n-1}(\Omega)$. The conclusion follows. □

Lemma 4.10 *Let f, Ω and u be as in Theorem 2.2, and let $b : \pi_{n-1}(\Omega) \to \mathbb{R}$ be the function defined in Lemma 4.8. Then $b \in W^{1,1}_{\rm loc}(\pi_{n-1}(\Omega))$.*

Proof. Owing to formula (4.41), it suffices to show that the function $h : \pi_{n-1}(\Omega) \to \mathbb{R}$, defined as

$$h(x') = \int_{\Omega_{x'}} y\, dy \qquad \text{for } x' \in \pi_{n-1}(\Omega)\,,$$

is in $W^{1,1}_{\rm loc}(\pi_{n-1}(\Omega))$, that $\mu_u(x', 0) \in W^{1,1}_{\rm loc}(\pi_{n-1}(\Omega))$, and that for every open set $\omega \subset\subset \pi_{n-1}(\Omega)$ there exists a positive constant c such that $\mu_u(x', 0) \geq c$ for every $x' \in \omega$. Consider the function h. It is clearly measurable, and, since Ω is bounded in the direction y, it is also bounded. Now, let L be a positive number such that $\Omega \subset \pi_{n-1}(\Omega) \times (-L, L)$. Fix any function $\varphi \in C^1_0(\pi_{n-1}(\Omega))$ and any function $\psi \in C^1_0(\mathbb{R})$ satisfying $\psi(y) = 1$ for all $y \in [-L, L]$. Then

$$\begin{aligned} \int_{\pi_{n-1}(\Omega)} \nabla_i\varphi(x')h(x')dx' &= \int_{\pi_{n-1}(\Omega)} \nabla_i\varphi(x')dx' \int_{\Omega_{x'}} ydy = \int_\Omega \nabla_i\varphi(x')ydx'dy \\ &= \int_{\pi_{n-1}(\Omega)\times\mathbb{R}_y} \nabla_i\varphi(x')y\chi_\Omega(x', y)\psi(y)dx'dy = -\int_{\pi_{n-1}(\Omega)\times\mathbb{R}_y} \varphi(x')ydD_i\chi_\Omega(x', y)\,. \end{aligned} \tag{4.45}$$

Notice that we have made use of the fact that Ω has locally finite perimeter in $\pi_{n-1}(\Omega) \times \mathbb{R}_y$. From (4.45) one immediately gets that $h \in BV_{\rm loc}(\pi_{n-1}(\Omega))$ and that, for every $\varphi \in C^1_0(\pi_{n-1}(\Omega))$,

$$\int_{\pi_{n-1}(\Omega)} \varphi(x')\, dD_ih(x') = \int_{\pi_{n-1}(\Omega)\times\mathbb{R}_y} \varphi(x')y\, dD_i\chi_\Omega(x', y), \qquad i = 1, \ldots, n-1\,. \tag{4.46}$$

An analogous argument as in the proof of Lemma 4.1 ensures that (4.46) holds, in fact, for any bounded Borel function $\varphi : \pi_{n-1}(\Omega) \to \mathbb{R}$. Equation (4.46) can be used to show that

$$|Dh|(B) \le L|D\chi_\Omega|(B \times \mathbb{R}_y) \tag{4.47}$$

for every open set $B \subset\subset \pi_{n-1}(\Omega)$. By approximation, (4.47) still holds for any Borel set $B \subset \pi_{n-1}(\Omega)$. Owing to (3.3), to the coarea formula (3.10) and to assumption (2.11),

$$|D\chi_\Omega|(B \times \mathbb{R}_y) = \mathcal{H}^{n-1}\left(\partial^*\Omega \cap (B \times \mathbb{R}_y)\right) = \int_B dx' \int_{\Omega_{x'}} \frac{d\mathcal{H}^0}{|\nu_y^\Omega|} . \tag{4.48}$$

Combining (4.47) and (4.48) tells us that $|Dh|(B) = 0$ whenever $\mathcal{L}^n(B) = 0$, whence $h \in W^{1,1}_{\rm loc}(\pi_{n-1}(\Omega))$. Owing to (4.44), a completely analogous argument shows that also $\mu_u(x', 0) \in W^{1,1}_{\rm loc}(\pi_{n-1}(\Omega))$. Finally, the fact that $\mu_u(x', 0)$ is bounded away from 0 on any open subset $\omega \subset\subset \pi_{n-1}(\Omega)$ is again a consequence of (4.44) and of the fact that $\mathcal{L}^1(\Omega_{x'})$ is a lower semicontinuous function of x' in $\pi_{n-1}(\Omega)$. □

Proof of Theorem 2.2. Owing to (2.4), it suffices to show that

$$(\mathcal{S}_u)^s \text{ is equivalent to } \mathcal{S}_u \tag{4.49}$$

up to translations along the y-axis. As above, throughout the proof u will stand for its extension u_0. Let $y_1(x', t)$ and $y_2(x', t)$ be defined as in Lemma 4.7, and let b be the function given by Lemma 4.8. Let us set

$$z_1(x', t) = b(x') - \frac{1}{2}\mu_u(x', t), \qquad z_2(x', t) = b(x') + \frac{1}{2}\mu_u(x', t)$$

for $(x', t) \in \pi_{n-1}(\Omega) \times \mathbb{R}^+_t$. Then, by Lemmas 4.1 and 4.10, $z_i \in BV_{\rm loc}(\pi_{n-1}(\Omega) \times \mathbb{R}^+_t)$, $i = 1, 2$. Moreover, by (4.38), (4.42) and (4.33), a set $N \subset \pi_{n-1,n+1}(\mathcal{S}_u)^+$ exists such that $\mathcal{L}^n(\pi_{n-1,n+1}(\mathcal{S}_u)^+ \setminus N) = 0$ and

$$z_i(x', t) = y_i(x', t)$$

for $(x', t) \in N$. Thus, thanks to Lemma 4.7, the set $\mathcal{S}_u$ is equivalent to the set E defined by

$$E = \left\{(x', y, t) : (x', t) \in \pi_{n-1,n+1}(\mathcal{S}_u)^+, \, z_1(x', t) < y < z_2(x', t)\right\} . \tag{4.50}$$

Now, define

$$E_1 = \left\{(x', y, t) : (x', t) \in \pi_{n-1}(\Omega) \times \mathbb{R}^+_t, \, y > z_1(x', t)\right\} ,$$

$$E_2 = \left\{(x', y, t) : (x', t) \in \pi_{n-1}(\Omega) \times \mathbb{R}^+_t, \, y < z_2(x', t)\right\} .$$

Observe that E is equivalent to $E_1 \cap E_2$. By Theorem A, the sets E, E_1 and E_2 are of finite perimeter in $U \times \mathbb{R}_y$ for every bounded open set $U \subset\subset \pi_{n-1}(\Omega) \times \mathbb{R}^+_t$, and hence, by Theorem D, Borel sets G_E, G_{E_1} and G_{E_2} exist such that

$$\mathcal{L}^n\left(\pi_{n-1,n+1}(E)^+ \setminus G_E\right) = 0, \qquad \mathcal{L}^n\left((\pi_{n-1}(\Omega) \times \mathbb{R}^+_t) \setminus G_{E_i}\right) = 0, \qquad i = 1, 2 ,$$

and (3.11)-(3.14) hold. In particular,

$$(\partial^* E)_{x',t} = \partial^*(E_{x',t}) = \{z_1(x', t), z_2(x', t)\} \qquad \text{for every } (x', t) \in G_E \tag{4.51}$$

$$(\partial^* E_i)_{x',t} = \partial^* (E_i)_{x',t} = \{z_i(x',t)\} \qquad \text{for every } (x',t) \in G_{E_i},\ i=1,2\,. \tag{4.52}$$

By Theorem B and by (3.14) of Theorem D, a Borel set S exists such that $\mathcal{H}^n(S) = 0$ and

$$\nu^E(x',y,t) = \nu^{E_i}(x',y,t) \qquad \text{for } (x',y,t) \in [\partial^* E \cap \partial^* E_i \setminus S] \cap [(G_E \cap G_{E_i}) \times \mathbb{R}_y]\,. \tag{4.53}$$

Notice that, since orthogonal projections do not increase Hausdorff measure (see [AFP, Proposition 2.49]), then $\mathcal{L}^n(\pi_{n-1,n+1}(S)) = 0$.
We next claim that a subset R of $\pi_{n-1,n+1}(E)^+$ exists such that $\mathcal{L}^n(\pi_{n-1,n+1}(E)^+ \setminus R) = 0$ and

$$\begin{cases} \nu_i^E(x', z_1(x',t), t) = \nu_i^E(x', z_2(x',t), t) & i = 1, \dots, n-1, t\,, \\ \nu_y^E(x', z_1(x',t), t) = -\nu_y^E(x', z_2(x',t), t) & \end{cases} \tag{4.54}$$

for $(x',t) \in R$. To verify this claim, recall from Theorem C that, since $u \in W^{1,1}(\omega \times \mathbb{R}_y)$ for every open set $\omega \subset\subset \pi_{n-1}(\Omega)$, a subset V of $\partial^* E \cap (\pi_{n-1}(\Omega) \times \mathbb{R}_y \times \mathbb{R}_t^+)$ exists such that

$$\mathcal{H}^n\left([\partial^* E \cap (\pi_{n-1}(\Omega) \times \mathbb{R}_y \times \mathbb{R}_t^+)] \setminus V\right) = 0 \tag{4.55}$$

and

$$\nu^E(x',y,t) = \left(\frac{\nabla_1 u(x',y)}{\sqrt{1+|\nabla u|^2}}, \dots, \frac{\nabla_{n-1} u(x',y)}{\sqrt{1+|\nabla u|^2}}, \frac{\nabla_y u(x',y)}{\sqrt{1+|\nabla u|^2}}, \frac{-1}{\sqrt{1+|\nabla u|^2}}\right) \tag{4.56}$$

for every $(x',y,t) \in V$. Set $Q = \pi_{n-1,n+1}([\partial^* E \cap (\pi_{n-1}(\Omega) \times \mathbb{R}_y \times \mathbb{R}_t^+)] \setminus V)$. Equation (4.55) entails that $\mathcal{L}^n(Q) = 0$. Next, observe that

$$(x', z_i(x',t), t) \in V \qquad \text{for } \mathcal{L}^n\text{-a.e. } (x',t) \in \pi_{n-1,n+1}(E)^+ \setminus Q\,. \tag{4.57}$$

Indeed, owing to (4.51), the points $(x', z_i(x',t), t)$, $i = 1,2$, belong to $\partial^* E \cap (\pi_{n-1}(\Omega) \times \mathbb{R}_y \times \mathbb{R}_t^+)$ for $\mathcal{L}^n$-a.e. $(x',t) \in \pi_{n-1,n+1}(E)^+$. Moreover, if $(x',t) \in \pi_{n-1,n+1}(E)^+ \setminus Q$, then both $(x', z_1(x',t), t)$ and $(x', z_2(x',t), t)$ have to be in V, since if either of them belongs to $[\partial^* E \cap (\pi_{n-1}(\Omega) \times \mathbb{R}_y \times \mathbb{R}_t^+)] \setminus V$, then $(x',t) \in Q$. Equations (4.54) follow from (4.57), (4.56) and from (4.31)-(4.32) of Lemma 4.7.
Finally, from equation (3.8) applied to z_1 and z_2, and from (4.51) we deduce that a set $T \subset \pi_{n-1}(\Omega) \times \mathbb{R}_t^+$ exists such that $\mathcal{L}^n((\pi_{n-1}(\Omega) \times \mathbb{R}_t^+) \setminus T) = 0$ and

$$\nu^{E_i}(x', z_i(x',t), t) = (-1)^i \left(\frac{\nabla_1 z_i(x',t)}{\sqrt{1+|\nabla z_i|^2}}, \dots, \frac{\nabla_{n-1} z_i(x',t)}{\sqrt{1+|\nabla z_i|^2}}, \frac{-1}{\sqrt{1+|\nabla z_i|^2}}, \frac{\nabla_t z_i(x',t)}{\sqrt{1+|\nabla z_i|^2}}\right) \qquad i = 1,2\,, \tag{4.58}$$

for $(x',t) \in T$. Now, set

$$Z = [\pi_{n-1,n+1}(E)^+ \cap N \cap G_E \cap G_{E_1} \cap G_{E_2} \cap R \cap T] \setminus \pi_{n-1,n+1}(S)\,,$$

and notice that $\mathcal{L}^n(\pi_{n-1,n+1}(E)^+ \setminus Z) = 0$. Combining (4.51), (4.52), (4.53), (4.54) and (4.58) we infer that

$$\nabla_{x',t} z_1(x',t) + \nabla_{x',t} z_2(x',t) = 0$$

for $(x',t) \in Z$, and hence for $\mathcal{L}^n$-a.e. $(x',t) \in \pi_{n-1,n+1}(\mathcal{S}_u)^+$. Consequently,

$$\nabla_{x',t} b(x') = 0 \qquad \text{for } \mathcal{L}^n\text{-a.e. } (x',t) \in \pi_{n-1,n+1}(\mathcal{S}_u)^+\,. \tag{4.59}$$

