Convergence of the parabolic Ginzburg-Landau equation to motion by mean curvature

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Abstract

For the complex parabolic Ginzburg-Landau equation, we prove that, asymptotically, vorticity evolves according to motion by mean curvature in Brakke's weak formulation. The only assumption is a natural energy bound on the initial data. In some cases, we also prove convergence to enhanced motion in the sense of Ilmanen.

Introduction

In this paper we study the asymptotic analysis, as the parameter ε goes to zero, of the complex-valued parabolic Ginzburg-Landau equation for functions $u_\varepsilon : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{C}$ in space dimension $N \geq 3$,

$$(\mathrm{PGL})_\varepsilon \qquad \begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} - \Delta u_\varepsilon = \dfrac{1}{\varepsilon^2} u_\varepsilon (1 - |u_\varepsilon|^2) & \text{on } \mathbb{R}^N \times (0, +\infty), \\ u_\varepsilon(x, 0) = u_\varepsilon^0(x) & \text{for } x \in \mathbb{R}^N . \end{cases}$$

It corresponds to the heat-flow for the Ginzburg-Landau energy

$$\mathcal{E}_\varepsilon(u) = \int_{\mathbb{R}^N} e_\varepsilon(u) = \int_{\mathbb{R}^N} \frac{|\nabla u|^2}{2} + V_\varepsilon(u) \qquad \text{for } u : \mathbb{R}^N \to \mathbb{C},$$

where V_ε denotes the non-convex potential

$$V_\varepsilon(u) = \frac{(1 - |u|^2)^2}{4\varepsilon^2}.$$

This energy plays an important role in physics, and has been studied extensively from the mathematical point of view in the last decades. It is well known that $(\mathrm{PGL})_\varepsilon$ is well-posed for initial datas in H^1_{loc} with finite Ginzburg-Landau energy $\mathcal{E}_\varepsilon(u_\varepsilon^0)$. Moreover, we have the energy identity

$$\mathcal{E}_\varepsilon(u_\varepsilon(\cdot, T_2)) + \int_{T_1}^{T_2} \int_{\mathbb{R}^N} \left| \frac{\partial u_\varepsilon}{\partial t} \right|^2 (x, t) dx\, dt = \mathcal{E}_\varepsilon(u_\varepsilon(\cdot, T_1)) \quad \forall\, 0 \leq T_1 \leq T_2 . \qquad \text{(I)}$$

We assume that the initial condition u_ε^0 verifies the bound, natural in this context,

$$(H_0) \qquad \mathcal{E}_\varepsilon(u_\varepsilon^0) \leq M_0|\log\varepsilon|,$$

where M_0 is a fixed positive constant. Therefore, in view of (I) we have

$$\mathcal{E}_\varepsilon(u_\varepsilon(\cdot,T)) \leq \mathcal{E}_\varepsilon(u_\varepsilon^0) \leq M_0|\log\varepsilon| \qquad \text{for all } T\geq 0. \tag{II}$$

The main emphasis of this paper is placed on the asymptotic limits of the Radon measures μ_ε defined on $\mathbb{R}^N\times[0,+\infty)$ by

$$\mu_\varepsilon(x,t) = \frac{e_\varepsilon(u_\varepsilon(x,t))}{|\log\varepsilon|}\,dx\,dt,$$

and of their time slices μ_ε^t defined on $\mathbb{R}^N\times\{t\}$ by

$$\mu_\varepsilon^t(x) = \frac{e_\varepsilon(u_\varepsilon(x,t))}{|\log\varepsilon|}\,dx,$$

so that $\mu_\varepsilon = \mu_\varepsilon^t\,dt$. In view of assumption (H_0) and (II), we may assume, up to a subsequence $\varepsilon_n\to 0$, that there exists a Radon measure μ_* defined on $\mathbb{R}^N\times[0,+\infty)$ such that

$$\mu_\varepsilon \rightharpoonup \mu_* \qquad \text{as measures.}$$

Actually, passing possibly to a further subsequence, we may also assume (see Lemma 1) that

$$\mu_\varepsilon^t \rightharpoonup \mu_*^t \qquad \text{as measures on } \mathbb{R}^N\times\{t\},\ \text{for all } t\geq 0.$$

Our main results describe the properties of the measures μ_*^t. We first have :

Theorem A. *There exist a subset Σ_μ in $\mathbb{R}^N\times(0,+\infty)$, and a smooth real-valued function Φ_* defined on $\mathbb{R}^N\times(0,+\infty)$ such that the following properties hold.*

i) Σ_μ is closed in $\mathbb{R}^N\times(0,+\infty)$ and for any compact subset $\mathcal{K}\subset\mathbb{R}^N\times(0,+\infty)\setminus\Sigma_\mu$

$$|u_\varepsilon(x,t)|\to 1 \qquad \textit{uniformly on } \mathcal{K} \textit{ as } \varepsilon\to 0.$$

ii) For any $t>0$, $\Sigma_\mu^t\equiv\Sigma_\mu\cap\mathbb{R}^N\times\{t\}$ verifies

$$\mathcal{H}^{N-2}(\Sigma_\mu^t)\leq KM_0.$$

iii) The function Φ_* verifies the heat equation on $\mathbb{R}^N\times(0,+\infty)$.

iv) For each $t>0$, the measure μ_*^t can be exactly decomposed as

$$\mu_*^t = |\nabla\Phi_*|^2\mathcal{H}^N + \Theta_*(x,t)\mathcal{H}^{N-2}\llcorner\Sigma_\mu^t, \tag{III}$$

where $\Theta_*(\cdot,t)$ is a bounded function.

v) There exists a positive function η defined on $\mathbb{R}_*^+$ such that, for almost every $t > 0$, the set Σ_μ^t is (N-2)-rectifiable and

$$\Theta_*(x,t) = \Theta_{N-2}(\mu_*^t, x) = \lim_{r\to 0} \frac{\mu_*^t(B(x,r))}{\omega_{N-2}r^{N-2}} \geq \eta(t),$$

for $\mathcal{H}^{N-2}$ a.e. $x \in \Sigma_\mu^t$.

Remark 1. Theorem A remains valid also for $N = 2$. In that case Σ_μ^t is therefore a finite set.

In view of the decomposition (III), μ_*^t can be split into two parts. A diffuse part $|\nabla\Phi_*|^2$, and a concentrated part

$$\nu_*^t = \Theta_*(x,t)\mathcal{H}^{N-2}\llcorner\Sigma_\mu^t.$$

By iii), the diffuse part is governed by the heat equation. Our next theorem focuses on the evolution of the concentrated part ν_*^t as time varies.

Theorem B. *The family $(\nu_*^t)_{t>0}$ is a mean curvature flow in the sense of Brakke [13].*

Comment. We recall that there exists a classical notion of mean curvature flow for smooth compact embedded manifolds. In this case, the motion corresponds basically to the gradient flow for the area functional. It is well known that such a flow exists for small times (and is unique), but develops singularities in finite time. Asymptotic behavior (for convex bodies) and formation of singularities have been extensively studied in particular by Huisken (see [27, 28] and the references therein). Brakke [13] introduced a weak formulation which allows to encompass singularities and makes sense for (rectifiable) measures. Whereas it allows to handle a large class of objects, an important and essential flaw of Brakke's formulation is that there is never uniqueness. Even though non uniqueness is presumably an intrinsic property of mean curvature flow when singularities appear, a major part of non uniqueness in Brakke's formulation is non intrinsic, and therefore allows for weird solutions. A stronger notion of solution will be discussed in Theorem D.

More precise definitions of the above concepts will be provided in the introduction of Part II.

The proof of Theorem B relies both on the measure theoretic analysis of Ambrosio and Soner [4], and on the analysis of the structure of μ_*, in particular the statements in Theorem A. In [4], Ambrosio and Soner proved the result in Theorem B under the additional assumption

$$(AS) \qquad \limsup_{r\to 0} \frac{\mu_*^t(B(x,r))}{\omega_{N-2}r^{N-2}} \geq \eta, \qquad \text{for } \mu_*^t\text{-a.e } x,$$

for some constant $\eta > 0$. In view of the decomposition (III), assumption (AS) holds if and only if $|\nabla\Phi_*|^2$ vanishes, i.e. there is no diffuse energy. If $|\nabla\Phi_*|^2$ vanishes, it follows therefore that Theorem B can be directly deduced from [4] Theorem 5.1 and statements *iv)* and *v)* in Theorem A.

In the general case where $|\nabla\Phi_*|^2$ does not vanish, their argument has to be adapted, however without major changes. Indeed, one of the important consequences of our analysis is that the concentrated and diffuse energies do not interfere.

In view of the previous discussion, one may wonder if some conditions on the initial data will guarantee that there is no diffuse part. In this direction, we introduce the conditions

$$(H_1) \qquad u_\varepsilon^0 \equiv 1 \qquad \text{in } \mathbb{R}^N \setminus B(R_1)$$

for some $R_1 > 0$, and

$$(H_2) \qquad \left\| u_\varepsilon^0 \right\|_{H^{\frac{1}{2}}(B(R_1))} \leq M_2.$$

Theorem C. *Assume that u_ε^0 verifies (H_0), (H_1) and (H_2). Then $|\nabla\Phi_*|^2 = 0$, and the family $(\mu_*^t)_{t>0}$ is a mean curvature flow in the sense of Brakke.*

In stating conditions (H_1) and (H_2) we have not tried to be exhaustive, and there are many ways to generalize them.

We now come back to the already mentioned difficulty related to Brakke's weak formulation, namely the strong non-uniqueness. To overcome this difficulty, Ilmanen [31] introduced the stronger notion of enhanced motion, which applies to a slightly smaller class of objects, but has much better uniqueness properties (see [31]). In this direction we prove the following.

Theorem D. *Let $\mathcal{M}_0$ be any given integer multiplicity (N-2)-current without boundary, with bounded support and finite mass. There exists a sequence $(u_\varepsilon^0)_{\varepsilon>0}$ and an integer multiplicity (N-1)-current $\mathcal{M}$ in $\mathbb{R}^N \times [0, +\infty)$ such that*

$$i)\ \partial\mathcal{M} = \mathcal{M}_0\,, \qquad ii)\ \mu_*^0 = \pi|\mathcal{M}_0|\,,$$

and such that the pair $\left(\mathcal{M}, \frac{1}{\pi}\mu_*^t\right)$ is an enhanced motion in the sense of Ilmanen [31].

Remark 2. Our result is actually a little stronger than the statement of Theorem D. Indeed, we will show that **any** sequence u_ε^0 satisfying $Ju_\varepsilon^0 \rightharpoonup \pi\mathcal{M}_0$ and $\mu_*^0 = \pi|\mathcal{M}_0|$ gives rise to an Ilmanen motion. [Ju_ε^0 denotes the Jacobian of u_ε^0 (see the introduction of Part II).]

The equation $(\mathrm{PGL})_\varepsilon$ has already been considered in recent years. In particular, the dynamics of vortices has been described in the two dimensional case (see [32, 36]). Concerning higher dimensions $N \geq 3$, under the assumption that the initial measure is concentrated on a smooth manifold, a conclusion similar to ours has been obtained first on a formal level by Pismen and Rubinstein [44], and then rigorously by Jerrard and

Soner [33] and Lin [37], in the time interval where the classical solution exists, that is only before the appearance of singularities. As already mentioned, a first convergence result past the singularities has been obtained by Ambrosio and Soner [4], under the crucial density assumption (AS) for the measures μ_*^t discussed above. Some important asymptotic properties for solutions of $(\mathrm{PGL})_\varepsilon$ were also considered in [40, 53, 9].

Beside these works, we had at least two important sources of inspiration in our study. The first one was the corresponding theory for the elliptic case, developed in the last decade, in particular in [7, 51, 11, 46, 38, 39, 8, 34, 12, 10]. The second one was the corresponding theory for the scalar case (i.e. the Allen-Cahn equation) developed in particular in [17, 21, 18, 22, 30, 49]. The outline of our paper bears some voluntary resemblance with the work of Ilmanen [30] (and Brakke [13]) : to stress this analogy, we will try to adopt their terminology as far as this is possible. In particular, the Clearing-Out Lemma is a step-stone in the proofs of Theorems A to D.

We divide the paper into two distinct parts. The first and longest one deals with the analysis of the functions u_ε, for fixed ε. This part involves mainly PDE techniques. The second part is devoted to the analysis of the limiting measures, and borrows some arguments of Geometric Measure Theory. The last step of the argument there will be taken directly from Ambrosio and Soner's work [4]. The transition between the two parts is realized through delicate pointwise energy bounds which allow to translate a clearing-out lemma for functions into one for measures.

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Contents

Part I

PDE Analysis of $(\mathbf{PGL})_\varepsilon$.

Introduction

In this part, we derive a number of properties of solutions u_ε of $(\mathrm{PGL})_\varepsilon$, which enter directly in the proof of the Clearing-Out Lemma (the proof of which will be completed at the beginning of Part II). We believe however that the techniques and results in this part have also an independent interest. Throughout this part, we will assume that $0 < \varepsilon < 1$. Unless explicitly stated, all the results here also hold in the two dimensional case $N = 2$. In our analysis, the sets

$$\mathcal{V}_\varepsilon = \left\{(x,t) \in \mathbb{R}^N \times (0,+\infty),\ |u_\varepsilon(x,t)| \leq \frac{1}{2}\right\},$$

as well as their time slices $\mathcal{V}_\varepsilon^t = \mathcal{V}_\varepsilon \cap (\mathbb{R}^N \times \{t\})$ will play a central role. We will loosely refer to $\mathcal{V}_\varepsilon$ as the **vorticity set**. [In the scalar case, such a set is often referred to as the "interfaces" or "jump set"].

The two main ingredients in the proof of the Clearing-Out are a clearing-out theorem for vorticity, as well as some precise pointwise (renormalized) energy bounds.

1 Clearing-out and annihilation for vorticity

The main result here is the following.

Theorem 1. *Let $0 < \varepsilon < 1$, u_ε be a solution of $(PGL)_\varepsilon$ with $\mathcal{E}_\varepsilon(u_\varepsilon^0) < +\infty$, and $\sigma > 0$ be given. There exists $\eta_1 = \eta_1(\sigma) > 0$ depending only on the dimension N and on σ such that if*

$$\int_{\mathbb{R}^N} e_\varepsilon(u_\varepsilon^0) \exp(-\frac{|x|^2}{4}) \leq \eta_1 |\log \varepsilon| \,, \tag{1}$$

then

$$|u_\varepsilon(0,1)| \geq 1 - \sigma \,. \tag{2}$$

Note that here we do not need assumption (H_0). This kind of result was obtained for $N = 3$ in [40], and for $N = 4$ in [53]. The corresponding result for the stationary case was established in [11, 51, 46, 38, 39, 8]. The restrictions on the dimension in [40, 53] seem essentially due to the fact that the term $\frac{\partial u}{\partial t}$ in $(\mathrm{PGL})_\varepsilon$ is treated there as a perturbation of the elliptic equation. Instead, our approach will be more parabolic in nature. Finally, let us mention that a result similar to Theorem 1 also holds in the scalar case, and enters in Ilmanen's framework (see [30] page 436) : the proof there is fairly direct and elementary.

Our (rather lengthy) proof of Theorem 1 involves a number of tools, some of which were already used in a similar context. In particular :

• A monotonicity formula which in our case was derived first by Struwe ([51], see also [19]), in its study of the heat-flow for harmonic maps. Similar monotonicity formulas were derived by Huisken [28] for the mean curvature flow, and Ilmanen [30] for the Allen-Cahn equation.

- A localization property for the energy (see Proposition 2.4) following a result of Lin and Rivière [40] (see also [37]).
- Refined Jacobian estimates due to Jerrard and Soner [34],

and many of the techniques and ideas that were introduced for the stationary equation.

Equation $(\mathrm{PGL})_\varepsilon$ has standard scaling properties. If u_ε is a solution to $(\mathrm{PGL})_\varepsilon$, then for $R > 0$ the function

$$v_\varepsilon(x,t) \equiv u_\varepsilon(Rx, R^2 t)$$

is a solution to $(\mathrm{PGL})_{R^{-1}\varepsilon}$. We may then apply Theorem 1 to v_ε. For this purpose, define, for $z_* = (x_*, t_*) \in \mathbb{R}^N \times (0, +\infty)$ the scaled weighted energy, taken at time $t = t_*$,

$$\tilde{\mathcal{E}}_{w,\varepsilon}(u_\varepsilon, z_*, R) \equiv \tilde{\mathcal{E}}_{w,\varepsilon}(z_*, R) = \frac{1}{R^{N-2}} \int_{\mathbb{R}^N} e_\varepsilon(u_\varepsilon(x,t_*)) \exp(-\frac{|x-x_*|^2}{4R^2}) dx\,.$$

We have the following

Proposition 1. *Let $T > 0$, $x_T \in \mathbb{R}^N$, and set $z_T = (x_T, T)$. Assume u_ε is a solution to $(PGL)_\varepsilon$ on $\mathbb{R}^N \times [0,T)$ and let $R > \sqrt{2\varepsilon}$. Assume moreover*

$$\tilde{\mathcal{E}}_{w,\varepsilon}(z_T, R) \leq \eta_1(\sigma)|\log \varepsilon|\,, \tag{3}$$

then

$$|u_\varepsilon(x_T, T + R^2)| \geq 1 - \sigma\,. \tag{4}$$

The condition in (3) involves an integral on the whole of $\mathbb{R}^N$. In some situations, it will be convenient to integrate on finite domains. From this point of view, **assuming** (H_0) we have the following (in the spirit of Brakke's original Clearing-Out [13], Lemma 6.3, but for vorticity here, not yet for the energy!).

Proposition 2. *Let u_ε be a solution of $(PGL)_\varepsilon$ verifying assumption (H_0) and $\sigma > 0$ be given. Let $x_T \in \mathbb{R}^N$, $T > 0$ and $R \geq \sqrt{2\varepsilon}$. There exists a positive continuous function λ defined on $\mathbb{R}_*^+$ such that, if*

$$\check{\eta}(x_T, T, R) \equiv \frac{1}{R^{N-2}|\log \varepsilon|} \int_{B(x_T, \lambda(T)R)} e_\varepsilon(u_\varepsilon(\cdot, T)) \leq \frac{\eta_1(\sigma)}{2}$$

then

$$|u_\varepsilon(x,t)| \geq 1 - \sigma \qquad \textit{for } t \in [T + T_0, T + T_1] \quad \textit{and } x \in B(x_T, \frac{R}{2})\,.$$

Here T_0 and T_1 are defined by

$$T_0 = \max(2\varepsilon, \left(\frac{2\check{\eta}}{\eta_1(\sigma)}\right)^{\frac{2}{N-2}} R^2), \qquad T_1 = R^2.$$

Remark 3. It follows from the proof that $\lambda(T)$ diverge as $T \to 0$. More precisely,

$$\lambda(T) \sim \sqrt{\frac{N-2}{2}|\log T|} \qquad \text{as } T \to 0,$$

if $N \geq 3$. A slightly improved version will be proved and used in Section 4.1.

Theorem 1 and Propositions 1 and 2 have many consequences. Some are of independent interest. For instance, the simplest one is the complete annihilation of vorticity for $N \geq 3$.

Proposition 3. *Assume that $N \geq 3$. Let u_ε be a solution of $(PGL)_\varepsilon$ verifying assumption (H_0). Then*

$$|u_\varepsilon(x,t)| \geq \frac{1}{2} \qquad \text{for any } t \geq T_f \equiv \left(\frac{M_0}{\eta_1}\right)^{\frac{2}{N-2}} \text{ and for all } x \in \mathbb{R}^N, \tag{5}$$

where $\eta_1 = \eta_1(\frac{1}{2})$.

In particular, there exists a function φ defined on $\mathbb{R}^N \times [T_0, +\infty)$ such that

$$u_\varepsilon = \rho \exp(i\varphi)\,, \qquad \rho = |u_\varepsilon|.$$

The equation for the phase φ is then the linear parabolic equation

$$\rho^2 \frac{\partial \varphi}{\partial t} - \operatorname{div}(\rho^2 \nabla \varphi) = 0. \tag{6}$$

From this equation (and the equation for ρ) one may prove that, for fixed ε,

$$\mathcal{E}_\varepsilon(u_\varepsilon(\cdot,t)) \to 0 \qquad \text{as } t \to +\infty\,, \tag{7}$$

and moreover,

$$u_\varepsilon(\cdot,t) \to C \qquad \text{as } t \to +\infty\,. \tag{8}$$

Remark 4. The result of Proposition 3 **does not** hold in dimension 2. This fact is related to the so-called "slow motion of vortices" as established in [36]: vortices essentially move with a speed of order $|\log \varepsilon|^{-1}$. Therefore, a time of order $|\log \varepsilon|$ is necessary to annihilate vorticity (compared with the time $T = O(1)$ in Proposition 3). On the other hand, long-time estimates, similar to (7) and (8) have been established, for $N = 2$, in [5].

2 Improved pointwise energy bounds

Assume for a moment that $|u_\varepsilon| = 1$ on $\mathbb{R}^N \times [0, +\infty)$ [and in particular $\mathcal{V}_\varepsilon = \emptyset$]. Then, we may write $u_\varepsilon = \exp(i\varphi_\varepsilon)$ and φ_ε is determined, up to an integer multiple of 2π, by the linear parabolic problem

$$\begin{cases} \frac{\partial\varphi}{\partial t} - \Delta\varphi = 0 & \text{on } \mathbb{R}^N \times (0, +\infty) \\ \varphi(x, 0) = \varphi_\varepsilon(x, 0) & \text{on } \mathbb{R}^N \times \{0\}. \end{cases} \tag{9}$$

By standard regularization properties of the heat equation, we deduce that for any compact $K \subset \mathbb{R}^N \times (0, \infty)$,

$$|\nabla\varphi_\varepsilon|^2_{L^\infty(K)} \leq C(K) \int_{\mathbb{R}^N} \frac{|\nabla\varphi_\varepsilon|^2}{2}(x, 0)\, dx = C(K)\mathcal{E}_\varepsilon(u^0_\varepsilon),$$

so that

$$\lim_{r\to 0} \frac{1}{r^N} \int_{B(x,r)\times\{t\}} \frac{|\nabla\varphi_\varepsilon|^2}{|\log\varepsilon|} \leq M_0 C(t), \quad \forall x \in \mathbb{R}^N,\ \forall t > 0.$$

In particular, going back to the discussion of the main introduction of this paper, it means that the measures μ^t_* are absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^N(\mathbb{R}^N)$, i.e. $\mu^t_* = g(x,t)\mathcal{H}^N$ for some diffuse density g. Since (9) is linear, one cannot expect that g vanishes without additional assumptions, for instance compactness assumptions on the initial data u^0_ε (see [15] for related remarks in the elliptic case).

In the general situation, it is of course impossible to impose $|u_\varepsilon| = 1$. However, on the complement of $\mathcal{V}_\varepsilon$, $|u_\varepsilon| \geq \frac{1}{2}$ and the situation is quite similar. More precisely, we have

Theorem 2. *Let $B(x_0, R)$ be a ball in $\mathbb{R}^N$ and $T > 0$, $\Delta T > 0$ be given. Consider the cylinder*

$$\Lambda = B(x_0, R) \times [T, T + \Delta T].$$

There exists a constant $0 < \sigma \leq \frac{1}{2}$, and $\beta > 0$ depending only on N, such that if

$$|u_\varepsilon| \geq 1 - \sigma \qquad \text{on } \Lambda, \tag{10}$$

then

$$e_\varepsilon(u_\varepsilon)(x,t) \leq C(\Lambda) \left[\int_\Lambda e_\varepsilon(u_\varepsilon) + M_0 \varepsilon^\beta \right], \tag{11}$$

for any $(x,t) \in \Lambda_{\frac{1}{2}} = B(x_0, \frac{R}{2}) \times [T + \frac{\Delta T}{4}, T + \Delta T]$. Moreover,

$$e_\varepsilon(u_\varepsilon) = |\nabla\Phi_\varepsilon|^2 + \kappa_\varepsilon \qquad \text{in } \Lambda_{\frac{1}{2}}, \tag{12}$$

where the functions Φ_ε and κ_ε are defined on $\Lambda_{\frac{1}{2}}$ and verify

$$\frac{\partial\Phi_\varepsilon}{\partial t} - \Delta\Phi_\varepsilon = 0 \qquad \text{in } \Lambda_{\frac{1}{2}}, \tag{13}$$

$$\|\kappa_\varepsilon\|_{L^\infty(\Lambda_{\frac{1}{2}})} \leq C(\Lambda)\varepsilon^\beta, \qquad \|\nabla\Phi_\varepsilon\|^2_{L^\infty(\Lambda_{\frac{1}{2}})} \leq C(\Lambda) M_0 |\log\varepsilon|. \tag{14}$$

Remark 5. Since $|u_\varepsilon| \geq \frac{1}{2}$ on Λ, we may write $u_\varepsilon = \rho_\varepsilon \exp(i\varphi_\varepsilon)$ where $\rho_\varepsilon = |u_\varepsilon|$ and where φ_ε is a smooth real-valued function. The proof of Theorem 2 shows actually that

$$\|\nabla\varphi_\varepsilon - \nabla\Phi_\varepsilon\|_{L^\infty(\Lambda_{\frac{1}{2}})} \leq C(\Lambda)\varepsilon^\beta. \tag{15}$$

The result of Theorem 2 is reminiscent of a result by Chen and Struwe [19] (see also [51],[33]) developed in the context of the heat flow for harmonic maps. This technique is based on an earlier idea of Schoen [47] developed in the elliptic case. Note however that a smallness assumption on the energy is needed there. This is not the case for Theorem 2, where even a divergence of the energy (as $|\log\varepsilon|$) is allowed. We would like also to emphasize that the proofs of Theorem 1 and 2 are completely disconnected.

Combining Theorem 1 and Theorem 2, we obtain the following immediate consequence.

Proposition 4. *There exist an absolute constant $\eta_2 > 0$ and a positive function λ defined on $\mathbb{R}_*^+$ such that if, for $x \in \mathbb{R}^N$, $t > 0$ and $r > \sqrt{2\varepsilon}$, we have*

$$\int_{B(x,\lambda(t)r)} e_\varepsilon(u_\varepsilon) \leq \eta_2 r^{N-2}|\log\varepsilon|,$$

then

$$e_\varepsilon(u_\varepsilon) = |\nabla\Phi_\varepsilon|^2 + \kappa_\varepsilon$$

in $\Lambda_{\frac{1}{4}}(x,t,r) \equiv B(x,\frac{r}{4}) \times [t + \frac{15}{16}r^2, t + r^2]$, where Φ_ε and κ_ε are as in Theorem 2. In particular,

$$\mu_\varepsilon = \frac{e_\varepsilon(u_\varepsilon)}{|\log\varepsilon|} \leq C(t,r) \qquad \text{on } \Lambda_{\frac{1}{4}}(x,t,r).$$

[The constant η_2 is actually defined as $\eta_2 = \eta_1(\sigma)$, where σ is the constant in Theorem 2 and η_1 is the function defined in Proposition 2]

3 Identifying sources of non compactness

In the previous discussion, we already identified one possible source of non compactness, namely oscillations in the phase. However, the analysis was carried out on the complement of $\mathcal{V}_\varepsilon$, i.e. away from vorticity. On the vorticity set on the other hand, u_ε may vanish, and this introduces some new contribution to the energy. Nevertheless, we will show that this new contribution is **not** a source of non compactness (at least for some weaker norm). More precisely,

Theorem 3. *Let $\mathcal{K} \subset \mathbb{R}^N \times (0,+\infty)$ be any compact set. There exist a real-valued function ϕ_ε and a complex-valued function w_ε, both defined on a neighborhood of $\mathcal{K}$, such that*

1. $u_\varepsilon = w_\varepsilon \exp(i\phi_\varepsilon)$ *on $\mathcal{K}$,*

2. *ϕ_ε verifies the heat equation on $\mathcal{K}$,*

3. $|\nabla\phi_\varepsilon(x,t)| \leq C(\mathcal{K})\sqrt{M_0|\log\varepsilon|}$ *for all* $(x,t) \in \mathcal{K}$,

4. $\|\nabla w_\varepsilon\|_{L^p(\mathcal{K})} \leq C(p,\mathcal{K})$, *for any* $1 \leq p < \frac{N+1}{N}$.

Here, $C(\mathcal{K})$ *and* $C(p,\mathcal{K})$ *are constants depending only on* $\mathcal{K}$, *and* $\mathcal{K}$, p *(and* M_0*) respectively.*

The proof extends an argument of [9] (see also [6] for the elliptic case), and relies once more on the refined Jacobian estimates of [34].

We would like to emphasize once more that Theorem 3 provides an exact splitting of the energy in two different modes

- The topological mode, i.e. the energy related to w_ε
- The linear mode, i.e. the energy of ϕ_ε.

More precisely, it follows easily from Theorem 3 that for any set $\mathcal{K}' \subset\subset \mathcal{K}$, we have

$$\int_{\mathcal{K}'} e_\varepsilon(u_\varepsilon) = \int_{\mathcal{K}'} e_\varepsilon(w_\varepsilon) + \int_{\mathcal{K}'} |\nabla\phi_\varepsilon|^2 + O(\sqrt{|\log\varepsilon|}).$$

We would also like to stress also that a new and important feature of Theorem 3 is that ϕ_ε is defined and smooth **even across the singular set**, and verifies globally (on $\mathcal{K}$) the heat flow. Going back to Theorem A, this fact will be determinant to define the function Φ_* globally. For Theorem B, it will allow us to prove that the linear mode **does not** perturb the topological mode, which undergoes its own (Brakke) motion.

One possible way to remove the linear mode is to impose additional compactness on the initial data. We will not try to find the most general assumptions in that direction, but instead give simple conditions which keep however the essential features of the problem. Assume next that u_ε^0 verifies the additional conditions

$$(H_1) \qquad u_\varepsilon^0 \equiv 1 \quad \text{in } \mathbb{R}^N \setminus B(R_1)$$

for some $R_1 > 0$, and

$$(H_2) \qquad \left\|u_\varepsilon^0\right\|_{H^{\frac{1}{2}}(B(R_1))} \leq M_2.$$

Then a stronger conclusion holds.

Theorem 4. *Assume that* u_ε^0 *verifies* (H_0), (H_1) *and* (H_2). *Then for any* $1 \leq p < \frac{N+1}{N}$ *and any compact set* $\mathcal{K} \subset \mathbb{R}^N \times (0,+\infty)$, *we have*

$$\|\nabla u_\varepsilon\|_{L^p(\mathcal{K})} \leq C(p,\mathcal{K}),$$

where $C(p,\mathcal{K})$ *is a constant depending only on* p, $\mathcal{K}$, M_0 *and* M_2.

Theorem 4 is of course of particular interest if one is interested in the asymptotic behavior of the **function** u_ε itself. We will not carry out this analysis here (see [9] for a related discussion for boundary value problems on compact domains).

Combining Theorem 1, Theorem 2 and Theorem 4 we finally derive the following, in the same spirit as Proposition 4.

Proposition 5. *Assume that* $(H_0), (H_1)$ *and* (H_2) *hold. There exist an absolute constant* $\eta_2 > 0$ *and a positive function* λ *defined on* $\mathbb{R}_*^+$ *such that if, for* $x \in \mathbb{R}^N$, $t > 0$ *and* $r > \sqrt{2\varepsilon}$, *we have*

$$\int_{B(x,\lambda(t)r)} e_\varepsilon(u_\varepsilon) \leq \eta_2 r^{N-2} |\log \varepsilon|, \tag{16}$$

then

$$e_\varepsilon(u_\varepsilon) \leq C(M_0, M_2) r^{-2}$$

in $\Lambda_{\frac{1}{8}}(x,t,r) \equiv B(x, \frac{r}{8}) \times [t + \frac{63}{64}r^2, t + r^2]$.

[Here $\eta_2 = \eta_1(\sigma)$ is the same constant as in Proposition 4].

1 Pointwise estimates

In this section we recall (standard) pointwise parabolic estimates. Although these estimates are presumably well known to the experts, we are not aware of precise statements in the (Ginzburg-Landau) literature. For the reader's convenience, we therefore provide complete proofs.

Proposition 1.1. *Let* u_ε *be a solution of* $(PGL)_\varepsilon$ *with* $\mathcal{E}_\varepsilon(u_\varepsilon^0) < +\infty$. *Then there exists a constant* $K > 0$ *depending only on* N *such that, for* $t \geq \varepsilon^2$ *and* $x \in \mathbb{R}^N$, *we have*

$$|u_\varepsilon(x,t)| \leq 3, \qquad |\nabla u_\varepsilon(x,t)| \leq \frac{K}{\varepsilon}, \qquad |\frac{\partial u_\varepsilon}{\partial t}(x,t)| \leq \frac{K}{\varepsilon^2}.$$

[Note in particular that K *is independent of the initial data].*

Proof. It is convenient to make the following change of variable, setting

$$U(x,t) = u_\varepsilon(\varepsilon x, \varepsilon^2 t)\,,$$

so that the function U verifies

$$\frac{\partial U}{\partial t} - \Delta U = U(1 - |U|^2) \qquad \text{on } \mathbb{R}^N \times [0, +\infty)\,. \tag{1.1}$$

It is therefore sufficient to prove that for $t \geq 1$ and $x \in \mathbb{R}^N$,

$$|U(x,t)| \leq K, \qquad |\nabla U(x,t)| \leq K, \qquad |\frac{\partial U}{\partial t}(x,t)| \leq K.$$

We begin with the L^∞ estimate for U. Set

$$\sigma(x,t) := |U(x,t)|^2 - 1.$$

Multiplying equation (1.1) by U we are led to the equation for σ,

$$\frac{\partial \sigma}{\partial t} - \Delta\sigma + 2|\nabla U|^2 + 2(\sigma+1)\sigma = 0. \tag{1.2}$$

Consider next the ODE

$$y'(t) + 2(y(t)+1)y(t) = 0\,, \tag{1.3}$$

and notice that (1.3) possesses an explicit solution y_0 which blows-up as t tends to zero, namely

$$y_0(t) := \frac{\exp(-t/2)}{1-\exp(-t/2)} \qquad \text{for } t > 0\,.$$

We claim that

$$\sigma(t,x) \le y_0(t), \qquad \text{for all } t>0 \text{ and } x \in \mathbb{R}^N, \tag{1.4}$$

so that, in particular,

$$|U(x,t)|^2 = \sigma(x,t) + 1 \le 9 \quad \text{ for all } t \ge \frac{1}{4} \text{ and } x \in \mathbb{R}^N.$$

Indeed, set $\tilde\sigma(x,t) = y_0(t)$. Then,

$$\frac{\partial\tilde\sigma}{\partial t} - \Delta\tilde\sigma + 2(\tilde\sigma+1)\tilde\sigma = 0,$$

and therefore by (1.2),

$$\frac{\partial}{\partial t}(\tilde\sigma - \sigma) - \Delta(\tilde\sigma-\sigma) + 2(\tilde\sigma-\sigma)(1+\tilde\sigma+\sigma) \ge 0.$$

Note that $1+\sigma+\tilde\sigma = |U|^2 + \tilde\sigma \ge 0$. The maximum principle implies that

$$\tilde\sigma(x,t) - \sigma(x,t) \ge 0 \qquad \text{for all } t>0 \text{ and } x\in\mathbb{R}^N,$$

which proves the claim (1.4).

We next turn to the space and time derivatives. Since $|U(x,t)| \le 3$ for $t \ge 1/4$, we have

$$\left|U(1-|U|^2)\right| \le 24 \qquad \text{for } t \ge \frac{1}{4}.$$

Let $p > N+1$ be fixed. It follows from the standard regularity theory for the linear heat equation (see e.g. [35]) that for each compact set $\mathcal{F} \subset \mathbb{R}^N \times [1/4, +\infty)$ we have

$$\|\partial_t U\|_{L^p(\mathcal{F})} \le K(\mathcal{F}) \quad \text{and} \quad \|D^2 U\|_{L^p(\mathcal{F})} \le K(\mathcal{F}).$$

In particular, by the Sobolev embedding and the L^∞ bound for U we obtain

$$\|U\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^N\times[1/2,+\infty))} \leq K, \tag{1.5}$$

where $\alpha = (1 - N/p)/2$. It follows from (1.5) that moreover

$$\|U(1-|U|^2)\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^N\times[1/2,+\infty))} \leq K.$$

Invoking the $\mathcal{C}^{0,\alpha}$ regularity theory (see e.g. [24]), we obtain

$$\|U\|_{\mathcal{C}^{1,\alpha/2}(\mathbb{R}^N\times[1,+\infty))} \leq K,$$

and the proof is complete. □

Remark 1.1. It follows from the proof of Proposition 1.1 that the bound

$$|u_\varepsilon(x,t)|^2 \leq 1 + C\exp(-\frac{t}{2\varepsilon^2})$$

holds for $t \geq \varepsilon^2$.

We have the following variant of Proposition 1.1.

Proposition 1.2. *Assume u_ε is a solution of $(PGL)_\varepsilon$ such that $\mathcal{E}_\varepsilon(u_\varepsilon^0) < +\infty$. and that for some constants $C_0 \geq 1$, $C_1 \geq 0$ and $C_2 \geq 0$,*

$$|u_\varepsilon^0(x)| \leq C_0\,, \qquad |\nabla u_\varepsilon^0(x)| \leq \frac{C_1}{\varepsilon}\,, \qquad |D^2 u_\varepsilon^0(x)| \leq \frac{C_2}{\varepsilon^2} \qquad \forall\, x \in \mathbb{R}^N\,.$$

Then, for any $x \in \mathbb{R}^N$ and any $t > 0$,

$$|u_\varepsilon(x,t)| \leq C_0, \qquad |\nabla u_\varepsilon(x,t)| \leq \frac{K}{\varepsilon}\,, \qquad |\frac{\partial u_\varepsilon}{\partial t}(x,t)| \leq \frac{K}{\varepsilon^2}\,,$$

where K depends only on C_0, C_1 and C_2.

Proof. As in the proof of Proposition 1.1, we work with the rescaled function U. It follows from (1.2) and the maximum principle that

$$|U(x,t)| \leq \sup_{x\in\mathbb{R}^N} |U(0,x)| \leq C_0.$$

It remains to prove the bounds on the space and time derivatives. Since these estimates are already known for $t \geq 1$ by Proposition 1.1, we only need to consider the case $t \in (0,1]$. For the space derivative, we use the following Bochner type inequality

$$\frac{\partial}{\partial t}(|\nabla U|^2) - \Delta(|\nabla U|^2) \leq K|\nabla U|^2\,, \tag{1.6}$$

so that

$$\frac{\partial}{\partial t}(\exp(Kt)|\nabla U|^2) - \Delta(\exp(Kt)|\nabla U|^2) \leq 0.$$

The conclusion then follows from the maximum principle.

For the time derivative, one argues similarly, using the inequality

$$\frac{\partial}{\partial t}(|\frac{\partial U}{\partial t}|^2) - \Delta(|\frac{\partial U}{\partial t}|^2) \leq K|\frac{\partial U}{\partial t}|^2$$

and the fact that, for $t = 0$, we have by assumption

$$\left|\frac{\partial U}{\partial t}\right|^2 = \left|\Delta U + U(1 - |U|^2)\right|^2 \leq K\,.$$

□

Proposition 1.1 above provides an upper bound for $|u_\varepsilon|$. Our next lemma provides a **local** lower bound on $|u_\varepsilon|$, when we know it is away from zero on some region.

