Γ-convergence for Beginners

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PREFACE

The object of Γ-convergence is the description of the asymptotic behaviour of families of minimum problems, usually depending on some parameters whose nature may be geometric or constitutive, deriving from a discretization argument, an approximation procedure, etc. Since its introduction by De Giorgi in the early 1970s Γ-convergence has gained an undiscussed role as the most flexible and natural notion of convergence for variational problems, and is now been widely used also outside the field of the Calculus of Variations and of Partial Differential Equations. Its flexibility is due to its being linked to no a priori *ansatz* on the form of minimizers, which is in a sense automatically described by a process of optimization. In this way Γ-convergence is not bound to any prescribed setting, and it can be applied to the study of problems with discontinuities in Computer Vision as well as to the description of the overall properties of nonlinear composites, to the formalization of the passage from discrete systems to continuum theories, to the modelling of thin films or plates, etc., and may be potentially of help in a great variety of situations where a variational limit intervenes or an approximation process is needed.

This books stems from the lecture notes of a course I gave at the SISSA in Trieste in Spring 1999 aimed at all PhD students in Applied Functional Analysis. The idea of the course was to describe all the main features of Γ-convergence to an audience interested in applications but not necessarily wishing to work in that field of the Calculus of Variations, and at the same time to give a simplified introduction to some topics of active research. After a brief presentation of the main abstract properties of Γ-convergence, the lectures were organized as a series of examples in a one-dimensional setting. This choice was aimed at separating those arguments proper of the variational convergence from the technicalities of higher dimensions that render the results at times much more interesting but often are not directly related to the general issues of the convergence process. This structure (with some changes in the order of the chapters) is kept also in the present book, with the addition of some final chapters, which are thought as an introduction to a selected choice of higher-dimensional problems. The scope of this final part of the book is showing how, contrary to what happens for differential equations where passing from Ordinary Differential Equations to Partial Differential Equations and then to systems involves a substation change of viewpoint, the main arguments of Γ-convergence essentially remain unchanged when passing from one-dimensional problems to higher-dimensional ones and from scalar to vector-valued functions. Apart from these chapters 'for the advanced beginner' (which require some notions on Sobolev spaces and whose title is marked by an asterisk) the rest of the book is reasonably self contained, requiring standard notions of Measure Theory and basic Functional Analysis.

I have tried to describe the principles of variational convergence rather than include the sharpest results. Hence, I have frequently chosen proofs that are not the most efficient for the specific result but illustrate most clearly the arguments that can be repeated elsewhere or the technical points that can be generalized to more complex situations. Conversely, I have frequently left minor details as an exercise. All chapters have a final section of comments where some more refined issues are addressed, an outline of the higher-dimensional problems is often sketched, and some bibliographical indication is given. Since this is not thought as a research book on each single subject treated (homogenization, phase transitions, free-discontinuity problems, etc.) I refer to other monographs for complete references on established results. On the contrary, I have chosen to include references to the most recent advances in some problems that may interest the research-oriented reader.

As an advice for the user, it must be mentioned that it is not by chance that no dynamical problem is treated: Γ-convergence is a purely-variational technique aimed at treating minimum problems, and, even though it may give some precious hints in particular situations, in general it is not designed to treat time-dependent cases. Furthermore, also in the 'static case' the generality of Γ-convergence does not allow to obtain the more accurate results of matched asymptotics techniques whenever a very accurate *ansatz* for optimal sequences is available (for example in linear homogenization).

Finally, I wish to thank the many friends that have fruitfully interacted with me during and after the course at SISSA and another course given at the University of Rome 'La Sapienza' in 2001, where part of the material was again presented. I am indebted to Roberto Alicandro, Nadia Ansini, Marco Cicalese, Lorenzo D'Ambrosio, Francesco Del Fra, Gianpietro Del Piero, Maria Stella Gelli and Chiara Leone for accurately reading parts of the manuscript of the book, and to Lev Truskinovsky for his enthusiastic support. The final form of the material much owes to the precious advices and critical comments of Giovanni Alberti, whom I regard as the invisible second author of the book. Thanks to Adriana Garroni for help in the figures and for being there.

Rome Andrea Braides
February 2002

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INTRODUCTION

Dear Sir or Madam will you read my book?
It took me years to write, will you take a look?
(Lennon and McCartney, *Paperback Writer*)

Cosa ne può importare alla casalinga di Treviso,
al bracciante lucano, al pastore abruzzese?
(Nanni Moretti, *Sogni d'oro*)

Why a variational convergence?

In many mathematical problems, may they come from the world of Physics, industrial applications or abstract mathematical questions, some parameter appears (small or large, of geometric or constitutive origin, coming from an approximation process or a discretization argument, at times more than a single parameter) which makes those problems increasingly complex or more and more degenerate. Nevertheless, as this parameter varies, it is often possible to foresee some 'limit' behaviour, and 'guess' that we may substitute the complex, degenerate problems we started with, with a new one, simpler and with a more understandable behaviour, possibly of a completely different type, where the parameters have disappeared, or appear in a more handy way.

Sometimes this type of questions may be studied in a variational framework. In this case, it can be rephrased as the study of the asymptotic behaviour of a family of minimum problems depending on a parameter; in an abstract notation,

$$\min\{F_\varepsilon(u) : \ u \in X_\varepsilon\}. \tag{0.1}$$

The next section provides a number of examples in which F_ε range from singularly-perturbed non-convex problems to highly-oscillating integrals, from discrete energies defined on varying lattices to functionals approximating combined bulk and interfacial energies. The form and the dependence on ε of the solutions in those examples as well as the way they convergence may be very different from case to case.

A way to describe the behaviour of the solutions of (0.1) is provided by substituting such a family by an 'effective problem' (not depending on ε)

$$\min\{F(u) : \ u \in X\}, \tag{0.2}$$

which captures the relevant behaviour of minimizers and for which a solution can be more easily obtained. Γ-convergence is a convergence on functionals which

loosely speaking amounts to requiring the convergence of minimizers and of minimum values of problems (0.1) and of their continuous perturbations to those of (0.2) with the same perturbations. In this way the relevant properties of the actual solutions of (0.1) can be approximately described by those of the solution of (0.2). Note that the function space X and the form of the energy F may be very different from those at level ε, so that the way this convergence is defined must be quite flexible.