On taking into account (4.33), equation (4.59) easily implies that

$$\nabla_{x'} b(x') = 0 \tag{4.60}$$

for $\mathcal{L}^{n-1}$-a.e. $x' \in \pi_{n-1}(\Omega)$. Thus, since $b \in W^{1,1}_{\mathrm{loc}}(\pi_{n-1}(\Omega))$ and satisfies (4.60), and $\pi_{n-1}(\Omega)$ is assumed to be connected, then a constant k exists such that $b(x') = k$ for $\mathcal{L}^{n-1}$-a.e. $x' \in \pi_{n-1}(\Omega)$ (see e.g. [Z, Corollary 2.1.9]). Hence, (4.49) follows. $\square$

5 Functionals of the normal

The present section is devoted to the proofs of Theorems 2.7-2.9. The general scheme of these proofs is analogous to that of corresponding results in the special case of perimeter contained in [CCF]. Several steps, however, require substantially new arguments. We begin by recalling formulas for the approximate gradient of the function ℓ_E, in the same spirit as (4.1)-(4.2), which are proved in [CCF, Lemmas 3.1, 3.2].

Lemma 5.1 *Let U be an open subset of $\mathbb{R}^{n-1} \times \mathbb{R}_t$. Let E be a set of finite perimeter in $U \times \mathbb{R}_y$ such that $\mathcal{L}^{n+1}(E \cap (U \times \mathbb{R}_y)) < +\infty$. Then $\ell_E \in BV(U)$ and*

$$(5.1) \qquad \int_U \psi(x', t)\, dD_i\ell_E(x', t) = \int_{U\times\mathbb{R}_y} \psi(x', t)\, dD_i\chi_E(x', y, t) \qquad i = 1, \dots, n-1, t\,,$$

for every bounded Borel function ψ in U. In particular

$$|D\ell_E|(A) \le |D\chi_E|(A \times \mathbb{R}_y)$$

for every Borel set $A \subset U$. Moreover, for $\mathcal{L}^n$-a.e. $(x', t) \in \pi_{n-1,n+1}(E)^+ \cap U$,

$$(5.2) \qquad \nabla_i\ell_E(x', t) = \int_{\partial^* E_{x',t}} \frac{\nu_i^E(x', y, t)}{|\nu_y^E(x', y, t)|}\, d\mathcal{H}^0(y) \qquad i = 1, \dots, n-1, t\,.$$

Remark 5.2 An application of formulas (5.2) to E^s yields, in particular, that

$$(5.3) \qquad \nabla_i\ell_E(x', t) = 2\frac{\nu_i^{E^s}}{|\nu_y^{E^s}|}\Big|_{\partial^*(E^s)_{x',t}} \qquad i = 1, \dots, n-1, t\,,$$

for $\mathcal{L}^n$-a.e. $(x', t) \in \pi_{n-1,n+1}(E)^+ \cap U$.

The next two lemmas play a role, in the proof of Theorem 2.7, in dealing with the subset of $\partial^* E^s$ where $\nu_y^{E^s} = 0$.

Lemma 5.3 *Let U be an open subset of $\mathbb{R}^{n-1} \times \mathbb{R}_t$. Let $F : \mathbb{R}^{n-1} \to [0, +\infty)$ be a convex function satisfying (2.21) and (2.24), and let E be a set of finite perimeter in $U\times\mathbb{R}_y$ such that $\mathcal{L}^{n+1}(E\cap(U\times\mathbb{R}_y)) < +\infty$. Then*

$$(5.4) \qquad \int_{\partial^* E^s \cap (A\times\mathbb{R}_y)} F(\nu^{E^s})\, d\mathcal{H}^n \le \int_A F\left(\frac{D_1\ell_E}{|D\ell_E|}, \dots, \frac{D_{n-1}\ell_E}{|D\ell_E|}, 0, \frac{D_t\ell_E}{|D\ell_E|}\right) d|D\ell_E| + F(0, \dots, 0, 1, 0)|D_y\chi_{E^s}|(A \times \mathbb{R}_y)$$

for every Borel set $A \subset U$.

Proof. We may limit ourselves to consider the case where A is a bounded open set in (5.4). In this case, we can find a sequence of functions $\{\ell_k\}$ from $C^\infty(A)$ such that $\ell_k(x', t) > 0$ for every $(x', t) \in A$ and every $k \in \mathbb{N}$, $\ell_k \to \ell_E$ in $L^1(A)$, $\nabla\ell_k\mathcal{L}^n \rightharpoonup D\ell_E$ weakly* in A in the sense of measures, and

$$(5.5) \qquad \int_A |\nabla\ell_k|\, dx'dt \to |D\ell_E|(A)\,.$$

Let us set, for $k \in \mathbb{N}$, $E_k = \{(x', y, t) : (x', t) \in A, |y| < \ell_k(x', t)/2\}$. Then, $\chi_{E_k} \to \chi_{E^s}$ in $L^1(A \times \mathbb{R}_y)$ and a constant C exists such that

$$|D\chi_{E_k}|(A \times \mathbb{R}_y) = P(E_k; A \times \mathbb{R}_y) = 2\int_A \sqrt{1 + \frac{|\nabla \ell_k|^2}{4}}\, dx'dt \le 2\mathcal{L}^n(A) + \int_A |\nabla \ell_k|\, dx'dt \le C$$

for every $k \in \mathbb{N}$. Hence, one deduces that

$$D\chi_{E_k} \rightharpoonup D\chi_{E^s} \qquad \text{weakly* in } A \times \mathbb{R}_y\,. \tag{5.6}$$

Our assumptions on F ensure that

$$\begin{aligned}
&\int_{\partial^* E^s \cap (A \times \mathbb{R}_y)} F(\nu^{E^s}) d\mathcal{H}^n = 2\int_{\partial^* E^s \cap (A \times \mathbb{R}_y^+)} F(\nu^{E^s}) d\mathcal{H}^n \qquad (5.7)\\
&\le 2\int_{\partial^* E^s \cap (A \times \mathbb{R}_y^+)} F(\nu_1^{E^s}, \ldots, \nu_{n-1}^{E^s}, 0, \nu_t^{E^s}) d\mathcal{H}^n + 2\int_{\partial^* E^s \cap (A \times \mathbb{R}_y^+)} F(0, \ldots, 0, \nu_y^{E^s}, 0) d\mathcal{H}^n\\
&= 2\int_{A \times \mathbb{R}_y^+} F\Big(\frac{D_1\chi_{E^s}}{|D\chi_{E^s}|}, \ldots, \frac{D_{n-1}\chi_{E^s}}{|D\chi_{E^s}|}, 0, \frac{D_t\chi_{E^s}}{|D\chi_{E^s}|}\Big) d|D\chi_{E^s}| + 2F(0, \ldots, 0, 1, 0)\int_{\partial^* E^s \cap (A \times \mathbb{R}_y^+)} |\nu_y^{E^s}| d\mathcal{H}^n.
\end{aligned}$$

Thanks to (5.6) and to a lower semicontinuity theorem by Reshetnyak (see e.g. [AFP, Theorem 2.38]), we have that

$$\begin{aligned}
&\int_{A \times \mathbb{R}_y^+} F\Big(\frac{D_1\chi_{E^s}}{|D\chi_{E^s}|}, \ldots, \frac{D_{n-1}\chi_{E^s}}{|D\chi_{E^s}|}, 0, \frac{D_t\chi_{E^s}}{|D\chi_{E^s}|}\Big)\, d|D\chi_{E^s}| \qquad (5.8)\\
&\qquad \le \liminf_{k\to\infty} \int_{A \times \mathbb{R}_y^+} F\Big(\frac{D_1\chi_{E_k}}{|D\chi_{E_k}|}, \ldots, \frac{D_{n-1}\chi_{E_k}}{|D\chi_{E_k}|}, 0, \frac{D_t\chi_{E_k}}{|D\chi_{E_k}|}\Big)\, d|D\chi_{E_k}|\\
&\qquad = \liminf_{k\to\infty} \int_{\partial^* E_k \cap (A \times \mathbb{R}_y^+)} F\Big(\nu_1^{E_k}, \ldots, \nu_{n-1}^{E_k}, 0, \nu_t^{E_k}\Big)\, d\mathcal{H}^n\,.
\end{aligned}$$

Since ℓ_k is a smooth function for every $k \in \mathbb{N}$, then

$$\nu_i^{E_k}(x', y, t) = \frac{\nabla_i \ell_k(x', t)}{2}\Big(1 + \frac{|\nabla \ell_k|^2}{4}\Big)^{-1/2} \qquad i = 1, \ldots, n-1, t\,,$$

for every $(x', y, t) \in \partial^* E_k \cap (A \times \mathbb{R}_y^+)$. Hence, combining (5.7)-(5.8) and recalling (3.4) yield

$$\begin{aligned}
&\int_{\partial^* E^s \cap (A \times \mathbb{R}_y)} F(\nu^{E^s}) d\mathcal{H}^{n-1} \qquad (5.9)\\
&\le \liminf_{k\to\infty} \int_{\partial^* E_k \cap (A \times \mathbb{R}_y^+)} F\Big(\frac{\nabla_1 \ell_k}{\sqrt{1 + |\nabla \ell_k|^2/4}}, \ldots, \frac{\nabla_{n-1} \ell_k}{\sqrt{1 + |\nabla \ell_k|^2/4}}, 0, \frac{\nabla_t \ell_k}{\sqrt{1 + |\nabla \ell_k|^2/4}}\Big)\, d\mathcal{H}^n\\
&\qquad + 2F(0, \ldots, 0, 1, 0)\int_{\partial^* E^s \cap (A \times \mathbb{R}_y^+)} |\nu_y^{E^s}|\, d\mathcal{H}^n\\
&= \liminf_{k\to\infty} \int_A F(\nabla_1 \ell_k, \ldots, \nabla_{n-1} \ell_k, 0, \nabla_t \ell_k)\, dx'dt + 2F(0, \ldots, 0, 1, 0)|D_y\chi_{E^s}|(A \times \mathbb{R}_y^+)\\
&= \liminf_{k\to\infty} \int_A F\Big(\frac{\nabla_1 \ell_k}{|\nabla \ell_k|}, \ldots, \frac{\nabla_{n-1} \ell_k}{|\nabla \ell_k|}, 0, \frac{\nabla_t \ell_k}{|\nabla \ell_k|}\Big)|\nabla \ell_k|\, dx'dt + 2F(0, \ldots, 0, 1, 0)|D_y\chi_{E^s}|(A \times \mathbb{R}_y^+)\,.
\end{aligned}$$