Since we have to deal with parabolic problems, it is natural to consider parabolic cylinders of the type

$$\Lambda_\alpha(x_0, T, R, \Delta T) = B(x_0, \alpha R) \times [T + (1 - \alpha^2)\Delta T, T + \Delta T].$$

Sometimes, it will be convenient to choose $\Delta T = R$ and write $\Lambda_\alpha(x_0, T, R)$. Finally if this is not misleading we will simply write Λ_α, and Λ if $\alpha = 1$.

Lemma 1.1. *Let u_ε be a solution of $(PGL)_\varepsilon$ verifying $\mathcal{E}_\varepsilon(u_\varepsilon^0) < +\infty$. Let $x_0 \in \mathbb{R}^N$, $R > 0$, $T \geq 0$ and $\Delta T > 0$ be given. Assume that*

$$|u_\varepsilon| \geq \frac{1}{2} \qquad \textit{on } \Lambda(x_0, T, R, \Delta T),$$

then

$$1 - |u_\varepsilon| \leq C(\alpha, \Lambda)\varepsilon^2 \left(\|\nabla\varphi_\varepsilon\|^2_{L^\infty(\Lambda)} + |\log\varepsilon|\right) \qquad \textit{on } \Lambda_\alpha,$$

where φ_ε is defined on Λ, up to a multiple of 2π, by $u_\varepsilon = |u_\varepsilon|\exp(i\varphi_\varepsilon)$.

Proof. We may always assume that $T \geq \varepsilon$, otherwise we consider a smaller cylinder. Set $\rho = |u_\varepsilon|$ and $\theta = 1 - \rho$. The function θ verifies the equation

$$\frac{\partial\theta}{\partial t} - \Delta\theta + \frac{\theta}{\varepsilon^2} = (1 - \theta)|\nabla\varphi_\varepsilon|^2 - \frac{1}{\varepsilon^2}\theta(\theta - 1)^2.$$

On the other hand, by Proposition 1.1, we already know that $\theta \geq -\exp(-\frac{1}{\varepsilon})$, so that

$$\frac{\partial\theta}{\partial t} - \Delta\theta + \frac{\theta}{\varepsilon^2} \leq 2|\nabla\varphi_\varepsilon|^2 + C\varepsilon^{-2}\exp(-\frac{1}{\varepsilon}). \tag{1.7}$$

We next construct an upper solution for (1.7). Let χ be a smooth cut-off function defined on $\mathbb{R}^N$ such that $0 \leq \chi \leq 1$ and

$$\chi \equiv 1 \text{ on } B(x_0, \alpha R), \qquad \chi \equiv 0 \text{ on } \mathbb{R}^N \setminus B(x_0, \frac{1+\alpha}{2}R).$$

Consider the function τ defined on $[T, T+\Delta T]$ by

$$\tau(t) = \frac{1}{2} - \frac{1}{2}\exp\left(\frac{t-T}{(1-\alpha^2)\Delta T}\log\varepsilon^2\right),$$

and set

$$\sigma_0(x,t) = \frac{1}{2} - \tau(t)\chi(x).$$

We have $\sigma_0 \geq 0$ and

$$|\partial_t\sigma_0| = |\tau'(t)|\chi(x) \leq \frac{1}{(1-\alpha^2)\Delta T}|\log\varepsilon|, \qquad |\Delta\sigma_0| \leq \tau(t)|\Delta\chi(x)| \leq C(\Lambda),$$

so that

$$\frac{\partial\sigma_0}{\partial t} - \Delta\sigma_0 + \frac{\sigma_0}{\varepsilon^2} \geq -C(\Lambda)|\log\varepsilon|.$$

Finally, set

$$\sigma = \sigma_0 + 2\varepsilon^2\left(\|\nabla\varphi_\varepsilon\|^2_{L^\infty(\Lambda)} + C(\Lambda)|\log\varepsilon|\right).$$

By construction,

$$\frac{\partial\sigma}{\partial t} - \Delta\sigma + \frac{\sigma}{\varepsilon^2} \geq 2\|\nabla\varphi_\varepsilon\|^2_{L^\infty(\Lambda)} + C(\Lambda)|\log\varepsilon| \geq \frac{\partial\theta}{\partial t} - \Delta\theta + \frac{\theta}{\varepsilon^2}.$$

On the other hand,

$$\sigma \geq \frac{1}{2} \geq \theta \qquad \text{on } B(x_0,R)\times\{T\} \cup \partial B(x_0,R)\times[T,T+\Delta T],$$

so that by the maximum principle $\theta \leq \sigma$ on Λ. Since $\chi \equiv 1$ on $B(x_0,\alpha R)$, we have on Λ_α

$$\begin{aligned}\sigma(x,t) &= \frac{1}{2}\exp\left(\frac{t-T}{(1-\alpha^2)\Delta T}\log\varepsilon^2\right) + 2\varepsilon^2\left(\|\nabla\varphi_\varepsilon\|^2_{L^\infty(\Lambda)} + C(\Lambda)|\log\varepsilon|\right) \\ &\leq \frac{1}{2}\varepsilon^2\left(\|\nabla\varphi_\varepsilon\|^2_{L^\infty(\Lambda)} + C(\Lambda)|\log\varepsilon|\right)\end{aligned}$$

and the proof is complete. □

2 Toolbox

The purpose of this section is to present a number of tools, which will enter directly in the proof of Theorem 1. As mentioned earlier, some of them are already available in the literature. We will adapt their statements to our needs. Note that all the results in this section remain valid for vector-valued maps $u_\varepsilon : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}^k$, for every $k \geq 1$, u_ε solution to $(\mathrm{PGL})_\varepsilon$.

2.1 Evolution of localized energies

Identity (I) of the introduction states a global decrease in time of the energy. In this section, we recall some classical results, describing the behavior of localized integrals of energy.

Lemma 2.1. *Let χ be a bounded lipschitz function on $\mathbb{R}^N$. Then, for any $T \geq 0$ we have*

$$\frac{d}{dt}\int_{\mathbb{R}^N\times\{T\}} e_\varepsilon(u_\varepsilon)\chi(x)\,dx = -\int_{\mathbb{R}^N\times\{T\}} |\partial_t u_\varepsilon|^2\chi(x)\,dx - \int_{\mathbb{R}^N\times\{T\}} \partial_t u_\varepsilon \nabla u_\varepsilon\cdot\nabla\chi\,dx. \quad (2.1)$$

In particular, for any $0 \leq T_1 \leq T_2$ we have

$$\int_{\mathbb{R}^N\times\{T_2\}} e_\varepsilon(u_\varepsilon)\chi(x)\,dx - \int_{\mathbb{R}^N\times\{T_1\}} e_\varepsilon(u_\varepsilon)\chi(x)\,dx = \\ -\int_{\mathbb{R}^N\times[T_1,T_2]} |\partial_t u_\varepsilon|^2\chi(x)\,dx\,dt - \int_{\mathbb{R}^N\times[T_1,T_2]} \partial_t u_\varepsilon\nabla u_\varepsilon\cdot\nabla\chi\,dx\,dt. \quad (2.2)$$

Proof. We have

$$\frac{d}{dt}\left([\frac{|\nabla u_\varepsilon|^2}{2} + V_\varepsilon(u_\varepsilon)]\chi\right) = \nabla u_\varepsilon\cdot\nabla(\partial_t u_\varepsilon)\,\chi + V_\varepsilon'(u_\varepsilon)\partial_t u_\varepsilon\,\chi.$$

Integrating by parts on $\mathbb{R}^N \times \{T\}$ we thus have

$$\frac{d}{dt}\int_{\mathbb{R}^N\times\{T\}} e_\varepsilon(u_\varepsilon)\chi(x)\,dx = -\int_{\mathbb{R}^N\times\{T\}} (-\Delta u_\varepsilon + V_\varepsilon'(u_\varepsilon))\,\partial_t u_\varepsilon\chi(x)\,dx \\ -\int_{\mathbb{R}^N\times\{T\}} \partial_t u_\varepsilon\nabla u_\varepsilon\cdot\nabla\chi\,dx$$

and the conclusion follows since u_ε verifies $(\mathrm{PGL})_\varepsilon$. □

As a straightforward consequence we obtain the following semi-decreasing property.

Corollary 2.1. *Let χ be as above, then*

$$\frac{d}{dt}\int_{\mathbb{R}^N\times\{T\}} e_\varepsilon(u_\varepsilon)\chi^2(x)\,dx \leq 2\,\|\nabla\chi\|^2_{L^\infty}\,\mathcal{E}_\varepsilon(u^0_\varepsilon).$$

2.2 The monotonicity formula

Let $u \equiv u_\varepsilon$ be a solution to $(\mathrm{PGL})_\varepsilon$ verifying (H_1). For simplicity, we will drop the subscripts ε when this is not misleading. For $(x_*, t_*) \in \mathbb{R}^N \times \mathbb{R}^+$ we set

$$z_* = (x_*, t_*)\,.$$

For $0 < R \leq \sqrt{t_*}$ we define the weighted energy

$$E_w(z_*, R) \equiv E_{w,\varepsilon}(u; z_*, R) \equiv \mathcal{E}_{w,\varepsilon}(u, x_*, t_* - R^2, R)\,, \quad (2.3)$$

that is,

$$E_w(z_*, R) = \int_{\mathbb{R}^N} e_\varepsilon(u(x, t_* - R^2)) \exp(-\frac{|x - x_*|^2}{4R^2}) dx\,, \tag{2.4}$$

and the corresponding scaled energy

$$\tilde{E}_w(z_*, R) = \frac{1}{R^{N-2}} E_w(z_*, R) = \frac{1}{R^{N-2}} \int_{\mathbb{R}^N} e_\varepsilon(u(x, t_* - R^2)) \exp(-\frac{|x - x_*|^2}{4R^2}) dx\,. \tag{2.5}$$

We emphasize the fact that the above integral is computed at the time $t = t_* - R^2$, and **not** at time $t = t_*$, as it is the case for $\mathcal{E}_\varepsilon$, i.e. a shift in time $\delta t = -R^2$ has been introduced. Note also that in (2.4) and (2.5) the weight becomes small outside the ball $B(x_*, R)$. Moreover, the following inequality holds

$$\exp(\frac{1}{4}) \tilde{E}_w(z_*, R) \geq \frac{1}{R^{N-2}} \int_{B(x_*, R)} e_\varepsilon(u(x, t_* - R^2)) dx\,. \tag{2.6}$$

The right-hand side of (2.6) arises naturally in the stationary equation, where its monotonicity properties (with respect to the radius R) play an important role. In our parabolic setting, we recall once more that the time t at which E_w and $\tilde{E}_w$ are computed is related to R by

$$t = t_* - R^2\,.$$

This is consistent with the usual parabolic scaling (for $\lambda > 0$)

$$\begin{cases} x & \to \lambda x \\ t & \to \lambda^2 t\,, \end{cases}$$

which leaves the linear heat equation invariant.

In this context, the following monotonicity formula for $\tilde{E}_w$ was derived first by Struwe [50] for the heat-flow of harmonic maps (see also [19, 28]). In a different context Giga and Kohn [26] used related ideas.

Proposition 2.1. *We have*

$$\begin{aligned} \frac{d\tilde{E}_w}{dR}(z_*, R)\Big|_{R=r} =& \frac{1}{r^{N-1}} \int_{\mathbb{R}^N \times \{t_* - r^2\}} \frac{1}{2r^2} [(x - x_*) \cdot \nabla u - 2r^2 \partial_t u]^2 \exp(-\frac{|x - x_*|^2}{4r^2}) dx \\ &+ \frac{1}{r^{N-1}} \int_{\mathbb{R}^N \times \{t_* - r^2\}} 2V_\varepsilon(u) \exp(-\frac{|x - x_*|^2}{4r^2}) dx \\ =& \frac{1}{2r} \int_{\mathbb{R}^N \times \{t_* - r^2\}} [(x - x_*) \cdot \nabla u + 2(t - t_*) \partial_t u]^2 G(x - x_*, t - t_*) dx \\ &+ r \int_{\mathbb{R}^N \times \{t_* - r^2\}} 2V_\varepsilon(u(x, t)) G(x - x_*, t - t_*) dx, \end{aligned} \tag{2.7}$$

where $G(x, t)$ denotes, up to a multiplicative factor $\pi^{-N/2}$, the heat kernel

$$\begin{cases} G(x, t) = & \frac{1}{t^{N/2}} \exp(-\frac{|x|^2}{4t}) \quad \textit{for } t > 0 \\ G(x, t) = & 0 \quad \textit{for } t \leq 0\,. \end{cases}$$

In particular,

$$\frac{d\tilde{E}_w}{dR}(z_*, R) \geq 0\,, \tag{2.8}$$

i.e. $\tilde{E}_w(z_*, R)$ *is a non-decreasing function of* R.

Proof. Set $\tilde{E}_w(R) \equiv \tilde{E}_w(z_*, R)$. Due to translation invariance, it is sufficient to consider the case $z_* = (x_*, t_*) = (0,0)$, so that u is defined on $\mathbb{R}^N \times [-t_*, +\infty)$. In order to keep the integration domain fixed with respect to R, we consider the following change of variables, for $z = (x, y) \in \mathbb{R}^N \times [-t_*, +\infty)$:

$$z = (x,t) = (Ry, R^2\tau) = \Phi_R(y, \tau) = \Phi_R(z')\,. \tag{2.9}$$

Set $u_R(z') = u \circ \Phi_R(z') = u(z)$, i.e. $u_R(y,\tau) = u(Ry, R^2\tau) = u(x,t)$, so that in particular

$$\nabla u_R(z') = R\nabla u_R(z)\,, \qquad \frac{\partial u_R}{\partial \tau}(z') = R^2 \frac{\partial u}{\partial t}(z)\,, \qquad \Delta u_R(z') = R^2 \Delta u(z)\,. \tag{2.10}$$

It follows that

$$\frac{\partial u_R}{\partial \tau} - \Delta u_R = \frac{R^2}{\varepsilon^2} u_R(1 - |u_R|^2) = -R^2 V_\varepsilon'(u_R)\,. \tag{2.11}$$

Moreover,

$$\frac{d}{dR} u_R(z') = \frac{d}{dR} u(Ry, R^2\tau) = y \cdot \nabla u(z) + 2R\tau \frac{\partial u}{\partial t}(z)\,. \tag{2.12}$$

From (2.12) and (2.10) we deduce the formula

$$R\frac{du_R}{dR}(z') = x \cdot \nabla u(z) + 2t\frac{\partial u}{\partial t}(z) = y \cdot \nabla u_R(z') + 2\tau \frac{\partial u_R}{\partial \tau}(z')\,. \tag{2.13}$$

The scaled energy $\tilde{E}_w(R)$ (defined by formula (2.5)) can be then expressed as follows

$$\tilde{E}_w(R) = \int_{\mathbb{R}^N \times \{-1\}} \left[\frac{|\nabla u_R(y,-1)|^2}{2} + R^2 V_\varepsilon(u_R(y,-1)) \right] \exp(-\frac{|y|^2}{4}) dy\,. \tag{2.14}$$

Taking into account (2.9), (2.11) and (2.13), we compute

$$\begin{aligned}
\left.\frac{d\tilde{E}_w}{dR}\right|_r &= \int_{\mathbb{R}^N \times \{-1\}} \left[\nabla u_r \cdot \nabla(\left.\frac{du_R}{dR}\right|_r) + r^2 V_\varepsilon'(u_r) \cdot \left.\frac{du_R}{dR}\right|_r + 2rV_\varepsilon(u_r) \right] \exp(-\frac{|y|^2}{4}) dy \\
&= \int_{\mathbb{R}^N \times \{-1\}} \left[(-\Delta u_r + \frac{y}{2} \cdot \nabla u_r + r^2 V_\varepsilon'(u_r)) \cdot \left.\frac{du_R}{dR}\right|_r + 2rV_\varepsilon(u_r) \right] \exp(-\frac{|y|^2}{4}) dy \\
&= \int_{\mathbb{R}^N \times \{-1\}} \left[(\frac{y}{2} \cdot \nabla u_r - \frac{\partial u_r}{\partial \tau})(\frac{1}{r}(y \cdot \nabla u_r - 2\frac{\partial u_r}{\partial \tau})) + 2rV_\varepsilon(u_r) \right] \exp(-\frac{|y|^2}{4}) dy \\
&= \int_{\mathbb{R}^N \times \{-1\}} \left[\frac{1}{2r}(y \cdot \nabla u_r - 2\frac{\partial u_r}{\partial \tau})^2 + 2rV_\varepsilon(u_r) \right] \exp(-\frac{|y|^2}{4}) dy \\
&= \int_{\mathbb{R}^N \times \{-r^2\}} \left[\frac{1}{2r}(x \cdot \nabla u - 2r^2\frac{\partial u}{\partial t})^2 + 2rV_\varepsilon(u) \right] r^{-N} \exp(-\frac{|x|^2}{4r^2}) dx \\
&= \int_{\mathbb{R}^N \times \{-r^2\}} \left[\frac{1}{2r}(x \cdot \nabla u + 2t\frac{\partial u}{\partial t})^2 + 2rV_\varepsilon(u) \right] G(x,t) dx\,.
\end{aligned} \tag{2.15}$$

The last formula in the above computation gives precisely (2.7) in the particular case $z_* = (x_*, t_*) = (0, 0)$. □

2.3 Space-time estimates and auxiliary functions

Let $u \equiv u_\varepsilon$ be a solution to $(\mathrm{PGL})_\varepsilon$ verifying $\mathcal{E}_\varepsilon(u_\varepsilon^0) < +\infty$.

Lemma 2.2. *For any $z_* = (x_*, t_*) \in \mathbb{R}^N \times \mathbb{R}^+$, the following equality holds, for $R_* = \sqrt{t_*}$.*

$$\begin{aligned}&\int_{\mathbb{R}^N \times [0,t_*]} (V_\varepsilon(u) + \Xi(u, z_*))\, G(x - x_*, t - t_*) dx dt \\ &= \frac{1}{t_*^{\frac{N-2}{2}}} \int_{\mathbb{R}^N \times \{0\}} e_\varepsilon(u(\cdot, 0)) \exp(-\frac{|x - x_*|^2}{4t_*}) dx = \tilde{E}_w(z_*, R_*)\,,\end{aligned} \tag{2.16}$$

where we have set

$$\Xi(u, z_*)(x, t) = \frac{1}{4|t - t_*|}[(x - x_*) \cdot \nabla u + 2(t - t_*)\partial_t u]^2. \tag{2.17}$$

Proof. Integrating equality (2.7) from zero to R_* we obtain

$$\begin{aligned}\tilde{E}_w(z_*, R_*) - \tilde{E}_w(z_*, 0) &= \int_0^{R_*} 2r dr \int_{\mathbb{R}^N \times \{t_* - r^2\}} V_\varepsilon(u(x, t)) G(x - x_*, t - t_*) dx \\ &+ \int_0^{R_*} 2r dr \int_{\mathbb{R}^N \times \{t_* - r^2\}} \frac{1}{4r^2}[(x - x_*) \cdot \nabla u - 2r^2 \partial_t u]^2 G(x - x_*, t - t_*) dx\,.\end{aligned} \tag{2.18}$$

Expressing the integral on the right-hand side of (2.18) in the variable $t \equiv t_* - r^2$ (so that $dt = -2r dr$) yields

$$\begin{aligned}\tilde{E}_w(z_*, R_*) - \tilde{E}_w(z_*, 0) &= -\int_{t_*}^0 dt \int_{\mathbb{R}^N \times \{t\}} V_\varepsilon(u(x, t)) G(x - x_*, t - t_*) dx \\ &- \int_{t_*}^0 dt \int_{\mathbb{R}^N \times \{t\}} \frac{1}{4|t - t_*|}[(x - x_*) \cdot \nabla u - 2r^2 \partial_t u]^2 G(x - x_*, t - t_*) dx\,.\end{aligned} \tag{2.19}$$

Finally, since u is smooth on $\mathbb{R}^N \times (0, +\infty)$ and with finite energy on each time slice, we obtain

$$\tilde{E}_w(z_*, 0) = 0\,,$$

so that the proof is complete. □

The following elementary lemma will be useful for further purposes.

Lemma 2.3. *Let $0 < t_* < T$, and $z_* = (x_*, t_*)$. We have*

$$\tilde{E}_{w,\varepsilon}(z_*, \sqrt{t_*}) \le \left(\frac{T}{t_*}\right)^{\frac{N}{2}} \exp\left(\frac{|x_T - x_*|^2}{4(T - t_*)}\right) \tilde{\mathcal{E}}_{w,\varepsilon}((x_T, 0), \sqrt{T})\,, \qquad \forall x_T \in \mathbb{R}^N\,. \tag{2.20}$$

Proof. By definition of $\tilde{E}_w$, we have

$$\begin{aligned}\tilde{E}_w(z_*,\sqrt{t_*}) &= \frac{1}{{t_*}^{N/2}}\int_{\mathbb{R}^N} e_\varepsilon(u(x,0))\exp(-\frac{|x-x_*|^2}{4t_*})dx \\ &= \left(\frac{T}{t_*}\right)^{N/2}\frac{1}{T^{N/2}}\int_{\mathbb{R}^N} e_\varepsilon(u(x,0))\exp(-\frac{|x-x_T|^2}{4T})Q(x)dx\,,\end{aligned} \tag{2.21}$$

where the function Q is defined on $\mathbb{R}^N$ as

$$Q(x) = \exp\left(\frac{|x-x_T|^2}{4T} - \frac{|x-x_*|^2}{4t_*}\right) \qquad \forall\, x\in\mathbb{R}^N\,. \tag{2.22}$$

Clearly Q is positive and bounded on $\mathbb{R}^N$. Its maximum is achieved at a point $x_0 \in \mathbb{R}^N$ such that

$$\frac{(x_0-x_T)}{T} = \frac{(x_0-x_*)}{t_*}\,,$$

so that

$$x_0 - x_* = \frac{(x_*-x_T)}{T-t_*}t_*\,, \qquad x_0 - x_T = \frac{(x_*-x_T)}{T-t_*}T\,. \tag{2.23}$$

Inserting (2.23) in (2.22), we are led to

$$\sup_{x\in\mathbb{R}^N} Q(x) = Q(x_0) = \exp(\frac{|x_*-x_T|^2}{4(T-t_*)})\,. \tag{2.24}$$

Hence, combining (2.24) with (2.21) we obtain

$$\tilde{E}_w(z_*,\sqrt{t_*}) \le \left(\frac{T}{t_*}\right)^{N/2}\exp(\frac{|x_*-x_T|^2}{4(T-t_*)})\int_{\mathbb{R}^N} e_\varepsilon(u(x,0))\exp(-\frac{|x-x_T|^2}{4T})dx\,, \tag{2.25}$$

and (2.20) follows. □

Next, let $T>0$ be given and let $f \in L^\infty(\mathbb{R}^N\times[0,T])$ be such that

$$|f(z)| \le V_\varepsilon(|u(z)|)\,, \qquad \text{for any } z=(x,t)\in\mathbb{R}^N\times[0,T]\,. \tag{2.26}$$

We consider the solution ω of the heat equation with source term f, i.e. ω solves

$$\begin{cases} \frac{\partial\omega}{\partial t} - \Delta\omega = f & \text{on } \mathbb{R}^N\times[0,T], \\ \omega(x,0) = 0 & \text{for } x\in\mathbb{R}^N\,. \end{cases} \tag{2.27}$$

The following L^∞-estimate, which already played a key role in the elliptic setting (see [8]), will enter similarly in the proof of Theorem 1.

Lemma 2.4. *For any $z_* = (x_*,t_*) \in \mathbb{R}^N\times[0,T]$, we have*

$$|\omega(z_*)| \le \pi^{-N/2}\tilde{E}_w(z_*,\sqrt{t_*})\,. \tag{2.28}$$

Proof. The function ω is given explicitly by Duhamel's formula

$$\omega(z_*) = \pi^{-N/2} \int_{\mathbb{R}^N \times [0,t_*]} f(x,t) G(x - x_*, t - t_*) dx dt \,,$$

so that, by (2.26),

$$|\omega(z_*)| \leq \pi^{-N/2} \int_{\mathbb{R}^N \times [0,t_*]} V_\varepsilon(u(x,t)) G(x - x_*, t - t_*) dx dt \,,$$

and the conclusion follows from (2.16). □

Combining Lemma 2.3 and Lemma 2.4 we obtain

Proposition 2.2. *Let $T > 0$, $x_T \in \mathbb{R}^N$. For any $z = (x,t) \in \mathbb{R}^N \times [0,T]$, the following estimate holds*

$$|\omega(z)| \leq \left(\frac{T}{t}\right)^{\frac{N}{2}} \exp\left(\frac{|x_T - x|^2}{4(T-t)}\right) \tilde{\mathcal{E}}_{w,\varepsilon}((x_T,0), \sqrt{T}) \,, \qquad \forall\, x_T \in \mathbb{R}^N \,. \tag{2.29}$$

2.4 Bounds for the scaled weighted energy $\tilde{\mathcal{E}}_{w,\varepsilon}$

Our next lemma provides an upper bound for $\tilde{\mathcal{E}}_{w,\varepsilon}(z,R)$ in terms of $\tilde{\mathcal{E}}_{w,\varepsilon}((x_T,0),\sqrt{T})$ provided $z < T$ and R is sufficiently small. More precisely, we have

Lemma 2.5. *Let $T > 0$, and $z = (x,t) \in \mathbb{R}^N \times [0,T)$. We have the inequality*

$$\tilde{\mathcal{E}}_{w,\varepsilon}(z,R) \leq \left(\frac{T}{t+R^2}\right)^{\frac{N}{2}} \exp\left(\frac{|x_T - x|^2}{4(T-t-R^2)}\right) \tilde{\mathcal{E}}_{w,\varepsilon}((x_T,0), \sqrt{T}) \,, \tag{2.30}$$

for any $x_T \in \mathbb{R}^N$, and for $0 < R < \sqrt{T-t}$.

Proof. In view of the monotonicity formula (2.8), we have the inequality

$$\tilde{\mathcal{E}}_{w,\varepsilon}(z,R) = \tilde{E}_w((x, t+R^2), R) \leq \tilde{E}_w((x,t+R^2), \sqrt{t+R^2}) \,. \tag{2.31}$$

By Lemma 2.3 applied to $z_* = (x, t+R^2)$, we obtain

$$\tilde{E}_w((x,t+R^2), \sqrt{t+R^2}) \leq \left(\frac{T}{t+R^2}\right)^{\frac{N}{2}} \exp\left(\frac{|x_T - x|^2}{4(T-t-R^2)}\right) \tilde{\mathcal{E}}_{w,\varepsilon}((x_T,0), \sqrt{T}) \,, \tag{2.32}$$

for any $x_T \in \mathbb{R}^N$. Combining (2.32) with (2.31) the conclusion follows. □

Comment. Note that (2.30) holds in particular for **small** R. It can therefore be understood as a regularizing property of $(\mathrm{PGL})_\varepsilon$. Indeed, starting with an arbitrary initial condition, the gradient of the solution at time t remains bounded in the Morrey space $\mathcal{L}^{2,N-2}$ (so that the solution itself remains bounded in BMO, locally).

2.5 Localizing the energy

In some of the proofs of the main results, it will be convenient to work on bounded domains for fixed time slices (in particular in view of the elliptic estimates needed there). On the other hand, the definition of $\tilde{\mathcal{E}}_{w,\varepsilon}$ and $\tilde{E}_w$ involves integration on the whole space (even though the weight has an extremely fast decay at infinity). In order to overcome this difficulty, we will make use of two kinds of localization methods. The first one is a fairly elementary consequence of the monotonicity formula and can be stated as follows.

Proposition 2.3. *Let $T > 0$, $x_T \in \mathbb{R}^N$ and $R > \sqrt{2\varepsilon}$. Assume u_ε is a solution to $(PGL)_\varepsilon$ verifying (H_0). Then the following inequality holds, for any $\lambda > 0$,*

$$\int_{\mathbb{R}^N\times\{T\}} e_\varepsilon(u_\varepsilon)\, e^{-\frac{|x-x_T|^2}{4R^2}}\,dx \le \int_{B(x_T,\lambda R)\times\{T\}} e_\varepsilon(u_\varepsilon) + \Big(\frac{\sqrt{2}R}{\sqrt{T+2R^2}}\Big)^{N-2} M_0 \exp\Big(-\frac{\lambda^2}{8}\Big)|\log\varepsilon|.$$

Proof. It suffices obviously to prove that

$$\int_{\{|x-x_T|\ge\lambda R\}\times\{T\}} e_\varepsilon(u_\varepsilon)\, e^{-\frac{|x-x_T|^2}{4R^2}}\,dx \le \Big(\frac{\sqrt{2}R}{\sqrt{T+2R^2}}\Big)^{N-2} M_0 \exp\Big(-\frac{\lambda^2}{8}\Big)|\log\varepsilon|. \qquad (2.33)$$

First, we write $\exp(-\frac{|x-x_T|^2}{4R^2}) = [\exp(-\frac{|x-x_T|^2}{8R^2})]^2$, so that on $\mathbb{R}^N \setminus B(x_T,\lambda R)$ we have

$$\exp\Big(-\frac{|x-x_T|^2}{4R^2}\Big) \le \exp\Big(-\frac{\lambda^2}{8}\Big)\exp\Big(-\frac{|x-x_T|^2}{8R^2}\Big). \qquad (2.34)$$

On the other hand, applying the monotonicity formula at the point $(x_T, T+2R^2)$, we obtain

$$\begin{aligned}\frac{1}{(\sqrt{2}R)^{N-2}}\int_{\mathbb{R}^N\times\{T\}} e_\varepsilon(u_\varepsilon)\, e^{-\frac{|x-x_T|^2}{8R^2}}\,dx &\le \frac{1}{(\sqrt{T+2R^2})^{N-2}}\int_{\mathbb{R}^N\times\{0\}} e_\varepsilon(u_\varepsilon)\, e^{-\frac{|x-x_T|^2}{4(T+2R^2)}}\,dx \\ &\le \frac{1}{(\sqrt{T+2R^2})^{N-2}} M_0|\log\varepsilon|.\end{aligned} \qquad (2.35)$$

Combining (2.33),(2.34) and (2.35) the conclusion follows. □

The idea of the second localization method originated in [40] and is based on a Pohozaev type inequality.

Proposition 2.4. *Let $0 < t < T$. The following inequality holds, for any $x_T \in \mathbb{R}^N$,*

$$\begin{aligned}\int_{\mathbb{R}^N\times\{t\}} e_\varepsilon(u)\frac{|x-x_T|^2}{4(T-t)}\exp\Big(-\frac{|x-x_T|^2}{4(T-t)}\Big)\,dx &\le \frac{N}{2}\int_{\mathbb{R}^N\times\{t\}} e_\varepsilon(u)\exp\Big(-\frac{|x-x_T|^2}{4(T-t)}\Big)\,dx \\ &+ \int_{\mathbb{R}^N\times\{t\}} [V_\varepsilon(u) + 3\,\Xi(u,z_T)]\exp\Big(-\frac{|x-x_T|^2}{4(T-t)}\Big)\,dx.\end{aligned} \qquad (2.36)$$

As a consequence,

$$\int_{\mathbb{R}^N\times\{t\}} e_\varepsilon(u)\exp\left(-\frac{|x-x_T|^2}{4(T-t)}\right)dx \le \int_{B(x_T,r_T)\times\{t\}} e_\varepsilon(u)\exp\left(-\frac{|x-x_T|^2}{4(T-t)}\right)dx$$
$$+\frac{2}{N}\int_{\mathbb{R}^N\times\{t\}}[V_\varepsilon(u)+3\,\Xi(u,z_T)]\exp\left(-\frac{|x-x_T|^2}{4(T-t)}\right)dx\,, \tag{2.37}$$

where $r_T = 2\sqrt{N(T-t)}$.

Note that the radius r_T of the ball $B(x_T, r_T)$ where the first integral of the r.h.s. of (2.37) is computed is proportional to $\sqrt{T-t}$, which is the width of the parabolic cone with vertex $z_T = (x_T, T)$.

The proof of Proposition 2.4 relies on the following inequality.

Lemma 2.6. *Let* $0 < T_1 \le T_2 < T$, $x_T \in \mathbb{R}^N$, $z_T = (x_T, T)$. *We have*

$$\int_{\mathbb{R}^N\times[T_1,T_2]} e_\varepsilon(u)\frac{|x-x_T|^2}{4(T-t)}\exp\left(-\frac{|x-x_T|^2}{4(T-t)}\right)dxdt$$
$$\le (T-T_1)^{N/2}\tilde{E}_w(z_T,\sqrt{T-T_1}) - (T-T_2)^{N/2}\tilde{E}_w(z_T,\sqrt{T-T_2}) \tag{2.38}$$
$$+\int_{\mathbb{R}^N\times[T_1,T_2]}\frac{1}{2(T-t)}[(x-x_T)\cdot\nabla u - 2(T-t)\partial_t u]^2\exp\left(-\frac{|x-x_T|^2}{4(T-t)}\right)dxdt\,.$$

Proof. The idea is to multiply $(\mathrm{PGL})_\varepsilon$ by the multiplier $2(T-t)\partial_t u\exp(-\frac{|x-x_T|^2}{4(T-t)})$ and integrate on $\mathbb{R}^N\times[T_1,T_2]$. One obtains, after integration by parts in the space variable,

$$\int_{T_1}^{T_2}\int_{\mathbb{R}^N} 2(T-t)|\partial_t u|^2\exp(-\frac{|x_T-x|^2}{4(T-t)})\,dx\,dt$$
$$=\int_{T_1}^{T_2}\int_{\mathbb{R}^N} 2(T-t)\Delta u\,\partial_t u\,\exp(-\frac{|x_T-x|^2}{4(T-t)})\,dx\,dt$$
$$-\int_{T_1}^{T_2}\int_{\mathbb{R}^N} 2(T-t)\frac{\partial}{\partial t}[V_\varepsilon(u)]\exp(-\frac{|x_T-x|^2}{4(T-t)})\,dx\,dt$$
$$=-\int_{T_1}^{T_2}\int_{\mathbb{R}^N}\nabla u\cdot\left(2(T-t)\frac{\partial}{\partial t}\nabla u-(T-t)\partial_t u\frac{x-x_T}{T-t}\right)\exp(-\frac{|x_T-x|^2}{4(T-t)})\,dx\,dt \tag{2.39}$$
$$-\int_{T_1}^{T_2}\int_{\mathbb{R}^N} 2(T-t)\frac{\partial}{\partial t}[V_\varepsilon(u)]\exp(-\frac{|x_T-x|^2}{4(T-t)})\,dx\,dt$$
$$=-\int_{T_1}^{T_2}\int_{\mathbb{R}^N}(T-t)\frac{\partial}{\partial t}\left[|\nabla u|^2+2V_\varepsilon(u)\right]\exp(-\frac{|x_T-x|^2}{4(T-t)})\,dx\,dt$$
$$+\int_{T_1}^{T_2}\int_{\mathbb{R}^N}((x-x_T)\cdot\nabla u)\frac{\partial u}{\partial t}\exp(-\frac{|x_T-x|^2}{4(T-t)})\,dx\,dt.$$

Integration by parts in the time variable now yields

$$
\begin{aligned}
&\int_{T_1}^{T_2}\int_{\mathbb{R}^N} 2(T-t)|\partial_t u|^2 \exp(-\frac{|x_T-x|^2}{4(T-t)})\,dx\,dt \\
&= -\int_{T_1}^{T_2}\int_{\mathbb{R}^N} (T-t)\frac{\partial}{\partial t}\left[|\nabla u|^2 + 2V_\varepsilon(u)\right] \exp(-\frac{|x_T-x|^2}{4(T-t)})\,dx\,dt \\
&- \int_{T_1}^{T_2}\int_{\mathbb{R}^N} \frac{|x_T-x|^2}{4(T-t)}\left[|\nabla u|^2 + 2V_\varepsilon(u)\right] \exp(-\frac{|x_T-x|^2}{4(T-t)})\,dx\,dt \\
&+ \int_{\mathbb{R}^N\times\{T_1\}} (T-T_1)\left[|\nabla u|^2 + 2V_\varepsilon(u)\right] \exp(-\frac{|x_T-x|^2}{4(T-T_1)})\,dx \\
&- \int_{\mathbb{R}^N\times\{T_2\}} (T-T_2)\left[|\nabla u|^2 + 2V_\varepsilon(u)\right] \exp(-\frac{|x_T-x|^2}{4(T-T_2)})\,dx \\
&+ \int_{T_1}^{T_2}\int_{\mathbb{R}^N} ((x-x_T)\cdot\nabla u)\frac{\partial u}{\partial t} \exp(-\frac{|x_T-x|^2}{4(T-t)})\,dx\,dt.
\end{aligned}
\tag{2.40}
$$

Adding the integral

$$
\int_{T_1}^{T_2}\int_{\mathbb{R}^N} \frac{1}{2(T-t)}\left|(x-x_T)\cdot\nabla u\right|^2 \exp(-\frac{|x_T-x|^2}{4(T-t)})\,dx\,dt
$$

to equation (2.40) we finally obtain

$$
\begin{aligned}
&\int_{T_1}^{T_2}\int_{\mathbb{R}^N} \frac{1}{2(T-t)}\left|(x-x_T)\cdot\nabla u - 2(T-t)\partial_t u\right|^2 \exp(-\frac{|x_T-x|^2}{4(T-t)})\,dx\,dt \\
&+ \int_{T_1}^{T_2}\int_{\mathbb{R}^N} \left(1+\frac{|x_T-x|^2}{4(T-t)}\right)\left[|\nabla u|^2 + 2V_\varepsilon(u)\right] \exp(-\frac{|x_T-x|^2}{4(T-t)})\,dx\,dt \\
&= + \int_{\mathbb{R}^N\times\{T_1\}} (T-T_1)\left[|\nabla u|^2 + 2V_\varepsilon(u)\right] \exp(-\frac{|x_T-x|^2}{4(T-T_1)})\,dx \\
&- \int_{\mathbb{R}^N\times\{T_2\}} (T-T_2)\left[|\nabla u|^2 + 2V_\varepsilon(u)\right] \exp(-\frac{|x_T-x|^2}{4(T-T_2)})\,dx \\
&+ \int_{T_1}^{T_2}\int_{\mathbb{R}^N} \frac{(x-x_T)\cdot\nabla u}{2(T-t)}\left[(x-x_T)\cdot\nabla u\right] \exp(-\frac{|x_T-x|^2}{4(T-t)})\,dx\,dt.
\end{aligned}
\tag{2.41}
$$