A fundamental remark is that the effectiveness of Γ-convergence is linked to the possibility of obtaining converging sequences (or subsequences) from minimizers (or almost-minimizers) of (0.1). A preliminary fundamental question is then compactness: the notion of convergence of functions u_ε must be given so that the existence of a limit of minimizers of (0.1) — assuming that they exist — is ensured beforehand. A too strong notion of convergence of functions will result in a useless definition of convergence of energies, simply because minimizers will not converge. The candidate space X for the limit problem is the space where this compactness argument leads.

Once a notion of convergence $u_\varepsilon \to u$ is agreed upon, the way the functional F in the limit problem (0.2) is obtained can be heuristically explained as an optimization between lower and upper bounds. A *lower bound* for F is an energy G such that

$$G(u) \le F_\varepsilon(u_\varepsilon) + o(1) \tag{0.3}$$

(or in other terms $G(u) \le \liminf_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon)$) whenever $u_\varepsilon \to u$. The requirement that this hold for all u and u_ε (and not only for $u_\varepsilon = u$ or for minimizers) is a characteristic of Γ-convergence that makes it 'stable under perturbations' and at the same time suggests some structure properties on the candidate G (i.e. lower semicontinuity). Condition (0.3) above implies that

$$\inf\{G(u) :\ u \in X\} \le \lim_{\varepsilon\to 0^+} \min\{F_\varepsilon(u) :\ u \in X_\varepsilon\}$$

(given the limit exists). The sharpest lower bound is then obtained by optimizing the role of G. The way this is obtained in practice differs greatly from case to case, but always involves some minimization argument: in the case of homogenization the minimization is done in classes of periodic functions, for phase transitions it consists in an optimal profile issue, in the study of non-convex discrete systems it amounts to optimize a 'separation of scales' argument, etc. (see the examples below). A crucial point at this stage is the study of (necessary) conditions for lower semicontinuity, that allows to restrict the class of competing G.

Once it is computed, the optimal G in this procedure suggests an *ansatz* for the form of the minimizing sequences: in the case of homogenization it suggests that minimizers oscillate close to their limit following an energetically-optimal locally-periodic pattern, for phase transitions that sharp phases are approximated by smoothened functions with an optimal profile, in the study of non-convex discrete systems that minimizers are obtained by an optimal two-scale

discretization, etc. Using this *ansatz* for each $u \in X$ (and not only for minimizers) we may construct a particular $\overline{u}_\varepsilon \to u$ and define $H(u) = \lim_{\varepsilon\to 0^+} F_\varepsilon(\overline{u}_\varepsilon)$. This H is an *upper bound* for the limit energy, and for such H we have

$$\lim_{\varepsilon\to 0^+} \min\{F_\varepsilon(u) : \ u \in X_\varepsilon\} \le \inf\{H(u) : \ u \in X\};$$

that is, to an *ansatz* on approximating sequences there corresponds an estimate 'from above' for the limit of the minimum problems.

The Γ-convergence of F_ε is precisely the requirement that these two bounds coincide, and hence it implies the convergence of (0.1) to (0.2). Having taken care of defining the upper and lower bound energies for all functions and not only for minimizers Γ-convergence enjoys important properties, such as:

— Γ-convergence itself implies the convergence of minimum problems (that for the sake of simplicity was assumed true in the argument above) and the convergence of (sub)sequences of (almost-)minimizers to minimizers of the Γ-limit,

— it is stable under continuous perturbations. This means that our analysis is still valid if we add to all problems any fixed continuous term. In this sense the Γ-limit F provides a 'limit theory' which describes all relevant features of F_ε and not only those related to a specific minimum problem,

— the Γ-limit F is a lower semicontinuous functional. This is a very useful structure property that usually implies existence of minimizers and helps in giving a better description of F through representation results.

Comparing this notion with others used for asymptotic expansion we note that the main issue here is the computation of the lower bound, which uniquely involves minimization and 'optimization' procedures and is totally *ansatz*-free. To this lower bound there corresponds an upper bound where the *ansatz* on minimizers is automatically driven by the lower bound itself. As a result Γ-convergence does not require the computation of minimizers of (0.1) — that indeed may or may not exist — nor the solution of the associated Euler–Lagrange equations, and it is not linked to any structure of X_ε and X.

It must be mentioned that, given the generality of applications of Γ-convergence, whenever a good *ansatz* for minimizers is reached, additional *ad hoc* techniques should be also used to give a more complete characterization of the convergence of minimum problems. This is the case, for example, of periodic (linear) homogenization where asymptotic expansion in locally-periodic functions provide a more complete description of the behaviour of minimizers, and finer issues can be fruitfully addressed by different methods. The same example of homogenization shows that we must be very careful when we start from an *ansatz* that looks completely natural but is not justified by a convergence result: in the vector-valued non-linear case minimizers are in general (locally) almost periodic (i.e. oscillations at all scales must be taken into account). This behaviour is natural from the viewpoint of Γ-convergence but it is easily missed if we start from the wrong assumptions on the (local) periodicity of minimizing sequences.

In the rest of this chapter we provide a series of examples, which serve also as an introduction to the core of the book, and a final section in which we introduce the definition of Γ-convergence as a 'natural' extension of the so-called direct methods of the Calculus of Variations.

Parade of examples

In this section we include a number of examples, in which we show how a notion of variational convergence must be sensible, as it must include cases where the limit problem is set on a space X completely different from all X_ε, and even when X is the same it may be very different from pointwise convergence. Furthermore, by describing the approximate forms of minimizers in these examples, which will be obtained as a final result in the Γ-convergence process and exhibit a variety of structures, we want to highlight how the convergence must not rely on any a priori *ansatz* on the asymptotic form of minimizers, and it should in a sense itself suggest the precise meaning of this asymptotic question, as this could not be supplied by problems (0.1). These examples will be dealt with in detail in the next chapters.