Since $\nabla \ell_k \mathcal{L}^n \rightharpoonup D\ell_E$ weakly* and (5.5) holds, then a continuity theorem by Reshetnyak (see [AFP, Theorem 2.39]) tells us that

$$\liminf_{k\to\infty}\int_A F\Big(\frac{\nabla_1 \ell_k}{|\nabla \ell_k|}, \ldots, \frac{\nabla_{n-1} \ell_k}{|\nabla \ell_k|}, 0, \frac{\nabla_t \ell_k}{|\nabla \ell_k|}\Big)|\nabla \ell_k| dx'dt = \int_A F\Big(\frac{D_1 \ell_E}{|D\ell_E|}, \ldots, \frac{D_{n-1} \ell_E}{|D\ell_E|}, 0, \frac{D_t \ell_E}{|D\ell_E|}\Big) d|D\ell_E|. \tag{5.10}$$

Inequality (5.4) follows from (5.9)-(5.10). □

Lemma 5.4 *Let U be an open subset of $\mathbb{R}^{n-1} \times \mathbb{R}_t$. Let $F : \mathbb{R}^{n+1} \to [0, +\infty)$ be a convex function satisfying (2.21), and let E be a set of finite perimeter in $U \times \mathbb{R}_y$ such that $\mathcal{L}^{n+1}(E \cap (U \times \mathbb{R}_y)) < +\infty$. Then*

$$\int_A F\Big(\frac{D_1\ell_E}{|D\ell_E|}, \dots, \frac{D_{n-1}\ell_E}{|D\ell_E|}, 0, \frac{D_t\ell_E}{|D\ell_E|}\Big) d|D\ell_E| \le \int_{\partial^* E \cap (A \times \mathbb{R}_y)} F(\nu_1^E, \dots, \nu_{n-1}^E, 0, \nu_t^E) d\mathcal{H}^n \tag{5.11}$$

for every Borel set $A \subset U$.

Proof. As in the proof of Lemma 5.3 we may assume, without loss of generality, that A is a bounded open set. Since F is a convex nonnegative function satisying (2.21), there exists a sequence $\{\alpha_j\}$ with $\alpha_j \in \mathbb{R}^{n+1}$ such that

$$F(\xi_1, \dots, \xi_{n-1}, 0, \xi_{n+1}) = \sup_{j \in \mathbb{N}} \{\langle \widehat{\alpha}_j, \widehat{\xi} \rangle^+\} \qquad \text{for every } \ \xi \in \mathbb{R}^{n+1}, \tag{5.12}$$

where $\widehat{\xi} = (\xi_1, \dots, \xi_{n-1}, \xi_{n+1}) \in \mathbb{R}^n$ and $\widehat{\alpha}_j$ is defined analogously. From the representation formula (5.12), we get that (see e.g. [AFP, Lemma 2.35])

$$\int_A F\Big(\frac{D_1\ell_E}{|D\ell_E|}, \dots, \frac{D_{n-1}\ell_E}{|D\ell_E|}, 0, \frac{D_t\ell_E}{|D\ell_E|}\Big)\, d|D\ell_E| = \sup\Big\{\sum_{j \in J} \int_{A_j} \langle \widehat{\alpha}_j, \frac{D\ell_E}{|D\ell_E|} \rangle^+ \, d|D\ell_E|\Big\}, \tag{5.13}$$

where the supremum is extended over all finite sets $J \subset \mathbb{N}$ and all families $\{A_j\}_{j \in J}$ of pairwise disjoint open subsets of A. Now, fix a family $\{A_j\}_{j \in J}$, fix $j \in J$ and define

$$P_j = \Big\{(x', t) \in A_j : \langle \widehat{\alpha}_j, \frac{D\ell_E}{|D\ell_E|}(x', t) \rangle \ge 0\Big\}.$$

On making use of (5.1) we get

$$\begin{aligned} \int_{A_j} \langle \widehat{\alpha}_j, \frac{D\ell_E}{|D\ell_E|} \rangle^+ d|D\ell_E| &= \int_U \chi_{P_j}(x', t) \Big[\sum_{i=1}^{n-1} (\alpha_j)_i \frac{D_i\ell_E}{|D\ell_E|} + (\alpha_j)_{n+1} \frac{D_t\ell_E}{|D\ell_E|}\Big] d|D\ell_E| \\ &= \sum_{i=1}^{n-1} \int_U (\alpha_j)_i \, \chi_{P_j}(x', t)\, dD_i\ell_E(x', t) + \int_U (\alpha_j)_{n+1} \, \chi_{P_j}(x', t)\, dD_t\ell_E(x', t) \\ &= \sum_{i=1}^{n-1} \int_{U \times \mathbb{R}_y} (\alpha_j)_i \, \chi_{P_j \times \mathbb{R}_y}(x', y, t) dD_i\chi_E + \int_{U \times \mathbb{R}_y} (\alpha_j)_{n+1} \, \chi_{P_j \times \mathbb{R}_y}(x', y, t)\, dD_t\chi_E. \end{aligned}$$

Hence, by (3.2), we infer that

$$\int_{A_j} \langle \widehat{\alpha}_j, \frac{D\ell_E}{|D\ell_E|} \rangle^+ d|D\ell_E| = \int_{\partial^* E} \chi_{P_j \times \mathbb{R}_y} \langle \widehat{\alpha}_j, \widehat{\nu}^E \rangle d\mathcal{H}^n \le \int_{\partial^* E} \chi_{A_j \times \mathbb{R}_y} \langle \widehat{\alpha}_j, \widehat{\nu}^E \rangle^+ d\mathcal{H}^n. \tag{5.14}$$

From (5.12) and (5.14) one deduces that

$$\begin{aligned} \sum_{j \in J} \int_{A_j} \langle \widehat{\alpha}_j, \frac{D\ell_E}{|D\ell_E|} \rangle^+ d|D\ell_E| &\le \sum_{j \in J} \int_{\partial^* E \cap (A_j \times \mathbb{R}_y)} F(\nu_1^E, \dots, \nu_{n-1}^E, 0, \nu_t^E)\, d\mathcal{H}^n \\ &\le \int_{\partial^* E \cap (A \times \mathbb{R}_y)} F(\nu_1^E, \dots, \nu_{n-1}^E, 0, \nu_t^E)\, d\mathcal{H}^n. \end{aligned} \tag{5.15}$$

Inequality (5.11) follows from (5.13) and (5.15). □

Proof of Theorem 2.7. Let us first assume that $\mathcal{L}^{n+1}(E \cap (U \times \mathbb{R}_y)) < +\infty$.
Let G_{E^s} be the set provided by Theorem D, with E replaced by E^s. Given any Borel set $A \subset U$, define $A_1 = A \setminus G_{E^s}$ and $A_2 = A \cap G_{E^s}$. Inequalities (5.4) and (5.11), and assumption (2.24), yield

$$\int_{\partial^* E^s \cap (A_1 \times \mathbb{R}_y)} F(\nu^{E^s})\, d\mathcal{H}^n \le \int_{\partial^* E \cap (A_1 \times \mathbb{R}_y)} F(\nu^E)\, d\mathcal{H}^n + F(0,\dots,0,1,0)|D_y \chi_{E^s}|(A_1 \times \mathbb{R}_y). \tag{5.16}$$

We have

$$|D_y \chi_{E^s}|(A_1 \times \mathbb{R}_y) = \int_{\partial^* E^s \cap (A_1 \times \mathbb{R}_y)} |\nu_y^{E^s}| d\mathcal{H}^n = \int_{A_1} \mathcal{H}^0((\partial^* E^s)_{x',t})\, dx' dt = \int_{A_1} \mathcal{H}^0(\partial^* (E^s)_{x',t})\, dx' dt = 0, \tag{5.17}$$

where the first equality is due to (3.4), the second to the coarea formula (3.10), the third to equation (3.12), which holds for $\mathcal{L}^n$-a.e. $(x',t) \in \mathbb{R}^{n-1} \times \mathbb{R}_t$ and the fourth to the fact that $\mathcal{L}^n(\pi_{n-1,n+1}(E)^+ \cap A_1) = 0$. From (5.16)-(5.17) we get

$$\int_{\partial^* E^s \cap (A_1 \times \mathbb{R}_y)} F(\nu^{E^s})\, d\mathcal{H}^n \le \int_{\partial^* E \cap (A_1 \times \mathbb{R}_y)} F(\nu^E)\, d\mathcal{H}^n . \tag{5.18}$$

As for A_2, we have the following chain of equalities and inequalities (below, G_E is the set associated with E as in Theorem D):

$$\begin{aligned}
\int_{\partial^* E^s \cap (A_2 \times \mathbb{R}_y)} F(\nu^{E^s}) d\mathcal{H}^n &= \int_{\partial^* E^s \cap (A_2 \times \mathbb{R}_y)} F\Big(\frac{\nu^{E^s}}{|\nu_y^{E^s}|}\Big) |\nu_y^{E^s}| d\mathcal{H}^n \qquad \text{(by (2.21))} \\
&= \int_{A_2} dx'dt \int_{\partial^*(E^s)_{x',t}} F\Big(\frac{\nu_1^{E^s}}{|\nu_y^{E^s}|}, \dots, \frac{\nu_{n-1}^{E^s}}{|\nu_y^{E^s}|}, 1, \frac{\nu_t^{E^s}}{|\nu_y^{E^s}|}\Big) d\mathcal{H}^0(y) \qquad \text{(by (2.24) and formula (3.10))} \\
&= \int_{A_2} F(\nabla_1 \ell_E(x',t), \dots, \nabla_{n-1} \ell_E(x',t), 2, \nabla_t \ell_E(x',t)) dx'dt \qquad \text{(by formulas (5.3))} \\
&= \int_{A_2 \cap G_E} F(\nabla_1 \ell_E(x',t), \dots, \nabla_{n-1} \ell_E(x',t), 2, \nabla_t \ell_E(x',t)) dx'dt \qquad (\text{since } \mathcal{L}^n(A_2 \setminus G_E) = 0) \\
&\le \int_{A_2 \cap G_E} F\Big(\int_{\partial^* E_{x',t}} \frac{\nu_1^E}{|\nu_y^E|} d\mathcal{H}^0, \dots, \int_{\partial^* E_{x',t}} \frac{\nu_{n-1}^E}{|\nu_y^E|} d\mathcal{H}^0, \int_{\partial^* E_{x',t}} d\mathcal{H}^0, \int_{\partial^* E_{x',t}} \frac{\nu_t^E}{|\nu_y^E|} d\mathcal{H}^0\Big) dx'dt \\
&\qquad\qquad \text{(by (5.2) and the isoperimetric inequality in } \mathbb{R}) \\
&\le \int_{A_2 \cap G_E} dx'dt \int_{\partial^* E_{x',t}} F\Big(\frac{\nu_1^E}{|\nu_y^E|}, \dots, \frac{\nu_{n-1}^E}{|\nu_y^E|}, 1, \frac{\nu_t^E}{|\nu_y^E|}\Big) d\mathcal{H}^0(y) \qquad \text{(by Jensen's inequality)} \\
&= \int_{\partial^* E \cap [(A_2 \cap G_E) \times \mathbb{R}_y]} F(\nu^E) d\mathcal{H}^n \qquad \text{(by the coarea formula (3.10))} \\
&\le \int_{\partial^* E \cap (A_2 \times \mathbb{R}_y)} F(\nu^E) d\mathcal{H}^n .
\end{aligned} \tag{5.19}$$

The conclusion follows from (5.18)-(5.19).
If the set E is such that $\ell_E < +\infty$ $\mathcal{L}^n$-a.e. in U, then (2.25) is a straightforward consequence of what we have already proved and of Lemma 5.5 below. Finally, if E is a set of finite perimeter in $\mathbb{R}^{n+1}$, the isoperimetric inequality (see [AFP, Theorem 3.46]) implies that either E or $\mathbb{R}^{n+1} \setminus E$ has finite measure, and in the latter case (2.26) immediately follows, since E^s is equivalent to $\mathbb{R}^{n+1}$, and hence $\partial^* E^s = \emptyset$. □

Lemma 5.5 *Let U be an open subset of $\mathbb{R}^{n-1} \times \mathbb{R}_t$ and let E be a set of finite perimeter in $U \times \mathbb{R}_y$ such that $\ell_E(x',t) < +\infty$ for $\mathcal{L}^n$-a.e. $(x',t) \in U$. Then*

$$\mathcal{L}^{n+1}(E \cap (U' \times \mathbb{R}_y)) < +\infty \tag{5.20}$$

for every open set $U' \subset\subset U$.