We bound the last term in (2.41), using the inequality $ab \leq \frac{a^2}{4} + b^2$, with the choice

$$
a = \frac{(x-x_T)\cdot\nabla u}{\sqrt{2(T-t)}} \exp(-\frac{|x_T-x|^2}{8(T-t)})
$$

and

$$
b = \frac{\left[(x-x_T)\cdot\nabla u - 2(T-t)\partial_t u\right]}{\sqrt{2(T-t)}} \exp(-\frac{|x_T-x|^2}{8(T-t)}),
$$

and the desired conclusion follows. □

Proof of Proposition 2.4. Let $0 < t < T$ be given and fixed and apply Lemma 2.6 with $T_1 = t$, $T_2 = t + \Delta t$, for $\Delta t > 0$. We divide by Δt and let Δt tend to zero in (2.38). This yields

$$\begin{aligned}
&\int_{\mathbb{R}^N \times \{t\}} e_\varepsilon(u) \frac{|x - x_T|^2}{4(T-t)} \exp(-\frac{|x - x_T|^2}{4(T-t)})\, dx \\
\leq &\int_{\mathbb{R}^N \times \{t\}} \frac{1}{2(T-t)} [(x - x_T) \cdot \nabla u - 2(T-t)\partial_t u]^2 \exp(-\frac{|x - x_T|^2}{4(T-t)})\, dx \\
&+ g'(T-t),
\end{aligned} \tag{2.42}$$

where we have set

$$g(s) := s^{N/2} \tilde{E}_w(z_T, \sqrt{s}). \tag{2.43}$$

Since

$$g'(T-t) = \frac{N}{2}(T-t)^{\frac{N-2}{2}} \tilde{E}_w(z_T, \sqrt{T-t}) + \frac{(T-t)^{\frac{N-1}{2}}}{2} \frac{d}{dR} \tilde{E}_w(\sqrt{T-t}),$$

we obtain, using the monotonicity formula in Proposition 2.1 and (2.42),

$$\begin{aligned}
\int_{\mathbb{R}^N \times \{t\}} e_\varepsilon(u) \frac{|x - x_T|^2}{4(T-t)} \exp(-\frac{|x - x_T|^2}{4(T-t)})\, dx &\leq \frac{N}{2} \int_{\mathbb{R}^N \times \{t\}} e_\varepsilon(u) \exp(-\frac{|x - x_T|^2}{4(T-t)})\, dx \\
&+ \int_{\mathbb{R}^N \times \{t\}} [V_\varepsilon(u) + 3\,\Xi(u, z_T)] \exp(-\frac{|x - x_T|^2}{4(T-t)})\, dx,
\end{aligned} \tag{2.44}$$

which proves (2.36). For (2.37), consider the region

$$A := \left\{ x \in \mathbb{R}^N \text{ s.t. } \frac{|x - x_T|^2}{8(T-t)} \leq \frac{N}{2} \right\}.$$

We deduce from (2.44) that

$$\begin{aligned}
\int_{\mathbb{R}^N \times \{t\}} e_\varepsilon(u) \frac{|x - x_T|^2}{4(T-t)} \exp(-\frac{|x - x_T|^2}{4(T-t)})\, dx &\leq \frac{N}{2} \int_{A \times \{t\}} e_\varepsilon(u) \exp(-\frac{|x - x_T|^2}{4(T-t)})\, dx \\
&+ \int_{(\mathbb{R}^N \setminus A) \times \{t\}} e_\varepsilon(u) \frac{|x - x_T|^2}{8(T-t)} \exp(-\frac{|x - x_T|^2}{4(T-t)})\, dx \\
&+ \int_{\mathbb{R}^N \times \{t\}} [V_\varepsilon(u) + 3\,\Xi(u, z_T)] \exp(-\frac{|x - x_T|^2}{4(T-t)})\, dx,
\end{aligned} \tag{2.45}$$

so that

$$\begin{aligned}
\int_{(\mathbb{R}^N \setminus A) \times \{t\}} \frac{N}{2} e_\varepsilon(u) \exp(-\frac{|x - x_T|^2}{4(T-t)})\, dx &\leq \frac{N}{2} \int_{A \times \{t\}} e_\varepsilon(u) \exp(-\frac{|x - x_T|^2}{4(T-t)})\, dx \\
&+ \int_{\mathbb{R}^N \times \{t\}} [V_\varepsilon(u) + 3\,\Xi(u, z_T)] \exp(-\frac{|x - x_T|^2}{4(T-t)})\, dx,
\end{aligned} \tag{2.46}$$

and finally

$$\int_{\mathbb{R}^N\times\{t\}} e_\varepsilon(u)\exp(-\frac{|x-x_T|^2}{4(T-t)})\,dx \le 2\int_{A\times\{t\}} e_\varepsilon(u)\exp(-\frac{|x-x_T|^2}{4(T-t)})\,dx$$
$$+\frac{2}{N}\int_{\mathbb{R}^N\times\{t\}} [V_\varepsilon(u)+3\,\Xi(u,z_T)]\exp(-\frac{|x-x_T|^2}{4(T-t)})\,dx. \quad (2.47)$$

This completes the proof. □

2.6 Choice of an appropriate scaling

Let $z_T=(x_T,T)$ as above, and set

$$E_{w,\varepsilon}(R)\equiv E_{w,\varepsilon}(z_T,R)\equiv E_{w,\varepsilon}(u_\varepsilon;z_T,R)\,,$$

and accordingly

$$\tilde E_{w,\varepsilon}(R)\equiv \tilde E_{w,\varepsilon}(z_T,R)\equiv \tilde E_{w,\varepsilon}(u_\varepsilon;z_T,R)\,.$$

Let $0<\delta<1/16$ be fixed. We have

Proposition 2.5. *There exists a constant $\varepsilon_1>0$ depending only on T and δ, such that, for $\varepsilon\le\varepsilon_1$, there exists $R_1>0$, with $R_1\in(\varepsilon^{1/2},\sqrt{T})$ such that*

$$0\le \tilde E_{w,\varepsilon}(R_1)-\tilde E_{w,\varepsilon}(\delta R_1)\le 4|\log\delta|\frac{\tilde E_{w,\varepsilon}(\sqrt{T})}{|\log\varepsilon|}\,, \quad (2.48)$$

and therefore

$$\int_{T-R_1^2}^{T-\delta^2R_1^2}\int_{\mathbb{R}^N}(V_\varepsilon(u)+\,\Xi(u,z_T))\exp(-\frac{|x-x_T|^2}{4\delta^2R_1^2})dxdt\le 4|\log\delta|\frac{\tilde E_{w,\varepsilon}(\sqrt{T})}{|\log\varepsilon|}\,. \quad (2.49)$$

Proof. Set $R=\sqrt{T}$, and for $n\in\mathbb{N}^*$ $R_n=\delta^n R$. Let k_0 be the largest integer such that

$$\delta^{k_0-1}R\ge\varepsilon 1/2.$$

We have

$$k_0=\left[\frac{(\log\varepsilon)/2-\log R}{\log\delta}\right],$$

where, for $\alpha\in\mathbb{R}$, $[\alpha]$ denotes the largest integer less or equal to α, so that, if $\varepsilon\le R^4\delta^{-8}$ then k_0 verifies

$$k_0-2\le\frac{|\log\varepsilon|}{|\log\delta|}. \quad (2.50)$$

On the other hand, we have the equality

$$\tilde E_w(\delta R)-\tilde E_w(\delta^{k_0}R)=\sum_{j=2}^{k_0}\left(\tilde E_w(\delta^{j-1}R)-\tilde E_w(\delta^jR)\right),$$

and all the terms of the sum of the right hand side of the equality are non negative. Therefore, there exists $k_1 \in \{2, ..., k_0\}$ such that

$$\tilde{E}_w(\delta^{k_1-1}R) - \tilde{E}_w(\delta^{k_1}R) \leq \frac{\tilde{E}_w(R)}{k_0 - 2} \leq 4|\log\delta|\frac{\tilde{E}_w(R)}{|\log\varepsilon|},$$

where we have used (2.50) for the last inequality. We therefore set $R_1 = \delta^{k_1-1}R$. Inequality (2.49) is a direct consequence of the monotonicity formula. □

Blowing-up. In view of Proposition 2.5 we perform the following change of variables

$$\tilde{x} = \frac{x - x_T}{R_1} \qquad \tilde{t} = \frac{t - T}{R_1^2} + 1$$

so that (x_T, T) becomes in new variables $(0, 1)$, and $(x_T, T - R_1^2)$ becomes $(0, 0)$. Set

$$\epsilon = \frac{\varepsilon}{R_1}$$

and define the map $\mathrm{v}_\epsilon : \mathbb{R}^N \times (0, +\infty) \to \mathbb{C}$ by

$$\mathrm{v}_\epsilon(\tilde{x}, \tilde{t}) = u_\varepsilon(x, t)\,,$$

so that v_ϵ verifies the equation

$$\frac{\partial \mathrm{v}_\epsilon}{\partial t} - \Delta \mathrm{v}_\epsilon = \frac{1}{\epsilon^2}\mathrm{v}_\epsilon(1 - |\mathrm{v}_\epsilon|^2) \qquad \text{on } \mathbb{R}^N \times (0, +\infty), \tag{2.51}$$

i.e. v_ϵ is a solution to $(\mathrm{PGL})_\epsilon$. Note that

$$\frac{\varepsilon}{T} \leq \epsilon \leq \varepsilon^{1/2}, \tag{2.52}$$

therefore $\epsilon \to 0$ as $\varepsilon \to 0$, $|\log\epsilon| \geq |\log\varepsilon|/2$ and the asymptotic analysis for $(\mathrm{PGL})_\varepsilon$ is also valid for $(\mathrm{PGL})_\epsilon$. In the sequel we skip the tildes on the new variables for simplicity.

Lemma 2.7. *We have,*

$$|\mathrm{v}_\epsilon(x)| \leq 3, \quad |\nabla \mathrm{v}_\epsilon(x)| \leq \frac{K}{\epsilon}, \quad |\partial_t \mathrm{v}_\epsilon(x)| \leq \frac{K}{\epsilon^2} \tag{2.53}$$

for any $(x, t) \in \mathbb{R}^N \times (0, +\infty)$. *Moreover,*

$$\tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon, (0, 1), 1) = \tilde{E}_{w,\varepsilon}(u_\varepsilon, z_T, R_1), \tag{2.54}$$

$$\tilde{E}_{w,\epsilon}((0, 1), 1) - \tilde{E}_{w,\epsilon}((0, 1), \delta) \leq 4|\log\delta|\frac{\tilde{E}_{w,\varepsilon}(z_T, \sqrt{T})}{|\log\varepsilon|}, \tag{2.55}$$

and

$$\int_{\mathbb{R}^N \times [0, 1-\delta^2]} [V_\epsilon(\mathrm{v}_\epsilon) + \Xi(\mathrm{v}_\epsilon, (0, 1))]\, G(x, t - 1)\, dx\, dt \leq 4|\log\delta|\frac{\tilde{E}_{w,\varepsilon}(z_T, \sqrt{T})}{|\log\varepsilon|}. \tag{2.56}$$

Proof. This is a direct consequence of the scaling invariance of each term. □

3 Proof of Theorem 1

3.1 Change of scale and improved energy decay

Let u_ε be a solution of $(\mathrm{PGL})_\varepsilon$ as in Theorem 1, i.e. satisfying the bounds

$$\mathcal{E}_\varepsilon(u_\varepsilon^0) \leq M_0|\log\varepsilon| \tag{3.1}$$

$$\int_{\mathbb{R}^N} e_\varepsilon(u_\varepsilon^0)\exp(-|x|^2/4) \leq \eta|\log\varepsilon|. \tag{3.2}$$

Let $0 < \delta < \frac{1}{16}$ be fixed, but to be determined later at the very end of the proof. Let also $T = 1$, and $z_T = (0,1)$. Recall that in section 2.6 we have constructed a rescaled map v_ϵ defined by

$$\mathrm{v}_\epsilon(x,t) = u_\varepsilon(R_1 x, R_1^2(t-1)+1), \quad \epsilon = \frac{\varepsilon}{R_1}, \quad \varepsilon \leq \epsilon \leq \varepsilon^{1/2},$$

for some appropriate choice of R_1. In particular, the function v_ϵ is a solution of $(\mathrm{PGL})_\epsilon$ and it follows from the monotonicity formula that

$$\tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon,(0,1),1) = \tilde{E}_{w,\varepsilon}(u_\varepsilon,(0,1),1) = \check{\eta}|\log\epsilon| \leq \eta|\log\varepsilon| \leq 2\eta|\log\epsilon|, \tag{3.3}$$

where we have set

$$\check{\eta}(\mathrm{v}_\epsilon) = \frac{\tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon)}{|\log\epsilon|}.$$

In view of Lemma 2.7, we have the estimates

$$|\mathrm{v}_\epsilon| \leq 3 \qquad \text{on } \mathbb{R}^N \times [0,+\infty), \tag{3.4}$$

$$|\nabla \mathrm{v}_\epsilon| \leq \frac{K}{\epsilon},\ |\partial_t \mathrm{v}_\epsilon| \leq \frac{K}{\epsilon^2} \qquad \text{on } \mathbb{R}^N \times [0,+\infty), \tag{3.5}$$

$$\tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon,(0,1),1) - \tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon,(0,1),\delta) \leq 4|\log\delta|\check{\eta} \leq 8|\log\delta|\eta, \tag{3.6}$$

$$\int_{\mathbb{R}^N\times[0,1-\delta^2]} V_\epsilon(\mathrm{v}_\epsilon)\frac{1}{(1-t)^{N/2}}\exp(-\frac{|x|^2}{4(1-t)})\,dx\,dt \leq 4|\log\delta|\check{\eta} \leq 8|\log\delta|\eta, \tag{3.7}$$

$$\int_{\mathbb{R}^N\times[0,1-\delta^2]} \Xi(\mathrm{v}_\epsilon,(0,1))\frac{1}{(1-t)^{N/2}}\exp(-\frac{|x|^2}{4(1-t)})\,dx\,dt \leq 4|\log\delta|\check{\eta} \leq 8|\log\delta|\eta. \tag{3.8}$$

Note that $\mathrm{v}_\epsilon(0,1) = u_\varepsilon(0,1)$. Thus, in order to prove Theorem 1 it suffices to establish that v_ϵ verifies

$$|\mathrm{v}_\epsilon(0,1)| \geq 1-\sigma. \tag{3.9}$$

Throughout this section, we will work with v_ϵ instead of u_ε. The main advantage to do so is that we have the additional estimates (3.4,3.6,3.7,3.8) which provide uniform bounds which are independent of ϵ. In the definition of $\tilde{E}_{w,\epsilon}$, $\mathcal{E}_{w,\epsilon}$, and the various quantities involved in the proof, we will thus skip the reference to v_ϵ or even ϵ if this is not misleading.

The main ingredient in the proof of (3.9), i.e. Theorem 1, is the following δ-energy decay estimate.

Proposition 3.1. *There exists constants* $0 < \delta_0 < \frac{1}{16}$, $0 < \epsilon_0 < \frac{1}{2}$, *and* $\eta_0 > 0$ *such that for* $0 < \eta \leq \eta_0$ *the following inequality holds*

$$\tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon, (0,1), \delta_0) \leq \frac{1}{2}\tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon, (0,1), 1) + \mathcal{R}(\eta), \tag{3.10}$$

where $\mathcal{R}(\eta)$ *tends to zero as* $\eta \to 0$.

We postpone the proof of Proposition 3.1 and show first how it implies Theorem 1.

3.2 Proposition 3.1 implies Theorem 1

Assume $0 < \eta \leq \eta_0$ and set $\lambda(\sigma) = \sqrt{\frac{\sigma}{2K}}$, where σ is the constant appearing in the statement of Theorem 1, whereas K is the constant appearing in (3.5). Set $r_\epsilon = \min(1, \lambda(\sigma)\epsilon)$ and $T_\epsilon = \max(0, 1 - \lambda^2(\sigma)\epsilon^2) = 1 - r_\epsilon^2$. We claim that

$$\frac{1}{\epsilon^N}\int_{B(\epsilon)}(1 - |\mathrm{v}_\epsilon(x, T_\epsilon)|^2)^2 \leq \mathcal{R}_1(\eta), \tag{3.11}$$

where $\mathcal{R}_1(\eta) \to 0$ as $\eta \to 0$.
Proof of the claim. Combining (3.6) and (3.10) we are led to

$$\tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon, (0,1), 1) \leq 8|\log\delta|\eta + 2\mathcal{R}(\eta). \tag{3.12}$$

Assume first that $\lambda(\sigma)\epsilon \leq 1$, so that $T_\epsilon = 1 - \lambda^2(\sigma)\epsilon^2$. We deduce from the monotonicity formula that

$$\tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon, (0,1), \lambda(\sigma)\epsilon) \leq \tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon, (0,1), 1) \tag{3.13}$$

so that, combining (3.12) and (3.13) we obtain

$$\tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon, (0,1), \lambda(\sigma)\epsilon) \leq 8|\log\delta|\eta + 2\mathcal{R}(\eta).$$

If $\lambda(\sigma)\epsilon \geq 1$, then $r_\epsilon = 1$, $T_\epsilon = 0$ so that

$$\tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon, (0,1), r_\epsilon) \leq \eta|\log\epsilon| \leq \eta|\log\lambda(\sigma)|.$$

In both cases the claim (3.11) follows from the inequality

$$\begin{aligned}\frac{1}{\epsilon^N}\int_{B(\epsilon)}(1 - |\mathrm{v}_\epsilon(x, T_\epsilon)|^2)^2 &\leq \frac{C(\sigma)}{r_\epsilon^{N-2}}\int_{B(\epsilon)}\frac{(1 - |\mathrm{v}_\epsilon(x, T_\epsilon)|^2)^2}{\epsilon^2}\exp(-\frac{|x|^2}{r_\epsilon^2})dx \\ &\leq C(\sigma)\tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon, (0,1), r_\epsilon)\,,\end{aligned} \tag{3.14}$$

valid for some constant $C(\sigma)$ depending only on σ and N.

Arguing as in the proof of Lemma III.2 in [8], we are led to

$$1 - |\mathrm{v}_\epsilon(0, T_\epsilon)| \leq C\left(\frac{1}{\epsilon^N}\int_{B(\epsilon)}(1 - |\mathrm{v}_\epsilon(x, T_\epsilon)|^2)^2\right)^{\frac{1}{N+2}} \leq C\mathcal{R}_1(\eta)^{\frac{1}{N+2}}. \tag{3.15}$$

On the other hand, by (3.5),

$$|\mathrm{v}_\epsilon(0,T_\epsilon)-\mathrm{v}_\epsilon(0,1)|\leq\frac{K}{\epsilon^2}(T_\epsilon-1)\leq\frac{\sigma}{2}. \tag{3.16}$$

Combining (3.15) and (3.16), we obtain

$$|1-|\mathrm{v}_\epsilon(0,1)||\leq\frac{\sigma}{2}+C\,\mathcal{R}_1(\eta)^{\frac{1}{N+2}}$$

so that the conclusion follows if η_0 is chosen sufficiently small, since $\mathcal{R}_1(\eta)\to 0$ as $\eta\to 0$.

3.3 Paving the way to Proposition 3.1

As in [8], let us first consider the ideal situation where

$$|\mathrm{v}_\epsilon|\equiv 1 \qquad \text{on } \mathbb{R}^N\times[0,1].$$

Then, we may write $\mathrm{v}_\epsilon=\exp(i\varphi)$ where the phase $\varphi:\mathbb{R}^N\times[0,1]\to\mathbb{R}$ is uniquely defined, up to a constant multiple of 2π. The equation for the phase φ is then the linear heat equation

$$\frac{\partial\varphi}{\partial t}-\Delta\varphi=0 \qquad \text{on } \mathbb{R}^N\times(0,1).$$

Notice that in that situation, $|\nabla\mathrm{v}_\epsilon|=|\nabla\varphi|$ so that $e_\epsilon(\mathrm{v}_\epsilon)=|\nabla\varphi|^2/2$ and $|\partial_t\mathrm{v}_\epsilon|=|\partial_t\varphi|$. Moreover, $|\nabla\varphi|^2$ verifies the equation

$$\frac{\partial|\nabla\varphi|^2}{\partial t}-\Delta(|\nabla\varphi|^2)=-2|\nabla\varphi|^2\leq 0$$

so that for any $0<\delta<1$, and any $x_*\in\mathbb{R}^N$,

$$|\nabla\varphi(x_*,1-\delta^2)|^2\leq\int_{\mathbb{R}^N}\frac{1}{\pi^{N/2}(1-\delta^2)^{N/2}}e_\epsilon(\mathrm{v}_\epsilon(x,0))\exp(-\frac{|x-x_*|^2}{4(1-\delta^2)})\,dx. \tag{3.17}$$

For $\mu\in\mathbb{R}^N$ and $\sigma>0$, consider the Gaussian $N(\mu,\sigma^2)=\frac{1}{(4\pi)^{N/2}\sigma^N}\exp(-\frac{|x-\mu|^2}{4\sigma^2})$. We deduce form (3.17) that

$$\begin{aligned}\frac{1}{(4\pi)^{N/2}\delta^N}&\int_{\mathbb{R}^N}|\nabla\varphi(x_*,1-\delta^2)|^2\exp(-\frac{|x_*|^2}{4\delta^2})\,dx_*\\ &\leq\int_{\mathbb{R}^N\times\mathbb{R}^N}N(0,\delta^2)(x_*)N(x,1-\delta^2)(x_*)e_\epsilon(\mathrm{v}_\epsilon(x,0))\,dx_*dx\\ &\leq\int_{\mathbb{R}^N}\left(N(0,\delta^2)*N(0,1-\delta^2)\right)(x)e_\epsilon(\mathrm{v}_\epsilon(x,0))\,dx\\ &=\int_{\mathbb{R}^N}N(0,1)e_\epsilon(\mathrm{v}_\epsilon(x,0))\,dx,\end{aligned} \tag{3.18}$$

i.e.

$$\tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon, (0,1), \delta) \leq \delta^2 \tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon, (0,1), 1)$$

so that (3.10) is verified for $\delta \leq \sqrt{1/2}$.

In the general case v_ϵ may vanish, so that it is not possible to find a phase φ which is globally defined. However, if locally we may write $\mathrm{v}_\epsilon = \rho \exp(i\varphi)$, then

$$\mathrm{v}_\epsilon \times \nabla \mathrm{v}_\epsilon = \rho^2 \nabla \varphi$$

so that when ρ is close to 1, $\mathrm{v}_\epsilon \times \nabla \mathrm{v}_\epsilon$ represents essentially the gradient of the phase. The quantity $\mathrm{v}_\epsilon \times \nabla \mathrm{v}_\epsilon$ is always globally defined, in contrast with the phase. The following decomposition formula is then the starting point of the analysis of $|\nabla \mathrm{v}_\epsilon|^2$

$$4|\mathrm{v}_\epsilon|^2 |\nabla \mathrm{v}_\epsilon|^2 = 4|\mathrm{v}_\epsilon \times \nabla \mathrm{v}_\epsilon|^2 + |\nabla |\mathrm{v}_\epsilon|^2| = 4|\mathrm{v}_\epsilon \times \nabla \mathrm{v}_\epsilon|^2 + 4\rho^2 |\nabla \rho|^2, \tag{3.19}$$

where $\rho = |\mathrm{v}_\epsilon|$ is the modulus.

In order to establish (3.10), it suffices to prove a similar inequality when δ_0 is replaced by some $\delta \in [\delta_0, 2\delta_0]$ That is, we will show that there exist $\delta \in [\delta_0, 2\delta_0]$ such that

$$\tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon, (0,1), \delta) \leq \frac{1}{2} \tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon, (0,1), 1) + \mathcal{R}(\eta). \tag{3.20}$$

We will determine δ using averaging arguments, for quantities which will be integrated on **constant time slices** [and bounded thanks to (3.6,3.7,3.8)]. For that purpose, we introduce first some notation. Set, for $t \in [0,1]$,

$$A(t) = \frac{1}{(1-t)^{N/2}} \int_{\mathbb{R}^N \times \{t\}} V_\epsilon(\mathrm{v}_\epsilon) \exp(-\frac{|x|^2}{4(1-t)})\, dx,$$

$$B(t) = \frac{1}{(1-t)^{N/2}} \int_{\mathbb{R}^N \times \{t\}} \Xi(\mathrm{v}_\epsilon, (0,1), (x,t)) \exp(-\frac{|x|^2}{4(1-t)})\, dx.$$

By (3.7) and (3.8) we have therefore

$$\int_0^{1-\delta^2} A(t)\, dt \leq 4|\log \delta_0| \eta \tag{3.21}$$

and

$$\int_0^{1-\delta^2} B(t)\, dt \leq 4|\log \delta_0| \eta. \tag{3.22}$$

We first observe that the left hand side of (3.20), i.e. $\tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon, (0,1), \delta)$, involves an integral on the whole $\mathbb{R}^N$. However, for "many" choices of δ, we may localize this integral.

3.4 Localizing the energy on appropriate time slices

Consider the set Θ_1 defined by

$$\Theta_1 = \left\{ t \in [1-4\delta_0^2, 1-\delta_0^2] \text{ s.t. } A(t)+B(t) \le \frac{32|\log\delta_0|\eta}{\delta_0^2} \right\}. \tag{3.23}$$

Lemma 3.1. *We have*

$$meas\,(\Theta_1) \ge \frac{3}{4} meas\,\left([1-4\delta_0^2, 1-\delta_0^2]\right).$$

Proof. The proof is an easy consequence of (3.21) and (3.22). □

Lemma 3.2. *The following inequality holds for any $t \in \Theta_1$:*

$$\tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon, (0,1), \delta) \le \frac{1}{\delta^{N-2}} \int_{B(2\sqrt{N}\delta)\times\{t\}} e_\epsilon(\mathrm{v}_\epsilon) + K|\log\delta|\eta,$$

where $\delta = \sqrt{1-t}$.

Proof. The proof is an immediate consequence of Proposition 2.4 and the definition of Θ_1. □

3.5 Improved energy decay estimate for the modulus

Set $\sigma_\epsilon = 1 - |\mathrm{v}_\epsilon|^2$. Recall that v_ϵ verifies the equation

$$\partial_t\sigma_\epsilon - \Delta\sigma_\epsilon = 2|\nabla \mathrm{v}_\epsilon|^2 - \frac{2}{\epsilon^2}\sigma_\epsilon(1-\sigma_\epsilon) \qquad \text{on } \mathbb{R}^N \times (0,+\infty). \tag{3.24}$$

Let $\delta \in [\delta_0, 2\delta_0]$ be given. Our first aim is to bound $\int_{B(1)\times\{t\}} |\nabla\sigma_\epsilon|^2$, where $t = 1-\delta^2$.

Lemma 3.3. *The following inequality holds*

$$\int_{B(1)\times\{t\}} |\nabla\sigma_\epsilon|^2 \le C(\delta_0) \left(\int_{\mathbb{R}^N\times\{t\}} V_\epsilon(\mathrm{v}_\epsilon) \exp(-\frac{|x|^2}{4\delta^2}) \right)^{1/2}$$
$$\cdot \left(\int_{\mathbb{R}^N\times\{t\}} (|\nabla \mathrm{v}_\epsilon|^2 + |\frac{x}{2\delta^2}\cdot\nabla \mathrm{v}_\epsilon - \partial_t \mathrm{v}_\epsilon|^2) \exp(-\frac{|x|^2}{4\delta^2}) \right)^{1/2}, \tag{3.25}$$

where $C(\delta_0) = K\delta_0^{-2}\exp(\frac{1}{\delta_0^2})$.

Proof. Let $r_1 \in [1,2]$, multiplying (3.24) by σ_ϵ and integrating by parts on $B(r_1)$ we obtain

$$\begin{aligned} \int_{B(r_1)} |\nabla \mathrm{v}_\epsilon|^2 = {} & 2\int_{B(r_1)} |\nabla \mathrm{v}_\epsilon|^2\sigma_\epsilon - \int_{B(r_1)} \partial_t\sigma_\epsilon \cdot \sigma_\epsilon \\ & - \int_{\partial B(r_1)} \partial_r\sigma_\epsilon\cdot\sigma_\epsilon - \frac{2}{\epsilon^2}\int_{B(r_1)} \sigma_\epsilon^2(1-\sigma_\epsilon) \\ \le {} & 2\int_{B(r_1)} |\nabla \mathrm{v}_\epsilon|^2 + \int_{B(r_1)} |\partial_t\sigma_\epsilon\cdot\sigma_\epsilon| + \int_{\partial B(r_1)} |\partial_r\sigma_\epsilon\cdot\sigma_\epsilon|. \end{aligned} \tag{3.26}$$

Here we have used for the last inequality the fact that $(1-\sigma_\epsilon)\sigma_\epsilon^2 \geq 0$. In order to bound the last term of the r.h.s. of the previous inequality, we choose $r_1 \in [1,2]$ so that

$$\int_{\partial B(r_1)\times\{t\}} |\nabla \mathrm{v}_\epsilon|^2 \exp(-\frac{|x|^2}{4\delta^2}) \leq \int_{\mathbb{R}^N\times\{t\}} |\nabla \mathrm{v}_\epsilon|^2 \exp(-\frac{|x|^2}{4\delta^2})$$

and

$$\int_{\partial B(r_1)\times\{t\}} V_\epsilon(\mathrm{v}_\epsilon) \exp(-\frac{|x|^2}{4\delta^2}) \leq \int_{\mathbb{R}^N\times\{t\}} V_\epsilon(\mathrm{v}_\epsilon) \exp(-\frac{|x|^2}{4\delta^2}).$$

For this choice of r_1, we have therefore

$$\begin{aligned} \int_{\partial B(r_1)\times\{t\}} |\partial_r\sigma_\epsilon\cdot\sigma_\epsilon| \quad &\leq \quad K\epsilon\left(\int_{\partial B(r_1)\times\{t\}} |\nabla \mathrm{v}_\epsilon|\cdot|\frac{1-|\mathrm{v}_\epsilon|^2}{\epsilon}|\right) \\ \leq K\epsilon\exp(\frac{1}{\delta_0^2})&\left(\int_{\partial B(r_1)\times\{t\}} |\nabla \mathrm{v}_\epsilon|^2 \exp(-\frac{|x|^2}{4\delta^2})\right)^{1/2}\left(\int_{\partial B(r_1)\times\{t\}} V_\epsilon(\mathrm{v}_\epsilon)\exp(-\frac{|x|^2}{4\delta^2})\right)^{1/2} \\ \leq K\epsilon\exp(\frac{1}{\delta_0^2})&\left(\int_{\mathbb{R}^N\times\{t\}} |\nabla \mathrm{v}_\epsilon|^2 \exp(-\frac{|x|^2}{4\delta^2})\right)^{1/2}\left(\int_{\mathbb{R}^N\times\{t\}} V_\epsilon(\mathrm{v}_\epsilon)\exp(-\frac{|x|^2}{4\delta^2})\right)^{1/2}. \end{aligned} \tag{3.27}$$

Finally, we estimate the remaining two terms on the r.h.s of (3.26). First, we have by (3.5)

$$\begin{aligned} \int_{B(r_1)\times\{t\}} |\nabla \mathrm{v}_\epsilon|^2\sigma_\epsilon \quad &\leq \quad K\int_{B(r_1)\times\{t\}} |\nabla \mathrm{v}_\epsilon|\cdot|\frac{1-|\mathrm{v}_\epsilon|^2}{\epsilon}| \\ \leq K\exp(\frac{1}{\delta_0^2})&\left(\int_{\mathbb{R}^N\times\{t\}} |\nabla \mathrm{v}_\epsilon|^2 \exp(-\frac{|x|^2}{4\delta^2})\right)^{1/2}\left(\int_{\mathbb{R}^N\times\{t\}} V_\epsilon(\mathrm{v}_\epsilon)\exp(-\frac{|x|^2}{4\delta^2})\right)^{1/2}. \end{aligned} \tag{3.28}$$

Similarly,

$$\begin{aligned} \int_{B(r_1)\times\{t\}} |\partial_t\sigma_\epsilon\cdot\sigma_\epsilon| \quad &\leq \quad K\epsilon\int_{B(r_1)\times\{t\}} |\partial_t \mathrm{v}_\epsilon||\frac{1-|\mathrm{v}_\epsilon|^2}{\epsilon}| \\ \leq K\epsilon\exp(\frac{1}{\delta_0^2})&\left(\int_{B(r_1)\times\{t\}} |\partial_t \mathrm{v}_\epsilon|^2 \exp(-\frac{|x|^2}{4\delta^2})\right)^{1/2}\left(\int_{B(r_1)\times\{t\}} V_\epsilon(\mathrm{v}_\epsilon)\exp(-\frac{|x|^2}{4\delta^2})\right)^{1/2} \\ \leq K\epsilon\exp(\frac{1}{\delta_0^2})&\left(\int_{B(r_1)\times\{t\}} V_\epsilon(\mathrm{v}_\epsilon)\exp(-\frac{|x|^2}{4\delta^2})\right)^{1/2}\left(\int_{B(r_1)\times\{t\}}(|\frac{x}{2\delta^2}\cdot\nabla \mathrm{v}_\epsilon|^2\right. \\ &\left.+|\frac{x}{2\delta^2}\cdot\nabla \mathrm{v}_\epsilon-\partial_t \mathrm{v}_\epsilon|^2)\exp(-\frac{|x|^2}{4\delta^2})\right)^{1/2}. \end{aligned} \tag{3.29}$$

Combining (3.27), (3.28) and (3.29) we derive the conclusion. □

The previous lemma allows us to estimate the contribution of the modulus to the energy on appropriate time slices. More precisely,

Proposition 3.2. *For any $t \in \Theta_1$ we have*

$$\int_{B(1)\times\{t\}} \frac{1}{2}|\nabla|\mathrm{v}_\epsilon|^2|^2 + \frac{(1-|\mathrm{v}_\epsilon|^2)^2}{4\epsilon^2} \le C_1(\delta_0)\left[\eta^{1/2}(\tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon,(0,1),1)+1)\right],$$

where $C_1(\delta_0) = K\delta_0^{N-4}\exp(\frac{1}{\delta_0^2})|\log\delta_0|$.

Proof. By (3.25), we have

$$\begin{aligned}\int_{B(1)\times\{t\}} \frac{1}{2}|\nabla|\mathrm{v}_\epsilon|^2|^2 &\le C(\delta_0)[\delta^N A(t)]^{1/2}\left[\delta^{(N-2)/2}\tilde{E}^{1/2}_{w,\epsilon}(\mathrm{v}_\epsilon,(0,1),\delta) + \delta^{(N-2)/2}B(t)^{1/2}\right] \\ &\le C(\delta_0)\delta^{N-1}A(t)^{1/2}\left[\tilde{E}^{1/2}_{w,\epsilon}(\mathrm{v}_\epsilon,(0,1),1) + B(t)^{1/2}\right],\end{aligned}$$

and we have made use of the monotonicity formula for the last inequality.

For $t \in \Theta_1$, $A(t) + B(t) \le 32\,|\log\delta_0|\delta_0^{-2}\eta$, so that

$$\begin{aligned}\int_{B(1)\times\{t\}} \frac{1}{2}|\nabla|\mathrm{v}_\epsilon|^2|^2 &\le K\,C(\delta_0)\delta_0^{N-2}|\log\delta_0|\eta^{1/2}\left(\tilde{E}^{1/2}_{w,\epsilon}(\mathrm{v}_\epsilon,(0,1),1) + \eta^{1/2}\right) \\ &\le K\,C(\delta_0)\delta_0^{N-2}|\log\delta_0|\left[\eta^{1/2}(\tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon,(0,1),1)+1)\right].\end{aligned}$$

Finally, we have for the potential and for $t \in \Theta_1$,

$$\frac{1}{4\epsilon^2}\int_{B(1)\times\{t\}}(1-|\mathrm{v}_\epsilon|^2)^2 \le \delta^N\exp(\frac{1}{\delta_0^2})A(t) \le K\delta_0^{N-2}|\log\delta_0|\exp(\frac{1}{\delta_0^2})\eta$$

and the conclusion follows. □

3.6 Hodge - de Rham decomposition of $\mathrm{v}_\epsilon \times d\mathrm{v}_\epsilon$

In view of (3.19) and the previous subsection, it remains to provide an improved decay estimate for $|\mathrm{v}_\epsilon \times d\mathrm{v}_\epsilon|^2$. For that purpose, we will introduce as for the elliptic case an appropriate Hodge - de Rham decomposition of $\mathrm{v}_\epsilon \times d\mathrm{v}_\epsilon$. We would like to emphasize the fact that the estimates obtained so far work equally well if we consider instead vector-valued maps $u_\varepsilon : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}^k$, $k \ge 1$. The techniques of the present subsection however heavily rely on the fact that $k = 2$, i.e. u_ε is complex-valued.

Let $\chi \in \mathcal{C}^\infty_c(\mathbb{R}^N)$ be such that $0 \le \chi \le 1$, $\chi \equiv 1$ on $B(2)$ and $\chi \equiv 0$ on $\mathbb{R}^N \setminus B(4)$. We assume moreover that $\|\nabla\chi\|_\infty \le 1$. Consider for $t > 0$ the two-form ψ_t defined on $\mathbb{R}^N \times \{t\}$ by

$$\psi_t = -G_N * d(\mathrm{v}_\epsilon \times d\mathrm{v}_\epsilon)\,\chi \qquad \text{on } \mathbb{R}^N \times \{t\} \tag{3.30}$$

where G_N denotes the Green's function of the Laplace operator in dimension N,

$$G_N(x) = -\frac{\omega_{N-1}}{|x|^{N-2}} \quad \text{for } N > 2 \qquad \text{and} \qquad G_2(x) = \frac{1}{2\pi}\log|x|.$$

Note in particular that

$$-\Delta\psi_t = d(\mathrm{v}_\epsilon \times d\mathrm{v}_\epsilon)\,\chi \qquad \text{on } \mathbb{R}^N \times \{t\} \tag{3.31}$$

and that, for $N \geq 3$,

$$|\psi_t|(x) \to 0 \quad \text{as } |x| \to +\infty.$$

Since $-\Delta = dd^* + d^*d$ and since $\chi \equiv 1$ on $B(2)$ it follows that

$$d(\mathrm{v}_\epsilon \times d\mathrm{v}_\epsilon - d^*\psi_t) = d^*d\psi_t \equiv \zeta_t \qquad \text{on } B(2) \times \{t\}. \tag{3.32}$$

We observe that

$$\Delta(d\psi_t) = 0 \qquad \text{on } B(2) \times \{t\}. \tag{3.33}$$

Indeed, we have

$$\Delta(d\psi_t) = d(\Delta\psi_t) = d\left(d(\mathrm{v}_\epsilon \times d\mathrm{v}_\epsilon)\right) = 0.$$

It follows that the two-form $\zeta_t = d^*d\psi_t$ is closed, since

$$d\zeta_t = d(d^*d\psi_t) = dd^*(d\psi_t) = -\Delta(d\psi_t) - d^*d(d\psi_t) = -\Delta(d\psi_t) = 0. \tag{3.34}$$

By Poincaré Lemma, there exists therefore a 1-form ξ_t defined on $B(3/2) \times \{t\}$ such that

$$\begin{cases} d\xi_t = \zeta_t & \text{on } B(3/2) \times \{t\} \\ d^*\xi_t = 0 & \text{on } B(3/2) \times \{t\}, \end{cases} \tag{3.35}$$

and

$$\|\xi_t\|_{L^2(B(3/2)\times\{t\})} \leq K\|\zeta_t\|_{L^2(B(7/4)\times\{t\})}. \tag{3.36}$$

[Note that such a form ξ_t is not uniquely defined]

Going back to (3.32), we may write

$$d(\mathrm{v}_\epsilon \times d\mathrm{v}_\epsilon - d^*\psi_t - \xi_t) = 0 \qquad \text{on } B(3/2) \times \{t\}.$$

Invoking once more the Poincaré Lemma, we deduce that there exists some function φ_t uniquely determined on $B(3/2) \times \{t\}$ (up to an additive constant) such that

$$\mathrm{v}_\epsilon \times d\mathrm{v}_\epsilon = d\varphi_t + d^*\psi_t + \xi_t \qquad \text{on } B(3/2) \times \{t\}. \tag{3.37}$$

This is precisely the Hodge - de Rham decomposition of $\mathrm{v}_\epsilon \times d\mathrm{v}_\epsilon$ which best fits our needs. We are going to estimate the L^2 norm of each of the three terms on the r.h.s. of (3.37) successively. As we will see, the most delicate estimate is for ψ_t. Although it will enter in the final estimates for ξ_t and φ_t, we will treat these last two terms first.

