GRAPHS OF BOUNDED VARIATION, EXISTENCE AND LOCAL BOUNDEDNESS OF NON-PARAMETRIC MINIMAL SURFACES IN HEISENBERG GROUPS

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Abstract. In the setting of the sub-Riemannian Heisenberg group $\mathbb{H}^n$, we introduce and study the classes of t- and intrinsic graphs of bounded variation. For both notions we prove the existence of non-parametric area-minimizing surfaces, i.e., of graphs with the least possible area among those with the same boundary. For minimal graphs we also prove a local boundedness result which is sharp at least in the case of t-graphs in $\mathbb{H}^1$.

1. Introduction and statement of the main results

In this paper we deal with the problem of existence and regularity for generalized non-parametric minimal hypersurfaces in the setting of the Heisenberg group $\mathbb{H}^n$, endowed with its sub-Riemannian (or Carnot-Carathéodory) metric structure. The classes of t- and intrinsic graphs of bounded variation will be introduced and studied. We prove existence and local boundedness results for those graphs locally minimizing the sub-Riemannian area (precisely: the $\mathbb{H}$-perimeter measure). Minimal graphs are tipically named *non-parametric* minimal surfaces in order to distinguish them from the more general *parametric* ones (see, for instance, [39]).

Let us recall some preliminary facts about the Heisenberg group; we refer to [12] for a more complete introduction. We denote the points of $\mathbb{H}^n \equiv \mathbb{R}^{2n+1}$ by

$$P = (x, y, t), \qquad x, y \in \mathbb{R}^n,\ t \in \mathbb{R}.$$

For $P = (x, y, t)$, $Q = (x', y', t') \in \mathbb{H}^n$, the group operation reads as

$$P \cdot Q := (x + x', y + y', t + t' - 2\langle x, y'\rangle + 2\langle x', y\rangle)$$

where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product of $\mathbb{R}^n$. The group identity is the origin 0 and $(x, y, t)^{-1} = (-x, -y, -t)$. In $\mathbb{H}^n$ there is a one-parameter group of non-isotropic dilations $\delta_r(x, y, t) := (rx, ry, r^2 t)$, $r > 0$. The Lie algebra $\mathfrak{h}_n$ of left invariant vector fields is linearly generated by

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad j = 1, \dots, n, \quad T = \frac{\partial}{\partial t}$$

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and the only nonvanishing commutation relationships between these generators are

$$[X_j, Y_j] = -4T, \qquad j = 1, \dots, n.$$

We also use the notation $X_j := Y_{j-n}$ for $j = n+1, \dots, 2n$.
The group $\mathbb{H}^n$ can be endowed with the homogeneous norm

$$\|P\|_\infty := \max\{|(x,y)|_{\mathbb{R}^{2n}}, |t|^{1/2}\}$$

and with the left-invariant and homogeneous distance

$$d_\infty(P,Q) := \|P^{-1} \cdot Q\|_\infty.$$

It is well-known that d_∞ is equivalent to the standard Carnot-Carathéodory (CC) distance, that will be denoted by d_c. The Hausdorff dimension of $(\mathbb{H}^n, d_\infty)$ is $Q := 2n+2$, whereas its topological dimension is $2n+1$.

Let $\Omega \subset \mathbb{H}^n$ be an open set and $\varphi = (\varphi_1, \dots, \varphi_{2n}) \in \mathbf{C}^1_c(\Omega; \mathbb{R}^{2n})$. The Heisenberg divergence of φ is

$$\mathrm{div}_{\mathbb{H}}\varphi := \sum_{j=1}^{2n} X_j \varphi_j. \tag{1.1}$$

Following the classical theory of sets with finite perimeter à la De Giorgi, the $\mathbb{H}$-perimeter in Ω of a measurable set $E \subset \mathbb{H}^n$ was introduced in [11] as

$$|\partial E|_{\mathbb{H}}(\Omega) := \sup\left\{ \int_E \mathrm{div}_{\mathbb{H}}\varphi \, d\mathcal{L}^{2n+1} : \varphi \in \mathbf{C}^1_c(\Omega, \mathbb{R}^{2n}), |\varphi| \leqslant 1 \right\}, \tag{1.2}$$

where $\mathcal{L}^{2n+1}$ denotes the Lebesgue measure on $\mathbb{H}^n \equiv \mathbb{R}^{2n+1}$, which is also the Haar measure of the group. It is well-known that, for smooth sets, the $\mathbb{H}$-perimeter coincides with (a multiple of) the $(Q-1)$-spherical Hausdorff measure, associated with d_∞, of the boundary, see also Proposition 2.10.
We want to study those graphs of bounded variation that are boundaries of sets minimizing the $\mathbb{H}$-perimeter measure. A set $E \subset \mathbb{H}^n$ is said to be a *(local) minimizer* of the $\mathbb{H}$-perimeter in an open set $\Omega \subset \mathbb{H}^n$ if it has locally finite $\mathbb{H}$-perimeter in Ω and for any open subset $\Omega' \Subset \Omega$

$$|\partial E|_{\mathbb{H}}(\Omega') \leqslant |\partial F|_{\mathbb{H}}(\Omega') \tag{1.3}$$

for any measurable $F \subset \mathbb{H}^n$ such that $E \Delta F \Subset \Omega'$; we hereafter denote by $E\Delta F := (E \setminus F) \cup (F \setminus E)$ the symmetric difference between E and F.
There is a huge variety of results concerning minimal-surfaces type problems (isoperimetric problem, existence and regularity of $\mathbb{H}$-perimeter minimizing sets, Bernstein problem, etc.). A general account of the many facets and contributions in this direction is far beyond the aim of this introduction and we refer to [45, 59, 49, 58, 21, 23, 48, 56, 9, 10, 40, 41, 25].

We can now introduce the classes of t- and intrinsic graphs of bounded variation in $\mathbb{H}^n$. A set $S \subset \mathbb{H}^n$ is called *t-graph* in $\mathbb{H}^n$ if it is a graph with respect to the *non horizontal* vector field T, i.e., if there exists a function $u : \mathcal{U} \to \mathbb{R}$ such that

$$S = \{(x, y, u(x,y)) \in \mathbb{H}^n : (x,y) \in \mathcal{U}\}.$$

Hereafter, by $\mathcal{U}$ we will denote a fixed open and bounded subset of the $2n$-dimensional plane

$$\Pi := \exp(\mathrm{span}\{X_j : j = 1, \dots, 2n\}) = \{(x, y, t) \in \mathbb{H}^n : t = 0\} .$$

When clear from the context we will canonically identify Π with $\mathbb{R}^{2n}$, and accordingly we will write (x, y) instead of $(x, y, 0)$. By $\mathcal{U} \times \mathbb{R}$ we will mean the t-cylinder

$$\mathcal{U} \times \mathbb{R} := \{(x, y, t) \in \mathbb{H}^n : (x, y) \in \mathcal{U}, t \in \mathbb{R}\} .$$

The *t-subgraph* E_u^t of $u : \mathcal{U} \to \mathbb{R}$ is defined as

$$E_u^t := \{(x, y, t) \in \mathbb{H}^n : (x, y) \in \mathcal{U}, t < u(x, y)\} . \tag{1.4}$$

For maps u with Sobolev regularity the area functional $\mathscr{A}_t : W^{1,1}(\mathcal{U}) \to \mathbb{R}$ reads as

$$\mathscr{A}_t(u) := |\partial E_u^t|_{\mathbb{H}}(\mathcal{U} \times \mathbb{R}) = \int_{\mathcal{U}} |\nabla u + \boldsymbol{X}^*| d\mathcal{L}^{2n} \tag{1.5}$$

where, following the notation in [17], $\boldsymbol{X}^* : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is defined by $\boldsymbol{X}^*(x, y) := 2(-y, x)$ (see Section 3). The formula (1.5) was proved in [11] for $u \in \mathbf{C}^1(\mathcal{U})$.

Definition 1.1. We say that $u \in L^1(\mathcal{U})$ belongs to the space $BV_t(\mathcal{U})$ of maps with bounded t-variation if $|\partial E_u^t|_{\mathbb{H}}(\mathcal{U} \times \mathbb{R}) < +\infty$.
We say that $u \in BV_{t,loc}(\mathcal{U})$ if E_u^t has finite $\mathbb{H}$-perimeter in $\mathcal{U}' \times \mathbb{R}$ for any open set $\mathcal{U}' \Subset \mathcal{U}$.

In Section 3 we will first study the structure of the space $BV_t(\mathcal{U})$ and several different notions of "area" for boundaries $\partial E_u^t \cap (\mathcal{U} \times \mathbb{R})$ of t-subgraphs of functions $u \in L^1(\mathcal{U})$. We will prove that these notions (among them: the perimeter $|\partial E_u^t|_{\mathbb{H}}(\mathcal{U} \times \mathbb{R})$ and the relaxed functional $\overline{\mathscr{A}_t}$ of $\mathscr{A}_t$) agree on $L^1(\mathcal{U})$: see Theorem 3.2. We introduce the notation

$$\int_{\mathcal{U}} |Du + \boldsymbol{X}^*| := |\partial E_u^t|_{\mathbb{H}}(\mathcal{U} \times \mathbb{R}) = \overline{\mathscr{A}_t}(u), \qquad u \in L^1(\mathcal{U}) .$$

It turns out that $BV_t(\mathcal{U})$, which is the finiteness domain of these functionals, coincides with the classical space $BV(\mathcal{U})$ of functions with bounded variation in $\mathcal{U}$. In particular, $BV(\mathcal{U})$ provides the appropriate framework, chosen for example in [14] and [55], for the study of area minimization problems for t-graphs.

Theorem 1.2. *Let $\mathcal{U} \subset \mathbb{R}^{2n}$ be a bounded open set. Then $BV_t(\mathcal{U}) = BV(\mathcal{U})$.*

In particular, each function in $BV_t(\mathcal{U})$ can be approximated with respect to the "strict" metric (see [2, pages 125-126]) by a sequence of C^∞ regular functions: see Corollary 3.3. Moreover, the space $BV_t(\mathcal{U})$ can be compactly embedded in $L^1(\mathcal{U})$ and the classical notion of trace $u_{|\partial\mathcal{U}}$ on $\partial\mathcal{U}$ is well defined provided $\mathcal{U}$ is bounded with Lipschitz regular boundary: see Theorem 3.4 and [39, Chapter 2].

In the second part of Section 3 we deal with the existence of t-minimizers.

Definition 1.3. Let $\mathcal{U} \subset \Pi$ be a bounded open set with Lipschitz regular boundary. We say that $u \in BV(\mathcal{U})$ is a t-minimizer of the area functional (briefly: *t-minimizer*) if

$$\int_{\mathcal{U}} |Du + \boldsymbol{X}^*| \leqslant \int_{\mathcal{U}} |Dv + \boldsymbol{X}^*|$$

for any $v \in BV(\mathcal{U})$ such that $v_{|\partial\mathcal{U}} = u_{|\partial\mathcal{U}}$.

Given a generic open set $\mathcal{U} \subset \Pi$, we say that $u \in BV_{loc}(\mathcal{U})$ is a *local t-minimizer* if

$$\int_{\mathcal{U}'} |Du + \boldsymbol{X}^*| \leqslant \int_{\mathcal{U}'} |Dv + \boldsymbol{X}^*|$$

for any $\mathcal{U}' \Subset \mathcal{U}$ and any $v \in L^1_{loc}(\mathcal{U})$ with $\{u \neq v\} \Subset \mathcal{U}'$. Equivalently (see Remark 3.12), if u is a t-minimizer on any open set $\mathcal{U}' \Subset \mathcal{U}$ with Lipschitz regular boundary.

A t-minimizer is also a local t-minimizer (see Remark 3.11). Moreover, it is easily seen that a t-subgraph E^t_u that is locally $\mathbb{H}$-perimeter minimizing in $\mathcal{U} \times \mathbb{R}$ must be associated with a local t-minimizer $u \in BV(\mathcal{U})$. Conversely, we will prove in Corollary 3.16 that a local t-minimizer $u \in BV(\mathcal{U})$ induces a t-subgraph E^t_u that is a local minimizer of the $\mathbb{H}$-perimeter in $\mathcal{U} \times \mathbb{R}$.
Local t-minimizers have been widely studied assuming $u \in W^{1,1}(\mathcal{U})$, the classical Sobolev space which is strictly contained in $BV(\mathcal{U})$. The functional (1.5) has good variational properties such as convexity and lower semicontinuity with respect to the L^1 topology. On the other hand, it is not coercive and differentiable due to the presence of the so called *characteristic points*, i.e., the points on the graph of u where the tangent hyperplane to the graph coincides with the horizontal plane. Equivalently, the set whose projection on Π is

$$\operatorname{Char}(u) := \{(x,y) \in \mathcal{U} : \nabla u(x,y) + \boldsymbol{X}^*(x,y) = 0\}. \tag{1.6}$$

Notice that, if $u \in W^{1,1}(\mathcal{U})$, the set $\operatorname{Char}(u)$ must be understood up to a $\mathcal{L}^{2n}$-negligible set.

Nevertheless, the existence of solutions to the Dirichlet problem with regular boundary conditions was obtained in [53] and [17], by means of an elliptic approximation argument, for $\mathcal{U}$ satisfying suitable convexity assumptions. The lack of coercivity for the functional (1.5) does not allow a first variation near the set $\operatorname{Char}(u)$. This and related questions have been studied in [53, 38, 15, 13, 59, 61] for $\mathbf{C}^2$ minimizers of $\mathscr{A}_t$, and in [18, 16] for $\mathbf{C}^1$ regular ones, also in connection with the Bernstein problem for t-graphs. A suitable *minimal surface equation* for t-graphs (see (3.1)) has been obtained in these papers; its solutions are called *H-minimal surfaces*. In particular, in [15] a deep analysis of $\operatorname{Char}(u)$ was carried out for local minimizers $u \in \mathbf{C}^2(\mathcal{U})$ together with other regularity properties like comparison principles and uniqueness for the associated Dirichlet problem. The study of the characteristic set has been performed in [16] for C^1 surfaces in $\mathbb{H}^1$ satisfying a constant mean curvature equation in a weak sense. The much more delicate case of minimizers $u \in W^{1,1}(\mathcal{U})$ was attacked in [17]. Several examples of t-minimizers with at most Lipschitz regularity have been provided in $\mathbb{H}^1$ (see [54, 17, 57]). Therefore, at least in the $\mathbb{H}^1$ setting, the problem of regularity for t-minimizers is very different from the Euclidean case, where minimal graphs of codimension one are analytic regular (see [39, Theorem 14.13]).
In the spirit of the previous results, we are able to establish an existence result for the Dirichlet minimum problem for the functional (1.5) on the class of t-graphs of bounded variation.

Theorem 1.4 (Existence of minimizers for a penalized functional)**.** *Let $\mathcal{U} \subset \mathbb{R}^{2n}$ be a bounded open set with Lipschitz regular boundary. Then, for any given $\varphi \in L^1(\partial\mathcal{U})$*

the functional

$$BV(\mathcal{U}) \ni u \longmapsto \int_{\mathcal{U}} |Du + \boldsymbol{X}^*| + \int_{\partial\mathcal{U}} |u_{|\partial\mathcal{U}} - \varphi| d\mathcal{H}^{2n-1} \tag{1.7}$$

attains its minimum and

$$\begin{aligned} &\inf\left\{\int_{\mathcal{U}} |Du + \boldsymbol{X}^*| : u \in BV(\mathcal{U}),\ u_{|\partial\mathcal{U}} = \varphi\right\} \\ &= \min\left\{\int_{\mathcal{U}} |Du + \boldsymbol{X}^*| + \int_{\partial\mathcal{U}} |u_{|\partial\mathcal{U}} - \varphi| d\mathcal{H}^{2n-1} : u \in BV(\mathcal{U})\right\}. \end{aligned}$$

We remark that the last integral in (1.7) equals both the Euclidean and sub-Riemannian areas of that part of the cylinder $\partial\mathcal{U} \times \mathbb{R}$ between the graphs of u and φ, hence it can be seen as a penalization for not taking the boundary values φ on $\partial\mathcal{U}$. See also Proposition 3.7 and Remark 3.8.

Theorem 1.4 extends the existence results contained in [53] and [17] in the sense that formulation (1.7) allows to consider more general domains $\mathcal{U}$. We point out that a minimizer of the penalized functional (1.7) might not take the prescribed boundary value φ: we illustrate this situation by explicitly constructing an example where t-minimizers do not exist, see Example 3.6. In particular, the existence of solutions for the Dirichlet minimum problem for $\mathscr{A}_t$ is not guaranteed even when the boundary $\partial\mathcal{U}$ and the datum φ are very regular: in this sense, Theorem 1.4 does not extend the results in [53] and [17].

An existence result for continuous BV t-minimizers, taking the prescribed boundary datum, has been obtained by J.-H. Cheng and J.-F. Hwang [14] for continuous boundary data on smooth parabolically convex domains. In the forthcoming paper [55] existence, uniqueness and Lipschitz regularity of t-minimizers (assuming the prescribed boundary datum) is proved under the assumption that the boundary datum φ satisfies the so-called Bounded Slope Condition: this result, in particular, extends Theorem 1.4 as well as some related results in [53] and [17].

In the third part of Section 3 we study the boundedness of local t-minimizer; our main result is the following.

Theorem 1.5 (Local boundedness of minimal t-graphs)**.** *Let $u \in BV_{loc}(\mathcal{U})$ be a local t-minimizer. Then $u \in L^\infty_{loc}(\mathcal{U})$.*

As a consequence, we obtain a local boundedness result for weak solutions of the minimal surface equation, see Theorem 3.17.

Theorem 1.5 is sharp at least in the first Heisenberg group $\mathbb{H}^1$. Indeed, we are able to provide a minimal t-graph induced by a function $u \in L^\infty_{loc}(\mathcal{U}) \setminus \mathbf{C}^0(\mathcal{U})$: see subsection 3.4. It is an open problem whether a similar example can be constructed also in $\mathbb{H}^n$, $n \geqslant 2$.

We want to stress also the following consequence of Theorem 1.2; we refer to Section 2 for the definition of $\mathbb{H}$-regular hypersurface.

Corollary 1.6. *Let S be an $\mathbb{H}$-regular hypersurface that is not (Euclidean) countably $\mathcal{H}^{2n}$-rectifiable; then S is not a t-graph.*

We are now going to introduce the notion of intrinsic graphs, i.e., graphs with respect to one of the *horizontal* vector fields X_i. This is not a pointless generalization:

without entering into motivations, we recall only that any $\mathbb{H}$-regular hypersurface is locally an intrinsic graph. For further details we refer to [34]. Without loss of generality, we will always consider X_1-graphs, i.e., intrinsic graphs along the X_1-direction.
Let us introduce some preliminary notation. If $n \geqslant 2$, we identify the maximal subgroup

$$\mathbb{W} := \exp(\mathrm{span}\{X_2, \dots, X_n, Y_1, \dots, Y_n, T\}) = \{(x, y, t) \in \mathbb{H}^n : x_1 = 0\}$$

with $\mathbb{R}^{2n}$ by writing $(x_2, \dots, x_n, y_1, \dots, y_n, t)$ instead of $(0, x_2, \dots, x_n, y_1, \dots, y_n, t)$; similarly $\mathbb{W} := \exp(\mathrm{span}\{Y_1, T\}) = \{(0, y, t) \in \mathbb{H}^1 : y, t \in \mathbb{R}\} \equiv \mathbb{R}^2_{y,t}$ if $n = 1$. Let ω denote a fixed open bounded subset of $\mathbb{W}$; the intrinsic cylinder $\omega \cdot \mathbb{R}$ is defined by

$$\omega \cdot \mathbb{R} := \{A \cdot s \in \mathbb{H}^n : A \in \omega, s \in \mathbb{R}\},$$

where, for $A \in \mathbb{W}$ and $s \in \mathbb{R}$ we write $A \cdot s$ to denote the Heisenberg product $A \cdot (s, 0, \dots, 0)$. In this way $I \cdot J = \{A \cdot s : A \in I, s \in J\}$ for any $I \subset \mathbb{W}$, $J \subset \mathbb{R}$. Similarly, we will write $s \cdot A$ to denote $(s, 0, \dots, 0) \cdot A$.
Given a function $\phi : \omega \to \mathbb{R}$, we denote by $\Phi : \omega \to \mathbb{H}^n$ the corresponding X_1-graph map

$$\Phi(A) := A \cdot \phi(A) \qquad A \in \omega. \tag{1.8}$$

A set $S \subset \mathbb{H}^n$ is called X_1-*graph* of $\phi : \omega \to \mathbb{R}$ if

$$S := \Phi(\omega) = \{A \cdot \phi(A) : A \in \omega\} .$$

The X_1-*subgraph* and the X_1-*epigraph* of ϕ are defined, respectively, as

$$E_\phi := \{A \cdot s : A \in \omega, s < \phi(A)\} , \tag{1.9}$$

and

$$E^\phi := \{A \cdot s : A \in \omega, s > \phi(A)\} . \tag{1.10}$$

Let *Lip*(ω) be the classical space of Lipschitz functions on $\omega \subset \mathbb{W} \equiv \mathbb{R}^{2n}$. The area functional $\mathscr{A}_{\mathbb{W}} : Lip(\omega) \to \mathbb{R}$ reads as

$$\mathscr{A}_{\mathbb{W}}(\phi) := |\partial E_\phi|_{\mathbb{H}}(\omega \cdot \mathbb{R}) = \int_\omega \sqrt{1 + |\nabla^\phi \phi|^2}\, d\mathcal{L}^{2n}, \tag{1.11}$$

where $\nabla^\phi \phi$ is the non-linear *intrinsic gradient* for X_1-graphs

$$\nabla^\phi \phi := \begin{cases} (X_2\phi, \dots, X_n\phi, W^\phi\phi, Y_2\phi, \dots, Y_n\phi) & \text{if } n \geqslant 2 \\ W^\phi \phi & \text{if } n = 1 \end{cases} \tag{1.12}$$

where

$$W^\phi \phi := Y_1\phi - 2T(\phi^2) = \partial_{y_1}\phi - 2\,\partial_t(\phi^2) . \tag{1.13}$$

We agree that, when ϕ is not regular, the differential operators appearing in (1.12) will be understood in the sense of distributions. The intrinsic gradient ∇^ϕ was introduced and studied in [3], see also [19, 8].

Definition 1.7. We say that $\phi \in L^1(\omega)$ belongs to the class $BV_{\mathbb{W}}(\omega)$ of functions with intrinsic bounded variation if $|\partial E_\phi|_{\mathbb{H}}(\omega \cdot \mathbb{R}) < +\infty$.
We say that ϕ belongs to $BV_{\mathbb{W},loc}(\omega)$ if E_ϕ is a set with finite $\mathbb{H}$-perimeter in $\omega' \cdot \mathbb{R}$ for any open set $\omega' \Subset \omega$.

The class $BV_{\mathbb{W}}(\omega)$ is deeply different from $BV(\omega)$: for instance, it is not even a vector space (see Remark 4.2). In spite of these differences, $BV_{\mathbb{W}}(\omega)$ shares with $BV(\omega)$ several properties:

- the functional $\phi \mapsto |\partial E_\phi|_{\mathbb{H}}(\omega \cdot \mathbb{R})$ coincides with the relaxed one $\overline{\mathscr{A}}_{\mathbb{W}}$ of $\mathscr{A}_{\mathbb{W}}$ on $L^1(\omega)$, see Theorem 4.7;
- each function in $BV_{\mathbb{W}}(\omega)$ can be approximated by a sequence of C^∞ regular functions $(\phi_j)_j$ such that
$$\phi_j \to \phi \text{ in } L^1(\omega) \quad \text{and} \quad |\partial E_{\phi_j}|_{\mathbb{H}}(\omega \cdot \mathbb{R}) \to |\partial E_\phi|_{\mathbb{H}}(\omega \cdot \mathbb{R}),$$
see Theorem 4.9;
- when ω has Lipschitz regular boundary, a *trace in generalized sense* exists at least for some large subclass of $BV_{\mathbb{W}}(\omega)$ (see Proposition 4.15). This notion of trace is related to the possibility of extending the set E_ϕ out of $\omega \cdot \mathbb{R}$ without "creating" perimeter on the boundary $\partial\omega \cdot \mathbb{R}$: see Definition 4.11. We conjecture that any ϕ in $BV_{\mathbb{W}}$ does have a trace in this sense. However, as shown in Remark 4.10, any meaningful notion of trace in $BV_{\mathbb{W}}$ cannot possess all the features of classical traces;
- any sequence $(\phi_j)_j \subset BV_{\mathbb{W}}(\omega)$ bounded in the $\|\cdot\|_{BV_{\mathbb{W}}}$ "norm" (see (4.43)) and such that
$$\sup_j \big(|\partial E_{\phi_j}|_{\mathbb{H}}(\mathbb{H}^n_+) + |\partial E^{\phi_j}|_{\mathbb{H}}(\mathbb{H}^n_-) \big) < +\infty \tag{1.14}$$
is compact with respect to the $L^1_{loc}(\omega \cdot \mathbb{R})$-convergence of its subgraphs $(E_{\phi_j})_j$ (see Proposition 4.18), where we have set
$$\mathbb{H}^n_+ := \{(x,y,t) \in \mathbb{H}^n : x_1 \geqslant 0\}, \quad \mathbb{H}^n_- := \{(x,y,t) \in \mathbb{H}^n : x_1 \leqslant 0\}. \tag{1.15}$$
Condition (1.14) is equivalent to
$$\sup_j \big(|\partial E_{\phi_j}|_{\mathbb{H}}(\partial\omega \cdot \mathbb{R}^+) + |\partial E^{\phi_j}|_{\mathbb{H}}(\partial\omega \cdot \mathbb{R}^-) \big) < +\infty ,$$
where $\mathbb{R}^+ := [0, +\infty)$, $\mathbb{R}^- := (-\infty, 0]$.

In Section 4 we also attack the problem of the existence and regularity of minimal X_1-intrinsic graphs. In the literature, the regularity problem has been studied assuming $\phi \in Lip(\omega)$ (see [9, 10]) or $\phi \in W^{1,1}_{\mathbb{W}}(\omega)$ (see [51]), a suitable class of intrinsic graphs with Sobolev regularity introduced in [51] (see Definition 2.8).
The area functional $\mathscr{A}_{\mathbb{W}}$ is lower semicontinuous with respect to the L^1 topology but it is not convex (see [24] and Proposition 4.1). Furthermore, it is differentiable and its first variation yields the *minimal surface equation* (4.4) for X_1-graphs. A study of the $\mathbf{C}^2$ minimizers of $\mathscr{A}_{\mathbb{W}}$ was carried out in [22], [7], [21] and [24] also in connection with the Bernstein problem for intrinsic graphs. First and second variations for minimizers in $W^{1,1}_{\mathbb{W}}(\omega)$ have been studied in [51]. The regularity of Lipschitz continuous vanishing viscosity solutions of the minimal surface equation for intrinsic graphs has been studied in [9, 10]. We have to mention that, in the first Heisenberg group $\mathbb{H}^1$, there are minimizers of $\mathscr{A}_{\mathbb{W}}$ whose regularity is not better than $\frac{1}{2}$-Hölder: see [51, Theorem 1.5].
We shall prove an existence result for minimal X_1-graphs on ω with prescribed "boundary datum". Let $\omega_0 \Supset \omega$ be a bounded open set and $\theta \in BV_{\mathbb{W}}(\omega_0)$ be such

that

$$|\partial E_\theta|_{\mathbb{H}}(\partial\omega_0 \cdot \mathbb{R}^+) + |\partial E^\theta|_{\mathbb{H}}(\partial\omega_0 \cdot \mathbb{R}^-) < \infty, \quad |\partial E_\theta|_{\mathbb{H}}(\partial\omega \cdot \mathbb{R}) = 0\,. \tag{1.16}$$

We consider the problem

$$\inf\ \left\{ |\partial E_\psi|_{\mathbb{H}}(\overline{\omega} \cdot \mathbb{R}) : \psi \in BV_{\mathbb{W}}(\omega_0), \psi = \theta \text{ on } \omega_0 \setminus \overline{\omega} \right\}. \tag{1.17}$$

When $\phi \in BV_{\mathbb{W}}(\omega)$ has a trace in generalized sense, then it possesses an extension $\theta \in BV_{\mathbb{W}}(\omega_0)$, on a suitable $\omega_0 \Supset \omega$, satisfying (1.16): if this is the case, the problem (1.17) can be viewed as that of minimizing area with boundary datum given by ϕ.

Theorem 1.8 (Existence of minimal X_1-graphs)**.** *The problem* (1.17) *attains a minimum in* $BV_{\mathbb{W}}(\omega_0)$.

Theorem 1.8 is proved in Section 4.2; in the subsequent Section 4.3 we obtain a local boundedness result for minimal X_1-graphs.

Theorem 1.9 (Local boundedness of minimal X_1-graphs)**.** *Let* $\phi \in L^{2n+1}_{loc}(\omega)$ *be such that* E_ϕ *is a local minimizer of the* $\mathbb{H}$*-perimeter in* $\omega \cdot \mathbb{R}$*. Then* $\phi \in L^\infty_{loc}(\omega)$.

This result is not the exact counterpart of Theorem 1.5 for minimal t-graphs. We do not know whether the additional $(2n+1)$-summability is only a technical problem or if there exist minimal X_1-graphs $\phi \notin L^\infty_{\mathrm{loc}}(\omega)$. Moreover, in Theorem 1.5 we prove the local boundedness of t-minimizers using the fact that their associated subgraphs are also $\mathbb{H}$-perimeter minimizing sets. In Theorem 1.9 we instead *require* the subgraph E_ϕ to be $\mathbb{H}$-perimeter minimizing: as far as we know, there is no geometric rearrangement, similar to the one for t-graphs given by Theorem 3.15, ensuring that the subgraph of a minimal intrinsic graph is also $\mathbb{H}$-perimeter minimizing. At any rate, the problem of further regularity for area minimizing intrinsic graphs is completely open.

Finally, we have to point out that our techniques have been strongly inspired from the important work [47]; we also refer to [39, 46].

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2. Preliminaries

By $\mathcal{H}^m_\infty, \mathcal{S}^m_\infty$ we denote, respectively, the m-dimensional Hausdorff and spherical Hausdorff measures associated with the distance d_∞, while $\mathcal{H}^m, \mathcal{S}^m$ refer to the corresponding Euclidean measures. Recall that (see [6])

$$S^{Q-1}_\infty \ll \mathcal{H}^{2n}\,. \tag{2.1}$$

By $U(P,r)$ and $U_c(P,r)$ we mean the open balls of center P and radius r, respectively, with respect to the d_∞ and the CC metric d_c; when centered at the origin, balls will be denoted by U_r and $U_{c,r}$.

Euclidean open balls in $\mathbb{R}^m$ will be denoted by $B(P,r)$ and B_r. The symbol $|\cdot|$ is reserved for the norm of elements of $\mathbb{R}^m$, while the Euclidean distance between two points $A, B \in \mathbb{R}^m$ is denoted by $dist(A, B)$. For $E \subset \mathbb{R}^m$ we write χ_E for the characteristic function of E and $|E|$ for its Lebesgue measure $\mathcal{L}^m(E)$ (of course, no confusion with the norm of a vector will arise). The identification $\mathbb{H}^n \equiv \mathbb{R}^{2n+1}$ is understood when the previous symbols involve elements or subsets of $\mathbb{H}^n$.

A real measurable function f defined on an open set $\Omega \subset \mathbb{H}^n$ is said to be of class $\mathbf{C}^1_{\mathbb{H}}(\Omega)$ if $f \in \mathbf{C}^0(\Omega)$ and the distributional horizontal gradient $\nabla_{\mathbb{H}} f := (X_1 f, \dots, X_n f, Y_1 f, \dots, Y_n f)$ is represented by a continuous function. The function f is said to be of class $Lip_{\mathbb{H}}(\Omega)$ if $f : (\Omega, d_\infty) \to \mathbb{R}$ is Lipschitz continuous. Each function $f \in Lip_{\mathbb{H}}(\Omega)$ also admits a distributional horizontal gradient $\nabla_{\mathbb{H}} f := (X_1 f, \dots, X_n f, Y_1 f, \dots, Y_n f) \in (L^\infty(\Omega))^{2n}$ (see, for instance, [30, Proposition 2.9]).

Given a function $f \in L^1(\Omega)$ we define

$$|D_{\mathbb{H}} f|(\Omega) := \sup \left\{ \int_\Omega f \operatorname{div}_{\mathbb{H}} \varphi : \varphi \in \mathbf{C}^1_c(\Omega),\ |\varphi| \leqslant 1 \right\}.$$

We say (see [11]) that f belongs to the space of functions with *bounded $\mathbb{H}$-variation* $BV_{\mathbb{H}}(\Omega)$ if $|D_{\mathbb{H}} f|(\Omega) < +\infty$. In this case $|D_{\mathbb{H}} f|$ defines a Radon measure that coincides with the total variation of the distributional horizontal derivatives $\nabla_{\mathbb{H}} f$. A measurable set has finite $\mathbb{H}$-perimeter in Ω if and only if $\chi_E \in BV_{\mathbb{H}}(\Omega)$; moreover, $|\partial E|_{\mathbb{H}} = |D_{\mathbb{H}} \chi_E|$. A norm in $BV_{\mathbb{H}}$ is defined by

$$\|f\|_{BV_{\mathbb{H}}(\Omega)} := \|f\|_{L^1(\Omega)} + |D_{\mathbb{H}} f|(\Omega).$$

The inclusion $BV_{\mathbb{H}}(U_c(P, r)) \subset L^1(U_c(P, r))$ is compact (see [37, Theorem 1.28]). Let us recall the following coarea formula (see [50, Theorem 4.2]).

Theorem 2.1. *Let $f \in Lip_{\mathbb{H}}(\mathbb{H}^n)$ and $u \in L^1(\mathbb{H}^n)$. Then*

$$\int_{\mathbb{H}^n} u |\nabla_{\mathbb{H}} f| \, d\mathcal{L}^{2n+1} = \int_{-\infty}^{+\infty} \int_{\{f = t\}} u \, d\mu_t \, dt$$

where $\mu_t := |\partial\{f < t\}|_{\mathbb{H}}$.

We say that a sequence of measurable subsets $(E_j)_j$ of $\mathbb{H}^n$ converges in $L^1(\Omega)$ (respectively in $L^1_{loc}(\Omega)$) to a measurable set $E \subset \mathbb{H}^n$, and we will write $E_j \to E$ in $L^1(\Omega)$ (respectively in $L^1_{loc}(\Omega)$), if $\chi_{E_j} \to \chi_E$ in $L^1(\Omega)$ (respectively in $L^1_{loc}(\Omega)$). An immediate consequence of definition (1.2) is the $L^1_{loc}(\Omega)$-lower semicontinuity of the $\mathbb{H}$-perimeter:

Proposition 2.2. *Let $\Omega \subset \mathbb{H}^n$ be an open set and let $(E_j)_j$ be a sequence of measurable subsets of $\mathbb{H}^n$ converging in $L^1_{loc}(\Omega)$ to $E \subset \mathbb{H}^n$. Then*

$$|\partial E|_{\mathbb{H}}(\Omega) \leqslant \liminf_{j \to \infty} |\partial E_j|_{\mathbb{H}}(\Omega).$$

We also recall the following properties of the $\mathbb{H}$-perimeter measure: they can be proved as in the classical case (see, for instance, [2, Proposition 3.38]).