Proof. Let us fix $U' \subset\subset U$ and set, for $h \in \mathbb{N}$, $E_h = E \cap (U' \times (-h,h))$. Thus, E_h is a set of finite perimeter in $U' \times \mathbb{R}_y$ and, by (3.1),

$$\partial^M E_h \cap (U' \times \mathbb{R}_y) \subset \left(\partial^M E \cup \{|y| = h\}\right) \cap (U' \times \mathbb{R}_y)\,. \tag{5.21}$$

Therefore, since $\mathcal{L}^{n+1}(E_h \cap (U' \times \mathbb{R}_y)) < +\infty$, from the first part of the proof of Theorem 2.7, and from (5.21),(3.3), (3.5) we have

$$P((E_h)^s; U' \times \mathbb{R}_y) \leq P(E_h; U' \times \mathbb{R}_y) \leq P(E; U' \times \mathbb{R}_y) + 2\mathcal{L}^n(U')\,. \tag{5.22}$$

Since we are assuming that $\ell_E(x',t) < +\infty$ for $\mathcal{L}^n$-a.e. $(x',t) \in U$, then $\chi_{(E_h)^s}(x',y,t) \to \chi_{E^s}(x',y,t)$ for $\mathcal{L}^{n+1}$-a.e. $(x',y,t) \in U' \times \mathbb{R}_y$. Thus, $\chi_{(E_h)^s} \to \chi_{E^s}$ in $L^1_{\rm loc}(U' \times \mathbb{R}_y)$ and, by the lower semicontinuity of the perimeter and by (5.22),

$$P(E^s; U' \times \mathbb{R}_y) \leq \liminf_{h\to\infty} P((E_h)^s; U' \times \mathbb{R}_y) \leq P(E; U \times \mathbb{R}_y) + 2\mathcal{L}^n(U')\,.$$

This inequality ensures that E^s is a set of finite perimeter in $U' \times \mathbb{R}_y$ and that

$$\int_{U'\times\mathbb{R}_y} \chi_{E^s} \operatorname{div}\psi\, dx \leq P(E^s; U' \times \mathbb{R}_y) < \infty \tag{5.23}$$

for every function $\psi \in [C^1_0(U' \times \mathbb{R}_y)]^{n+1}$ satisfying $\|\psi\|_\infty \leq 1$. Let us fix $k \in \mathbb{N}$ and choose $\psi = (0,\ldots,0,g(y),0)$ in (5.23), where g is any function form $C^1_0(\mathbb{R})$ fulfilling $0 \leq g(y) \leq 1$ for $y \in \mathbb{R}$ and vanishing outside $(0,k)$. Thus,

$$\int_{U'} g\left(\frac{\ell_E(x',t)}{2}\right) dx'dt = \int_{U'} dx'dt \int_{-\ell_E(x',t)/2}^{\ell_E(x',t)/2} g'(y)\, dy \leq P(E^s; U' \times \mathbb{R}_y) < \infty\,. \tag{5.24}$$

On approximating first $\chi_{(0,k)}(y)$ monotonically by functions g as above, and then letting k go to $+\infty$, we get from (5.24)

$$\frac{1}{2}\int_{U'} \ell_E(x',t)\, dx'dt \leq P(E^s; U' \times \mathbb{R}_y) < \infty\,,$$

whence (5.20) immediately follows □

Proof of Theorem 2.8. Assumption (2.30) and inequality (2.25) entail that

$$\int_{\partial^* E^s \cap (A_2 \times \mathbb{R}_y)} F(\nu^{E^s})\, d\mathcal{H}^n = \int_{\partial^* E \cap (A_2 \times \mathbb{R}_y)} F(\nu^E)\, d\mathcal{H}^n \tag{5.25}$$

for every Borel subset A_2 of A. Thus, by Lemma 5.5, on replacing, if necessary, A by $A \cap U'$, where $U' \subset\subset U$, we may assume, without loss of generality, that $\mathcal{L}^{n+1}(E \cap (U \times \mathbb{R}_y)) < +\infty$. Therefore, on choosing $A_2 = A \cap G_E \cap G_{E^s}$ in (5.25), we get that all the inequalities in (5.19) hold, in fact, as equalities.

The first one, combined with assumption (2.27), ensures that $\mathcal{H}^0(\partial^* E_{x',t}) = 2$ for $\mathcal{L}^n$-a.e. $(x',t) \in A_2$, and hence for $\mathcal{L}^n$-a.e. $(x',t) \in \pi_{n-1,n+1}(E)^+ \cap A$. By the isoperimetric theorem in $\mathbb{R}$, assertion (2.31) follows. If the second of the inequalities in (5.19) holds as an equality and (2.28), (2.29) are in force, then

$$\frac{\nu_i^E(x', y_1(x',t), t)}{|\nu_y^E(x', y_1(x',t), t)|} = \frac{\nu_i^E(x', y_2(x',t), t)}{|\nu_y^E(x', y_2(x',t), t)|} \qquad i = 1, \dots, n-1, t\,, \tag{5.26}$$

for $\mathcal{L}^n$-a.e. $(x',t) \in A_2$, and hence for $\mathcal{L}^n$-a.e. $(x',t) \in \pi_{n-1,n+1}(E)^+ \cap A$. Since $|\nu^E(x',y,t)| = 1$ for $(x',y,t) \in \partial^* E$, equation (5.26) implies (2.32) and

$$|\nu_y^E(x', y_1(x',t), t)| = |\nu_y^E(x', y_2(x',t), t)| \tag{5.27}$$

for $\mathcal{L}^n$-a.e. $(x',t) \in \pi_{n-1,n+1}(E)^+ \cap A$. Equation (2.33) easily follows from (5.27), owing to (3.14), Theorem D. □

The remaining part of this section is concerned with the proof of Theorem 2.9. The following two lemmas deal with conditions (2.34)-(2.36).

Lemma 5.6 *Let U be an open subset of $\mathbb{R}^{n-1} \times \mathbb{R}_t$. Let E be a set of finite perimeter in $U \times \mathbb{R}_y$ and let A be a Borel subset of U. Let $F : \mathbb{R}^{n+1} \to [0, +\infty)$ be a Borel function. If*

$$\mathcal{H}^n\Big(\{(x',y,t) \in \partial^* E : \nu_y^E(x',y,t) = 0\} \cap (A \times \mathbb{R}_y)\Big) = 0\,, \tag{5.28}$$

then

$$\int_{\partial^* E \cap (A' \times \mathbb{R}_y)} F(\nu^E)\, d\mathcal{H}^n = 0 \qquad \text{for every Borel subset } A' \text{ of } A \text{ such that } \mathcal{L}^n(A') = 0\,. \tag{5.29}$$

Assume, in addition, that (2.34) is fulfilled. Then (5.29) implies (5.28).

Proof. Assume that E and A satisfy (5.28), and let A' be as in (5.29). Then

$$\begin{aligned}\int_{\partial^* E \cap (A' \times \mathbb{R}_y)} F(\nu^E)\, d\mathcal{H}^n &= \int_{\partial^* E} \frac{1}{|\nu_y^E|} \chi_{\{\nu_y^E \neq 0\} \cap (A' \times \mathbb{R}_y)} F(\nu^E) |\nu_y^E|\, d\mathcal{H}^n + \int_{\partial^* E} \chi_{\{\nu_y^E = 0\} \cap (A' \times \mathbb{R}_y)} F(\nu^E)\, d\mathcal{H}^n \\ &= \int_{A'} dx'dt \int_{(\partial^* E)_{x',t}} \chi_{\{\nu_y^E \neq 0\}} \frac{F(\nu^E)}{|\nu_y^E|}\, d\mathcal{H}^n + \int_{\{\nu_y^E = 0\} \cap (A' \times \mathbb{R}_y)} F(\nu^E)\, d\mathcal{H}^n,\end{aligned}$$

where the second equality holds because of the coarea formula (3.10). Hence, (5.29) follows, since $\mathcal{L}^n(A') = 0$ and (5.28) is in force.
Conversely, (5.29) and (2.34) imply that

$$\mathcal{H}^n\Big(\{\nu_y^E = 0\} \cap (A' \times \mathbb{R}_y)\Big) = 0 \tag{5.30}$$

for every A' as in (5.29). In particular, we may choose $A' = A \setminus G_E$, where G_E is the set given by Theorem D. Since $\nu_y^E \neq 0$ on $G_E \times \mathbb{R}_y$, then $\{\nu_y^E = 0\} \cap (A \times \mathbb{R}_y) = \{\nu_y^E = 0\} \cap (A' \times \mathbb{R}_y)$. Hence, by (5.30),

$$\mathcal{H}^n\Big(\{\nu_y^E = 0\} \cap (A \times \mathbb{R}_y)\}\Big) = \mathcal{H}^n\Big(\{\nu_y^E = 0\} \cap (A' \times \mathbb{R}_y)\}\Big) = 0\,.$$

□

Lemma 5.7 *Let U be an open subset of $\mathbb{R}^{n-1} \times \mathbb{R}_t$. Let E be a set of finite perimeter in $U \times \mathbb{R}_y$ and let A be a Borel subset of U. Then (5.28) implies that*

$$\mathcal{H}^n \left(\{(x', y, t) \in \partial^* E^s : \nu_y^{E^s}(x', y, t) = 0\} \cap (A \times \mathbb{R}_y) \right) = 0 . \tag{5.31}$$

Conversely, if (2.30) and (2.34) are fulfilled, then (5.31) implies (5.28).

Proof. Assume that (5.28) is satisfied, and choose any convex function $F : \mathbb{R}^{n+1} \to [0, +\infty)$ satisfying (2.21), (2.24) and (2.34) (e.g. $F(\xi) = |\xi|$). By Lemma 5.6, $\int_{\partial^* E \cap (A' \times \mathbb{R}_y)} F(\nu^E) d\mathcal{H}^n = 0$ for every Borel set $A' \subset A$ such that $\mathcal{L}^n(A') = 0$. Inequality (2.25) ensures that $\int_{\partial^* E^s \cap (A' \times \mathbb{R}_y)} F(\nu^{E^s}) d\mathcal{H}^n = 0$ as well. Owing to Lemma 5.6, equation (5.31) follows.
In order to establish the reverse implication, observe that, if (2.30) holds, then, by (2.25), we have in fact that

$$\int_{\partial^* E \cap (A' \times \mathbb{R}_y)} F(\nu^E)\, d\mathcal{H}^n = \int_{\partial^* E^s \cap (A' \times \mathbb{R}_y)} F(\nu^{E^s})\, d\mathcal{H}^n \tag{5.32}$$

for every Borel set $A' \subset A$. Lemma 5.6 applied to E^s, and equations (5.31)-(5.32) imply that

$$\int_{\partial^* E \cap (A' \times \mathbb{R}_y)} F(\nu^E)\, d\mathcal{H}^n = \int_{\partial^* E^s \cap (A' \times \mathbb{R}_y)} F(\nu^{E^s})\, d\mathcal{H}^n = 0 \tag{5.33}$$

for every Borel set $A' \subset A$ such that $\mathcal{L}^n(A') = 0$. Owing to (2.34), equation (5.33) yields (5.28), via Lemma 5.6. □

The purpose of Lemma 5.8 below is to show that, if E is any set of finite perimeter in $U \times \mathbb{R}_y$ satisfying (2.36) and (2.31) for some open set $U \subset \mathbb{R}^{n-1} \times \mathbb{R}_t$, and F is a convex function as in Theorem 2.7, then the functional $\int_{\partial^* E \cap (U \times \mathbb{R}_y)} F(\nu^E) d\mathcal{H}^n$ is invariant under *polarization* (also called *two-point symmetrization*) of E about any hyperplane of the form $\{y = \lambda\}$, with $\lambda \in \mathbb{R}$. This result plays a role in the proof of Theorem 2.9, in the case where E is not necessarily bounded along the y-axis.