3.7 Estimate for ξ_t

Since $d\psi_t$ is harmonic on $B(2)$ by (3.32), we have for any $k \in \mathbb{N}$,

$$\|d\psi_t\|_{C^k(B(3/2)\times\{t\})} \leq K_k\|d\psi_t\|_{L^2(B(2)\times\{t\})} \leq K_k\|\nabla\psi_t\|_{L^2(B(2)\times\{t\})}. \tag{3.38}$$

On the other hand, since $\zeta_t = d^*d\psi_t$, it follows that

$$\|\zeta_t\|_{C^k(B(7/4)\times\{t\})} \leq K_k\|\nabla\psi_t\|_{L^2(B(2)\times\{t\})}$$

and going back to (3.36) we obtain the estimate :

Lemma 3.4. *We have,*

$$\|\xi_t\|_{L^2(B(3/2)\times\{t\})} \leq K\|\nabla\psi_t\|_{L^2(B(2)\times\{t\})}.$$

3.8 Estimate for φ_t

The first step is to derive an elliptic equation for φ_t. This equation involves a linear elliptic operator (with a first order term) which appears naturally in the context of parabolic equations (see [26]). In a second step we provide some simple linear estimates for this operator. We finally use them to complete the estimates for φ_t.

The equation for φ_t. Taking the external product of $(\text{PGL})_\epsilon$ for v_ϵ with v_ϵ, we obtain

$$\text{v}_\epsilon \times \partial_t \text{v}_\epsilon + d^*(\text{v}_\epsilon \times d\text{v}_\epsilon) = 0 \qquad \text{on } \mathbb{R}^N \times (0, +\infty). \tag{3.39}$$

[Note that if $\text{v}_\epsilon = \rho \exp(i\phi)$ then (3.39) is equivalent to (6)] The term $d^*(\text{v}_\epsilon \times d\text{v}_\epsilon)$ can be computed using the Hodge - de Rham decomposition (3.37). We have, since $d^*\xi_t = 0$,

$$d^*(\text{v}_\epsilon \times d\text{v}_\epsilon) = -\Delta\varphi_t \qquad \text{on } B(3/2) \times \{t\}.$$

On the other hand, we may write

$$\text{v}_\epsilon \times \partial_t \text{v}_\epsilon = -\text{v}_\epsilon \times \left(\frac{x}{2\delta^2} \cdot \nabla \text{v}_\epsilon - \partial_t \text{v}_\epsilon\right) + \frac{x}{2\delta^2} \cdot (\text{v}_\epsilon \times \nabla \text{v}_\epsilon)$$

and

$$\frac{x}{2\delta^2} \cdot (\text{v}_\epsilon \times \nabla \text{v}_\epsilon) = \frac{x}{2\delta^2}(\nabla\varphi_t + d^*\psi_t + \xi_t).$$

Going back to (3.39) we thus obtain

$$\begin{aligned} -\Delta\varphi_t + \frac{x}{2\delta^2} \cdot \nabla\varphi_t = {}& \text{v}_\epsilon \times \left(\frac{x}{2\delta^2} \cdot \nabla \text{v}_\epsilon - \partial_t \text{v}_\epsilon\right) \\ & - (d^*\psi_t + \xi_t) \cdot \frac{x}{2\delta^2} \qquad \text{on } B(3/2) \times \{t\}. \end{aligned} \tag{3.40}$$

In view of (3.40), we are led to consider the linear elliptic operator

$$L_\delta \equiv -\Delta + \frac{x}{2\delta^2} \cdot \nabla = -\exp(\frac{|x|^2}{4\delta^2})\text{div}\left(\exp(-\frac{|x|^2}{4\delta^2})\nabla\right).$$

Linear estimates for L_δ. Let $r > 0$ and consider functions v and f on $B(r)$ such that

$$L_\delta v = f \qquad \text{on } B(r). \tag{3.41}$$

The next lemma corresponds to the of Pohozaev's identity for the operator L_δ.

Lemma 3.5. *Let v and f satisfy (3.41), then the following equality holds*

$$\begin{aligned} \int_{B(r)} \left[(\frac{N-2}{2} - \frac{|x|^2}{4\delta^2})|\nabla v|^2 \exp(-\frac{|x|^2}{4\delta^2})\right] + \int_{B(r)} x \cdot \nabla v f \exp(-\frac{|x|^2}{4\delta^2}) \\ = \frac{r}{2}\int_{\partial B(r)} |\nabla v|^2 \exp(-\frac{|x|^2}{4\delta^2}) - r\int_{\partial B(r)} |\partial_r v|^2 \exp(-\frac{|x|^2}{4\delta^2}). \end{aligned} \tag{3.42}$$

Proof. We multiply Δv by $x\cdot\nabla v\exp(-\frac{|x|^2}{4\delta^2})$ and integrate by parts on $B(r)$. This yields,

$$
\begin{aligned}
&\int_{B(r)}\Delta v\,(\sum_i x_i\partial_i v)\exp(-\frac{|x|^2}{4\delta^2})\\
=&-\int_{B(r)}\nabla v\cdot\nabla(\sum_i x_i\partial_i v)\exp(-\frac{|x|^2}{4\delta^2})+\int_{B(r)}\frac{x}{2\delta^2}\cdot\nabla v(\sum_i x_i\partial_i v)\exp(-\frac{|x|^2}{4\delta^2})\\
&+\int_{\partial B(r)}\partial_r v(\sum_i x_i\partial_i v)\exp(-\frac{|x|^2}{4\delta^2})\\
=&-\int_{B(r)}\sum_j\partial_j v(\sum_i x_i\partial_i v)\exp(-\frac{|x|^2}{4\delta^2})+\int_{B(r)}\frac{x}{2\delta^2}\cdot\nabla v(\sum_i x_i\partial_i v)\exp(-\frac{|x|^2}{4\delta^2})\\
&+\int_{\partial B(r)}\partial_r v(\sum_i x_i\partial_i v)\exp(-\frac{|x|^2}{4\delta^2})\\
=&-\int_{B(r)}\sum_j|\partial_j v|^2\exp(-\frac{|x|^2}{4\delta^2})-\int_{B(r)}\sum_{i,j}\frac{x_i}{2}\partial_i(|\partial_j v|^2)\exp(-\frac{|x|^2}{4\delta^2})\\
&+\int_{B(r)}\frac{x}{2\delta^2}\cdot\nabla v(\sum_i x_i\partial_i v)\exp(-\frac{|x|^2}{4\delta^2})+\int_{\partial B(r)}\partial_r v(\sum_i x_i\partial_i v)\exp(-\frac{|x|^2}{4\delta^2})\\
=&-\int_{B(r)}\sum_j|\partial_j v|^2\exp(-\frac{|x|^2}{4\delta^2})+\int_{B(r)}\sum_{i,j}\frac{1}{2}|\partial_j v|^2\exp(-\frac{|x|^2}{4\delta^2})\\
&-\int_{B(r)}\frac{|x|^2}{4\delta^2}|\nabla v|^2\exp(-\frac{|x|^2}{4\delta^2})+\int_{B(r)}\frac{x}{2\delta^2}\cdot\nabla v(\sum_i x_i\partial_i v)\exp(-\frac{|x|^2}{4\delta^2})\\
&-\int_{\partial B(r)}\sum_j\frac{r}{2}|\partial_j v|^2\exp(-\frac{|x|^2}{4\delta^2})+\int_{\partial B(r)}\partial_r v(\sum_i x_i\partial_i v)\exp(-\frac{|x|^2}{4\delta^2})\\
=&\int_{B(r)}(\frac{N-2}{2}-\frac{|x|^2}{4\delta^2})|\nabla v|^2\exp(-\frac{|x|^2}{4\delta^2})+\int_{B(r)}\frac{x}{2\delta^2}\cdot\nabla v(\sum_i x_i\partial_i v)\exp(-\frac{|x|^2}{4\delta^2})\\
&-\frac{r}{2}\int_{\partial B(r)}|\nabla v|^2\exp(-\frac{|x|^2}{4\delta^2})+r\int_{\partial B(r)}|\partial_r v|^2\exp(-\frac{|x|^2}{4\delta^2}).
\end{aligned}
\tag{3.43}
$$

The conclusion then follows from (3.41). ☐

Corollary 3.1. *We have, if v and f satisfy (3.41), the inequality*

$$
\int_{\partial B(r)}|\nabla_\top v|^2\exp(-\frac{|x|^2}{4\delta^2})\le\frac{N-2}{r}\int_{B(r)}|\nabla v|^2\exp(-\frac{|x|^2}{4\delta^2})+\frac{1}{r}\int_{B(r)}f^2\exp(-\frac{|x|^2}{4\delta^2})
+\int_{\partial B(r)}|\partial_r v|^2\exp(-\frac{|x|^2}{4\delta^2})\tag{3.44}
$$

Proof. It suffices to note that

$$\int_{B(r)} x \cdot \nabla v f \exp(-\frac{|x|^2}{4\delta^2}) - \int_{B(r)} \frac{|x|^2}{4\delta^2} |\nabla v|^2 \exp(-\frac{|x|^2}{4\delta^2})$$
$$\leq \left(\int_{B(r)} |x|^2 |\nabla v|^2 \exp(-\frac{|x|^2}{4\delta^2})\right)^{\frac{1}{2}} \left(\int_{B(r)} f^2 \exp(-\frac{|x|^2}{4\delta^2})\right)^{\frac{1}{2}} - \int_{B(r)} \frac{|x|^2}{4\delta^2} |\nabla v|^2 \exp(-\frac{|x|^2}{4\delta^2})$$
$$\leq \frac{1}{2} \int_{B(r)} f^2 \exp(-\frac{|x|^2}{4\delta^2}),$$

since $4\delta^2 \leq 1$. □

Next, we consider the boundary value problem

$$\begin{cases} L_\delta v = f & \text{on } B(r) \\ \frac{\partial v}{\partial r} = g & \text{on } \partial B(r). \end{cases} \tag{3.45}$$

Lemma 3.6. *There exists some constants $C(\delta, r)$ depending only and continuously on δ and r, such that if v, f, g verify (3.45) then*

$$\int_{B(r)} |\nabla v|^2 \exp(-\frac{|x|^2}{4\delta^2}) \leq C(\delta, r) \left[\int_{B(r)} f^2 \exp(-\frac{|x|^2}{4\delta^2}) + \left(\int_{B(r)} f^2 \exp(-\frac{|x|^2}{4\delta^2})\right)^{1/2} \cdot \left(\int_{\partial B(r)} g^2 \exp(-\frac{|x|^2}{4\delta^2})\right)^{1/2}\right] + Kr \int_{\partial B(r)} g^2 \exp(-\frac{|x|^2}{4\delta^2}), \tag{3.46}$$

where K depends only possibly on N but not on δ or r.

Proof. Note that (3.46) involves only the gradient of v, whereas if v is a solution to (3.45) so is $v + c$ for every $c \in \mathbb{R}$. Therefore we may assume that

$$\int_{\partial B(r)} v = 0. \tag{3.47}$$

It is convenient to use the divergence form of the equation, namely

$$-\text{div} \left(\exp(-\frac{|x|^2}{4\delta^2}) \nabla v\right) = \exp(-\frac{|x|^2}{4\delta^2}) f. \tag{3.48}$$

We multiply (3.48) by v and integrate by parts on $B(r)$ to obtain

$$\int_{B(r)} |\nabla v|^2 \exp(-\frac{|x|^2}{4\delta^2}) = \int_{B(r)} f v \exp(-\frac{|x|^2}{4\delta^2}) - \int_{B(r)} \partial_r v \cdot v \exp(-\frac{|x|^2}{4\delta^2})$$
$$\leq \left(\int_{B(r)} f^2 \exp(-\frac{|x|^2}{4\delta^2})\right)^{1/2} \left(\int_{B(r)} v^2 \exp(-\frac{|x|^2}{4\delta^2})\right)^{1/2} + \left(\int_{\partial B(r)} g^2 \exp(-\frac{|x|^2}{4\delta^2})\right)^{1/2} \left(\int_{\partial B(r)} v^2 \exp(-\frac{|x|^2}{4\delta^2})\right)^{1/2}. \tag{3.49}$$

In view of (3.47), we have by the Poincaré-Wirtinger inequality

$$\int_{\partial B(r)} v^2 \le \frac{r^2}{N-1}\int_{\partial B(r)} |\nabla_\top v|^2. \tag{3.50}$$

By (3.44) we thus have

$$\begin{aligned}\int_{\partial B(r)} v^2 \exp(-\frac{|x|^2}{4\delta^2}) \le r\int_{B(r)} |\nabla v|^2 \exp(-\frac{|x|^2}{4\delta^2}) + \frac{r}{N-1}\int_{B(r)} f^2 \exp(-\frac{|x|^2}{4\delta^2})\\ + \frac{r^2}{N-1}\int_{\partial B(r)} g^2 \exp(-\frac{|x|^2}{4\delta^2}).\end{aligned} \tag{3.51}$$

On the other hand, standard elliptic estimates yield

$$\int_{B(r)} v^2 \exp(-\frac{|x|^2}{4\delta^2}) \le C(\delta, r)\left[\int_{B(r)} f^2 \exp(-\frac{|x|^2}{4\delta^2}) + \int_{\partial B(r)} g^2 \exp(-\frac{|x|^2}{4\delta^2})\right] \tag{3.52}$$

where the constant $C(\delta, r)$ may depend (strongly) on δ and r. Going back to (3.49) we bound the second term on the r.h.s. by

$$\begin{aligned}\left(\int_{\partial B(r)} g^2 \exp(-\frac{|x|^2}{4\delta^2})\right)^{1/2}\left(\int_{\partial B(r)} v^2 \exp(-\frac{|x|^2}{4\delta^2})\right)^{1/2} &\le \frac{r}{2}\int_{\partial B(r)} g^2 \exp(-\frac{|x|^2}{4\delta^2})\\ &\quad + \frac{1}{2r}\int_{\partial B(r)} v^2 \exp(-\frac{|x|^2}{4\delta^2})\\ \le r\int_{\partial B(r)} g^2 \exp(-\frac{|x|^2}{4\delta^2}) + \frac{1}{2}\int_{B(r)} |\nabla v|^2 \exp(-\frac{|x|^2}{4\delta^2}) &+ \frac{1}{2}\int_{B(r)} f^2 \exp(-\frac{|x|^2}{4\delta^2}).\end{aligned}$$

The first term on the r.h.s. of (3.49) is estimated as follows

$$\begin{aligned}\left(\int_{B(r)} f^2 \exp(-\frac{|x|^2}{4\delta^2})\right)^{1/2}\left(\int_{B(r)} v^2 \exp(-\frac{|x|^2}{4\delta^2})\right)^{1/2} \le C(\delta, r)\Bigg[\int_{B(r)} f^2 \exp(-\frac{|x|^2}{4\delta^2})\\ +(\int_{B(r)} f^2 \exp(-\frac{|x|^2}{4\delta^2}))^{1/2}(\int_{\partial B(r)} g^2 \exp(-\frac{|x|^2}{4\delta^2}))^{1/2}\Bigg]\end{aligned}$$

and the conclusion follows. ☐

We are now in position to complete the estimates for φ_t.

Estimates for $\boldsymbol{\varphi_t}$. Recall that for every $0 < r < 3/2$, φ_t verifies the equation

$$\begin{cases} L_\delta \varphi_t = f & \text{on } B(r) \times \{t\} \\ \frac{\partial \varphi_t}{\partial r} = g & \text{on } \partial B(r) \times \{t\} \end{cases}$$

where f and g are defined by

$$f = \mathrm{v}_\epsilon \times \left(\frac{x}{2\delta^2}\cdot \nabla \mathrm{v}_\epsilon - \partial_t \mathrm{v}_\epsilon\right) - (d^*\psi_t + \xi_t)\cdot \frac{x}{2\delta^2} \qquad \text{on } B(3/2)\times\{t\} \tag{3.53}$$

and

$$g = \mathrm{v}_\epsilon \times \frac{\partial \mathrm{v}_\epsilon}{\partial r} - (d^*\psi_t + \xi_t)_N \qquad \text{on } \partial B(r) \times \{t\}. \tag{3.54}$$

In view of Lemma 3.6 we choose $r \in [1, 3/2]$ such that

$$\int_{\partial B(r)\times\{t\}} |\nabla \mathrm{v}_\epsilon|^2 \exp(-\frac{|x|^2}{4\delta^2}) \le 12 \int_{(B(3/2)\setminus B(1))\times\{t\}} |\nabla \mathrm{v}_\epsilon|^2 \tag{3.55}$$

$$\int_{\partial B(r)\times\{t\}} |\nabla \psi_t|^2 \exp(-\frac{|x|^2}{4\delta^2}) \le 12 \int_{(B(3/2)\setminus B(1))\times\{t\}} |\nabla \psi_t|^2 \tag{3.56}$$

$$\int_{\partial B(r)\times\{t\}} |\xi_t|^2 \exp(-\frac{|x|^2}{4\delta^2}) \le 12 \int_{(B(3/2)\setminus B(1))\times\{t\}} |\xi_t|^2 \tag{3.57}$$

so that

$$\int_{\partial B(r)} g^2 \exp(-\frac{|x|^2}{4\delta^2}) \le K \int_{(B(3/2)\setminus B(1))\times\{t\}} \left(|\nabla \mathrm{v}_\epsilon|^2 + |\nabla \psi_t|^2 + |\xi_t|^2\right) \exp(-\frac{|x|^2}{4\delta^2}). \tag{3.58}$$

Our main estimate for φ_t is the following proposition.

Proposition 3.3. *We have*

$$\begin{aligned}\int_{B(1)} |\nabla \varphi_t|^2 \exp(-\frac{|x|^2}{4\delta^2}) \le K\delta^N \tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon, (0,1), \delta) \\ + C(\delta_0)\left[R(t) + R(t)^{\frac{1}{2}} \tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon, (0,1), \delta)^{\frac{1}{2}}\right],\end{aligned} \tag{3.59}$$

where $C(\delta_0)$ is a constant depending only on δ_0, and $R(t)$ is defined as

$$R(t) = \int_{\mathbb{R}^N\times\{t\}} \left[\Xi(\mathrm{v}_\epsilon, (0,1)) + V_\epsilon(\mathrm{v}_\epsilon) + (|\nabla \psi_t|^2 + |\xi_t|^2) 1_{B(3/2)}\right] \exp(-\frac{|x|^2}{4\delta^2})\, dx.$$

Proof. We apply Lemma 3.6 to φ_t. Clearly, in view of the definition (3.53) of f,

$$\int_{B(r)} f^2 \exp(-\frac{|x|^2}{4\delta^2}) \le C(\delta)\, R(t). \tag{3.60}$$

On the other hand, by (3.58),

$$\int_{B(r)} g^2 \exp(-\frac{|x|^2}{4\delta^2}) \le K \int_{(B(3/2)\setminus B(1))\times\{t\}} |\nabla \mathrm{v}_\epsilon|^2 \exp(-\frac{|x|^2}{4\delta^2}) + K\, R(t). \tag{3.61}$$

The important observation is that

$$\begin{aligned}\int_{(B(3/2)\setminus B(1))\times\{t\}} |\nabla \mathrm{v}_\epsilon|^2 \exp(-\frac{|x|^2}{4\delta^2}) &\le 4\delta^2 \int_{(B(3/2)\setminus B(1))\times\{t\}} \frac{|x|^2}{4\delta^2} |\nabla \mathrm{v}_\epsilon|^2 \exp(-\frac{|x|^2}{4\delta^2}) \\ &\le 2N\delta^n \tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon, (0,1), \delta) + 3R(t),\end{aligned}$$

where we have used (2.44) for the last inequality. The conclusion then follows from Lemma 3.6. ☐

The following is a direct consequence of Proposition 3.3 and the definition of Θ_1.

Corollary 3.2. *For $t \in \Theta_1$, we have*

$$\int_{B(2\sqrt{N}\delta)} |\nabla \varphi_t|^2 \le K\delta_0^N \tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon, (0,1), 1) + C(\delta_0)\left[\eta + \eta^{\frac{1}{2}} \tilde{E}_{w,\epsilon}^{\frac{1}{2}}(\mathrm{v}_\epsilon, (0,1), 1) + R_2(t)\right],$$

where

$$R_2(t) = C(\delta_0)\left[\int_{B(3/2)} (|\nabla \psi_t|^2 + |\xi_t|^2) + (\int_{B(3/2)} |\nabla \psi_t|^2 + |\xi_t|^2)^{\frac{1}{2}} E_{w,\epsilon}^{\frac{1}{2}}(\mathrm{v}_\epsilon, (0,1), 1)\right],$$

and $C(\delta_0) = K \exp(-\frac{4}{\delta_0^2})$.

3.9 Splitting ψ_t

We turn next to the estimate for ψ_t. As already announced, this is the key part, and our main contribution in the proof of Theorem 1.

Recall that ψ_t verifies the equation

$$-\Delta \psi_t = 2d(\mathrm{v}_\epsilon \times d\mathrm{v}_\epsilon)\,\chi \qquad \text{on } \mathbb{R}^N \times \{t\}, \tag{3.62}$$

where $t = 1 - \delta^2$, $\delta \in [\delta_0, 2\delta_0]$ is fixed but to be determined later, and the cut-off function χ depends only on x, verifies $0 \le \chi \le 1$, $\chi \equiv 1$ on $B(2)$, $\chi \equiv 0$ on $\mathbb{R}^N \setminus B(4)$, and $|\nabla \chi| \le 1$.

First, as in [8], we define a reprojection of v_ϵ in the following way. Let τ be the rel-valued function defined on $\mathbb{R}^N \times (0, +\infty)$ by

$$\tau(x,t) = p(|\mathrm{v}_\epsilon(x,t)|)$$

where $p(\cdot)$ is a function : $[0,3] \to [\frac{1}{3}, 2]$ verifying the properties

$$\begin{cases} p(s) = \frac{1}{s} & \text{if } \frac{1}{2} \le s \\ p(s) = 1 & \text{if } 0 \le s \le \frac{1}{4} \\ |p'(s)| \le 4 & \text{for all } s. \end{cases} \tag{3.63}$$

By construction, τ verifies the inequality

$$\left|1 - \tau^2(x)\right| \le K\left|1 - |\mathrm{v}_\epsilon(x)|^2\right|. \tag{3.64}$$

Set $\tilde{\mathrm{v}}_\epsilon = \tau \mathrm{v}_\epsilon$, so that

$$\begin{cases} \tilde{\mathrm{v}}_\epsilon = \mathrm{v}_\epsilon & \text{if } |\mathrm{v}_\epsilon| \le \frac{1}{4} \\ |\tilde{\mathrm{v}}_\epsilon| = 1 & \text{if } |\mathrm{v}_\epsilon| \ge \frac{1}{2}. \end{cases}$$

The main motivation for the previous construction is the following observation.

Lemma 3.7. *We have*

$$d(\tilde{\mathrm{v}}_\epsilon \times d\tilde{\mathrm{v}}_\epsilon)(x) = 2\sum_{i<j}(\partial_i\tilde{\mathrm{v}}_\epsilon \times \partial_j\tilde{\mathrm{v}}_\epsilon)\, dx_i \wedge dx_j. \tag{3.65}$$

In particular,

$$d(\tilde{\mathrm{v}}_\epsilon \times d\tilde{\mathrm{v}}_\epsilon)(x) = 0 \quad \text{if } |\mathrm{v}_\epsilon(x)| \geq \frac{1}{2} \tag{3.66}$$

and therefore

$$|d(\tilde{\mathrm{v}}_\epsilon \times d\tilde{\mathrm{v}}_\epsilon)| \leq K\frac{(1-|\mathrm{v}_\epsilon|^2)^2}{4\epsilon^2} = K\, V_\epsilon(\mathrm{v}_\epsilon) \quad \text{on } \mathbb{R}^N \times (0,+\infty). \tag{3.67}$$

Proof. The identity (3.65) follows easily from the definition of d and the identity $d^2 \equiv 0$. For (3.66), we notice that if $|\mathrm{v}_\epsilon(x)| \geq \frac{1}{2}$ then $|\tilde{\mathrm{v}}_\epsilon(x)| = 1$ so that $\partial_i\tilde{\mathrm{v}}_\epsilon$ and $\partial_j\tilde{\mathrm{v}}_\epsilon$ are collinear on the set

$$\mathcal{O} = \left\{x \in \mathbb{R}^N \text{ s.t. } |\mathrm{v}_\epsilon(x)| \geq \frac{1}{2}\right\}.$$

Finally (3.67) follows from (3.66) and the bound (3.5). □

We decompose ψ_t as

$$\psi_t = \psi_{1,t} + \psi_{2,t} \qquad \text{on } \mathbb{R}^N \times \{t\}$$

where

$$\begin{cases} \psi_{1,t} = -G_N * d(\tilde{\mathrm{v}}_\epsilon \times d\tilde{\mathrm{v}}_\epsilon)\chi & \text{on } \mathbb{R}^N \times \{t\} \\ \psi_{2,t} = -G_N * d((1-\tau^2)v_\epsilon \times d\mathrm{v}_\epsilon)\chi & \text{on } \mathbb{R}^N \times \{t\} \end{cases}$$

so that

$$\begin{cases} -\Delta\psi_{1,t} = d(\tilde{\mathrm{v}}_\epsilon \times d\tilde{\mathrm{v}}_\epsilon)\,\chi & \text{on } \mathbb{R}^N \times \{t\} \\ -\Delta\psi_{2,t} = d((1-\tau^2)\mathrm{v}_\epsilon \times d\mathrm{v}_\epsilon)\,\chi & \text{on } \mathbb{R}^N \times \{t\}. \end{cases} \tag{3.68}$$

In view of its definition, $\psi_{2,t}$ is an error term arising from the projection $\tilde{\mathrm{v}}_\epsilon$ of v_ϵ. This term can be handled easily as we see next.

3.10 L^2 estimate for $\nabla\psi_{2,t}$

The following inequality holds for $\psi_{2,t}$.

Lemma 3.8. *We have*

$$\int_{\mathbb{R}^N\times\{t\}} |\nabla\psi_{2,t}|^2 \leq C(\delta_0)\int_{\mathbb{R}^N\times\{t\}} V_\epsilon(\mathrm{v}_\epsilon)\exp(-\frac{|x|^2}{4\delta^2})\, dx. \tag{3.69}$$

where $C(\delta_0) = K\exp(4/\delta_0^2)$ and K is a constant depending only on N.

Proof. We multiply the second equation of (3.68) by $\psi_{2,t}$ and integrate by parts on $\mathbb{R}^N$. This yields

$$\int_{\mathbb{R}^N\times\{t\}} |\nabla\psi_{2,t}|^2 \le \int_{\mathbb{R}^N\times\{t\}} |1-\tau^2|\,|\mathrm{v}_\epsilon\times d\mathrm{v}_\epsilon|\,(\chi|\nabla\psi_{2,t}|+|\nabla\chi|\,|\psi_{2,t}|)$$
$$\le K\left(\int_{B(4)\times\{t\}} V_\epsilon(\mathrm{v}_\epsilon)\right)^{\frac{1}{2}}\left[\left(\int_{\mathbb{R}^N\times\{t\}}|\nabla\psi_{2,t}|^2\right)^{\frac{1}{2}}+\left(\int_{\mathbb{R}^N\times\{t\}}|\nabla\chi|^N\right)^{\frac{1}{N}}\left(\int_{\mathbb{R}^N\times\{t\}}|\psi_{2,t}|^{2^*}\right)^{\frac{1}{2^*}}\right]$$
$$\le K\left(\int_{B(4)\times\{t\}} V_\epsilon(\mathrm{v}_\epsilon)\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^N\times\{t\}}|\nabla\psi_{2,t}|^2\right)^{\frac{1}{2}}$$

where we have used (3.5), (3.64) and the Sobolev inequality. It follows that

$$\int_{\mathbb{R}^N\times\{t\}} |\nabla\psi_{2,t}|^2 \le K\exp(\frac{4}{\delta_0^2})\int_{\mathbb{R}^N\times\{t\}} V_\epsilon(\mathrm{v}_\epsilon)\exp(-\frac{|x|^2}{4\delta^2})\,dx$$

and the proof is complete. □

We next turn to the estimate for $\psi_{1,t}$. We will first present a simple proof in dimension two, and then give the proof for $N\ge 3$. Although this proof might be adapted for the case $N=2$, we believe that the simple arguments in case $N=2$ will shed some insight for the general case.

3.11 L^2 estimate for $\nabla\psi_{1,t}$ when $N=2$

The following estimate holds.

Lemma 3.9. *For every $t\in[1-4\delta_0^2, 1-\delta_0^2]$ we have*

$$\int_{B(2)\times\{t\}} |\nabla\psi_{1,t}|^2 \le C(\delta_0)\left[\int_{\mathbb{R}^2\times\{t\}} V_\epsilon(\mathrm{v}_\epsilon)\exp(-\frac{|x|^2}{4\delta^2})\,dx\right]^2 |\log\epsilon|.$$

Proof. In view of Lemma 3.7 we have

$$\|\Delta\psi_{1,t}\|_{L^1(\mathbb{R}^2)} \le K\int_{B(4)\times\{t\}} V_\epsilon(\mathrm{v}_\epsilon)$$

and by standard elliptic estimates

$$\|\psi_{1,t}\|_{W^{1,p}(B(4))} \le K_p\int_{B(4)\times\{t\}} V_\epsilon(\mathrm{v}_\epsilon)$$

for any $1\le p<2$. On the other hand, we have

$$\Delta(\psi_{1,t}\chi) = (\Delta\psi_{1,t})\chi + 2\nabla\psi_{1,t}\nabla\chi + \psi_{1,t}\Delta\chi$$

so that

$$\|\Delta(\psi_{1,t}\chi)\|_{L^1(\mathbb{R}^2)} \le K\int_{B(4)\times\{t\}} V_\epsilon(\mathrm{v}_\epsilon). \tag{3.70}$$

To complete the proof, we present an unpublished argument of a preliminary version of [8], which relies on the following inequality, due to [14] (see also [16] and [54]).

Lemma 3.10. *We have, for any $u \in H^2(\mathbb{R}^2)$,*

$$\|u\|_{L^\infty(\mathbb{R}^2)} \leq K\|u\|_{H^1(\mathbb{R}^2)} \left[1 + \log^{\frac{1}{2}}(1 + \|u\|_{H^2(\mathbb{R}^2)})\right].$$

We apply Lemma 3.10 to $\psi_{1,t}\chi$. Since

$$\|\Delta(\psi_{1,t}\chi)\|_{L^\infty(\mathbb{R}^2)} \leq \frac{K}{\epsilon^2}$$

by (3.5), we deduce that

$$\|\psi_{1,t}\chi\|_{H^2(\mathbb{R}^2)} \leq \frac{K}{\epsilon^2}$$

and therefore

$$\|\psi_{1,t}\chi\|_{L^\infty(\mathbb{R}^2)} \leq K\|\psi_{1,t}\|_{H^1(\mathbb{R}^2)} \, |\log\epsilon|^{\frac{1}{2}}. \tag{3.71}$$

On the other hand, form standard elliptic estimates and using (3.71) we obtain

$$\begin{aligned} \|\psi_{1,t}\chi\|^2_{H^1(\mathbb{R}^2)} &\leq K\|\Delta(\psi_{1,t}\chi)\|_{L^1(\mathbb{R}^2)} \, \|\psi_{1,t}\chi\|_{L^\infty(\mathbb{R}^2)} \\ &\leq K\|\Delta(\psi_{1,t}\chi)\|_{L^1(\mathbb{R}^2)} \, \|\psi_{1,t}\chi\|_{H^1(\mathbb{R}^2)}|\log\epsilon|^{\frac{1}{2}}. \end{aligned} \tag{3.72}$$

The conclusion then follows from (3.70). □

Remark 3.1. i) The main point here is the L^∞ estimate for $\psi_{1,t}$. The only property of the equation which is used is the pointwise L^∞ bound on $\nabla \mathrm{v}_\epsilon$ in (3.5). A similar type of L^∞ estimate is also used in an essential way for the elliptic case in [8]. The proof there uses, besides (3.5), the monotonicity formula.

ii) Recall that $H^s(\mathbb{R}^2) \hookrightarrow L^\infty_{\mathrm{loc}}(\mathbb{R}^2)$ for $s > 1$. This is however not true for $s = 1$, which is therefore critical for the previous embedding. Lemma 3.10 can thus be interpreted as an interpolation inequality in the critical dimension. There are generalizations of Lemma 3.10 for higher dimension (see [14, 16, 54]), nevertheless they involve critical Sobolev spaces for the corresponding dimension, which require more regularity than H^1.

iii) The proof of Lemma 3.10 can be obtained in the Fourier variable by a decomposition in high and low frequencies. This idea will be used also in our estimate of $\psi_{1,t}$ in the next section, however we have to use additional ingredients related to the nonlinear parabolic nature of $(\mathrm{PGL})_\epsilon$.

3.12 L^2 estimate for $\psi_{1,t}$ when $N \geq 3$

The analog of Lemma 3.9 in higher dimension is the following.

Proposition 3.4. *There exists a subset $\Theta_2 \subseteq [1 - 4\delta_0^2, 1 - \delta_0^2]$ such that*

$$\mathrm{meas}\,(\Theta_2) \geq \frac{3}{4}\mathrm{meas}\left([1 - 4\delta_0^2, 1 - \delta_0^2]\right) \tag{3.73}$$

and for each $t \in \Theta_2$,

$$\int_{\mathbb{R}^N \times \{t\}} |\nabla \psi_{1,t}|^2 \leq C(\delta_0) \epsilon^{\frac{1}{6}} \tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon, (0,1), 1)$$
$$+ C(\delta_0) \int_{\mathbb{R}^N \times \{t\}} V_\epsilon(\mathrm{v}_\epsilon) \exp(-\frac{|x|^2}{4\delta^2})\, dx\ \Big(\tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon, (0,1), 1) + 1\Big), \quad (3.74)$$

where $C(\delta_0) = K \exp(\frac{34^2}{\delta_0^2})$ *and* K *is a constant depending only on* N.

Comment. In contrast with Lemma 3.9, we are only able to establish inequality (3.74) for appropriate time slices.

The proof of Proposition 3.4 is rather involved. We divide it therefore in several steps.

Step 1 : Splitting $\boldsymbol{\psi_1}$**.**

In view of the proof for the elliptic case in [8], as well as in view of Lemma 3.9, it is tempting to believe that a similar L^∞ bound for $\psi_{1,t}$ can be derived for $N \geq 3$. Nevertheless, this may not be true (see however [53] for $N = 4$). To overcome this difficulty, we perform a splitting of $\psi_{1,t}$ in high and low frequencies,

$$\psi_{1,t} = \psi_{1,t}^i + \psi_{1,t}^e \qquad \text{on } \mathbb{R}^N \times \{t\}, \quad (3.75)$$

We will derive an L^∞ estimate for the low frequency part $\psi_{1,t}^e$ and a smallness property for the (weaker) L^2 norm of $\psi_{1,t}^i$. For the sake of simplicity, we write ψ_1 instead of $\psi_{1,t}$ and similarly ψ_1^i and ψ_1^e, whenever this does not lead to a confusion. The high frequencies are essentially contained in ψ_1^i, whereas ψ_1^e stands for the low frequency range. Since

$$\psi_1 = G_N * d(\mathrm{v}_\epsilon \times d\mathrm{v}_\epsilon)\, \chi,$$

we define the splitting (3.75)introducing an appropriate splitting of the kernel G_N. More precisely, we write

$$G_N = G_N^i + G_N^e \equiv m(|x|)\, G_N + (1 - m(|x|))\, G_N$$

where m is some non negative function with compact support which we will define now. Choose $\alpha \in (\frac{2}{3}, \frac{3}{4})$ and consider the non negative function l defined on $\mathbb{R}^+$ by

$$l(s) = \begin{cases} 0 & \text{if } s \leq \epsilon^\alpha \\ ((\frac{s}{\epsilon^\alpha})^{N-1} - 1)(2^{N-1} - 1)^{-1} & \text{if } \epsilon^\alpha \leq s \leq 2\epsilon^\alpha \\ 1 & \text{if } 2\epsilon^\alpha \leq s \leq 16 \\ (2^{N-1} - (\frac{s}{16})^{N-1})(2^{N-1} - 1)^{-1} & \text{if } 16 \leq s \leq 32 \\ 0 & \text{if } s \geq 32. \end{cases}$$

We set

$$m(s) = \begin{cases} 1 - l(s) & \text{if } 0 \leq s \leq 16 \\ 0 & \text{if } s \geq 16. \end{cases}$$

In particular, m is lipschitz with compact support, and

$$\begin{cases} m(s) \equiv 1 & \text{for } s \in (0, \epsilon^\alpha) \\ m(s) \equiv 0 & \text{for } s \in (2\epsilon^\alpha, +\infty) \\ |m'(s)| \leq K\epsilon^{-\alpha}. \end{cases}$$

Finally, we define

$$\psi_1^i = G_N^i * d(\tilde{\mathrm{v}}_\epsilon \times d\tilde{\mathrm{v}}_\epsilon)\, \chi \qquad \text{on } \mathbb{R}^N \times \{t\},$$

and

$$\psi_1^e = G_N^e * d(\tilde{\mathrm{v}}_\epsilon \times d\tilde{\mathrm{v}}_\epsilon)\, \chi \qquad \text{on } \mathbb{R}^N \times \{t\}.$$

The following properties of the kernel G_N^i will be useful. The proofs are elementary and left to the reader.

Lemma 3.11. *We have*

$$\|\nabla G_N^i\|_{L^1(\mathbb{R}^N)} \leq K\varepsilon^\alpha,$$

and

$$\|\Delta G_N^i\|_{\mathcal{M}(\mathbb{R}^N)} \leq K,$$

where $\mathcal{M}$ denotes the set of finite Radon measures on $\mathbb{R}^N$ and K is a constant depending only on N. □

We first begin with the L^∞ estimate for ψ_1^e.

Step 2 : L^∞ estimate for ψ_1^e.

First, notice that $G_N^e(x) = G_N(x)$ for $|x| \geq 2\epsilon^\alpha$. In particular, since χ has compact support in $B(4)$, it follows that

$$\psi_1^e = G_N * d(\tilde{\mathrm{v}}_\epsilon \times d\tilde{\mathrm{v}}_\epsilon)\, \chi \qquad \text{on } \Big(\mathbb{R}^N \setminus B(4 + 2\epsilon^\alpha)\Big) \times \{t\}. \tag{3.76}$$

Therefore, ψ_1^e is harmonic on $(\mathbb{R}^N \setminus B(5))$ (provided ϵ is sufficiently small). Hence, by the maximum principle,

$$\|\psi_1^e\|_{L^\infty(\mathbb{R}^N \times \{t\})} \leq \|\psi_1^e\|_{L^\infty(B(5) \times \{t\})}\,. \tag{3.77}$$

On the other hand, on the larger ball $B(12)$, one has by the definition of m, and in view of the support of χ,

$$\psi_1^e = l(|x|) G_N * d(\tilde{\mathrm{v}}_\epsilon \times d\tilde{\mathrm{v}}_\epsilon)\, \chi \qquad \text{on } B(12) \times \{t\}. \tag{3.78}$$

Recall also that $\mathrm{supp}(l) \subseteq B(32)$ so that $l(|x|) G_N * d(\tilde{\mathrm{v}}_\epsilon \times d\tilde{\mathrm{v}}_\epsilon)\, \chi$ has compact support in $B(36)$. Combining (3.77) and (3.78), we obtain

$$\|\psi_1^e\|_{L^\infty(\mathbb{R}^N \times \{t\})} \leq \|l(|x|) G_N * d(\tilde{\mathrm{v}}_\epsilon \times d\tilde{\mathrm{v}}_\epsilon)\, \chi\|_{L^\infty(\mathbb{R}^N \times \{t\})}\,. \tag{3.79}$$

In order to estimate the r.h.s. of (3.79), we invoke the following lemma, which motivated the precise definition of l. A similar construction was already used in our previous work on the NLS equation [10].