Example 0.1 (gradient theory of phase transitions). The simplest example that shows a dramatic change of type in the passage to the limit, is perhaps that of the *gradient theory of phase transitions* for a homogeneous isothermal fluid contained in a bounded region Ω. If we denote the concentration of the fluid with a function $u : \Omega \to [0,1]$, then the equilibrium configurations are described as minimizing a suitable energy depending on u under a mass constraint:

$$\min\Big\{E(u) : \ u : \Omega \to [0,1], \ \int_\Omega u\,dx = C\Big\}, \tag{0.4}$$

where the energy is of the form

$$E(u) = \int_\Omega W(u)\,dx. \tag{0.5}$$

The energy density $W : (0, +\infty) \to \mathbf{R}$ is a non-convex function given by the Van der Waals Cahn Hilliard theory, whose graph is of the form as in Fig. 0.1.

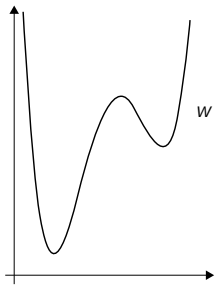

FIG. 0.1. The van der Waals energy density

In order to make problem (0.4) clearer and understand the properties of minimizers, we may add an affine change of variable to W; that is, consider $W(u)+c_1u+c_2$ in place of W. Note that this change in the energy density does not affect the minimum problem (0.4) since it amounts to add the fixed quantity

$$\int_\Omega (c_1u+c_2)\,dx = c_1C + c_2|\Omega|$$

to $E(u)$. At this point, we may choose c_1 and c_2 so that the new energy density, which we still denote by W, is non-negative and has precisely two zeros at points α and β, as in Fig. 0.2.

It is clear now that, if this is allowed by the mass constraint, minimizers of (0.4) will be simply given by (all!) functions u which take only the values α and β and still satisfy the constraint $\int_\Omega u\,dx = C$. For such a function the two regions $\{u=\alpha\}$ and $\{u=\beta\}$ are called the two *phases* of the fluid and form a partition of Ω. Note that minimizing problem (0.4) does not provide *any* information about the interface between the two phases, which may be irregular or even dense in Ω. This is not what is observed in those equilibrium phenomena: among these minimizers some special configuration are preferred, instead, and precisely those with minimal interface between the phases. This *minimal-interface criterion* is interpreted as a consequence of higher-order terms: in order to prevent the appearance of irregular interfaces, we add a term containing the derivative of u as a *singular perturbation*, which may be interpreted as giving a (small) *surface tension* between the phases. The new problem, in which we see the appearance of a small positive parameter ε, takes the form

$$\min\Big\{\int_\Omega (W(u)+\varepsilon^2|Du|^2)\,dx : \int_\Omega u\,dx = C\Big\} \tag{0.6}$$

(the power ε^2 comes from dimensional considerations), where now some more regularity on u is required. The solutions to this problem indeed have the form

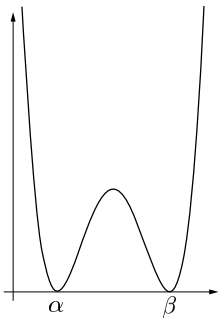

FIG. 0.2. The energy density after the affine translation

$$u_\varepsilon(x) \approx u(x) + u_1\Big(\frac{\operatorname{dist}(x,S)}{\varepsilon}\Big),$$

where $u : \Omega \to \{\alpha, \beta\}$ is a phase-transition function with minimal interface S in Ω, and $u_1 : \mathbf{R} \to \mathbf{R}$ is a function with limit 0 at infinity, which gives the optimal profile between the phases at $\varepsilon > 0$. Fig. 0.3 picture a minimizer u_ε corresponding to a minimal u with a minimal (linear) interface between the phases.

This is a natural *ansatz* and is proved rigorously by a Γ-convergence arguments. We can picture this behaviour in the one-dimensional case, where, then, u is simply a function with a single discontinuity point. In Fig. 0.4 are represented functions u_ε for various values of ε.

The behaviour of u_ε cannot be read out directly by examining small-energy functions for problem (0.6), but may be more easily deduced if that problem is rewritten as

$$\min\Big\{\int_\Omega \Big(\frac{W(u)}{\varepsilon} + \varepsilon|Du|^2\Big)\,dx : \int_\Omega u\,dx = C\Big\}. \tag{0.7}$$

In this way it may be seen that the contributions of the two terms in the integral have the same order as ε tends to 0 for minimizing sequences; the qualitative

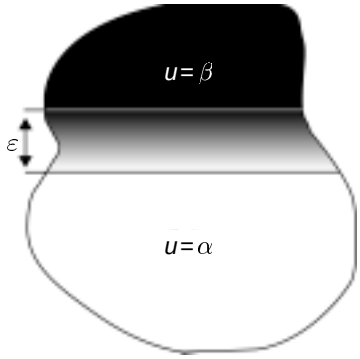

FIG. 0.3. Approximate phase transition with a minimal interface

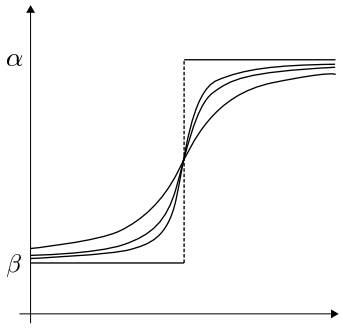

FIG. 0.4. Behaviour of approximate phase transitions

effect of the first term is that u tends to get closer to α or β, while the effect of the second term is to penalize unnecessary interfaces. It can (and it will) be seen that problem (0.7) is well approximated as ε gets small by a *minimal interface problem*:

$$\min\Big\{\mathrm{Per}(\{u=\alpha\},\Omega) : u:\Omega\to\{\alpha,\beta\}, \int_\Omega u\,dx = C\Big\}, \tag{0.8}$$

where $\mathrm{Per}(A,\Omega)$ denotes the (suitably defined) perimeter of A in Ω. In this case we have a complete change of type in the problems: in particular, while problem (0.7) involves only (sufficiently) smooth functions, its limit counterpart (0.8) gets into play only discontinuous functions. The treatment by Γ-convergence of this example will be done in Chapters 6 and 14.