Lemma 5.8 *Let U be an open subset of $\mathbb{R}^{n-1} \times \mathbb{R}_t$. Let E be a set of finite perimeter in $U \times \mathbb{R}_y$ having the property that there exist two functions $y_1, y_2 : U \to \mathbb{R}$ such that, for $\mathcal{L}^n$-a.e. $(x', t) \in U$, $y_1(x', t) \le y_2(x', t)$ and $E_{x',t}$ is equivalent to $(y_1(x', t), y_2(x', t))$. Given any $\lambda \in \mathbb{R}$, set*

$$\widehat{y}_1^\lambda(x', t) = \max\{y_1(x', t) - \lambda, \lambda - y_2(x', t)\}, \quad \widehat{y}_2^\lambda(x', t) = \max\{\lambda - y_1(x', t), y_2(x', t) - \lambda\}$$

for $(x', t) \in U$, and

$$\widehat{E}_{(\lambda)} = \{(x', y, t) : (x', t) \in U \ \text{and} \ \widehat{y}_1^\lambda(x', t) < y < \widehat{y}_2^\lambda(x', t)\} . \tag{5.34}$$

Then $\widehat{E}_{(\lambda)}$ is a set of finite perimeter in $U \times \mathbb{R}_y$ and

$$\ell_{\widehat{E}_{(\lambda)}} \equiv \ell_E . \tag{5.35}$$

Assume, in addition, that (2.36) is fulfilled. Then the same condition is fulfilled with E replaced by $\widehat{E}_{(\lambda)}$, and

$$\int_{\partial^* \widehat{E}_{(\lambda)} \cap (U \times \mathbb{R}_t)} F(\nu^{\widehat{E}_{(\lambda)}})\, d\mathcal{H}^n = \int_{\partial^* E \cap (U \times \mathbb{R}_t)} F(\nu^E)\, d\mathcal{H}^n . \tag{5.36}$$

Moreover, if E satisfies (2.37), then the same condition is satisfied with E replaced by $\widehat{E}_{(\lambda)}$.

Proof. We assume that $\lambda = 0$, the other cases being completely analogous, and denote $\widehat{y}_1^\lambda$, $\widehat{y}_2^\lambda$ and $\widehat{E}_{(\lambda)}$ simply by $\widehat{y}_1$, $\widehat{y}_2$ and $\widehat{E}$. We may also assume, without loss of generality that, for every $(x',t) \in U$, $y_1(x',t) \le y_2(x',t)$ and $E_{x',t} = (y_1(x',t), y_2(x',t))$. Definition (5.34) immediately implies (5.35), and hence the assertion concerning (2.37) trivially follows.
Now, define $\widetilde{E} = \{(x',y,t) : (x',-y,t) \in E\}$ and observe that

$$\widehat{E} \text{ is equivalent to } \Big[(E \cup \widetilde{E}) \cap (U \times (\mathbb{R}_y^+ \cup \{0\}))\Big] \cup \Big[(E \cap \widetilde{E}) \cup (U \times \mathbb{R}_y^-)\Big] . \tag{5.37}$$

Thus, $\widehat{E}$ is a set of finite perimeter in $U' \times \mathbb{R}_y$ for every open set $U' \subset\subset U$. Moreover, from (5.37) and (3.1) we infer that $\partial^M \widehat{E} \subset \partial^M (E \cup \widetilde{E}) \cup \partial(E \cap \widetilde{E}) \cup \{y = 0\} \subset \partial^M E \cup \partial^M \widetilde{E} \cup \{y = 0\}$, whence condition (2.36) for $\widehat{E}$ easily follows, owing to the same condition for E and to (3.5).
Let us prove (5.36). Let G_E, $G_{\widetilde{E}}$, and $G_{\widehat{E}}$ be the subsets of U associated with E, $\widetilde{E}$ and $\widehat{E}$ as in Theorem D. Clearly $G_E = G_{\widetilde{E}}$. Set $G = G_E \cap G_{\widehat{E}}$. Then

$$(\partial^* E)_{x',t} = \partial^*(E_{x',t}), \quad (\partial^* \widetilde{E})_{x',t} = \partial^*(\widetilde{E}_{x',t}), \quad (\partial^* \widehat{E})_{x',t} = \partial^*(\widehat{E}_{x',t}) \tag{5.38}$$

for every $(x',t) \in G$. By the very definition of $\widehat{E}$, either $\widehat{E}_{x',t} = E_{x',t}$, or $\widehat{E}_{x',t} = \widetilde{E}_{x',t}$. Thus, equations (5.38) imply that

$$\text{either } (\partial^* \widehat{E})_{x',t} = (\partial^* E)_{x',t} \text{ or } (\partial^* \widehat{E})_{x',t} = (\partial^* \widetilde{E})_{x',t} \text{ for each } (x',t) \in G . \tag{5.39}$$

By (5.39), by (2.36) for $\widehat{E}$, and by the coarea formula (3.10), we have that $\widehat{E}$ is, in fact, a set of finite perimeter in $U \times \mathbb{R}_y$. We claim that there exists a set $N \subset \mathbb{R}^{n+1}$ such that $\mathcal{H}^n(N) = 0$ and

$$\begin{cases} \nu^{\widehat{E}}(x',y,t) = \nu^E(x',y,t) & \text{if } (x',y,t) \in \Big(\partial^* \widehat{E} \cap \partial^* E \cap (U \times \mathbb{R}_y)\Big) \setminus N \\ \nu^{\widehat{E}}(x',y,t) = \nu^{\widetilde{E}}(x',y,t) & \text{if } (x',y,t) \in \Big(\partial^* \widehat{E} \cap \partial^* \widetilde{E} \cap (U \times \mathbb{R}_y)\Big) \setminus N . \end{cases} \tag{5.40}$$

Actually, the fact that (5.40) holds with the equalities replaced by $\nu^{\widehat{E}} = \pm \nu^E$ and $\nu^{\widehat{E}} = \pm \nu^{\widetilde{E}}$ is a consequence of Theorem B. On the other hand, if $(x',y,t) \in \partial^* \widehat{E} \cap \partial^* E$, namely if $\widehat{E}_{x',t} = (y_1(x',t), y_2(x',t)) = E_{x',t}$, then $\lim\limits_{y \to y_1(x',t)^+} \chi^*_{\widehat{E}}(x',y,t) = 1$ and $\lim\limits_{y \to y_2(x',t)^-} \chi^*_{\widehat{E}}(x',y,t) = 1$, whence, by (3.14), $\nu_y^{\widehat{E}}(x', y_1(x',t), t) > 0$ and $\nu_y^{\widehat{E}}(x', y_2(x',t), t) < 0$. Thus, the first equality in (5.40) holds true. An analogous argument proves the second equality in (5.40) when $(x',y,t) \in \partial^* \widehat{E} \cap \partial^* \widetilde{E}$. From (5.40) we get that

$$\begin{cases} \nu^{\widehat{E}}(x',\cdot,t) = \nu^E(x',\cdot,t) & \text{if } (x',t) \in (G \setminus \pi_{n-1,n+1}(N)) \cap \pi_{n-1,n+1}(\partial^* \widehat{E} \cap \partial^* E) \\ \nu^{\widehat{E}}(x',\cdot,t) = \nu^{\widetilde{E}}(x',\cdot,t) & \text{if } (x',t) \in (G \setminus \pi_{n-1,n+1}(N)) \cap \pi_{n-1,n+1}(\partial^* \widehat{E} \cap \partial^* \widetilde{E}) , \end{cases} \tag{5.41}$$

where $\pi_{n-1,n+1}(N)$ satisfies $\mathcal{L}^n(\pi_{n-1,n+1}(N)) = 0$. Thus,

$$\begin{aligned} \int_{\partial^* \widehat{E} \cap (U \times \mathbb{R}_y)} F(\nu^{\widehat{E}}) d\mathcal{H}^n &= \int_{\partial^* \widehat{E} \cap ((G \setminus \pi_{n-1,n+1}(N)) \times \mathbb{R}_y)} F(\nu^{\widehat{E}})\, d\mathcal{H}^n = \int_{G \setminus \pi_{n-1,n+1}(N)} dx'dt \int_{\partial^*(\widehat{E}_{x',t})} \frac{F(\nu^{\widehat{E}})}{|\nu_y^{\widehat{E}}|} d\mathcal{H}^0 \\ &= \sum_{i=1}^2 \int_{G \setminus \pi_{n-1,n+1}(N)} \frac{F(\nu^{\widehat{E}}(x', \widehat{y}_i(x',t), t))}{|\nu_y^{\widehat{E}}(x', \widehat{y}_i(x',t), t)|} dx'dt \\ &= \int_{G \setminus \pi_{n-1,n+1}(N)} dx'dt \int_{\partial^*(E_{x',t})} \frac{F(\nu^E)}{|\nu_y^E|} d\mathcal{H}^0 = \int_{\partial^* E \cap (U \times \mathbb{R}_y)} F(\nu^E) d\mathcal{H}^n . \end{aligned}$$

Note that the first equality is due to (2.36) with E replaced by $\widehat{E}$, to Lemma 5.6 and to the fact that $\mathcal{L}^n\left((\pi_{n-1,n+1}(E)^+ \cap U) \setminus (G \setminus \pi_{n-1,n+1}(N))\right) = 0$; the second is an application of the coarea formula (3.10); the fourth is a consequence of (5.39), (5.41) and of the fact that if $\widehat{E}_{x',t} = \widetilde{E}_{x',t}$, then $\nu_j^E(x', y_i(x',t), t) = \nu_j^{\widehat{E}}(x', -y_i(x',t), t)$ $(j = 1, \dots, n-1, t)$ and $\nu_y^E(x', y_i(x',t), t) = -\nu_y^{\widehat{E}}(x', -y_i(x',t), t)$ for $i = 1, 2$ and for $(x', t) \in G$; the fifth is due to the first three equalities applied with $\widehat{E}$ replaced by E.