Lemma 3.12. *Let $f \in L^1(\mathbb{R}^N)$. The following equality holds for any $y \in \mathbb{R}^N$.*

$$(l(|x|)G_N * f)(y) = \int_{\epsilon^\alpha}^{16} r^{-1}[\frac{1}{r^{N-2}} \int_{B(y,2r)\times\{t\}} f(x)h(|x-y|,r)\,dx]\,dr + \left[\frac{r^{2-N}}{N-2}\int_{B(y,2r)\times\{t\}} f(x)h(|x-y|,r)\,dx\right]_{\epsilon^\alpha}^{16}, \quad (3.80)$$

where the lipschitz cut-off function h is defined on $\mathbb{R}^+ \times \mathbb{R}^+$ by

$$h(s,r) = \omega_{N-1}\frac{(N-1)(N-2)}{2^{N-1}} \cdot \begin{cases} 1 & \text{if } 0 \le s \le r, \\ \frac{2r-s}{r} & \text{if } r \le s \le 2r, \\ 0 & \text{if } s \ge 2r. \end{cases}$$

Proof. We start with the r.h.s. of the equality. Integrating by parts in the variable r, we obtain

$$\int_{\epsilon^\alpha}^{16} r^{-1}[\frac{1}{r^{N-2}} \int_{B(y,2r)\times\{t\}} f(x)h(|x-y|,r)\,dx]\,dr + \left[\frac{r^{2-N}}{N-2}\int_{B(y,2r)\times\{t\}} f(x)h(|x-y|,r)\,dx\right]_{\epsilon^\alpha}^{16}$$
$$= \int_{\epsilon^\alpha}^{16} \frac{r^{2-N}}{N-2}\int_{B(y,2r)\times\{t\}} f(x)\frac{\partial h}{\partial r}(|x-y|,r)\,dx]\,dr. \quad (3.81)$$

Here, we have used the fact that $h(2r,r) = 0$ for each $r > 0$. Notice that

$$\frac{\partial h}{\partial r}(|x-y|,r) = \omega_{N-1}\frac{(N-1)(N-2)}{2^{N-1}}\frac{|x-y|}{r^2} \qquad \text{for } x \in B(y,2r)\setminus B(y,r)$$

and is equal to zero elsewhere. The last term in (3.81) can thus be rewritten as

$$\int_{\epsilon^\alpha}^{16} r^{-N}\omega_{N-1}\frac{N-1}{2^{N-1}}\int_{(B(y,2r)\setminus B(y,r))\times\{t\}} f(x)|x-y|\,dx]\,dr, \quad (3.82)$$

and therefore also as

$$\omega_{N-1}\frac{N-1}{2^{N-1}}\int_{\epsilon^\alpha}^{16} r^{-N}\int_r^{2r} s[\int_{\partial B(y,s)\times\{t\}} f(x)\,dx]\,ds\,dr \equiv \mathcal{I}(y). \quad (3.83)$$

Using Fubini's theorem, we obtain

$$\mathcal{I}(y) = \omega_{N-1}\frac{N-1}{2^{N-1}}\int_{\epsilon^\alpha}^{32} s[\int_{\partial B(y,s)\times\{t\}} f(x)\,dx]\int_{\min(s/2,\epsilon^\alpha)}^{\max(s,16)} r^{-N}\,dr\,ds. \quad (3.84)$$

Note that by construction, l verifies

$$\frac{N-1}{2^{N-1}}\int_{\min(s/2,\epsilon^\alpha)}^{\max(s,16)} r^{-N}\,dr \equiv l(s)s^{1-N}.$$

Therefore, since $\mathrm{supp}(l) \subseteq [\epsilon^\alpha, 32]$ we can rewrite

$$\mathcal{I}(y) = \int_{\epsilon^\alpha}^{32} \omega_{N-1} s^{2-N} [\int_{\partial B(y,s)\times\{t\}} f(x)\,dx] l(s)\,ds$$
$$= \int_{\mathbb{R}^N\times\{t\}} \frac{\omega_{N-1}}{|y-x|^{N-2}} l(|y-x|) f(x)\,dx,$$

and the proof is complete. ☐

In view of the previous lemma, we have, for any $y \in \mathbb{R}^N$,

$$\left(l(|x|)G_N * d(\tilde{\mathrm{v}}_\epsilon \times d\tilde{\mathrm{v}}_\epsilon)\,\chi\right)(y) = \int_{\epsilon^\alpha}^{16} \frac{1}{r}\mathcal{J}_r(y)\,dr + \frac{1}{N-2}[\mathcal{J}_{16}(y) - \mathcal{J}_{\epsilon^\alpha}(y)],$$

where we set

$$\mathcal{J}_r(y) = \frac{1}{r^{N-2}} \int_{B(y,2r)\times\{t\}} d(\tilde{\mathrm{v}}_\epsilon \times d\tilde{\mathrm{v}}_\epsilon) h(|x-y|, r)\,\chi(x)\,dx.$$

In particular, for any $y \in \mathbb{R}^N$

$$\begin{aligned} |(l(|x|)G_N * d(\tilde{\mathrm{v}}_\epsilon \times d\tilde{\mathrm{v}}_\epsilon)\,\chi)(y)| &\le K \sup_{r\in[\epsilon^\alpha,16]} \left[\int_{\epsilon^\alpha}^{16} \frac{dr}{r} + \frac{2}{N-2}\right] \\ &\le K \sup_{r\in[\epsilon^\alpha,16]} |\mathcal{J}_r(y)|\,[|\log\epsilon| + 1]. \end{aligned} \tag{3.85}$$

In view of (3.65) and the monotonicity formula, one may derive the bound

$$|\mathcal{J}_r(y)| \le K\tilde{E}_{w,\epsilon}(0,(0,1),1).$$

This bound however is far from being satisfactory for our purposes. To proceed further, we argue as in [10], and use a refined estimate due to Jerrard and Soner [34] which relies on the special structure of the Jacobian

$$J\tilde{\mathrm{v}}_\epsilon = \frac{1}{2} d(\tilde{\mathrm{v}}_\epsilon \times d\tilde{\mathrm{v}}_\epsilon).$$

More precisely, we have

Lemma 3.13 (Jerrard & Soner). *Let $w \in H^1_{loc}(\mathbb{R}^N)$, $\varphi \in \mathcal{C}^{0,1}_c(\mathbb{R}^N, \Lambda^2\mathbb{R}^N)$ and set $\mathcal{K} = \mathrm{supp}(\varphi)$. Then there exist some constants $K > 0$ and $0 < \beta < 1$ depending only on N such that*

$$\left|\int_{\mathbb{R}^N} \langle Jw, \varphi\rangle\right| \le \frac{K}{|\log\epsilon|}\|\varphi\|_{L^\infty} \int_{\mathcal{K}} e_\epsilon(w) + K\epsilon^\beta \|\varphi\|_{\mathcal{C}^{0,1}}(1 + \int_{\mathcal{K}} e_\epsilon(w))(1 + |\mathcal{K}|^2). \tag{3.86}$$

With the help of the previous lemma and of the analysis in Section 2.4, we obtain the following.

Lemma 3.14. *Let $\beta > 0$ given in Lemma 3.13. We have, for any $y \in B(36)$,*

$$\sup_{r\in[\epsilon^\alpha,16]} |\mathcal{J}_r(y)| \leq K \exp(\frac{34^2}{\delta_0^2}) \left(\frac{\tilde{E}_{w,\epsilon}(v_\epsilon,(0,1),1)}{|\log\epsilon|} + \epsilon^{\frac{\beta}{4}} \right).$$

Proof. Define the rescaled functions $\tilde{\mathrm{v}}_{\epsilon,y,r}$ and $\chi_{y,r}$ by

$$\tilde{\mathrm{v}}_{\epsilon,y,r}(x) = \tilde{\mathrm{v}}_\epsilon(rx+y) \qquad \text{and} \qquad \chi_{y,r} = \chi(rx+y).$$

Define also $\epsilon_r = \frac{\epsilon}{r}$, and notice that for $r \in [\epsilon^\alpha, 16]$ we have

$$|\log \epsilon_r| \geq (1-\alpha)|\log\epsilon| \geq \frac{1}{4}|\log\epsilon|. \tag{3.87}$$

By scaling and the definition of h, we obtain

$$\begin{aligned} \mathcal{J}_r(y) \equiv \frac{1}{r^{N-2}} \int_{B(y,2r)\times\{t\}} d(\tilde{\mathrm{v}}_\epsilon \times d\tilde{\mathrm{v}}_\epsilon) h(|x-y|,r)\,\chi(x)\,dx \\ = \int_{B(2)\times\{t\}} 2J\tilde{\mathrm{v}}_{\epsilon,y,r}(x)\,h(x,1)\,\chi_{y,r}(x)\,dx. \end{aligned} \tag{3.88}$$

Note that since χ has compact support and $r \leq 16$,

$$\|2\,h(\cdot,1)\chi_{y,r}(\cdot)\|_{\mathcal{C}^{0,1}(B(2))} \leq K$$

where the constant K depends only on N. We apply Jerrard-Soner estimate (3.86) to $w = \tilde{\mathrm{v}}_{\epsilon,y,r}$ and $\varphi = 2h(\cdot,1)\chi_{y,r}$. In view of (3.87), this yields

$$\begin{aligned} \left| \int_{B(2)\times\{t\}} 2J\tilde{\mathrm{v}}_{\epsilon,y,r}(x)\,h(x,1)\,\chi_{y,r}(x)\,dx \right| &\leq K \left(\frac{\int_{B(2)\times\{t\}} e_{\epsilon_r}(\tilde{\mathrm{v}}_{\epsilon,y,r})}{|\log\epsilon_r|} + {\epsilon_r}^{\beta} \right) \\ &\leq K \left(\frac{\int_{B(2)\times\{t\}} e_{\epsilon_r}(\tilde{\mathrm{v}}_{\epsilon,y,r})}{|\log\epsilon|} + \epsilon^{\beta/4} \right). \end{aligned} \tag{3.89}$$

On the one hand, for $\epsilon^\alpha \leq r \leq \frac{\delta}{2}$,

$$\begin{aligned} \int_{B(2)\times\{t\}} e_{\epsilon_r}(\tilde{\mathrm{v}}_{\epsilon,y,r}) &= \frac{1}{r^{N-2}} \int_{B(y,2r)\times\{t\}} e_\epsilon(\tilde{\mathrm{v}}_\epsilon) \\ &\leq K \frac{1}{r^{N-2}} \int_{\mathbb{R}^N\times\{t\}} e_\epsilon(\mathrm{v}_\epsilon) \exp(-\frac{|x-y|^2}{4r^2})\,dx \\ &= K\tilde{\mathcal{E}}_{w,\epsilon}(\mathrm{v}_\epsilon,(y,1-\delta^2),r) \leq K\exp(\frac{1}{12})\tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon,(0,1),1), \end{aligned} \tag{3.90}$$

where we have used Lemma 2.5 for the last inequality.
On the other hand, for $\frac{\delta_2}{2} \leq r \leq 16$,

$$\begin{aligned} \int_{B(2)\times\{t\}} e_{\epsilon_r}(\tilde{\mathrm{v}}_{\epsilon,y,r}) &= \frac{1}{r^{N-2}} \int_{B(y,2r)\times\{t\}} e_\epsilon(\tilde{\mathrm{v}}_\epsilon) \\ &\leq K\exp(\frac{(|y|+32)^2}{4\delta_0^2}) \frac{1}{\delta^{N-2}} \int_{\mathbb{R}^N\times\{t\}} e_\epsilon(\tilde{\mathrm{v}}_\epsilon) \exp(-\frac{|x|^2}{4\delta^2})\,dx \\ &\leq K\exp(\frac{34^2}{\delta_0^2})\tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon,(0,1),1), \end{aligned} \tag{3.91}$$

where we have used the monotonicity formula for the last inequality. The conclusion then follows from (3.88), (3.89), (3.90) and (3.91). ☐

We are now in position to derive our L^∞ estimate for ψ_1^e.

Lemma 3.15. *There exists a constant K depending only on N such that*

$$\|\psi_1^e\|_{L^\infty(\mathbb{R}^N\times\{t\})} \leq C(\delta_0)\left(\tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon,(0,1),1)+1\right), \tag{3.92}$$

where $C(\delta_0)=K\exp(\frac{34^2}{\delta_0^2})$.

Proof. Recall that by (3.79) we have

$$\|\psi_1^e\|_{L^\infty(\mathbb{R}^N\times\{t\})} \leq \|l(|x|)G_N * d(\tilde{\mathrm{v}}_\epsilon\times d\tilde{\mathrm{v}}_\epsilon)\,\chi\|_{L^\infty(\mathbb{R}^N\times\{t\})}.$$

Since $\mathrm{supp}(l)\subseteq B(32)$ and $\mathrm{supp}(\chi)\subseteq B(4)$, we also have

$$\Big(l(|x|)G_N * d(\tilde{\mathrm{v}}_\epsilon\times d\tilde{\mathrm{v}}_\epsilon)\,\chi\Big)(y)=0 \qquad \text{for } y\in\mathbb{R}^N\setminus B(36).$$

Therefore we only need to consider the case $y\in B(36)$, and the conclusion follows using (3.85) and Lemma 3.14. ☐

We next turn to the estimates for the high frequency part of ψ_1, namely ψ_1^i.

Step 3 : L^2 estimate for ψ_1^i.

Since $\psi_1^i = G_n^i * d(\tilde{\mathrm{v}}_\epsilon\times d\tilde{\mathrm{v}}_\epsilon)\chi$ and since $\|\nabla G_N^i\|_{L^1}\leq K\epsilon^\alpha$, a few computations yield the following lemma.

Lemma 3.16. *There exists a constant K depending only on N such that*

$$\int_{\mathbb{R}^N\times\{t\}} |\psi_1^i|^2 \leq C(\delta_0)\epsilon^{2\alpha}\tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon,(0,1),1), \tag{3.93}$$

where $C(\delta_0)=K\exp(\frac{4}{\delta_0^2})$.

Proof. We have

$$\begin{aligned}\psi_1^i &= m(|x|)\frac{\omega_{N-1}}{|x|^{N-2}} * d(\tilde{\mathrm{v}}_\epsilon\times d\tilde{\mathrm{v}}_\epsilon)\,\chi\\ &= m(|x|)\frac{\omega_{N-1}}{|x|^{N-2}} * d(\tilde{\mathrm{v}}_\epsilon\times d\tilde{\mathrm{v}}_\epsilon\,\chi) - m(|x|)\frac{\omega_{N-1}}{|x|^{N-2}} * (\tilde{\mathrm{v}}_\epsilon\times d\tilde{\mathrm{v}}_\epsilon\cdot d\chi)\\ &= d\Big(m(|x|)\frac{\omega_{N-1}}{|x|^{N-2}}\Big) * (\tilde{\mathrm{v}}_\epsilon\times d\tilde{\mathrm{v}}_\epsilon\,\chi) - m(|x|)\frac{\omega_{N-1}}{|x|^{N-2}} * (\tilde{\mathrm{v}}_\epsilon\times d\tilde{\mathrm{v}}_\epsilon\cdot d\chi). \end{aligned} \tag{3.94}$$

Note that by Lemma 3.11,

$$\left\|d\Big(m(|x|)\frac{\omega_{N-1}}{|x|^{N-2}}\Big)\right\|_{L^1(\mathbb{R}^N)} \leq K\epsilon^\alpha, \tag{3.95}$$

and that

$$\left\| m(|x|)\frac{\omega_{N-1}}{|x|^{N-2}} \right\|_{L^1(\mathbb{R}^N)} \leq K\epsilon^{2\alpha} \leq K\epsilon^{\alpha}. \tag{3.96}$$

From (3.94) we thus infer that

$$\left\| \psi_1^i \right\|_{L^2(\mathbb{R}^N\times\{t\})} \leq K\epsilon^{\alpha} \left(\int_{B(4)\times\{t\}} |\nabla \tilde{\mathrm{v}}_\epsilon|^2 \right)^{\frac{1}{2}} \leq K \exp(\frac{2}{\delta_0^2})\epsilon^{\alpha} \tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon, (0,1), \delta)^{\frac{1}{2}} \tag{3.97}$$

and the conclusion follows using the monotonicity formula. □

Step 4 : Introducing an auxiliary parabolic problem.

Recall that

$$-\Delta\psi_1 = d(\tilde{\mathrm{v}}_\epsilon \times d\tilde{\mathrm{v}}_\epsilon)\,\chi \qquad \text{on } \mathbb{R}^N \times \{t\}. \tag{3.98}$$

In view of the result of Section 2.3, it is tempting to compare ψ_1 with the solution ψ_1^* of the parabolic problem

$$\begin{cases} \partial_t\psi_1^* - \Delta\psi_1^* = d(\tilde{\mathrm{v}}_\epsilon \times d\tilde{\mathrm{v}}_\epsilon)\,\chi & \text{on } \mathbb{R}^N \times [0,+\infty) \\ \psi_1^*(.,0) = 0 & \text{on } \mathbb{R}^N \times \{0\}. \end{cases} \tag{3.99}$$

In view of Lemma 3.7, we have

$$|d(\tilde{\mathrm{v}}_\epsilon \times d\tilde{\mathrm{v}}_\epsilon)\,\chi| \leq K\frac{(1-|\mathrm{v}_\epsilon|^2)^2}{4\epsilon^2}\chi \qquad \text{on } \mathbb{R}^N \times [0,\infty), \tag{3.100}$$

where the constant K depends only on N, and the results of Section 2.3 apply directly to ψ_1^*. This yields

Lemma 3.17. *We have, for any $\delta \in [1-4\delta_0^2, 1-\delta^2]$,*

$$\left\| \psi_1^*(.,1-\delta^2) \right\|_{L^\infty(\mathbb{R}^N)} \leq C(\delta_0)\tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon,(0,1),1), \tag{3.101}$$

and

$$\int_{\mathbb{R}^N\times[0,1-\delta_0^2]} |\nabla\psi_1^*|^2 \leq C(\delta_0)\tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon,(0,1),1), \tag{3.102}$$

where $C(\delta_0) = K\exp(4/\delta_0^2)$ and the constant K depends only on N.

Proof. For (3.101), consider the function f defined by

$$f(x,t) = \frac{(1-|\mathrm{v}_\epsilon(x,t)|^2)^2}{4\epsilon^2}\chi(x) \qquad \text{on } \mathbb{R}^N \times [0,\infty)$$

and let ω be the solution of

$$\begin{cases} \frac{\partial\omega}{\partial t} - \Delta\omega = f & \text{on } \mathbb{R}^N \times [0,\infty), \\ \omega(x,0) = 0 & \text{for } x \in \mathbb{R}^N. \end{cases}$$

It follows from the maximum principle and (3.100) that

$$|\psi_1^*(x, 1-\delta^2)| \leq K\omega(x, 1-\delta^2) \qquad \forall\, x \in \mathbb{R}^N. \tag{3.103}$$

We deduce from Proposition 2.2 with $T = 1$ and $z_T = (0,1)$ that

$$\omega(x, 1-\delta^2) \leq K \left(\frac{1}{1-\delta^2}\right)^{N/2} \exp(\frac{|x|^2}{4\delta^2})\, \tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon, (0,1), 1).$$

On the other hand, since χ is supported in $B(4)$, we deduce from Duhamel's representation formula for ω that

$$\sup_{x\in\mathbb{R}^N} \omega(x, 1-\delta^2) = \sup_{x\in B(4)} \omega(x, 1-\delta^2),$$

and therefore

$$\sup_{x\in\mathbb{R}^N} \omega(x, 1-\delta^2) \leq K \exp(\frac{4}{\delta_0^2})\, \tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon, (0,1), 1).$$

The estimate (3.101) then follows from (3.103).

We next turn to (3.102). Multiplying (3.99) by ψ_1^* and integrating by parts we obtain

$$\frac{1}{2}\int_{\mathbb{R}^N\times[1-\delta_0^2]} |\psi_1^*|^2 + \int_{\mathbb{R}^N\times[0,1-\delta_0^2]} |\nabla\psi_1^*|^2 = \int_{\mathbb{R}^N\times[0,1-\delta_0^2]} \langle d(\tilde{\mathrm{v}}_\epsilon \times d\tilde{\mathrm{v}}_\epsilon)\chi, \psi_1^*\rangle. \tag{3.104}$$

Therefore,

$$\begin{aligned}
\int_{\mathbb{R}^N\times[0,1-\delta_0^2]} |\nabla\psi_1^*|^2 &\leq \int_{\mathbb{R}^N\times[0,1-\delta_0^2]} |\tilde{\mathrm{v}}_\epsilon \times d\tilde{\mathrm{v}}_\epsilon|\ (|\nabla\chi|\cdot|\psi_1^*| + \chi|\nabla\psi_1^*|) \\
&\leq K\exp(\frac{2}{\delta_0^2}) \left(\int_{\mathbb{R}^N\times[0,1-\delta_0^2]} |\nabla\psi_1^*|^2\right)^{\frac{1}{2}} \tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon, (0,1), 1)^{\frac{1}{2}} \qquad (3.105)\\
&\leq \frac{1}{2}\int_{\mathbb{R}^N\times[0,1-\delta_0^2]} |\nabla\psi_1^*|^2 + K\exp(\frac{4}{\delta_0^2})\tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon, (0,1), 1).
\end{aligned}$$

Here, we have used the Sobolev inequality and the monotonicity of $\tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon, (0,1), \cdot)$. Estimate (3.102) follows and the proof is complete. □

Comment : Estimate (3.102) seems a little disappointing, since it does not offer any improvement for the energy (in the spirit of Proposition 3.4). However, a little more computations show, using (3.101), that

$$\int_{\mathbb{R}^N}\int_{1-4\delta_0^2}^{1-\delta_0^2} |\nabla\psi_1^*|^2 \leq C(\delta_0)\left(\int_{\mathbb{R}^N}\int_{1-4\delta_0^2}^{1-\delta_0^2} V_\epsilon(\mathrm{v}_\epsilon)\exp(-\frac{|x|^2}{4(1-t)})\,dx\right)(\tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon, (0,1), 1)+1).$$

Notice that this inequality involves only integration on $\mathbb{R}^N \times [1-4\delta_0^2, 1-\delta_0^2]$ whereas (3.102) involves integration on $\mathbb{R}^N \times [0, 1-\delta_0^2]$. We will not make use of the previous bound.

Step 5 : L^2 estimate for $\partial_t\psi_1^*$ on appropriate time slices.

In order to compare ψ_1^* with ψ_1, it seems natural to try to derive some bound on the time derivative $\partial_t\psi_1^*$. In this direction, we have the following estimate.

Lemma 3.18. *We have,*

$$\int_{\mathbb{R}^N\times[0,1-\delta_0^2]} |\partial_t \psi_1^*|^2 \leq C(\delta_0)\epsilon^{-1}\tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon,(0,1),1), \tag{3.106}$$

where $C(\delta_0) = K\delta_0^{-1}\exp(\frac{4}{\delta_0^2})$ *and* K *is a constant depending only on* N.

A straightforward corollary is the following.

Corollary 3.3. *There exists a set* $\Theta_2 \subseteq [1-4\delta_0^2, 1-\delta_0^2]$ *such that*

$$\text{meas}\,(\Theta_2) \geq \frac{3}{4}\text{meas}\left([1-4\delta_0^2, 1-\delta_0^2]\right) \tag{3.107}$$

and for each $t \in \Theta_2$,

$$\int_{\mathbb{R}^N\times\{t\}} |\partial_t \psi_1^*|^2 \leq C(\delta_0)\epsilon^{-1}\tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon,(0,1),1), \tag{3.108}$$

where $C(\delta_0) = K\delta_0^{-3}\exp(\frac{4}{\delta_0^2})$ *and* K *is a constant depending only on* N.

Comment. At first sight, this estimate seems rather poor, since the r.h.s. diverges as $|\log\epsilon|\epsilon^{-1}$, whereas for v_ϵ we already know that

$$\int_{\mathbb{R}^N\times[0,1-\delta_0^2]} |\partial_t \mathrm{v}_\epsilon|^2 \leq \int_{\mathbb{R}^N\times\{0\}} e_\epsilon(\mathrm{v}_\epsilon). \tag{3.109}$$

If one assumes (H_1) then the r.h.s. of the previous inequality behaves as $|\log\epsilon|$. However, estimate (3.109) is deeply related to the fact that $(\mathrm{PGL})_\epsilon$ is the heat flow for the Ginzburg-Landau energy. Linear estimates based on the pointwise bound $|\nabla\tilde{\mathrm{v}}_\epsilon| \leq K\epsilon^{-1}$ would lead only to estimates of order ϵ^{-2}. In this respect, (3.108) presents a substantial improvement which is again related to the divergence structure of the term $d(\tilde{\mathrm{v}}_\epsilon \times d\tilde{\mathrm{v}}_\epsilon)$. This improvement will be crucial for estimate (3.134).

In order to prove Lemma 3.18, we begin with the following estimate for the time derivative $\partial_t \mathrm{v}_\epsilon$.

Lemma 3.19. *We have*

$$\int_{\mathbb{R}^N\times[0,1-\delta_0^2]} |\partial_t \mathrm{v}_\epsilon|^2 \exp(-\frac{|x|^2}{4(1-t)})\,dx\,dt \leq K\delta_0^{-2}\tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon,(0,1),1). \tag{3.110}$$

Proof. By definition of Ξ, we have

$$\begin{aligned}\int_{\mathbb{R}^N\times[0,1-\delta_0^2]} &|\partial_t \mathrm{v}_\epsilon|^2 \exp(-\frac{|x|^2}{4(1-t)})\,dx\,dt \\ &\leq \int_{\mathbb{R}^N\times[0,1-\delta_0^2]} \frac{1}{1-t}\left(\Xi(\mathrm{v}_\epsilon,(0,1)) + \frac{|x|^2}{4(1-t)}|\nabla \mathrm{v}_\epsilon|^2\right)\exp(-\frac{|x|^2}{4(1-t)})\,dx\,dt.\end{aligned}$$

The conclusion follows, using (2.36), (3.7), (3.8) and the monotonicity formula. □

Proof of Lemma 3.18. We multiply the equation for ψ_1^*, namely

$$\partial_t \psi_1^* - \Delta \psi_1^* = d(\tilde{\mathrm{v}}_\epsilon \times d\tilde{\mathrm{v}}_\epsilon)\,\chi \qquad \text{on } \mathbb{R}^N \times [0,\infty)\,, \tag{3.111}$$

by $\partial_t \psi_1^*$ and integrate by parts on $\mathbb{R}^N \times [0, 1-\delta_0^2]$. We obtain

$$\begin{aligned}\int_{\mathbb{R}^N\times[0,1-\delta_0^2]} |\partial_t \psi_1^*|^2 &\leq -\frac{1}{2}\int_0^{1-\delta_0^2} \frac{d}{dt}\Big(\int_{\mathbb{R}^N} |\nabla \psi_1^*|^2\, dx\Big)dt + \int_{\mathbb{R}^N\times[0,1-\delta_0^2]} \langle d(\tilde{\mathrm{v}}_\epsilon \times d\tilde{\mathrm{v}}_\epsilon)\chi, \partial_t\psi_1^*\rangle \\ &= -\frac{1}{2}\int_{\mathbb{R}^N\times\{1-\delta_0^2\}} |\nabla\psi_1^*|^2 + \int_{\mathbb{R}^N\times[0,1-\delta_0^2]} \langle d(\tilde{\mathrm{v}}_\epsilon \times d\tilde{\mathrm{v}}_\epsilon)\chi, \partial_t\psi_1^*\rangle.\end{aligned} \tag{3.112}$$

Since the first term in the r.h.s. of (3.112) is non positive, we only need to concentrate on the term

$$\int_{\mathbb{R}^N\times[0,1-\delta_0^2]} \langle d(\tilde{\mathrm{v}}_\epsilon \times d\tilde{\mathrm{v}}_\epsilon)\chi, \partial_t\psi_1^*\rangle.$$

The main idea is to exchange space and time derivatives of $\tilde{\mathrm{v}}_\epsilon$ and ψ_1^*, and for that purpose we proceed by two successive integrations by parts. Set $\mathcal{U} = \mathbb{R}^N \times [0, 1-\delta_0^2]$. We first have

$$\begin{aligned}\int_{\mathcal{U}} \langle d(\tilde{\mathrm{v}}_\epsilon \times d\tilde{\mathrm{v}}_\epsilon)\chi, \partial_t\psi_1^*\rangle &= \sum_{i<j}\int_{\mathcal{U}} \left(\partial_i(\tilde{\mathrm{v}}_\epsilon\times\partial_j\tilde{\mathrm{v}}_\epsilon) - \partial_j(\tilde{\mathrm{v}}_\epsilon\times\partial_i\tilde{\mathrm{v}}_\epsilon)\right)\chi\partial_t\psi_{1,ij}^* \\ = \sum_{i<j}\int_{\mathcal{U}} -\tilde{\mathrm{v}}_\epsilon\times\partial_j\tilde{\mathrm{v}}_\epsilon &\Big(\partial_t(\partial_i\psi_{1,ij}^*)\chi + \partial_i\chi\,\partial_t\psi_{1,ij}^*\Big) + \tilde{\mathrm{v}}_\epsilon\times\partial_i\tilde{\mathrm{v}}_\epsilon\Big(\partial_t(\partial_j\psi_{1,ij}^*)\,\chi + \partial_j\chi\partial_t\psi_{1,ij}^*\Big) \\ = \sum_{i<j}\int_{\mathcal{U}} \partial_t(\tilde{\mathrm{v}}_\epsilon\times\partial_j\tilde{\mathrm{v}}_\epsilon)\,\chi\,\partial_i\psi_{1,ij}^* &- \partial_t(\tilde{\mathrm{v}}_\epsilon\times\partial_i\tilde{\mathrm{v}}_\epsilon)\,\chi\,\partial_j\psi_{1,ij}^* \\ &+ \sum_{i<j}\int_{\mathcal{U}} -(\tilde{\mathrm{v}}_\epsilon\times\partial_j\tilde{\mathrm{v}}_\epsilon)\,\partial_i\chi\,\partial_t\psi_{1,ij}^* + (\tilde{\mathrm{v}}_\epsilon\times\partial_i\tilde{\mathrm{v}}_\epsilon)\,\partial_j\chi\,\partial_t\psi_{1,ij}^* \\ &- \sum_{i<j}\int_{\mathbb{R}^N\times\{1-\delta_0^2\}} (\tilde{\mathrm{v}}_\epsilon\times\partial_j\tilde{\mathrm{v}}_\epsilon)\,\partial_i\psi_{1,ij}^*\,\chi - (\tilde{\mathrm{v}}_\epsilon\times\partial_i\tilde{\mathrm{v}}_\epsilon)\,\partial_j\psi_{1,ij}^*\,\chi.\end{aligned} \tag{3.113}$$

[Here, we write $\psi_1^* = \sum_{i<j}\psi_{1,ij}^*\, dx_i \wedge dx_j$] Notice that

$$\begin{aligned}&\int_{\mathcal{U}} \partial_j\left(\tilde{\mathrm{v}}_\epsilon\times\partial_t\tilde{\mathrm{v}}_\epsilon\right)\,\partial_i\psi_{1,ij}^*\,\chi \\ &= -\int_{\mathcal{U}} \left(\tilde{\mathrm{v}}_\epsilon\times\partial_t\tilde{\mathrm{v}}_\epsilon\right)\,\partial_{ij}\psi_{1,ij}^*\,\chi + \int_{\mathcal{U}} \left(\tilde{\mathrm{v}}_\epsilon\times\partial_t\tilde{\mathrm{v}}_\epsilon\right)\,\partial_i\psi_{1,ij}^*\,\partial_j\chi \\ &= \int_{\mathcal{U}} \partial_i\left(\tilde{\mathrm{v}}_\epsilon\times\partial_t\tilde{\mathrm{v}}_\epsilon\right)\,\partial_j\psi_{1,ij}^*\,\chi + \int_{\mathcal{U}} \left(\tilde{\mathrm{v}}_\epsilon\times\partial_t\tilde{\mathrm{v}}_\epsilon\right)\,\Big(\partial_j\psi_{1,ij}^*\,\partial_i\chi - \partial_i\psi_{1,ij}^*\,\partial_j\chi\Big)\,.\end{aligned} \tag{3.114}$$

Combining (3.113) and (3.114) we obtain, after some easy algebra,

$$\int_{\mathcal{U}} \langle d(\tilde{\mathrm{v}}_\epsilon \times d\tilde{\mathrm{v}}_\epsilon)\chi, \partial_t\psi_1^*\rangle = T_1 + T_2 + T_3 + T_4, \tag{3.115}$$

where

$$T_1 = \sum_{i<j} \int_{\mathcal{U}} (\partial_t \tilde{\mathrm{v}}_\epsilon \times \partial_j \tilde{\mathrm{v}}_\epsilon)\, \partial_i \psi^*_{1,ij}\, \chi + (\partial_i \tilde{\mathrm{v}}_\epsilon \times \partial_t \tilde{\mathrm{v}}_\epsilon)\, \partial_j \psi^*_{1,ij}\, \chi,$$

$$T_2 = \sum_{i<j} \int_{\mathcal{U}} (\tilde{\mathrm{v}}_\epsilon \times \partial_t \tilde{\mathrm{v}}_\epsilon)\, \partial_j \psi^*_{1,ij}\, \partial_i \chi - (\tilde{\mathrm{v}}_\epsilon \times \partial_t \tilde{\mathrm{v}}_\epsilon)\, \partial_i \psi^*_{1,ij}\, \partial_j \chi,$$

$$T_3 = \sum_{i<j} \int_{\mathcal{U}} -(\tilde{\mathrm{v}}_\epsilon \times \partial_j \tilde{\mathrm{v}}_\epsilon)\, \partial_t \psi^*_{1,ij}\, \partial_i \chi + (\tilde{\mathrm{v}}_\epsilon \times \partial_i \tilde{\mathrm{v}}_\epsilon)\, \partial_t \psi^*_{1,ij}\, \partial_j \chi,$$

and

$$T_4 = -\sum_{i<j} \int_{\mathbb{R}^N \times \{1-\delta_0^2\}} (\tilde{\mathrm{v}}_\epsilon \times \partial_j \tilde{\mathrm{v}}_\epsilon)\, \partial_i \psi^*_{1,ij}\, \chi - (\tilde{\mathrm{v}}_\epsilon \times \partial_i \tilde{\mathrm{v}}_\epsilon)\, \partial_j \psi^*_{1,ij}\, \chi.$$

We first estimate T_1. By (3.5) we obtain,

$$T_1 \leq \frac{K}{\epsilon} \left(\int_{\mathcal{U}} |\partial_t \tilde{\mathrm{v}}_\epsilon|^2 \chi^2 \right)^{\frac{1}{2}} \left(\int_{\mathcal{U}} |\nabla \psi^*_1|^2 \right)^{\frac{1}{2}}. \tag{3.116}$$

It follows from (3.110) and the definition of $\tilde{\mathrm{v}}_\epsilon$ that

$$\int_{\mathcal{U}} |\partial_t \tilde{\mathrm{v}}_\epsilon|^2 \chi^2 \leq K \exp(\frac{4}{\delta_0^2}) \int_{\mathcal{U}} |\partial_t \mathrm{v}_\epsilon|^2 \exp(-\frac{|x|^2}{4(1-t)}) \leq K \delta_0^{-2} \exp(\frac{4}{\delta_0^2}) \tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon, (0,1), 1), \tag{3.117}$$

and from (3.102) that

$$\int_{\mathbb{R}^N \times [0, 1-\delta_0^2]} |\nabla \psi^*_1|^2 \leq K \exp(\frac{4}{\delta_0^2}) \tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon, (0,1), 1). \tag{3.118}$$

Combining (3.116), (3.117) and (3.118), the estimate for T_1 can be completed as

$$T_1 \leq K \delta_0^{-1} \exp(\frac{4}{\delta_0^2}) \epsilon^{-1} \tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon, (0,1), 1). \tag{3.119}$$

We turn next to T_2, which is estimated exactly as T_1 except that we don't need to invoke estimate (3.5). This yields

$$\begin{aligned} T_2 &\leq K \left(\int_{\mathcal{U}} |\partial_t \tilde{\mathrm{v}}_\epsilon|^2 |\nabla \chi|^2 \right)^{\frac{1}{2}} \left(\int_{\mathcal{U}} |\nabla \psi^*_1|^2 \right)^{\frac{1}{2}} \\ &\leq K \delta_0^{-1} \exp(\frac{4}{\delta_0^2}) \tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon, (0,1), 1). \end{aligned} \tag{3.120}$$

For T_3 we obtain, using the monotonicity formula,

$$\begin{aligned} T_3 &\leq K \left(\int_{\mathcal{U}} |\nabla \tilde{\mathrm{v}}_\epsilon|^2 |\nabla \chi|^2 \right)^{\frac{1}{2}} \left(\int_{\mathcal{U}} |\partial_t \psi^*_1|^2 \right)^{\frac{1}{2}} \\ &\leq K \exp(\frac{4}{\delta_0^2}) \tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon, (0,1), 1) + \frac{1}{2} \int_{\mathcal{U}} |\partial_t \psi^*_1|^2. \end{aligned} \tag{3.121}$$

[Notice that the factor 1/2 in front of the last term on the r.h.s. of (3.121) will allow us to absorb it in the l.h.s. of (3.112)]. Finally, for T_4, we obtain, using once more the monotonicity formula,

$$\begin{aligned} T_4 &\leq K \left(\int_{\mathbb{R}^N \times \{1-\delta_0^2\}} |\nabla \tilde{\mathrm{v}}_\epsilon|^2 |\chi|^2 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N \times \{1-\delta_0^2\}} |\nabla \psi_1^*|^2 \right)^{\frac{1}{2}} \\ &\leq K \exp(\frac{4}{\delta_0^2}) \tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon, (0,1), 1) + \frac{1}{4} \int_{\mathbb{R}^N \times \{1-\delta_0^2\}} |\nabla \psi_1^*|^2. \end{aligned} \tag{3.122}$$

[Here again, the presence of the factor 1/4 in front of the last term in (3.122), will allow us to absorb it in the l.h.s. of (3.112)].

Combining (3.112) and (3.115) with the estimates (3.119), (3.120), (3.121) and (3.122), we finally obtain

$$\int_{\mathbb{R}^N \times [0,1-\delta_0^2]} |\partial_t \psi_1^*|^2 \leq K \delta_0^{-1} \exp(\frac{4}{\delta_0^2}) \epsilon^{-1} \tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon, (0,1), 1), \tag{3.123}$$

and the proof is completed.

Step 6 : Proof of Proposition 3.4 completed. Let us recall the estimates that we have obtained so far for $\psi_1 = \psi_1^i + \psi_1^e$ and ψ_1^*.