Proposition 2.3. *Let $\Omega \subset \mathbb{H}^n$ be an open set and let E and F be measurable subsets of $\mathbb{H}^n$. Then*

(i) $\operatorname{spt} |\partial E|_{\mathbb{H}} \subset \partial E$, *where* $\operatorname{spt} |\partial E|_{\mathbb{H}}$ *denotes the support of the measure* $|\partial E|_{\mathbb{H}}$*;*
(ii) $|\partial E|_{\mathbb{H}}(\Omega) = |\partial(\mathbb{H}^n \setminus E)|_{\mathbb{H}}(\Omega)$*;*
(iii) *(locality of $\mathbb{H}$- perimeter measure)* $|\partial E|_{\mathbb{H}}(\Omega) = |\partial(E \cap \Omega)|_{\mathbb{H}}(\Omega)$*;*

(iv) $|\partial(E\cup F)|_{\mathbb{H}}(\Omega) + |\partial(E\cap F)|_{\mathbb{H}}(\Omega) \leqslant |\partial E|_{\mathbb{H}}(\Omega) + |\partial F|_{\mathbb{H}}(\Omega)$.

An isoperimetric inequality holds in the Heisenberg group, see [37, Theorem 1.18]:

Theorem 2.4. *There is a positive constant $c_I > 0$ such that for any set E with finite $\mathbb{H}$-perimeter, for all $x \in \mathbb{H}^n$ and $r > 0$*

$$\min\{|E\cap U_c(x,r)|, |U_c(x,r)\setminus E|\}^{\frac{Q-1}{Q}} \leqslant c_I |\partial E|_{\mathbb{H}}(U_c(x,r)) \tag{2.2}$$

and

$$\min\{|E|, |\mathbb{H}^n\setminus E|\}^{\frac{Q-1}{Q}} \leqslant c_I |\partial E|_{\mathbb{H}}(\mathbb{H}^n). \tag{2.3}$$

By Riesz' representation Theorem, if E has finite $\mathbb{H}$-perimeter in Ω then $|\partial E|_{\mathbb{H}}$ is a Radon measure on Ω for which there exists a unique $|\partial E|_{\mathbb{H}}$-measurable function $\nu_E : \Omega \to \mathbb{R}^{2n}$ such that

$$\begin{aligned} &|\nu_E| = 1 \quad |\partial E|_{\mathbb{H}}\text{-a.e. in } \Omega \\ &\int_E \mathrm{div}_{\mathbb{H}}\varphi \, d\mathcal{L}^{2n+1} = -\int_\Omega \langle \varphi, \nu_E\rangle \, d|\partial E|_{\mathbb{H}} \quad \text{for all } \varphi \in \mathbf{C}^1_c(\Omega,\mathbb{R}^{2n}). \end{aligned} \tag{2.4}$$

We call ν_E the *horizontal inward normal to E* (see [29]); the distributional derivatives $\nabla_{\mathbb{H}}\chi_E$ are represented by the vector measure $\nu_E|\partial E|_{\mathbb{H}}$.
It is well-known that the $\mathbb{H}$-perimeter measure of a set $E \subset \mathbb{H}^n$ does not change under modifications of E on sets of null $2n+1$-dimensional Lebesgue measure. Let us define the *interior, exterior* and *boundary (in measure)* of E, respectively, by

$$\begin{aligned} \mathrm{int}_m E &:= \{P \in \mathbb{H}^n : \exists \varrho > 0 \text{ with } |E\cap U(P,\varrho)| = |U(P,\varrho)|\}\ , \\ \mathrm{ext}_m E &:= \{P \in \mathbb{H}^n : \exists \varrho > 0 \text{ with } |E\cap U(P,\varrho)| = 0\}\,, \\ \partial_m E &:= \{P \in \mathbb{H}^n : 0 < |E\cap U(P,\varrho)| < |U(P,\varrho)| \ \forall \varrho > 0\}\ . \end{aligned}$$

It is easily seen that $\mathrm{int}_m E$, $\mathrm{ext}_m E$ and $\partial_m E$ are stable under replacing the metric d_∞ with an equivalent one. In particular, we can equivalently define them by means of CC balls.

Proposition 2.5. *Let $E \subset \mathbb{H}^n$ be a Borel set and define*

$$\tilde{E} := (E \cup \mathrm{int}_m E) \setminus \mathrm{ext}_m E\,. \tag{2.5}$$

Then $\tilde{E}$ is a Borel set with $|\tilde{E}\Delta E| = 0$ and its topological boundary $\partial\tilde{E}$ coincides with $\partial_m \tilde{E}$. In particular, $|\partial\tilde{E}|_{\mathbb{H}} = |\partial E|_{\mathbb{H}}$.

The proof of Proposition 2.5 is perfectly analogous to that of the corresponding Euclidean result, see [39, Proposition 3.1]. Without loss of generality, in the following we will always suppose that E coincide with the associated set $\tilde{E}$ in (2.5).

At this point we have to summarize some of the results of [31]. For a set E with finite $\mathbb{H}$-perimeter it is possible to introduce the *reduced boundary* $\partial^*_{\mathbb{H}}E$ as the set of those points P such that

$$\begin{aligned} &|\partial E|_{\mathbb{H}}(U(P,r)) > 0 \text{ for any } r > 0 \\ &\text{the limit } \lim_{r\to 0} \frac{\int_{U(P,r)} \nu_E \, d|\partial E|_{\mathbb{H}}}{|\partial E|_{\mathbb{H}}(U(P,r))} \text{ exists and is a unit vector.} \end{aligned}$$

It turns out that

$$|\partial E|_{\mathbb{H}} = c_n \mathcal{S}^{Q-1}_\infty \llcorner \partial^*_{\mathbb{H}} E, \tag{2.6}$$

where c_n is a positive constant depending on n. The blow-up properties of E at points of the reduced boundary (see [31]) ensure that $\partial^*_{\mathbb{H}} E \subset E^{1/2}$, where for given $\alpha \in [0,1]$ we set E^α to be the set of points with density α

$$E^\alpha := \left\{ P \in \mathbb{H}^n : \lim_{r\to 0} \frac{|E \cap U(P,r)|}{|U(P,r)|} = \alpha \right\} .$$

The measure theoretic boundary $\partial_{*,\mathbb{H}} E$ was introduced in [31, Definition 7.4]; it coincides with $\mathbb{H}^n \setminus (E^1 \cup E^0)$.

The following result is implicitly contained in [31]:

Theorem 2.6. *Let E be a set with locally finite $\mathbb{H}$-perimeter; then*

$$\mathcal{S}^{Q-1}_\infty\big(\mathbb{H}^n \setminus (E^1 \cup E^0 \cup E^{1/2})\big) = 0.$$

Moreover, $|\partial E|_{\mathbb{H}} = c_n \mathcal{S}^{Q-1}_\infty \llcorner E^{1/2} = c_n \mathcal{S}^{Q-1}_\infty \llcorner \partial_{*,\mathbb{H}} E$.

Proof. Since $\partial_{*,\mathbb{H}} E = \mathbb{H}^n \setminus (E^1 \cup E^0)$, one has

$$\mathcal{S}^{Q-1}_\infty(\mathbb{H}^n \setminus (E^1 \cup E^0 \cup E^{1/2})) = \mathcal{S}^{Q-1}_\infty(\partial_{*,\mathbb{H}} E \setminus E^{1/2}) \leqslant \mathcal{S}^{Q-1}_\infty(\partial_{*,\mathbb{H}} E \setminus \partial^*_{\mathbb{H}} E) = 0\,,$$

the last equality following from [31, Lemma 7.5]. The second part of the statement follows from (2.6) and

$$\mathcal{S}^{Q-1}_\infty \llcorner \partial^*_{\mathbb{H}} E \leqslant \mathcal{S}^{Q-1}_\infty \llcorner E^{1/2} \leqslant \mathcal{S}^{Q-1}_\infty \llcorner \partial_{*,\mathbb{H}} E = \mathcal{S}^{Q-1}_\infty \llcorner \partial^*_{\mathbb{H}} E\,.$$

□

We say that $S \subset \mathbb{H}^n$ is an $\mathbb{H}$-*regular hypersurface* if for every $P \in S$ there exist a neighbourhood Ω of P and a function $f \in \mathbf{C}^1_{\mathbb{H}}(\Omega)$ such that $\nabla_{\mathbb{H}} f \neq 0$ and $S \cap \Omega = \{Q \in \Omega : f(Q) = 0\}$. The *horizontal normal* to S at P is

$$\nu_S(P) := -\frac{\nabla_{\mathbb{H}} f(P)}{|\nabla_{\mathbb{H}} f(P)|}.$$

An $\mathbb{H}$-regular hypersurface can be highly irregular from the Euclidean viewpoint as it can be a fractal set [43]. This not being restrictive, we will deal only with hypersurfaces S that are level sets of functions $f \in \mathbf{C}^1_{\mathbb{H}}$ with $X_1 f \neq 0$. The importance of $\mathbb{H}$-regular hypersurfaces is clear in the theory of rectifiability in $\mathbb{H}^n$. The reduced boundary of a set with finite $\mathbb{H}$-perimeter is $\mathbb{H}$-*rectifiable* (see [31]), i.e., it is contained, up to $\mathcal{S}^{Q-1}_\infty$-negligible sets, in a countable union of $\mathbb{H}$-regular hypersurfaces.

The following equalities

$$\begin{aligned}
&|E_\phi \cap \mathbb{H}^n_+| = \textstyle\int_\omega \phi^+ \, d\mathcal{L}^{2n}, \quad |\mathbb{H}^n_- \setminus E_\phi| = \int_\omega \phi^- \, d\mathcal{L}^{2n}\,,\\
&|E_\phi \Delta \mathbb{H}^n_-| = \textstyle\int_\omega |\phi| \, d\mathcal{L}^{2n}\,,\\
&|E_{\phi_1} \Delta E_{\phi_2}| = \textstyle\int_{\mathbb{H}^n} |\chi_{E_{\phi_1}} - \chi_{E_{\phi_2}}| \, d\mathcal{L}^{2n+1} = \int_\omega |\phi_1 - \phi_2| \, d\mathcal{L}^{2n}\,,\\
&|E^t_{u_1} \Delta E^t_{u_2}| = \textstyle\int_{\mathbb{H}^n} |\chi_{E^t_{u_1}} - \chi_{E^t_{u_2}}| \, d\mathcal{L}^{2n+1} = \int_{\mathcal{U}} |u_1 - u_2| \, d\mathcal{L}^{2n}\,,
\end{aligned} \tag{2.7}$$

hold for any measurable functions $\phi, \phi_1, \phi_2 : \omega \to \mathbb{R}$, $u_1, u_2 : \mathcal{U} \to \mathbb{R}$, where $\mathbb{H}^n_+$ and $\mathbb{H}^n_-$ are the half-spaces of $\mathbb{H}^n$ introduced in (1.15) and

$$\phi^+ := \max\{\phi, 0\}, \qquad \phi^- := \max\{-\phi, 0\}\,. \tag{2.8}$$

The first three equalities in (2.7) can be easily proved because the smooth map

$$\begin{aligned} \omega \times \mathbb{R} &\longrightarrow \mathbb{H}^n \equiv \mathbb{R}^{2n+1} \\ (A, s) &\longmapsto A \cdot s \end{aligned}$$

has Jacobian determinant equal to 1.
Given $\phi : \omega \to \mathbb{R}$, the associated graph map $\Phi : \omega \to \mathbb{H}^n$ was defined in (1.8). Similarly, we agree to denote Φ_ϵ, Φ_j, etc. the graph maps associated with ϕ_ϵ, ϕ_j, etc. The projection $\pi_{\mathbb{W}} : \mathbb{H}^n \to \mathbb{W}$ is defined by

$$\pi_{\mathbb{W}}(x, y, t) := (x, y, t) \cdot (-x_1, 0, \dots, 0) = (0, x_2, \dots, x_n, y_1, \dots, y_n, t - 2x_1y_1) \tag{2.9}$$

so that $(x, y, t) = \pi_{\mathbb{W}}(x, y, t) \cdot x_1$. Observe that $\Phi^{-1} = \pi_{\mathbb{W}\,|\Phi(\omega)}$ and that

$$\pi_{\mathbb{W}}(P \cdot s) = \pi_{\mathbb{W}}(P), \quad \pi_{\mathbb{W}}(s \cdot P) = s \cdot \pi_{\mathbb{W}}(P) \cdot (-s) \qquad \forall P \in \mathbb{H}^n, \forall s \in \mathbb{R}\,.$$

It is easily seen that the map $\pi_{\mathbb{W}}$ is open and there exists $c = c(n) > 0$ such that $\|\pi_{\mathbb{W}}(P)\|_\infty \leqslant c\|P\|_\infty$ for any $P \in \mathbb{H}^n$ (see [35, Proposition 3.2 and Remark 4.2]).

An $\mathbb{H}$-regular hypersurface S with $\nu_S^1 < 0$ is locally an X_1-graph, see [31]; a characterization of the functions ϕ such that $\Phi(\omega)$ is an $\mathbb{H}$-regular hypersurface was given in [3] (see also [19]). We define

$$\mathbf{C}^1_{\mathbb{W}}(\omega) := \{\phi \in \mathbf{C}^0(\omega) : \Phi(\omega) \text{ is } \mathbb{H}\text{-regular and } \nu^1_{\Phi(\omega)}(\Phi(A)) < 0\ \forall A \in \omega\}.$$

Functions in the class $\mathbf{C}^1_{\mathbb{W}}$ have been characterized in [8] improving some previous results obtained in [3, 19]. Moreover, for such functions an area-type formula was obtained in [3]. We summarize these results in the following

Theorem 2.7. *A function $\phi : \omega \to \mathbb{R}$ belongs to $\mathbf{C}^1_{\mathbb{W}}(\omega)$ if and only if $\phi \in \mathbf{C}^0(\omega)$ and the distributional derivatives $\nabla^\phi \phi$ are represented by continuous functions. Moreover*

$$|\partial E_\phi|_{\mathbb{H}}(\omega \cdot \mathbb{R}) = c_n\, \mathcal{S}_\infty^{Q-1}(\Phi(\omega)) = \int_\omega \sqrt{1 + |\nabla^\phi \phi|^2}\, d\mathcal{L}^{2n}. \tag{2.10}$$

The area-type formula (2.10) has been extended to the more general class of intrinsic Sobolev graphs.

Definition 2.8. A function $\phi \in L^2(\omega)$ belongs to the class $W^{1,1}_{\mathbb{W}}(\omega)$ if there exist a sequence $(\phi_j)_j \subset \mathbf{C}^1(\omega)$ and a vector valued map $w \in L^1(\omega; \mathbb{R}^{2n-1})$ such that, as $j \to +\infty$,

$$\phi_j \to \phi, \quad \phi_j^2 \to \phi^2 \quad \text{and} \quad \nabla^{\phi_j}\phi_j \to w \quad \text{in } L^1(\omega). \tag{2.11}$$

We say that $\phi \in L^2_{loc}(\omega)$ belongs to the class $W^{1,1}_{\mathbb{W},loc}(\omega)$ if there exist $(\phi_j)_j \subset \mathbf{C}^1(\omega)$ and $w \in L^1_{loc}(\omega; \mathbb{R}^{2n-1})$ such that all the convergences in (2.11) hold in $L^1_{loc}(\omega)$.

For a function $\phi \in W^{1,1}_{\mathbb{W},loc}(\omega)$, the distribution $\nabla^\phi \phi$ is represented by a vector valued map $w \in L^1_{loc}(\omega, \mathbb{R}^{2n-1})$ and namely by the function in (2.11). It was proved in [51] that

$$|\partial E_\phi|_{\mathbb{H}}(\omega \cdot \mathbb{R}) = \int_\omega \sqrt{1 + |\nabla^\phi \phi|^2} d\mathcal{L}^{2n}\,.$$

for any $\phi \in W^{1,1}_{\mathbb{W}}(\omega)$.

Remark 2.9. As proved in Remark 4.2, the classes $\mathbf{C}^1_{\mathbb{W}}$ and $W^{1,1}_{\mathbb{W}}$ are not vector spaces. By definition, the inclusion $W^{1,1}_{\mathbb{W}} \subset L^2$ holds as well as the inclusions of the corresponding local classes. We also have

$$\mathbf{C}^1_{\mathbb{W}} \subset W^{1,1}_{\mathbb{W},loc}, \quad Lip \subset W^{1,1}_{\mathbb{W}}\,,$$

see [51, Remark 3.2 and Proposition 3.6]. An example of a function in $\mathbf{C}^1_{\mathbb{W}} \setminus W^{1,1}_{loc}$ is given in [43].

It is well-known that a set $E \subset \mathbb{R}^{2n+1}$ with locally finite Euclidean perimeter has also locally finite $\mathbb{H}$-perimeter (see for instance [31, Remark 2.13]). For such a set one can represent its $\mathbb{H}$-perimeter measure with respect to the Hausdorff measures $\mathcal{H}^{2n}$ and $\mathcal{S}^{Q-1}_\infty$. This representation is already well-known when ∂E is regular (see [11] and [31]). We denote by $\partial^* E$ and $\boldsymbol{n}_E(P)$, respectively, the classical reduced boundary of E and the generalized Euclidean inward normal to E at $P \in \partial^* E$ (see e.g. [39]).

Proposition 2.10. *Let E be a set with locally finite Euclidean perimeter. Then E has locally finite $\mathbb{H}$-perimeter and*

$$|\partial E|_{\mathbb{H}} = |\boldsymbol{n}^{\mathbb{H}}_E| \mathcal{H}^{2n} \llcorner \partial^* E = c_n\, \mathcal{S}^{Q-1}_\infty \llcorner \partial^* E\,, \tag{2.12}$$

where $\boldsymbol{n}^{\mathbb{H}}_E := (\langle X_1, \boldsymbol{n}_E\rangle, \dots, \langle X_{2n}, \boldsymbol{n}_E\rangle) \in \mathbb{R}^{2n}$.

Proof. It is well-known that

$$X_j \chi_E = \langle X_j, \boldsymbol{n}_E\rangle |\partial E| = \langle X_j, \boldsymbol{n}_E\rangle \mathcal{H}^{2n} \llcorner \partial^* E$$

holds in the sense of distributions for any $j = 1, \dots, 2n$, $|\partial E|$ being the Euclidean perimeter of E. The first equality in (2.12) immediately follows. Moreover

$$|\boldsymbol{n}^{\mathbb{H}}_E| \mathcal{H}^{2n} \llcorner \partial^* E = |\partial E|_{\mathbb{H}} = c_n \mathcal{S}^{Q-1}_\infty \llcorner \partial^*_{\mathbb{H}} E \tag{2.13}$$

and thus the second equality in (2.12) will follow if we show that

$$\mathcal{S}^{Q-1}_\infty (\partial^*_{\mathbb{H}} E \Delta \partial^* E) = 0\,. \tag{2.14}$$

Notice that, by (2.13), we have

$$\mathcal{S}^{Q-1}_\infty (\partial^*_{\mathbb{H}} E \setminus \partial^* E) = 0$$

and (2.14) follows provided we show that $\mathcal{S}^{Q-1}_\infty (\partial^* E \setminus \partial^*_{\mathbb{H}} E) = 0$.
To this aim, notice that from (2.13) we obtain

$$\boldsymbol{n}^{\mathbb{H}}_E = 0 \qquad \mathcal{H}^{2n}\text{-a.e. on } \partial^* E \setminus \partial^*_{\mathbb{H}} E\,. \tag{2.15}$$

Since $\partial^* E$ is locally $2n$-rectifiable in the Euclidean sense, there exists a family $(S_j)_{j\in\mathbb{N}}$ of (Euclidean) $\mathbf{C}^1$ surfaces in $\mathbb{H}^n$ such that $\mathcal{H}^{2n}(\partial^* E \setminus \cup_{j=0}^\infty S_j) = 0$ (whence also $\mathcal{S}^{Q-1}_\infty(\partial^* E \setminus \cup_{j=0}^\infty S_j) = 0$ because of (2.1)) and

$$\boldsymbol{n}_E = \boldsymbol{n}_{S_j} \qquad \mathcal{H}^{2n}\text{-a.e. on } \partial^* E \cap S_j\,, \tag{2.16}$$

$\boldsymbol{n}_{S_j}$ being the Euclidean unit normal to S_j. The well-known result by Z. Balogh [5] ensures that for any j

$$\mathcal{S}^{Q-1}_\infty \big(\{P \in S_j : \langle \boldsymbol{n}_{S_j}(P), X_1(P)\rangle = \dots = \langle \boldsymbol{n}_{S_j}(P), X_{2n}(P)\rangle = 0\}\big) = 0\,.$$

Taking into account the fact that (2.16) holds also $\mathcal{S}^{Q-1}_\infty$-a.e. on $\partial^* E \cap S_j$ (recall (2.1)), we deduce that $\mathcal{S}^{Q-1}_\infty\big(\{P \in \partial^* E \cap S_j : \boldsymbol{n}^{\mathbb{H}}_E(P) = 0\}\big) = 0$ for any j, i.e.,

$\mathcal{S}^{Q-1}_\infty\big(\{P\in\partial^*E : \boldsymbol{n}^{\mathbb{H}}_E(P)=0\}\big)=0$. The desired equality $\mathcal{S}^{Q-1}_\infty(\partial^*E\setminus\partial^*_{\mathbb{H}}E)=0$ follows from the fact that (2.15) holds also $\mathcal{S}^{Q-1}_\infty$-a.e. on $\partial^*E\setminus\partial^*_{\mathbb{H}}E$. □

Remark 2.11. An immediate consequence of Proposition 2.10 is the negligibility of the characteristic points of ∂^*E

$$\mathcal{S}^{Q-1}_\infty\big(\{P\in\partial^*E : \boldsymbol{n}^{\mathbb{H}}_E(P)=0\}\big)=0\,. \tag{2.17}$$

The following relationships hold between $\mathcal{S}^{Q-1}_\infty$ and $\mathcal{H}^{2n}$.

Lemma 2.12. *Let $\mathcal{U}\subset\Pi\equiv\mathbb{R}^{2n}$ and $\omega\subset\mathbb{W}\equiv\mathbb{R}^{2n}$ be open sets.*

(i) *If $\mathcal{U}$ is bounded, there exists a constant $C=C(\mathcal{U})>0$ such that*

$$\mathcal{S}^{Q-1}_\infty\llcorner(\mathcal{U}\times\mathbb{R})\leqslant C\mathcal{H}^{2n}\llcorner(\mathcal{U}\times\mathbb{R}).$$

(ii) *For each $s\in\mathbb{R}$ one has*

$$c_n\,\mathcal{S}^{Q-1}_\infty\llcorner(\omega\cdot s)=\mathcal{H}^{2n}\llcorner(\omega\cdot s).$$

Proof. (i) It has been proved in [6] that for any $r>0$ there exists $c=c(r,n)>0$ such that

$$\mathcal{S}^{Q-1}_\infty\llcorner U(0,r)\leqslant c\mathcal{H}^{2n}\llcorner U(0,r).$$

In particular, there exists $C=C(\mathcal{U})>0$ such that

$$\mathcal{S}^{Q-1}_\infty\llcorner(\mathcal{U}\times[-1,1])\leqslant C\mathcal{H}^{2n}\llcorner(\mathcal{U}\times[-1,1]).$$

Since vertical translations are isometries in $\mathbb{H}^n$, we have also

$$\mathcal{S}^{Q-1}_\infty\llcorner(\mathcal{U}\times[h-1,h+1])\leqslant C\mathcal{H}^{2n}\llcorner(\mathcal{U}\times[h-1,h+1])$$

for any $h\in\mathbb{R}$. Our claim easily follows.

(ii) Let $\phi:\omega\to\mathbb{R}$ be the constant function taking value s; by Proposition 2.10

$$c_n\mathcal{S}^{Q-1}_\infty\llcorner(\omega\cdot s)=|\partial E_\phi|_{\mathbb{H}}\llcorner(\omega\cdot\mathbb{R})=|\boldsymbol{n}^{\mathbb{H}}_{E_\phi}|\mathcal{H}^{2n}\llcorner(\omega\cdot s)\,.$$

The statement easily follows noticing that $\boldsymbol{n}_{E_\phi}=(1,0,\dots,0)$, i.e., $|\boldsymbol{n}^{\mathbb{H}}_{E_\phi}|=1$. □

The following localization estimates for the $\mathbb{H}$-perimeter measure have been proved in [1, Lemma 3.5] and [32, Lemma 2.21].

Lemma 2.13. *Let E be a set with locally finite $\mathbb{H}$-perimeter; for given $P\in\mathbb{H}^n$ and $r>0$ set $m_E(P,r):=|E\cap U_c(P,r)|$. Then for a.e. $r>0$*

$$|\partial(E\setminus U_c(P,r))|_{\mathbb{H}}(\partial U_c(P,r))\leqslant m'_E(P,r) \tag{2.18}$$

and

$$|\partial(E\cap U_c(P,r))|_{\mathbb{H}}(\mathbb{H}^n)\leqslant|\partial E|_{\mathbb{H}}(U_c(P,r))+m'_E(P,r)\,. \tag{2.19}$$

Let us recall once more our assumption that E coincides with the set $\tilde{E}$ in (2.5). In particular $|E\cap U(P,r)|>0$ for all $P\in E$ and $r>0$.

Proposition 2.14. *Let $E\subset\mathbb{H}^n$ be $\mathbb{H}$-perimeter minimizing in an open set $\Omega\subset\mathbb{H}^n$. Then there exists a constant $C=C(n)>0$ such that*

$$|E\cap U(P,r)|\geqslant C\,r^Q\qquad \textit{for any } P\in E\cap\Omega, 0<r<d_\infty(P,\partial\Omega)\,. \tag{2.20}$$

Proof. By the equivalence of d_∞ and d_c, it will be sufficient to prove that there exists $C = C(n) > 0$ such that

$$|E \cap U_c(P,r)| \geqslant C\, r^Q \qquad \text{for any } 0 < r < d_c(P, \partial\Omega)\,. \tag{2.21}$$

Up to a left translation we can suppose that P coincides with the identity 0. Since E is $\mathbb{H}$-perimeter minimizing, we have

$$|\partial E|_{\mathbb{H}}(\Omega) \leqslant |\partial(E \setminus U_{c,r})|_{\mathbb{H}}(\Omega)$$

and so, by subtracting $|\partial E|_{\mathbb{H}}(\Omega \setminus \overline{U_{c,r}}) = |\partial(E \setminus U_{c,r})|(\Omega \setminus \overline{U_{c,r}})$,

$$|\partial E|_{\mathbb{H}}(\overline{U_{c,r}}) \leqslant |\partial(E \setminus U_{c,r})|_{\mathbb{H}}(\partial U_{c,r})\,.$$

For a.e. $r > 0$ one has $|\partial E|_{\mathbb{H}}(\partial U_{c,r}) = 0$ because $|\partial E|_{\mathbb{H}}$ is a Radon measure. By (2.18) we achieve for such r

$$|\partial E|_{\mathbb{H}}(U_{c,r}) \leqslant m_E'(r)\,, \tag{2.22}$$

where $m_E(r) := |E \cap U_{c,r}|$. Taking into account (2.22), (2.19) and the isoperimetric inequality (2.2), we obtain

$$m_E(r)^{\frac{Q-1}{Q}} = |E \cap U_{c,r}|^{\frac{Q-1}{Q}} \leqslant c_I |\partial(E \cap U_{c,r})|_{\mathbb{H}}(\Omega) \leqslant 2c_I\, m_E'(r)\,.$$

Since $m_E(r) > 0$ for any $r > 0$ we have

$$m_E(r)^{\frac{1-Q}{Q}} m_E'(r) = \left(m_E^{1/Q}\right)'(r) \geqslant \frac{1}{2c_I} \qquad \text{for a.e. } r > 0$$

and (2.21) follows by integration because m_E is locally Lipschitz continuous

$$|m_E(r_1) - m_E(r_2)| \leqslant |U_{c,r_1} \setminus U_{c,r_2}| = c|r_1^Q - r_2^Q|\,.$$

□

We will need in the sequel the well-known notion of convolution between functions in the Heisenberg group (see [28]): for given $g \in L^1(\mathbb{H}^n), f \in L^p(\mathbb{H}^n)$ we set $g \star f \in L^p(\mathbb{H}^n)$ as the function defined by

$$(g \star f)(P) := \int_{\mathbb{H}^n} g(P \cdot Q^{-1}) f(Q)\, dQ = \int_{\mathbb{H}^n} g(Q) f(Q^{-1} \cdot P)\, dQ\,. \tag{2.23}$$

The symbol $*$ will be instead used to denote the classical Euclidean convolution $g * f$ between f and g. Recall that in general $f \star g \neq g \star f$; moreover

$$\nabla_{\mathbb{H}}(g \star f) = g \star (\nabla_{\mathbb{H}} f) \neq (\nabla_{\mathbb{H}} g) \star f$$

whenever $f, g \in W^{1,1}(\mathbb{H}^n)$. We will often consider a fixed mollification kernel $\varrho \in \mathbf{C}_c^\infty(U_1)$ such that

$$\int_{\mathbb{H}^n} \varrho\, d\mathcal{L}^{2n+1} = 1, \quad \varrho \geqslant 0 \quad \text{and} \quad \varrho(P) = \varrho(P^{-1}) \tag{2.24}$$

and write $\varrho_\alpha(P) := \alpha^{-Q} \varrho(\delta_{1/\alpha}(P))$ for any $\alpha > 0$. For any $f \in L^p(\mathbb{H}^n)$ the mollified functions $\varrho_\alpha \star f \in \mathbf{C}^\infty(\mathbb{H}^n)$ converge to f in $L^p(\mathbb{H}^n)$ as $\alpha \to 0$. Notice that the convolution $\varrho_\alpha \star f$ is well-defined and smooth also for $f \in L^1_{loc}(\mathbb{H}^n)$. Moreover

$$\operatorname{spt}(\varrho_\alpha \star f) \subset U_\alpha \cdot \operatorname{spt} f \tag{2.25}$$

and

$$
\begin{aligned}
\int_{\mathbb{H}^n} (\varrho_\alpha \star f) g \, d\mathcal{L}^{2n+1} &= \int_{\mathbb{H}^n} \left(\int_{\mathbb{H}^n} \varrho_\alpha(P \cdot Q^{-1}) f(Q) \, dQ \right) g(P) \, dP \\
&= \int_{\mathbb{H}^n} \left(\int_{\mathbb{H}^n} \varrho_\alpha(Q \cdot P^{-1}) g(P) \, dP \right) f(Q) \, dQ \\
&= \int_{\mathbb{H}^n} f(\varrho_\alpha \star g) \, d\mathcal{L}^{2n+1}
\end{aligned} \tag{2.26}
$$

for any $f \in L^p(\mathbb{H}^n)$ and $g \in L^{p'}(\mathbb{H}^n)$ with $\frac{1}{p} + \frac{1}{p'} = 1$.

Finally, we recall the following calibration result proved in [7, Theorem 2.1] in the setting of CC spaces.

Theorem 2.15. *Let $\Omega \subset \mathbb{H}^n$ be an open set and E a set with locally finite $\mathbb{H}$-perimeter in Ω. Suppose there are two sequences $(\Omega_h)_h$ and $(\nu_h)_h$ such that*

(i) *Ω_h is open, $\Omega_h \Subset \Omega_{h+1}$, $\cup_{h=1}^{\infty} \Omega_h = \Omega$;*
(ii) *$\nu_h \in \mathbf{C}^1(\Omega; \mathbb{R}^{2n})$, $|\nu_h(x)| \leqslant 1$ for all $x \in \Omega$, $h \in \mathbb{N}$;*
(iii) *$\mathrm{div}_{\mathbb{H}} \nu_h = 0$ in Ω_h for each h;*
(iv) *$\nu_h(x) \to \nu_E(x)$ $|\partial E|_{\mathbb{H}}$-a.e. $x \in \Omega$.*

Then E is a minimizer for the $\mathbb{H}$-perimeter in Ω.

3. Existence and local boundedness of minimal t-graphs

3.1. **Bounded variation for t-graphs.** When u is a function in the Sobolev space $W^{1,1}(\mathcal{U})$ it is possible to write

$$
\mathscr{A}_t(u) = \int_{\mathcal{U}} \mathscr{L}(z, \nabla u(z)) \, d\mathcal{L}^{2n}(z)
$$

where $\mathscr{L} : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to [0, +\infty)$ is defined by $\mathscr{L}((x,y), \xi) = |\xi + \boldsymbol{X}^*(x,y)|$. The functional $\mathscr{A}_t$ is convex since $\mathscr{L}(z, \cdot) : \mathbb{R}^{2n} \to \mathbb{R}$ is convex; $\mathscr{L}(z, \cdot) : \mathbb{R}^{2n} \to \mathbb{R}$ is not strictly convex.

When $u \in \mathbf{C}^2(\mathcal{U})$ is a local minimizer of $\mathscr{A}_t$, a first variation of the functional yields the minimal surface equation for t-graphs

$$
\mathrm{div}(N(u)) = 0 \quad \text{in } \mathcal{U}_{\mathrm{nc}}(u) \, . \tag{3.1}
$$

We have defined

$$
N(u) := \frac{\nabla u + \boldsymbol{X}^*}{|\nabla u + \boldsymbol{X}^*|} \qquad \text{on } \mathcal{U}_{\mathrm{nc}}(u) \tag{3.2}
$$

where $\mathcal{U}_{\mathrm{nc}}(u) := \mathcal{U} \setminus \mathrm{Char}(u)$ and $\mathrm{Char}(u)$ is the set of characteristic points of u defined in (1.6). The solutions of (3.1) are called *H-minimal.* One is not allowed to deduce that (3.1) is satisfied on $\mathcal{U}$ in the sense of distributions even when $\mathcal{L}^{2n}(\mathrm{Char}(u)) = 0$; moreover, the size of $\mathrm{Char}(u)$ may be large even for $u \in W^{1,1}(\mathcal{U})$ (see [5]). These problems have been studied with details in [17] and a suitable minimal surface equation was obtained.

We are going to study the relaxed functional $\overline{\mathscr{A}_t} : L^1(\mathcal{U}) \to [0, +\infty]$ of $\mathscr{A}_t$ with respect to the L^1-topology and to give a representation formula on its domain. We therefore introduce

$$\overline{\mathscr{A}_t}(u) := \inf\left\{\liminf_{k\to\infty}\int_{\mathcal{U}} |\nabla u_k + \boldsymbol{X}^*|\, d\mathcal{L}^{2n} : u_k \in W^{1,1}(\mathcal{U}), u_k \to u \text{ in } L^1(\mathcal{U})\right\}.$$

In the sequel we will consider also the following $[0,+\infty]$-valued functionals on $L^1(\mathcal{U})$

$$I_t(u) := \inf\left\{\liminf_{k\to\infty}\int_{\mathcal{U}} |\nabla u_k + \boldsymbol{X}^*|\, d\mathcal{L}^{2n} : u_k \in \mathbf{C}^1(\mathcal{U}), u_k \to u \text{ in } L^1(\mathcal{U})\right\}$$

$$S_t(u) := \sup\left\{\int_{\mathcal{U}} \big(- u\,\mathrm{div} g + \langle \boldsymbol{X}^*, g\rangle\big)\, d\mathcal{L}^{2n} : g \in \mathbf{C}^1_c(\mathcal{U};\mathbb{R}^{2n}), |g| \leqslant 1\right\}.$$

Routine arguments ensure the L^1-lower semicontinuity of $\overline{\mathscr{A}_t}$, I_t and S_t and that they coincide on $\mathbf{C}^1(\mathcal{U})$ or $W^{1,1}(\mathcal{U})$. Moreover, if $u \in \mathbf{C}^1(\mathcal{U})$ or $W^{1,1}(\mathcal{U})$

$$|\partial E^t_u|_{\mathbb{H}}(\mathcal{U}\times\mathbb{R}) = \int_{\mathcal{U}} |\nabla u + \boldsymbol{X}^*|\, d\mathcal{L}^{2n} = \overline{\mathscr{A}_t}(u) = I_t(u) = S_t(u),$$

the first equality following from [11].