Example 0.2 (homogenization of variational problems). Another class of problems, which can be (partly) set in this framework are some types of *homogenization problems*. 'Homogenization' is a general term which underlines the asymptotic description of problems with increasingly oscillating solutions. In its simplest form it regards the description of static phenomena involving the study of minimum points of some energy functional whose energy density is periodic on a very small scale (see Fig. 0.5). The simplest case is related to the stationary heat equation in a (locally isotropic) composite medium of $\mathbf{R}^n$ of thermal conductivity $a(x/\varepsilon)$ occupying a region Ω. The function a is periodic (say, of period one) in each coordinate direction, so that the integrand above is periodic of period ε. To fix ideas we may assume that a takes only two values (say, α and β). In this case the medium we have in mind is a *composite* of two materials whose 'microscopic pattern' is described by the function a. If f is a source term and we impose a boundary condition (for simplicity homogeneous) the temperature u_ε will satisfy

$$\begin{cases} -\displaystyle\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a\Big(\frac{x}{\varepsilon}\Big)\frac{\partial u}{\partial x_j}\Big) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

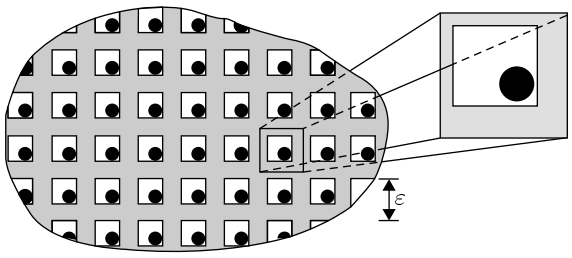

FIG. 0.5. A 'composite medium' and its microscopic pattern

or, equivalently, can be characterized as the minimizer of

$$\min\Big\{E_\varepsilon(u) - 2\int_\Omega f u\,dx : u = 0 \text{ on } \partial\Omega\Big\}, \tag{0.9}$$

where

$$E_\varepsilon(u) = \int_\Omega a\Big(\frac{x}{\varepsilon}\Big)|Du|^2\,dx. \tag{0.10}$$

If the dimensions of the set Ω are very large with respect to ε we may expect that the overall 'macroscopic' behaviour of the medium described above is 'very similar' to a (now, possibly anisotropic) homogeneous material. Indeed the solutions u_ε of (0.9) 'oscillate' close to the solution of a limit problem as we let ε tend to 0; that is, they have the form, at least locally in Ω, (see Figure 0.6)

$$u_\varepsilon(x) \approx u(x) + \varepsilon u_1\Big(x, \frac{x}{\varepsilon}\Big). \tag{0.11}$$

The function u is the solution of a problem of the type

$$\min\Big\{\int_\Omega \sum\nolimits_{i,j} q_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\,dx - 2\int_\Omega f u\,dx : u = 0 \text{ on } \partial\Omega\Big\}. \tag{0.12}$$

The constant 'homogenized' coefficients q_{ij} do not depend on f and Ω, and can be computed directly from a through some auxiliary minimum problems on sets of periodic functions. It is instructive to look at the one-dimensional case, where $\Omega \subset \mathbf{R}$; in this case the *pointwise* limit of the functionals E_ε exists and is given simply by

$$E(u) = \overline{a}\int_\Omega |u'|^2\,dt, \tag{0.13}$$

where the coefficient $\overline{a}$ is the *mean value* of a, $\overline{a} = \int_0^1 a(s)\,ds$, but the coefficient $\widehat{a}$ $(= q_{11}$ in this simple case) appearing in (0.12) is given by the *harmonic mean* of a:

$$\widehat{a} = \Big(\int_0^1 \frac{1}{a(s)}\,ds\Big)^{-1}. \tag{0.14}$$

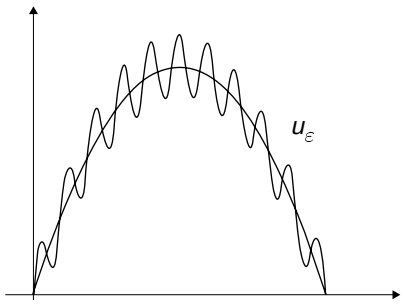

FIG. 0.6. Oscillating solutions and their limit

This observation shows an interesting non-trivial effect of the oscillations in the minimizing sequences, that interact with those of $a(x/\varepsilon)$. By optimizing this interaction we obtain the value of $\widehat{a}$. Some issues of this type of homogenization will be addressed in Chapters 3 and 12.

Another 'homogenization' problem intervenes as a question concerning the convergence of distances. A Riemannian distance is characterized (in local coordinates) by minimum problems of the form

$$\min\Big\{\int_0^1 \sum\nolimits_{i,j} a_{ij}(u)u_i'u_j'\,dx : u(0)=u_0,\ u(1)=u_1\Big\}, \tag{0.15}$$

where $u:[0,1]\to\mathbf{R}^n$ vary among all (regular) curves joining u_0 and u_1. In some problems (e.g. when dealing with families of *viscosity solutions*) it is necessary to characterize the limit of *oscillating* Riemannian metrics of the form

$$\min\Big\{\int_0^1 \sum\nolimits_{i,j} a_{ij}\Big(\frac{u}{\varepsilon}\Big)u_i'u_j'\,dx : u(0)=u_0,\ u(1)=u_1\Big\}. \tag{0.16}$$

In this case we may still characterize the limit of these problems, but it can be seen that it is not related to a Riemannian metric anymore; that is, such problems behave as $\varepsilon\to 0$ as

$$\min\Big\{\int_0^1 \psi(u')\,dx : u(0)=u_0,\ u(1)=u_1\Big\}, \tag{0.17}$$

but in general ψ is not a quadratic form. To understand this behaviour it is instructive to consider the case when $n=2$ and $a_{ij}(u)=a(u)\delta_{ij}$ (δ_{ij} denotes Kronecker's delta), and a models a 'chessboard structure' with two values α, β with $\sqrt{2}\,\alpha<\beta$. With this condition, it is 'not convenient' for the competing curves u in (0.16) to cross the β region, and with this constraint in mind it is easy to find the exact form of ψ and to check that it is not a quadratic form (see Chapter 3). The solutions to (0.16) are pictured in Fig. 0.7.