$\square$

Proof of Theorem 2.9. Owing to Theorem 2.8, we may assume, without loss of generality, that there exist functions $y_1, y_2 : U \to \mathbb{R}$ such that

$$y_1(x',t) \le y_2(x',t) \quad \text{and} \quad E_{x',t} = (y_1(x',t), y_2(x',t)) \tag{5.42}$$

for every $(x',t) \in U$. The set E satisfies (2.37) and, by Lemma 5.7, it certainly satisfies (2.36) as well. Suppose, for a moment, that E fulfills the additional assumption that a constant k exists such that

$$\text{either } y_1(x',t) \le k \text{ for } \mathcal{L}^n\text{-a.e. } (x',t) \in U, \text{ or } y_2(x',t) \ge k \text{ for } \mathcal{L}^n\text{-a.e. } (x',t) \in U. \tag{5.43}$$

Lemma 3.3 of [CCF] ensures that, under (5.42), (2.36), (2.37) and (5.43), the functions $y_1, y_2 \in W^{1,1}_{\text{loc}}(U)$. Let us define

$$E^1 = \{(x', y, t) : (x',t) \in U, \ \ y > y_1(x',t)\}$$
$$E^2 = \{(x', y, t) : (x',t) \in U, \ \ y < y_2(x',t)\}.$$

Then, by Theorem A, both E^1 and E^2 are sets of finite perimeter in $U' \times \mathbb{R}_y$ for every open set $U' \subset\subset U$. The same argument as in the proof of Theorem 2.2 tells us that (4.51), (4.52), (4.53), (4.54) and (4.58) are fulfilled , with E_1, E_2 replaced by E^1, E^2 and with z_1, z_2 replaced by y_1, y_2, for $\mathcal{L}^n$-a.e. $(x',t) \in U$. Notice that, after these replacements, equations (4.54) turn into (2.32)-(2.33) of Theorem 2.8. We can thus conclude that

$$\nabla y_1(x',t) + \nabla y_2(x',t) = 0 \qquad \text{for } \mathcal{L}^n\text{-a.e. } (x',t) \in U\,. \tag{5.44}$$

Since, $y_1, y_2 \in W^{1,1}_{\text{loc}}(U)$, equation (5.44) entails that, for every connected component U_α of U, a constant c_α exists such that

$$y_1(x',t) + y_2(x',t) = c_\alpha \qquad \text{for } \mathcal{L}^n\text{-a.e. } (x',t) \in U_\alpha\,,$$

and hence the conclusion follows.

We have now to remove assumption (5.43). Given any $\lambda \in \mathbb{R}$, let $\widehat{E}_{(\lambda)}$ be the set defined as in (5.34). Thanks to Lemma 5.8, the set $\widehat{E}_{(\lambda)}$ is of finite perimeter in $U \times \mathbb{R}_y$, and (2.36)-(2.37) are satisfied with E replaced by $\widehat{E}_{(\lambda)}$; the additional assumption (5.43) is also satisfied with y_1, y_2 replaced by $\widehat{y}_1^\lambda$, $\widehat{y}_2^\lambda$, since $\widehat{y}_2^\lambda(x',t) \ge 0$ f for $(x',t) \in U$. Furthermore

$$\int_{\partial^* \widehat{E}_{(\lambda)} \cap (U \times \mathbb{R}_y)} F(\nu^{\widehat{E}_{(\lambda)}})\, d\mathcal{H}^n = \int_{\partial^* E \cap (U \times \mathbb{R}_y)} F(\nu^E)\, d\mathcal{H}^n$$
$$= \int_{\partial^* E^s \cap (U \times \mathbb{R}_y)} F(\nu^{E^s})\, d\mathcal{H}^n = \int_{\partial^* (\widehat{E}_{(\lambda)})^s \cap (U \times \mathbb{R}_y)} F(\nu^{(\widehat{E}_{(\lambda)})^s})\, d\mathcal{H}^n\,,$$

where the first equality is due to (5.36), the second to (2.30) and to (2.25), and the third to the fact that $E^s = (\widehat{E}_{(\lambda)})^s$. Thus, from Theorem 2.8, we infer that (2.32)-(2.33) are fulfilled, with E replaced by $\widehat{E}_{(\lambda)}$

and y_1, y_2 replaced by $\widehat{y}_1^\lambda$, $\widehat{y}_2^\lambda$, for $\mathcal{L}^n$-a.e. $(x', t) \in U$ (notice that $\pi_{n-1,n+1}(E)^+ \cap U$ is equivalent to U, owing to (2.37)). Thus, the same argument as above can be applied to conclude that, for each connected component U_α of U, a constant $c_{\alpha,\lambda}$ exists such that

$$\widehat{y}_1^\lambda(x', t) + \widehat{y}_2^\lambda(x', t) = c_{\alpha,\lambda} \qquad \text{for } \mathcal{L}^n\text{-a.e. } (x', t) \in U_\alpha\,. \tag{5.45}$$

Choosing any two different values of λ in (5.45) easily entails that $y_1(x', t) + y_2(x', t)$ has to be constant $\mathcal{L}^n$-a.e. in U. The proof is complete. □

6 Functionals of BV functions

Here, we accomplish the proofs of Theorems 2.4 and 2.5. The former relies upon Theorem 2.7 and Proposition 2.6. The proof of this proposition makes use of Theorem F below, an easy consequence of well known relaxation results for integral functionals depending on BV functions – see e.g. [But] and [AFP, Theorem 5.47].

Theorem F *Let f be a convex function satisfying (2.14). Let Ω be an open subset of $\mathbb{R}^n$, and let J_f be the functional defined as in (2.15). If $u \in BV(\Omega)$ and $\{u_k\}$ is any sequence in $BV(\Omega)$ such that $u_k \to u$ in $L^1_{\mathrm{loc}}(\Omega)$, then*

$$J_f(u; \Omega) \le \liminf_{k\to+\infty} J_f(u_k; \Omega)\,. \tag{6.1}$$

Moreover, for every $u \in BV(\Omega)$, there exists a sequence $\{u_h\}$ in $C^1(\Omega)$ such that $u_h \to u$ in $L^1_{\mathrm{loc}}(\Omega)$ and

$$J_f(u; \Omega) = \lim_{h\to+\infty} J_f(u_h; \Omega)\,.$$

The next result tells us how properties of the integrand f are inherited by F_f.

Lemma 6.1 *Let $f : \mathbb{R}^n \to [0, +\infty)$ be a convex function vanishing at 0. Then, the function F_f defined by (2.22) is a convex function satisfying (2.21). If, in addition, f satisfies the same assumptions as in Theorem 2.5, then F_f satisfies (2.24), (2.27) and (2.28) with $K = \mathbb{R}^{n-1} \times (\mathbb{R}^-_t \cup \{0\})$.*

Proof.
Since f is a convex function such that $f(0) = 0$, then (4.25) holds with $b_j \le 0$ for every $j \in \mathbb{N}$. Therefore,

$$F(\xi) = \sup_{j\in\mathbb{N}} \Big\{ \sum_{i=1}^n (a_j)_i \xi_i + (b_j \xi_{n+1})^+ \Big\}$$

for every $\xi \in \mathbb{R}^{n+1}$. Thus, F is a convex function satisfying (2.21). If f satisfies also (2.6), then we infer from (4.26) that,

$$F(\xi) = \sup_{j\in\mathbb{N}} \Big\{ \sum_{i=1}^n (a_j)_i \xi_i + (b_j \xi_{n+1})^+, \sum_{i=1}^n (\overline{a}_j)_i \xi_i + (b_j \xi_{n+1})^+ \Big\}\,,$$

for every $\xi \in \mathbb{R}^{n+1}$. Hence, (2.24) follows. The remaining assertions follow by elementary considerations. □

Proof of Proposition 2.6. Since $\mathcal{S}_u \subset \mathbb{R}^n \times \mathbb{R}_t^+$, then $\partial^* \mathcal{S}_u \cap (\Omega \times \mathbb{R}_t^-) = \emptyset$. Moreover, $\nu^{\mathcal{S}_u}(x', y, 0) = (0, \dots, 0, 1)$ for $\mathcal{H}^n$-a.e. $(x', y, 0) \in \partial^* \mathcal{S}_u \cap \{t = 0\}$, by Theorem B, and $F_f(\nu^{\mathcal{S}_u}) = 0$ for the same $(x', y, 0)$, since $f(0) = 0$. Thus, for every Borel set $B \subset \Omega$,

$$\int_{\partial^* \mathcal{S}_u \cap (B \times \mathbb{R}_t)} F_f(\nu^{\mathcal{S}_u})\, d\mathcal{H}^n = \int_{\partial^* \mathcal{S}_u \cap (B \times \mathbb{R}_t^+)} F_f(\nu^{\mathcal{S}_u})\, d\mathcal{H}^n .$$

Similarly, we have

$$\int_{\partial^* \mathcal{S}_u^- \cap (B \times \mathbb{R}_t)} F_f(\nu^{\mathcal{S}_u})\, d\mathcal{H}^n = \int_{\partial^* \mathcal{S}_u^- \cap (B \times \mathbb{R}_t^+)} F_f(\nu^{\mathcal{S}_u^-})\, d\mathcal{H}^n .$$

Consequently, by (3.9),

$$\int_{\partial^* \mathcal{S}_u \cap (B \times \mathbb{R}_t)} F_f(\nu^{\mathcal{S}_u})\, d\mathcal{H}^n = \int_{\partial^* \mathcal{S}_u^- \cap (B \times \mathbb{R}_t)} F_f(\nu^{\mathcal{S}_u^-})\, d\mathcal{H}^n . \tag{6.2}$$

Let us first assume that $u \in W^{1,1}_{\text{loc}}(\Omega)$.
By Theorem C, $\nu^{\mathcal{S}_u}(x', y, t) < 0$ for for $\mathcal{H}^n$-a.e. $(x', y, t) \in \partial^* \mathcal{S}_u^-$. Thus, owing to formulas (3.8) and to the coarea formula (3.10),

$$\begin{aligned} \int_{\partial^* \mathcal{S}_u^- \cap (B \times \mathbb{R}_t)} F_f(\nu^{\mathcal{S}_u^-})\, d &= \int_{\partial^* \mathcal{S}_u^- \cap (B \times \mathbb{R}_t)} f\left(\frac{(\nu_1^{\mathcal{S}_u^-}, \dots, \nu_n^{\mathcal{S}_u^-})}{-\nu_t^{\mathcal{S}_u^-}} \right) (-\nu_t^{\mathcal{S}_u^-})\, d\mathcal{H}^n \\ &= \int_{\partial^* \mathcal{S}_u^- \cap (B \times \mathbb{R}_t)} f(\nabla u) |\nu_t^{\mathcal{S}_u^-}|\, d\mathcal{H}^n = \int_B f(\nabla u)\, dx = J_f(u; B) . \end{aligned} \tag{6.3}$$

Equation (2.23) is established.
Let us next take into account the case where $u \in BV_{\text{loc}}(\Omega)$ and let us fix an open set $\Omega' \subset\subset \Omega$. By Theorem F applied with $f(\xi) = \sqrt{1 + |\xi|^2}$, we may choose a sequence of functions $u_h \in C^1(\Omega')$ such that $u_h \to u$ in $L^1_{\text{loc}}(\Omega')$ and

$$\int_{\Omega'} \sqrt{1 + |\nabla u|^2}\, dx + |D^s u|(\Omega') = \lim_{h \to \infty} \int_{\Omega'} \sqrt{1 + |\nabla u_h|^2}\, dx . \tag{6.4}$$

Denote, for simplicity, $\mathcal{S}_{u_h}^-$ by $\mathcal{S}_h^-$, and observe that $\chi_{\mathcal{S}_h^-} \to \chi_{\mathcal{S}_u^-}$ in $L^1_{\text{loc}}(\Omega' \times \mathbb{R}_t)$ and that, owing to (6.4), $D\chi_{\mathcal{S}_h^-} \rightharpoonup D\chi_{\mathcal{S}_u^-}$ weakly* in $\Omega' \times \mathbb{R}_t$ and $|D\chi_{\mathcal{S}_h^-}|(\Omega' \times \mathbb{R}_t) \to |D\chi_{\mathcal{S}_u^-}|(\Omega' \times \mathbb{R}_t)$ as $h \to \infty$. From (6.1) and from the continuity theorem of Reshetnyak ([AFP, Theorem 2.39]) we get

$$\begin{aligned} J_f(u; \Omega') &\le \liminf_{h \to \infty} J_f(u_h; \Omega') = \liminf_{h \to \infty} \int_{\partial^* \mathcal{S}_h^- \cap (\Omega' \times \mathbb{R}_t)} F_f(\nu^{\mathcal{S}_h^-})\, d\mathcal{H}^n \\ &= \int_{\partial^* \mathcal{S}_u^- \cap (\Omega' \times \mathbb{R}_t)} F_f(\nu^{\mathcal{S}_u^-})\, d\mathcal{H}^n = \int_{\partial^* \mathcal{S}_u \cap (\Omega' \times \mathbb{R}_t)} F_f(\nu^{\mathcal{S}_u})\, d\mathcal{H}^n . \end{aligned} \tag{6.5}$$