Remark 3.1. It follows from the definition that S_t is the total variation of $Du + \boldsymbol{X}^*\mathcal{L}^{2n}$, where Du is the gradient of u in the sense of distributions: it is sufficient to apply Riesz' Theorems (see e.g. [2, Teorema 1.54]).

The following is one of the crucial results of this Section.

Theorem 3.2. *Let $\mathcal{U} \subset \mathbb{R}^{2n}$ be a bounded open set. The equalities*

$$|\partial E^t_u|_{\mathbb{H}}(\mathcal{U}\times\mathbb{R}) = \overline{\mathscr{A}_t}(u) = I_t(u) = S_t(u). \tag{3.3}$$

hold for any $u \in L^1(\mathcal{U})$.

Proof. For the reader's convenience, we divide the proof into several steps.

Step 1: $\overline{\mathscr{A}_t}(u) \leqslant I_t(u)$. We may suppose that $I_t(u) < +\infty$. By definition, there exists a sequence $(u_k)_k \subset \mathbf{C}^1(\mathcal{U}) \cap L^1(\mathcal{U})$ such that

$$\lim_{k\to\infty} u_k = u \text{ in } L^1(\mathcal{U}) \qquad \text{and} \qquad \lim_{k\to\infty}\int_{\mathcal{U}} |\nabla u_k + \boldsymbol{X}^*|\, d\mathcal{L}^{2n} = I_t(u)\,.$$

Since

$$\int_{\mathcal{U}} |\nabla u_k|\, d\mathcal{L}^{2n} \leqslant \int_{\mathcal{U}} |\nabla u_k + \boldsymbol{X}^*|\, d\mathcal{L}^{2n} + \int_{\mathcal{U}} |\boldsymbol{X}^*|\, d\mathcal{L}^{2n}\,,$$

the sequence $(u_k)_k$ is definitely in $W^{1,1}(\mathcal{U})$ because $\mathcal{U}$ is bounded.

Step 2: $\overline{\mathscr{A}_t}(u) \geqslant I_t(u)$. We can suppose $\overline{\mathscr{A}_t}(u) < +\infty$. By definition there exists a sequence $(u_k)_k \subset W^{1,1}(\mathcal{U})$ such that $u_k \to u$ in $L^1(\mathcal{U})$ and $\int_{\mathcal{U}} |\nabla u_k + \boldsymbol{X}^*|\, d\mathcal{L}^{2n} \to \overline{\mathscr{A}_t}(u)$. By the density of smooth functions in Sobolev spaces, for each given k there exists a function $v_k \in \mathbf{C}^\infty(\mathcal{U}) \cap W^{1,1}(\mathcal{U})$ such that $\|u_k - v_k\|_{W^{1,1}(\mathcal{U})} < 1/k$. On the other hand

$$\begin{aligned}
\int_{\mathcal{U}} |\nabla v_k + \boldsymbol{X}^*|\, d\mathcal{L}^{2n} &\leqslant \int_{\mathcal{U}} |\nabla u_k + \boldsymbol{X}^*|\, d\mathcal{L}^{2n} + \int_{\mathcal{U}} |\nabla (v_k - u_k)|\, d\mathcal{L}^{2n} \\
&\leqslant \int_{\mathcal{U}} |\nabla u_k + \boldsymbol{X}^*|\, d\mathcal{L}^{2n} + \frac{1}{k} \quad \forall k
\end{aligned}$$

which allows to conclude because $v_k \to u$ in L^1.

Step 3: $S_t(u) \leqslant I_t(u)$. We may assume $I_t(u) < \infty$. By definition, for any $\epsilon > 0$ there exists a sequence $(u_k)_k$ of $\mathbf{C}^1$ functions such that

$$u_k \to u \text{ in } L^1(\mathcal{U}) \quad \text{and} \quad \liminf_{k\to\infty} \int_{\mathcal{U}} |\nabla u_k + \boldsymbol{X}^*| \leqslant I_t(u) + \epsilon.$$

Let $g \in \mathbf{C}^1_c(\mathcal{U}, \mathbb{R}^{2n})$, $|g| \leqslant 1$ be fixed; then

$$\begin{aligned}
\int_{\mathcal{U}} [-u\mathrm{div} g + \langle \boldsymbol{X}^*, g\rangle]\, d\mathcal{L}^{2n} &= \lim_{k\to\infty} \int_{\mathcal{U}} [-u_k \mathrm{div} g + \langle \boldsymbol{X}^*, g\rangle]\, d\mathcal{L}^{2n} \\
&= \lim_{k\to\infty} \int_{\mathcal{U}} \langle \nabla u_k + \boldsymbol{X}^*, g\rangle\, d\mathcal{L}^{2n} \\
&\leqslant \liminf_{k\to\infty} \int_{\mathcal{U}} |\nabla u_k + \boldsymbol{X}^*|\, d\mathcal{L}^{2n} \\
&\leqslant I_t(u) + \epsilon\,.
\end{aligned}$$

By taking the supremum on g we immediately conclude.

Step 4: $I_t(u) \leqslant S_t(u)$. We closely follow a classical argument by Anzellotti-Giaquinta (see [39, Theorem 1.17]). As before, we can suppose $S_t(u) < \infty$. Fix $\epsilon > 0$ and consider a sequence of open sets $(\mathcal{U}_i)_i$ with $\mathcal{U}_i \Subset \mathcal{U}_{i+1}$ and $\mathcal{U}_i \uparrow \mathcal{U}$. We additionally require that

$$\sup\left\{ \int_{\mathcal{U}\setminus\overline{\mathcal{U}_1}} [-u\mathrm{div} g + \langle \boldsymbol{X}^*, g\rangle]\, d\mathcal{L}^{2n} : g \in \mathbf{C}^1_c(\mathcal{U}\setminus\overline{\mathcal{U}_1}, \mathbb{R}^{2n}), |g| \leqslant 1 \right\} < \epsilon\,; \tag{3.4}$$

this is possible thanks to the boundedness of $S_t(u)$, i.e., the fact that $Du + \boldsymbol{X}^*\mathcal{L}^{2n}$ is a Radon vector valued measure on $\mathcal{U}$ (see Remark 3.1). Set $\mathcal{A}_1 := \mathcal{U}_1$ and $\mathcal{A}_i := \mathcal{U}_{i+1} \setminus \overline{\mathcal{U}_{i-1}}$ for $i \geqslant 2$, and consider a partition of the unity in $\mathcal{U}$ subordinate to the covering $\mathcal{A}_i$, i.e., a family of functions $(\psi_i)_i$ such that

$$\psi_i \in \mathbf{C}^\infty_c(\mathcal{A}_i), \quad 0 \leqslant \psi_i \leqslant 1 \quad \text{and} \quad \sum_{i=1}^\infty \psi_i = 1\,.$$

Let ϱ be a standard smooth mollifier with support in $B(0,1) \subset \Pi = \mathbb{R}^{2n}$, and define $\varrho_\alpha(x) := \alpha^{-2n}\varrho(\alpha^{-2n}x)$ for $\alpha > 0$. It is possible to fix numbers $\alpha_i > 0$ such that $\mathrm{spt}\big(\varrho_{\alpha_i} * (u\psi_i)\big) \subset \mathcal{A}_i$ and

$$\int_{\mathcal{U}} |\varrho_{\alpha_i} * (u\psi_i) - u\psi_i|\, d\mathcal{L}^{2n} < 2^{-i}\epsilon \tag{3.5}$$

$$\int_{\mathcal{U}} |\varrho_{\alpha_i} * (u\nabla\psi_i) - u\nabla\psi_i|\, d\mathcal{L}^{2n} < 2^{-i}\epsilon \tag{3.6}$$

$$\int_{\mathcal{U}} |\varrho_{\alpha_i} * (\psi_i\boldsymbol{X}^*) - \psi_i\boldsymbol{X}^*|\, d\mathcal{L}^{2n} < 2^{-i}\epsilon\,. \tag{3.7}$$

Finally, we set $u_\epsilon := \sum_{i=1}^\infty \varrho_{\alpha_i} * (u\psi_i)$; condition (3.5) ensures that $u_\epsilon \to u$ in $L^1(\mathcal{U})$ as $\epsilon \to 0$.

Fix $g \in \mathbf{C}^1_c(\mathcal{U}, \mathbb{R}^{2n})$, $|g| \leqslant 1$; it is a matter of computations that

$$\int_{\mathcal{U}} -u_\epsilon \mathrm{div}\, g \, d\mathcal{L}^{2n} = \int_{\mathcal{U}} -u \, \mathrm{div}(\psi_1(\varrho_{\alpha_1} * g)) \, d\mathcal{L}^{2n} + \sum_{i=2}^{\infty} \int_{\mathcal{U}} -u \, \mathrm{div}(\psi_i(\varrho_{\alpha_i} * g)) \, d\mathcal{L}^{2n}$$
$$+ \sum_{i=1}^{\infty} \int_{\mathcal{U}} \langle g, \varrho_{\alpha_i} * (u \nabla \psi_i) - u \nabla \psi_i \rangle \, d\mathcal{L}^{2n}.$$

Thus

$$\begin{aligned}
&\int_{\mathcal{U}} [-u_\epsilon \mathrm{div} g + \langle \boldsymbol{X}^*, g \rangle] \, d\mathcal{L}^{2n} \\
&= \int_{\mathcal{U}} [-u \, \mathrm{div}(\psi_1(\varrho_{\alpha_1} * g)) + \langle \boldsymbol{X}^*, \psi_1(\varrho_{\alpha_1} * g) \rangle] \, d\mathcal{L}^{2n} \\
&\quad + \sum_{i=2}^{\infty} \int_{\mathcal{U}} [-u \, \mathrm{div}(\psi_i(\varrho_{\alpha_i} * g)) + \langle \boldsymbol{X}^*, \psi_i(\varrho_{\alpha_i} * g) \rangle] \, d\mathcal{L}^{2n} \\
&\quad - \sum_{i=1}^{\infty} \int_{\mathcal{U}} \langle g, \varrho_{\alpha_i} * (u \nabla \psi_i) - u \nabla \psi_i \rangle \, d\mathcal{L}^{2n} + \sum_{i=1}^{\infty} \int_{\mathcal{U}} \langle \boldsymbol{X}^*, \psi_i (g - \varrho_{\alpha_i} * g) \rangle \, d\mathcal{L}^{2n} \\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Notice that $|\psi_i(\varrho_{\alpha_i} * g)| \leqslant 1$, whence $I_1 \leqslant S_t(u)$. Moreover

$$I_2 = \lim_{N \to +\infty} \int_{\mathcal{U}} [-u \, \mathrm{div}\big(\textstyle\sum_{i=2}^N \psi_i(\varrho_{\alpha_i} * g) \big) + \big\langle \boldsymbol{X}^*, \textstyle\sum_{i=2}^N \psi_i(\varrho_{\alpha_i} * g) \big\rangle] \, d\mathcal{L}^{2n} \leqslant 2\epsilon;$$

this follows from (3.4) and the fact that $|\sum_{i=2}^N \psi_i(\varrho_{\alpha_i} * g)| \leqslant 2$, which in turn is justified by $|\varrho_{\alpha_i} * g| \leqslant 1$ and $\mathcal{A}_i \cap \mathcal{A}_j = \emptyset$ for $|i - j| \geqslant 2$.
Estimate (3.6) yields $I_3 \leqslant \epsilon$. Finally, using (3.7) we obtain

$$I_4 = \sum_{i=1}^{\infty} \int_{\mathcal{U}} \langle \psi_i \boldsymbol{X}^*, g - \varrho_{\alpha_i} * g \rangle \, d\mathcal{L}^{2n} = \sum_{i=1}^{\infty} \int_{\mathcal{U}} \langle (\psi_i \boldsymbol{X}^*) - \varrho_{\alpha_i} * (\psi_i \boldsymbol{X}^*), g \rangle \, d\mathcal{L}^{2n} \leqslant \epsilon \, .$$

On taking the supremum among $g \in \mathbf{C}^1_c(\mathcal{U}, \mathbb{R}^{2n})$ we obtain

$$\int_{\mathcal{U}} |\nabla u_\epsilon + \boldsymbol{X}^*| \, d\mathcal{L}^{2n} \leqslant S_t(u) + 4\epsilon$$

and the desired inequality follows.

Step 5: $|\partial E^t_u|_{\mathbb{H}}(\mathcal{U} \times \mathbb{R}) \leqslant I_t(u)$. Fix a sequence $(u_k)_k \subset \mathbf{C}^1(\mathcal{U})$ with $u_k \to u$ in L^1 and

$$I_t(u) = \liminf_{k \to \infty} \int_{\mathcal{U}} |\nabla u_k + \boldsymbol{X}^*| \, d\mathcal{L}^{2n}.$$

By (2.7) we have $\chi_{E^t_{u_k}} \to \chi_{E^t_u}$ in $L^1(\mathcal{U} \times \mathbb{R})$ and thus

$$|\partial E^t_u|_{\mathbb{H}}(\mathcal{U} \times \mathbb{R}) \leqslant \liminf_{k \to \infty} |\partial E^t_{u_k}|_{\mathbb{H}}(\mathcal{U} \times \mathbb{R}) = \liminf_{k \to \infty} \int_{\mathcal{U}} |\nabla u_k + \boldsymbol{X}^*| \, d\mathcal{L}^{2n} = I_t(u)$$

by the semicontinuity of the $\mathbb{H}$-perimeter.

Step 6: $S_t(u) \leqslant |\partial E_u^t|_{\mathbb{H}}(\mathcal{U}\times\mathbb{R})$. It is enough to prove that, for any fixed $g \in \mathbf{C}_c^1(\mathcal{U},\mathbb{R}^{2n})$, $|g|\leqslant 1$, there holds

$$|\partial E_u^t|_{\mathbb{H}}(\mathcal{U}\times\mathbb{R}) \geqslant \int_{\mathcal{U}} [-u\mathrm{div} g + \langle \boldsymbol{X}^*, g\rangle]\, d\mathcal{L}^{2n}. \tag{3.8}$$

For fixed $M>0$ let $h_M \in \mathbf{C}_c^\infty(\mathbb{R})$ be such that

$$h_M \equiv 1 \text{ on } [-M,M],\ \mathrm{spt}\, h_M \subset [-M-1, M+1],\ 0\leqslant h_M \leqslant 1,\ |h_M'|\leqslant 2. \tag{3.9}$$

We may assume that $J := \int_{-M-1}^{-M} h_M(t)dt = \int_M^{M+1} h_M(t)dt$ and that J does not depend on M: it is sufficient to fix a suitable "profile" that h must assume on $[-M-1,-M]$ and $[M,M+1]$. We explicitly compute the following integrals: if $z:=(x,y)$, then

$$\begin{aligned}
&\int_{-\infty}^{u(z)} h_M(t)dt = J + u(z) + M && \text{if } |u(z)|\leqslant M,\\
&\int_{-\infty}^{u(z)} h_M(t)dt = J + 2M + \int_M^{u(z)} h_M(t)dt && \text{if } u(z) > M,\\
&\int_{-\infty}^{u(z)} h_M(t)dt = \int_{-M-1}^{u(z)} h_M(t)dt && \text{if } u(z) < -M,\\
&\int_{-\infty}^{u(z)} h_M'(t)dt = 1 && \text{if } |u(z)|\leqslant M,\\
&\int_{-\infty}^{u(z)} h_M'(t)dt = h_M(u(z)) && \text{if } |u(z)| > M.
\end{aligned}$$

Define $\varphi_M \in \mathbf{C}_c^1(\mathcal{U}\times\mathbb{R},\mathbb{R}^{2n})$ by $\varphi_M(x,y,t) := -h_M(t)g(z)$; it follows that

$$\begin{aligned}
|\partial E_u^t|_{\mathbb{H}}(\mathcal{U}\times\mathbb{R}) \ &\geqslant \ \int_{E_u^t} \mathrm{div}_{\mathbb{H}}\varphi_M \, d\mathcal{L}^{2n+1}\\
= &\int_{\mathcal{U}} \sum_{j=1}^n \int_{-\infty}^{u(z)} \big[-h_M(t)\partial_{x_j} g_j(z) - 2y_j h_M'(t) g_j(z)\\
&\qquad - h_M(t)\partial_{y_j} g_{n+j}(z) + 2x_j h_M'(t) g_{n+j}(z)\big] dt\, dz\\
= &\int_{\mathcal{U}} \big[-(J+u(z)+M)\mathrm{div}\, g(z) + \langle \boldsymbol{X}^*(z), g(z)\rangle\big] dz\\
&+ \int_{\{u>M\}} \big[-(J+2M+\textstyle\int_M^{u(z)} h_M(t)dt - J - u(z) - M)\mathrm{div}\, g(z)\\
&\qquad + \langle \boldsymbol{X}^*(z), g(z)\rangle \big(h_M(u(z))-1\big)\big] dz\\
&+ \int_{\{u<-M\}} \big[-(\textstyle\int_{-M-1}^{u(z)} h_M(t)dt - J - u(z) - M)\mathrm{div}\, g(z)\\
&\qquad + \langle \boldsymbol{X}^*(z), g(z)\rangle \big(h_M(u(z))-1\big)\big] dz\\
=: &R_M + S_M + T_M\, .
\end{aligned}$$

Since g is compactly supported, $R_M = \int_{\mathcal{U}}[-u\,\mathrm{div} g + \langle \boldsymbol{X}^*, g\rangle]\, d\mathcal{L}^{2n}$; inequality (3.8) will follow if we prove that $\lim_{M\to\infty} S_M = \lim_{M\to\infty} T_M = 0$.

Let us rewrite S_M as

$$S_M = \int_{\{u>M\}} \left[-\left(M + \textstyle\int_M^{u(z)} h_M(t)dt - u(z)\right)\operatorname{div} g(z) + \langle \boldsymbol{X}^*(z), g(z)\rangle \left(h_M(u(z)) - 1\right)\right] dz.$$

We point out the implication

$$u(z) > M \implies \left|M + \textstyle\int_M^{u(z)} h_M(t)dt - u(z)\right| \leqslant |u(z) - M| + 1 < |u(z)| + 1.$$

which gives the existence of a positive constant $c = c(\mathcal{U}, g)$ such that

$$|S_M| \leqslant c \int_{\{u>M\}} (|u| + 1)\, d\mathcal{L}^{2n}.$$

Since $u \in L^1(\mathcal{U})$ it follows $\lim_{M\to\infty} S_M = 0$. A similar argument gives $\lim_{M\to\infty} T_M = 0$ and the proof is accomplished. $\square$

From now on, for any $u \in BV_t(\mathcal{U})$ we will use the notation

$$\int_{\mathcal{U}} |Du + \boldsymbol{X}^*|$$

to denote any of the quantities $I_t(u), S_t(u), \overline{\mathscr{A}_t}$ and $|\partial E_u^t|_{\mathbb{H}}(\mathcal{U} \times \mathbb{R})$.

The following result has been obtained along the proof of Theorem 3.2.

Corollary 3.3. *Let $\mathcal{U} \subset \mathbb{R}^{2n}$ be a bounded open set. Let $u \in BV_t(\mathcal{U})$; then there exists a sequence $(u_k)_k \subset \mathbf{C}^\infty(\mathcal{U})$ converging to u in $L^1(\mathcal{U})$ and such that*

$$\int_{\mathcal{U}} |Du + \boldsymbol{X}^*| = \lim_{k\to\infty} \int_{\mathcal{U}} |\nabla u_k + \boldsymbol{X}^*|\, d\mathcal{L}^{2n}$$

Other important consequences of Theorem 3.2 are Theorem 1.2 and the compact embedding of $BV_t(\mathcal{U})$ in $L^1(\mathcal{U})$.

Proof of Theorem 1.2. It will be sufficient to show that $BV_t(\mathcal{U}) = BV(\mathcal{U})$. Recalling that an equivalent definition for the Euclidean variation of a map $u : \mathcal{U} \to \mathbb{R}$ is

$$|Du|(\mathcal{U}) := \sup\left\{ \int_{\mathcal{U}} u \operatorname{div} g\, d\mathcal{L}^{2n} : g \in \mathbf{C}^1_c(\mathcal{U}, \mathbb{R}^{2n}), |g| \leqslant 1 \right\},$$

the result will immediately follow from Theorem 3.2, the definition of the functional S_t and the fact that $\mathcal{U}$ is bounded. $\square$

Proof of Corollary 1.6. Reasoning by contradiction we prove that, if S is an $\mathbb{H}$-regular hypersurface that coincides with the t-graph of a map $u : \mathcal{U} \to \mathbb{R}$ defined on some open bounded set $\mathcal{U}$, then S is (Euclidean) countably $\mathcal{H}^{2n}$-rectifiable.

Let us prove that u is continuous. For any $z \in \mathcal{U}$ there exists a neighbourhood Ω of $P = (z, u(z)) \in S$ and $f \in \mathbf{C}^1_{\mathbb{H}}(\Omega)$ such that $S \cap \Omega = \{f = 0\}$. We may assume that $\Omega = \mathcal{U}' \times (a, b)$ for some $a < u(z) < b$ and some open set $\mathcal{U}' \subset \mathcal{U}$ with $z \in \mathcal{U}'$; in this way we have $S \cap \Omega = \{f = 0\} \cap \Omega = \{(z', u(z')) : z' \in \mathcal{U}'\}$. Possibly replacing f with $-f$, the continuity of f gives

$$f(z,t) > 0\ \forall\, t \in (u(z), b) \quad \text{and} \quad f(z,t) < 0\ \forall\, t \in (a, u(z)).$$

Again by the continuity of f, it follows that for any $\epsilon > 0$ there exists an open set $\mathcal{U}'' \subset \mathcal{U}'$, $z \in \mathcal{U}''$, such that

$$f(z', u(z) + \epsilon) > 0 \quad \text{and} \quad f(z', u(z) - \epsilon) < 0 \qquad \forall\, z' \in \mathcal{U}'',$$

i.e., $u(z)-\epsilon < u(z') < u(z)+\epsilon$ for any $z' \in \mathcal{U}''$. This proves that u is continuous and, in particular, that E_u^t is open. By Theorem 1.2, u is a continuous function belonging to $BV(\mathcal{U})$. Thus, it is enough to prove that, for any $u \in C^0(\mathcal{U}) \cap BV(\mathcal{U})$, its graph $S = \{(z,u(z)) : z \in \mathcal{U}\}$ is countably $\mathcal{H}^{2n}$-rectifiable.

First, assume there exists a sequence $(\mathcal{U}_h)_h$ of bounded open sets in $\mathbb{R}^{2n}$ satisfying the following properties:

$$\mathcal{U}_h \Subset \mathcal{U}\ \forall h \quad \text{and} \quad \mathcal{U} = \cup_{h=1}^{\infty} \mathcal{U}_h \tag{3.10}$$

$$\text{each } \mathcal{U}_h \text{ is finitely, rectilinearly, triangulable according to [26]} \tag{3.11}$$

$$|\partial E_u^t|(\partial \mathcal{U}_h \times \mathbb{R}) = 0 \quad \forall h, \tag{3.12}$$

$|\partial E|$ denoting the Euclidean perimeter of a set $E \subset \mathbb{R}^{2n+1}$. Then, by the first assumption in (3.10), (3.12) and [47, Theorems 1.3 and 1.8], we obtain

$$\mathrm{L}_{2n}(S_h) = |\partial E_u^t|(\mathcal{U}_h \times \mathbb{R}) < \infty\,, \tag{3.13}$$

for each h, where $S_h := \{(z,u(z)) : z \in \mathcal{U}_h\}$ and L_{2n} denotes the $2n$-dimensional Lebesgue area. On the other hand, by (3.11), (3.13) and [26], it follows that, for each h, S_h is countably $\mathcal{H}^{2n}$-rectifiable. Because of the second assumption in (3.10), we also obtain that $S = \cup_{h=1}^{\infty} S_h$. Thus, S is countably $\mathcal{H}^{2n}$-rectifiable.

Finally, we have to prove the existence of a sequence $(\mathcal{U}_h)_h$ satisfying (3.10), (3.11) and (3.12). For each $z \in \mathbb{R}^{2n}$, $r > 0$, let $Q(z,r)$ denote the (open) cube in $\mathbb{R}^{2n}$ centered at z with sides of length $2r$. Such a cube is trivially a finitely, rectilinearly, triangulable set in $\mathbb{R}^{2n}$. For any $z \in \mathcal{U}$ let $r(z) > 0$ be such that $\overline{Q(z,r(z))} \subset \mathcal{U}$. Since $|\partial E_u^t|(\mathcal{U} \times \mathbb{R}) < \infty$, without loss of generality we can choose $r(z)$ so that

$$|\partial E_u^t|(\partial Q(z,r(z)) \times \mathbb{R}) = 0\,.$$

By standard arguments, there exists a sequence of cubes $\mathcal{U}_h := Q(z_h, r(z_h)) \subset \mathcal{U}$ such that (3.10), (3.11) and (3.12) hold. $\square$

Theorem 3.4. *Let $\mathcal{U} \subset \mathbb{R}^{2n}$ be a bounded open set with Lipschitz regular boundary. Then the inclusion $BV_t(\mathcal{U}) \hookrightarrow L^1(\mathcal{U})$ is compact.*

Proof. Let the sequence $(u_j)_j$ be bounded in BV_t. Since

$$|Du_j|(\mathcal{U}) \leqslant \int_{\mathcal{U}} |Du_j + \boldsymbol{X}^*| + \int_{\mathcal{U}} |\boldsymbol{X}^*|\, d\mathcal{L}^{2n}$$

the sequence is bounded in BV too. The result follows from the compact inclusion of BV in L^1. $\square$

Finally, an explicit representation of the t-area functional is available on its finiteness domain. Recall that for any $u \in BV$ one can decompose the distributional derivatives Du as $\nabla u\, \mathcal{L}^{2n} + (Du)_s$, where $\nabla u \in L^1(\mathcal{U})$ is the approximate gradient of u and $(Du)_s$ is the singular part of the $\mathbb{R}^{2n}$-valued Radon measure Du with respect to $\mathcal{L}^{2n}$.

Theorem 3.5. *For any $u \in BV(\mathcal{U})$*

$$\int_{\mathcal{U}} |Du + \boldsymbol{X}^*| = \int_{\mathcal{U}} |\nabla u + \boldsymbol{X}^*|\, d\mathcal{L}^{2n} + |(Du)_s|(\mathcal{U})\,. \tag{3.14}$$

Proof. By Remark 3.1, $\int_{\mathcal{U}} |Du + \boldsymbol{X}^*|$ coincides with the total variation of $Du + \boldsymbol{X}^*\mathcal{L}^{2n} = (\nabla u + \boldsymbol{X}^*)\mathcal{L}^{2n} + (Du)_s$. Since $(\nabla u + \boldsymbol{X}^*)\mathcal{L}^{2n}$ and $(Du)_s$ are mutually singular, the total variation of their sum coincides with the sum of their total variations, and (3.14) follows. □

3.2. **Existence of minimal t-graphs.** The open set $\mathcal{U}$ is henceforth supposed to be open, bounded and with Lipschitz regular boundary. In particular, the notion of trace $u_{|\partial\mathcal{U}}$ of $u \in BV(\mathcal{U})$ on $\partial\mathcal{U}$ is well defined (see e.g. [39, Chapter 2]). If $\mathcal{U} \Subset \mathcal{U}_0$ and $u \in BV(\mathcal{U}_0)$ we denote by $u^-_{|\partial\mathcal{U}}$ and $u^+_{|\partial\mathcal{U}}$, respectively, the inner and outer traces of u on $\partial\mathcal{U}$ defined according to [39, Remark 2.13].

As the following Example 3.6 shows, the existence of minimizers with given boundary datum is a delicate matter even for smooth data. In particular, the existence of minimizers is not guaranteed for the functional (1.5). This example was inspired by similar Euclidean ones that can be found e.g. in [27, 42], see also [39, Example 12.15].

Example 3.6. Let $n = 1$ and $\mathcal{U} := \{z = (x,y) \in \Pi : 1 < |(x,y)| < 2\}$; consider the Dirichlet problem of minimizing the t-area functional $\int_{\mathcal{U}} |Du + \boldsymbol{X}^*|$ among those functions $u \in BV(\mathcal{U})$ with boundary datum

$$\varphi(z) = \begin{cases} 0 & \text{if } |z| = 2 \\ M & \text{if } |z| = 1. \end{cases}$$

We will show that this problem admits no minimizer when M is large enough.

We begin by proving that, if a minimizer exists, then there exists a rotationally invariant one. To this aim, it is enough to prove that for any $u \in BV(\mathcal{U})$ we have

$$\int_{\mathcal{U}} |D\tilde{u} + \boldsymbol{X}^*| \leqslant \int_{\mathcal{U}} |Du + \boldsymbol{X}^*| \tag{3.15}$$

where, after setting R_θ to be the rotation in $\Pi = \mathbb{R}^2$ of an angle θ, we define the rotationally symmetric function $\tilde{u} : \mathcal{U} \to \mathbb{R}$ by

$$\tilde{u}(z) := \fint_0^{2\pi} (u \circ R_\theta)(z)\, d\theta = \fint_0^{2\pi} u(|z|\cos\theta, |z|\sin\theta)\, d\theta\,.$$

Indeed, when $u \in \mathbf{C}^1(\mathcal{U})$ one has

$$\nabla(u \circ R_\theta) = R_{-\theta} \circ (\nabla u) \circ R_\theta \qquad \text{and} \qquad \boldsymbol{X}^* = R_{-\theta} \circ \boldsymbol{X}^* \circ R_\theta\,,$$

for any $\theta \in [0, 2\pi]$, the second equality following from $\boldsymbol{X}^*(z) = 2R_{\pi/2}(z)$. Therefore

$$\begin{aligned} \int_{\mathcal{U}} |\nabla\tilde{u} + \boldsymbol{X}^*|\, d\mathcal{L}^2 &= \int_{\mathcal{U}} \left| \fint_0^{2\pi} \nabla(u \circ R_\theta)\, d\theta + \boldsymbol{X}^* \right| d\mathcal{L}^2 \\ &= \int_{\mathcal{U}} \left| \fint_0^{2\pi} R_{-\theta} \circ (\nabla u + \boldsymbol{X}^*) \circ R_\theta\, d\theta \right| d\mathcal{L}^2 \\ &\leqslant \fint_0^{2\pi} \int_{\mathcal{U}} |R_{-\theta} \circ (\nabla u + \boldsymbol{X}^*) \circ R_\theta|\, d\mathcal{L}^2\, d\theta \\ &= \int_{\mathcal{U}} |\nabla u + \boldsymbol{X}^*|\, d\mathcal{L}^2 \end{aligned}$$

and (3.15) is proved for u of class $\mathbf{C}^1$. When $u \in BV(\mathcal{U}) \setminus \mathbf{C}^1(\mathcal{U})$ it is sufficient to consider a sequence $(u_k)_k$ as in Corollary 3.3 and to observe that $\widetilde{u_k} \to \tilde{u}$ in L^1, whence

$$\int_{\mathcal{U}} |D\tilde{u} + \boldsymbol{X}^*| \leqslant \liminf_{k\to\infty} \int_{\mathcal{U}} |D\widetilde{u_k} + \boldsymbol{X}^*| \leqslant \liminf_{k\to\infty} \int_{\mathcal{U}} |Du_k + \boldsymbol{X}^*| = \int_{\mathcal{U}} |Du + \boldsymbol{X}^*| \,.$$

Moreover, it is not difficult to show that $\tilde{u}_{|\partial\mathcal{U}} = \varphi$ for any $u \in BV(\mathcal{U})$ with $u_{|\partial\mathcal{U}} = \varphi$. Indeed, let us extend u and $\tilde{u}$ to take value M on $\{z : |z| \leqslant 1\}$ and 0 on $\{z : 2 \leqslant |z| < 3\}$; set $A_\epsilon := \{z : 1-\epsilon < |z| < 1+\epsilon \text{ or } 2-\epsilon < |z| < 2+\epsilon\}$. Reasoning as before we can prove the inequality

$$\int_{A_\epsilon} |D\tilde{u}| \leqslant \int_{A_\epsilon} |Du| \qquad \forall \epsilon \in (0,1)$$

first in case u is of class $\mathbf{C}^1(B(0,3))$ and then, by an approximation argument, for any $u \in BV(\mathcal{U})$. This implies that $|D\tilde{u}|(\partial\mathcal{U}) = 0$ and, in particular, that $\tilde{u}_{|\partial\mathcal{U}} = \varphi$.

We are now going to exclude the existence of rotationally invariant minimizers. Let $u(z) = u(x,y) = v(\sqrt{x^2+y^2})$ be fixed and consider $g \in \mathbf{C}^1_c(\mathcal{U}, \mathbb{R}^2)$ with $|g| \leqslant 1$. Let us decompose g in its radial and angular components $\overline{g}_r, \overline{g}_\theta$ by

$$g(z) = \overline{g}_r(z)\tfrac{z}{|z|} + \overline{g}_\theta(z)\tfrac{(-y,x)}{|z|}.$$

Let $g_r(r,\theta) := \overline{g_r}(r\cos\theta, r\sin\theta)$ and $g_\theta(r,\theta) := \overline{g_\theta}(r\cos\theta, r\sin\theta)$. Using polar coordinates, according to which $\operatorname{div} g = \partial_r g_r + \frac{1}{r}(\partial_\theta g_\theta + g_r)$, we obtain

$$\begin{aligned}
&\int_{\mathcal{U}} [-u \operatorname{div} g + \langle \boldsymbol{X}^*, g\rangle]\, d\mathcal{L}^2 \\
&= \int_1^2 \int_0^{2\pi} r\big[-v(r)\big(\partial_r g_r(r,\theta) + \tfrac{1}{r}\partial_\theta g_\theta(r,\theta) + \tfrac{1}{r} g_r(r,\theta)\big) + 2r g_\theta(r,\theta)\big]\, d\theta\, dr \\
&= \int_1^2 \int_0^{2\pi} \big[-v(r)\partial_r(r g_r)(r,\theta) + 2r^2 g_\theta(r,\theta)\big]\, d\theta\, dr\,.
\end{aligned}$$

In particular, $u \in BV(\mathcal{U})$ if and only if $v \in BV(1,2)$. Integration by parts gives

$$\begin{aligned}
&\int_{\mathcal{U}} [-u \operatorname{div} g + \langle \boldsymbol{X}^*, g\rangle]\, d\mathcal{L}^2 \\
&= \int_0^{2\pi} \left[\int_1^2 r\big[v'(r) g_r(r,\theta) + 2r g_\theta(r,\theta)\big]\, dr + \int_1^2 r g_r(r,\theta) d(Dv)_s(r)\right] d\theta.
\end{aligned}$$

On passing to the supremum among functions $g \in \mathbf{C}^1_c(\mathcal{U})$ with $|(\overline{g}_r, \overline{g}_\theta)| \leqslant 1$, we obtain

$$\int_{\mathcal{U}} |Du + \boldsymbol{X}^*| = 2\pi \int_1^2 r\sqrt{v'(r)^2 + 4r^2}\, dr + 2\pi |r(Dv)_s|(1,2) =: L(v)$$

where, for a given measure μ on $(1,2)$, we denote by $r\mu$ the measure $r\mu(I) = \int_I r\, d\mu(r)$. Notice, in particular, that a rotationally symmetric function $u(z) = v(|z|)$ belongs to $BV(\mathcal{U})$ if and only if the associated function v belongs to $BV(1,2)$.