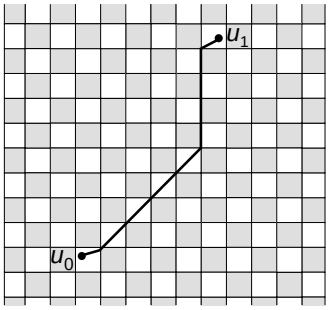

FIG. 0.7. Curves of minimal distance on the 'chessboard' are not line segments

The choice of the scaling (i.e. the dependence on ε) is not always as obvious as above. Another 'classical' example of problems in a periodic setting is that of Dirichlet problems in *perforated domains*. In this case the problem we encounter is of the form

$$\min\Big\{\int_{\Omega_\varepsilon} |Du|^2\,dx - 2\int_{\Omega_\varepsilon} fu\,dx : u = 0 \text{ on } \partial\Omega_\varepsilon\Big\}, \tag{0.18}$$

where Ω_ε is a 'perforation' of a fixed bounded open set $\Omega \subset \mathbf{R}^n$. The simplest case is when Ω_ε is obtained by removing from Ω a periodic array of closed balls of equal radius $\delta = \delta(\varepsilon)$ with centres placed on a regular lattice of spacing ε; that is, of the form

$$\Omega_\varepsilon = \Omega \setminus \bigcup_{\mathbf{i}\in\mathbf{Z}^n} \overline{B(\varepsilon\mathbf{i}, \delta(\varepsilon))} \tag{0.19}$$

(see Fig. 0.8). In terms of the corresponding stationary heat equation, the condition $u = 0$ can be interpreted as the presence of evenly distributed small particles at a fixed temperature (it is suggestive to think of ice mixed with water) in the interior of Ω.

The behaviour as ε gets smaller is trivial when $n = 1$ (the solutions simply tend to 0 since they are equi-continuous and vanish on a set which tends to be dense), but gives rise to an interesting phenomenon when $n \geq 2$ and $\delta = \delta(\varepsilon)$ is appropriately chosen. Let $n \geq 3$ for the sake of simplicity; in this case the interesting case is when

$$\delta(\varepsilon) \approx \varepsilon^{n/(n-2)}, \tag{0.20}$$

all other cases giving trivial results: either the effect of the perforation is negligible, and the boundary condition in the interior of Ω disappears, or it is too strong, and it forces the solutions to tend to zero. The case (0.20) is the intermediate situation where the effect of the perforation is of the same order as that of the Dirichlet energy and it penalizes the distance of the solution from 0 in a very precise manner. The overall effect as ε tends to 0 is that u_ε 'behave approximately' as the solution u of the problem

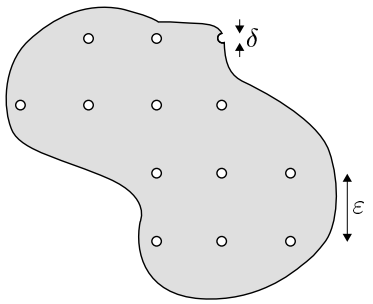

FIG. 0.8. A 'perforated' domain

$$\min\Big\{\int_\Omega (|Du|^2 + C|u|^2)\,dx - 2\int_\Omega fu\,dx : u \in H_0^1(\Omega)\Big\}, \tag{0.21}$$

meaning that (at least in the interior of Ω)

$$u_\varepsilon(x) \approx u(x)\Big(1 - \sum_{\mathbf{i}} u_1\Big(\frac{x - \varepsilon\mathbf{i}}{\varepsilon^{n/n-2}}\Big)\Big), \tag{0.22}$$

where u_1 is a 'capacitary potential' decreasing to 0 at infinity and with $u_1 = 1$ on the unit ball, and the constant C is computed explicitly and does not depend on f. Figure 0.9 pictures the behaviour of the solutions on a one-dimensional section passing through the perforation.

Note that even though we remain in the same functional space the form of the limit energy is different from the approximating ones and it has an additional 'strange term coming from nowhere' (as baptized by Cioranescu and Murat). An explanation in terms of Γ-convergence is given in Chapter 13.

Example 0.3 (dimension reduction). Other problems where a small parameter ε appears are asymptotic theories of elastic plates, shells, films and rods.

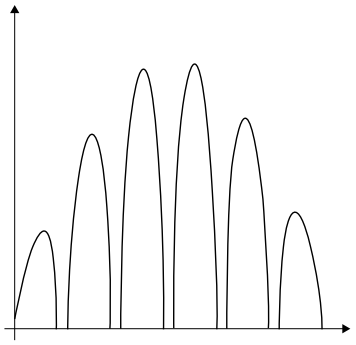

FIG. 0.9. Behaviour of oscillating solutions ('cross section')