In order to prove the reverse inequality, fix any $\varepsilon > 0$ and apply Theorem F with $f_\varepsilon(\xi) = f(\xi) + \varepsilon|\xi|$. Thus a sequence $\{u_h\}$ of functions from $C^1(B)$ can be chosen in such a way that $u_h \to u$ in $L^1_{\text{loc}}(B)$ and

$$J_{f_\varepsilon}(u; \Omega') = \lim_{h \to \infty} J_{f_\varepsilon}(u_h; \Omega') . \tag{6.6}$$

Since $f_\varepsilon(\xi) \ge \varepsilon|\xi|$ for every $t \ge 0$, then (6.6) entails that the sequence $|\nabla u_h|$ is bounded in $L^1(\Omega')$. Hence, by Theorem A, on passing if necessary to a subsequence, we may assume that $D\chi_{\mathcal{S}_h^-} \rightharpoonup D\chi_{\mathcal{S}_u^-}$ weakly* in $\Omega' \times \mathbb{R}_t$, where $\mathcal{S}_h^-$ is defined as above. Thus, from (6.6) and from the semicontinuity theorem of Reshetnyak ([AFP, Theorem 2.38]) we deduce that

$$(6.7) \quad \int_{\partial^* \mathcal{S}_u^- \cap (\Omega' \times \mathbb{R}_t)} F_{f_\varepsilon}(\nu^{\mathcal{S}_u^-})\, d\mathcal{H}^n \le \liminf_{h \to \infty} \int_{\partial^* \mathcal{S}_h^- \cap (\Omega' \times \mathbb{R}_t)} F_{f_\varepsilon}(\nu^{\mathcal{S}_h^-})\, d\mathcal{H}^n = \lim_{h \to \infty} J_{f_\varepsilon}(u_h; \Omega') = J_f(u; \Omega')\,.$$

On letting ε go to 0^+, we immediately get from (6.7) and (6.2) that

$$(6.8) \qquad \int_{\partial^* \mathcal{S}_u \cap (\Omega' \times \mathbb{R}_t)} F_f(\nu^{\mathcal{S}_u})\, d\mathcal{H}^n \le J(u; \Omega')\,.$$

Equation (2.23) with $B = \Omega'$ follows from (6.5) and (6.8). The general case where B is any Borel set in Ω follows by approximation. □

Proof of Theorem 2.4. We shall prove that if B is any Borel subset of $\pi_{n-1}(\Omega)$, then

$$(6.9) \qquad J_f(u^s; B \times \mathbb{R}_y) \le J_f(u; B \times \mathbb{R}_y)\,,$$

where we are again denoting u_0 by u.

First, notice that if u is any nonnegative function from $BV_{0,y}(\Omega)$, then $u^s \in BV(\omega \times \mathbb{R}_y)$ for every open set $\omega \subset\subset \pi_{n-1}(\Omega)$. Indeed, since $u \in BV(\omega \times \mathbb{R}_y)$, then there exists a sequence $\{u_h\}$ of nonnegative functions $u_h \in C^1(\omega \times \mathbb{R}_y)$, such that $u_h \to u$ in $L^1(U \times \mathbb{R}_y)$ and such that $\lim_{h \to \infty} \int_{\omega \times \mathbb{R}_y} |\nabla u_h| dx = |Du|(\omega \times \mathbb{R}_y)$. Thus, by the continuity of Steiner symmetrization ([K1, pag 23]), $u_h^s \to u^s$ in $L^1(U \times \mathbb{R}_y)$ and, by (4.16), the sequence $\int_{\omega \times \mathbb{R}_y} |\nabla u_h^s| dx$ is bounded. Hence, one infers that $u^s \in BV(\omega \times \mathbb{R}_y)$.

Let us next establish (6.9) under the assumption that u has compact support in Ω. By Theorem A, $\mathcal{S}_u$ is a set of finite perimeter in $\mathbb{R}^{n+1}$. Thus, by Proposition 2.6, Theorem 2.7 and (2.4),

$$J_f(u^s; B \times \mathbb{R}_y) = \int_{\partial^* \mathcal{S}_{u^s} \cap (B \times \mathbb{R}_y \times \mathbb{R}_t)} F_f(\nu^{\mathcal{S}_{u^s}})\, d\mathcal{H}^n \le \int_{\partial^* \mathcal{S}_u \cap (B \times \mathbb{R}_y \times \mathbb{R}_t)} F_f(\nu^{\mathcal{S}_u})\, d\mathcal{H}^n = J_f(u; B \times \mathbb{R}_y)$$

for every Borel set $B \subset \pi_{n-1}(\Omega)$. Inequality (6.9) follows.

Finally, consider the general case where u is any nonnegative function from $BV_{0,y}(\Omega)$. Fixed any open set $\omega \subset\subset \pi_{n-1}(\Omega)$, choose any function $\varphi \in C^1(\mathbb{R}^{n-1})$ having compact support in $\pi_{n-1}(\Omega)$ and such that $\varphi \equiv 1$ in B, and any function $\eta \in C_0^1(\mathbb{R})$ such that $\eta(y) \equiv 1$ in $[-1, 1]$. Define the function v in $\mathbb{R}^n$ as $v = u\varphi$ and, for $h \in \mathbb{N}$, the functions v_h as $v_h(x', y) = v(x', y)\eta(\frac{y}{h})$. Since $v \in BV(\mathbb{R}^n)$ and $v_h \to v$ in $L^1(\mathbb{R}^n)$, then

$$(6.10) \qquad J_f(u^s; \omega \times \mathbb{R}_y) = J_f(v^s; \omega \times \mathbb{R}_y) \le \liminf_{h \to \infty} J_f(v_h^s; \omega \times \mathbb{R}_y)\,.$$

Moreover, it is easily verified that $|D(v - v_h)|(\mathbb{R}^n) \to 0$ as $h \to \infty$. Thus,

$$(6.11) \qquad \liminf_{h \to \infty} J_f(v_h; \omega \times \mathbb{R}_y) = J_f(v; \omega \times \mathbb{R}_y) = J_f(u; \omega \times \mathbb{R}_y)\,.$$

Inequality (6.9) with $B = \omega$ follows from (6.10), (6.11) and from the same inequality for v_h. The general case where B is a Borel set can be derived by approximation. □

We now pass to the proof of Theorem 2.5. The next result plays an analogous role as Lemma 4.7 in Section 4.

Lemma 6.2 *Let f be as in Theorem 2.5. Let Ω be an open subset of $\mathbb{R}^n$ satisfying (2.10) and let u be a nonnegative function from $BV_{0,y}(\Omega)$ satisfying (2.19). Then there exist two functions $y_1, y_2 : \pi_{n-1,n+1}(\mathcal{S}_u)^+ \to \mathbb{R}$ such that, for $\mathcal{L}^n$-a.e. $(x',t) \in \pi_{n-1,n+1}(\mathcal{S}_u)^+$,*

$$(\mathcal{S}_u)_{x',t} \text{ is equivalent to } (y_1(x',t), y_2(x',t)) \tag{6.12}$$

$$\nu_i^{\mathcal{S}_u}(x', y_1(x',t), t) = \nu_i^{\mathcal{S}_u}(x', y_2(x',t), t) \qquad i = 1, \dots, n-1, t \tag{6.13}$$

$$\nu_y^{\mathcal{S}_u}(x', y_1(x',t), t) = -\nu_y^{\mathcal{S}_u}(x', y_2(x',t), t) \,. \tag{6.14}$$

Proof. Combining (2.19) and (6.9) tells us that equality holds in (6.9) for every Borel set $B \subset \pi_{n-1}(\Omega)$. Given any open set $\omega \subset\subset \pi_{n-1}(\Omega)$, let φ be any smooth function in $\mathbb{R}^{n-1}$, compactly supported in $\pi_{n-1}(\Omega)$, and such that $\varphi \equiv 1$ in ω. Thus, defined v in $\mathbb{R}^n$ as $v = u_0\varphi$, we have that

$$J_f(v^s; \omega \times \mathbb{R}_y) = J_f(v; \omega \times \mathbb{R}_y) \,.$$

Thus, by Proposition 2.6,

$$\int_{\partial^* \mathcal{S}_{v^s} \cap (\omega \times \mathbb{R}_y \times \mathbb{R}_t)} F(\nu^{\mathcal{S}_{v^s}})\, d\mathcal{H}^n = \int_{\partial^* \mathcal{S}_v (\omega \times \mathbb{R}_y \times \mathbb{R}_t)} F(\nu^{\mathcal{S}_v})\, d\mathcal{H}^n \,. \tag{6.15}$$

Since v is a nonnegative function from $BV(\mathbb{R}^n)$ which, owing to (2.10), has compact support, then $\mathcal{S}_v$ is a set of finite perimeter in $\mathbb{R}^{n+1}$. By Theorem C,

$$\left(\frac{\nu_1^{\mathcal{S}_v}}{|\nu_y^{\mathcal{S}_v}|}, \dots, \frac{\nu_{n-1}^{\mathcal{S}_v}}{|\nu_y^{\mathcal{S}_v}|}, \frac{\nu_t^{\mathcal{S}_v}}{|\nu_y^{\mathcal{S}_v}|} \right) \in \mathbb{R}^{n-1} \times (\mathbb{R}_t^- \cup \{0\}) \qquad \mathcal{H}^n\text{-a.e. on } \partial^* \mathcal{S}_v \cap (\mathbb{R}^{n-1} \times \mathbb{R}_y \times \mathbb{R}_t^+) \,. \tag{6.16}$$

By (6.15)–(6.16) and by Lemma 6.1, the conclusion follows from Theorem 2.8, thanks to the arbitrariness of ω. □

The BV counterpart of lemma of Lemma 4.8, contained in Lemma 6.4 below, requires the following result.