We are going to show that for $|M| \gg 1$ the functional L does not admit minimizers in the class of functions in $BV(1,2)$ with trace M at 1 and trace 0 at 2. We start by proving that any possible minimizer should belong to $W^{1,1}(1,2)$. More precisely, for any $v \in BV \setminus W^{1,1}$ with that boundary datum we can construct a function $w \in BV$ with the same trace and $L(w) < L(v)$. Recall that

$$Dv = v'\, d\mathcal{L}^1 + (Dv)_s, \qquad v' \in L^1(1,2) \text{ and } (Du)_s \perp \mathcal{L}^1.$$

By hypothesis there exists $\delta > 1$ such that $|(Dv)_s|(\delta, 2) > 0$; we can assume that $|(Dv)_s|(\{\delta\}) = 0$ (i.e., δ is not a jump point of v). Define $w \in BV(1,2)$ by

$$w = v \text{ in } (1,\delta), \qquad w(r) = \int_2^r v'(s)ds = -\int_r^2 v'(s)ds \quad \text{if } r \in (\delta, 2).$$

By construction, the trace of w is M at 1 and 0 at 2 and

$$\begin{aligned}
&Dw = v'\mathcal{L}^1 + (Dw)_s,\\
&(Dw)_s \llcorner (1,\delta) = (Dv)_s \llcorner (1,\delta), \quad (Dw)_s \llcorner (\delta, 2) = 0,\\
&(Dw)_s(\{\delta\}) = (Dv)_s(\delta, 2).
\end{aligned}$$

In particular

$$\begin{aligned}
|r(Dw)_s|(1,2) &= |r(Dw)_s|(1,\delta) + \delta|(Dw)_s(\{\delta\})|\\
&\leqslant |r(Dw)_s|(1,\delta) + \delta|(Dv)_s|(\delta, 2)\\
&< |r(Dw)_s|(1,\delta) + |r(Dv)_s|(\delta, 2)\\
&= |r(Dv)_s|(1,2)\,.
\end{aligned}$$

and this gives that $L(w) < L(v)$.

Suppose now that v is a minimizer in $W^{1,1}(1,2)$; the Euler equation for the functional L gives

$$\frac{r\,v'(r)}{\sqrt{4r^2 + v'(r)^2}} = C \tag{3.16}$$

for a suitable $C \in \mathbb{R}$. In particular for a.e. $r \in [1,2]$

$$|C| = \left| \frac{r\,v'(r)}{\sqrt{4r^2 + v'(r)^2}} \right| \leqslant |r|$$

and so $|C| \leqslant 1$. From (3.16) and taking into account that $\operatorname{sgn} v' = \operatorname{sgn} C$, we obtain

$$v'(r) = 2C \frac{r}{\sqrt{r^2 - C^2}}\,.$$

Since $v(2) = 0$, the solution is given by

$$v(r) = 2C\big(\sqrt{r^2 - C^2} - \sqrt{4 - C^2}\big).$$

and thus

$$|M| = |v(1)| \leqslant \sup_{C \in [-1,1]} \left| 2C\big(\sqrt{1 - C^2} - \sqrt{4 - C^2}\big) \right| < \infty\,.$$

This proves that a solution cannot exist for large enough $|M|$.

Our approach to the existence of t-minimizers follows the one outlined in [39, Chapter 14].

Proposition 3.7. *Let $\varphi \in L^1(\partial\mathcal{U})$ and consider an open set $\mathcal{U}_0 \Supset \mathcal{U}$ and a function $u_0 \in BV(\mathcal{U}_0)$ with $u^+_{0|\partial\mathcal{U}} = u^-_{0|\partial\mathcal{U}} = \varphi$. Then*

$$
\begin{aligned}
&\inf\left\{\int_{\mathcal{U}} |Du + \boldsymbol{X}^*| : u \in BV(\mathcal{U}),\ u_{|\partial\mathcal{U}} = \varphi\right\}\\
&= \inf\left\{\int_{\mathcal{U}} |Du + \boldsymbol{X}^*| + \int_{\partial\mathcal{U}} |u_{|\partial\mathcal{U}} - \varphi| d\mathcal{H}^{2n-1}\ :\ u \in BV(\mathcal{U})\right\}\\
&= \inf\left\{\ |\partial E^t_u|_{\mathbb{H}}(\overline{\mathcal{U}} \times \mathbb{R}) : u \in BV(\mathcal{U}_0),\ u = u_0 \text{ on } \mathcal{U}_0 \setminus \mathcal{U}\right\}\\
&= \inf\left\{\int_{\mathcal{U}} |Du + \boldsymbol{X}^*| + \big|\partial\big((E^t_u \Delta E^t_{u_0}) \cap (\mathcal{U} \times \mathbb{R})\big)\big|_{\mathbb{H}}(\partial\mathcal{U} \times \mathbb{R}) : u \in BV(\mathcal{U})\right\}.
\end{aligned}
\tag{3.17}
$$

Proof. The proof of the first equality verbatim follows the one of [39, Proposition 14.3]. The second one follows because, for any $u \in BV(\mathcal{U}_0)$ with $u = u_0$ on $\mathcal{U}_0 \setminus \mathcal{U}$, one has

$$
|\partial E^t_u|_{\mathbb{H}}(\partial\mathcal{U} \times \mathbb{R}) = \int_{\partial\mathcal{U}} |u^-_{|\partial\mathcal{U}} - \varphi| d\mathcal{H}^{2n-1}\,. \tag{3.18}
$$

This is in turn due to the fact that E^t_u is a set with locally finite Euclidean perimeter on $\mathcal{U}_0 \times \mathbb{R}$ and by Proposition 2.10

$$
|\partial E^t_u|_{\mathbb{H}}(\partial\mathcal{U} \times \mathbb{R}) = \int_{\partial^* E^t_u \cap (\partial\mathcal{U} \times \mathbb{R})} |\boldsymbol{n}^{\mathbb{H}}_{E^t_u}|\, d\mathcal{H}^{2n}\,.
$$

Since

$$
\boldsymbol{n}_{E^t_u} = \boldsymbol{n}_{\partial\mathcal{U} \times \mathbb{R}} = (\boldsymbol{n}_{\partial\mathcal{U}}, 0) \qquad \mathcal{H}^{2n}\text{-a.e. on } \partial^* E^t_u \cap (\partial\mathcal{U} \times \mathbb{R})
$$

one has $|\boldsymbol{n}^{\mathbb{H}}_{E^t_u}| = 1$ $\mathcal{H}^{2n}$-a.e. on $\partial^* E^t_u \cap \partial(\mathcal{U} \times \mathbb{R})$. Therefore

$$
|\partial E^t_u|_{\mathbb{H}}(\partial\mathcal{U} \times \mathbb{R}) = \mathcal{H}^{2n}(\partial^* E^t_u \cap (\partial\mathcal{U} \times \mathbb{R})) = \int_{\partial\mathcal{U}} |u^-_{|\partial\mathcal{U}} - \varphi|\, d\mathcal{H}^{2n-1},
$$

the last equality following e.g. from [39, Remark 2.13].

Let us prove the equality between the second and fourth term in (3.17). Notice that $F := (E^t_u \Delta E^t_{u_0}) \cap (\mathcal{U} \times \mathbb{R})$ is a set with locally finite Euclidean perimeter. Reasoning as before we have

$$
\boldsymbol{n}_F = (\boldsymbol{n}_{\partial\mathcal{U}}, 0) \quad \mathcal{H}^{2n}\text{-a.e. on } \partial^* F \cap (\partial\mathcal{U} \times \mathbb{R})
$$

and by Proposition 2.10

$$
|\partial F|_{\mathbb{H}}(\partial\mathcal{U} \times \mathbb{R}) = \mathcal{H}^{2n}(\partial^* F \cap (\partial\mathcal{U} \times \mathbb{R})) = \int_{\partial\mathcal{U}} |u_{|\partial\mathcal{U}} - \varphi|\, d\mathcal{H}^{2n-1}.
$$

This concludes the proof. □

Remark 3.8. We point out that the penalization term $\int_{\partial\mathcal{U}} |u - \varphi| d\mathcal{H}^{2n-1}$ is "natural" from the viewpoint of the geometry of $\mathbb{H}^n$, its geometric meaning being given by (3.18).

Remark 3.9. For any $\varphi \in L^1(\partial\mathcal{U})$, there exist an open set $\mathcal{U}_0$ and a function u_0 as in the statement of Proposition 3.7. Moreover, it is possible (see [36]) to choose $\mathcal{U}_0$ bounded with Lipschitz regular boundary and u_0 so that

$$
u_{0|\mathcal{U}_0 \setminus \overline{\mathcal{U}}} \in W^{1,1}(\mathcal{U}_0 \setminus \overline{\mathcal{U}}) \quad \text{and} \quad u_{0|\partial\mathcal{U}_0} = 0.
$$

With such a choice, in analogy with the terminology that will be introduced in Definition 4.11, we can consider $u_{0|\mathcal{U}_0\setminus\overline{\mathcal{U}}}$ as a *trace in generalized sense* for any $u \in BV(\mathcal{U})$ with $u_{|\partial\mathcal{U}} = \varphi$. We observe (compare with (4.47), (4.48), (4.49)) that

$$|\partial E^t_{u_0}|_{\mathbb{H}}(\partial\mathcal{U}\times\mathbb{R}) = |\partial E^t_{u_0}|_{\mathbb{H}}(\partial\mathcal{U}_0\times\mathbb{R}^+) = |\partial((\mathcal{U}_0\times\mathbb{R})\setminus E^t_{u_0})|_{\mathbb{H}}(\partial\mathcal{U}_0\times\mathbb{R}^-) = 0.$$

We are now in a position to prove Theorem 1.4; the latter, thanks to Proposition 3.7, has to be understood as an existence result for minimal t-graphs.

Proof of Theorem 1.4. As in Remark 3.9, let us fix $\mathcal{U}_0$ with Lipschitz boundary and $u_0 \in BV(\mathcal{U}_0)$ such that

$$u_{0|\partial\mathcal{U}_0} = 0 \quad \text{and} \quad u^+_{0|\partial\mathcal{U}} = u^-_{0|\partial\mathcal{U}} = \varphi.$$

For $u \in BV(\mathcal{U})$ define $v_u \in BV(\mathcal{U}_0)$ by $v_u := u$ on $\mathcal{U}$, $v_u := u_0$ on $\mathcal{U}_0\setminus\overline{\mathcal{U}}$; it was proved in Proposition 3.7 that

$$\begin{aligned} &|\partial E^t_{v_u}|_{\mathbb{H}}(\mathcal{U}_0\times\mathbb{R})\\ =&|\partial E^t_{v_u}|_{\mathbb{H}}(\mathcal{U}\times\mathbb{R}) + |\partial E^t_{v_u}|_{\mathbb{H}}(\partial\mathcal{U}\times\mathbb{R}) + |\partial E^t_{v_u}|_{\mathbb{H}}((\mathcal{U}_0\setminus\overline{\mathcal{U}})\times\mathbb{R})\\ =&\int_{\mathcal{U}} |Du + \boldsymbol{X}^*| + \int_{\partial\mathcal{U}} |u_{|\partial\mathcal{U}} - \varphi| d\mathcal{H}^{2n-1} + \int_{\mathcal{U}_0\setminus\overline{\mathcal{U}}} |Du_0 + \boldsymbol{X}^*|\,. \end{aligned} \tag{3.19}$$

Consider the family

$$\mathcal{F} := \{v \in BV(\mathcal{U}_0) : v \equiv u_0 \text{ in } \mathcal{U}_0\setminus\mathcal{U}\}\,;$$

by (3.19), the functional (1.7) and

$$\mathcal{F} \ni v \longmapsto |\partial E^t_v|_{\mathbb{H}}(\mathcal{U}_0\times\mathbb{R}) \tag{3.20}$$

share the same minimizers and it is sufficient to prove that (3.20) attains its minimum in $\mathcal{F}$.

Let $(v_h)_h \subset \mathcal{F}$ be a minimizing sequence for the functional (3.20). The proof will be accomplished if we show that $(v_h)_h$ is bounded in $BV(\mathcal{U}_0)$, since in this case (up to a subsequence) $v_h \to v$ in L^1. In particular, v would belong to $\mathcal{F}$ and the semicontinuity of (3.20) would allow to conclude.
For any h we have

$$\int_{\mathcal{U}_0} |Dv_h| \leqslant |\partial E^t_{v_h}|_{\mathbb{H}}(\mathcal{U}_0\times\mathbb{R}) + \int_{\mathcal{U}_0} |\boldsymbol{X}^*|\, d\mathcal{L}^{2n} \tag{3.21}$$

while by [39, Theorem 1.28 and Remark 2.14]

$$\left(\int_{\mathcal{U}_0} |v_h| d\mathcal{L}^{2n}\right)^{\frac{n-1}{n}} \leqslant c\left(\int_{\mathcal{U}_0} |Dv_h| + \int_{\partial\mathcal{U}_0} |v_h| d\mathcal{H}^{2n-1}\right) = c\int_{\mathcal{U}_0} |Dv_h|$$

and the boundedness of $(v_h)_h$ in $BV(\mathcal{U}_0)$ follows. □

Remark 3.10. It is a routine task to prove that any minimizer u of the functional (1.7) is a t-minimizer according to Definition 1.3.

We owe the reader two remarks concerning the definitions of t-minimizers and local t-minimizers.

Remark 3.11. Let us prove that, if $u \in BV(\mathcal{U})$ is a t-minimizer, then it is also a local t-minimizer. Let $\mathcal{U}' \Subset \mathcal{U}$ and $v \in L^1_{loc}(\mathcal{U})$ with $\{v \neq u\} \Subset \mathcal{U}'$ be fixed. Then $v_{|\partial\mathcal{U}} = u_{|\partial\mathcal{U}}$ and thus

$$|\partial E^t_u|_{\mathbb{H}}(\mathcal{U} \times \mathbb{R}) \leqslant |\partial E^t_v|_{\mathbb{H}}(\mathcal{U} \times \mathbb{R})\,. \tag{3.22}$$

Since $u = v$ in a neighbourhood of $\mathcal{U} \setminus \mathcal{U}'$ we have $E^t_u = E^t_v$ in a neighbourhood of $(\mathcal{U} \setminus \mathcal{U}') \times \mathbb{R}$ and thus

$$|\partial E^t_u|_{\mathbb{H}}((\mathcal{U} \setminus \mathcal{U}') \times \mathbb{R}) = |\partial E^t_v|_{\mathbb{H}}((\mathcal{U} \setminus \mathcal{U}') \times \mathbb{R})$$

which, together with (3.22), gives $|\partial E^t_u|_{\mathbb{H}}(\mathcal{U}' \times \mathbb{R}) \leqslant |\partial E^t_v|_{\mathbb{H}}(\mathcal{U}' \times \mathbb{R})$, as desired.

Remark 3.12. Let $\mathcal{U}$ be a fixed (not necessarily bounded) open set; let us prove that $u \in BV_{loc}(\mathcal{U})$ is a local t-minimizer if and only if u is a t-minimizer on any compact subset of $\mathcal{U}$ with Lipschitz regular boundary. Namely, that the following two conditions are equivalent:

(a) for any $\mathcal{U}' \Subset \mathcal{U}$ and any $v \in L^1_{loc}(\mathcal{U})$ with $\{u \neq v\} \Subset \mathcal{U}'$ there holds

$$|\partial E^t_u|_{\mathbb{H}}(\mathcal{U}' \times \mathbb{R}) \leqslant |\partial E^t_v|_{\mathbb{H}}(\mathcal{U}' \times \mathbb{R})\,; \tag{3.23}$$

(b) for any $\mathcal{U}'' \Subset \mathcal{U}$ with Lipschitz boundary and any $w \in BV(\mathcal{U}'')$ with $w_{|\partial\mathcal{U}''} = u^-_{|\partial\mathcal{U}''}$ there holds

$$|\partial E^t_u|_{\mathbb{H}}(\mathcal{U}'' \times \mathbb{R}) \leqslant |\partial E^t_w|_{\mathbb{H}}(\mathcal{U}'' \times \mathbb{R})\,. \tag{3.24}$$

Let us prove that $(a) \Rightarrow (b)$. Let $\mathcal{U}'', w$ be as in (b); we choose $\mathcal{U}'$ such that $\mathcal{U}'' \Subset \mathcal{U}' \Subset \mathcal{U}$ and define $v \in BV_{loc}(\mathcal{U})$ by

$$v := w \text{ on } \mathcal{U}'', \qquad v := u \text{ on } \mathcal{U} \setminus \mathcal{U}''\,.$$

Since

$$|\partial E^t_u|_{\mathbb{H}}(\partial\mathcal{U}'' \times \mathbb{R}) = \int_{\partial\mathcal{U}''} |u^+_{|\partial\mathcal{U}''} - u^-_{|\partial\mathcal{U}''}| d\mathcal{H}^{2n-1} = |\partial E^t_v|_{\mathbb{H}}(\partial\mathcal{U}'' \times \mathbb{R})\,,$$

where the second equality can be proved similarly to (3.18), from (3.23) we obtain that

$$\begin{aligned}
&|\partial E^t_u|_{\mathbb{H}}(\mathcal{U}'' \times \mathbb{R}) + |\partial E^t_u|_{\mathbb{H}}(\partial\mathcal{U}'' \times \mathbb{R}) + |\partial E^t_u|_{\mathbb{H}}((\mathcal{U}' \setminus \overline{\mathcal{U}''}) \times \mathbb{R})\\
\leqslant&|\partial E^t_v|_{\mathbb{H}}(\mathcal{U}'' \times \mathbb{R}) + |\partial E^t_v|_{\mathbb{H}}(\partial\mathcal{U}'' \times \mathbb{R}) + |\partial E^t_v|_{\mathbb{H}}((\mathcal{U}' \setminus \overline{\mathcal{U}''}) \times \mathbb{R})\\
=&|\partial E^t_w|_{\mathbb{H}}(\mathcal{U}'' \times \mathbb{R}) + |\partial E^t_u|_{\mathbb{H}}(\partial\mathcal{U}'' \times \mathbb{R}) + |\partial E^t_u|_{\mathbb{H}}((\mathcal{U}' \setminus \overline{\mathcal{U}''}) \times \mathbb{R})
\end{aligned}$$

and (3.24) follows.

Let us prove that $(b) \Rightarrow (a)$. Let $\mathcal{U}', v$ be as in (a) and choose $\mathcal{U}''$ with Lipschitz regular boundary and such that $\{u \neq v\} \Subset \mathcal{U}'' \Subset \mathcal{U}'$. If $v_{|\mathcal{U}''} \notin BV(\mathcal{U}'')$ we have $|\partial E^t_v|_{\mathbb{H}}(\mathcal{U}' \times \mathbb{R}) = +\infty$ and (3.23) follows; otherwise, set $w := v_{|\mathcal{U}''} \in BV(\mathcal{U}'')$. Notice that $w_{|\partial\mathcal{U}''} = u^-_{|\partial\mathcal{U}''}$, thus by (3.24) we obtain

$$\begin{aligned}
|\partial E^t_u|_{\mathbb{H}}(\mathcal{U}'' \times \mathbb{R}) + |\partial E^t_u|_{\mathbb{H}}((\mathcal{U}' \setminus \mathcal{U}'') \times \mathbb{R}) \leqslant&|\partial E^t_w|_{\mathbb{H}}(\mathcal{U}'' \times \mathbb{R}) + |\partial E^t_u|_{\mathbb{H}}((\mathcal{U}' \setminus \mathcal{U}'') \times \mathbb{R})\\
=&|\partial E^t_v|_{\mathbb{H}}(\mathcal{U}'' \times \mathbb{R}) + |\partial E^t_v|_{\mathbb{H}}((\mathcal{U}' \setminus \mathcal{U}'') \times \mathbb{R})
\end{aligned}$$

where the last equality is due to the locality of the $\mathbb{H}$-perimeter and the fact that $u = v$ in a neighbourhood of $\mathcal{U}' \setminus \mathcal{U}''$. This gives (3.23).

3.3. **Local boundedness of minimal t-graphs.** We start with the following lemma, which will be used in the sequel in the particular case of t-subgraphs.

Lemma 3.13. *Let $\mathcal{U} \subset \Pi$ be a bounded open set and let $F \subset \mathcal{U} \times \mathbb{R}$ be a measurable set. Assume that*

$$|F \,\Delta\, (\mathcal{U} \times \mathbb{R}^-)| < \infty \tag{3.25}$$

and, for any positive real number k, set $F_k := \big(F \cup (\mathcal{U} \times (-\infty, -k))\big) \setminus \big(\mathcal{U} \times [k, +\infty)\big)$. Then there exists a sequence $(k_j)_j$ such that $k_j \to +\infty$ and

$$\lim_{j \to +\infty} |\partial F_{k_j}|_{\mathbb{H}}(\mathcal{U} \times \mathbb{R}) = |\partial F|_{\mathbb{H}}(\mathcal{U} \times \mathbb{R})\,. \tag{3.26}$$

Proof. If $|\partial F|_{\mathbb{H}}(\mathcal{U} \times \mathbb{R}) = +\infty$ we immediately conclude by noticing that

$$|\partial F_k|_{\mathbb{H}}(\mathcal{U} \times \mathbb{R}) \geqslant |\partial F_k|_{\mathbb{H}}(\mathcal{U} \times (-k, k)) = |\partial F|_{\mathbb{H}}(\mathcal{U} \times (-k, k)) \;\longrightarrow\; +\infty.$$

We assume from now on that $|\partial F|_{\mathbb{H}}(\mathcal{U} \times \mathbb{R}) < +\infty$; in particular, by Proposition 2.3 any of the sets F_k has finite $\mathbb{H}$-perimeter. Since $|\partial F_k|_{\mathbb{H}}(\mathcal{U} \times (\mathbb{R} \setminus [-k, k])) = 0$ we have

$$|\partial F_k|_{\mathbb{H}}(\mathcal{U} \times \mathbb{R}) = \underbrace{|\partial F|_{\mathbb{H}}(\mathcal{U} \times (-k, k))}_{\to |\partial F|_{\mathbb{H}}(\mathcal{U} \times \mathbb{R}) \text{ as } k \to \infty} + |\partial F_k|_{\mathbb{H}}(\mathcal{U} \times \{k\}) + |\partial F_k|_{\mathbb{H}}(\mathcal{U} \times \{-k\});$$

in particular, it will be enough to show that

$$\lim_{j \to +\infty} |\partial F_{k_j}|_{\mathbb{H}}(\mathcal{U} \times \{k_j, -k_j\}) = 0$$

for some sequence $k_j \to +\infty$. By Theorem 2.6, this is equivalent to find $(k_j)_j$ so that

$$\lim_{j \to +\infty} \mathcal{S}_\infty^{Q-1}(F_{k_j}^{1/2} \cap \mathcal{U} \times \{k_j, -k_j\}) = 0 \tag{3.27}$$

From the inclusion $F_k \cap (\mathcal{U} \times \mathbb{R}^+) \subset F$ we infer, for any $k > 0$, the implication

$$P \in F_k^{1/2} \cap (\mathcal{U} \times \{k\}) \Longrightarrow \liminf_{r \to 0} \frac{|F \cap U(P, r)|}{|U(P, r)|} \geqslant \frac{1}{2}$$

while the inclusion $F \cap (\mathcal{U} \times \mathbb{R}^-) \subset F_k$ gives

$$P \in F_k^{1/2} \cap (\mathcal{U} \times \{-k\}) \Longrightarrow \limsup_{r \to 0} \frac{|F \cap U(P, r)|}{|U(P, r)|} \leqslant \frac{1}{2}\,.$$

Recalling that $\mathcal{S}_\infty^{Q-1}(\mathbb{H}^n \setminus (F^0 \cup F^{1/2} \cup F^1)) = 0$ we obtain that the implications

$$\begin{aligned} P \in F_k^{1/2} \cap (\mathcal{U} \times \{k\}) &\Longrightarrow P \in (F^{1/2} \cup F^1) \cap (\mathcal{U} \times \{k\}) \\ P \in F_k^{1/2} \cap (\mathcal{U} \times \{-k\}) &\Longrightarrow P \in (F^{1/2} \cup F^0) \cap (\mathcal{U} \times \{-k\}) \end{aligned}$$

hold for any $k > 0$ and $|\partial F_k|_{\mathbb{H}}$-a.e. $P \in (\mathcal{U} \times \{k, -k\})$. In particular

$$\begin{aligned} &\mathcal{S}_\infty^{Q-1}(F_k^{1/2} \cap \mathcal{U} \times \{k, -k\}) \\ &\leqslant \mathcal{S}_\infty^{Q-1}(F^1 \cap (\mathcal{U} \times \{k\})) + \mathcal{S}_\infty^{Q-1}(F^{1/2} \cap (\mathcal{U} \times \{k, -k\})) + \mathcal{S}_\infty^{Q-1}(F^0 \cap (\mathcal{U} \times \{-k\})) \end{aligned}$$

and, recalling (3.27), it will suffice to find $(k_j)_j$ so that

$$\begin{aligned}
&\lim_{j\to+\infty} \mathcal{S}_\infty^{Q-1}(F^{1/2}\cap(\mathcal{U}\times\{k_j,-k_j\}))=0,\\
&\lim_{j\to+\infty} \mathcal{S}_\infty^{Q-1}(F^{1}\cap(\mathcal{U}\times\{k_j\}))=0,\\
&\lim_{j\to+\infty} \mathcal{S}_\infty^{Q-1}(F^{0}\cap(\mathcal{U}\times\{-k_j\}))=0.
\end{aligned}\tag{3.28}$$

Since $|\partial F|_{\mathbb{H}} = c_n\mathcal{S}_\infty^{Q-1}\llcorner F^{1/2}$ is a finite measure and $(\mathcal{U}\times\{k,-k\})\cap(\mathcal{U}\times\{k',-k'\})=\emptyset$ for any $k\neq k'$, we obtain that

$$\lim_{k\to+\infty} \mathcal{S}_\infty^{Q-1}(F^{1/2}\cap(\mathcal{U}\times\{k,-k\}))=0\,;$$

in particular, the first statement in (3.28) holds for any sequence $k_j\to\infty$.
Let us find $(k_j)_j$ satisfying the second and third equalities in (3.28). By Lemma 2.12 there exists $C(\mathcal{U})>0$ such that

$$\mathcal{S}_\infty^{Q-1}\llcorner(\mathcal{U}\times\{k,-k\})\leqslant C(\mathcal{U})\,\mathcal{H}^{2n}\llcorner(\mathcal{U}\times\{k,-k\})\tag{3.29}$$

and thus it is enough to find $(k_j)_j$ so that

$$\lim_{j\to\infty}\big(\mathcal{H}^{2n}(F^1\cap(\mathcal{U}\times\{k_j\}))+\mathcal{H}^{2n}(F^0\cap(\mathcal{U}\times\{-k_j\}))\big)=0.\tag{3.30}$$

By Fubini's theorem and assumption (3.25) there holds

$$\begin{aligned}
&\int_0^{+\infty}\big[\mathcal{H}^{2n}(F^1\cap(\mathcal{U}\times\{k\}))+\mathcal{H}^{2n}(F^0\cap(\mathcal{U}\times\{-k\}))\big]\,dk\\
&=|F^1\cap(\mathcal{U}\times\mathbb{R}^+)|+|F^0\cap(\mathcal{U}\times\mathbb{R}^-)|\\
&=|F\setminus(\mathcal{U}\times\mathbb{R}^-)|+|(\mathcal{U}\times\mathbb{R}^-)\setminus F|\\
&=|F\,\Delta\,(\mathcal{U}\times\mathbb{R}^-)|\ <\ \infty
\end{aligned}$$

and the existence of $(k_j)_j, k_j\to+\infty$ satisfying (3.30) easily follows. □

Remark 3.14. A result analogous to Lemma 3.13 holds for intrinsic graphs: see Lemma 4.6. The proofs of the two results are completely analogous: it is sufficient to replace the symbols "$\times$" and "$\mathcal{U}$", respectively, by "$\cdot$" and "ω", so that any t-product $\mathcal{U}\times(a,b)$ becomes an intrinsic one $\omega\cdot(a,b)$. None of the arguments change except (3.29) where, according to Lemma 2.12, the constant $C(\mathcal{U})$ has to be replaced by $1/c_n$.

The following result provides a geometric rearrangement for sets in $\mathbb{H}^n$ which decreases their $\mathbb{H}$-perimeter. An analogous rearrangement exists in the Euclidean setting, see e.g. [47].

Theorem 3.15. *Let F be a set with finite $\mathbb{H}$-perimeter in $\mathcal{U}\times\mathbb{R}$ satisfying*

(i) $\lim_{t\to+\infty}\chi_F(z,t)=0$ *and* $\lim_{t\to-\infty}\chi_F(z,t)=1$ *for a.e.* $z\in\mathcal{U}$,

(ii) $|F\,\Delta\,(\mathcal{U}\times\mathbb{R}^-)|<\infty$.

Then the function

$$w(z):=\lim_{k\to\infty}\Big(\int_{-k}^{k}\chi_F(z,t)dt-k\Big)\tag{3.31}$$

belongs to $BV(\mathcal{U})$ *and*

$$|\partial E^t_w|_{\mathbb{H}}(\mathcal{U}\times\mathbb{R}) \leqslant |\partial F|_{\mathbb{H}}(\mathcal{U}\times\mathbb{R})\,. \tag{3.32}$$

Proof. We preliminarily observe that w is well defined almost everywhere on $\mathcal{U}$. In fact, $w_k(z) := \int_{-k}^k \chi_F(z,t)dt - k$ coincides with $w_{k'}(z)$ for large enough k,k': more precisely, whenever

$$k,k' \;\geqslant\; \tau(z) := \inf\{t>0 : \chi_F(z,s)=0\ \forall s>t \text{ and } \chi_F(z,s)=1\ \forall s<-t\}\,. \tag{3.33}$$

In particular

$$w(z) = \int_{-\tau(z)}^{\tau(z)} \chi_F(z,t)dt - \tau(z) \qquad \text{for any } z\in\mathcal{U}. \tag{3.34}$$

First, let us assume that there exists $M\in\mathbb{R}^+$ such that

$$\mathcal{U}\times(-\infty,-M) \,\subset\, F \,\subset \mathcal{U}\times(-\infty,M). \tag{3.35}$$

This assumption corresponds to the map τ being uniformly bounded by M; notice also that $w=w_M$. We are going to closely follow the approach and computations in the proof of Theorem 3.2, step 6: let us fix $h\in\mathbf{C}^1_c(\mathbb{R})$ which satisfies (3.9). By (3.35)

$$\int_{\mathbb{R}} \chi_F(z,t)h(t)dt = J + w(z) + M \quad \text{and} \quad \int_{\mathbb{R}} \chi_F(z,t)h'(t)dt = 1$$

for a.e. $z\in\mathcal{U}$, where $J := \int_{-M-1}^{-M} h(t)dt$. For fixed $g\in\mathbf{C}^1_c(\mathcal{U},\mathbb{R}^{2n})$, with $|g|\leqslant 1$, set $\varphi(z,t) := -h(t)g(z)$. Computations analogous to those in Theorem 3.2, step 6, yield

$$\begin{aligned}
|\partial F|_{\mathbb{H}}(\mathcal{U}\times\mathbb{R}) &\geqslant \int_F \mathrm{div}_{\mathbb{H}}\varphi \\
&= \int_{\mathcal{U}}\int_{\mathbb{R}} \chi_F(z,t)\big[-h(t)\,\mathrm{div}g(z) + h'(t)\langle \boldsymbol{X}^* z, g(z)\rangle\big]dt\,dz \\
&= \int_{\mathcal{U}} [-(J+w+M)\,\mathrm{div}g + \langle \boldsymbol{X}^*, g\rangle]dz \\
&= \int_{\mathcal{U}} [-w\,\mathrm{div}g + \langle \boldsymbol{X}^*, g\rangle]dz\,.
\end{aligned}$$

Recalling that $|\partial E^t_w|_{\mathbb{H}}(\mathcal{U}\times\mathbb{R}) = S_t(w)$, the inequality (3.32) follows by taking the supremum on g in the previous inequality.

If (3.35) does not hold we consider the sets $(F_{k_j})_j$ given by Lemma 3.13,

$$F_k := F\cup\big(\mathcal{U}\times(-\infty,-k)\big)\setminus\big(\mathcal{U}\times[k,+\infty)\big)\,.$$

Notice that $w_k(x,y) = \int_{-k}^k \chi_{F_k}(x,y,t)dt - k$ for any $k>0$. We will prove later that $w_k\to w$ in $L^1(\mathcal{U})$: assuming this to hold, by semicontinuity we obtain

$$\begin{aligned}
|\partial E^t_w|_{\mathbb{H}}(\mathcal{U}\times\mathbb{R}) = \int_{\mathcal{U}} |Dw+\boldsymbol{X}^*| &\leqslant \liminf_{j\to\infty}\int_{\mathcal{U}} |Dw_{k_j}+\boldsymbol{X}^*| \\
&\leqslant \liminf_{j\to\infty} |\partial F_{k_j}|_{\mathbb{H}}(\mathcal{U}\times\mathbb{R}) = |\partial F|_{\mathbb{H}}(\mathcal{U}\times\mathbb{R})\,.
\end{aligned}$$

We have used Lemma 3.13 and the fact that any set F_{k_j} satisfies (3.35). This would conclude the proof of the theorem.