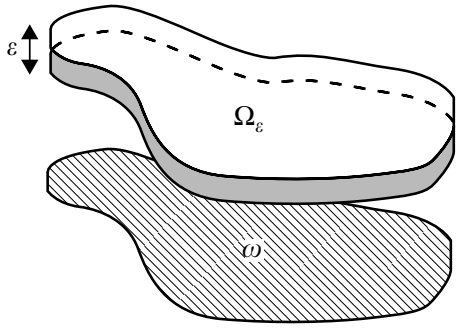

FIG. 0.10. A 'thin' domain

In this case the goal is a rigorous derivation of a lower (one- or two-)dimensional theory (for elastic plates, shells, etc.) from the corresponding three-dimensional one. The starting point (e.g. for thin films) is then to consider energies of the form

$$E_\varepsilon(u) = \int_{\Omega_\varepsilon} f(Du)\,dy, \tag{0.23}$$

where $u : \Omega \to \mathbf{R}^3$ and the domain, in the simplest case of flat films, is of the form

$$\Omega_\varepsilon = \{(y', y_3) : y' \in \omega,\ 0 < y_3 < \varepsilon\} \tag{0.24}$$

and ω is a fixed bounded open subset of $\mathbf{R}^2$ (see Figure 0.10). The simplest type of problems related to such energies are of the form

$$\min\{E_\varepsilon(u) : u = \phi \text{ on } (\partial\omega) \times (0, \varepsilon)\}, \tag{0.25}$$

where $\phi = \phi(y')$ and the boundary conditions are given only on the 'vertical' boundary. After scaling (dividing by ε) the energy E_ε and the change of variables $x' = y'$, $\varepsilon x_3 = y_3$, we have the equivalent energies

$$F_\varepsilon(u) = \int_{\omega \times (0,1)} f\Big(D_1 u, D_2 u, \frac{1}{\varepsilon} D_3 u\Big)\,dy. \tag{0.26}$$

We now have a family of scaled energies, which are defined on a common space of functions, but which tend to degenerate with respect to the derivative in the third direction as ε tends to 0. Problems (0.25) can be rewritten as

$$\min\{F_\varepsilon(u) : u = \phi \text{ on } (\partial\omega) \times (0, 1)\}. \tag{0.27}$$

If u_ε are solutions to such problems, in view of (0.26), one expects that $D_3 u_\varepsilon$ tend to 0 and hence the limit actually to be independent of the third variable. Indeed we have that

$$u_\varepsilon(x) \approx u(x') + \varepsilon x_3 b(x'), \tag{0.28}$$

where u minimizes a two-dimensional problem

$$\min\Big\{ \int_\omega \tilde{f}(D_1 u, D_2 u)\,dx' : u = \phi \text{ on } \partial\omega \Big\}. \tag{0.29}$$

The function $\tilde{f}$ is independent of the boundary datum ϕ and it is obtained, heuristically, by minimizing the contribution of the function b in (0.26). In this case both the problems at fixed ε and at the limit have the same form, but on domains of different dimension. Minimizing sequences do not necessarily develop oscillations, but the limit lower dimensional theory may not be derived in a trivial way from the full three-dimensional one.

An outline of the approach by Γ-convergence to dimension reduction is given in Chapter 14.

Example 0.4 (approximation of free-discontinuity problems). The terminology 'free-discontinuity problems' (as opposed to free-*boundary* problems) denotes a class of problems in the Calculus of Variations where the unknown is a pair (u, K) with K varying in a class of (sufficiently smooth) closed hypersurfaces contained in a fixed open set $\Omega \subset \mathbf{R}^n$ and $u : \Omega \setminus K \to \mathbf{R}^m$ belonging to a class of (sufficiently smooth) functions. Such problems usually consist in minimizing an energy with competing *volume* and *surface* energies. The main examples in this framework are variational theories in Image Reconstructions and Fracture Mechanics. In the first case $n = 2, m = 1$ and the so-called Mumford Shah functional is taken into account

$$E(u, K) = \int_{\Omega \setminus K} |Du|^2 \, dx + c_1 \, \mathrm{length}(K) + c_2 \int_{\Omega \setminus K} |u - g|^2 \, dx. \tag{0.30}$$

Here, the function g is interpreted as the input picture taken from a camera, u is the 'restored' image, and K is the relevant contour of the objects in the picture; c_1 and c_2 are contrast parameters. Note that the problem is meaningful also adding the constraint $Du = 0$ outside K, in which case we have a *minimal partitioning* problem. In the case of *fractured hyperelastic media* $m = n = 3$ and the volume and surface energies taken into account are very similar (with the area of K in place of the length), Ω is interpreted as the reference configuration of an elastic body, K is the crack surface, and u represents the elastic deformation in the unfractured part of the body.

Functionals arising in free-discontinuity problems present some drawbacks; for example, numerical difficulties arise in the detection of the unknown discontinuity surface. To bypass these drawbacks, a considerable effort has been spent to provide variational approximations, in particular of the Mumford Shah functional E defined above, with differentiable energies defined on smooth functions. An approximation was given by Ambrosio and Tortorelli, who followed the idea of the gradient theory of phase transitions introducing an approximation with an auxiliary variable v. A family of approximating functionals is the following:

$$\begin{aligned} G_\varepsilon(u, v) = &\int_\Omega v^2 |Du|^2 \, dx + \frac{1}{2} \int_\Omega \Big(\varepsilon |Dv|^2 + \frac{1}{\varepsilon} (1 - v)^2 \Big) \, dx \\ &+ c_2 \int_\Omega |u - g|^2 \, dx \,, \end{aligned} \tag{0.31}$$

defined on regular functions u and v with $0 \le v \le 1$. Heuristically, the new variable v in the limit as $\varepsilon \to 0$ approaches $1 - \chi_K$ and introduces a penalty on the length of K in the same way as a phase transition. Since the functionals G_ε are elliptic, even though non-convex, numerical methods can be applied to them. It is interesting to note that the functionals G_ε may have also an interpretation in terms of Fracture Mechanics, as v can be thought as a pointwise *damage parameter*.

Free-discontinuity problems and their approximations are dealt with in Chapters 7 and 8.