For $t \in \Theta_2$ (Θ_2 given by Lemma 3.3), we have

$$\|\psi_1^e\|_{L^\infty(\mathbb{R}^N \times \{t\})} \leq C(\delta_0) \left(\tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon, (0,1), 1) + 1 \right), \tag{3.124}$$

$$\int_{\mathbb{R}^N \times \{t\}} |\psi_1^i|^2 \leq C(\delta_0) \epsilon^{2\alpha} \tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon, (0,1), 1), \tag{3.125}$$

$$\|\psi_1^*(.,t)\|_{L^\infty(\mathbb{R}^N)} \leq C(\delta_0) \tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon, (0,1), 1), \tag{3.126}$$

$$\int_{\mathbb{R}^N \times \{t\}} |\partial_t \psi_1^*|^2 \leq C(\delta_0) \epsilon^{-1} \tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon, (0,1), 1), \tag{3.127}$$

and

$$|d(\tilde{\mathrm{v}}_\epsilon \times d\tilde{\mathrm{v}}_\epsilon)\, \chi| \leq K \frac{(1 - |\mathrm{v}_\epsilon|^2)^2}{4\epsilon^2} \chi \qquad \text{on } \mathbb{R}^N \times [0, \infty), \tag{3.128}$$

where $C(\delta_0) \leq K \exp(\frac{34^2}{\delta_0^2})$ and K is a constant depending only on N. We also recall that ψ_1 and ψ_1^* verify the equations

$$-\Delta \psi_1 = d(\tilde{\mathrm{v}}_\epsilon \times d\tilde{\mathrm{v}}_\epsilon)\, \chi \qquad \text{on } \mathbb{R}^N \times \{t\}, \tag{3.129}$$

$$\partial_t \psi_1^* - \Delta \psi_1^* = d(\tilde{\mathrm{v}}_\epsilon \times d\tilde{\mathrm{v}}_\epsilon)\, \chi \qquad \text{on } \mathbb{R}^N \times [0, \infty). \tag{3.130}$$

In order to complete the estimate for $|\nabla \psi_1|^2$, we write

$$|\nabla \psi_1|^2 = \nabla \psi_1 \cdot \nabla \psi_1^e + \nabla \psi_1 \cdot \nabla \psi_1^i$$

and integrate each of the terms of the r.h.s separately. Multiplying (3.129) by ψ_1^e we obtain :

$$\left|\int_{\mathbb{R}^N\times\{t\}} \nabla\psi_1\cdot\nabla\psi_1^e\right| \leq C(\delta_0)\int_{\mathbb{R}^N\times\{t\}} V_\epsilon(\mathrm{v}_\epsilon)\exp(-\frac{|x|^2}{4\delta^2})\,dx \Big(\tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon,(0,1),1)+1\Big). \quad (3.131)$$

where we have used the L^∞ estimate (3.124) and the L^1 estimate (3.128). Similarly, multiplying (3.129) by ψ_1^i we are led to

$$\left|\int_{\mathbb{R}^N\times\{t\}} \nabla\psi_1\cdot\nabla\psi_1^i\right| \leq K\left|\int_{\mathbb{R}^N\times\{t\}}\langle d(\tilde{\mathrm{v}}_\epsilon\times d\tilde{\mathrm{v}}_\epsilon)\,\chi,\psi_1^i\rangle\right|. \quad (3.132)$$

We bound the r.h.s. of (3.132) using the equation for ψ_1^*. We obtain, multiplying (3.130) by ψ_1^i and integrating by parts on $\mathbb{R}^N\times\{t\}$, the equality

$$\int_{\mathbb{R}^N\times\{t\}}\langle d(\tilde{\mathrm{v}}_\epsilon\times d\tilde{\mathrm{v}}_\epsilon)\,\chi,\psi_1^i\rangle = \int_{\mathbb{R}^N\times\{t\}}\partial_t\psi_1^*\cdot\psi_1^i + \int_{\mathbb{R}^N\times\{t\}}\nabla\psi_1^*\cdot\psi_1^i. \quad (3.133)$$

For the first term of (3.133) we invoke Lemma 3.18 (i.e. estimate (3.127)) and Lemma 3.16 (i.e. estimate (3.125)). By the Cauchy-Schwarz inequality, we therefore obtain

$$\left|\int_{\mathbb{R}^N\times\{t\}}\langle\partial_t\psi_1^*,\psi_1^i\rangle\right| \leq C(\delta_0)\epsilon^{\alpha-\frac{1}{2}}\tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon,(0,1),1) \leq C(\delta_0)\epsilon^{\frac{1}{6}}\tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon,(0,1),1). \quad (3.134)$$

Finally, we turn to the last term in (3.133), that is

$$\int_{\mathbb{R}^N\times\{t\}}\nabla\psi_1^*\cdot\nabla\psi_1^i = -\int_{\mathbb{R}^N\times\{t\}}\Delta\psi_1^i\cdot\psi_1^*. \quad (3.135)$$

Notice that

$$-\Delta\psi_1^i = \Delta G_N^i * d(\tilde{\mathrm{v}}_\epsilon\times d\tilde{\mathrm{v}}_\epsilon)\,\chi.$$

By standard estimates for convolutions we have

$$\begin{aligned}\left\|\Delta G_N^i * d(\tilde{\mathrm{v}}_\epsilon\times d\tilde{\mathrm{v}}_\epsilon)\,\chi\right\|_{L^1(\mathbb{R}^N\times\{t\})} &\leq \left\|\Delta G_N^i\right\|_{\mathcal{M}(\mathbb{R}^N)}\cdot\|d(\tilde{\mathrm{v}}_\epsilon\times d\tilde{\mathrm{v}}_\epsilon)\,\chi\|_{L^1(\mathbb{R}^N\times\{t\})}\\ &\leq C(\delta_0)\int_{\mathbb{R}^N\times\{t\}} V_\epsilon(\mathrm{v}_\epsilon)\exp(-\frac{|x|^2}{4\delta^2})\,dx,\end{aligned}$$

where we have used Lemma 3.11 and (3.129). Going back to (3.135) we obtain, by (3.126),

$$\left|\int_{\mathbb{R}^N\times\{t\}}\nabla\psi_1^*\cdot\nabla\psi_1^i\right| \leq C(\delta_0)\int_{\mathbb{R}^N\times\{t\}}V_\epsilon(\mathrm{v}_\epsilon)\exp(-\frac{|x|^2}{4\delta^2})\,dx\,\tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon,(0,1),1). \quad (3.136)$$

Combining (3.132), (3.133), (3.134) and (3.136) we obtain

$$\left|\int_{\mathbb{R}^N\times\{t\}} \nabla\psi_1\cdot\nabla\psi_1^i\right| \le C(\delta_0)\epsilon^{\frac{1}{6}}\tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon,(0,1),1) + C(\delta_0)\int_{\mathbb{R}^N\times\{t\}} V_\epsilon(\mathrm{v}_\epsilon)\exp(-\frac{|x|^2}{4\delta^2})\,dx\,\tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon,(0,1),1). \quad (3.137)$$

Finally, adding (3.131) to (3.137) we obtain the estimate for $\nabla\psi_1$,

$$\int_{\mathbb{R}^N\times\{t\}} |\nabla\psi_1|^2 \le C(\delta_0)\epsilon^{\frac{1}{6}}\tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon,(0,1),1) + C(\delta_0)\int_{\mathbb{R}^N\times\{t\}} V_\epsilon(\mathrm{v}_\epsilon)\exp(-\frac{|x|^2}{4\delta^2})\,dx\,\left(\tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon,(0,1),1)+1\right),$$

which ends the proof. □

3.13 Proof of Proposition 3.1 completed

Recall that

$$\mathrm{v}_\epsilon\times d\mathrm{v}_\epsilon = d\varphi_t + d\psi_{1,t} + d\psi_{2,t} + \xi_t \qquad \text{on } B(1)\times\{t\},$$

and that by (3.19),

$$4|\mathrm{v}_\epsilon|^2|\nabla\mathrm{v}_\epsilon|^2 = 4|\mathrm{v}_\epsilon\times\nabla\mathrm{v}_\epsilon|^2 + |\nabla|\mathrm{v}_\epsilon|^2| = 4|\mathrm{v}_\epsilon\times\nabla\mathrm{v}_\epsilon|^2 + 4\rho^2|\nabla\rho|^2,$$

where $\rho = |\mathrm{v}_\epsilon|$ denotes the modulus. Using the fact that

$$4|(1-|\mathrm{v}_\epsilon|^2)|\cdot|\nabla\mathrm{v}_\epsilon|^2 \le K\frac{|1-|\mathrm{v}_\epsilon|^2|}{\epsilon}|\nabla\mathrm{v}_\epsilon| \le 2|\nabla\mathrm{v}_\epsilon|^2 + KV_\epsilon(\mathrm{v}_\epsilon),$$

we therefore obtain, using the Hodge - de Rham decomposition (3.37),

$$e_\epsilon(\mathrm{v}_\epsilon) \le K\left[|\nabla\varphi_t|^2 + |\nabla\rho|^2 + |\nabla\psi_{1,t}|^2 + |\nabla\psi_{2,t}|^2 + |\xi_t|^2 + V_\epsilon(\mathrm{v}_\epsilon)\right] \quad \text{on } B(1)\times\{t\}. \quad (3.138)$$

On the other hand, we have by Lemma 3.2, for $t\in\Theta_1$,

$$\tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon,(0,1),\delta) \le \frac{1}{\delta^{N-2}}\int_{B(2\sqrt{N}\delta)\times\{t\}} e_\epsilon(\mathrm{v}_\epsilon) + K|\log\delta|\eta, \quad (3.139)$$

where $\delta = \sqrt{1-t}\in[\delta_0, 2\delta_0]$.

We emphasize the fact that at this stage δ_0 **has not been determined** yet. In order to use (3.138), we first impose the condition

$$4\sqrt{N}\delta_0 \le 1, \quad (3.140)$$

so that, if (3.140) is verified, we have, for $t \in \Theta_1$,

$$\int_{B(2\sqrt{N}\delta)\times\{t\}} e_\epsilon(\mathrm{v}_\epsilon) \leq K \int_{B(2\sqrt{N}\delta)\times\{t\}} |\nabla\varphi_t|^2 + |\nabla\rho|^2 + |\nabla\psi_{1,t}|^2 + |\nabla\psi_{2,t}|^2 + |\xi_t|^2 + V_\epsilon(\mathrm{v}_\epsilon). \tag{3.141}$$

For each of the terms on the r.h.s. of (3.141), we may safely replace the small ball $B(2\sqrt{N}\delta)$ by the larger ball $B(1)$, except for the term involving φ_t for which it is crucial to integrate on a ball of radius of order δ (see Corollary 3.2).

Notice that $\Theta_1 \cap \Theta_2 \neq \emptyset$. Indeed

$$\mathrm{meas}(\Theta_1 \cap \Theta_2) \geq \frac{3}{2}\delta_0^2,$$

by Lemma 3.1 and Corollary 3.3. Therefore, combining the estimates in Proposition 3.2, Lemma 3.4, Corollary 3.2, Lemma 3.8 and Proposition 3.4, we obtain, for $t \in \Theta_1 \cap \Theta_2$,

$$\int_{B(2\sqrt{N}\delta)\times\{t\}} e_\epsilon(\mathrm{v}_\epsilon) \leq K(\delta_0^N + C(\delta_0)\eta^{\frac{1}{2}})\tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon, (0,1), 1) + C(\delta_0)\eta^{\frac{1}{2}}.$$

Hence,

$$\frac{1}{\delta^{N-2}} \int_{B(2\sqrt{N}\delta)\times\{t\}} e_\epsilon(\mathrm{v}_\epsilon) \leq K(\delta_0^2 + C(\delta_0)\eta^{\frac{1}{2}})\tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon, (0,1), 1) + C(\delta_0)\eta^{\frac{1}{2}}, \tag{3.142}$$

where $C(\delta_0)$ depends only on δ_0 and K depends only on N.

We **fix** δ_0 such that (3.140) holds and such that

$$K\delta_0^2 \leq \frac{1}{4}.$$

From now on, δ_0 is completely determined. So is $C(\delta_0)$ in (3.142). Therefore, choosing η_0 such that

$$C(\delta_0)\eta_0^{\frac{1}{2}} \leq \frac{1}{4},$$

we have for $0 \leq \eta \leq \eta_0$, combining (3.142) with (3.139),

$$\tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon, (0,1), \delta) \leq \frac{1}{2}\tilde{E}_{w,\epsilon}(\mathrm{v}_\epsilon, (0,1), 1) + \mathcal{R}(\eta),$$

where

$$\mathcal{R}(\eta) = C(\delta_0)\eta^{\frac{1}{2}} + K|\log\delta_0|\eta.$$

This finishes the proof of Proposition 3.1, and hence the proof of Theorem 1 is completed. □

4 Consequences of Theorem 1

In this section, we prove some consequences of Theorem 1 which were announced in the introduction. Proposition 1 is immediate and we leave the proof to the reader. We present the proofs of Proposition 2 and 3, and we add another consequence, which allows to localize vorticity under some additional compactness properties of the initial data u_ε^0.

4.1 Proof of Proposition 2

Let x_0 be any given point in $B(x_T, \frac{R}{2})$. We claim that we can find $0 < \lambda(T)$ such that

$$\tilde{\mathcal{E}}_{w,\varepsilon}((x_0,T),r) \leq \eta_1 |\log\varepsilon|, \qquad \text{for every } \sqrt{T_0} < r < \sqrt{T_1} = R, \tag{4.1}$$

provided $\check{\eta} \leq \frac{\eta_1}{2}$ [recall that λ enters in the definition of $\check{\eta}$].
Proof of the claim. We invoke Proposition 2.3. Let $\lambda > 0$ and $\sqrt{T_0} < r < \sqrt{T_1} = R$, we have

$$\tilde{\mathcal{E}}_{w,\varepsilon}((x_0,T),r) \leq \frac{1}{r^{N-2}} \int_{B(x_0,\lambda r)\times\{T\}} e_\varepsilon(u_\varepsilon) + (\frac{\sqrt{2}}{\sqrt{T+2r^2}})^{N-2} M_0 \exp(-\frac{\lambda^2}{8})|\log\varepsilon|. \tag{4.2}$$

First we choose $\lambda_0(T)$ such that

$$(\frac{2}{T})^{\frac{N-2}{2}} M_0 \exp(-\frac{\lambda^2}{8}) \leq \frac{\eta_1}{2}. \tag{4.3}$$

Set

$$\lambda(T) = \max(2, 2\lambda_0(T)).$$

Since x_0 belongs to $B(x_T, \frac{R}{2})$ and $r < R$, it follows that

$$B(x_0, \lambda_0(T)r) \subset B(x_T, \lambda(T)R).$$

Therefore,

$$\begin{aligned}\frac{1}{r^{N-2}} \int_{B(x_0,\lambda_0(T)r)\times\{T\}} e_\varepsilon(u_\varepsilon) &\leq (\frac{R}{r})^{N-2} \frac{1}{R^{N-2}} \int_{B(x_T,\lambda(T)R))\times\{T\}} e_\varepsilon(u_\varepsilon) \\ &= (\frac{R}{r})^{N-2} \check{\eta}|\log\varepsilon| \leq (\frac{R}{\sqrt{T_0}})^{N-2}\check{\eta}|\log\varepsilon|.\end{aligned}$$

Choosing T_0 of the form $T_0 = K\check{\eta}^{\frac{2}{N-2}} R^2$, we obtain

$$\frac{1}{r^{N-2}} \int_{B(x_0,\lambda_0(T)r)\times\{T\}} e_\varepsilon(u_\varepsilon) \leq K^{-\frac{N-2}{2}} |\log\varepsilon|. \tag{4.4}$$

It suffices then to fix the constant K as

$$K = (\frac{\eta_1}{2})^{\frac{N-2}{2}},$$

so that combining (4.2),(4.3) and (4.4) we obtain (4.1) and the claim is proved. The conclusion then follows from Proposition 1. □

In the next section, we will make use of the following easy variant of Proposition 2.

Proposition 4.1. *Let u_ε be a solution of $(PGL)_\varepsilon$ verifying assumption (H_0). Let $x_T \in \mathbb{R}^N$, $T \geq 0$ and $R \geq \sqrt{2\varepsilon}$. There exists a positive continuous function λ defined on $(\mathbb{R}_*^+)^2$ such that, if*

$$\check{\eta}(x_T, T, R) \equiv \frac{1}{R^{N-2}|\log \varepsilon|} \int_{B(x_T, \lambda(T,R)R)} e_\varepsilon(u_\varepsilon(\cdot, T)) \leq \frac{\eta_1}{2}$$

then

$$|u_\varepsilon(x,t)| \geq \frac{1}{2} \qquad \text{for } t \in [T + T_0, T + T_1] \quad \text{and } x \in B(x_T, \frac{R}{2}) \, .$$

The function $\lambda(T, R)$ verifies

$$\lambda(T,R) \sim \sqrt{\frac{N-2}{2} |\log(T + R^2)|}, \qquad \text{for } (T,R) \to (0,0),$$

and in particular $\lambda(T,R)R$ remains bounded as $R \to 0$, for any T.

4.2 Proof of Proposition 3

We have, for any $x_0 \in \mathbb{R}^N$ and $t \geq T_f$,

$$\begin{aligned} \tilde{\mathcal{E}}_{w,\varepsilon}(x_0, 0), \sqrt{t}) \frac{1}{\sqrt{t}^{N-2}} \int_{\mathbb{R}^N} e_\varepsilon(u_\varepsilon^0) \exp(-\frac{|y-x|^2}{4t}) \, dy \\ \leq \frac{1}{\sqrt{t}^{N-2}} M_0 |\log \varepsilon| \leq T_f^{-\frac{N-2}{2}} M_0 |\log \varepsilon| \\ \leq \eta_1 |\log \varepsilon|, \end{aligned}$$

in view of the definition of T_f. The conclusion follows from Proposition 1. □

4.3 Localizing vorticity

In this section, we assume that u_ε^0 is localized in some large ball $B(R_1)$. More precisely, we will assume that there exists $R_1 > 0$ such that

$$(H_1) \qquad\qquad u_\varepsilon^0 \equiv 1 \qquad \text{on } \mathbb{R}^N \setminus B(R_1).$$

In particular, there is no vorticity outside $B(R_1)$ at time zero. In this situation, we will show that $\mathcal{V}_\varepsilon \cap \{t \geq 2\varepsilon\}$ remains confined in a bounded region of $\mathbb{R}^N \times (0, +\infty)$. In view of Proposition 3, we already know that

$$\mathcal{V}_\varepsilon \subset \mathbb{R}^N \times [0, T_f], \qquad \text{where } T_f = (\frac{M_0}{\eta_1})^{\frac{2}{N-2}}.$$

We thus need to prove that, under assumption (H_1), horizontal spreading is excluded. More precisely, we have

Proposition 4.2. *Assume u_ε^0 verifies (H_0) and (H_1). Then there exists $\bar{R} > 0$ depending on M_0 and R_1, but not on ε, such that*

$$|u_\varepsilon(x,t)| \geq \frac{1}{2} \qquad \text{for all } x \in \mathbb{R}^N \setminus B(\bar{R}) \text{ and } t \geq 2\varepsilon. \tag{4.5}$$

Proof. In view of Proposition 3, (4.5) is already established for $T \geq T_f$. We therefore assume $t \leq T_f$. Set

$$\tau = \max_{0<R\leq\sqrt{T_f}} \lambda(0,R)R,$$

where λ is the function defined in Proposition 4.1. Note that τ is finite in view of the last remark in Proposition 4.1. Let $x_0 \in \mathbb{R}^N \setminus B(R_1+\tau)$, and $\sqrt{2\varepsilon} < R < \sqrt{T_f}$. We have

$$\breve{\eta}(x_0,0,R) \leq \frac{1}{R^{N-2}|\log\varepsilon|}\int_{B(x_0,\tau)} e_\varepsilon(u_\varepsilon^0) = 0,$$

where we have used (H_1). Applying Proposition 4.1 for $T=0$, $x_T = x_0$ and R we obtain the desired conclusion setting $\bar{R} = R_1 + \tau$. □

5 Improved pointwise bounds and compactness

The aim of this section is to provide proofs to Theorems 2, 3 and 4.

5.1 Proof of Theorem 2

Since by assumption (10), $|u_\varepsilon| \geq 1-\sigma \geq \frac{1}{2}$ on Λ, there is some real-value function φ_ε defined on Λ such that

$$u_\varepsilon = \rho_\varepsilon \exp(i\varphi_\varepsilon) \qquad \text{in } \Lambda, \tag{5.1}$$

where $\rho_\varepsilon = |u_\varepsilon|$. Changing u_ε possibly by a constant phase, we may impose the additional condition

$$\frac{1}{|\Lambda|}\int_\Lambda \varphi_\varepsilon = 0. \tag{5.2}$$

We split as previously the estimates for the phase φ_ε and for the modulus ρ_ε, and we begin with the phase. Inserting (5.1) into $(\mathrm{PGL})_\varepsilon$ we are led to the parabolic equation

$$\rho_\varepsilon^2 \frac{\partial \varphi_\varepsilon}{\partial t} - \mathrm{div}(\rho_\varepsilon \nabla \varphi_\varepsilon) = 0 \qquad \text{in } \Lambda. \tag{5.3}$$

In contrast with the equation for the modulus, (5.3) has the advantage that the explicit dependence on ε has been removed. We will handle (5.3) as a linear equation for the function φ_ε, ρ_ε being considered as a coefficient. In the sequel, we write $\varphi = \varphi_\varepsilon$ and $\rho = \rho_\varepsilon$ when this is not misleading. In order to work on a finite domain, we consider the truncated function $\tilde{\varphi}$ defined on $\mathbb{R}^N \times [T, T+\Delta T]$ by

$$\tilde{\varphi}(x,t) = \varphi(x,t)\,\chi(x),$$

where χ is a smooth cut-off function such that

$$\chi \equiv 1 \text{ on } B(\frac{4}{5}R) \qquad \text{and} \qquad \chi \equiv 0 \text{ on } \mathbb{R}^N \setminus B(\frac{5}{6}R).$$

The function $\tilde{\varphi}$ then verifies the equation

$$\rho^2 \frac{\partial \tilde{\varphi}}{\partial t} - \operatorname{div}(\rho \nabla \tilde{\varphi}) = \operatorname{div}(\rho^2 \varphi \nabla \chi) + \rho^2 \nabla \chi \cdot \nabla \varphi \qquad \text{in } \Lambda. \tag{5.4}$$

Moreover, by construction

$$\operatorname{supp}(\tilde{\varphi}) \subset B(\frac{4}{5}R) \times [T, T + \Delta T],$$

and in particular $\tilde{\varphi} = 0$ on the vertical part of the boundary of Λ. By a mean value argument, we may choose some $t_0 \in [T + T + \frac{\Delta T}{4}]$ such that

$$\int_{B(R) \times \{t_0\}} e_\varepsilon(u_\varepsilon) \leq \frac{4}{\Delta T} \int_\Lambda e_\varepsilon(u_\varepsilon)$$

and we set

$$\Lambda_0 = B(R) \times [t_0, T + \Delta T] \supset \Lambda_{\frac{3}{4}} = B(\frac{3}{4}R) \times [T + \frac{\Delta T}{4}, T + \Delta T].$$

Since by assumption ρ is close to 1, it is natural to treat the l.h.s. of (5.4) as a perturbation of a heat operator, and to rewrite (5.4) as follows

$$\frac{\partial \tilde{\varphi}}{\partial t} - \Delta \tilde{\varphi} = \operatorname{div}((\rho^2 - 1) \nabla \tilde{\varphi}) + (1 - \rho^2) \frac{\partial \tilde{\varphi}}{\partial t} + \operatorname{div}(\rho^2 \varphi \nabla \chi) + \rho^2 \nabla \chi \cdot \nabla \varphi \qquad \text{in } \Lambda.$$

We introduce the function φ_0 defined on Λ_0 as the solution of

$$\left\{ \begin{array}{rcll} \frac{\partial \varphi_0}{\partial t} - \Delta \varphi_0 &=& \operatorname{div}(\rho^2 \varphi \nabla \chi) + \rho^2 \nabla \chi \cdot \nabla \varphi & \text{in } \Lambda_0, \\ \varphi_0(x, t_0) &=& \tilde{\varphi}(x, t_0) & \text{on } B(R) \times \{t_0\}, \\ \varphi_0(x, t) &=& 0 & \forall x \in \partial B(R), \ \forall t \geq t_0. \end{array} \right. \tag{5.5}$$

In particular, since $\chi \equiv 1$ on $B(\frac{4}{5}R)$, we have

$$\frac{\partial \varphi_0}{\partial t} - \Delta \varphi_0 = 0 \qquad \text{in } B(\frac{4}{5}R) \times [t_0, T + \Delta T]. \tag{5.6}$$

We set $\varphi_1 = \tilde{\varphi} - \varphi_0$, i.e.

$$\tilde{\varphi} = \varphi_0 + \varphi_1.$$

We will show that φ_1 is essentially a perturbation term.

At this stage, we divide the estimates into several steps. We start with linear estimates for φ_0.

Step 1 : Estimates for φ_0. We claim that

$$\|\nabla\varphi_0\|_{L^2L^{2^*}(\Lambda_0)} \leq C_1(\Lambda)\left[\int_\Lambda e_\varepsilon(u_\varepsilon)\right] \tag{5.7}$$

and

$$\|\nabla\varphi_0\|_{L^\infty(\Lambda_{\frac{3}{4}})} \leq C_2(\Lambda)\left[\int_\Lambda e_\varepsilon(u_\varepsilon)\right], \tag{5.8}$$

where $2^* = \frac{2N}{N-2}$ is the Sobolev exponent in dimension N, and, for $1 < p, q < +\infty$,

$$L^pL^q(\Lambda_0) = \{f \text{ measurable on } \Lambda_0 \text{ s.t. } \int_{t_0}^{T+\Delta T}\left[\int_{B(R)}|f|^q\right]^{\frac{p}{q}} < +\infty\}$$

[We recall the obvious identity $L^pL^p(\Lambda_0) = L^p(\Lambda_0)$].

Proof. We write $\varphi_0 = \varphi_0^0 + \varphi_0^1$, where φ_0^0 is defined by

$$\begin{cases} \frac{\partial\varphi_0^0}{\partial t} - \Delta\varphi_0^0 &= 0 \qquad \text{in } \Lambda, \\ \varphi_0^0(x,t_0) &= \tilde{\varphi}(x,t_0) \quad \text{on } B(R)\times\{t_0\}, \\ \varphi_0^0(x,t) &= 0 \qquad \forall x\in\partial B(R),\ \forall t\geq t_0. \end{cases}$$

By standard estimates for the heat equation, we have

$$\|\nabla^2\varphi_0^0\|_{L^2(\Lambda_0)} \leq C(\Lambda)\|\nabla\tilde{\varphi}\|_{L^2(B(R)\times\{t_0\}} \leq C(\Lambda)\|e_\varepsilon(u_\varepsilon)\|_{L^1(\Lambda)},$$

and therefore by Sobolev embedding

$$\|\nabla\varphi_0^0\|_{L^2L^{2^*}(\Lambda_0)} \leq C(\Lambda)\|e_\varepsilon(u_\varepsilon)\|_{L^1(\Lambda)}. \tag{5.9}$$

We turn next to φ_0^1. Let $\mathcal{T}$ be the linear mapping which, to any function f defined on Λ_0, associates the unique solution $v = \mathcal{T}f$ of the problem

$$\begin{cases} \frac{\partial v}{\partial t} - \Delta v &= f \qquad \text{in } \Lambda, \\ v &= 0 \qquad \text{on } B(R)\times\{t_0\}, \\ v &= 0 \qquad \forall x\in\partial B(R),\ \forall t\geq t_0. \end{cases}$$

It is well known that the operators $f\mapsto\nabla^2(\mathcal{T}f)$, $f\mapsto\frac{\partial}{\partial t}(\mathcal{T}f)$ and $g\mapsto\nabla(\mathcal{T}(\operatorname{div} g))$ are linear continuous on $L^pL^q(\Lambda_0)$ (see e.g. [35]). With these notations, we may write

$$\varphi_0^1 = \mathcal{T}f + \mathcal{T}(\operatorname{div} g)$$

where

$$f = \rho^2\nabla\chi\cdot\nabla\varphi, \qquad g = \rho^2\varphi\nabla\chi.$$

We have the easy estimate

$$\|f\|_{L^2(\Lambda_0)} \leq C(\Lambda)\|\nabla\varphi\|_{L^2(\Lambda_0)} \leq C(\Lambda)\|e_\varepsilon(u_\varepsilon)\|_{L^1(\Lambda)},$$

and in view of (5.2) and Sobolev embedding,

$$\|g\|_{L^2L^{2^*}(\Lambda_0)} \leq C\|\varphi\|_{L^2L^{2^*}(\Lambda_0)} \leq C(\Lambda)\|\nabla\varphi\|_{L^2L^2(\Lambda_0)} \leq C(\Lambda)\|e_\varepsilon(u_\varepsilon)\|_{L^1(\Lambda)}.$$

Therefore, by the linear theory for $\mathcal{T}$ mentioned above,

$$\begin{aligned}
\|\nabla\varphi_0^1\|_{L^2L^{2^*}(\Lambda_0)} &\leq \|\nabla(\mathcal{T}f)\|_{L^2L^{2^*}(\Lambda_0)} + \|\nabla(\mathcal{T}(\operatorname{div} g))\|_{L^2L^{2^*}(\Lambda_0)} \\
&\leq C(\Lambda)\left[\|(\nabla^2 + I)\mathcal{T}f\|_{L^2L^2(\Lambda_0)} + \|\nabla(\mathcal{T}(\operatorname{div} g))\|_{L^2L^{2^*}(\Lambda_0)}\right] \\
&\leq C(\Lambda)\left[\|f\|_{L^2L^2(\Lambda_0)} + \|g\|_{L^2L^{2^*}(\Lambda_0)}\right] \\
&\leq C(\Lambda)\|e_\varepsilon(u_\varepsilon)\|_{L^1(\Lambda)}.
\end{aligned} \tag{5.10}$$

Combining (5.9) and (5.10) we obtain (5.7). Finally, (5.8) follows from (5.6), (5.7) and standard estimates for the homogeneous heat equation. □

Step 2 : The equation for φ_1. The function φ_1 verifies the evolution problem

$$\left\{\begin{array}{rcll}
\frac{\partial\varphi_1}{\partial t} - \Delta\varphi_1 &=& \operatorname{div}((\rho^2-1)\nabla\tilde{\varphi}) + (1-\rho^2)\frac{\partial\tilde{\varphi}}{\partial t} & \text{in } \Lambda, \\
\varphi_1(x,t_0) &=& 0 & \text{on } B(R)\times\{t_0\}, \\
\varphi_1(x,t) &=& 0 & \forall x\in\partial B(R),\ \forall t\geq t_0.
\end{array}\right. \tag{5.11}$$

It is convenient to rewrite equation (5.11) as

$$\frac{\partial\varphi_1}{\partial t} - \Delta\varphi_1 = \operatorname{div}((\rho^2-1)\nabla\varphi_1) + f_0 + \operatorname{div}(g_0), \tag{5.12}$$

where we have set

$$f_0 = (1-\rho^2)\frac{\partial\tilde{\varphi}}{\partial t} \qquad \text{and} \qquad g_0 = (\rho^2-1)\nabla\varphi_0.$$

We have, for any $1 \leq p < 2$,

$$\|f_0\|^p_{L^2L^p(\Lambda_0)} \leq C(\Lambda)M_0\varepsilon^{2-p}|\log\varepsilon|. \tag{5.13}$$

Indeed, for any $t\in[t_0, T+\Delta T]$

$$\begin{aligned}
\int_{B(R)\times\{t\}}|f_0|^p = \int_{B(R)\times\{t\}}|1-\rho^2|^p\left|\frac{\partial\tilde{\varphi}}{\partial t}\right|^p &\leq \left(\int_{B(R)\times\{t\}}\left|\frac{\partial\tilde{\varphi}}{\partial t}\right|^2\right)^{\frac{p}{2}}\left(\int_{B(R)\times\{t\}}(1-\rho^2)^{\frac{2p}{2-p}}\right)^{\frac{2-p}{2}} \\
&\leq C(\Lambda)\left(\int_{B(R)\times\{t\}}\left|\frac{\partial\tilde{\varphi}}{\partial t}\right|^2\right)^{\frac{p}{2}}\varepsilon^{2-p}(M_0|\log\varepsilon|)^{\frac{2-p}{2}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\int_{t_0}^{T+\Delta T}\left(\int_{B(R)\times\{t\}}|h_0|^p\right)^{\frac{2}{p}} &\leq C(\Lambda)\left[\int_{\Lambda_0}\left|\frac{\partial\tilde{\varphi}}{\partial t}\right|^2\right]\varepsilon^{\frac{2}{p}(2-p)}(M_0|\log\varepsilon|)^{\frac{2-p}{p}} \\
&\leq C(\Lambda)\varepsilon^{\frac{2}{p}(2-p)}(M_0|\log\varepsilon|)^{\frac{2}{p}},
\end{aligned}$$

and (5.13) follows. Similarly using (5.7), we obtain, for any $2 \leq q < 2^*$, the estimate for g_0

$$\|g_0\|^q_{L^2 L^q(\Lambda_0)} \leq C(\Lambda) M_0 \varepsilon^{2^*-q} |\log \varepsilon|. \tag{5.14}$$

We now estimate φ_1 from (5.12) through a fixed point argument.

Step 3 : The fixed point argument. Equation (5.12) may be rewritten as

$$\varphi_1 = \mathcal{T}(\operatorname{div}((\rho^2 - 1)\nabla\varphi_1)) + \mathcal{T}(f_0 + \operatorname{div} g_0),$$

which is of the form

$$(\mathrm{Id} - A)\varphi_1 = b$$

where A is the linear operator $v \mapsto \mathcal{T}(\operatorname{div}((\rho^2 - 1)\nabla v))$ and $b = \mathcal{T}(f_0 + \operatorname{div} g_0)$. To go further we need to specify the function space on which we consider this operator. Set $I = [t_0, T + \Delta T]$. Fix p and q such that they verify the conditions

$$1 < p < 2, \qquad q = p^* = \frac{Np}{N-p}, \qquad \text{and} \qquad 2 < q < 2^*$$

[Although the choice of possible p and q verifying the previous conditions shrinks as N increases, it never becomes void!]. Consider the Banach space

$$X_q = \left\{ v \in W^{1,2}(I, W^{-1,q}(B(R))) \cap L^2(I, W^{1,q}(B(R))) \text{ s.t. } v(0) = 0 \right\}.$$

It follows from the linear theory for $\mathcal{T}$ mentioned earlier that $A : X_q \to X_q$ is linear continuous and that

$$\|A\|_{\mathcal{L}(X_q)} \leq C(q) \|1 - \rho\|_{L^\infty(\Lambda_0)}.$$

In particular, we may fix $\sigma > 0$ such that

$$C(q)\|1 - \rho\|_{L^\infty(\Lambda_0)} \leq C(q)\sigma < \frac{1}{2}.$$

With this choice of σ, we deduce that $I - A$ is invertible on X_q and

$$\|\varphi_1\|_{X_q} \leq C\|b\|_{X_q}. \tag{5.15}$$

Finally, by (5.13), (5.14) and Sobolev embedding we obtain

$$\begin{aligned}
\|b\|_{X_q} &\leq \|\mathcal{T} f_0\|_{X_q} + \|\mathcal{T}(\operatorname{div} g_0)\|_{X_q} \\
&\leq \|\nabla \mathcal{T} f_0\|_{L^2 L^{q=p^*}} + \|\partial_t \mathcal{T} f_0\|_{L^2 W^{-1,q}} + \|g_0\|_{L^2 L^q} \\
&\leq C(\Lambda) \left[\|(\nabla^2 + Id)\mathcal{T} f_0\|_{L^2 L^p} + \|\partial_t \mathcal{T} f_0\|_{L^2 L^p} + \|g_0\|_{L^2 L^q} \right] \\
&\leq C(\Lambda) \left[\|f_0\|_{L^2 L^p} + \|g_0\|_{L^2 L^q} \right] \\
&\leq C(\Lambda)(\varepsilon^{\frac{2-p}{p}} + \varepsilon^{\frac{2^*-q}{q}})(M_0 + 1)|\log \varepsilon|.
\end{aligned}$$

For the third inequality, we have used the fact that $L^p \hookrightarrow W^{-1,q}$ (recall our choice $q = p^*$). This, in turn, is a consequence of the Sobolev embedding $W^{1,q'} \hookrightarrow L^{p'}$ which follows from the identity $(q')^* = ((p^*)')^* = p'$, where stars and primes refer to Hölder and Sobolev conjugates in dimension N.

The following estimate for φ_1 then follows from (5.15)

$$\|\nabla\varphi_1\|_{L^2L^q(\Lambda_0)} \leq C(\Lambda)(\varepsilon^{\frac{2-p}{p}} + \varepsilon^{\frac{2^*-q}{q}})(M_0+1)|\log\varepsilon|. \tag{5.16}$$

We now combine the estimates for φ_0 and φ_1.

Step 4 : Improved integrability of $\nabla\tilde{\varphi}$. Combining (5.7) and (5.16) we obtain

$$\|\nabla\tilde{\varphi}\|_{L^2L^q(\Lambda_0)} \leq C(\Lambda)(M_0+1)|\log\varepsilon|. \tag{5.17}$$

Comment. Since $q > 2$, the previous estimate presents a substantial improvement over the corresponding inequality with q replaced by 2, which follows directly from (H_0). This improvement is crucial in order to prove the smallness of both the modulus and potential terms in the energy, which we derive now.

Step 5 : Estimates for the modulus and potential terms.

The function ρ satisfies the equation

$$\frac{\partial\rho}{\partial t} - \Delta\rho + \rho|\nabla\varphi|^2 = \rho\frac{(1-\rho^2)}{\varepsilon^2}. \tag{5.18}$$

Since $\chi \equiv 1$ on $B(\frac{4}{5}R)$, we have $\varphi = \tilde{\varphi}$ on $B(\frac{4}{5}R)$. Let ξ be a non-negative cut-off function such that $\xi \equiv 1$ on $B(\frac{3}{4}R)$ and $\xi \equiv 0$ outside $B(\frac{4}{5}R)$. Multiplying (5.18) by $(1-\rho^2)\xi$ and integrating by parts we obtain

$$\int_{\Lambda_0} 2\rho|\nabla\rho|^2\xi + \int_{\Lambda_0}\rho\frac{(1-\rho^2)^2}{\varepsilon^2} = \int_{\Lambda_0}\frac{\partial\rho}{\partial t}(1-\rho^2)\xi + \int_{\Lambda_0}\nabla\rho\cdot\nabla\xi(1-\rho^2) + \int_{\Lambda_0}\rho(1-\rho^2)|\nabla\tilde{\varphi}|^2\xi.$$

Hence, since $\rho \geq \frac{1}{2}$ on Λ we obtain

$$\int_{\Lambda_{\frac{3}{4}}} |\nabla\rho|^2 + V_\varepsilon(u_\varepsilon) \leq K\varepsilon\left(\int_{\Lambda_0}|\nabla\rho|^2\right)^{\frac{1}{2}}\left(\int_{\Lambda_0}V_\varepsilon(u_\varepsilon)\right)^{\frac{1}{2}} + K\int_{t_0}^{T+\Delta T}\left(\int_{B(\frac{4}{5}R)\times\{t\}}|\nabla\tilde{\varphi}|^q\right)^{\frac{2}{q}}\left(\int_{B(\frac{4}{5}R)\times\{t\}}(1-\rho^2)^{\frac{q}{q-2}}\right)^{\frac{q-2}{q}}dt$$

so that using (5.17) we finally infer that

$$\int_{\Lambda_{\frac{3}{4}}}[\,\big|\nabla|u_\varepsilon|\big|^2 + V_\varepsilon(u_\varepsilon)] \leq C(\Lambda)(M_0+1)(\varepsilon^{\frac{2}{q}(q-2)} + \varepsilon)|\log\varepsilon|^2. \tag{5.19}$$

To summarize, we have proved at this stage that

$$e_\varepsilon(u_\varepsilon) \leq |\nabla \varphi_0|^2 + r_\varepsilon, \tag{5.20}$$

for some $r_\varepsilon \geq 0$ which verifies

$$\int_{\Lambda_{\frac{3}{4}}} r_\varepsilon \leq C(\Lambda) M_0 \varepsilon^\alpha, \tag{5.21}$$

for some small $\alpha > 0$ depending only on N. Therefore, we set

$$\Phi_\varepsilon = \varphi_0.$$

Step 6 : Proof of the L^∞ bound (11) for the energy.