We prove that $w_k \to w$ in $L^1(\mathcal{U})$ by thedominated convergence theorem, where we already know that the convergence holds pointwise almost everywhere. Since

$$w_k(z) = \int_0^k \chi_F(z,t)dt - \int_{-k}^0 \chi_{\mathbb{H}^n\setminus F}(z,t)dt\,,$$

it follows that

$$\begin{aligned} w_k(z) \geqslant 0 \implies\ & 0 \leqslant w_k(z) \leqslant \mathcal{L}^1\big(\{t\in[0,k] : (z,t)\in F\}\big)\\ & \qquad\qquad \leqslant \mathcal{L}^1\big(\{t\in\mathbb{R} : (z,t)\in F\setminus(\mathcal{U}\times\mathbb{R}^-)\}\big)\\ w_k(z) \leqslant 0 \implies\ & 0 \leqslant -w_k(z) \leqslant \mathcal{L}^1\big(\{t\in[-k,0] : (z,t)\notin F\}\big)\\ & \qquad\qquad \leqslant \mathcal{L}^1\big(\{t\in\mathbb{R} : (z,t)\in (\mathcal{U}\times\mathbb{R}^-)\setminus F\}\big)\,. \end{aligned}$$

Therefore

$$|w_k(z)| \leqslant \mathcal{L}^1\big(\{t\in\mathbb{R} : (z,t)\in F\,\Delta\,(\mathcal{U}\times\mathbb{R}^-)\}\big)$$

and then the sequence $(w_k)_k$ is dominated in $L^1(\mathcal{U})$ by Fubini's theorem and hypothesis (ii). □

Corollary 3.16. *Let $u\in BV_{loc}(\mathcal{U})$ be a local t-minimizer of the area functional. Then E_u^t is a local minimizer of the $\mathbb{H}$-perimeter in $\mathcal{U}\times\mathbb{R}$.*

Proof. Without loss of generality, we may assume that $u\in BV(\mathcal{U})$. Let $F\subset\mathcal{U}\times\mathbb{R}$ be such that $F\Delta E_u^t \Subset \mathcal{U}\times\mathbb{R}$; in particular $F\Delta E_u^t\subset\mathcal{U}'\times\mathbb{R}$ for some $\mathcal{U}'\Subset\mathcal{U}$. It is clear that F satisfies hypothesis (i) of Theorem 3.15. Hypothesis (ii) is also verified because

$$|F\Delta(\mathcal{U}\times\mathbb{R}^-)| \leqslant |F\Delta E_u^t| + |E_u^t\Delta(\mathcal{U}\times\mathbb{R}^-)| = |F\Delta E_u^t| + \int_{\mathcal{U}}|u|\,d\mathcal{L}^{2n} < +\infty.$$

Defining τ and w as in (3.33) and (3.31), one has $\tau(z)=|u(z)|$ for any $z\in\mathcal{U}\setminus\mathcal{U}'$: this is due to the fact that $F=E_u^t$ in $(\mathcal{U}\setminus\mathcal{U}')\times\mathbb{R}$. By (3.34) we get $w=u$ on $\mathcal{U}\setminus\mathcal{U}'$ and

$$|\partial E_u^t|_{\mathbb{H}}(\mathcal{U}''\times\mathbb{R}) \leqslant |\partial E_w^t|_{\mathbb{H}}(\mathcal{U}''\times\mathbb{R}) \leqslant |\partial F|_{\mathbb{H}}(\mathcal{U}''\times\mathbb{R})$$

for any $\mathcal{U}''$ such that $\mathcal{U}'\Subset\mathcal{U}''\Subset\mathcal{U}$. This is sufficient to conclude. □

Proof of Theorem 1.5. By Corollary 3.16, E_u^t is a local minimizer of the $\mathbb{H}$-perimeter. Suppose by contradiction that there exists a compact set $K\subset\mathcal{U}$ such that $\|u\|_{L^\infty(K)} = +\infty$. Without loss of generality we may assume that $\mathcal{L}^{2n}(K\cap\{u>M\})>0$ for any $M>0$.
Given $M>0$ we can find $z(M)\in K$ such that $u(z(M))>M$ and

$$\mathcal{L}^{2n}\big(\mathcal{V}\cap K\cap\{u>M\}\big)>0 \tag{3.36}$$

for any neighbourhood $\mathcal{V}\subset\mathcal{U}$ of $z(M)$. For instance, it is sufficient to consider a Lebesgue point for the set $K\cap\{u>M\}$.
Fix a sequence $(M_j)_j\subset\mathbb{R}^+$ increasing to $+\infty$ rapidly enough, so that the balls $U((z_j,M_j),R)$ are pairwise disjoint, where $z_j:=z(M_j)$ and

$$R := d_\infty(K\times\mathbb{R},\partial\mathcal{U}\times\mathbb{R}) = dist(K,\partial\mathcal{U})>0. \tag{3.37}$$

By condition (3.36), the point $(z_j,M_j)\in E_u^t$ is such that

$$|E_u^t\cap U((z_j,M_j),r)|>0 \qquad \text{for any } r>0\,.$$

Proposition 2.14 ensures that for any j

$$|E_u^t \cap U((z_j, M_j), R)| \geqslant cR^Q$$

whence

$$\int_{\mathcal{U}} |u| d\mathcal{L}^{2n} \geqslant |E_u^t \cap (\mathcal{U} \times \mathbb{R}^+)| = +\infty\,,$$

which gives a contradiction. □

We stress the importance of (3.37) and mention that, on the contrary, the distance $d_\infty(\omega' \cdot \mathbb{R}, \partial\omega \cdot \mathbb{R})$ between "intrinsic cylinders" is in general null for any $\omega' \Subset \omega$ with ω bounded. See Proposition 4.20.

As a consequence of Theorem 1.5 we obtain a local boundedness result for weak solutions of (3.1).

Theorem 3.17. *Let $u \in W^{1,1}_{loc}(\mathcal{U})$ and define $N(u) : \mathcal{U} \to \mathbb{R}^{2n}$ by extending the vector defined in* (3.2) *so that $N(u) = 0$ on* Char(u). *Assume that the equation*

$$\operatorname{div}(N(u)) = 0 \quad \text{in } \mathcal{U} \tag{3.38}$$

holds in the sense of distributions. Then E_u^t is locally $\mathbb{H}$-perimeter minimizing in $\mathcal{U} \times \mathbb{R}$ and $u \in L^\infty_{loc}(\mathcal{U})$.

Proof. The subgraph $E := E_u^t$ has locally finite Euclidean perimeter in $\mathcal{U} \times \mathbb{R}$ and

$$\boldsymbol{n}_E(z,t) = \frac{(-\nabla u(z), 1)}{\sqrt{1 + |\nabla u(z)|^2}} \quad \text{for } \mathcal{H}^{2n}\text{-a.e. } (z,t) \in \partial^* E \cap (\mathcal{U} \times \mathbb{R}).$$

In particular, by Proposition 2.10

$$|\partial E|_{\mathbb{H}} \llcorner (\mathcal{U} \times \mathbb{R}) = \left| \frac{\nabla u + \boldsymbol{X}^*}{\sqrt{1 + |\nabla u|^2}} \circ \pi_\Pi \right| \mathcal{H}^{2n} \llcorner (\partial^* E \cap (\mathcal{U} \times \mathbb{R})) \tag{3.39}$$

where $\pi_\Pi : \mathbb{H}^n \to \Pi$ is defined by $\pi_\Pi(z,t) := z$. It follows that $|\partial E|_{\mathbb{H}}(\mathrm{Char}(u) \times \mathbb{R}) = 0$ and

$$\nu_E(z,t) = N(u)(z) \quad \text{for } |\partial E|_{\mathbb{H}}\text{-a.e. } (z,t) \in \partial^* E \cap (\mathcal{U} \times \mathbb{R})\,. \tag{3.40}$$

Define $\bar{N} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ by $\bar{N} := N(u)$ in $\mathcal{U}$ and $\bar{N} := 0$ in $\mathbb{R}^{2n} \setminus \mathcal{U}$. Let ϱ_ϵ be a standard family of mollifiers in $\mathbb{R}^{2n}$ with support in $B(0,\epsilon)$ and define

$$N_\epsilon(z,t) := (\varrho_\epsilon * \bar{N})(z) = (\varrho_\epsilon * \bar{N}_1, \dots, \varrho_\epsilon * \bar{N}_{2n})(z), \quad z \in \mathbb{R}^{2n}\,.$$

It follows that

$$\operatorname{div}_{\mathbb{H}}(N_\epsilon) = 0 \text{ in } \{(z,t) : dist(z, \partial\mathcal{U}) < \epsilon\}\,. \tag{3.41}$$

We claim that E is a local minimizer for the $\mathbb{H}$-perimeter; we are going to use Theorem 2.15. Fix a sequence $(\Omega_h)_h$ of open sets such that $\Omega_h \Subset \Omega_{h+1} \uparrow \mathcal{U} \times \mathbb{R}$. By (3.41) there exists a sequence $\epsilon_h \to 0$ such that the functions

$$\nu_h : \mathcal{U} \times \mathbb{R} \to \mathbb{R}^{2n}, \qquad \nu_h(z,t) := N_{\epsilon_h}(z,t)$$

satisfy the assumptions of Theorem 2.15. Only hypothesis (iv) therein is not immediate; to this aim, notice that for $\mathcal{L}^{2n}$-a.e. $z \in \mathcal{U}$ the property

$$\nu_h(z,t) \to N(u)(z) \quad \text{for any } t \in \mathbb{R}$$

holds. Taking into account (3.40) and the fact that $|\partial E|_{\mathbb{H}} \ll \mathcal{S}_\infty^{2n+1} \ll \mathcal{H}^{2n}$, our claim easily follows. $\square$

3.4. **A discontinuous minimal t-graph in $\mathbb{H}^1$.** We are able to provide an example of a local t-minimizer in $\mathbb{H}^1$ that is not continuous, thus proving that the regularity result of Theorem 1.5 cannot be improved at least in the case $n = 1$. Let us consider $\mathcal{U} := \{(x, y) \in \mathbb{R}^2 : y > 0\}$ and the discontinuous function

$$\begin{aligned} u \ : \ &\mathcal{U} \longrightarrow \mathbb{R} \\ &(x,y) \longmapsto \begin{cases} -2xy - 1 & \text{if } x \geqslant 0 \\ 0 & \text{if } x < 0. \end{cases} \end{aligned}$$

We are going to prove that $E := E_u^t$ is a local minimizer for the $\mathbb{H}$-perimeter in $\mathcal{U}\times\mathbb{R}$, which implies that u is a local t-minimizer.

The open set E is a piecewise $\mathbf{C}^\infty$ domain, its Euclidean inward unit normal being

$$\boldsymbol{n}_E(x,y,t) = \begin{cases} (0,0,-1) & \text{if } (x,y,t) \in \partial E \cap \{x < 0\} \\ (-1,0,0) & \text{if } (x,y,t) \in \partial E \cap \{x = 0\} \\ -\frac{(2y,2x,1)}{\sqrt{1+4(x^2+y^2)}} & \text{if } (x,y,t) \in \partial E \cap \{x > 0\} \end{cases}$$

for $\mathcal{H}^2$-a.e. $(x, y, t) \in \partial E$. It follows that the horizontal normal to E at a point (x, y, t) is

$$\nu_E(x,y,t) = \frac{\boldsymbol{n}_E^{\mathbb{H}}(x,y,t)}{|\boldsymbol{n}_E^{\mathbb{H}}(x,y,t)|} = \begin{cases} (-1,0) & \text{if } x \geqslant 0 \\ \left(-\frac{y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}}\right) & \text{if } x < 0 \end{cases}$$

where $\boldsymbol{n}_E^{\mathbb{H}}$ is defined as in Proposition 2.10; in particular, ν_E is continuous on ∂E. Defining

$$\nu(x,y,t) := \begin{cases} (-1,0) & \text{if } (x,y,t) \in \mathcal{U}\times\mathbb{R} \cap \{x \geqslant 0\} \\ \left(-\frac{y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}}\right) & \text{if } (x,y,t) \in \mathcal{U}\times\mathbb{R} \cap \{x < 0\} \end{cases}$$

and taking into account Theorem 2.15, it will be sufficient to prove that $\operatorname{div}_{\mathbb{H}} \nu = 0$ in the sense of distributions in $\mathcal{U} \times \mathbb{R}$, i.e. that

$$\int_{\mathcal{U}\times\mathbb{R}} \langle \nu, \nabla_{\mathbb{H}}\varphi\rangle d\mathcal{L}^3 = 0 \tag{3.42}$$

for any $\varphi = (\varphi_1, \varphi_2) \in \mathbf{C}_c^1(\mathcal{U} \times \mathbb{R}, \mathbb{R}^2)$. We have

$$\operatorname{spt} \varphi \subset \mathcal{R} := \{(x, y, t) \in \Omega : -a < x < a, -a < t < a \text{ and } 1/a < y < a\}$$

for a suitable $a > 0$. Let $\mathcal{R}_\epsilon := \mathcal{R} \setminus \{(x, y, t) \in \mathcal{U} \times \mathbb{R} : |x| < \epsilon\}$. Writing $\nu(x, y, t) = \big(\nu_1(x, y), \nu_2(x, y)\big)$, by Gauss-Green formula

$$\begin{aligned} \int_{\mathcal{R}_\epsilon} \nu_1\, X\varphi_1\, d\mathcal{L}^3 \ &= \int_{\mathcal{R}_\epsilon} \nu_1 \frac{\partial\varphi_1}{\partial x}\, d\mathcal{L}^3 = \int_{\partial\mathcal{R}_\epsilon} \varphi_1\nu_1\, d\mathcal{H}^2 - \int_{\mathcal{R}_\epsilon} \frac{\partial\nu_1}{\partial x}\varphi_1\, d\mathcal{L}^3 \\ &= \int_{1/a}^a\int_{-a}^a \big(\varphi_1(\epsilon,y,t)\nu_1(\epsilon,y,t) - \varphi_1(-\epsilon,y,t)\nu_1(-\epsilon,y,t)\big)\, dt\, dy - \int_{\mathcal{R}_\epsilon} \frac{\partial\nu_1}{\partial x}\varphi_1\, d\mathcal{L}^3. \end{aligned}$$

Since ν is continuous and smooth in $\mathcal{U}\times\mathbb{R}\setminus\{x=0\}$ we get

(3.43)

$$\begin{aligned}&\int_{\mathcal{U}\times\mathbb{R}}\nu_1 X\varphi_1\,d\mathcal{L}^3 \;=\; \int_{\mathcal{R}}\nu_1 X\varphi_1\,d\mathcal{L}^3 \;=\; \lim_{\epsilon\to 0}\int_{\mathcal{R}_\epsilon}\nu_1\,X\varphi_1\,d\mathcal{L}^3\\ &=\lim_{\epsilon\to 0}\left[\int_{1/a}^{a}\int_{-a}^{a}\big[\varphi_1(\epsilon,y,t)\nu_1(\epsilon,y,t)-\varphi_1(-\epsilon,y,t)\nu_1(-\epsilon,y,t)\big]\,dt\,dy-\int_{\mathcal{R}_\epsilon}\varphi_1\frac{\partial\nu_1}{\partial x}d\mathcal{L}^3\right]\\ &=-\int_{(\mathcal{U}\times\mathbb{R})\setminus\{x=0\}}\varphi_1\frac{\partial\nu_1}{\partial x}\,d\mathcal{L}^3 \;=\; -\int_{(\mathcal{U}\times\mathbb{R})\setminus\{x=0\}}\varphi_1 X\nu_1\,d\mathcal{L}^3.\end{aligned}$$

Analogously we obtain

$$\int_{\mathcal{U}\times\mathbb{R}}\nu_2\,Y\varphi_2\,d\mathcal{L}^3=-\int_{(\mathcal{U}\times\mathbb{R})\setminus\{x=0\}}\varphi_2 Y\nu_2\,d\mathcal{L}^3. \tag{3.44}$$

Equation (3.42) follows from (3.43), (3.44) and the fact that $\operatorname{div}_{\mathbb{H}}\nu=0$ in $(\mathcal{U}\times\mathbb{R})\setminus\{x=0\}$.

Remark 3.18. The set E coincides with the intrinsic subgraph E_ϕ of the map $\phi:\{(y,t)\in\mathbb{W}:y>0\}\to\mathbb{R}$ defined by

$$\phi(y,t):=\begin{cases}-\frac{1+t}{4y} & \text{if } t\leqslant -1\\ 0 & \text{if } -1\leqslant t\leqslant 0\\ -\frac{t}{2y} & \text{if } t\geqslant 0.\end{cases}$$

Since the horizontal normal ν_E can be extended continuously to $\mathcal{U}\times\mathbb{R}$, the boundary ∂E is an $\mathbb{H}$-regular hypersurface (see [52]) and ϕ is of class $\mathbf{C}^1_{\mathbb{W}}$.

4. Existence and local boundedness of minimal intrinsic graphs

4.1. **Bounded variation for intrinsic graphs.** Here and in the following, ω will denote a fixed bounded open subset of $\mathbb{W}$. We will denote by $\mathscr{A}_{\mathbb{W}}:W^{1,1}_{\mathbb{W}}(\omega)\to\mathbb{R}$ the *area functional for X_1-graphs* introduced in (1.11)

$$\mathscr{A}_{\mathbb{W}}(\phi):=\int_\omega\sqrt{1+|\nabla^\phi\phi|^2}\,d\mathcal{L}^{2n}\,. \tag{4.1}$$

When ϕ is Lipschitz continuous we can write

$$\mathscr{A}_{\mathbb{W}}(\phi)=\int_\omega\mathscr{L}_{\mathbb{W}}(A,\phi(A),\nabla\phi(A))\,d\mathcal{L}^{2n}(A),$$

where $\mathscr{L}_{\mathbb{W}}:\mathbb{W}\times\mathbb{R}\times\mathbb{R}^{2n}\to[0,+\infty)$ is defined by

$$\mathscr{L}_{\mathbb{W}}(A,\phi,\xi):=\left(1+\langle X_{n+1}(A)-4\,\phi\,T(A),\xi\rangle^2+\sum_{j=2,j\neq n+1}^{2n}\langle X_j(A),\xi\rangle^2\right)^{1/2} \tag{4.2}$$

if $n\geqslant 2$, while

$$\mathscr{L}_{\mathbb{W}}(A,\phi,\xi):=\big(1+\langle Y_1(A)-4\,\phi\,T(A),\xi\rangle^2\big)^{1/2} \tag{4.3}$$

if $n=1$. The vector fields X_j $(j=2,\dots,n), Y_j$ $(j=1,\dots,n)$ and T are tangent to $\mathbb{W}\equiv\mathbb{R}^{2n}$ and therefore can be viewed as elements of $\mathbb{R}^{2n}$. The scalar products in (4.2) and (4.3) are the usual ones between vectors in $\mathbb{R}^{2n}$.

When $\phi \in \mathbf{C}^2(\omega)$ is a local minimizer of the functional $\mathscr{A}_{\mathbb{W}}$, the first variation of the functional $\mathscr{A}_{\mathbb{W}}$ gives the *minimal surface equation for X_1-graphs* (see e.g. [51])

$$\nabla^{\phi} \cdot \left(\frac{\nabla^{\phi}\phi}{\sqrt{1+|\nabla^{\phi}\phi|^2}} \right) = 0 \quad \text{in } \omega. \tag{4.4}$$

It was pointed out in [24] that $\mathscr{A}_{\mathbb{W}}$ is not convex for $n = 1$. Indeed, for any $\alpha > 0$ the function

$$\phi(y,t) := -\frac{\alpha\, y\, t}{1 + 2\,\alpha\, y^2}$$

satisfies (4.4) on $\mathbb{R}^2$, while (see [22]) ϕ is not a local minimizer for $\mathscr{A}_{\mathbb{W}} : Lip(\omega) \to \mathbb{R}$ on a suitable bounded open set $\omega \subset \mathbb{R}^2$. In particular, $\mathscr{A}_{\mathbb{W}}$ cannot be convex on $Lip(\omega)$ because the stationary point ϕ is not a minimum. The presence of stationary points that are not minimizers for $\mathscr{A}_{\mathbb{W}}$ is an interesting open question in the case $n \geqslant 2$. Nevertheless, the nonconvexity of $\mathscr{A}_{\mathbb{W}} : Lip(\omega) \to \mathbb{R}$ can occur also in the higher dimensional case.

Proposition 4.1. *For any $n \geqslant 2$ there exists a bounded open set $\omega \subset \mathbb{R}^{2n}$ for which the functional $\mathscr{A}_{\mathbb{W}} : Lip(\omega) \to \mathbb{R}$ is not convex.*

Proof. For $A = (x_2, \dots, x_n, y_1, \dots, y_n, t) \in \mathbb{W}$ we set

$$\hat{\boldsymbol{X}}^*(A) := 2\,(-y_2, \dots, -y_n, x_2, \dots, x_n) \in \mathbb{R}^{2n-2}\,.$$

Define

$$\omega := \left\{ A \in \mathbb{W} : |\hat{\boldsymbol{X}}^*(A)| < 1, y_1 > 0, |t| < 1, (y_1+1)^2 < \frac{t^2}{4} \right\}$$

and $\phi, \phi_s : \omega \to \mathbb{R}$ by

$$\phi(A) = -\frac{t}{4(y_1+1)}, \qquad \phi_s := s\phi,\ s \in \mathbb{R}.$$

Letting

$$g_1(A) := \frac{|\hat{\boldsymbol{X}}^*(A)|^2}{16\,(y_1+1)^2}, \qquad g_2(A) := \frac{t^2}{16\,(y_1+1)^4},$$
$$g(s;A) := 1 + g_1(A)s^2 + g_2(A)s^2(1-s)^2$$

it is easy to verify that

$$f(s) := \mathscr{A}_{\mathbb{W}}(\phi_s) = \int_\omega \left[1 + |\hat{\boldsymbol{X}}^*|^2 \left(\frac{\partial \phi_s}{\partial t}\right)^2 + (W^{\phi_s}\phi_s)^2 \right]^{1/2} d\mathcal{L}^{2n}$$
$$= \int_\omega \sqrt{g(s;A)}\, d\mathcal{L}^{2n}(A).$$

For our purposes, it will be sufficient to show that

$$f''(s) = \frac{1}{4} \int_\omega \frac{2\,g''(s;A)\,g(s;A) - \,g'(s;A)^2}{g(s;A)^{3/2}}\, d\mathcal{L}^{2n}(A) < 0 \tag{4.5}$$

for s belonging to some interval $I \subset \mathbb{R}$. In turn, it is enough to prove that

$$g''(s;A) = 2\big(6g_2(A)s^2 - 6g_2(A)s + g_1(A) + g_2(A)\big) < 0 \quad \forall s \in I, A \in \omega. \tag{4.6}$$

Since $g_2 > 0$ on ω, inequality (4.6) holds for

$$s \in \left(\frac{1}{2}\left(1 - \sqrt{\frac{1}{3} - \frac{2}{3}\frac{g_1(A)}{g_2(A)}}\right), \frac{1}{2}\left(1 + \sqrt{\frac{1}{3} - \frac{2}{3}\frac{g_1(A)}{g_2(A)}}\right) \right). \tag{4.7}$$

The open interval in (4.7) contains $I := \left(\frac{1}{2}\big(1 - \frac{1}{\sqrt{3}}\big), \frac{1}{2}\big(1 - \frac{1}{\sqrt{6}}\big)\right)$ because

$$\frac{g_1}{g_2} \geqslant 0 \qquad \text{and} \qquad \sup_{A \in \omega} \frac{g_1(A)}{g_2(A)} = \sup_{A \in \omega} \left\{ |\hat{\boldsymbol{X}}^*(A)|^2 \frac{(y_1+1)^2}{t^2} \right\} < \frac{1}{4}$$

and (4.5) follows. □

Following Section 3.1, we define the relaxed functional $\overline{\mathscr{A}_{\mathbb{W}}} : L^1(\omega) \to [0, +\infty]$ as

$$\overline{\mathscr{A}_{\mathbb{W}}}(\phi) := \inf \left\{ \liminf_{j \to \infty} \mathscr{A}_{\mathbb{W}}(\phi_j) : \phi_j \in W^{1,1}_{\mathbb{W}}(\omega), \phi_j \to \phi \text{ in } L^1(\omega) \right\}$$

and $I_{\mathbb{W}} : L^1(\omega) \to [0, +\infty]$ as

$$I_{\mathbb{W}}(\phi) := \inf \left\{ \liminf_{j \to \infty} \int_\omega \mathscr{A}_{\mathbb{W}}(\phi_j) : \phi_j \in \mathbf{C}^1(\omega), \phi_j \to \phi \text{ in } L^1(\omega) \right\}.$$

It is known (see [3]) that for any $\phi \in \mathbf{C}^1_{\mathbb{W}}(\omega)$

$$\mathscr{A}_{\mathbb{W}}(\phi) = \overline{\mathscr{A}_{\mathbb{W}}}(\phi) = I_{\mathbb{W}}(\phi) = \int_\omega \sqrt{1 + |\nabla^\phi \phi|^2}\, d\mathcal{L}^{2n} = |\partial E_\phi|_{\mathbb{H}}(\omega \cdot \mathbb{R}).$$

The same equalities hold also for $\phi \in W^{1,1}_{\mathbb{W}}(\omega)$: the only non immediate one is that between $I_{\mathbb{W}}(\phi)$ and the others functionals. However, by definition one has $I_{\mathbb{W}}(\phi) \geqslant \overline{\mathscr{A}_{\mathbb{W}}}(\phi)$; for the reverse inequality it is sufficient to consider the sequence $(\phi_j)_j \subset \mathbf{C}^1(\omega)$ given by (2.11).

We will prove later (see Theorem 4.7) that $BV_{\mathbb{W}}(\omega)$ is the finiteness domain of the functionals $\overline{\mathscr{A}_{\mathbb{W}}}$ and $I_{\mathbb{W}}$ and that $\overline{\mathscr{A}_{\mathbb{W}}}(\phi) = I_{\mathbb{W}}(\phi) = |\partial E_\phi|_{\mathbb{H}}(\omega \cdot \mathbb{R})$ for any $\phi \in BV_{\mathbb{W}}(\omega)$. Taking this into account, some remarks on $BV_{\mathbb{W}}$ are in order. We start by noticing that $\mathbf{C}^1_{\mathbb{W}} \subset W^{1,1}_{\mathbb{W},loc}$ and $W^{1,1}_{\mathbb{W}} \subset BV_{\mathbb{W}}$.

Remark 4.2. The class $BV_{\mathbb{W}}$ is not a vector space. Consider a function $\phi \in \mathbf{C}^1_{\mathbb{W}}(\mathbb{W})$ whose subgraph E_ϕ has not locally finite Euclidean perimeter: such a function can be easily found taking into account the results in [43]. Consider the set $F := E_{\phi+1} = E_\phi \cdot (1, 0, \dots, 0)$. It is not difficult to show that the distributional derivative $Y_1 \chi_F$ satisfies

$$\langle Y_1 \chi_F, \varphi \rangle = \left\langle Y_1 \chi_{E_\phi} - 4T \chi_{E_\phi}\,,\, \varphi \circ R_{(1,0,\dots,0)} \right\rangle$$

for any $\varphi \in \mathbf{C}^1_c(\mathbb{H}^n)$. Here, $\langle \cdot, \cdot \rangle$ denotes the duality between $\mathbf{C}^1_c$ functions and first-order distributions, while $R_{(1,0,\dots,0)}$ is the right translation by the element $(1, 0, \dots, 0)$. Since $T\chi_{E_\phi}$ is not a Radon measure (otherwise E_ϕ would have locally finite Euclidean perimeter), $Y_1 \chi_F$ cannot be a Radon measure, i.e., F has not locally finite $\mathbb{H}$-perimeter. In particular, $\phi + 1 \notin BV_{\mathbb{W}}(\omega)$ for any $\omega \Subset \mathbb{W}$, while $\phi, 1 \in BV_{\mathbb{W}}(\omega)$. This same example allows to conclude that neither $\mathbf{C}^1_{\mathbb{W}}$ nor $W^{1,1}_{\mathbb{W}}$ are vector spaces.

Remark 4.3. The inclusion $BV(\omega) \subset BV_{\mathbb{W}}(\omega)$ does not hold. Consider in fact $\phi(A) := 1/|A|^\alpha$, $n - \frac{1}{2} < \alpha < 2n-1$, defined on the Euclidean unit ball $B(0,1) \subset \mathbb{W} \equiv \mathbb{R}^{2n}$. One has $\phi \in BV(B(0,1))$ while

$$|\partial E_\phi|_{\mathbb{H}}(B(0,1)\cdot\mathbb{R}) \geqslant \int_{B(0,1)} \sqrt{1+|\nabla^\phi\phi|^2}\,d\mathcal{L}^{2n} \geqslant \int_{B(0,1)} |W^\phi\phi|\,d\mathcal{L}^{2n} = +\infty$$

because (see (1.13))

$$W^\phi\phi(A) = \frac{y_1}{|A|^{\alpha+2}} - 4\frac{t}{|A|^{2\alpha+2}}, \quad A = (x_2,\dots,y_n,t) \in B(0,1).$$

In particular, ϕ does not belong to $BV_{\mathbb{W}}(B(0,1))$.

Remark 4.4. The inclusion $BV(\omega)\cap L^\infty(\omega) \subset BV_{\mathbb{W}}(\omega)$ holds for any bounded domain ω. In fact, for each $\phi \in BV(\omega)\cap L^\infty(\omega)$ the reduced boundary $\partial^* E_\phi \cap \omega\cdot\mathbb{R}$ is contained in some bounded subset K of $\omega\cdot\mathbb{R}$. This, together with Proposition 2.10, allows to conclude that $|\partial E_\phi|_{\mathbb{H}} \llcorner \omega\cdot\mathbb{R} \leqslant c|\partial E_\phi| \llcorner \omega\cdot\mathbb{R}$ for some $c = c(K)$. In particular, $\phi \in BV_{\mathbb{W}}(\omega)$.

Remark 4.5. Following Section 3.1, one could introduce the functional $S_{\mathbb{W}} : L^1(\omega) \to [0,\infty]$ defined by

$$S_{\mathbb{W}}(\phi) := \sup\Big\{ \int_\omega \Big[g_1 + \big(\phi\, Y_1 g_{n+1} - 2\phi^2\, T g_{n+1}\big) + \sum_{j=2}^n \phi\big(X_j g_j + Y_j g_{n+j}\big)\Big] d\mathcal{L}^{2n} : \\ g = (g_1,\dots,g_{2n}) \in \mathbf{C}^1_c(\omega,\mathbb{R}^{2n}),\, |g| \leqslant 1 \Big\}$$

if $n \geqslant 2$, and

$$S_{\mathbb{W}}(\phi) := \sup\Big\{ \int_\omega \Big[g_1 + \big(\phi\, Y g_2 - 2\phi^2\, T g_2\big)\Big] d\mathcal{L}^2 : g = (g_1,g_2) \in \mathbf{C}^1_c(\omega,\mathbb{R}^2),\, |g| \leqslant 1 \Big\}.$$

if $n = 1$. It is natural to ask whether $S_{\mathbb{W}}$ coincides with $\overline{\mathscr{A}_{\mathbb{W}}}$ and $I_{\mathbb{W}}$ on $L^1(\omega)$. This occurs in $W^{1,1}_{\mathbb{W}}$: in fact, an integration by parts gives

$$\int_\omega \Big[g_1 + \big(\phi\, Y_1 g_{n+1} - 2\phi^2\, T g_{n+1}\big) + \sum_{j=2}^n \phi\big(X_j g_j + Y_j g_{n+j}\big)\Big] d\mathcal{L}^{2n} = \int_\omega \langle (1, -\nabla^\phi\phi), g\rangle d\mathcal{L}^{2n}$$

for any $g \in \mathbf{C}^1_c(\omega,\mathbb{R}^{2n})$ with $|g| \leqslant 1$. Defining $b \in L^\infty(\omega,\mathbb{R}^{2n})$ by

$$b := \frac{(1,-\nabla^\phi\phi)}{\sqrt{1+|\nabla^\phi\phi|^2}}$$

it will be sufficient to construct a sequence $(g_j)_j \subset \mathbf{C}^1_c(\omega,\mathbb{R}^{2n})$ such that $|g| \leqslant 1$ and $g_j \overset{*}{\rightharpoonup} b$ in $L^\infty(\omega,\mathbb{R}^{2n})$ weak* $\sigma(L^\infty, L^1)$. This can be done with classical tools.

Since $S_{\mathbb{W}}$ is lower semicontinuous and coincides with $\overline{\mathscr{A}_{\mathbb{W}}} = I_{\mathbb{W}}$ on $W^{1,1}_{\mathbb{W}}$, the inequality

$$S_{\mathbb{W}}(\phi) \leqslant \overline{\mathscr{A}_{\mathbb{W}}}(\phi) \tag{4.8}$$

can be easily proved. However, the inequality (4.8) can in general be strict, as the following example shows. Consider $\omega := (0,1)^{2n}$, $n \geqslant 2$ (but this example works also for $n = 1$) and the function

$$\phi(x_2, \dots, y_n, t) := \begin{cases} 1 & \text{if } t \in [\frac{1}{2}, 1) \\ -1 & \text{if } t \in (0, \frac{1}{2})\,. \end{cases} \tag{4.9}$$

The maps ϕ belongs to $BV_{\mathbb{W}}(\omega)$ because of Remark 4.4. We have $Y_1\phi - 2T\phi^2 = 0$ in the sense of distributions and so

$$\begin{aligned} &\int_\omega \Big[g_1 + \big(\phi\, Y_1 g_{n+1} - 2\phi^2\, T g_{n+1}\big) + \sum_{j=2}^n \phi\big(X_j g_j + Y_j g_{n+j}\big)\Big] d\mathcal{L}^{2n} \\ &= \int_\omega \Big[g_1 - 4\sum_{j=2}^n \big(y_j\, g_j(x_2, \dots, y_n, \tfrac{1}{2}) - x_j\, g_j(x_2, \dots, y_n, \tfrac{1}{2})\big)\Big] d\mathcal{L}^{2n}. \end{aligned}$$

By taking the supremum among $g = (g_1, \dots, g_{2n}) \in \mathbf{C}^1_c(\omega, \mathbb{R}^{2n})$, $|g| \leqslant 1$ we obtain

$$S_{\mathbb{W}}(\phi) = \mathcal{L}^{2n}(\omega) + 4\int_{\omega \cap \{t=1/2\}} |\hat{\boldsymbol{X}}^*|\, d\mathcal{L}^{2n-1} \tag{4.10}$$

where we have set

$$\hat{\boldsymbol{X}}^*(x_2, \dots, x_n, y_2, \dots, y_n, t) := (-y_2, \dots, -y_n, x_2, \dots x_n) \in \mathbb{R}^{2n-2}.$$

It is easily seen that E_ϕ is an open set whose piecewise smooth boundary $\partial E_\phi \cap \omega \cdot \mathbb{R}$ can be written, up to $\mathcal{S}^{Q-1}_\infty$-negligible sets, as a (disjoint) union $S_1 \cup S_2 \cup S_3$, where

$$\begin{aligned} S_1 =& \Phi\big((0,1)^{2n-1} \times (\tfrac{1}{2}, 1)\big) \\ S_2 =& \Phi\big((0,1)^{2n-1} \times (0, \tfrac{1}{2})\big) \\ S_3 =& \{(x, y, t) : -1 < x_1 < 1, (x_2, \dots, y_n) \in (0,1)^{2n-1}, t = \tfrac{1}{2} + 2x_1 y_1\}\,. \end{aligned}$$

We have

$$\begin{aligned} c_n \mathcal{S}^{Q-1}_\infty(S_1 \cup S_2) &= |\partial E_\phi|_{\mathbb{H}}\big((\omega \setminus \{t = 1/2\}) \cdot \mathbb{R}\big) \\ &= \int_{\omega \setminus \{t = \frac{1}{2}\}} \sqrt{1 + |\nabla^\phi \phi|^2}\, d\mathcal{L}^{2n} \ = \ \mathcal{L}^{2n}(\omega)\,. \end{aligned} \tag{4.11}$$

In order to compute the measure of S_3, it is convenient to see it as the t-graph of the map $u(x, y) := \frac{1}{2} + 2x_1 y_1$ defined on the open set

$$\mathcal{U} := \{(x, y) \in \mathbb{R}^{2n} : -1 \leqslant x_1 \leqslant 1, (x_2, \dots, y_n) \in (0,1)^{2n-1}\}.$$

In this way

$$
\begin{aligned}
c_n\mathcal{S}^{Q-1}_\infty(S_3) &= \int_{\mathcal{U}} |\nabla u + \boldsymbol{X}^*|\, d\mathcal{L}^{2n}\\
&= \int_{\mathcal{U}} 2\big|(0,-y_2,\dots,-y_n,2x_1,x_2,\dots,x_n)\big|\, d\mathcal{L}^{2n}\\
&> \int_{(0,1)^{2n-1}}\int_{-1}^{1} 2\big|(0,-y_2,\dots,-y_n,0,x_2,\dots,x_n)\big|\, dx_1\, d\mathcal{L}^{2n-1}(x_2,\dots,y_n)\\
&= 4\int_{(0,1)^{2n-1}} \big|(-y_2,\dots,-y_n,x_2,\dots,x_n)\big|\, d\mathcal{L}^{2n-1}\\
&= 4\int_{\omega\cap\{t=1/2\}} |\hat{\boldsymbol{X}}^*|\, d\mathcal{L}^{2n-1}.
\end{aligned}
\tag{4.12}
$$

Using (4.10), (4.11) and (4.12) we finally obtain

$$|\partial E_\phi|_{\mathbb{H}}(\omega\cdot\mathbb{R}) = c_n\mathcal{S}^{Q-1}_\infty(S_1\cup S_2) + c_n\mathcal{S}^{Q-1}_\infty(S_3) > S_{\mathbb{W}}(\phi)$$

as claimed.