Lemma 6.3 *Let u be a nonnegative function from $BV_{0,y}(\Omega)$ satisfying the conclusion of Lemma 6.2. Assume, in addition, that u fulfills (2.8). Then, for $\mathcal{L}^{n-1}$-a.e. $x' \in \pi_{n-1}(\Omega)$,*

$$\frac{\chi_{(\mathcal{D}_u^+)_{x'}}(y_1(x',t))}{|\nabla_y u(x', y_1(x',t))|} = \frac{\chi_{(\mathcal{D}_u^+)_{x'}}(y_2(x',t))}{|\nabla_y u(x', y_2(x',t))|} \qquad \text{for } \mathcal{L}^1\text{-a.e. } t \in (0, M(x')) \,. \tag{6.17}$$

Proof. Our assumption on u ensures that there exists a subset G of $\pi_{n-1}(\Omega)$ satisfying $\mathcal{L}^{n-1}(\pi_{n-1}(\Omega) \backslash G) = 0$ and such that, for every $x' \in G$, $M(x') > 0$ and (6.12)-(6.14) hold for $\mathcal{L}^1$-a.e. $t \in (0, M(x'))$. Thus, on setting $L = (G \times \mathbb{R}_t^+) \cap \pi_{n-1,n+1}(\mathcal{S}_u)^+$, we have that, for $\mathcal{L}^n$-a.e. $(x',t) \in L$, the set $(\mathcal{S}_u)_{x',t}$ is equivalent to $(y_1(x',t), y_2(x',t))$ and $(\partial^* \mathcal{S}_u)_{x',t} = \{y_1(x',t), y_2(x',t)\}$. Hence, we deduce that

$$\pi_{n-1,n+1}(\partial^* \mathcal{S}_u) \cap L \text{ is equivalent to } L \,. \tag{6.18}$$

By Theorem C, there exists a Borel subset P of $\partial^*\mathcal{S}_u\cap(\mathcal{D}_u\times\mathrm{I\!R}^+_t)$ such that $\mathcal{H}^n((\partial^*\mathcal{S}_u\cap(\mathcal{D}_u\times\mathrm{I\!R}^+_t))\setminus P)=0$ and

$$\nu^{\mathcal{S}_u}(x',y,t)=\Big(\frac{\nabla_1 u(x',y)}{\sqrt{1+|\nabla u|^2}},\dots,\frac{\nabla_{n-1}u(x',y)}{\sqrt{1+|\nabla u|^2}},\frac{\nabla_y u(x',y)}{\sqrt{1+|\nabla u|^2}},\frac{-1}{\sqrt{1+|\nabla u|^2}}\Big) \tag{6.19}$$

for every $(x',y,t)\in P$, and there exists a Borel subset Q of $\partial^*\mathcal{S}_u\cap((\Omega\setminus\mathcal{D}_u)\times\mathrm{I\!R}^+_t)$ such that $\mathcal{H}^n((\partial^*\mathcal{S}_u\cap((\Omega\setminus\mathcal{D}_u)\times\mathrm{I\!R}^+_t)))\setminus Q)=0$ and

$$\nu^{\mathcal{S}_u}_t(x',y,t)=0 \qquad \text{for every } (x',y,t)\in Q\,. \tag{6.20}$$

Notice that

$$\pi_{n-1,n+1}(P\cup Q)\cap L \text{ is equivalent to } L\,, \tag{6.21}$$

as a consequence of the fact that $\pi_{n-1,n+1}(\partial^*\mathcal{S}_u)\setminus\pi_{n-1,n+1}(P\cup Q)\subset\pi_{n-1,n+1}(\partial^*\mathcal{S}_u\setminus(P\cup Q))$, of the fact that $\mathcal{L}^n(\pi_{n-1,n+1}(\partial^*\mathcal{S}_u\setminus(P\cup Q)))=0$ and of (6.18). Obviously, (6.21) implies that

$$(\pi_{n-1,n+1}(P)\cap L)\cup(\pi_{n-1,n+1}(Q)\cap L) \text{ is equivalent to } L\,. \tag{6.22}$$

Owing to (6.22) and to (4.33), we have that $(x',t)\in(\pi_{n-1,n+1}(P)\cap L)\cup(\pi_{n-1,n+1}(Q)\cap L)$ for $\mathcal{L}^{n-1}$-a.e. $x'\in G$ and for $\mathcal{L}^1$-a.e. $t\in(0,M(x'))$. Fix any such x'. Then, for $\mathcal{L}^1$-a.e. $t\in(0,M(x'))$ satisfying $(x',t)\in\pi_{n-1,n+1}(P)\cap L$, we have, by (6.19)-(6.20) and (6.13), that either $(x',y_1(x',t),t),(x',y_2(x',t),t)\in\partial^*\mathcal{S}_u\cap(\mathcal{D}^+_u\times\mathrm{I\!R}^+_t)$, or $(x',y_1(x',t),t),(x',y_2(x',t),t)\in\partial^*\mathcal{S}_u\cap(\mathcal{D}^0_u\times\mathrm{I\!R}^+_t)$. Hence, either $y_1(x',t),y_2(x',t)\in(\mathcal{D}^+_u)_{x'}$ or $y_1(x',t),y_2(x',t)\in(\mathcal{D}^0_u)_{x'}$ and, thanks to (6.14), equation (6.17) holds. On the other hand, for $\mathcal{L}^1$-a.e. $t\in(0,M(x'))$ such that $(x',t)\in\pi_{n-1,n+1}(Q)\cap L$, we have by (6.19)-(6.20) and (6.13) that both $(x',y_1(x',t),t)$ and $(x',y_2(x',t),t)$ belong to $\partial^*\mathcal{S}_u\cap((\Omega\setminus\mathcal{D}_u)\times\mathrm{I\!R}^+_t)$, whence $y_1(x',t),y_2(x',t)\notin(\mathcal{D}_u)_{x'}$, and (6.17) holds also in this case. □

Lemma 6.4 *Let f be as in Theorem 2.5. Let Ω be an open subset of $\mathrm{I\!R}^n$ and let u be a nonnegative function from $BV_{0,y}(\Omega)$ satisfying (2.8) and (2.19). Let y_1,y_2 be the functions appearing in Lemma 6.2. Then there exists a function $b:\pi_{n-1}(\Omega)\to\mathrm{I\!R}$ such that, for $\mathcal{L}^{n-1}$-a.e. $x'\in\pi_{n-1}(\Omega)$,*

$$\frac{1}{2}(y_1(x',t)+y_2(x',t))=b(x') \qquad \textit{for } \mathcal{L}^1\textit{-a.e. } t\in(0,M(x'))\,.$$

Proof. On adopting the same notations and proceeding along the same lines as in the proof of Lemma 4.8, one arrives at

$$\begin{aligned}\mu_{v_i^{x'}}(t)&=\mathcal{L}^1\left((\mathcal{D}^0_u)_{x'}\cap\{v_i^{x'}>t\}\right)+\mathcal{L}^1\left((\mathcal{D}^+_u)_{x'}\cap\{v_i^{x'}>t\}\right)\\&=\mathcal{L}^1\left((\mathcal{D}^+_u)_{x'}\cap\{v_i^{x'}>t\}\right)=\mathcal{L}^1\left((\mathcal{D}^+_u)_{x'}\cap\{u(x',\cdot)^*>t\}\cap I_i\right) \qquad i=1,2\,,\end{aligned} \tag{6.23}$$

for $\mathcal{L}^{n-1}$-a.e. $x'\in\pi_{n-1}(\Omega)$. Notice that the second equality in (6.23) holds because of (4.39). Owing to (3.17), we have that $|D^s v_i^{x'}|\llcorner(\mathcal{D}_u)_{x'}=0$ for $\mathcal{L}^{n-1}$-a.e. $x'\in\pi_{n-1}(\Omega)$. Hence,

$$\begin{aligned}&\int_{I_i}\frac{\chi_{\{u(x',\cdot)^*>t\}\cap(\mathcal{D}^+_u)_{x'}}(y)}{|\nabla_y u|}\,d|Dv_i^{x'}|\\&=\int_{I_i}\frac{\chi_{\{u(x',\cdot)^*>t\}\cap(\mathcal{D}^+_u)_{x'}}(y)}{|\nabla_y u|}\,d|D^a v_i^{x'}|+\int_{I_i}\frac{\chi_{\{u(x',\cdot)^*>t\}\cap(\mathcal{D}^+_u)_{x'}}(y)}{|\nabla_y u|}\,d|D^s v_i^{x'}|\\&=\int_{I_i}\frac{\chi_{\{u(x',\cdot)^*>t\}\cap(\mathcal{D}^+_u)_{x'}}(y)}{|\nabla_y u|}\left|\frac{d}{dy}v_i^{x'}(y)\right|dy=\mathcal{L}^1\left((\mathcal{D}^+_u)_{x'}\cap\{u(x',\cdot)^*>t\}\cap I_i\right) \qquad i=1,2,\end{aligned} \tag{6.24}$$

for $\mathcal{L}^{n-1}$-a.e. $x' \in \pi_{n-1}(\Omega)$. An application of the coarea formula (3.6) yields

$$(6.25)\quad \int_{I_i} \frac{\chi_{\{u(x',\cdot)^*>t\}\cap(\mathcal{D}_u^+)_{x'}}(y)}{|\nabla_y u|}\, d|Dv_i^{x'}| = \int_0^{M(x')} d\tau \int_{I_i\cap\partial^*\{v_i^{x'}>\tau\}} \frac{\chi_{\{u(x',\cdot)^*>t\}\cap(\mathcal{D}_u^+)_{x'}}(y)}{|\nabla_y u|}\, d\mathcal{H}^0(y)$$
$$= \int_0^{M(x')} d\tau \int_{I_i\cap\partial^*\{u(x',\cdot)>\tau\}} \frac{\chi_{\{u(x',\cdot)^*>t\}\cap(\mathcal{D}_u^+)_{x'}}(y)}{|\nabla_y u|}\, d\mathcal{H}^0(y)\,,$$

for $\mathcal{L}^{n-1}$-a.e. $x' \in \pi_{n-1}(\Omega)$. Observe that

$$\partial^M\{u(x',\cdot) > \tau\} \cap C_{u(x',\cdot)} \subset \{u(x',\cdot)^* = \tau\} \qquad \text{for } \tau > 0\,,$$

and hence, by (3.5),

$$\partial^*\{u(x',\cdot) > \tau\} \cap C_{u(x',\cdot)} \subset \{u(x',\cdot)^* = \tau\} \qquad \text{for } \mathcal{L}^1\text{-a.e. } \tau > 0\,.$$

Thus, since $(\mathcal{D}_u^+)_{x'} \subset (\mathcal{D}_u)_{x'} \subset (C_u)_{x'}$ and since, by (3.15), $(C_u)_{x'} \subset C_{u(x',\cdot)}$ for $\mathcal{L}^{n-1}$-a.e. $x' \in \pi_{n-1}(\Omega)$, then, for $\mathcal{L}^{n-1}$-a.e. $x' \in \pi_{n-1}(\Omega)$, the set $\{u(x',\cdot)^* > t\} \cap \partial^*\{u(x',\cdot) > \tau\} \cap (\mathcal{D}_u^+)_{x'}$ equals $\partial^*\{u(x',\cdot) > \tau\} \cap (\mathcal{D}_u^+)_{x'}$ for $\mathcal{L}^1$-a.e. $\tau \in (t,\infty)$ and is empty for $\mathcal{L}^1$-a.e. $\tau \in (0,t]$. Consequently,

$$(6.26)\quad \int_0^{M(x')} d\tau \int_{I_i\cap\partial^*\{u(x',\cdot)>\tau\}} \frac{\chi_{\{u(x',\cdot)^*>t\}\cap(\mathcal{D}_u^+)_{x'}}(y)}{|\nabla_y u|} d\mathcal{H}^0(y) = \int_t^{M(x')} d\tau \int_{I_i\cap\partial^*\{u(x',\cdot)>\tau\}} \frac{\chi_{(\mathcal{D}_u^+)_{x'}}(y)}{|\nabla_y u|} d\mathcal{H}^0(y)$$

for $\mathcal{L}^{n-1}$-a.e. $x' \in \pi_{n-1}(\Omega)$. Combining (6.23)-(6.26) and recalling (6.12) yield that, for $\mathcal{L}^{n-1}$-a.e. $x' \in \pi_{n-1}(\Omega)$,

$$(6.27)\quad \mu_{v_i^{x'}}(t) = \int_t^{M(x')} d\tau \int_{I_i\cap\partial^*\{u(x',\cdot)>\tau\}} \frac{\chi_{(\mathcal{D}_u^+)_{x'}}(y)}{|\nabla_y u|} d\mathcal{H}^0(y) = \int_t^{M(x')} \frac{\chi_{(\mathcal{D}_u^+)_{x'}}(y_i(x',\tau))}{|\nabla_y u(x', y_i(x',\tau))|} d\tau \qquad i = 1, 2,$$

for $t \in (0, M(x'))$. Thanks to (6.17), the conclusion follows from (6.27) as in Lemma 4.8. □

Proof of Theorem 2.5. Lemmas 4.9 and 4.10 continue to hold, with exactly the same proof, also if $u \in BV_{0,y}(\Omega)$. Thus, after replacing Lemmas 4.7 and 4.8 by Lemmas 6.2 and 6.3, respectively, the proof of the present theorem follows along the same lines as that of Theorem 2.2. The only difference is that equations (4.54) do not even require a proof, since they agree with (6.13)-(6.14). □

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