Example 0.5 (continuous limits of difference schemes). Another interesting problem is that of the description of variational limits of discrete problems (for the sake of brevity in a one-dimensional setting). Given $n \in \mathbf{N}$ and points $x_i^n = i\lambda_n$ ($\lambda_n = L/n$ is the *lattice spacing*, which plays the role of the small parameter ε) we consider energies of the general form

$$E_n(\{u_i\}) = \sum_{j=1}^{n} \sum_{i=0}^{n-j} \lambda_n \psi_n^j \Big(\frac{u_{i+j} - u_i}{j\lambda_n} \Big).$$

If we picture the set $\{x_i^n\}$ as the reference configuration of an array of material points interacting through some forces, and u_i represents the displacement of the i-th point, then ψ_n^j can be thought as the energy density of the interaction of points with distance $j\lambda_n$ (j lattice spacings) in the reference lattice. Note that the only assumption we make is that ψ_n^j depends on $\{u_i\}$ through the differences $u_{i+j} - u_i$, but we find it more convenient to highlight its dependence on 'discrete difference quotients'. For a quite general class of energies it is possible to describe the behaviour of solutions of problems of the form

$$\min\Big\{ E_n(\{u_i\}) - \sum_{i=0}^{n} \lambda_n u_i f_i : \ u_0 = U_0, \ u_n = U_L \Big\}$$

(and similar), and to show that these problems have a limit continuous counterpart. Their solutions then can be though of (non-trivial) discretizations of the corresponding solution on the continuum. Here $\{f_i\}$ represent the external forces and U_0, U_L are the boundary conditions at the endpoints of the interval $(0, L)$. More general statement and different problems can also be treated. Under some growth conditions, minimizers of the problem above are 'very close' to minimizers of a classical problem of the Calculus of Variations

$$\min\Big\{ \int_0^L \big(\psi(u') - fu \big)\, dt : \ u(0) = U_0, \ u(L) = U_L \Big\}.$$

The energy densities ψ can be explicitly identified by a series of operations on the functions ψ_n^j. The case when only *nearest-neighbour interactions* are taken into account,

$$E_n(\{u_i\}) = \sum_{i=0}^{n-1} \lambda_n \psi_n \Big(\frac{u_{i+1} - u_i}{\lambda_n} \Big),$$

is particularly simple. In this case, the limit energy density is given by the limit of the convex envelopes of the functions $\psi_n(z)$, which exists up to subsequences. The description of the limit energy gets much more complex when not only nearest-neighbour interactions come into play and involves arguments of homogenization type which highlight that the overall behaviour of a system of interacting points will behave as *clusters* of large arrays of neighbouring points interacting through their 'extremities' (see Chapters 4 and 11).

A maieutic approach to Γ-convergence. Direct methods

The scope of this section is to show that we may 'naturally' derive the definition of Γ-convergence for functionals from the requirements that

— it implies the convergence of minimizers and minimum values (under suitable assumptions),

— it is stable under continuous perturbations, and

— it is given in *local* terms (i.e. we can also speak of convergence 'at one point').

The starting point will be the examination of the so-called *direct methods of the Calculus of Variations*. For the sake of simplicity, from here onwards all our problems will be set on metric spaces, so that the topology is described by just using sequences. The idea is very simple: in order to prove the existence of a minimizer of a problem of the form

$$\min\{F(u) : \ u \in X\}, \tag{0.32}$$

we examine the behaviour of a *minimizing sequence*; that is, a sequence $(\overline{u}_j)$ such that

$$\lim_j F(\overline{u}_j) = \inf\{F(u) : \ u \in X\}, \tag{0.33}$$

which clearly always exists. Such a sequence, in general might lead nowhere. The first thing to check is then that we may find a *converging* minimizing sequence. This property may be at times checked by hand, but it is often more convenient to check that an *arbitrary* minimizing sequence lies in a *compact* subset K of X (i.e. since X is metric, that for any sequence (u_j) in K we can extract a subsequence (u_{j_k}) converging to some $u \in K$). This property is clearly stronger than requiring that there exists *one* converging minimizing sequence, but its verification often may rely on a number of characterizations of compact sets in different spaces. In its turn this compactness requirement can be directly made on the functional F by asking that it be *coercive*; that is, that for all t its sub-level sets $\{F < t\} = \{u \in X : F(u) < t\}$ are *pre-compact* (this means that for fixed t there exists a compact set K_t containing $\{F < t\}$, or, equivalently, in terms of sequences, that for all sequences (u_j) with $\sup_j F(u_j) < +\infty$ there exists a converging subsequence). Again, this is an even stronger requirement, but it may be derived directly from the form of the functional F and not from special properties of minimizing sequences. Once some compactness properties of a minimizing sequence are established, we may extract a (minimizing) subsequence, that we still denote by $(\overline{u}_j)$, converging to some $\overline{u}$.

At this stage, the point $\overline{u}$ is a candidate to be a minimizer of F; we have to prove that indeed

$$F(\overline{u}) = \inf\{F(u) : \ u \in X\}. \tag{0.34}$$

One inequality is trivial, since $\overline{u}$ can be used as a test function in (0.34) to obtain an *upper inequality*: $\inf\{F(u) : \ u \in X\} \leq F(\overline{u})$.

To obtain a *lower inequality* we have to link the value at $\overline{u}$ to those computed at $\overline{u}_j$, to obtain the right inequality

$$F(\overline{u}) \leq \lim_j F(\overline{u}_j) = \inf\{F(u) : \ u \in X\}. \tag{0.35}$$

Since we do not want to rely on special properties of $\overline{u}$ or of the approximating sequence $(\overline{u}_j)$, but instead we would like to isolate properties of the functional F, we require that *for all* $u \in X$ and *for all* sequences (u_j) tending to u we have the inequality

$$F(u) \leq \liminf_j F(u_j). \tag{0.36}$$

This property is called the *lower semicontinuity* of F. It is much stronger than requiring (0.35), but it may be interpreted as a structure condition on F and often derived from general considerations.

At this point we have not only proven that F admits a minimum, but we have also found a minimizer $\overline{u}$ by following a minimizing sequence. We may condensate the reasoning above in the following formula

$$\text{coerciveness} \quad + \quad \text{lower semicontinuity} \quad \Rightarrow \quad \text{existence of minimizers,} \tag{0.37}$$

which summarizes the direct methods of the Calculus of Variations. It is worth noticing that the coerciveness of F is easier to verify if we have *many* converging sequences, while the lower semicontinuity of F is more easily satisfied if we have *few* converging sequences. These two opposite requirements will result in a balanced choice of the metric on X, which is in general not given a priori, but in a sense forms a part of the problem.