The following lemma will be used in the proof of Theorem 4.7; its statement is perfectly analogous to that of Lemma 3.13 for t-graphs.

Lemma 4.6. *Let $\omega\subset\mathbb{W}$ be a bounded open set and let $F\subset\omega\cdot\mathbb{R}$ be a measurable set. Assume that*

$$|F\,\Delta\,(\omega\cdot\mathbb{R}^-)| < \infty \tag{4.13}$$

and, for any positive real number k, set $F_k := \big(F\cup(\omega\cdot(-\infty,-k))\big)\setminus\big(\omega\cdot[k,+\infty)\big)$. Then there exists a sequence $(k_j)_j$ such that $k_j\to+\infty$ and

$$\lim_{j\to+\infty} |\partial F_{k_j}|_{\mathbb{H}}(\omega\cdot\mathbb{R}) = |\partial F|_{\mathbb{H}}(\omega\cdot\mathbb{R})\,. \tag{4.14}$$

Proof. See Remark 3.14. □

We can now prove one of the main results of this Section.

Theorem 4.7. *Let $\phi\in L^1(\omega)$. Then*

$$|\partial E_\phi|_{\mathbb{H}}(\omega\cdot\mathbb{R}) = \overline{\mathscr{A}_{\mathbb{W}}}(\phi) = I_{\mathbb{W}}(\phi)\,. \tag{4.15}$$

Proof. The equality between $\overline{\mathscr{A}_{\mathbb{W}}}$ and $I_{\mathbb{W}}$ is rather immediate. We have by definition $\overline{\mathscr{A}_{\mathbb{W}}}(\phi)\leqslant I_{\mathbb{W}}(\phi)$; the reverse inequality follows by considering a sequence $(\phi_j)_j\subset W^{1,1}_{\mathbb{W}}(\omega)$ such that

$$\phi_j\to\phi \text{ in } L^1(\omega) \quad\text{and}\quad \mathscr{A}_{\mathbb{W}}(\phi_j)\to\overline{\mathscr{A}_{\mathbb{W}}}(\phi)$$

and another sequence $(\psi_j)_j\subset\mathbf{C}^1(\omega)$ such that

$$\|\phi_j-\psi_j\|_{L^1} < 1/j \quad\text{and}\quad |\mathscr{A}_{\mathbb{W}}(\phi_j)-\mathscr{A}_{\mathbb{W}}(\psi_j)| < 1/j\,.$$

This gives $I_{\mathbb{W}}(\phi)\leqslant\lim_{j\to+\infty}\mathscr{A}_{\mathbb{W}}(\phi_j)=\overline{\mathscr{A}_{\mathbb{W}}}(\phi)$.

Also the proof of the inequality $|\partial E_\phi|_{\mathbb{H}}(\omega\cdot\mathbb{R})\leqslant I_{\mathbb{W}}(\phi)$ is not difficult. We know that there exists a sequence $(\phi_j)_j\subset\mathbf{C}^1(\omega)$ such that $\phi_j\to\phi$ in $L^1(\omega)$ and

$$I_{\mathbb{W}}(\phi) = \lim_{j\to\infty}\int_\omega\sqrt{1+|\nabla^{\phi_j}\phi_j|^2} = \lim_{j\to\infty}|\partial E_{\phi_j}|_{\mathbb{H}}(\omega\cdot\mathbb{R})\,. \tag{4.16}$$

By (2.7) and the fact that $\phi_j \to \phi$ in $L^1(\omega)$ we deduce that $E_{\phi_j} \to E_\phi$ in $L^1(\omega \cdot \mathbb{R})$. From the semicontinuity of the perimeter

$$|\partial E_\phi|_{\mathbb{H}}(\omega \cdot \mathbb{R}) \leqslant \liminf_{j\to\infty} |\partial E_{\phi_j}|_{\mathbb{H}}(\omega \cdot \mathbb{R})$$

which, together with (4.16), gives the desired inequality.

We divide the proof of the remaining inequality $|\partial E_\phi|_{\mathbb{H}}(\omega \cdot \mathbb{R}) \geqslant I_{\mathbb{W}}(\phi)$ into several steps.

Step 0. We claim that it is sufficient to prove the inequality $|\partial E_\phi|_{\mathbb{H}}(\omega \cdot \mathbb{R}) \geqslant I_{\mathbb{W}}(\phi)$ for ϕ bounded. Once this has been done, by Lemma 4.6 we would get for a generic ϕ

$$|\partial E_\phi|_{\mathbb{H}}(\omega \cdot \mathbb{R}) = \lim_{j\to\infty} |\partial (E_\phi)_{k_j}|_{\mathbb{H}}(\omega \cdot \mathbb{R})$$

where $k_j \to \infty$ is the sequence given as in Lemma 4.6 (with $F = E_\phi$) and

$$(E_\phi)_{k_j} := \big(E_\phi \cup (\omega \cdot (-\infty, -k_j))\big) \setminus \big(\omega \cdot [k_j, +\infty)\big)\,.$$

It is easily seen that $(E_\phi)_{k_j} = E_{\phi_j}$ where $\phi_j := \max\{\min\{\phi, k_j\}, -k_j\}$ and thus we would have

$$|\partial E_\phi|_{\mathbb{H}}(\omega \cdot \mathbb{R}) = \lim_{j\to\infty} I_{\mathbb{W}}(\phi_{k_j}) \geqslant I_{\mathbb{W}}(\phi),$$

as desired. We also used the lower semicontinuity of $I_{\mathbb{W}}$ and the fact that $\phi_j \to \phi$ in $L^1(\omega)$.

Step 1. Without loss of generality we may assume that ϕ is bounded and $|\partial E_\phi|_{\mathbb{H}}(\omega \cdot \mathbb{R}) < +\infty$. Let us fix a smooth function $h : \mathbb{R} \to (0,1)$ such that

$$h' > 0, \quad \lim_{x\to-\infty} h(x) = 0, \quad \lim_{x\to+\infty} h(x) = 1 \quad \text{and} \quad \int_{-\infty}^0 h(x)dx < \infty\,,$$

and set $\tilde h(x,y,t) := h(x_1)$. For $\delta > 0$ consider $\chi_\delta := (1 - \delta \tilde h)\chi_{E_\phi}$. We are going to prove that

$$\chi_\delta - \chi_{E_\phi} \to 0 \text{ in } L^1(\omega \cdot \mathbb{R}) \quad \text{and} \quad |D_{\mathbb{H}}\chi_\delta|(\omega \cdot \mathbb{R}) \to |\partial E_\phi|_{\mathbb{H}}(\omega \cdot \mathbb{R}) \tag{4.17}$$

as $\delta \to 0$. One has

$$\begin{aligned}
\|\chi_{E_\phi} - \chi_\delta\|_{L^1} &= \delta \int_{\omega\cdot\mathbb{R}} \tilde h\, \chi_{E_\phi}\, d\mathcal{L}^{2n+1} \\
&\leqslant \delta \int_{\omega\cdot\mathbb{R}^-} \tilde h\, d\mathcal{L}^{2n+1} + \delta \int_{\omega\cdot\mathbb{R}^+} |\chi_{E_\phi}|\, d\mathcal{L}^{2n+1} \\
&= \delta \mathcal{L}^{2n}(\omega) \int_{-\infty}^0 h(t)\, dt + \delta \int_{\omega\cap\{\phi>0\}} |\phi|\, d\mathcal{L}^{2n} = O(\delta)\,.
\end{aligned}$$

Moreover, for any $\varphi \in \mathbf{C}^1_c(\omega\cdot\mathbb{R},\mathbb{R}^{2n})$ with $|\varphi|\leqslant 1$ one has

$$
\begin{aligned}
\left|\int_{\omega\cdot\mathbb{R}}(\chi_{E_\phi}-\chi_\delta)\mathrm{div}_{\mathbb{H}}\varphi\, d\mathcal{L}^{2n+1}\right| \;&=\; \left|\int_{\omega\cdot\mathbb{R}}\delta\tilde h\chi_{E_\phi}\mathrm{div}_{\mathbb{H}}\varphi\, d\mathcal{L}^{2n+1}\right|\\
&= \delta\left|\int_{\omega\cdot\mathbb{R}}\chi_{E_\phi}\mathrm{div}_{\mathbb{H}}(\tilde h\varphi)\, d\mathcal{L}^{2n+1}-\int_{\omega\cdot\mathbb{R}}\chi_{E_\phi}\langle\nabla_{\mathbb{H}}\tilde h,\varphi\rangle\, d\mathcal{L}^{2n+1}\right|\\
&\leqslant \delta|\partial E_\phi|_{\mathbb{H}}(\omega\cdot\mathbb{R})+\delta\left|\int_{\omega\cdot\mathbb{R}}\chi_{E_\phi}(x,y,t)\varphi_1(x,y,t)h'(x_1)\,dx\,dy\,dt\right|\\
&\leqslant \delta|\partial E_\phi|_{\mathbb{H}}(\omega\cdot\mathbb{R})+\delta\int_{\omega\cdot\mathbb{R}}|h'(x_1)|\,dx\,dy\,dt\\
&= \delta|\partial E_\phi|_{\mathbb{H}}(\omega\cdot\mathbb{R})+\delta\mathcal{L}^{2n}(\omega)\,.
\end{aligned}\tag{4.18}
$$

This proves that $\chi_\delta-\chi_{E_\phi}\to 0$ in $BV_{\mathbb{H}}(\omega\cdot\mathbb{R})$ as $\delta\to 0$ and, in particular, that (4.17) holds.

We also observe the following. If $M>0$ is such that $|\phi|\leqslant M$ on ω, then

$$\omega\cdot(-\infty,-M)\subset E_\phi\quad\text{and}\quad E_\phi\cap\omega\cdot(M,+\infty)=\emptyset\,,$$

whence

$$|\partial E_\phi|_{\mathbb{H}}(\omega\cdot(-\infty,-M))=|\partial E_\phi|_{\mathbb{H}}(\omega\cdot(M,+\infty))=0\,.\tag{4.19}$$

By (4.18), on taking the supremum over $\varphi\in\mathbf{C}^1_c(\omega\cdot(-\infty,-M),\mathbb{R}^{2n})$ (resp. on $\varphi\in\mathbf{C}^1_c(\omega\cdot(M,+\infty),\mathbb{R}^{2n})$) we get

$$
\begin{aligned}
&|D_{\mathbb{H}}\chi_\delta|(\omega\cdot(-\infty,-M))<\delta\big(|\partial E_\phi|_{\mathbb{H}}(\omega\cdot\mathbb{R})+\mathcal{L}^{2n}(\omega)\big)\quad\text{and}\\
&|D_{\mathbb{H}}\chi_\delta|(\omega\cdot(M,+\infty))<\delta\big(|\partial E_\phi|_{\mathbb{H}}(\omega\cdot\mathbb{R})+\mathcal{L}^{2n}(\omega)\big).
\end{aligned}\tag{4.20}
$$

Step 2. From now on we fix $\epsilon>0$ and $\delta=\delta(\epsilon)\leqslant 1/2$ such that

$$\|\chi_{E_\phi}-\chi_\delta\|_{L^1}<\epsilon\qquad\text{and}\qquad|D_{\mathbb{H}}(\chi_{E_\phi}-\chi_\delta)|(\omega\cdot\mathbb{R})<\epsilon\,;\tag{4.21}$$

Thanks to (4.20) we may also require that

$$|D_{\mathbb{H}}\chi_\delta|(\omega\cdot(-\infty,-M))<\epsilon\quad\text{and}\quad|D_{\mathbb{H}}\chi_\delta|(\omega\cdot(M,+\infty))<\epsilon.\tag{4.22}$$

Fix open sets ω_j, $j=0,1,2,\dots$ such that $\omega_j\Subset\omega_{j+1}$ and $\omega_j\uparrow\omega$; we may suppose that

$$|D_{\mathbb{H}}\chi_\delta|\big((\omega\setminus\omega_5)\cdot\mathbb{R}\big)<\epsilon\,.\tag{4.23}$$

For $j\in\mathbb{N}$ define $\mathcal{A}_0=\omega_5$ and $\mathcal{A}_j:=\omega_{j+5}\setminus\overline{\omega_j}$ if $j\geqslant 1$. We fix a partition of the unity $(\psi_j)_j$ subordinate to $(\mathcal{A}_j)_j$, i.e., a family of functions such that

$$\psi_j\in\mathbf{C}^\infty_c(\mathcal{A}_j),\quad 0\leqslant\psi_j\leqslant 1,\quad \sum_{j=0}^\infty\psi_j=1\quad\text{on }\omega\,.$$

Without loss of generality, we may construct ψ_j so that

$$\psi_0\geqslant 1/5\text{ on }\omega_4\qquad\text{and}\qquad\psi_j\geqslant 1/5\text{ on }\omega_{j+4}\setminus\overline{\omega_{j+1}},\ j=1,2,\dots\tag{4.24}$$

Indeed, it is enough to consider $\zeta_j\in\mathbf{C}^\infty_c(\mathcal{A}_j,[0,1])$ such that $\zeta_j=1$ on $\omega_{j+4}\setminus\overline{\omega_{j+1}}$, and to set $\zeta:=\sum_j\zeta_j$. Since $1\leqslant\zeta\leqslant 5$ on ω, the functions $\psi_j:=\zeta_j/\zeta$ provide a partition of the unity satisfying (4.24).

Finally, define a partition of the unity on $\omega \cdot \mathbb{R}$ by setting

$$\overline{\psi}_j(x,y,t) := \psi_j(\pi_{\mathbb{W}}(x,y,t)) = \psi_j(y, t - 2x_1y_1)\,,$$

so that $\overline{\psi}_j(A \cdot s) = \psi_j(A)$ for any $A \in \omega$, $s \in \mathbb{R}$.

Fix a smooth mollifier $\varrho \in \mathbf{C}_c^\infty(U_1)$ satisfying (2.24). Inspired by the classical proof by Anzellotti-Giaquinta (see [39, Theorem 1.17]) we fix numbers $\alpha_j > 0$ and define the functions

$$\begin{aligned} u_{\epsilon,j}(P) := \varrho_{\alpha_j} \star f_{\delta,j}(P) &= \int_{\mathbb{H}^n} \varrho_{\alpha_j}(P \cdot Q^{-1}) f_{\delta,j}(Q)\, d\mathcal{L}^{2n+1}(Q) \\ &= \int_{\mathbb{H}^n} \varrho_{\alpha_j}(Q) f_{\delta,j}(Q^{-1} \cdot P)\, d\mathcal{L}^{2n+1}(Q)\,. \end{aligned}$$

where $f_{\delta,j} : \mathbb{H}^n \to \mathbb{R}$ is the function defined by $f_{\delta,j} := \chi_\delta\, \overline{\psi}_j$ on $\omega \cdot (-4M, 4M)$ and $f_{\delta,j} \equiv 0$ outside. Several conditions are to be imposed on the α_j's: they will be listed along the proof, in order to clarify the role played by each of them. Observe that $f_{\delta,j} \in L^\infty(\mathbb{H}^n) \cap L^1(\mathbb{H}^n)$ and

$$\operatorname{spt} f_{\delta,j} \subset (\operatorname{spt} \psi_j) \cdot [-4M, M] \Subset \omega \cdot (-\infty, -2M).$$

If $\alpha_j > 0$ is so small that $U_{\alpha_j} \cdot P \subset \omega \cdot (-\infty, 2M)$ for each $P \in \operatorname{spt} f_{\delta,j}$, then $u_{\epsilon,j} \in \mathbf{C}^\infty(\mathbb{H}^n)$ and, by (2.25),

$$\operatorname{spt} u_{\epsilon,j} \subset U_{\alpha_j} \cdot \operatorname{spt} f_{\delta,j} \subset \omega \cdot (-\infty, 2M)\,. \tag{4.25}$$

The implication

$$\begin{aligned} &Q \in (\omega_{j+4} \setminus \overline{\omega_{j+1}}) \cdot (-\infty, -M), j \geqslant 1 \\ &\implies \chi_\delta(Q)\overline{\psi}_j(Q) = (1 - \delta\tilde{h}(Q))\overline{\psi}_j(Q) \geqslant \frac{1-\delta}{5} \geqslant \frac{1}{10} \end{aligned}$$

holds by (4.24). Similarly, we have

$$Q \in \omega_4 \cdot (-\infty, -M) \implies \chi_\delta(Q)\overline{\psi}_0(Q) \geqslant \frac{1}{10}.$$

This implies that, for sufficiently small α_j, one has

$$\begin{aligned} u_{\epsilon,j}(P) &\geqslant \frac{1}{10} \qquad \forall P \in \left(\omega_{j+3} \setminus \overline{\omega_{j+2}}\right) \cdot (-3M, -2M],\ j = 1, 2, \ldots \\ u_{\epsilon,0}(P) &\geqslant \frac{1}{10} \qquad \forall P \in \overline{\omega_3} \cdot (-3M, -2M] \end{aligned} \tag{4.26}$$

On the other hand, by (4.25),

$$u_{\epsilon,j}(P) = 0 \qquad \forall P \in \omega \cdot [2M, 3M), \forall j \in \mathbb{N}\,. \tag{4.27}$$

By the uniform continuity of $\overline{\psi}_j$ we may assume that

$$f_{\delta,j}(Q^{-1} \cdot P) \leqslant \overline{\psi}_j(Q^{-1} \cdot P) \leqslant \overline{\psi}_j(P) + \epsilon 2^{-j-1} \qquad \forall P \in \mathcal{A}_j \cdot (-3M, 3M), \forall Q \in U(0, \alpha_j)$$

whence

$$u_{\epsilon,j}(P) \leqslant \overline{\psi}_j(P) + \epsilon 2^{-j-1} \qquad \forall P \in \omega \cdot (-3M, 3M). \tag{4.28}$$

Let us define $u_\epsilon \in \mathbf{C}^\infty(\omega\cdot(-3M,3M))$ by $u_\epsilon := \sum_{j=0}^\infty u_{\epsilon,j}$. One has $0 \leqslant u_\epsilon \leqslant 1+\epsilon$ because of (4.28) and the fact that $\sum_{j=0}^\infty \overline{\psi}_j = 1$ on $\omega\cdot\mathbb{R}$. Moreover, by (4.26) and (4.27)

$$u_\epsilon \geqslant \frac{1}{10} \text{ on } \omega\cdot(-3M,-2M], \quad u_\epsilon = 0 \text{ on } \omega\cdot[2M,3M)\,. \tag{4.29}$$

If α_j is chosen so that

$$\|u_{\epsilon,j} - \chi_\delta\overline{\psi}_j\|_{L^1(\mathcal{A}_j\cdot(-3M,3M))} < \epsilon 2^{-j}$$

then

$$\|u_\epsilon - \chi_\delta\|_{L^1(\omega\cdot(-3M,3M))} = \left\| \sum_{j=1}^\infty (u_{\epsilon,j} - \chi_\delta\overline{\psi}_j) \right\|_{L^1(\omega\cdot(-3M,3M))} < \epsilon \tag{4.30}$$

We claim that

$$\int_{\omega\cdot(-2M,2M)} |\nabla_{\mathbb{H}} u_\epsilon|\, d\mathcal{L}^{2n+1} \leqslant |D_{\mathbb{H}}\chi_\delta|(\omega\cdot(-2M,2M)) + 9\epsilon\,. \tag{4.31}$$

By (2.26), for any $\varphi \in \mathbf{C}^1_c(\omega\cdot(-2M,2M))$ with $|\varphi| \leqslant 1$ we have

$$\begin{aligned}
\int_{\omega\cdot(-2M,2M)} u_\epsilon \mathrm{div}_{\mathbb{H}}\varphi\, d\mathcal{L}^{2n+1} &= \sum_{j=0}^\infty \int_{\omega\cdot(-2M,2M)} \big(\varrho_{\alpha_j} \star f_{\delta,j}\big)\mathrm{div}_{\mathbb{H}}\varphi\, d\mathcal{L}^{2n+1} \\
&= \sum_{j=0}^\infty \int_{\mathbb{H}^n} \big(\varrho_{\alpha_j} \star f_{\delta,j}\big)\mathrm{div}_{\mathbb{H}}\varphi\, d\mathcal{L}^{2n+1} \\
&= \sum_{j=0}^\infty \int_{\mathbb{H}^n} f_{\delta,j}\big(\varrho_{\alpha_j} \star (\mathrm{div}_{\mathbb{H}}\varphi)\big)\, d\mathcal{L}^{2n+1}.
\end{aligned}$$

If we choose $\alpha_j \leqslant M$ for any j we have

$$\mathrm{spt}\, f_{\delta,j}\big(\varrho_{\alpha_j} \star (\mathrm{div}_{\mathbb{H}}\varphi)\big) \subset \mathrm{spt}\, f_{\delta,j} \,\cap\, U_{\alpha_j}\cdot \mathrm{spt}\,\varphi \subset \mathcal{A}_j\cdot(-3M,3M)$$

and thus, setting $\Omega := \omega\cdot(-3M,3M)$,

$$\int_{\omega\cdot(-2M,2M)} u_\epsilon \mathrm{div}_{\mathbb{H}}\varphi\, d\mathcal{L}^{2n+1} = \sum_{j=0}^\infty \int_\Omega \chi_\delta\overline{\psi}_j \mathrm{div}_{\mathbb{H}}(\varrho_{\alpha_j} \star \varphi)\, d\mathcal{L}^{2n+1}\,.$$

In particular

$$
\begin{aligned}
&\int_{\omega\cdot(-2M,2M)} u_\epsilon \operatorname{div}_{\mathbb{H}}\varphi \, d\mathcal{L}^{2n+1} \\
&= \sum_{j=0}^{\infty}\int_\Omega \chi_\delta \operatorname{div}_{\mathbb{H}}(\overline{\psi}_j(\varrho_{\alpha_j}\star\varphi))\, d\mathcal{L}^{2n+1} - \sum_{j=0}^{\infty}\int_\Omega \big\langle \chi_\delta \nabla_{\mathbb{H}}\overline{\psi}_j,\ \varrho_{\alpha_j}\star\varphi\big\rangle \, d\mathcal{L}^{2n+1} \\
&= \sum_{j=0}^{\infty}\int_\Omega \chi_\delta \operatorname{div}_{\mathbb{H}}(\overline{\psi}_j(\varrho_{\alpha_j}\star\varphi))\, d\mathcal{L}^{2n+1} - \sum_{j=0}^{\infty}\int_\Omega \big\langle \varrho_{\alpha_j}\star(\chi_\delta \nabla_{\mathbb{H}}\overline{\psi}_j),\ \varphi\big\rangle \, d\mathcal{L}^{2n+1} \\
&= \sum_{j=0}^{\infty}\int_\Omega \chi_\delta \operatorname{div}_{\mathbb{H}}(\overline{\psi}_j(\varrho_{\alpha_j}\star\varphi))\, d\mathcal{L}^{2n+1} \\
&\quad - \sum_{j=0}^{\infty}\int_\Omega \big\langle \varrho_{\alpha_j}\star(\chi_\delta \nabla_{\mathbb{H}}\overline{\psi}_j) - (\chi_\delta \nabla_{\mathbb{H}}\overline{\psi}_j),\ \varphi\big\rangle \, d\mathcal{L}^{2n+1}
\end{aligned}
\tag{4.32}
$$

where we used the equality $\sum_j \nabla_{\mathbb{H}}\overline{\psi}_j = 0$ on $\omega\cdot\mathbb{R}$. If α_j is sufficiently small, we have

$$
\|\varrho_{\alpha_j}\star(\chi_\delta\nabla_{\mathbb{H}}\overline{\psi}_j) - \chi_\delta\nabla_{\mathbb{H}}\overline{\psi}_j\|_{L^1(\Omega)} < \epsilon 2^{-j}
$$

whence

$$
\left|\sum_{j=0}^{\infty}\int_\Omega \big\langle \varrho_{\alpha_j}\star(\chi_\delta \nabla_{\mathbb{H}}\overline{\psi}_j) - (\chi_\delta \nabla_{\mathbb{H}}\overline{\psi}_j),\ \varphi\big\rangle \, d\mathcal{L}^{2n+1}\right| < \epsilon. \tag{4.33}
$$

It is a good point to notice that the choice of the α_j's is independent of the particular function φ. Since $\overline{\psi}_j(\varrho_{\alpha_j}\star\varphi) \in \mathbf{C}^1_c(\mathcal{A}_j\cdot(-3M,3M))$, $|\overline{\psi}_j(\varrho_{\alpha_j}\star\varphi)| \leqslant 1$ and $\mathcal{A}_j\cdot(-3M,3M)\subset\Omega$, one has

$$
\begin{aligned}
&\left|\sum_{j=1}^{\infty}\int_\Omega \chi_\delta \operatorname{div}_{\mathbb{H}}(\overline{\psi}_j(\varrho_{\alpha_j}\star\varphi))\, d\mathcal{L}^{2n+1}\right| \\
&\leqslant \sum_{j=1}^{\infty} |D_{\mathbb{H}}\chi_\delta|(\mathcal{A}_j\cdot\mathbb{R}) \leqslant 6|D_{\mathbb{H}}\chi_\delta|\big((\omega\setminus\omega_5)\cdot\mathbb{R}\big) \leqslant 6\epsilon
\end{aligned}
\tag{4.34}
$$

where we used (4.23) and the fact that the intersection of more than any six of the sets $\mathcal{A}_j$'s is empty. By (4.32), (4.33) and (4.34) we obtain

$$
\int_{\omega\cdot(-2M,2M)} u_\epsilon \operatorname{div}_{\mathbb{H}}\varphi \, d\mathcal{L}^{2n+1} \leqslant \int_\Omega \chi_\delta \operatorname{div}_{\mathbb{H}}(\overline{\psi}_0(\varrho_{\alpha_0}\star\varphi))\, d\mathcal{L}^{2n+1} + 7\epsilon. \tag{4.35}
$$

Since $\operatorname{spt}\varphi\subset\mathbb{W}\cdot(-2M,2M)$ and $\alpha_0\leqslant M$ we have $\operatorname{spt}(\varrho_{\alpha_0}\star\varphi)\subset\mathbb{W}\cdot(-3M,3M)$, thus $\operatorname{spt}\overline{\psi}_0(\varrho_{\alpha_0}\star\varphi)\subset\omega\cdot(-3M,3M)=\Omega$. In particular, (4.35) becomes

$$
\begin{aligned}
\int_{\omega\cdot(-2M,2M)} u_\epsilon \operatorname{div}_{\mathbb{H}}\varphi \, d\mathcal{L}^{2n+1} &\leqslant |D_{\mathbb{H}}\chi_\delta|(\omega\cdot(-3M,3M)) + 7\epsilon \\
&\leqslant |D_{\mathbb{H}}\chi_\delta|(\omega\cdot(-2M,2M)) + 9\epsilon
\end{aligned}
$$

i.e., (4.31); in the last inequality we used assumption (4.22).

Step 3. For any $P, Q \in \omega \cdot \mathbb{R}$ and $s > 0$ one has

$$\chi_{E_\phi}(Q^{-1} \cdot P \cdot s) \leqslant \chi_{E_\phi}(Q^{-1} \cdot P), \text{ whence } \chi_\delta(Q^{-1} \cdot P \cdot s) \leqslant \chi_\delta(Q^{-1} \cdot P),$$
$$\overline{\psi}_j(Q^{-1} \cdot P \cdot s) = \overline{\psi}_j(Q^{-1} \cdot P) \geqslant 0\,.$$

This implies that $u_{\epsilon,j}(P \cdot s) \leqslant u_{\epsilon,j}(P)$ for any $P \in \omega \cdot (-2M, 2M)$ and $s > 0$ such that $P \cdot s \in \omega \cdot (-2M, 2M)$; in particular it is $u_\epsilon(P \cdot s) \leqslant u_\epsilon(P)$ and

$$X_1 u_\epsilon(P) = \lim_{s \to 0^+} \frac{u_\epsilon(P \cdot s) - u_\epsilon(P)}{s} \leqslant 0 \qquad \forall P \in \omega \cdot (-2M, 2M)\,. \tag{4.36}$$

The monotonicity in (4.36) can be improved: we claim in fact that the implication

$$P \in \omega \cdot (-2M, 2M),\ u_\epsilon(P) > 0 \Longrightarrow X_1 u_\epsilon(P) < 0 \tag{4.37}$$

holds. By definition, if $u_\epsilon(P) > 0$ there exists an index k such that

$$u_{\epsilon,k}(P) = \int_{U(0,\alpha_k)} \varrho_{\alpha_k}(Q) \chi_\delta(Q^{-1} \cdot P) \overline{\psi}_k(Q^{-1} \cdot P)\, d\mathcal{L}^{2n+1}(Q) > 0.$$

Let $c = c(P) > 0$ be such that

$$X_1 \tilde{h}(Q^{-1} \cdot P \cdot s) \geqslant c \qquad \text{for any } Q \in U(0, \alpha_k) \text{ and any } s \in [0, 1].$$

For any $s \in [0, 1]$ one has the implication

$$Q^{-1} \cdot P \notin E_\phi \ \Rightarrow\ \chi_\delta(Q^{-1} \cdot P \cdot s) = \chi_\delta(Q^{-1} \cdot P) = 0$$

while

$$\begin{aligned} Q^{-1} \cdot P \in E_\phi \ \Longrightarrow\ \chi_\delta(Q^{-1} \cdot P \cdot s) &= \chi_{E_\phi}(Q^{-1} \cdot P \cdot s)\big(1 - \delta \tilde{h}(Q^{-1} \cdot P \cdot s)\big) \\ &\leqslant \chi_{E_\phi}(Q^{-1} \cdot P)\big(1 - \delta\big(\tilde{h}(Q^{-1} \cdot P) + cs\big)\big) \\ &= \chi_\delta(Q^{-1} \cdot P) - \delta cs \chi_{E_\phi}(Q^{-1} \cdot P) \\ &\leqslant (1 - \delta cs) \chi_\delta(Q^{-1} \cdot P). \end{aligned}$$

We then have

$$\begin{aligned} u_\epsilon(P \cdot s) &= \sum_{j \neq k} u_{\epsilon,j}(P \cdot s) + \int_{U(0,\alpha_k)} \varrho_{\alpha_k}(Q) \chi_\delta(Q^{-1} \cdot P \cdot s) \overline{\psi}_k(Q^{-1} \cdot P \cdot s)\, d\mathcal{L}^{2n+1}(Q) \\ &\leqslant \sum_{j \neq k} u_{\epsilon,j}(P) + \int_{U(0,\alpha_k)} (1 - \delta cs) \varrho_{\alpha_k}(Q) \chi_\delta(Q^{-1} \cdot P) \overline{\psi}_k(Q^{-1} \cdot P)\, d\mathcal{L}^{2n+1}(Q) \\ &= u_\epsilon(P) - \delta\, c\, s\, u_{\epsilon,k}(P) \end{aligned}$$

whence $X_1 u_\epsilon(P) \leqslant -\delta\, c\, u_{\epsilon,k}(P) < 0$.