This step relies on a result by Chen and Struwe [19] (see also [51] and [47]), which provides an L^∞ bound for the Ginzburg Landau energy on a cylinder, provided the L^1 norm of the energy on a larger cylinder is small. More precisely we have

Proposition 5.1. *(see [19]) Let $0 < \epsilon < 1$ and let v_ϵ be a solution of $(PGL)_\varepsilon$ on the cylinder $\Lambda_R^0 = B(R) \times [0, R^2]$ for some $R > 0$. Then there exists a constant $\gamma_0 > 0$, depending only on N such that if $R > \sqrt{\epsilon}$ and*

$$\frac{1}{R^N} \int_{\Lambda_R} e_\epsilon(v_\epsilon) \leq \gamma_0 \tag{5.22}$$

then

$$e_\epsilon(v_\epsilon)(x,t) \leq K \frac{1}{R^{N+2}} \int_{\Lambda_R} e_\epsilon(v_\epsilon) \tag{5.23}$$

for any $(x,t) \in B(\frac{R}{2}) \times [\frac{3}{4}R, R]$.

In our situation, (5.22) is not met, in general, for the function u_ε itself. However, we will use Proposition 5.2 for a suitably scaled version of u_ε, for which (5.22) applies.

Let $\sqrt{\varepsilon} < r_0 < \frac{1}{8}R$, to be determined later, set $\epsilon = \frac{\varepsilon}{r_0}$ and let $(x_0, t_0) \in \Lambda_{\frac{5}{8}}$ be fixed. Consider the map v_ϵ defined on $\Lambda_1^0 = B(1) \times [0,1]$ by

$$v_\epsilon(x,t) = u_\varepsilon\left(\frac{x - x_0}{r_0}, \frac{(t - t_0) + r_0^2}{r_0^2}\right)$$

so that

$$u_\varepsilon(x_0, t_0) = v_\epsilon(0, 1).$$

By scaling, we have

$$\int_{\Lambda_1^0} e_\epsilon(v_\epsilon) = \frac{1}{r_0^N} \int_{\Lambda_{r_0}(x_0,t_0)} e_\varepsilon(u_\varepsilon) \tag{5.24}$$

where we set $\Lambda_{r_0}(x_0,t_0) = B(x_0, r_0) \times [t_0 - r_0^2, t_0]$. Note in particular, since $r_0 < \frac{1}{8}R$, that

$$\Lambda_{r_0}(x_0, t_0) \subset \Lambda_{\frac{3}{4}},$$

and we may apply the decomposition (5.20) to assert that

$$\int_{\Lambda_{r_0}(x_0,t_0)} e_\varepsilon(u_\varepsilon) \leq \text{meas}(\Lambda_{r_0}(x_0,t_0)\,) \ \cdot \|\nabla \Phi_\varepsilon\|^2_{L^\infty} + \int_{\Lambda_{\frac{3}{4}}} r_\varepsilon$$
$$\leq \omega_N C_2(\Lambda) r_0^{N+2} \|e_\varepsilon(u_\varepsilon)\|_{L^1(\Lambda)} + C(\Lambda) M_0 \varepsilon^\alpha.$$

Hence, going back to (5.24)

$$\int_{\Lambda_1^0} e_\epsilon(v_\epsilon) \leq \omega_N C_2(\Lambda) r_0^2 \|e_\varepsilon(u_\varepsilon)\|_{L^1(\Lambda)} + C(\Lambda) M_0 r_0^{-N} \varepsilon^\alpha. \tag{5.25}$$

Therefore, we choose

$$r_0 = \inf\left\{\frac{1}{8}R, (\frac{2\omega_N C_2(\Lambda)\|e_\varepsilon(u_\varepsilon)\|_{L^1(\Lambda)}}{\gamma_0})^{-\frac{1}{2}}\right\}.$$

Note in particular that r_0^{-N} diverges at most as $|\log\varepsilon|^{\frac{N}{2}}$. Hence, for ε sufficiently small,

$$C(\Lambda) M_0 r_0^{-N} \varepsilon^\alpha \leq \frac{\gamma_0}{2}.$$

On the other hand, by construction,

$$\omega_N C_2(\Lambda) r_0^2 \|e_\varepsilon(u_\varepsilon)\|_{L^1(\Lambda)} \leq \frac{\gamma_0}{2}.$$

Applying Proposition 5.2 to v_ϵ, together with $R = 1$, we therefore deduce

$$r_0^2 e_\varepsilon(u_\varepsilon)(x_0,t_0) = e_\epsilon(v_\epsilon)(0,1) \leq K \int_{\Lambda_1^0} e_\epsilon(v_\epsilon)$$
$$\leq K\omega_N C_2(\Lambda) r_0^2 \|e_\varepsilon(u_\varepsilon)\|_{L^1(\Lambda)} + C(\Lambda) M_0 r_0^{-N} \varepsilon^\alpha,$$

which leads to

$$e_\varepsilon(u_\varepsilon)(x_0,t_0) \leq C(\Lambda) \int_\Lambda e_\varepsilon(u_\varepsilon) + C(\Lambda) M_0 \varepsilon^\beta,$$

for some constant $0 < \beta < \alpha$. This proves (11), for every $(x_0,t_0) \in \Lambda_{\frac{5}{8}}$.

The remainder of the proof is devoted to the L^∞ estimates for κ_ε. We start with the modulus and potential terms.

Step 7 : Improved estimates for $\nabla\rho$ and $V_\varepsilon(u_\varepsilon)$. Set $\theta = 1 - \rho$. Applying Lemma 1.1 to the cylinder $\Lambda_{\frac{5}{8}}$, we obtain

$$|\theta| \leq C(\Lambda)\varepsilon^2 \left(\|\nabla\varphi\|^2_{L^\infty(\Lambda_{\frac{5}{8}})} + |\log\varepsilon|\right) \leq C(\Lambda)\varepsilon^2 |\log\varepsilon| \qquad \text{on } \Lambda_{\frac{9}{16}}, \tag{5.26}$$

where we invoke (11) for the last inequality. Going back to (5.18) and using once more (11), we infer that

$$|\partial_t\theta - \Delta\theta| \leq C(\Lambda)|\log\varepsilon| \qquad \text{on } \Lambda_{\frac{9}{16}}. \tag{5.27}$$

Since (5.27) is an L^∞ bound, we deduce by standard linear theory that, for every $1 < q_1 < +\infty$ and $1 < q_2 < +\infty$,

$$\|\theta\|_{W^{1,q_1}(I,L^{q_2}(B))} \le C(\Lambda)|\log\varepsilon|, \qquad \|\theta\|_{L^{q_1}(I,W^{2,q_2}(B))} \le C(\Lambda)|\log\varepsilon|,$$

where $I = [T + \frac{1}{2}\Delta T, T + \Delta T]$ and $B = B(x_0, \frac{1}{2}R)$. By interpolation (see e.g. [42, 35]), we obtain

$$\|\theta\|_{W^{\frac{1}{3},q_1}(I,W^{\frac{4}{3},q_2}(B))} \le C(\Lambda)|\log\varepsilon|.$$

Choosing q_1 and q_2 sufficiently large (in particular $q_1 > 3$, $q_2 > 3N$), we obtain that for every $0 < \gamma < 1$,

$$\|\theta\|_{\mathcal{C}^{0,\frac{1}{4}}(I,\mathcal{C}^{1,\gamma}(B))} \le C(\gamma,\Lambda)|\log\varepsilon|.$$

On the other hand, from (5.26) we have

$$\|\theta\|_{L^\infty(I,L^\infty(B))} \le C(\Lambda)\varepsilon^2|\log\varepsilon|,$$

and therefore by interpolation again

$$\|\theta\|_{\mathcal{C}^{0,\frac{1}{5}}(I,\mathcal{C}^{1,\beta}(B))} \le C(\Lambda)\varepsilon^\alpha \tag{5.28}$$

for some (small) $\alpha > 0$. In particular, we have

$$|\nabla\rho|_{L^\infty(\Lambda_{\frac{1}{2}})} = |\nabla\theta|_{L^\infty(\Lambda_{\frac{1}{2}})} \le C(\Lambda)\varepsilon^\alpha. \tag{5.29}$$

Finally, in view of (5.26) once more, we obtain

$$V_\varepsilon(u_\varepsilon) \le K\frac{\theta^2}{\varepsilon^2} \le C(\Lambda)\varepsilon^2|\log\varepsilon|^2 \qquad \text{on } \Lambda_{\frac{1}{2}}$$

so that

$$|\nabla\rho| + V_\varepsilon(u_\varepsilon) \le C(\Lambda)\varepsilon^\alpha \qquad \text{on } \Lambda_{\frac{1}{2}}. \tag{5.30}$$

Step 8 : Improved L^∞ estimates for $\nabla\varphi_1$. Going back to (5.12), we estimate again f_0 and g_0, but now with the help of the improved estimates for ρ. First, for g_0, we have by (5.28)

$$\|g_0\|_{\mathcal{C}^{0,\frac{1}{5}}(I,\mathcal{C}^{1,\beta}(B))} \le C(\Lambda)\varepsilon^\alpha|\log\varepsilon|.$$

For f_0, first notice that since we work on $\Lambda_{\frac{1}{2}}$, $\varphi = \tilde\varphi$ and therefore $f_0 = (1-\rho^2)\partial_t\varphi$. From the equation for φ,

$$\rho^2\frac{\partial\varphi}{\partial t} - \mathrm{div}(\rho^2\nabla\varphi) = 0,$$

and from the α-Hölder regularity bound for ρ, we infer α-Hölder regularity bounds for $\partial_t\varphi$, of the order $|\log\varepsilon|$. Since on the other hand (5.28) holds, we deduce that

$$\|f_0\|_{\mathcal{C}^{0,\alpha}(\Lambda_{\frac{1}{2}})} \le C(\Lambda)\varepsilon^\alpha|\log\varepsilon|^2.$$

Going back to (5.12) and invoking Schauder theory, we obtain

$$\|\nabla\varphi_1\|_{C^{0,\alpha}(\Lambda_{\frac{1}{2}})} \leq C(\Lambda)\varepsilon^{\beta}, \tag{5.31}$$

for some $\beta > 0$.

Step 9 : Estimate (14) completed. We write on $\Lambda_{\frac{1}{2}}$

$$\begin{aligned} e_\varepsilon(u_\varepsilon) &= |\nabla u_\varepsilon|^2 + V_\varepsilon(u_\varepsilon) \\ &= |\nabla\rho|^2 + \rho^2|\nabla\varphi|^2 + V_\varepsilon(u_\varepsilon) \\ &= |\nabla\rho|^2 + (\rho^2-1)|\nabla\varphi|^2 + |\nabla\Phi_\varepsilon|^2 + 2\nabla\Phi_\varepsilon\cdot\nabla\varphi_1 + |\nabla\varphi_1|^2 + V_\varepsilon(u_\varepsilon) \\ &\equiv |\nabla\Phi_\varepsilon|^2 + \kappa_\varepsilon, \end{aligned}$$

and the conclusion follows directly from our previous estimates. The proof of Theorem 2 is thus completed. □

In order to prove Theorem 3, we turn next to a new Hodge - de Rham decomposition which is specially tailored for situations where wild oscillations in the phase are present. This decomposition will later help us proving that the linear and topological modes do not interact.

5.2 Hodge - de Rham decomposition without compactness

Let $k \in \mathbb{N}$, $k \geq 3$, and consider a smooth bounded domain Ω in $\mathbb{R}^k$, such that $\pi_1(\partial\Omega) = 0$. [since $k \geq 3$, this is the case for instance if Ω is topologically a ball]. Let δ and δ^* denote respectively the exterior differentiation operator for differential forms on $\mathbb{R}^k$, and its formal adjoint [since in the sequel we will take $k = N+1$, we do not use the notations d and d^*, which we restrict to $\mathbb{R}^N$ for the ease of reading]. Let v_ε be a smooth complex-valued function defined on $\bar{\Omega}$. We assume that, for some constant $M_2 > 0$, v_ε verifies the bounds

$$\int_\Omega e_\varepsilon(v_\varepsilon) \leq M_2|\log\varepsilon|, \tag{5.32}$$

$$\int_{\partial\Omega} e_\varepsilon(v_\varepsilon) \leq M_2|\log\varepsilon|, \tag{5.33}$$

and

$$|v_\varepsilon| \leq 3. \tag{5.34}$$

Then we have

Proposition 5.2. *Assume that v_ε verifies (5.32),(5.33), (5.34). Then there exists a smooth function Φ, a smooth 1-form ζ, and a smooth 2-form Ψ defined on $\bar{\Omega}$, such that*

$$v_\varepsilon \times \delta v_\varepsilon = \delta\Phi + \delta^*\Psi + \zeta, \qquad \delta\Psi = 0 \ \textit{in}\ \Omega, \qquad \Psi_\top = 0 \ \textit{on}\ \partial\Omega, \tag{5.35}$$

and

$$\|\nabla\Phi\|_{L^2(\Omega)} + \|\nabla\Psi\|_{L^2(\Omega)} \leq C(\Omega)M_2|\log\varepsilon|. \tag{5.36}$$

Moreover, for any $1 \le p < \frac{k}{k-1}$ *we have*

$$\begin{cases} \|\nabla\Psi\|_{L^p(\Omega)} \le C(p,\Omega)M_2, \\ \|\zeta\|_{L^p(\Omega)} \le C(p,\Omega)M_2\varepsilon^{\frac{1}{2}}, \end{cases} \tag{5.37}$$

where $C(p,\Omega)$ *is a constant depending only on* p *and* Ω.

Comment. The terms Ψ and ζ in the decomposition (5.35) are bounded in suitable norms. Notice however that it is **not possible** to find a uniform bound on Φ in any reasonable norm. In vague terms, one might say that the possible lack of compactness of $v_\varepsilon \times \delta v_\varepsilon$ has been completely "locked" into Φ.

Proof. We split the proof into two steps. In the first step, we take care of the boundary $\Sigma = \partial\Omega$ (which is by assumption a smooth (k-1)-dimensional manifold), and of the Hodge - de Rham decomposition of the restriction $(v_\varepsilon \times \delta v_\varepsilon)_\top$ to Σ. Then, we "gauge away" the possible lack of compactness.

Step 1 : HdR decomposition on Σ. Since by assumption Σ is simply connected, we may write

$$(v_\varepsilon \times \delta v_\varepsilon)_\top = v_\varepsilon \times d_\Sigma v_\varepsilon = d_\Sigma \Phi_\varepsilon^\Sigma + d_\Sigma^* \Psi_\varepsilon^\Sigma \quad \text{on } \Sigma, \text{ with } d_\Sigma \Psi_\varepsilon^\Sigma = 0 \text{ on } \Sigma, \tag{5.38}$$

where d_Σ denotes the exterior derivative for forms on Σ, and d_Σ^* its formal adjoint. Moreover, by orthogonality, we have

$$\|\nabla_\Sigma \Phi_\varepsilon^\Sigma\|_{L^2}^2 + \|\nabla_\Sigma \Psi_\varepsilon^\Sigma\|_{L^2}^2 \le KM_2|\log\varepsilon|. \tag{5.39}$$

On the other hand, we claim that for $1 \le p < \frac{k}{k-1}$,

$$\|\nabla_\Sigma \Psi_\varepsilon^\Sigma\|_{L^p} \le C(p,\Omega)M_2. \tag{5.40}$$

Indeed, applying d_Σ to (5.38) we obtain

$$-\Delta_\Sigma \Psi_\varepsilon^\Sigma = d_\Sigma(v_\varepsilon \times d_\Sigma v_\varepsilon) = 2J_\Sigma v_\varepsilon \qquad \text{on } \Sigma. \tag{5.41}$$

By the Jerrard-Soner estimate [34], we know that, for any $0 < \alpha < 1$,

$$\|J_\Sigma v_\varepsilon\|_{[\mathcal{C}^{0,\alpha}]^*} \le C(\alpha,\Omega)\frac{1}{|\log\varepsilon|}\int_{\partial\Omega} e_\varepsilon(v_\varepsilon) \le C(\alpha,\Omega)M_2. \tag{5.42}$$

By the Sobolev embedding, if $q > k-1$ we have $W^{1,q}(\Sigma) \hookrightarrow \mathcal{C}^{0,\alpha}(\Sigma)$ for $\alpha = 1 - \frac{k-1}{q}$, so that by duality $[\mathcal{C}^{0,\alpha}(\Sigma)]^* \hookrightarrow [W^{1,q}(\Sigma)]^* = W^{-1,p}(\Sigma)$ where $\frac{1}{p} + \frac{1}{q} = 1$. By elliptic regularity theory, we therefore deduce from (5.41) and (5.42) that, for $1 \le p < \frac{k}{k-1}$,

$$\|\Psi_\varepsilon^\Sigma\|_{W^{1,p}} \le C(p,\Omega)\|J_\Sigma v_\varepsilon\|_{W^{-1,p}} \le C(p,\Omega)\|J_\Sigma v_\varepsilon\|_{[\mathcal{C}^{0,\alpha}(\Sigma)]^*} \le C(p,\Omega)M_2. \tag{5.43}$$

We consider next the harmonic extension Φ^0_ε of Φ^Σ_ε to Ω, i.e.

$$\begin{cases} \Delta\Phi^0_\varepsilon = 0 & \text{in } \Omega, \\ \Phi^0_\varepsilon = \Phi^\Sigma_\varepsilon & \text{on } \Sigma. \end{cases}$$

In view of (5.39), we have

$$\|\nabla\Phi^0_\varepsilon\|^2_{L^2(\Omega)} \leq KM_2|\log\varepsilon|. \tag{5.44}$$

Step 2 : "Gauge transformation" of v_ε. On Ω we consider the map w_ε defined by

$$w_\varepsilon = v_\varepsilon \exp(-i\Phi^0_\varepsilon) \qquad \text{in } \Omega.$$

Notice that $|w_\varepsilon| = |v_\varepsilon|$. Moreover, a simple computation shows that

$$w_\varepsilon \times \delta w_\varepsilon = v_\varepsilon \times \delta v_\varepsilon - |v_\varepsilon|^2\delta\Phi^0_\varepsilon = v_\varepsilon \times \delta v_\varepsilon - \delta\Phi^0_\varepsilon + (1-|v_\varepsilon|^2)\delta\Phi^0_\varepsilon. \tag{5.45}$$

Since by assumption (5.34) $|v_\varepsilon| \leq 3$, we have

$$|\nabla w_\varepsilon| \leq |\nabla v_\varepsilon| + 3|\nabla\Phi^0_\varepsilon| \tag{5.46}$$

and hence

$$\|\nabla w_\varepsilon\|^2_{L^2(\Omega)} + \varepsilon^{-2}\|(1-|w_\varepsilon|^2)^2\|^2_{L^2(\Omega)} \leq KM_2|\log\varepsilon|. \tag{5.47}$$

By Hölder inequality, we have for $1 \leq p < 2$,

$$\|(1-|v_\varepsilon|^2)\delta\Phi^0_\varepsilon\|^p_{L^p(\Omega)} \leq KM_2\varepsilon^{2-p}|\log\varepsilon| \tag{5.48}$$

and similarly

$$\|(1-|v_\varepsilon|^2)d_\Sigma\Phi^\Sigma_\varepsilon\|^p_{L^p(\Sigma)} \leq KM_2\varepsilon^{2-p}|\log\varepsilon|. \tag{5.49}$$

Next, we apply the Hodge - de Rham decomposition to $w_\varepsilon \times \delta w_\varepsilon$ on Ω so that

$$\begin{cases} w_\varepsilon \times \delta w_\varepsilon = \delta\Phi^1_\varepsilon + \delta^*\Psi_\varepsilon & \text{in } \Omega, \\ \delta\Psi_\varepsilon = 0 & \text{in } \Omega, \\ \Phi^1_\varepsilon = 0, \ (\Psi_\varepsilon)_\top = 0 & \text{on } \Sigma. \end{cases}$$

By orthogonality, we have

$$\|\nabla\Phi^1_\varepsilon\|_{L^2(\Omega)} \leq KM_2|\log\varepsilon|. \tag{5.50}$$

Arguing as above, we are led to the elliptic problem

$$\begin{cases} -\Delta\Psi_\varepsilon = \omega_\varepsilon \equiv 2Jw_\varepsilon & \text{in } \Omega \\ (\Psi_\varepsilon)_\top = 0 & \text{on } \Sigma, \\ (\delta^*\Psi_\varepsilon)_\top = A_\varepsilon \equiv (w_\varepsilon \times \delta w_\varepsilon)_\top = d^*_\Sigma\Psi^\Sigma_\varepsilon + (1-|v_\varepsilon|^2)d_\Sigma\Phi^\Sigma_\varepsilon & \text{on } \Sigma. \end{cases} \tag{5.51}$$

In view of (5.43) and (5.49) we have, for any $1 \leq p < \frac{k}{k-1}$,

$$\|A_\varepsilon\|_{L^p(\Sigma)} \leq C(p,\Omega)M_2.$$

On the other hand, by (5.33) we may invoke Proposition II.1 case ii) in [9] to conclude that for any $0 < \alpha < 1$,

$$\|\omega_\varepsilon\|_{[\mathcal{C}^{0,\alpha}(\Omega)]^*} \leq C(\alpha, \Omega) M_2.$$

[The previous inequality does not follow immediately from Jerrard-Soner's work [34] since ω_ε does not have compact support in Ω.]

In order to conclude, we first invoke the following linear estimate [for a proof see e.g. [8] and the references therein].

Lemma 5.1. *Let $1 < p < +\infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let $l \in \mathbb{N}, 1 \leq l \leq k$. Let Ψ and ω be l-forms on Ω and A be an (l-1)-form on $\Sigma = \partial\Omega$. Assume that*

$$\begin{cases} -\Delta \Psi = \omega & & \text{in } \Omega \\ \Psi_\top = 0, & (\delta^* \Psi)_\top = A & \text{on } \Sigma. \end{cases}$$

There exists some constant $C(p, \Omega)$, depending only on p and Ω such that

$$\|\Psi\|_{W^{1,p}(\Omega)} \leq C(p, \Omega) \left[\|\omega\|_{[W^{1,q}(\Omega)]^*} + \|A\|_{[W^{1-\frac{1}{q},q}(\Sigma)]^*} \right].$$

Proof of Proposition 5.2 completed. For any $1 \leq p < \frac{k}{k-1}$ and $\frac{1}{p} + \frac{1}{q} = 1$ we have,

$$\|A_\varepsilon\|_{[W^{1-\frac{1}{q},q}(\Sigma)]^*} \leq \|A_\varepsilon\|_{[L^q(\Sigma)]^*} = \|A_\varepsilon\|_{L^p(\Sigma)} \leq C(p, \Omega) M_2.$$

Arguing as for (5.43), we obtain

$$\|\omega_\varepsilon\|_{[W^{1,q}(\Omega)]^*} \leq C(p, \Omega) \|\omega_\varepsilon\|_{[\mathcal{C}^{0,\alpha}(\Omega)]^*} \leq C(\alpha, \Omega) M_2.$$

Therefore, we deduce from Lemma 5.1 that

$$\|\Psi_\varepsilon\|_{W^{1,p}} \leq C(p, \Omega) M_2. \tag{5.52}$$

Set

$$\Psi = \Psi_\varepsilon, \qquad \Phi = \Phi^0_\varepsilon + \Phi^1_\varepsilon, \qquad \text{and} \qquad \zeta(|v_\varepsilon|^2 - 1)\delta\Phi^0_\varepsilon.$$

Then,

$$\begin{aligned} v_\varepsilon \times \delta v_\varepsilon = w_\varepsilon \times \delta w_\varepsilon + |v_\varepsilon|^2 \delta \Phi^0_\varepsilon &= \delta\Phi^1_\varepsilon + \delta^*\Psi_\varepsilon + \delta\Phi^0_\varepsilon - (1 - |v_\varepsilon|^2)\delta\Phi^0_\varepsilon \\ &= \delta\Phi + \delta^*\Psi + \zeta, \end{aligned}$$

and the conclusion follows from (5.44),(5.48),(5.50) and (5.52). ☐

5.3 Evolution of the phase

Let u_ε be a solution of $(\mathrm{PGL})_\varepsilon$ verifying (H_0). Let K be any compact subset of $\mathbb{R}^N \times (0, +\infty)$. We first choose a parabolic cylinder Λ which contains K

$$K \subset \Lambda \equiv B \times (T_0, T_1) \subset \mathbb{R}^N \times (0, +\infty).$$

Here B is some open ball in $\mathbb{R}^N$ and $0 < T_0 < T_1$. Next, let Ω be a smooth bounded domain with simply connected boundary, such that

$$K \subset \Lambda \subset \Omega \subset \mathbb{R}^N \times (0, +\infty).$$

Without loss of generality, we may assume that for ε sufficiently small

$$\int_\Omega e_\varepsilon(u_\varepsilon) \le M_2 |\log \varepsilon|, \qquad \int_{\partial\Omega} e_\varepsilon(u_\varepsilon) \le M_2 |\log \varepsilon|, \qquad \text{and} \qquad |u_\varepsilon| \le 3,$$

where $M_2 = C(K)M_0$ and $C(K)$ depends only on K. We apply Proposition 5.2 to u_ε. This yields

$$u_\varepsilon \times \delta u_\varepsilon = \delta\Phi + \delta^*\Psi + \zeta, \qquad \delta\Psi = 0 \text{ in } \Omega, \qquad \Psi_\top = 0 \text{ on } \partial\Omega, \tag{5.53}$$

where Φ, Ψ and ζ verify the bounds (5.37). In view of (5.37), we have already obtained good estimates for Ψ and ζ. In order to handle Φ, we first prove that it solves an evolution equation.

Lemma 5.2. *The function Φ in (5.53) verifies the equation*

$$\frac{\partial\Phi}{\partial t} - \Delta\Phi = d^*(\delta^*\Psi + \zeta - P_t(\delta^*\Psi + \zeta)dt) - P_t(\delta^*\Psi + \zeta) \quad \text{in } \Omega. \tag{5.54}$$

Here, for a 1-form ω on Λ, we denote by $P_t(\omega)$ its dt component.

Proof. Taking the exterior product of $(\mathrm{PGL})_\varepsilon$ with u_ε, we are led to

$$u_\varepsilon \times \frac{\partial u_\varepsilon}{\partial t} - \operatorname{div}(u_\varepsilon \times \nabla u_\varepsilon) = 0 \qquad \text{in } \Lambda. \tag{5.55}$$

On the other hand, in view of the decomposition (5.53),

$$\begin{cases} u_\varepsilon \times du_\varepsilon &= d\Phi + (\delta^*\Psi + \zeta) - P_t(\delta^*\Psi + \zeta)dt, \\ u_\varepsilon \times \dfrac{\partial u_\varepsilon}{\partial t} &= \dfrac{\partial\Phi}{\partial t} + P_t(\delta^*\Psi + \zeta). \end{cases} \tag{5.56}$$

Combining (5.55) and (5.56) leads to the conclusion. □

5.4 Proof of Theorem 3

Let u_ε be a solution of $(\mathrm{PGL})_\varepsilon$ verifying (H_0), and let Λ, K, Ψ, Φ and ζ be as in Section 5.3. Without loss of generality, we may assume that

$$\int_{\partial\Lambda} |\nabla_{x,t}\Phi|^2 \le C(K)M_0|\log\varepsilon|, \tag{5.57}$$

where $\nabla_{x,t}$ denotes the gradient with respect to both space and time coordinates. Indeed, since by Proposition 5.2

$$\|\nabla_{x,t}\Phi\|_{L^2(\Omega)} \le C(\Omega)M_2|\log\varepsilon|,$$

if (5.57) were not verified for our original Λ, we could shrink it to a smaller cylinder, still containing K and verifying (5.57). We decompose next the proof in two steps.

Step 1 : Defining φ_ε. We set $\partial\Lambda = \mathcal{O}_0 \cup \mathcal{O}_1$, where

$$\mathcal{O}_0 = (B \times \{T_0\}) \cup (\partial B \times [T_0, T_1]) \qquad \text{and} \qquad \mathcal{O}_1 = B \times \{T_1\}.$$

Let Φ_1 be the unique solution of the parabolic problem

$$\begin{cases} \frac{\partial \Phi_1}{\partial t} - \Delta\Phi_1 & = d^*(\delta^*\Psi + \zeta - P_t(\delta^*\Psi + \zeta)dt) - P_t(\delta^*\Psi + \zeta) \quad \text{in } \Lambda, \\ \qquad \Phi_1 & = 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{on } \mathcal{O}_0 \end{cases} \tag{5.58}$$

Since by (5.37) we have

$$\|\delta^*\Psi + \zeta - P_t(\delta^*\Psi + \zeta)dt\|_{L^p(\Lambda)} + \|P_t(\delta^*\Psi + \zeta)\|_{L^p(\Lambda)} \leq C(p, K)M_0,$$

it follows from standard estimates for the non-homogeneous heat equation that

$$\|\nabla\Phi_1\|_{L^p(\Lambda)} \leq C(p, K)M_0. \tag{5.59}$$

Finally, we set

$$\varphi_\varepsilon = \Phi - \Phi_1$$

and

$$w_\varepsilon = u_\varepsilon \exp(-i\varphi_\varepsilon).$$

By construction, φ_ε verifies the homogeneous heat equation

$$\begin{cases} \frac{\partial \varphi_\varepsilon}{\partial t} - \Delta\varphi_\varepsilon & = 0 \quad \text{in } \Lambda, \\ \qquad \varphi_\varepsilon & = \Phi \quad \text{on } \mathcal{O}_1. \end{cases} \tag{5.60}$$

From standard regularity theory for the heat equation, we have

$$\|\nabla\varphi_\varepsilon\|^2_{L^\infty(K)} \leq C(K)\|\nabla_{x,t}\Phi\|^2_{L^2(\mathcal{O}_0)} \leq C(K)M_0|\log\varepsilon|. \tag{5.61}$$

This establishes the third statement of Theorem 3. We next turn to the fourth and last one.

Step 2 : $W^{1,p}$ estimates for w_ε. First notice that

$$|w_\varepsilon|^2|\nabla w_\varepsilon|^2 = |w_\varepsilon|^2|\nabla|w_\varepsilon||^2 + |w_\varepsilon \times \nabla w_\varepsilon|^2,$$

and hence, since $|w_\varepsilon| = |u_\varepsilon|$,

$$\int_{K\cap\{|u_\varepsilon|\geq\frac{1}{2}\}} |\nabla w_\varepsilon|^p \leq C(p)\left[\int_K |w_\varepsilon \times dw_\varepsilon|^p + \int_K |\nabla|w_\varepsilon||^p\right]. \tag{5.62}$$

On the other hand, since $|\nabla w_\varepsilon| \leq |\nabla u_\varepsilon| + 3|\nabla\varphi_\varepsilon| \leq C(K)M_0\varepsilon^{-1}$, we have

$$\int_{K\cap\{|u_\varepsilon|\leq\frac{1}{2}\}} |\nabla w_\varepsilon|^p \leq C(K)M_0\varepsilon^{2-p}\int_K \frac{(1-|u_\varepsilon|^2)^2}{4\varepsilon^2} \leq C(p, K)M_0. \tag{5.63}$$

By construction, we have

$$w_\varepsilon \times \delta w_\varepsilon = u_\varepsilon \times \delta u_\varepsilon - |u_\varepsilon|^2 \delta\varphi_\varepsilon = \delta^*\Psi + \delta\Phi_1 + (1-|u_\varepsilon|^2)\delta\varphi_\varepsilon. \tag{5.64}$$

Since by the Hölder and Cauchy-Schwarz inequalities

$$\|(1-|u_\varepsilon|^2)\delta\varphi_\varepsilon\|_{L^p(K)} \leq C(p,K)\|(1-|u_\varepsilon|^2)\|_{L^{\frac{2p}{2-p}}(K)} \|\delta\varphi_\varepsilon\|_{L^2(K)} \leq C(p,K)\varepsilon^{2-p}|\log\varepsilon| M_0,$$

it follows from (5.64), (5.37) and (5.59) that

$$\int_K |w_\varepsilon \times \delta w_\varepsilon|^p \leq C(p,K)(M_0+1). \tag{5.65}$$

It remains to bound the L^p norm of the gradient of the modulus. For that purpose, we use the following lemma.

Lemma 5.3. *Set* $\rho = |u_\varepsilon|$. *The following bound holds, for any compact subset* $K \subset \mathbb{R}^N \times (0,+\infty)$, *and any* $1 \leq p < 2$,

$$\int_K |\nabla |u_\varepsilon||^p \leq C(K)\,(M_0+1)\varepsilon^{1-\frac{p}{2}}|\log\varepsilon|,$$

where the constant $C(K)$ *depends only on* K.

Proof. The function ρ satisfies the equation

$$\frac{\partial \rho^2}{\partial t} - \Delta\rho^2 + 2|\nabla u_\varepsilon|^2 = \frac{2}{\varepsilon^2}\rho^2(1-\rho^2). \tag{5.66}$$

Let us introduce the set

$$A = \{(x,t) \in \Omega,\ \rho(x,t) > 1-\varepsilon^{1/2}\}$$

and the function

$$\bar\rho = \max\{\rho, 1-\varepsilon^{1/2}\},$$

so that $\bar\rho = \rho$ on A and $0 \leq 1-\bar\rho \leq \varepsilon^{1/2}$ in Ω.

Next, let χ_K be a cut-off function in $\mathcal{D}(\Omega)$ such that $0 \leq \chi_K \leq 1$ on Ω, $\chi_K \equiv 1$ on K, and $|\nabla\chi_K| \leq C(K)$.

Multiplying equation (5.66) by $\chi_K(\bar\rho^2-1)$ (which is compactly supported in Ω), and integrating over Ω we obtain

$$\int_\Omega \nabla\rho^2\nabla\bar\rho^2\chi_K + \int_\Omega \frac{2\rho(1-\rho^2)(1-\bar\rho^2)}{\varepsilon^2}\chi_K = \int_\Omega (1-\bar\rho^2)|\nabla u_\varepsilon|^2 + \int_\Omega \nabla\rho^2\nabla\chi_K(1-\bar\rho^2) - \int_\Omega \frac{\partial\rho^2}{\partial t}(\bar\rho^2-1)\chi_K.$$

It follows that on the set $A_K = A \cap K$ we have

$$\begin{aligned}
&\int_{A_K} |\nabla\rho^2|^2 = \int_{A_K} \nabla\rho^2 \nabla\bar{\rho}^2 \\
&\leq 2\varepsilon^{1/2} \int_\Omega |\nabla u_\varepsilon|^2 + C(K) \int_\Omega |\nabla\rho||1-\rho^2| + C(K) M_0 \varepsilon^{\frac{1}{2}} |\log\varepsilon| \\
&\leq 2\varepsilon^{1/2} \int_\Omega |\nabla u_\varepsilon|^2 + C(K)\varepsilon \left[\int_\Omega |\nabla\rho|^2 + \int_\Omega \frac{(1-\rho^2)^2}{4\varepsilon^2}\right] + C(K) M_0 \varepsilon^{\frac{1}{4}},
\end{aligned}$$

hence, since $\rho \geq 1 - \varepsilon^{1/2}$ on A_K, we have, for $\varepsilon \leq 1/4$,

$$\int_{A_K} |\nabla\rho|^2 \leq 4 \int_{A_K} |\nabla\rho^2|^2 \leq C(K) M_0 \varepsilon^{\frac{1}{4}}. \tag{5.67}$$

On the other hand, on $B_K = K \setminus A$ we have $\int_{B_K} (1-\rho^2)^2 \leq C(K) M_0 \varepsilon^2 |\log\varepsilon|$ and hence, since $(1-\rho) \geq \varepsilon^{1/2}$ on B_K, it follows that $|B_K| \leq C(K) M_0 \varepsilon |\log\varepsilon|$. Thus

$$\int_{B_K} |\nabla\rho|^p \leq \left(\int_\Omega |\nabla\rho|^2\right)^{p/2} |B_K|^{1-p/2} \leq C(K)(M_0+1)\varepsilon^{1-p/2}|\log\varepsilon|. \tag{5.68}$$

Adding (5.67) and (5.68) we complete the estimate. □

Combining (5.62), (5.63), (5.65) and Lemma 5.3, the proof of Theorem 3 is complete. □

5.5 Hodge - de Rham decomposition with compactness

In this section, we adapt the strategy of the proof of Section 5.2 assuming the compactness assumptions (H_1) and (H_2). We are going to work in the domain

$$\Omega_\varepsilon = \mathbb{R}^N \times (2\varepsilon, +\infty).$$

Notice that in contrast to the results of Section 5.2 the domain here is **not** compact (but the initial data possesses compactness!). Notice also that for technical reasons (in particular in view of Proposition 1.1 and Proposition 4.2) that we have removed the **boundary layer** present in $\mathbb{R}^N \times [0, 2\varepsilon]$. This however introduces a new difficulty, namely we have to keep track of the compactness across this boundary layer.

As in the proof of Theorem 1, we begin with a reprojection of u_ε. Let $p : \mathbb{R}^+ \to \mathbb{R}^+$ verify (3.63), let $\tau = p(|u_\varepsilon|)$ and set

$$\tilde{u}_\varepsilon = \tau u_\varepsilon\,,$$

so that $|\tilde{u}_\varepsilon| \leq 1$ on $\mathbb{R}^N \times (0, +\infty)$,

$$\tilde{u}_\varepsilon = u_\varepsilon \text{ if } |u_\varepsilon| \leq \frac{1}{4}, \qquad |\tilde{u}_\varepsilon| = 1 \text{ if } |u_\varepsilon| \geq \frac{1}{2},$$

and

$$J\tilde{u}_\varepsilon = 0 \text{ on } \mathcal{V}_\varepsilon = \{(x,t),\ |u_\varepsilon(x,t)| \geq \frac{1}{2}\}.$$

In particular,

$$\text{supp}\,(J\tilde{u}_\varepsilon) \subset B(R_1) \times [0, T_f].$$

We have

Proposition 5.3. *Assume that u_ε verifies $(PGL)_\varepsilon$, (H_0), (H_1) and (H_2). Then there exists a smooth function Φ and a smooth 2-form Ψ defined on $\bar{\Omega}_\varepsilon$ such that*

$$\begin{array}{ll} \tilde{u}_\varepsilon \times \delta\tilde{u}_\varepsilon = \delta\Phi + \delta^*\Psi & \text{in } \Omega_\varepsilon, \\ \delta\Psi = 0 & \text{in } \Omega_\varepsilon, \\ \Phi = 0,\ \Psi_\top = 0 & \text{on } \partial\Omega_\varepsilon = \mathbb{R}^N \times \{2\varepsilon\}. \end{array} \tag{5.69}$$

Moreover,

$$\|\nabla_{x,t}\Phi\|^2_{L^2(\mathbb{R}^N\times[0,T])} + \|\nabla_{x,t}\Psi\|^2_{L^2(\mathbb{R}^N\times[0,T])} \leq CM_0|\log\varepsilon|T, \qquad \text{for all } T \geq 0, \tag{5.70}$$

and for any $1 \leq p < \frac{N+1}{N}$ and any compact subset $K \subset \mathbb{R}^N \times (2\varepsilon, +\infty)$ we have

$$\|\nabla_{x,t}\Phi\|_{L^p(K)} + \|\nabla_{x,t}\Psi\|_{L^p(K)} \leq C(p, K, M_0). \tag{5.71}$$

Proof. The Hodge - de Rham decomposition of the 1-form $\tilde{u}_\varepsilon \times \delta\tilde{u}_\varepsilon$ on Ω_ε leads directly to (5.69) and (5.70). Moreover, applying the δ^* operator to (5.69) on Ω_ε we are led to the elliptic problem

$$\left\{ \begin{array}{ll} -\Delta_{x,t}\Psi = 2J_{x,t}\tilde{u}_\varepsilon & \text{in } \Omega_\varepsilon \\ \Psi_\top = 0, \quad (\delta^*\Psi)_\top = (\tilde{u}_\varepsilon \times \delta\tilde{u}_\varepsilon)_\top & \text{on } \partial\Omega_\varepsilon. \end{array} \right. \tag{5.72}$$

In the sequel we write simply Δ instead of $\Delta_{x,t}$ and similarly for J, when this is not misleading. Since Ψ is a 2-form it has $\frac{N(N+1)}{2}$ different scalar components

$$\Psi = \sum_{1\leq i\leq j\leq N} \Psi_{i,j}\, dx_i \wedge dx_j + \sum_{1\leq j\leq N} \Psi_j\, dt \wedge dx_j.$$

Going back to (5.72), we see that the boundary condition on $\partial\Omega_\varepsilon$ decouple into Neumann conditions for the functions Ψ_j, namely

$$\frac{\partial\Psi_j}{\partial x_N} = \tilde{u}_\varepsilon \times \frac{\partial u_\varepsilon}{\partial x_j} \qquad \text{on } \partial\Omega_\varepsilon,$$

whereas for the functions $\Psi_{i,j}$ we have homogeneous Dirichlet conditions

$$\Psi_{i,j} = 0 \qquad \text{on } \partial\Omega_\varepsilon.$$

We divide the proof of (5.71) in several steps.