We now turn our attention to the problem of describing the behaviour of a family of minimum problems depending on a parameter. In order to simplify the notation we deal with the case of a sequence of problems

$$\inf\{F_j(u) : \ u \in X_j\} \tag{0.38}$$

depending on a discrete parameter $j \in \mathbf{N}$; the case of a family depending on a continuous parameter ε introduces only a little extra complexity in the notation. As j increases we would like these problems to be approximated by a 'limit theory' described by a problem of the form

$$\min\{F(u) : \ u \in X\}. \tag{0.39}$$

In order to make this notion of 'convergence' precise we try to follow closely the direct approach outlined above. In this case we start by examining a minimizing sequence for the family F_j; that is, a sequence $(\overline{u}_j)$ such that

$$\lim_j \Big(F_j(\overline{u}_j) - \inf\{F_j(u) : \ u \in X_j\} \Big) = 0, \tag{0.40}$$

and try to follow this sequence.

In many problems the space X_j indeed varies with j, so that now we have to face a preliminary problem of defining the convergence of a sequence of functions which belong to different spaces. This is usually done by choosing X large enough so that it contains the domain of the candidate limit and all X_j. We can always consider all functionals F_j as defined on this space X by identifying them with the functionals

$$\widetilde{F}_j(u) = \begin{cases} F_j(u) & \text{if } u \in X_j \\ +\infty & \text{if } u \in X \setminus X_j. \end{cases} \tag{0.41}$$

This type of identification is customary in dealing with minimum problems and is very useful to include constraints directly in the functional. We may therefore suppose that all $X_j = X$. If one is not used to dealing with functionals which take the value $+\infty$, one may regard this as a technical tool; if the limit functional is not finite on the whole X it will always be possible to restrict it to its domain $\operatorname{dom} F = \{u \in X : \ F(u) < +\infty\}$.

As in the case of a minimizing sequence for a single problem, it is necessary to find a converging minimizing (sub)sequence. In general it will be possible to find a minimizing sequence lying in a compact set of X as before, or prove that the functionals themselves satisfy an *equi-coerciveness* property: for all t there exists a compact K_t such that for all j we have $\{F_j < t\} \subset K_t$.

If a compactness property as above is satisfied, then we may suppose that the whole sequence $(\overline{u}_j)$ converges to some $\overline{u}$ (this is a technical point that will be made clear in the next section). The function $\overline{u}$ is a good candidate as a minimizer.

First, we want to obtain an *upper bound* for the limit behaviour of the sequence of minima, of the form

$$\limsup_j \inf\{F_j(u) : \ u \in X\} \le \inf\{F(u) : \ u \in X\} \le F(\overline{u}). \tag{0.42}$$

The second inequality is trivially true; the first inequality means that for all $u \in X$ we have

$$\limsup_j \inf\{F_j(v) : \ v \in X\} \le F(u). \tag{0.43}$$

This is a requirement of global type; we can 'localize' it in the neighbourhood of the point u by requiring a stronger condition: that for all $\delta > 0$ we have

$$\limsup_j \inf\{F_j(v) : \ d(u,v) < \delta\} \le F(u). \tag{0.44}$$

By the arbitrariness of δ we can rephrase this condition as a condition on sequences converging to u as:

(*limsup inequality*) for all $u \in X$ there exists a sequence (u_j) converging to u such that

$$\limsup_j F_j(u_j) \le F(u). \tag{0.45}$$

This condition can be considered as a local version of (0.42); it clearly implies all conditions above and (0.42) in particular.

Next, we want to obtain a *lower bound* for the limit behaviour of the sequence of minima of the form

$$F(\overline{u}) \leq \liminf_j F_j(\overline{u}_j). \tag{0.46}$$

As we do not want to rely on particular properties of minimizers we regard $\overline{u}$ as an arbitrary point in X and $(\overline{u}_j)$ as any converging sequence; hence, condition (0.46) can be deduced from the more general requirement:

(*liminf inequality*) for all $u \in X$ and for all sequences (u_j) converging to u we have

$$F(u) \leq \liminf_j F_j(u_j). \tag{0.47}$$

This condition is the analog of the lower semicontinuity hypothesis in the case of a single functional.

From the considerations above, if we can find a functional F such that the liminf and limsup inequalities are satisfied and if we have a converging sequence of minimizers, from (0.46) and (0.42) we deduce the chain of inequalities

$$\begin{aligned} \limsup_j \inf\{F_j(u) : \ u \in X\} &\leq \inf\{F(u) : \ u \in X\} \\ &\leq F(\overline{u}) \leq \liminf_j F_j(\overline{u}_j) \\ &= \liminf_j \inf\{F_j(u) : \ u \in X\}. \end{aligned} \tag{0.48}$$

As the last term is clearly not greater than the first, all inequalities are indeed equalities; that is, we deduce that

(i) (*existence*) the limit problem $\min\{F(u) : \ u \in X\}$ admits a solution,

(ii) (*convergence of minimum values*) the sequence of infima $\inf\{F_j(u) : \ u \in X\}$ converges to this minimum value,

(iii) (*convergence of minimizers*) up to subsequences, the minimizing sequence for (F_j) converges to a minimizer of F on X.

Therefore, if we define the Γ-convergence of (F_j) to F as the requirement that the limsup and the liminf inequalities above both hold, then we may summarize the considerations above in the formula

$$\text{equi-coerciveness} + \Gamma\text{-convergence} \ \Rightarrow \ \text{convergence of minimum problems.} \tag{0.49}$$

As in the case of the application of the direct methods, a crucial role will be played by the type of metric we choose on X. In this case, again, it will be a matter of balance between the convenience of a stronger notion of convergence, that will make the liminf inequality easier to verify, and a weaker one, which would be more convenient both to satisfy an equi-coerciveness condition and to find sequences satisfying the limsup inequality.

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