Step 4. By (4.21) and (4.30) one has

$$\|u_\epsilon - \chi_{E_\phi}\|_{L^1(\omega \cdot (-2M, 2M))} < 2\epsilon;$$

this inequality yields $\|u_\epsilon\|_{L^1(\omega \cdot (-2M,2M))} \leqslant C$ for small ϵ. Set $v_\epsilon := u_\epsilon/(1 + \epsilon)$; then $v_\epsilon \in \mathbf{C}^\infty_c(\omega \cdot (-3M, 3M))$ and $0 \leqslant v_\epsilon \leqslant 1$. By Minkovski inequality

$$\|v_\epsilon - \chi_{E_\phi}\|_{L^1(\omega \cdot (-2M,2M))} \leqslant \|v_\epsilon - u_\epsilon\|_{L^1} + \|u_\epsilon - \chi_{E_\phi}\|_{L^1} < (C + 2)\epsilon\,; \tag{4.38}$$

in particular

$$|\partial E_\phi|_{\mathbb{H}}(\omega \cdot (-2M, 2M)) \leqslant \liminf_{\epsilon \to 0} \int_{\omega \cdot (-2M,2M)} |\nabla_{\mathbb{H}} v_\epsilon|\, d\mathcal{L}^{2n+1}\,. \tag{4.39}$$

From (4.19) and (4.21) it follows that

$$|D_{\mathbb{H}}\chi_\delta|(\omega\cdot(-2M,2M)) \leqslant |\partial E_\phi|_{\mathbb{H}}(\omega\cdot(-2M,2M)) + \epsilon$$

which, together with (4.31), gives

$$\int_{\omega\cdot(-2M,2M)} |\nabla_{\mathbb{H}} u_\epsilon|\, d\mathcal{L}^{2n+1} \leqslant |\partial E_\phi|_{\mathbb{H}}(\omega\cdot(-2M,2M)) + 10\epsilon.$$

Therefore

$$\begin{aligned}\int_{\omega\cdot(-2M,2M)} |\nabla_{\mathbb{H}} v_\epsilon|\, d\mathcal{L}^{2n+1} &\leqslant \int_{\omega\cdot(-2M,2M)} |\nabla_{\mathbb{H}} u_\epsilon|\, d\mathcal{L}^{2n+1}\\ &\leqslant |\partial E_\phi|_{\mathbb{H}}(\omega\cdot(-2M,2M)) + 10\epsilon\end{aligned}$$

and recalling (4.39)

$$\lim_{\epsilon\to 0}\int_{\omega\cdot(-2M,2M)} |\nabla_{\mathbb{H}} v_\epsilon|\, d\mathcal{L}^{2n+1} = |\partial E_\phi|_{\mathbb{H}}(\omega\cdot(-2M,2M))\,. \tag{4.40}$$

For any $c \in (0,1)$ define

$$E_{\epsilon,c} := \{v_\epsilon > c\}\cap(\omega\cdot(-2M,2M)) \quad \text{and} \quad E_\phi^{2M} := E_\phi \cap \omega\cdot(-2M,2M);$$

we have

$$v_\epsilon - \chi_{E_\phi} > c \text{ on } E_{\epsilon,c}\setminus E_\phi^{2M}, \qquad \chi_{E_\phi} - v_\epsilon \geqslant 1-c \text{ on } E_\phi^{2M}\setminus E_{\epsilon,c}.$$

From (4.38)

$$\begin{aligned}(C+2)\epsilon &> \int_{\omega\cdot(-2M,2M)} |v_\epsilon - \chi_{E_\phi}|\, d\mathcal{L}^{2n+1}\\ &\geqslant c\,|E_{\epsilon,c}\setminus E_\phi^{2M}| + (1-c)|E_\phi^{2M}\setminus E_{\epsilon,c}|\\ &\geqslant \min\{c,1-c\}|E_\phi^{2M}\Delta E_{\epsilon,c}|\,.\end{aligned}$$

In other words, for any $c\in(0,1)$ one has $E_{\epsilon,c}\to E_\phi$ in $L^1(\omega\cdot(-2M,2M))$ as $\epsilon\to 0$. In particular

$$|\partial E_\phi|_{\mathbb{H}}(\omega\cdot(-2M,2M)) \leqslant \liminf_{\epsilon\to 0} |\partial E_{\epsilon,c}|_{\mathbb{H}}(\omega\cdot(-2M,2M))\,. \tag{4.41}$$

By Fatou's Lemma, the coarea formula of [29, Theorem 2.3.5] and (4.39)

$$\begin{aligned}|\partial E_\phi|_{\mathbb{H}}(\omega\cdot(-2M,2M)) &\leqslant \int_0^1 \liminf_{\epsilon\to 0} |\partial E_{\epsilon,c}|_{\mathbb{H}}(\omega\cdot(-2M,2M))\, dc\\ &\leqslant \liminf_{\epsilon\to 0}\int_0^1 |\partial E_{\epsilon,c}|_{\mathbb{H}}(\omega\cdot(-2M,2M))\, dc\\ &= \liminf_{\epsilon\to 0}\int_{\omega\cdot(-2M,2M)} |\nabla_{\mathbb{H}} v_\epsilon|\, d\mathcal{L}^{2n+1}\\ &= |\partial E_\phi|_{\mathbb{H}}(\omega\cdot(-2M,2M))\end{aligned}$$

Using again (4.41) we obtain that for $\mathcal{L}^1$-a.e. $c\in(0,1)$

$$\liminf_{\epsilon\to 0} |\partial E_{\epsilon,c}|_{\mathbb{H}}(\omega\cdot(-2M,2M)) = |\partial E_\phi|_{\mathbb{H}}(\omega\cdot(-2M,2M)).$$

In particular there exists $\bar c \in (0,1/10)$ and a sequence $\epsilon_k \to 0$ such that, setting $E_k := E_{\epsilon_k,\bar c}$,

$$\begin{aligned} &\lim_{k\to\infty} \mathcal{L}^{2n+1}\big(E_k \,\Delta\, \big(E_\phi \cap \omega\cdot(-2M,2M)\big)\big) = 0 \\ &\lim_{k\to\infty} |\partial E_k|_{\mathbb{H}}(\omega\cdot(-2M,2M)) = |\partial E_\phi|_{\mathbb{H}}(\omega\cdot(-2M,2M)) \end{aligned} \tag{4.42}$$

We have $X_1 v_{\epsilon_k} \leqslant 0$ on $\omega\cdot(-2M,2M)$; recalling (4.29), for large enough k one has

$$v_{\epsilon_k} \geqslant \frac{1}{(1+\epsilon_k)10} > \bar c \text{ on } \omega\cdot(-3M,2M], \qquad v_{\epsilon_k}(P) = 0 \text{ on } \omega\cdot[2M,3M)\,.$$

By the following Lemma 4.8, the boundary ∂E_k is the X_1-graph of a smooth function $\phi_k : \omega \to (-2M,2M)$. One has $\phi_k \to \phi$ in $L^1(\omega)$ because of (4.42), (2.7) and the fact that

$$\begin{aligned} E_{\phi_k} \setminus \big(\omega\cdot(-2M,2M)\big) &= E_\phi \setminus \big(\omega\cdot(-2M,2M)\big) \\ E_{\phi_k} \cap \big(\omega\cdot(-2M,2M)\big) &= E_k \cap \big(\omega\cdot(-2M,2M)\big). \end{aligned}$$

Therefore

$$\begin{aligned} |\partial E_\phi|_{\mathbb{H}}(\omega\cdot\mathbb{R}) = |\partial E_\phi|_{\mathbb{H}}(\omega\cdot(-2M,2M)) &= \lim_{k\to\infty} |\partial E_k|_{\mathbb{H}}(\omega\cdot(-2M,2M)) \\ &= \lim_{k\to\infty} \int_\omega \sqrt{1+|\nabla^{\phi_k}\phi_k|^2}\, d\mathcal{L}^{2n} \geqslant I_{\mathbb{W}}(\phi) \end{aligned}$$

as requested. □

Lemma 4.8. *Suppose that $a,b\in\mathbb{R}, c>0$ and $v \in \mathbf{C}^1_{\mathbb{H}}(\omega\cdot(a,b)) \cap \mathbf{C}^0(\omega\cdot[a,b])$ are such that $X_1 v \leqslant 0$ and*

$$v(A\cdot a) > c, \quad v(A\cdot b) \leqslant 0 \qquad \forall A \in \omega.$$

Assume also that $X_1 v(P) < 0$ whenever $v(P) = c$. Then there exists $\phi : \omega \to (a,b)$ such that $\phi \in \mathbf{C}^1_{\mathbb{W}}(\omega)$ and $\{v > c\} \cap \omega\cdot(a,b) = E_\phi \cap \omega\cdot(a,b)$.

Moreover, if v belongs also to $\mathbf{C}^\infty(\omega\cdot(a,b))$, then ϕ is $\mathbf{C}^\infty(\omega)$.

Proof. Define $\tilde v : \omega\times[a,b] \to \mathbb{R}$ by $\tilde v(A,s) := v(A\cdot s)$. By construction, $\tilde v$ is continuous and

$$\tilde v(A,a) > c, \qquad \tilde v(A,b) \leqslant 0\,.$$

In particular, for any $A \in \omega$ there exists $s_A \in (a,b)$ such that $\tilde v(A,s_A) = c$. Since

$$\frac{\partial \tilde v}{\partial s}(A,s) = X_1 v(A\cdot s) \leqslant 0 \quad \text{and} \quad \frac{\partial \tilde v}{\partial s}(A,s_A) = X_1 v(A\cdot s_A) < 0,$$

s_A is unique and $\{s \in (a,b) : v(A\cdot s) > c\} = (a,s_A)$. Define $\phi(A) := s_A$; then $\{v>c\}\cap\omega\cdot(a,b) = E_\phi\cap\omega\cdot(a,b)$. Moreover, ϕ is of class $\mathbf{C}^1_{\mathbb{W}}$ because $\Phi(\omega) = \{v = c\}$ is an $\mathbb{H}$-regular surface with $\nu^1_{\Phi(\omega)} = \frac{X_1 v}{|\nabla_{\mathbb{H}} v|} < 0$.

If v is $\mathbf{C}^\infty(\omega\cdot(a,b))$ also $\tilde v$ is smooth; the classical Implicit Function Theorem allows to conclude that also ϕ is $\mathbf{C}^\infty(\omega)$. □

In the following, for any $\phi \in L^1(\omega)$ we will use the notation

$$\int_\omega \sqrt{1+|D^\phi\phi|^2} := |\partial E_\phi|_{\mathbb{H}}(\omega\cdot\mathbb{R}) = \overline{\mathscr{A}_{\mathbb{W}}}(\phi) = I_{\mathbb{W}}(\phi).$$

We also introduce the quantity

$$\|\phi\|_{BV_{\mathbb{W}}(\omega)} := \|\phi\|_{L^1(\omega)} + \int_\omega \sqrt{1+|D^\phi\phi|^2}. \tag{4.43}$$

With abuse of terminology, we refer to $\|\phi\|_{BV_{\mathbb{W}}(\omega)}$ as the $BV_{\mathbb{W}}$ *norm* of ϕ: recall that $BV_{\mathbb{W}}$ is not even a vector space, thus $\|\cdot\|_{BV_{\mathbb{W}}}$ is far from being subadditive or homogeneous.

We single out the following result, which was obtained along the proof of Theorem 4.7.

Theorem 4.9. *Let $\phi \in BV_{\mathbb{W}}(\omega)$. Then there exists a sequence of smooth functions $\phi_j : \omega \to \mathbb{R}$, converging to ϕ in $L^1(\omega)$ and such that*

$$\int_\omega \sqrt{1+|D^\phi\phi|^2} = \lim_{j\to\infty}\int_\omega \sqrt{1+|\nabla^{\phi_j}\phi_j|^2}\, d\mathcal{L}^{2n}\,.$$

4.2. **Existence of minimal X_1-graphs.** We want to deal with the problem of minimizing the area functional among intrinsic graphs with prescribed boundary datum. In order to attack this problem, one of the main difficulties comes from the absence of a trace notion in $BV_{\mathbb{W}}$, which is also due to the fact that the theory of traces for $BV_{\mathbb{H}}$ functions in $\mathbb{H}^n$ is only at an early stage, see [20, 60, 62]. We will introduce (see Definition 4.11) a generalized notion of trace which makes sense, at least, for a large subclass of $BV_{\mathbb{W}}$ functions (see Proposition 4.15).

The following Remark 4.10, while showing another peculiar difference between $BV_{\mathbb{W}}$ and the classical BV space, underlines one more time the difficulty of defining a trace for $BV_{\mathbb{W}}$ functions.

Remark 4.10. It is well-known that a function $u \in BV(\mathcal{U})$ admits a trace $\varphi \in L^1(\partial\mathcal{U})$ when $\mathcal{U} \subset \mathbb{R}^N$ has compact and Lipschitz regular boundary. Moreover, we have $\|\varphi\|_{L^1(\partial\mathcal{U})} \leqslant c(\mathcal{U})\|u\|_{BV(\mathcal{U})}$ for some $c = c(\mathcal{U}) > 0$. A geometric interpretation of $\|\varphi\|_{L^1}$ is given by the equality (see [39])

$$\|\varphi\|_{L^1(\partial\mathcal{U})} = \int_{\partial\mathcal{U}} |\varphi|\, d\mathcal{H}^{N-1} = |\partial\,\mathscr{E}_u|(\partial\mathcal{U}\times\mathbb{R}^+) + |\partial\,\mathscr{E}^u|(\partial\mathcal{U}\times\mathbb{R}^-),$$

Here, $\mathscr{E}_u, \mathscr{E}^u \subset \mathbb{R}^{N+1}$ are, respectively, the Euclidean subgraph and epigraph of u in $\mathcal{U}\times\mathbb{R}$

$$\mathscr{E}_u := \{(x,t) : x \in \mathcal{U}, t < u(x)\}, \qquad \mathscr{E}^u := \{(x,t) : x \in \mathcal{U}, t > u(x)\},$$

while $|\partial\,\mathscr{E}_u|, |\partial\,\mathscr{E}^u|$ stand for their Euclidean perimeter measures. We have therefore

$$|\partial\,\mathscr{E}_u|(\partial\mathcal{U}\times\mathbb{R}^+) + |\partial\,\mathscr{E}^u|(\partial\mathcal{U}\times\mathbb{R}^-) \leqslant c(\mathcal{U})\|u\|_{BV(\mathcal{U})}\,. \tag{4.44}$$

We are going to show that (4.44) admits no counterpart in $BV_{\mathbb{W}}(\omega)$ and more precisely that in general there is no positive c such that

$$|\partial E_\phi|_{\mathbb{H}}(\partial\omega\cdot\mathbb{R}^+) + |\partial E^\phi|_{\mathbb{H}}(\partial\omega\cdot\mathbb{R}^-) \leqslant c\|\phi\|_{BV_{\mathbb{W}}(\omega)}\,. \tag{4.45}$$

Consider $\omega := (0,1)^{2n}$ (but the domain could be easily made smooth) and the functions, which were suggested to us by G. P. Leonardi [44],

$$\begin{array}{ll} \phi_k(x_2,\dots,x_n,y_1,\dots,y_n,t) := e^{ky_1} + \frac{k}{4}t & \text{if } n \geqslant 2,\\ \phi_k(y,t) := e^{ky} + \frac{k}{4}t & \text{if } n = 1. \end{array}$$

Easy computations give

$$\|\phi_k\|_{BV_{\mathbb{W}}(\omega)} = \int_\omega |\phi_k|\, d\mathcal{L}^{2n} + \int_\omega \sqrt{1+|\nabla^{\phi_k}\phi_k|^2}\, d\mathcal{L}^{2n} = e^k/k + O(k^2)\,.$$

For $n \geqslant 2$ we have instead

$$\begin{aligned}
&|\partial E_{\phi_k}|_{\mathbb{H}}(\partial\omega\cdot\mathbb{R}^+)\\
&\geqslant c_n \mathcal{S}_\infty^{Q-1}\big(\{(x,y,t)\in\mathbb{H}^n : y_1 = 1, 0 < x_1 < \phi_k(\pi_{\mathbb{W}}(x,y,t))\}\big)\\
&= \mathcal{H}^{2n}\big(\{(x,y,t)\in\mathbb{H}^n : y_1 = 1, 0 < x_1 < \phi_k(\pi_{\mathbb{W}}(x,y,t)))\}\big)\\
&= \int_{\omega\cap\{y_1=1\}} \phi_k(x_2,\dots,x_n,1,y_2,\dots,y_n,t)\, d\mathcal{L}^{2n-1}\,.
\end{aligned}$$

The first of the two equalities follows from $c_n\mathcal{S}_\infty^{Q-1}\llcorner\{y_1 = 1\} = \mathcal{H}^{2n}\llcorner\{y_1 = 1\}$ (see also Lemma 2.12). Similar computations can be carried out also for $n = 1$; in both cases we obtain

$$|\partial E_{\phi_k}|_{\mathbb{H}}(\partial\omega\cdot\mathbb{R}^+) > e^k$$

and our claim follows.

We are now ready to introduce our generalized notion of trace. From now on, ω is supposed to have Lipschitz regular boundary.

Definition 4.11. Let ω_0 be an open bounded set such that $\omega \Subset \omega_0$. We will say that $\phi_0 \in BV_{\mathbb{W}}(\omega_0 \setminus \overline{\omega})$ is a *trace in generalized sense* (briefly: TGS) of $\phi \in BV_{\mathbb{W}}(\omega)$ if, after defining $\theta : \omega_0 \to \mathbb{R}$ as

$$\theta := \phi \text{ on } \omega, \qquad \theta := \phi_0 \text{ on } \omega_0 \setminus \omega, \tag{4.46}$$

then

$$|\partial E_\theta|_{\mathbb{H}}(\partial\omega\cdot\mathbb{R}) = 0\,, \tag{4.47}$$

$$|\partial E_\theta|_{\mathbb{H}}(\partial\omega_0\cdot\mathbb{R}^+) = |\partial E_\theta|_{\mathbb{H}}(\mathbb{H}^n_+ \setminus (\omega_0\cdot\mathbb{R})) < +\infty\,, \tag{4.48}$$

$$|\partial E^\theta|_{\mathbb{H}}(\partial\omega_0\cdot\mathbb{R}^-) = |\partial E^\theta|_{\mathbb{H}}(\mathbb{H}^n_- \setminus (\omega_0\cdot\mathbb{R})) < +\infty\,. \tag{4.49}$$

Roughly speaking, ϕ_0 (or, which is the same, θ) gives the *trace* of ϕ on $\partial\omega\cdot\mathbb{R}$ because of (4.47), which says that χ_{E_θ} takes on $\partial\omega\cdot\mathbb{R}$ the same “boundary value” both from the “outside” $(\omega_0\setminus\overline{\omega})\cdot\mathbb{R}$ and from the “inside” $\omega\cdot\mathbb{R}$, where it coincides with χ_{E_ϕ}. In some sense, $\chi_{E_\theta} \in BV_{\mathbb{H}}(\omega_0\cdot\mathbb{R})$ can be thought of as a sort of “tracing extension” of $\chi_{E_\phi} \in BV_{\mathbb{H}}(\omega\cdot\mathbb{R})$. See also [4] for the problem of the extension of BV functions in metric spaces.

Remark 4.12. The function θ is not defined on the $\mathcal{L}^{2n}$-negligible set $\partial\omega$ and so it is well defined in $L^1(\omega_0)$. It follows from (4.47) that $\theta \in BV_{\mathbb{W}}(\omega_0)$.

We will prove later the following

Proposition 4.13. *Let $\phi_0 \in BV_{\mathbb{W}}(\omega_0\setminus\overline{\omega})$ be a TGS for both $\phi,\psi \in BV_{\mathbb{W}}(\omega)$ and suppose that $\phi_1 \in BV(\omega_1\setminus\overline{\omega})$ is another TGS for ϕ, where $\omega_1 \subset \mathbb{W}$ is open and bounded and $\omega \Subset \omega_1$. Then ϕ_1 is a TGS also for ψ.*

The previous result allows to introduce an equivalence relation among traces in generalized sense as follows. If $\phi_0 \in BV_{\mathbb{W}}(\omega_0 \setminus \overline{\omega})$ and $\phi_1 \in BV_{\mathbb{W}}(\omega_1 \setminus \overline{\omega})$, we set $\phi_0 \sim \phi_1$ if for any $\phi \in BV_{\mathbb{W}}(\omega)$

$$\phi_0 \text{ is a TGS for } \phi \iff \phi_1 \text{ is a TGS for } \phi\,.$$

Equivalence classes are denoted by $[\phi_0]$; if ϕ_0 is a TGS of ϕ we write $\phi_{|\partial\omega} = [\phi_0]$ to underline the fact that, rather than by ϕ_0, the trace of ϕ is given by the equivalence class $[\phi_0]$.

Remark 4.14. In the classical case, when $\mathcal{U}$ has Lipschitz regular boundary any function $u \in BV(\mathcal{U})$ admits a (Euclidean) TGS. Namely, there exists $\mathcal{U}_0 \Supset \mathcal{U}$ and $\theta \in BV(\mathcal{U}_0)$ such that

$$|\partial\mathscr{E}_\theta|(\partial\mathcal{U}\times\mathbb{R}) = 0, \quad |\partial\mathscr{E}_\theta|(\partial\mathcal{U}_0\times\mathbb{R}^+) + |\partial\mathscr{E}^\theta|(\partial\mathcal{U}_0\times\mathbb{R}^-) < +\infty\,.$$

In fact it is sufficient to consider a smooth compact domain $\mathcal{U}_0 \Supset \mathcal{U}$ and to extend u to $\theta \in BV(\mathcal{U}_0)$ with $\theta_{|\partial\mathcal{U}_0} = 0$ and the inner and outer traces of θ on $\partial\mathcal{U}$ are the same: $\theta^+_{|\partial\mathcal{U}} = \theta^-_{|\partial\mathcal{U}}$. See [36].

We conjecture that, when $\partial\omega$ is compact and Lipschitz continuous, any function in $BV_{\mathbb{W}}(\omega)$ admits a TGS. We are able to prove this result for quite large subclasses of $BV_{\mathbb{W}}(\omega)$. Let us recall that the class $Lip_{\mathbb{W}}(\omega)$ of *intrinsic Lipschitz* functions in the Heisenberg group was introduced in [33] in terms of comparison with suitable intrinsic cones in $\mathbb{H}^n$; see also [35] and [62].

Proposition 4.15. *Suppose that ω has Lipschitz regular boundary and ϕ belongs to one of the following classes: $W^{1,2}(\omega)$, $BV(\omega)\cap L^\infty(\omega)$ or $Lip_{\mathbb{W}}(\omega)$. Then ϕ admits a trace in generalized sense.*

Proof. When $\phi \in W^{1,2}(\omega)$ (respectively $\phi \in BV(\omega)\cap L^\infty(\omega)$) we fix a bounded open set $\omega_0 \Supset \omega$ with smooth boundary. As in [36], it is possible to consider $\phi_0 \in W^{1,2}(\omega_0 \setminus \overline{\omega})$ (resp. $\phi_0 \in W^{1,1}(\omega_0 \setminus \overline{\omega}) \cap L^\infty(\omega_0 \setminus \overline{\omega})$) such that

$$\phi_{0|\partial\omega} = \phi_{|\partial\omega} \quad \text{and} \quad \phi_{0|\partial\omega_0} = 0; \tag{4.50}$$

here the traces are the classical ones. Define θ as in (4.46). The first equality in (4.50) implies that the Euclidean perimeter $|\partial E_\theta|(\partial\omega\cdot\mathbb{R})$ is zero, whence (4.47). The second equality in (4.50) gives

$$|\partial E_\theta|(\partial\omega_0\cdot\mathbb{R}^+) = |\partial E^\theta|(\partial\omega_0\cdot\mathbb{R}^-) = 0,$$

whence (4.48) and (4.49) by Proposition 2.10. Notice that $\phi_0 \in BV_{\mathbb{W}}(\omega_0\setminus\overline{\omega})$ because $W^{1,2}(\omega_0\setminus\overline{\omega}) \subset W^{1,1}_{\mathbb{W}}(\omega_0\setminus\overline{\omega})$ (respectively, because of Remark 4.4).

Finally, if $\phi \in Lip_{\mathbb{W}}(\omega)$ we can fix ω_0 as before and extend ϕ to $\theta \in Lip_{\mathbb{W}}(\omega_0)$: see [35]. In the latter work it is also proved that

$$(\pi_{\mathbb{W}})_\#(|\partial E_\theta|_{\mathbb{H}} \llcorner (\omega_0\cdot\mathbb{R})) \leqslant c\mathcal{L}^{2n}\llcorner\omega_0$$

for a suitable $c = c(\theta) > 0$. From this, (4.47) follows because $\mathcal{L}^{2n}(\partial\omega) = 0$. Moreover, θ is bounded in ω_0 (see [35]) and we have

$$E_\theta \cap \{x_1 > c\} = E^\theta \cap \{x_1 < -c\} = \emptyset.$$

In particular

$$|\partial E_\theta|_{\mathbb{H}}(\partial\omega_0\cdot\mathbb{R}^+) + |\partial E^\theta|_{\mathbb{H}}(\partial\omega_0\cdot\mathbb{R}^-) \leqslant c_n\mathcal{S}^{Q-1}_\infty(\partial\omega_0\cdot[-c,c]) < \infty$$

and also (4.48) and (4.49) are satisfied. □

Let us consider the minimization problem (1.17): fix $\omega_0 \Supset \omega$ and $\theta \in BV_{\mathbb{W}}(\omega_0)$. Assume that θ satisfies (4.47), (4.48) and (4.49); then $\phi_0 := \theta|_{\omega_0 \setminus \overline{\omega}}$ is a TGS for $\phi := \theta|_{\omega}$. In other words, θ is an extension of ϕ which determines its "trace" $\phi_{|\partial\omega}$. We are going to show that the infimum in the minimization problem (1.17) does not depend on θ but only on the "boundary behaviour" $\phi_{|\partial\omega}$ of ϕ, i.e, on $[\phi_0]$.

Proposition 4.16. *Suppose that* $\theta \in BV_{\mathbb{W}}(\omega_0)$ *satisfies* (4.47), (4.48) *and* (4.49)*; set* $\phi_0 := \theta_{|\omega_0 \setminus \overline{\omega}}$. *Then the infimum in* (1.17) *depends only on the equivalence class* $[\phi_0]$.

Proof. Suppose that $\phi_0 \in BV_{\mathbb{W}}(\omega_0 \setminus \overline{\omega})$ and $\phi_1 \in BV_{\mathbb{W}}(\omega_1 \setminus \overline{\omega})$ are both TGS of $\phi \in BV_{\mathbb{W}}(\omega)$. Let $\psi \in L^1(\omega)$ and set

$$\psi_0 := \begin{cases} \psi & \text{on } \omega \\ \phi_0 & \text{on } \omega_0 \setminus \overline{\omega} \end{cases} \qquad \text{and} \qquad \psi_1 := \begin{cases} \psi & \text{on } \omega \\ \phi_1 & \text{on } \omega_1 \setminus \overline{\omega}. \end{cases} \tag{4.51}$$

It will be enough to prove that $|\partial E_{\psi_0}|_{\mathbb{H}}(\overline{\omega} \cdot \mathbb{R}) = |\partial E_{\psi_1}|_{\mathbb{H}}(\overline{\omega} \cdot \mathbb{R})$; equivalently, by the locality of $\mathbb{H}$- perimeter (see Proposition 2.3), that

$$|\partial E_{\psi_0}|_{\mathbb{H}}(\partial\omega \cdot \mathbb{R}) = |\partial E_{\psi_1}|_{\mathbb{H}}(\partial\omega \cdot \mathbb{R}). \tag{4.52}$$

Let θ_0, θ_1 be defined by

$$\theta_0 := \begin{cases} \phi & \text{on } \omega \\ \phi_0 & \text{on } \omega_0 \setminus \overline{\omega} \end{cases} \qquad \text{and} \qquad \theta_1 := \begin{cases} \phi & \text{on } \omega \\ \phi_1 & \text{on } \omega_1 \setminus \overline{\omega}. \end{cases} \tag{4.53}$$

By (4.47), for any $\epsilon > 0$ we can find an open neighbourhood $\upsilon_\epsilon \subset \omega_0 \cap \omega_1$ of $\partial\omega$ such that

$$\left| \int_{E_{\theta_0}} \mathrm{div}_{\mathbb{H}} \varphi \, d\mathcal{L}^{2n+1} \right| < \epsilon \ \text{ and } \ \left| \int_{E_{\theta_1}} \mathrm{div}_{\mathbb{H}} \varphi \, d\mathcal{L}^{2n+1} \right| < \epsilon \quad \forall \varphi \in \mathbf{C}^1_c(\upsilon_\epsilon \cdot \mathbb{R}, \mathbb{R}^{2n}), |\varphi| \leqslant 1\,.$$

It is not restrictive to suppose that $\upsilon_\epsilon \downarrow \partial\omega$ as $\epsilon \to 0$. From the previous inequalities, for any $\varphi \in \mathbf{C}^1_c(\upsilon_\epsilon \cdot \mathbb{R}, \mathbb{R}^{2n})$ we obtain

$$\begin{aligned} \left| \int_{\upsilon_\epsilon \cdot \mathbb{R}} (\chi_{E_{\psi_0}} - \chi_{E_{\psi_1}}) \mathrm{div}_{\mathbb{H}} \varphi \, d\mathcal{L}^{2n+1} \right| &= \left| \int_{(\upsilon_\epsilon \setminus \omega) \cdot \mathbb{R}} (\chi_{E_{\phi_0}} - \chi_{E_{\phi_1}}) \mathrm{div}_{\mathbb{H}} \varphi \, d\mathcal{L}^{2n+1} \right| \\ &= \left| \int_{\upsilon_\epsilon \cdot \mathbb{R}} (\chi_{E_{\theta_0}} - \chi_{E_{\theta_1}}) \mathrm{div}_{\mathbb{H}} \varphi \, d\mathcal{L}^{2n+1} \right| \ < \ 2\epsilon. \end{aligned}$$

In particular

$$\big| |\partial E_{\psi_0}|_{\mathbb{H}}(\upsilon_\epsilon \cdot \mathbb{R}) - |\partial E_{\psi_1}|_{\mathbb{H}}(\upsilon_\epsilon \cdot \mathbb{R}) \big| < 2\epsilon$$

and (4.52) follows as $\epsilon \to 0$. □

We owe the reader the proof of Proposition 4.13.

Proof of Proposition 4.13. Define $\psi_0, \psi_1, \theta_0, \theta_1$ as in (4.51) and (4.53). Our aim is to prove that (4.47), (4.48), (4.49) are fulfilled with $\theta = \psi_1$; notice that the last two of them are satisfied by assumption. It is possible to follow the proof of Proposition 4.16 to obtain (4.52); in particular

$$|\partial E_{\psi_1}|_{\mathbb{H}}(\partial\omega \cdot \mathbb{R}) = |\partial E_{\psi_0}|_{\mathbb{H}}(\partial\omega \cdot \mathbb{R}) = 0$$

which is (4.47). □

In the spirit of Proposition 3.7, another natural formulation of the minimal area problem for X_1-graphs could be

$$\inf\ \left\{|\partial E_\psi|_{\mathbb{H}}(\omega\cdot\mathbb{R}) + |\partial(E_\psi\Delta E_\phi)|_{\mathbb{H}}(\partial\omega\cdot\mathbb{R}) : \psi\in BV_{\mathbb{W}}(\omega)\right\} \tag{4.54}$$

for a fixed "boundary datum" $\phi\in BV_{\mathbb{W}}(\omega)$. It will be precisely the possibility of extending ϕ by means of a generalized "trace" that will provide semicontinuity for this functional (see (4.76) in the proof of Theorem 1.8).

The problem (4.54) is equivalent to our formulation (1.17) at least when ϕ admits a trace in generalized sense and $\partial\omega$ is Lipschitz regular.

Proposition 4.17. *Let $\omega,\omega_0\subset\mathbb{W}$ be bounded open sets with $\omega\Subset\omega_0$ and $\partial\omega$ Lipschitz regular. Let $\phi\in BV_{\mathbb{W}}(\omega)$ and assume that $\theta\in BV_{\mathbb{W}}(\omega_0)$ with $\theta_{|\omega}=\phi$ is an extension of ϕ such that $\phi_0:=\theta_{|\omega_0\setminus\overline{\omega}}$ is a TGS for ϕ. Then*

$$\begin{aligned}&\inf\ \left\{|\partial E_\psi|_{\mathbb{H}}(\omega\cdot\mathbb{R}) + |\partial(E_\psi\Delta E_\phi)|_{\mathbb{H}}(\partial\omega\cdot\mathbb{R}) : \psi\in BV_{\mathbb{W}}(\omega)\right\}\\ =\ &\inf\ \left\{|\partial E_\psi|_{\mathbb{H}}(\overline{\omega}\cdot\mathbb{R}) : \psi\in BV_{\mathbb{W}}(\omega_0),\ \psi=\theta \text{ on } \omega_0\setminus\overline{\omega}\right\}.\end{aligned}$$

Proof. Given $\psi\in BV_{\mathbb{W}}(\omega)$ define $\widetilde{\psi}:\omega_0\to\mathbb{R}$ as

$$\widetilde{\psi}:=\begin{cases}\psi & \text{on } \omega\\ \theta & \text{on } \omega_0\setminus\omega\,.\end{cases}$$

It will suffice to prove that

$$|\partial E_{\widetilde{\psi}}|_{\mathbb{H}}(\overline{\omega}\cdot\mathbb{R}) = |\partial E_\psi|_{\mathbb{H}}(\omega\cdot\mathbb{R}) + |\partial(E_\psi\Delta E_\phi)|_{\mathbb{H}}(\partial\omega\cdot\mathbb{R})$$

or, equivalently, that

$$|\partial E_{\widetilde{\psi}}|_{\mathbb{H}}(\partial\omega\cdot\mathbb{R}) = |\partial(E_\psi\Delta E_\phi)|_{\mathbb{H}}(\partial\omega\cdot\mathbb{R}). \tag{4.55}$$

Notice that $\omega\cdot\mathbb{R}$ has locally finite Euclidean perimeter; thus it has also locally finite $\mathbb{H}$-perimeter and $\mathcal{S}_\infty^{Q-1}(\partial\omega\cdot\mathbb{R}\ \Delta\ \partial^*_{\mathbb{H}}(\omega\cdot\mathbb{R}))=0$. By Theorem 2.6 one has

$$\begin{aligned}|\partial(E_\psi\Delta E_\phi)|_{\mathbb{H}}(\partial\omega\cdot\mathbb{R}) &= c_n\mathcal{S}_\infty^{Q-1}\big(\partial\omega\cdot\mathbb{R}\cap(E_\psi\Delta E_\phi)^{1/2}\big)\\ &= c_n\mathcal{S}_\infty^{Q-1}\big((\omega\cdot\mathbb{R})^{1/2}\cap(E_\psi\Delta E_\phi)^{1/2}\big).\end{aligned} \tag{4.56}$$

Let us define (whenever they exist) the inner and outer density $\Theta_i(E,P)$ and $\Theta_o(E,P)$ of a set E at $P\in\mathbb{H}^n$ with respect to the intrinsic cylinder $\omega\cdot\mathbb{R}$ respectively by

$$\Theta_i(E,P):=\lim_{r\to0^+}\frac{|E\cap U(P,r)\cap\omega\cdot\mathbb{R}|}{|U(P,r)\cap\omega\cdot\mathbb{R}|},\quad \Theta_o(E,P):=\lim_{r\to0^+}\frac{|E\cap U(P,r)\setminus\omega\cdot\mathbb{R}|}{|U(P,r)\setminus\omega\cdot\mathbb{R}|}.$$

Since $E_\psi,E_\phi\subset\omega\cdot\mathbb{R}$ it is easy to observe that

$$P\in(\omega\cdot\mathbb{R})^{1/2}\cap(E_\psi\Delta E_\phi)^{1/2} \iff P\in(\omega\cdot\mathbb{R})^{1/2} \text{ and } \Theta_i(E_\psi\Delta E_\phi,P)=1 \tag{4.57}$$

and that

$$(\omega\cdot\mathbb{R})^{1/2}\cap E^1_\phi = (\omega\cdot\mathbb{R})^{1/2}\cap E^1_\psi=\emptyset\,.$$

Similarly, we have $E_{\phi_0}\subset\mathbb{H}^n\setminus(\omega\cdot\mathbb{R})$ and, noticing that $(\omega\cdot\mathbb{R})^{1/2}=(\mathbb{H}^n\setminus\omega\cdot\mathbb{R})^{1/2}$, we get

$$(\omega\cdot\mathbb{R})^{1/2}\cap E^1_{\phi_0}=\emptyset\,.$$

In particular, by Theorem 2.6,

$$
\begin{aligned}
\mathcal{S}_\infty^{Q-1}\big((\omega\cdot\mathbb{R})^{1/2}\setminus(E_\phi^0\cup E_\phi^{1/2})\big) &= \mathcal{S}_\infty^{Q-1}\big((\omega\cdot\mathbb{R})^{1/2}\setminus(E_\psi^0\cup E_\psi^{1/2})\big)\\
&= \mathcal{S}_\infty^{Q-1}\big((\omega\cdot\mathbb{R})^{1/2}\setminus(E_{\phi_0}^0\cup E_{\phi_0}^{1/2})\big) = 0\,.
\end{aligned}
\tag{4.58}
$$

Notice also that

$$
\begin{aligned}
&\text{if } P\in(\omega\cdot\mathbb{R})^{1/2}\cap E_\phi^0, \text{ then } \Theta_i(E_\phi,P)=0,\\
&\text{if } P\in(\omega\cdot\mathbb{R})^{1/2}\cap E_\phi^{1/2}, \text{ then } \Theta_i(E_\phi,P)=1,\\
&\text{if } P\in(\omega\cdot\mathbb{R})^{1/2}\cap E_\psi^0, \text{ then } \Theta_i(E_\psi,P)=0,\\
&\text{if } P\in(\omega\cdot\mathbb{R})^{1/2}\cap E_\psi^{1/2}, \text{ then } \Theta_i(E_\psi,P)=1
\end{aligned}
$$

whence

$$
\Theta_i(E_\phi,P)\in\{0,1\} \text{ and } \Theta_i(E_\psi,P)\in\{0,1\} \text{ for } \mathcal{S}_\infty^{Q-1}\text{-a.e. } P\in(\omega\cdot\mathbb{R})^{1/2}. \tag{4.59}
$$

We also point out the implications

$$
\begin{aligned}
\Theta_i(E_\phi,P)=\Theta_i(E_\psi,P)\in\{0,1\} &\Longrightarrow \Theta_i(E_\phi\Delta E_\psi,P)=0\,,\\
\{\Theta_i(E_\phi,P),\Theta_i(E_\psi,P)\}=\{0,1\} &\Longrightarrow \Theta_i(E_\phi\Delta E_\psi,P)=1\,.
\end{aligned}
\tag{4.60}
$$

Therefore, by (4.56), (4.57), (4.59) and (4.60) we obtain

$$
\begin{aligned}
&|\partial(E_\psi\Delta E_\phi)|_{\mathbb{H}}(\partial\omega\cdot\mathbb{R})\\
&= c_n\mathcal{S}_\infty^{Q-1}\big(\{P\in(\omega\cdot\mathbb{R})^{1/2}:\Theta_i(E_\psi\Delta E_\phi,P)=1\}\big)\\
&= c_n\mathcal{S}_\infty^{Q-1}\left(\left\{P\in(\omega\cdot\mathbb{R})^{1/2}:\begin{array}{l}\Theta_i(E_\phi,P)\in\{0,1\},\Theta_i(E_\psi,P)\in\{0,1\}\\ \text{and } \Theta_i(E_\psi\Delta E_\phi,P)=1\end{array}\right\}\right)\\
&= c_n\mathcal{S}_\infty^{Q-1}\left(\left\{P\in(\omega\cdot\mathbb{R})^{1/2}:\begin{array}{l}\big[\Theta_i(E_\psi,P)=1 \text{ and } \Theta_i(E_\phi,P)=0\big] \text{ or}\\ \big[\Theta_i(E_\psi,P)=0 \text{ and } \Theta_i(E_\phi,P)=1\big]\end{array}\right\}\right).
\end{aligned}
\tag{4.61}
$$

We now use the fact that $|\partial E_\theta|_{\mathbb{H}}((\omega\cdot\mathbb{R})^{1/2})=|\partial E_\theta|_{\mathbb{H}}(\partial\omega\cdot\mathbb{R})=0$, which implies (see Theorem 2.6) that

$$
P\in E_\theta^1\cup E_\theta^0 \text{ for } \mathcal{S}_\infty^{Q-1}\text{-a.e. } P\in(\omega\cdot\mathbb{R})^{1/2}. \tag{4.62}
$$

This in turn implies that for $\mathcal{S}_\infty^{Q-1}$-a.e. $P\in(\omega\cdot\mathbb{R})^{1/2}$

$$
\Theta_i(E_\phi,P)=\Theta_i(E_\theta,P)=\Theta_o(E_\theta,P)=\Theta_o(E_{\widetilde\psi},P)\in\{0,1\}\,. \tag{4.63}
$$

Indeed, the first and third equalities in (4.63) are clear. The second one can be obtained noticing that, if $P\in(\omega\cdot\mathbb{R})^{1/2}$, then

$$
\begin{aligned}
\lim_{r\to0}\frac{|E_\theta\cap U(P,r)|}{|U(P,r)|} &= \lim_{r\to0}\frac{|E_\theta\cap U(P,r)\cap\omega\cdot\mathbb{R}|}{|U(P,r)|}+\frac{|E_\theta\cap U(P,r)\setminus\omega\cdot\mathbb{R}|}{|U(P,r)|}\\
&= \lim_{r\to0}\frac{|E_\theta\cap U(P,r)\cap\omega\cdot\mathbb{R}|}{2|U(P,r)\cap\omega\cdot\mathbb{R}|}+\frac{|E_\theta\cap U(P,r)\setminus\omega\cdot\mathbb{R}|}{2|U(P,r)\setminus\omega\cdot\mathbb{R}|}\\
&= \frac{1}{2}(\Theta_i(P,r)+\Theta_o(P,r))\,.
\end{aligned}
\tag{4.64}
$$

By (4.62), the left hand side in (4.64) must be either 0 or 1 for $\mathcal{S}_\infty^{Q-1}$-a.e. $P\in(\omega\cdot\mathbb{R})^{1/2}$, thus $\frac{1}{2}(\Theta_i(P,r)+\Theta_o(P,r))\in\{0,1\}$ for $\mathcal{S}_\infty^{Q-1}$-a.e. $P\in(\omega\cdot\mathbb{R})^{1/2}$. This is possible only

if $\Theta_i(E_\theta, P)$ and $\Theta_o(E_\theta, P)$ are both 0 or both 1, which gives the second equality (as well as the last inclusion) in (4.63) for $\mathcal{S}_\infty^{Q-1}$-a.e. $P \in (\omega \cdot \mathbb{R})^{1/2}$.
Combining (4.61) and (4.63) one gets

$$\begin{aligned}
&|\partial(E_\psi \Delta E_\phi)|_{\mathbb{H}}(\partial\omega \cdot \mathbb{R})\\
&= c_n \mathcal{S}_\infty^{Q-1}\left(\left\{P \in (\omega\cdot\mathbb{R})^{1/2} : \begin{array}{l}\big(\Theta_i(E_{\widetilde{\psi}}, P) = 1 \text{ and } \Theta_o(E_{\widetilde{\psi}}, P) = 0\big) \text{ or}\\ \big(\Theta_i(E_{\widetilde{\psi}}, P) = 0 \text{ and } \Theta_o(E_{\widetilde{\psi}}, P) = 1\big)\end{array}\right\}\right)\\
&\overset{(4.63)}{=} c_n \mathcal{S}_\infty^{Q-1}\left(\left\{P \in (\omega\cdot\mathbb{R})^{1/2} : P \in E_{\widetilde{\psi}}^{1/2}\right\}\right)\\
&= |\partial E_{\widetilde{\psi}}|_{\mathbb{H}}(\partial\omega\cdot\mathbb{R})
\end{aligned}$$

and (4.55) follows. □

Proposition 4.18. *Let $\omega \subset \mathbb{W}$ be a bounded open set and $(\phi_j)_j$ be a bounded sequence in $BV_{\mathbb{W}}(\omega)$; assume that*

$$\sup_j \left\{|\partial E_{\phi_j}|_{\mathbb{H}}(\mathbb{H}^n_+) + |\partial E^{\phi_j}|_{\mathbb{H}}(\mathbb{H}^n_-)\right\} < +\infty\,. \tag{4.65}$$

Then there exists $\phi \in BV_{\mathbb{W}}(\omega)$ such that, up to a subsequence,

$$\phi_j \to \phi \ \mathcal{L}^{2n}\text{-a.e. on } \omega \quad \text{and} \quad E_{\phi_j} \to E_\phi \text{ in } L^1_{loc}(\omega\cdot\mathbb{R}). \tag{4.66}$$

In particular

$$|\partial E_\phi|_{\mathbb{H}}(\omega\cdot\mathbb{R}) \leqslant \liminf_{j\to\infty} |\partial E_{\phi_j}|_{\mathbb{H}}(\omega\cdot\mathbb{R})\,. \tag{4.67}$$

Proof. Since

$$\begin{aligned}
&|\partial(E_{\phi_j} \cap \mathbb{H}^n_+)|_{\mathbb{H}}(\mathbb{H}^n) + |\partial(E^{\phi_j} \cap \mathbb{H}^n_-)|_{\mathbb{H}}(\mathbb{H}^n)\\
&\leqslant |\partial E_{\phi_j}|_{\mathbb{H}}(\mathbb{H}^n_+) + |\partial E^{\phi_j}|_{\mathbb{H}}(\mathbb{H}^n_-) + 2c_n\mathcal{S}_\infty^{Q-1}(\omega)
\end{aligned}$$

we have

$$\sup_j |\partial(E_{\phi_j} \cap \mathbb{H}^n_+)|_{\mathbb{H}}(\mathbb{H}^n) < +\infty \quad \text{and} \quad \sup_j |\partial(E^{\phi_j} \cap \mathbb{H}^n_-)|_{\mathbb{H}}(\mathbb{H}^n) < +\infty\,. \tag{4.68}$$

Moreover, by (2.7),

$$\sup_j \left\{|E_{\phi_j} \cap \mathbb{H}^n_+| + |E^{\phi_j} \cap \mathbb{H}^n_-|\right\} \leqslant c \tag{4.69}$$

because $\|\phi_j\|_{L^1(\omega)}$ is bounded uniformly in j.

For any $i \in \mathbb{N}$ let $r_i > 0$ be such that $\omega\cdot[-i,i] \subset U_c(0,r_i)$. Using (4.68), (4.69) and the compact inclusion $BV_{\mathbb{H}}(U_c(0,r_i)) \hookrightarrow L^1(U_c(0,r_i))$, for any fixed i there exists a subsequence $(\phi_{j_\ell})_\ell$ and a set $E_i \subset U_c(0,r_i) \cap (\omega\cdot\mathbb{R}^+)$ such that

$$E_{\phi_{j_\ell}} \cap \mathbb{H}^n_+ \to E_i \quad \text{in } L^1(U_c(0,r_i)) \text{ as } \ell \to \infty\,.$$

Using a diagonal argument we can actually suppose that there exists $E^+ \subset \omega\cdot\mathbb{R}^+$ such that (up to a subsequence which we do not relabel)

$$E_{\phi_j} \cap \mathbb{H}^n_+ \to E^+ \text{ in } L^1_{loc}(\omega\cdot\mathbb{R}^+). \tag{4.70}$$

Notice that $E^+ = E_{\phi^+} \cap \mathbb{H}^n_+$ for a suitable $\phi^+ : \omega \to \mathbb{R}^+$. Indeed, let $i \in \mathbb{N}$ be fixed; since $E_{\phi_j} \to E^+$ in $L^1(\omega\cdot(0,i))$, by Fubini Theorem we obtain that, for $\mathcal{L}^{2n}$-a.e.

$A \in \omega$, the sequence $(s \mapsto \chi_{E_{\phi_j}\cap\mathbb{H}^n_+}(A\cdot s))_j$ converges to $s \mapsto \chi_{E^+}(A\cdot s)$ in $L^1(0,i)$. In particular, for $\mathcal{L}^{2n}$-a.e. $A \in \omega$

(4.71) the sequence $\big(s \mapsto \chi_{E_{\phi_j}\cap\mathbb{H}^n_+}(A\cdot s)\big)_j$ converges to $s \mapsto \chi_{E^+}(A\cdot s)$ in $L^1_{loc}(\mathbb{R}^+)$.

Since

$$\forall s \in \mathbb{R}^+ \quad \chi_{E_{\phi_j}\cap\mathbb{H}^n_+}(A\cdot s) = \chi_{[0,\phi_j^+(A))}(s)\,,$$

where ϕ_j^+ is the function defined in (2.8), the convergence in (4.71) holds (for a fixed $A \in \omega$) if and only if $s \mapsto \chi_{E^+}(A\cdot s)$ is ($\mathcal{L}^1$-a.e. equivalent to) a characteristic function $\chi_{[0,\phi^+(A))}$ for some $\phi^+(A) \geqslant 0$ such that $\lim_{j\to\infty}\phi_j^+(A) = \phi^+(A)$. In particular, $E^+ = E_{\phi^+}\cap\mathbb{H}^n_+$ (up to negligible sets) and $\phi_j^+ \to \phi^+$ $\mathcal{L}^{2n}$-a.e. on ω. By (4.70) we have also

$$E_{\phi_j}\cap\mathbb{H}^n_+ \to E_{\phi^+}\cap\mathbb{H}^n_+ \text{ in } L^1_{loc}(\omega\cdot\mathbb{R}^+). \tag{4.72}$$

Since for any $i \in \mathbb{N}$ one has

$$|E^+\cap U_c(0,r_i)| = \lim_{j\to\infty}|E_{\phi_j}\cap\mathbb{H}^n_+\cap U_c(0,r_i)|$$

we have

$$\int_\omega \phi^+\,d\mathcal{L}^{2n} = |E^+| \leqslant \liminf_{j\to\infty}|E_{\phi_j}\cap\mathbb{H}^n_+| = \liminf_{j\to\infty}\int_\omega \phi_j^+\,d\mathcal{L}^{2n} \leqslant \sup_j \|\phi_j\|_{L^1(\omega)} < \infty\,,$$

i.e., $\phi^+ \in L^1(\omega)$.

With similar considerations, involving $E^{\phi_j}\cap\mathbb{H}^n_-$ and ϕ_j^- in place of, respectively, $E_{\phi_j}\cap\mathbb{H}^n_+$ and ϕ_j^+, one can prove that (possibly after a further subsequence) there exist

$$E^- \subset \omega\cdot\mathbb{R}^- \quad\text{and}\quad \phi^- \in L^1(\omega), \phi^- \geqslant 0$$

such that $E^- = E^{(-\phi^-)}\cap\mathbb{H}^n_-$ and

$$E^{\phi_j}\cap\mathbb{H}^n_- \to E^- \text{ in } L^1_{loc}(\omega\cdot\mathbb{R}^-) \quad\text{and}\quad \phi_j^- \to \phi^- \ \mathcal{L}^{2n}\text{-a.e. on } \omega\,.$$

Therefore we have also

$$E_{\phi_j}\cap\mathbb{H}^n_- = (\omega\cdot\mathbb{R}^-)\setminus E^{\phi_j} \to (\omega\cdot\mathbb{R}^-)\setminus E^- = E_{(-\phi^-)}\cap\mathbb{H}^n_- \text{ in } L^1_{loc}(\omega\cdot\mathbb{R}^-)\,, \tag{4.73}$$

where the equalities have to be understood up to negligible sets. Setting $\phi := \phi^+ - \phi^-$in $L^1(\omega)$, by (4.72) and (4.73) we obtain (4.66). The map ϕ belongs to $BV_{\mathbb{W}}(\omega)$ because of the lower semicontinuity of $|\partial E_\phi|_{\mathbb{H}}(\omega\cdot\mathbb{R})$ with respect to the L^1_{loc} convergence of sets (see Proposition 2.2), which gives also (4.67). □

Proof of Theorem 1.8. Let $(\psi_j)_j \subset BV_{\mathbb{W}}(\omega_0)$ be a minimizing sequence for (1.17). We claim that

$$\sup_j |\partial(E_{\psi_j}\cap\mathbb{H}^n_+)|_{\mathbb{H}}(\mathbb{H}^n) < +\infty \quad\text{and}\quad \sup_j |\partial(E^{\psi_j}\cap\mathbb{H}^n_-)|_{\mathbb{H}}(\mathbb{H}^n) < +\infty\,. \tag{4.74}$$

We have in fact, by Proposition 2.3, Theorem 2.6 and Lemma 2.12,

$$\begin{aligned}
&|\partial(E_{\psi_j}\cap\mathbb{H}^n_+)|_{\mathbb{H}}(\mathbb{H}^n)\\
&= |\partial(E_{\psi_j}\cap\mathbb{H}^n_+)|_{\mathbb{H}}(\partial\omega_0\cdot\mathbb{R}^+) + |\partial(E_{\psi_j}\cap\mathbb{H}^n_+)|_{\mathbb{H}}(\omega_0\cdot\mathbb{R}^+)\\
&= |\partial E_\theta|_{\mathbb{H}}(\partial\omega_0\cdot\mathbb{R}^+) + |\partial E_{\psi_j}|_{\mathbb{H}}(\omega_0\cdot(0,+\infty)) + |\partial(E_{\psi_j}\cap\mathbb{H}^n_+)|_{\mathbb{H}}(\mathbb{W})\\
&\leqslant |\partial E_\theta|_{\mathbb{H}}(\partial\omega_0\cdot\mathbb{R}^+) + |\partial E_\theta|_{\mathbb{H}}((\omega_0\setminus\overline{\omega})\cdot(0,+\infty)) + |\partial E_{\psi_j}|_{\mathbb{H}}(\overline{\omega}\cdot\mathbb{R}) + c_n\mathcal{S}^{Q-1}_\infty(\omega_0),
\end{aligned}$$

whence $|\partial(E_{\psi_j} \cap \mathbb{H}^n_+)|_{\mathbb{H}}(\mathbb{H}^n)$ can be bounded uniformly in j. Similar computations can be carried out also for $|\partial(E^{\psi_j} \cap \mathbb{H}^n_-)|_{\mathbb{H}}(\mathbb{H}^n)$, and (4.74) follows.
By the isoperimetric inequality (2.3) we obtain the bound

$$\sup_j \left\{|E_{\psi_j} \cap \mathbb{H}^n_+| + |E^{\psi_j} \cap \mathbb{H}^n_-|\right\} \leqslant c \tag{4.75}$$

for some $c \geqslant 0$. Using (2.7)

$$\int_{\omega_0} |\psi_j|\, d\mathcal{L}^{2n} = |E_{\psi_j} \Delta \mathbb{H}^n_-| = |E_{\psi_j} \cap \mathbb{H}^n_+| + |E^{\psi_j} \cap \mathbb{H}^n_-| \leqslant c.$$

Therefore, the sequence $(\psi_j)_j$ is bounded in $BV_{\mathbb{W}}(\omega_0)$; moreover we have

$$\begin{aligned} &|\partial E_{\psi_j}|_{\mathbb{H}}(\mathbb{H}^n_+) + |\partial E^{\psi_j}|_{\mathbb{H}}(\mathbb{H}^n_-) \\ &\leqslant |\partial(E_{\psi_j} \cap \mathbb{H}^n_+)|_{\mathbb{H}}(\mathbb{H}^n) + |\partial(E^{\psi_j} \cap \mathbb{H}^n_-)|_{\mathbb{H}}(\mathbb{H}^n) + 2c_n \mathcal{S}^{Q-1}_\infty(\omega_0) \end{aligned}$$

and thus (4.65) holds because of (4.74). We can now apply Proposition 4.18 and get the existence of $\psi \in BV_{\mathbb{W}}(\omega_0)$ such that, up to subsequences, $\psi_j \to \psi$ $\mathcal{L}^{2n}$-a.e. on ω and $E_{\psi_j} \to E_\psi$ in $L^1_{loc}(\omega_0 \cdot \mathbb{R})$; in particular, $\psi = \theta$ on $\omega_0 \setminus \overline{\omega}$. By (4.67) we obtain

$$\begin{aligned} |\partial E_\psi|_{\mathbb{H}}(\overline{\omega} \cdot \mathbb{R}) &= |\partial E_\psi|_{\mathbb{H}}(\omega_0 \cdot \mathbb{R}) - |\partial E_\theta|_{\mathbb{H}}((\omega_0 \setminus \overline{\omega}) \cdot \mathbb{R}) \\ &\leqslant \liminf_{j\to\infty} \left\{|\partial E_{\psi_j}|_{\mathbb{H}}(\omega_0 \cdot \mathbb{R}) - |\partial E_\theta|_{\mathbb{H}}((\omega_0 \setminus \overline{\omega}) \cdot \mathbb{R})\right\} \\ &= \liminf_{j\to\infty} |\partial E_{\psi_j}|_{\mathbb{H}}(\overline{\omega} \cdot \mathbb{R}), \end{aligned} \tag{4.76}$$

i.e., ψ is a minimum for the problem (1.17). □

We single out the following result, implicitly obtained along the proof of Theorem 1.8. The assumptions on ω_0 given in Definition 4.11 are understood.

Proposition 4.19. *Let the sequence $(\phi_j)_{j\geqslant 1} \subset BV_{\mathbb{W}}(\omega)$ be bounded in the $BV_{\mathbb{W}}$ norm; assume that, for any j, $\phi_{j|\partial\omega} = [\phi_0]$ for a fixed TGS $\phi_0 : \omega_0 \setminus \overline{\omega} \to \mathbb{R}$. Then there exists $\phi \in BV_{\mathbb{W}}(\omega)$ such that, up to subsequences, $\phi_j \to \phi$ $\mathcal{L}^{2n}$-a.e. on ω and $E_{\phi_j} \to E_\phi$ in $L^1_{loc}(\omega \cdot \mathbb{R})$. Moreover, the function $\theta : \omega_0 \to \mathbb{R}$ defined by $\theta := \phi$ on ω, $\theta := \phi_0$ on $\omega_0 \setminus \overline{\omega}$ belongs to $BV_{\mathbb{W}}(\omega_0)$.*

Proof. For any $j \geqslant 1$ define $\theta_j \in BV_{\mathbb{W}}(\omega_0)$ by

$$\theta_j := \phi_j \text{ on } \omega,\ \theta_j := \phi_0 \text{ on } \omega_0 \setminus \overline{\omega}\,.$$

We have

$$\begin{aligned} |\partial E_{\theta_j}|_{\mathbb{H}}(\mathbb{H}^n_+) \leqslant\ & |\partial E_{\phi_j}|_{\mathbb{H}}(\omega \cdot \mathbb{R}^+) + |\partial E_{\phi_0}|_{\mathbb{H}}((\omega_0 \setminus \overline{\omega}) \cdot \mathbb{R}^+) \\ &+ |\partial E_{\phi_0}|_{\mathbb{H}}(\partial\omega_0 \cdot \mathbb{R}^+) + c_n \mathcal{S}^{Q-1}_\infty(\omega_0) \end{aligned}$$

and, since similar computations can be carried out in order to bound $|\partial E^{\theta_j}|_{\mathbb{H}}(\mathbb{H}^n_-)$, we get (4.65). The boundedness of $(\theta_j)_j$ in $BV_{\mathbb{W}}(\omega_0)$ can be easily achieved and Proposition 4.18 provides a function $\theta \in BV_{\mathbb{W}}(\omega_0)$ such that, up to subsequences, $\theta_j \to \theta$ $\mathcal{L}^{2n}$-a.e. on ω_0. Notice that $\theta = \phi_0$ on $\omega_0 \setminus \overline{\omega}$. The function $\phi := \theta_{|\omega}$ is the desired one. □

4.3. **Local boundedness of minimal X-graphs.** In the proof of Theorem 1.5 (see (3.37)) we used the fact that for any $K \Subset \mathcal{U}$ one has

$$R := d_\infty(K \times \mathbb{R}, \partial\mathcal{U} \times \mathbb{R}) > 0\,. \tag{4.77}$$

In general the inequality in (4.77) is not strict for intrinsic cylinders. This is stated in the following Proposition 4.20, where the open set $\omega \subset \mathbb{W}$ is not necessarily bounded.

Proposition 4.20. *Let ω, ω' be two open subsets of $\mathbb{R}^{2n}$ with $\omega' \Subset \omega$. Then the following conditions are equivalent:*

(*i*) *$d_\infty(\omega' \cdot \mathbb{R}, \partial\omega \cdot \mathbb{R}) > 0$;*

(*ii*) *the Euclidean distance between $\partial\omega \subset \mathbb{W} \equiv \mathbb{R}^{2n}$ and the open strip*

$$\Sigma := \big\{(x_2, \dots, y_n, t) \in \mathbb{W} : \exists \bar t \in \mathbb{R} \text{ such that } (x_2, \dots, y_n, \bar t) \in \omega'\big\} \subset \mathbb{W}$$

is not zero.

Proof. The implication $(ii) \Rightarrow (i)$ follows immediately by observing that

$$d_\infty\big((x_2', \dots, y_n', t') \cdot s', (x_2, \dots, y_n, t) \cdot s\big) \geqslant |(x_2' - x_2, \dots, y_n' - y_n)| \geqslant dist(\Sigma, \partial\omega)$$

for any $(x_2', \dots, y_n', t') \in \omega', (x_2, \dots, y_n, t) \in \partial\omega,\ s, s' \in \mathbb{R}$.

Concerning the reverse implication, we will provide the proof only for $n \geqslant 2$, the case $n = 1$ being analogous. By contradiction, we assume that for any $\epsilon > 0$ there exist $A = (x_2, \dots, y_n, t) \in \partial\omega$ and $A' = (x_2', \dots, y_n', t') \in \omega'$ such that

$$|x_i' - x_i| < \epsilon \quad \text{and} \quad |y_j' - y_j| < \epsilon, \qquad i = 2, \dots, n,\ j = 1, \dots n.$$

Since ω' is open, it is not restrictive to suppose $y_1 \neq y_1'$. Choose

$$s := \frac{t - t' - 2\sum_{j=2}^n (x_j y_j' - y_j x_j')}{4(y_1' - y_1)}$$

and consider the points

$$P := A \cdot s \in \partial\omega \cdot \mathbb{R}, \qquad Q := A' \cdot s \in \omega' \cdot \mathbb{R}$$

It is a matter of fact that

$$Q = P \cdot \big(0, x_2' - x_2, \dots, x_n' - x_n, y_1' - y_1, \dots, y_n' - y_n, 0\big),$$

whence $d_\infty(P, Q) < \sqrt{2n-1}\epsilon$ and $d_\infty(\omega' \cdot \mathbb{R}, \partial\omega \cdot \mathbb{R}) \leqslant \sqrt{2n-1}\epsilon$. By the arbitrariness of ϵ, this contradicts assumption (i). □

It is easy to see that, when the open set ω is bounded, there is no nonempty subset $\omega' \subset \omega$ for which condition (ii) in Proposition 4.20 is satisfied. In particular, the distance $d_\infty(\omega' \cdot \mathbb{R}, \partial\omega \cdot \mathbb{R})$ is always zero: the two intrinsic cylinders $\omega' \cdot \mathbb{R}$ and $\partial\omega \cdot \mathbb{R}$ get closer and closer at infinity. The following Lemma 4.21 allows to control how fast these cylinders get close at infinity.

Lemma 4.21. *Let $\omega \subset \mathbb{W}$ be a bounded open set. Then there exists $c = c(\omega) > 0$ such that*

$$d_\infty(A \cdot s, \partial\omega \cdot \mathbb{R}) \geqslant c\frac{dist(A, \partial\omega)}{|s|} \qquad \forall A \in \omega, \forall s \in \mathbb{R}, |s| \geqslant 1.$$

Proof. We accomplish the proof only for $n \geqslant 2$. Let $A \in \omega, s \in \mathbb{R}, |s| \geqslant 1$ be fixed; set $P := A \cdot s \in \omega \cdot \mathbb{R}$ and $d := dist(A, \partial\omega)$. We have to prove that, for a suitable $c > 0$ that will be determined later, $U(P, c\,d/|s|) \subset \omega \cdot \mathbb{R}$. Namely, that

$$P \cdot (x, y, t) \in \omega \cdot \mathbb{R} \quad \text{for any } (x, y, t) \in U(0, cd/|s|)$$

or, equivalently, that $\pi_{\mathbb{W}}(P \cdot (x, y, t)) \in \omega$ for any such (x, y, t). Therefore, it will be sufficient that

$$dist\big(A, \pi_{\mathbb{W}}(P \cdot (x, y, t))\big) < d \quad \text{for any } (x, y, t) \in U(0, cd/|s|).$$

Setting $A = (0, \bar{x}_2, \ldots, \bar{y}_n, \bar{t})$, by (2.9), one has

$$\begin{aligned}
&\pi_{\mathbb{W}}(P \cdot (x, y, t)) \\
&= (0, \bar{x}_2, \ldots, \bar{y}_n, \bar{t}) \cdot s \cdot (x, y, t) \cdot -(s + x_1) \\
&= \big(0, \bar{x}_2 + x_2, \ldots, \bar{y}_n + y_n, \bar{t} + t - 4sy_1 - 2x_1y_1 + 2\textstyle\sum_{j=2}^n (x_j\bar{y}_j - y_j\bar{x}_j)\big)\,.
\end{aligned}$$

Setting $M := \sup_{A\in\omega} |A|$ and using $d \leqslant 2M$, we have

$$\begin{aligned}
&dist\big(A, \pi_{\mathbb{W}}(P \cdot (x, y, t))\big) \\
&\leqslant |x_2| + \cdots + |y_n| + |t - 4sy_1 - 2x_1y_1 + 2\textstyle\sum_{j=2}^n (x_j\bar{y}_j - y_j\bar{x}_j)| \\
&\leqslant \frac{(2n-1)cd}{|s|} + |t| + 4s|y_1| + 2|x_1y_1| + 2\textstyle\sum_{j=2}^n (|x_j\bar{y}_j| + |y_j\bar{x}_j|) \\
&\leqslant (2n-1)cd + \frac{c^2d^2}{s^2} + 4cd + 2\frac{c^2d^2}{s^2} + 4(n-1)M\frac{cd}{|s|} \\
&\leqslant cd\big(2n - 1 + 2cM + 4 + 4cM + 4(n-1)M\big)
\end{aligned}$$

where we also used that $|s| \geqslant 1$. It is then sufficient to choose $c > 0$ so small that $c(2n + 3 + 6cM + 4(n-1)M) \leqslant 1/2$. □

We also need to recall the following result, which extends [35, Lemma 4.3].

Lemma 4.22. *Suppose that $P \in \mathbb{H}^n, \alpha > 0, r > 0$ and $E \subset \mathbb{H}^n$ are such that $|E \cap U(P, r)| \geqslant \alpha r^Q$; then $\mathcal{L}^{2n}(\pi_{\mathbb{W}}(E \cap U(P, r))) \geqslant \frac{\alpha}{2}\, r^{Q-1}$.*

Proof. For any $A \in \mathbb{W} \equiv \mathbb{R}^{2n}$ we define $l(A) := \mathcal{L}^1\left(\{s \in \mathbb{R} : A \cdot s \in E \cap U(P, r)\}\right)$; let us prove that

$$\mathcal{L}^{2n+1}(E \cap U(P, r)) = \int_{\pi_{\mathbb{W}}(E\cap U(P,r))} l(A)\, d\mathcal{L}^{2n}(A)\,, \tag{4.78}$$

then the thesis easily follows being $l(A) \leqslant 2r$. Let us introduce a change of variables in order to apply the classical Fubini Theorem for the splitting $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{R}$. Let $\Psi : \mathbb{R}^{2n+1} \to \mathbb{H}^n \equiv \mathbb{R}^{2n+1}$ be defined by $\Psi(\xi_1, \ldots, \xi_{2n+1}) = (0, \xi_2, \ldots, \xi_{2n+1}) \cdot \xi_1$. It is easy to see that Ψ satisfies the following properties:

$$\Psi \text{ is a diffeomorphism and } \det\frac{\partial\Psi}{\partial\xi} = 1, \tag{4.79}$$

$$\text{if } \Psi_0 := \Psi_{|\{\xi_1=0\}}, \text{ then } \Psi_0 : \mathbb{R}^{2n} \to \mathbb{W} \equiv \mathbb{R}^{2n} \text{ is the identity map}\,, \tag{4.80}$$

where $\frac{\partial\Psi}{\partial\xi}$ denotes the Jacobian matrix of Ψ.

If $F \subset \mathbb{H}^n$, denote $\hat{F} := \Psi^{-1}(F)$. By (4.79)

$$\mathcal{L}^{2n+1}(E \cap U(P,r)) = \mathcal{L}^{2n+1}(\hat{E} \cap \widehat{U(P,r)})\,. \tag{4.81}$$

Set $\omega := \pi_{\mathbb{W}}(E \cap U(P,r))$ and notice that $\hat{\omega} = \omega$. For a fixed $A \in \mathbb{R}^{2n}$ define

$$L_A := \{s \in \mathbb{R} :\ (s,A) \in \Psi^{-1}(E \cap U(P,r)) = \hat{E} \cap \widehat{U(P,r)}\}.$$

From (4.79), (4.80) and Fubini Theorem

$$\mathcal{L}^{2n+1}(\hat{E} \cap \widehat{U(P,r)}) = \int_{\hat{\omega}} \mathcal{L}^1(L_A)\, d\mathcal{L}^{2n}(A) = \int_{\omega} l(A)\, d\mathcal{L}^{2n}(A)\,. \tag{4.82}$$

Equalities (4.81) and (4.82) give (4.78). $\square$

We have now all the tools to prove our local boundedness result for minimal intrinsic graphs.

Proof of Theorem 1.9. Up to a localization argument, it is not restrictive to assume $\phi \in L^{Q-1}(\omega)$. We reason by contradiction: assume there exists a compact set $K \Subset \omega$ such that $\|\phi\|_{L^\infty(K)} = +\infty$. Without loss of generality we assume that $\mathcal{L}^{2n}\big(K \cap \{\phi > M\}\big) > 0$ for any $M > 0$. Given $M > 0$ we can find $A(M) \in \omega$ such that $\phi(A(M)) > M$ and

$$\mathcal{L}^{2n}\big(\mathcal{V} \cap K \cap \{\phi > M\}\big) > 0 \tag{4.83}$$

for any neighbourhood $\mathcal{V}$ of $A(M)$. For instance, it is sufficient to consider a Lebesgue point $A(M)$ for the set $K \cap \{\phi > M\}$.

For $j \in \mathbb{N}$ consider the sequence $A_j := A(2^j) \in K$. By (4.83) the point $P_j := A_j \cdot 2^j$ is such that $|E_\phi \cap U(P_j, r)| > 0$ for any $r > 0$. Lemma 4.21 ensures that $U(P_j, c2^{-j}d) \subset \omega \cdot \mathbb{R}$, where $c = c(\omega)$ and $d := dist(K, \partial\omega)$. By Proposition 2.14 one has $|E_\phi \cap U(P_j, c2^{-j}d)| \geqslant \beta 2^{-jQ}$ for some $\beta > 0$. By Lemma 4.22

$$\mathcal{L}^{2n}\big(\pi_{\mathbb{W}}(E_\phi \cap U(P_j, c2^{-j}d))\big) \geqslant \kappa 2^{-j(Q-1)}$$

for some $\kappa > 0$. In particular, for large j (precisely: for those such that $2^j - c2^{-j}d > 2^{j-1}$) we will have $\pi_{\mathbb{W}}(E_\phi \cap U(P_j, c2^{-j}d)) \subset \{\phi > 2^{j-1}\}$ and so

$$\mathcal{L}^{2n}\big(\{\phi > 2^{j-1}\}\big) \geqslant \kappa 2^{-j(Q-1)}\,,$$

where $\kappa > 0$ depends only on n, K and ω. This implies that

$$\int_{\omega \cap \{\phi > 2^{j-1}\}} |\phi|^{Q-1}\, d\mathcal{L}^{2n} \geqslant 2^{1-Q}\kappa$$

for any large j and contradicts the hypothesis $\phi \in L^{Q-1}(\omega)$. $\square$

Theorem 1.9 is not the exact counterpart of Theorem 1.5 or of the classical Euclidean one (see for instance [39, Theorem 14.10]). We are presently not aware whether the additional L^{Q-1}_{loc} summability is only a technical problem, or if it can be relaxed into the weaker L^1_{loc} assumption.

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