Step 1 : L^p estimate for $\nabla\Psi_{i,j}$. We introduce the reflection operator P_ε, which, to any function f defined on $\bar\Omega_\varepsilon$, associates its reflected function $P_\varepsilon f$ defined on $\mathbb{R}^N \times (-\infty, 2\varepsilon)$ by

$$P_\varepsilon f(x,t) = f(x, -t+4\varepsilon) \qquad \forall x \in \mathbb{R}^N,\ t < 2\varepsilon.$$

We extend $\Psi_{i,j}$ to $\mathbb{R}^{N+1}$ by setting

$$\Psi_{i,j}(x,t) = -P_\varepsilon \Psi_{i,j}(x,t)$$

so that

$$-\Delta\Psi_{i,j} = 2\,\tilde u_{x_i} \wedge \tilde u_{x_j} - P_\varepsilon(2\,\tilde u_{x_i}\wedge \tilde u_{x_j}) \qquad \text{in } \mathbb{R}^{N+1}. \tag{5.73}$$

Invoking Proposition II.1, case ii) of [9], we deduce that the right-hand side of (5.73) is bounded in $[\mathcal{C}^{0,\alpha}(\mathbb{R}^{N+1})]^*$, and arguing as in the proof of Proposition 5.2 we deduce that

$$\|\nabla\Psi_{i,j}\|_{L^p(K)} \le C(p, K, M_0). \tag{5.74}$$

The corresponding estimate for Ψ_j is less direct. The compactness assumption on the initial data will be determinant in the computation. We are going to use the following.
Step 2 : Compactness at the initial time. Let χ be any function in $\mathcal{C}^1_c(\mathbb{R}^N)$. We have, for any $q > N$,

$$\left|\int_{\mathbb{R}^N} u^0_\varepsilon \times (\frac{\partial u^0_\varepsilon}{\partial x_j})\,\chi\right| \le C(q)\left[\|u^0_\varepsilon\|^2_{\dot H^{\frac12}(\mathbb{R}^N)} + 1 + M_0\varepsilon|\log\varepsilon|\right]\,\|\chi\|_{W^{1-\frac1q,q}(\mathbb{R}^N)}. \tag{5.75}$$

Proof. First notice that since u^0_ε is constant outside $B(R_1)$, we only need to consider the case $\chi \in \mathcal{C}^1_c(B(2R_1))$. For the same reason, we also have

$$\|u^0_\varepsilon\|_{H^{\frac12}(B(2R_1))} \le C\,\left(\|u^0_\varepsilon\|_{\dot H^{\frac12}(\mathbb{R}^N)} + 1\right).$$

Consider the function $\check u^0_\varepsilon$ defined on $\mathbb{R}^N$ by $\check u^0_\varepsilon = u^0_\varepsilon$ if $|u^0_\varepsilon| \le 1$, $\check u^0_\varepsilon = u^0_\varepsilon/|u^0_\varepsilon|$ otherwise. We have $\check u^0_\varepsilon \times d\check u^0_\varepsilon = u^0_\varepsilon \times du^0_\varepsilon$ if $|u^0_\varepsilon| \le 1$, and $\check u^0_\varepsilon \times d\check u^0_\varepsilon = \frac{1}{|u^0_\varepsilon|^2} u^0_\varepsilon \times du^0_\varepsilon$, if $|u^0_\varepsilon| \ge 1$. Next, we use the embedding $W^{1-\frac1q,q} \hookrightarrow C^0 \cap H^{\frac12}$, and the fact that $H^{\frac12}\cap L^\infty$ is an algebra. Since $|\check u^0_\varepsilon| \le 1$, $\chi\check u^0_\varepsilon$ belongs to $H^{\frac12}\cap L^\infty$ and therefore

$$\begin{aligned}
\|\chi\check u^0_\varepsilon\|_{H^{\frac12}(B(2R_1))} &\le C\left(\|\chi\|_{L^\infty(B(2R_1))}\|\check u^0_\varepsilon\|_{H^{\frac12}(B(2R_1))} + \|\check u^0_\varepsilon\|_{L^\infty(B(2R_1))}\|\chi\|_{H^{\frac12}(B(2R_1))}\right)\\
&\le C(q)\left(\|\chi\|_{W^{1-\frac1q,q}(B(2R_1))}\|\check u^0_\varepsilon\|_{H^{\frac12}(B(2R_1))} + \|\chi\|_{W^{1-\frac1q,q}(B(2R_1))}\right)\\
&\le C(q)\left(1 + \|\check u^0_\varepsilon\|_{H^{\frac12}(B(2R_1))}\right)\|\chi\|_{W^{1-\frac1q,q}(B(2R_1))}.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
\left|\int_{B(2R_1)} \chi(\check u^0_\varepsilon \times d\check u^0_\varepsilon)\right| &\le \|d\check u^0_\varepsilon\|_{H^{-\frac12}(B(2R_1))}\|\chi\check u^0_\varepsilon\|_{H^{\frac12}(B(2R_1))}\\
&\le C(q)\|\check u^0_\varepsilon\|_{H^{\frac12}(B(2R_1))}\|\chi\|_{W^{1-\frac1q,q}(B(2R_1))}\left(1 + \|\check u^0_\varepsilon\|_{H^{\frac12}(B(2R_1))}\right).
\end{aligned} \tag{5.76}$$

On the other hand, by construction

$$|u_\varepsilon^0 \times du_\varepsilon^0 - \check{u}_\varepsilon^0 \times d\check{u}_\varepsilon^0| \leq \left||u_\varepsilon^0|^2 - 1\right| \cdot |du_\varepsilon^0| \leq \sqrt{2}\varepsilon \left(\frac{(1-|u_\varepsilon^0|^2)^2}{4\varepsilon^2} + \frac{|\nabla u_\varepsilon^0|^2}{2}\right).$$

so that

$$\left|\int_{\mathbb{R}^N} \chi(\check{u}_\varepsilon^0 \times d\check{u}_\varepsilon^0 - u_\varepsilon^0 \times du_\varepsilon^0)\right| \leq C(q) M_0\, \varepsilon |\log \varepsilon|\, \|\chi\|_{W^{1-\frac{1}{q},q}(\mathbb{R}^N)}. \tag{5.77}$$

Combining (5.76) with (5.77) we derive the conclusion. □

Step 3 : Propagating compactness. We claim that

$$\|u_\varepsilon(\cdot, 2\varepsilon) - u_\varepsilon^0(\cdot)\|_{L^2(\mathbb{R}^N)} \leq C M_0^{\frac{1}{2}} \varepsilon^{\frac{1}{2}} |\log \varepsilon|^{\frac{1}{2}}, \tag{5.78}$$

and that for any $\chi \in \mathcal{C}_c^1(\mathbb{R}^N)$,

$$\left|\int_{\mathbb{R}^N \times \{2\varepsilon\}} (u_\varepsilon \times \frac{\partial u_\varepsilon}{\partial x_j})\,\chi - \int_{\mathbb{R}^N} (u_\varepsilon^0 \times \frac{\partial u_\varepsilon^0}{\partial x_j})\,\chi\right| \leq C M_0 \varepsilon^{\frac{1}{2}} |\log \varepsilon| \left(\|\chi\|_{W^{1,\infty}} + |\mathrm{supp}(\chi)|^{\frac{1}{2}}\right). \tag{5.79}$$

Proof. Define on $\mathbb{R}^N$ the function $u_\varepsilon^f(x) = u_\varepsilon(x, 2\varepsilon)$. We have, for $x \in \mathbb{R}^N$,

$$|u_\varepsilon^f(x) - u_\varepsilon^0(x)|^2 \leq (\int_0^{2\varepsilon} \frac{\partial u_\varepsilon}{\partial t}(x,s)\, ds)^2 \leq 2\varepsilon \int_0^{2\varepsilon} |\frac{\partial u_\varepsilon}{\partial t}(x,s)|^2\, ds.$$

Hence, by integration,

$$\|u_\varepsilon^f - u_\varepsilon^0\|_{L^2(\mathbb{R}^N)}^2 \leq 2\varepsilon \int_{\mathbb{R}^N \times (0,2\varepsilon)} |\frac{\partial u_\varepsilon}{\partial t}|^2 \leq 2M_0 \varepsilon |\log \varepsilon|,$$

and (5.78) follows. For (5.79) we write

$$u_\varepsilon^f \times \frac{\partial u_\varepsilon^f}{\partial x_j} - u_\varepsilon^0 \times \frac{\partial u_\varepsilon^0}{\partial x_j} = (u_\varepsilon^f - u_\varepsilon^0) \times \frac{\partial u_\varepsilon^f}{\partial x_j} + u_\varepsilon^0 \times \frac{\partial}{\partial x_j}(u_\varepsilon^f - u_\varepsilon^0).$$

For the first term on the r.h.s., we obtain by Cauchy-Schwarz inequality

$$\begin{aligned} \left|\int_{\mathbb{R}^N} (u_\varepsilon^f - u_\varepsilon^0) \times \frac{\partial u_\varepsilon^f}{\partial x_j}\chi\right| &\leq \|u_\varepsilon^f - u_\varepsilon^0\|_{L^2(\mathbb{R}^N)} \|\frac{\partial u_\varepsilon^f}{\partial x_j}\|_{L^2(\mathbb{R}^N)} \|\chi\|_{L^\infty} \\ &\leq C\|\chi\|_{L^\infty} M_0 \varepsilon^{\frac{1}{2}} |\log \varepsilon|. \end{aligned} \tag{5.80}$$

For the second term, we integrate by parts

$$\begin{aligned} \left|\int_{\mathbb{R}^N} u_\varepsilon^0 \times \frac{\partial}{\partial x_j}(u_\varepsilon^f - u_\varepsilon^0)\chi\right| &\leq \int_{\mathbb{R}^N} \left|u_\varepsilon^f - u_\varepsilon^0\right| |\nabla \chi| + \int_{\mathbb{R}^N} \left|\frac{\partial u_\varepsilon^0}{\partial x_j}(u_\varepsilon^f - u_\varepsilon^0)\chi\right| \\ &\leq \|u_\varepsilon^f - u_\varepsilon^0\|_{L^2(\mathbb{R}^N)} \|u_\varepsilon^0\|_{L^2(\mathrm{supp}(\chi))} \|\nabla\chi\|_{L^\infty} \\ &\quad + \|\nabla u_\varepsilon^0\|_{L^2(\mathbb{R}^N)} \|u_\varepsilon^f - u_\varepsilon^0\|_{L^2(\mathbb{R}^N)} \|\chi\|_{L^\infty} \\ &\leq C M_0 \varepsilon^{\frac{1}{2}} |\log \varepsilon| \left(\|\chi\|_{W^{1,\infty}} + |\mathrm{supp}(\chi)|^{\frac{1}{2}}\right), \end{aligned} \tag{5.81}$$

where we have used the bound

$$\|u_\varepsilon^0\|_{L^2(\mathrm{supp}(\chi))} \le C\|(1-|u_\varepsilon^0|^2)\|_{L^2(\mathbb{R}^N)} + C|\mathrm{supp}(\chi)|^{\frac{1}{2}} \le C\varepsilon E_\varepsilon(u_\varepsilon^0)^{\frac{1}{2}} + C|\mathrm{supp}(\chi)|^{\frac{1}{2}}.$$

Combining (5.80) and (5.81) we deduce (5.79). □

Combining (5.79) and (5.75) we are led to

$$\left|\int_{\mathbb{R}^N\times\{2\varepsilon\}} u_\varepsilon \times (\frac{\partial u_\varepsilon}{\partial x_j})\,\chi\right| \le C\left[\|u_\varepsilon^0\|^2_{\dot H^{\frac{1}{2}}(\mathbb{R}^N)} + 1 + M_0\varepsilon|\log\varepsilon|\right]\left(\|\chi\|_{W^{1,\infty}} + |\mathrm{supp}(\chi)|^{\frac{1}{2}}\right),$$

and arguing as in Step 2 we conclude that

$$\left|\int_{\mathbb{R}^N\times\{2\varepsilon\}} \tilde u_\varepsilon \times (\frac{\partial \tilde u_\varepsilon}{\partial x_j})\,\chi\right| \le C\left[\|u_\varepsilon^0\|^2_{\dot H^{\frac{1}{2}}(\mathbb{R}^N)} + 1 + M_0\varepsilon|\log\varepsilon|\right]\left(\|\chi\|_{W^{1,\infty}} + |\mathrm{supp}(\chi)|^{\frac{1}{2}}\right). \tag{5.82}$$

Step 4 : Improving (5.82). Let χ be in $\mathcal{C}^1(\mathbb{R}^N)$ such that

$$\|\nabla\chi\|_{L^\infty(\mathbb{R}^N)} + \|\chi\|_{L^2(\mathbb{R}^N)} < +\infty.$$

Then we claim that

$$\left|\int_{\mathbb{R}^N\times\{2\varepsilon\}} \tilde u_\varepsilon \times (\frac{\partial \tilde u_\varepsilon}{\partial x_j})\,\chi\right| \le C\left[\|u_\varepsilon^0\|^2_{\dot H^{\frac{1}{2}}(\mathbb{R}^N)} + 1 + M_0\varepsilon^{\frac{1}{2}}|\log\varepsilon|^{\frac{1}{2}}\right](\|\nabla\chi\|_{L^\infty} + \|\chi\|_{L^2}). \tag{5.83}$$

Proof. Let ξ be a smooth non negative cut-off function such that $\xi \equiv 1$ on $B(R_1)$ and $\xi \equiv 0$ outside $B(R_2)$. We write $\chi = \chi\xi + \chi(1-\xi)$. By (5.82) we have

$$\left|\int_{\mathbb{R}^N\times\{2\varepsilon\}} \tilde u_\varepsilon \times (\frac{\partial \tilde u_\varepsilon}{\partial x_j})\,\chi\xi\right| \le C\left[\|u_\varepsilon^0\|^2_{\dot H^{\frac{1}{2}}(\mathbb{R}^N)} + 1 + M_0\varepsilon|\log\varepsilon|\right](\|\nabla\chi\|_{L^\infty} + \|\chi\|_{L^2} + 1).$$

On the other hand, we have

$$\left|\int_{\mathbb{R}^N\times\{2\varepsilon\}} \tilde u_\varepsilon \times (\frac{\partial \tilde u_\varepsilon}{\partial x_j})\,\chi(1-\xi)\right| \le C\left[\int_{\mathbb{R}^N\times\{2\varepsilon\}} e_\varepsilon(u_\varepsilon)(1-\xi)^2\right]^{\frac{1}{2}}\|\chi\|_{L^2(\mathbb{R}^N)}.$$

By Corollary 2.1 and assumption (H_1),

$$\begin{aligned}\int_{\mathbb{R}^N\times\{2\varepsilon\}} e_\varepsilon(u_\varepsilon)(1-\xi)^2 &\le \int_{\mathbb{R}^N\times\{0\}} e_\varepsilon(u_\varepsilon)(1-\xi)^2 + 4\varepsilon|\log\varepsilon|M_0\|\nabla\xi\|^2_{L^\infty}\\ &\le 4\varepsilon|\log\varepsilon|M_0\|\nabla\xi\|^2_{L^\infty},\end{aligned}$$

so that the conclusion follows. □

Step 5 : L^p estimate for $\nabla\Psi_j$. Let K be any compact set in $\mathbb{R}^N\times(2\varepsilon,+\infty)$. Then, for any $1 \le p < \frac{N+1}{N}$,

$$\|\nabla\Psi_j\|_{L^p(K)} \le C(p,K)M_0. \tag{5.84}$$

Proof. We argue by duality. Let $\frac{1}{q}+\frac{1}{p}=1$, so that in particular $q>N+1$, and let h be any vector field in $L^q(\mathbb{R}^N)$ with compact support in K. We introduce the solution ζ of the dual problem

$$\begin{cases} -\Delta\zeta = \operatorname{div} h & \text{in } \mathbb{R}^N \times \{2\varepsilon, +\infty), \\ \frac{\partial \zeta}{\partial x_N} = 0 & \text{on } \mathbb{R}^N \times \{2\varepsilon\} \end{cases} \tag{5.85}$$

(Actually, ζ is uniquely defined up to a constant). Extending ζ by reflection on the whole of $\mathbb{R}^N$, i.e. setting

$$\zeta = P_\varepsilon(\zeta) \qquad \text{on } \mathbb{R}^N \times (-\infty, 2\varepsilon),$$

then ζ solves the equation

$$-\Delta\zeta = \operatorname{div} h + \operatorname{div}(P_\varepsilon h) \qquad \text{on } \mathbb{R}^{N+1}.$$

It follows by standard elliptic estimates that

$$\|\nabla\zeta\|_{L^q(\mathbb{R}^N)} \leq C\|h\|_{L^q(\mathbb{R}^N)},$$

and

$$\zeta \in \mathcal{C}^\infty(\mathbb{R}^{N+1} \setminus (K \cup P_\varepsilon K)),$$

where $P_\varepsilon K$ obviously denotes the reflection of K. Moreover, since ζ is defined up to an additive constant, we may also assume that

$$|\zeta(x,t)| \leq C(K)\frac{\|h\|_{L^q}}{\operatorname{dist}((x,t), K \cup P_\varepsilon K)^{N-1}}$$

and that

$$|\nabla\zeta(x,t)| \leq C(K)\frac{\|h\|_{L^q}}{\operatorname{dist}((x,t), K \cup P_\varepsilon K)^{N}}.$$

We turn back to the system (5.72). Multiplying the equation of (5.72) corresponding to Ψ_j by ζ and integrating by parts on Ω_ε we obtain

$$\int_{\Omega_\varepsilon} \nabla\Psi_j \nabla\zeta = \int_{\Omega_\varepsilon} 2(\tilde{u}_t \times \tilde{u}_{x_j})\zeta + \int_{\mathbb{R}^N \times \{2\varepsilon\}} \tilde{u}_\varepsilon \times \frac{\partial \tilde{u}_\varepsilon}{\partial x_j}\zeta.$$

On the other hand, multiplying (5.70) by Ψ_j and integrating by parts, we obtain similarly

$$\int_{\Omega_\varepsilon} \nabla\Psi_j \nabla\zeta = \int_{\Omega_\varepsilon} h \cdot \nabla\Psi_j$$

Hence, combining the previous relations,

$$\left|\int_{\Omega_\varepsilon} h \cdot \nabla\Psi_j\right| \leq \left|\int_{\Omega_\varepsilon} 2(\tilde{u}_t \times \tilde{u}_{x_j})\zeta\right| + \left|\int_{\mathbb{R}^N \times \{2\varepsilon\}} \tilde{u}_\varepsilon \times \frac{\partial \tilde{u}_\varepsilon}{\partial x_j}\zeta\right|. \tag{5.86}$$

Arguing as in Step 1, we are led to the inequality

$$\left|\int_{\Omega_\varepsilon} 2(\tilde{u}_t \times \tilde{u}_{x_j})\zeta\right| \leq CM_0\|\nabla\zeta\|_L^q(\mathbb{R}^N) \leq CM_0\|h\|_{L^q(\mathbb{R}^N)}.$$

For the second term on the r.h.s. of (5.86) we invoke Step 4. This yields

$$\begin{aligned}\left|\int_{\mathbb{R}^N\times\{2\varepsilon\}} \tilde{u}_\varepsilon \times \frac{\partial \tilde{u}_\varepsilon}{\partial x_j}\zeta\right| &\leq C\,(1+M_2+M_0)\,[\|\nabla\zeta\|_{L^\infty} + \|\zeta\|_{L^2}]\\ &\leq C(K)\,(1+M_0+M_2)\|h\|_{L^q}.\end{aligned}$$

Going back to (5.86) we thus obtain

$$\left|\int_{\Omega_\varepsilon} h\cdot\nabla\Psi_j\right| \leq C(K)\,(1+M_0+M_2)\|h\|_{L^q(\Omega_\varepsilon)},$$

and since h was arbitrary the conclusion follows. □

Step 6 : Estimate for $\nabla\Phi$. As in Section 5.3 we derive a parabolic equation for the phase, using (5.55) once more. We have, recalling that $\tau = p(|u_\varepsilon|)$,

$$u_\varepsilon \times \delta u_\varepsilon = \tau^{-2}\tilde{u}_\varepsilon \times \delta\tilde{u}_\varepsilon = \tau^{-2}\delta\Phi + \tau^{-2}\delta^*\Psi.$$

Hence,

$$\begin{aligned} u_\varepsilon \times du_\varepsilon &= \tau^{-2}d\Phi + \tau^{-2}\left(\delta^*\Psi - P_t(\delta^*\Psi)dt\right)\\ u_\varepsilon \times \frac{\partial u_\varepsilon}{\partial t} &= \tau^{-2}\frac{\partial\Phi}{\partial t} + \tau^{-2}P_t(\delta^*\Psi),\end{aligned}$$

and (5.55) leads to

$$\tau^{-2}\frac{\partial\Phi}{\partial t} - \operatorname{div}(\tau^{-2}\nabla\Phi) = d^*\left(\tau^{-2}\delta*\Psi - P_t(\delta^*\Psi)dt\right) - \tau^{-2}P_t(\delta^*\Psi).$$

We obtain therefore

$$\frac{\partial\Phi}{\partial t} - \Delta\Phi = d^*\left(\tau^{-2}\delta*\Psi - P_t(\delta^*\Psi)dt\right) - \tau^{-2}P_t(\delta^*\Psi) + (1-\tau^{-2})\frac{\partial\Phi}{\partial t} + \operatorname{div}((1-\tau^2)\nabla\Phi).$$

By (3.64), $|1-\tau^{-2}| \leq C|1-|u_\varepsilon|^2|$. Given the L^p estimate for $\nabla\Psi$ obtained in Steps 1 and 5, and arguing as in Step 1 of the proof of Theorem 3, we finally derive the bound

$$\|\nabla\Phi\|_{L^p(K)} \leq C(p,K)$$

for any $1 \leq p < \frac{N+1}{N}$, and the proof of Proposition 5.3 is complete. □

5.6 Proof of Theorem 4

First notice that the same way we obtained (5.62) and (5.63) we have here

$$\int_K |\nabla u_\varepsilon|^p \le C(p)\left[\int_K |u_\varepsilon \times du_\varepsilon|^p + \int_K |\nabla |u_\varepsilon||^p\right] + C(p,K)M_0.$$

By Lemma 5.3 we have

$$\int_K |\nabla |u_\varepsilon||^p \le C(K)\,(M_0+1)\varepsilon^{1-\frac{p}{2}}|\log\varepsilon|.$$

On the other hand,

$$|u_\varepsilon \times du_\varepsilon|^p = \tau^{-2p}|\tilde{u}_\varepsilon \times d\tilde{u}_\varepsilon|^p \le C(p)\left(|\nabla\Phi|^p + |\nabla\Psi|^p\right).$$

Hence, by Proposition 5.3

$$\int_K |u_\varepsilon \times du_\varepsilon|^p \le C(p,K)$$

and the conclusion follows. □

5.7 Proof of Proposition 5

Since we assume (H_1) and (H_2), we may apply Theorem 4 (with $p=1$), so that

$$\int_{\Lambda_{\frac{1}{2}}} |\nabla u_\varepsilon| \le C, \tag{5.87}$$

where $\Lambda_{\frac{1}{2}} = \Lambda_{\frac{1}{2}}(x,t,r)$. Since u_ε verifies (16), we may also apply Proposition 4 so that

$$e_\varepsilon(u_\varepsilon) = |\nabla\Phi_\varepsilon|^2 + \kappa_\varepsilon \qquad \text{on } \Lambda_{\frac{1}{4}}, \tag{5.88}$$

where κ_ε is bounded in L^∞ and Φ_ε verifies the heat flow on $\Lambda_{\frac{1}{4}}$. Recall that Φ_ε was constructed in the proof of Theorem 2 and verifies (5.5). Notice that on $B(r)\times\{t_0\}$ we may impose the additional condition

$$\int_{B(r)\times\{t_0\}} |\nabla u_\varepsilon| \le C. \tag{5.89}$$

Going back to (5.5), we verify that all the terms on the r.h.s are bounded in some suitable norm, say L^1. On the other hand, the initial value is also bounded by (5.89). Since $\varphi_0 = \Phi_\varepsilon$ solves the heat equation on $\Lambda_{\frac{1}{4}}$, we therefore deduce that

$$|\nabla\Phi_\varepsilon| \le C \qquad \text{on } \Lambda_{\frac{1}{8}}$$

and the conclusion follows from (5.88).

Part II

Analysis of the measures μ_*^t

Introduction

As mentioned in our main Introduction, the focus of this paper is on the asymptotic limits, as $\varepsilon \to 0$, of the Radon measures μ_ε defined on $\mathbb{R}^N \times [0, +\infty)$ by

$$\mu_\varepsilon(x,t) = \frac{e_\varepsilon(u_\varepsilon(x,t))}{|\log \varepsilon|}\, dx\, dt,$$

where for $0 < \varepsilon < 1$, the functions u_ε are solutions of $(\mathrm{PGL})_\varepsilon$ satisfying assumption (H_0). We are specially interested in the properties of the time slices μ_ε^t defined by

$$\mu_\varepsilon^t(x) = \frac{e_\varepsilon(u_\varepsilon(x,t))}{|\log \varepsilon|}\, dx.$$

In view of assumption (H_0) and inequality (II), we may assume that for a subsequence $\varepsilon_n \to 0$, there exists a Radon measure μ_* defined on $\mathbb{R}^N \times [0, +\infty)$ such that

$$\mu_{\varepsilon_n} \rightharpoonup \mu_* \qquad \text{as measures.} \tag{1}$$

Following Brakke [13], we may also assume weak convergence of $\mu_{\varepsilon_n}^t$ for all $t > 0$, in the sense of measures.

Lemma 1. *There exist a subsequence of ε_n (still denoted ε_n) such that*

$$\mu_{\varepsilon_n}^t \rightharpoonup \mu_*^t \qquad \textit{for all } t \geq 0,$$

where μ_*^t is a finite Radon measure on $\mathbb{R}^N$ for all $t \geq 0$. Moreover, $\mu_* = \mu_*^t\, dt$.

The proof in [31] carries over word for word. We fix such a sequence ε_n, and we will therefore write ε instead of ε_n in the sequel, when this is not misleading. We also identify in some places the measure μ_*^t with a measure on $\mathbb{R}^N \times \{t\}$, and we will even sometimes identify $\mathbb{R}^N$ and $\mathbb{R}^N \times \{t\}$.

Some properties of the functions u_ε can be translated directly in the language of the measure μ_*. Firstly, an easy consequence of the monotonicity formula (for u_ε) is,

Lemma 2. *For each $t > 0$ and $x \in \mathbb{R}^N$, the function $\mathfrak{E}_\mu((x,t),\cdot)$ defined on $\mathbb{R}_*^+$ by*

$$r \mapsto \mathfrak{E}_\mu((x,t),r) \equiv \frac{1}{r^{N-2}} \int_{\mathbb{R}^N} \exp(-\frac{|x-y|^2}{4r^2})\, d\mu_*^{t-r^2}(y)$$

is non-decreasing for $0 < r < \sqrt{t}$.

Secondly, important consequences of the analysis developed in Part I are given by the following.

Theorem 5. *i) There exist an absolute constant $\eta_2 > 0$, and a positive continuous function λ defined on $\mathbb{R}_*^+$ such that if, for $x \in \mathbb{R}^N$, $t > 0$ and $r > 0$ we have*

$$\mu_*^t(B(x, \lambda(t)r)) < \eta_2 r^{N-2}, \tag{2}$$

then for every $s \in [t + \frac{15}{16}r^2, t + r^2]$, μ_*^t is absolutely continuous with respect to the Lebesgue measure on the ball $B(x, \frac{1}{4}r)$. More precisely,

$$\mu_*^s = |\nabla \Phi_*|^2 \, dx \qquad \text{on } B(x, \frac{1}{4}R),$$

where Φ_* satisfies the heat equation in $\Lambda_{\frac{1}{4}} = B(x_0, \frac{1}{4}r) \times [t + \frac{15}{16}r^2, t + r^2]$.

ii) If u_ε verifies the conditions (H_1) and (H_2) in addition to (2), then for every $s \in [t + \frac{15}{16}r^2, t + r^2]$,

$$\mu_*^s \equiv 0 \qquad \text{on } B(x, \frac{1}{4}R).$$

Remark 1. Note that the constant η_2 and the function λ are the same as in Proposition 4 of Part I. Notice also that $\mu_* = |\nabla \Phi_*|^2 \, dx \, dt$ on $\Lambda_{\frac{1}{4}}$, and that $|\nabla \Phi_*|^2$ is a smooth function.

We briefly sketch the proof of Theorem 5, which is a rather direct consequence of Theorems 1, 2 and 4 of Part I. We begin with case i). If (2) is verified, then for $\varepsilon = \varepsilon_n$ small enough

$$\int_{B(x,\lambda(t)r)} e_\varepsilon(u_\varepsilon) \leq \eta_2 r^{N-2} |\log \varepsilon|,$$

so that we may invoke Proposition 4. This yields

$$e_\varepsilon(u_\varepsilon) = |\nabla \Phi_\varepsilon|^2 + \kappa_\varepsilon \qquad \text{in } \Lambda_{\frac{1}{4}},$$

where Φ_ε verifies the heat equation in $\Lambda_{\frac{3}{8}}$ and

$$|\nabla \Phi_\varepsilon|^2 \leq C(\Lambda)|\log \varepsilon|, \qquad |\kappa_\varepsilon| \leq C(\Lambda)\varepsilon^\beta \qquad \text{in } \Lambda_{\frac{1}{4}}.$$

Extracting possibly a further subsequence we may assume that

$$\frac{\Phi_\varepsilon}{\sqrt{|\log \varepsilon|}} \to \Phi_* \qquad \text{uniformly on } \Lambda_{\frac{5}{16}}.$$

Since Φ_ε verifies the heat equation, it follows that for every $k \in \mathbb{N}$

$$\frac{\Phi_\varepsilon}{\sqrt{|\log \varepsilon|}} \to \Phi_* \qquad \text{in } \mathcal{C}^k(\Lambda_{\frac{1}{4}}),$$

and Φ_* verifies the heat equation on $\Lambda_{\frac{1}{4}}$. On the other hand,

$$\kappa_\varepsilon \to 0 \qquad \text{uniformly on } \Lambda_{\frac{1}{4}},$$

so that

$$\frac{e_\varepsilon(u_\varepsilon)}{|\log \varepsilon|} \to |\nabla \Phi_*|^2 \qquad \text{uniformly on } \Lambda_{\frac{5}{16}}.$$

For case ii), we argue similarly, invoking Proposition 5.

1 Densities and concentration set

In order to analyse geometric properties of the measures μ_* and μ_*^t, an important concept is that of densities. For a given Radon measure ν on $\mathbb{R}^N$, we have the classical definition.

Definition 1. *For $m \in \mathbb{N}$, the m-dimensional lower density of ν at the point x is defined by*

$$\Theta_{*,m}(\nu, x) = \liminf_{r\to 0} \frac{\nu(B(x,r))}{\omega_m r^m},$$

where ω_m denotes the volume of the unit ball B^m. Similarly, the m-dimensional upper density $\Theta^*_m(\nu^t, x)$ is given by

$$\Theta^*_m(\nu, x) = \limsup_{r\to 0} \frac{\nu(B(x,r))}{\omega_m r^m}.$$

When both quantities coincide, ν admits a m-dimensional density $\Theta_m(\nu, x)$ at the point x, defined as the common value.

Since the energy measure is expected to concentrate on (N-2)-dimensional objects, our main efforts will be devoted to the study of the density $\Theta_{*,N-2}(\mu_*^t, \cdot)$. Invoking the monotonicity formula once more, we have

Lemma 3. *For all $x \in \mathbb{R}^N$ and for all $t > 0$,*

$$\Theta_{*,N-2}(\mu_*^t, x) \leq \Theta^*_{N-2}(\mu_*^t, x) \leq K M_0 t^{\frac{2-N}{2}} < +\infty.$$

The previous provides an upper-bound. For regularity properties (of the concentration set) it is well known that lower bounds play a key role. However, there are some conceptual difficulties to attack $\Theta_{*,N-2}(\mu_*^t, \cdot)$ directly (since the equation depends on time). Instead, we will first work on the space-time measure μ_*, and recall the notion of parabolic density, which is natural in view of Lemma 2.

Definition 2. *Let ν be a Radon measure on $\mathbb{R}^N \times [0, +\infty)$ such that $\nu = \nu^t\, dt$. For $t > 0$ and $m \in \mathbb{N}$, the parabolic m-dimensional lower density of ν at the point (x,t) is defined by*

$$\Theta^P_{*,m}(\nu, (x,t)) = \liminf_{r\to 0} \frac{1}{r^m} \int_{\mathbb{R}^N} \exp(-\frac{|x-y|^2}{4r^2})\, d\nu^{t-r^2}(y).$$

The parabolic upper density and parabolic density are defined accordingly, and denoted respectively by $\Theta^{P,*}_m$ and Θ^P_m.

Remark 2. Notice that Θ^P is **not** the classical density, in the spirit of Definition 1, for the parabolic metric defined by $d_P((x,t),(x',t')) = \max(|x-x'|, |t-t'|^{\frac{1}{2}})$.

It clearly follows from monotonicity that the limit in Definition 2 is decreasing, so that $\Theta^P_{N-2}(\mu^*, (x,t))$ exists everywhere in $\mathbb{R}^N \times (0,+\infty)$. Another consequence, which we will prove later (see Section 6.2) is that

$$\Theta^P_{N-2}(\mu^*, (x,t)) \geq K\Theta_{*,N-2}(\mu^t_*, x) \tag{3}$$

for some explicit constant K. Motivated by this inequality, we set

$$\Sigma_\mu = \left\{(x,t) \in \mathbb{R}^N \times (0,+\infty) \ \text{ s.t. } \ \Theta^P_{N-2}(\mu_*, (x,t)) > 0\right\},$$

and for $t > 0$,

$$\Sigma^t_\mu = \Sigma_\mu \cap \left(\mathbb{R}^N \times \{t\}\right).$$

An obvious consequence of (3) is that

$$\Theta_{*,N-2}(\mu^t_*, x) \equiv 0 \qquad \text{on } \mathbb{R}^N \setminus \Sigma^t_\mu. \tag{4}$$

2 First properties of Σ_μ

As in Brakke's and Ilmanen's works ([13, 30]) the main tool in the study of geometric properties of Σ_μ is the following.

Theorem 6 (Clearing-Out). *There exist a positive continuous function η_3 defined on $\mathbb{R}^+_*$, such that for any $(x,t) \in \mathbb{R}^N \times (0,+\infty)$ and any $0 < r < \sqrt{t}$, if*

$$\mathfrak{E}_\mu((x,t),r) \equiv \frac{1}{r^{N-2}} \int_{\mathbb{R}^N} \exp(-\frac{|x-y|^2}{4r^2})\, d\mu^{t-r^2}_*(y) \leq \eta_3(t-r^2)$$

then

$$(x,t) \notin \Sigma_\mu.$$

Theorem 6 is a consequence of Theorem 5. An immediate corollary is

Corollary 1. *For any $(x,t) \in \Sigma_\mu$, we have*

$$\Theta^P_{N-2}(\mu_*, (x,t)) \geq \eta_3(t).$$

At this stage, we are in position to derive the following, without invoking any further property of the equation $(\text{PGL})_\varepsilon$.

Proposition 6. *i) The set Σ_μ is closed in $\mathbb{R}^N \times (0,+\infty)$.*

ii) For any $t > 0$,

$$\mathcal{H}^{N-2}(\Sigma^t_\mu) \leq KM_0 < +\infty.$$

iii) For any $t > 0$, the measure μ^t_* can be decomposed as

$$\mu^t_* = g(x,t)\mathcal{H}^N + \Theta_*(x,t)\mathcal{H}^{N-2} \llcorner \Sigma^t_\mu,$$

where g is some smooth function defined on $\mathbb{R}^N \times (0,+\infty) \setminus \Sigma_\mu$ and Θ_* verifies the bound $\Theta_*(x,t) \leq KM_0 t^{\frac{2-N}{2}}$.

Comment. a) The function Θ_* in decomposition *iii)* is the Radon-Nikodym derivative of $\mu_*^t \llcorner \Sigma_\mu^t$ with respect to $\mathcal{H}^{N-2}$; at this stage we may just assert that it lies between the lower and upper densities.

b) Concerning g, it can be locally defined as $|\nabla \Phi_*|^2$ for some smooth Φ_* verifying the heat equation. The function Φ_* however is not yet globally defined.

3 Regularity of Σ_μ^t

As already mentioned, lower bounds for $\Theta_{*,N-2}$ will play an important role for regularity issues : however, up to now we have only lower bounds for Θ_{N-2}^P (see Corollary 1). The next result provides the reverse inequality to (3).

Proposition 7. *For almost every $t > 0$, the following inequality holds*

$$\Theta_{*,N-2}(\mu_*^t, x) \geq K \theta_{N-2}^P(\mu_*, (x,t)) \tag{5}$$

for $\mathcal{H}^{N-2}$ almost every $x \in \mathbb{R}^N$.

Combining Corollary 1 and Proposition 7 we are led to

Corollary 2. *For almost every $t \geq 0$,*

$$\Theta_{*,N-2}(\mu_*^t, x) \geq K\eta_3(t), \qquad \text{for } \mathcal{H}^{N-2}\text{–a.e. } x \in \Sigma_\mu^t. \tag{6}$$

At this point, combining Theorem 5 with Corollary 2 and Ambrosio-Soner's work [4], **the proof of Theorem C is complete.** Indeed, since $\nabla \Phi_* = 0$, there is no diffuse part and (AS) holds.

To proceed further towards the proofs of Theorem A and Theorem B, we have to deal with the diffuse part, and different kinds of arguments could then lead to regularity for Σ_μ^t. One way is to follow the arguments of [4] (as above for Theorem C), which rely on a curvature equation for μ_*^t and Allard's first rectifiability theorem (see [48]). Another possible way is to prove the existence of the density Θ_{N-2} ($\mathcal{H}^{N-2}$ almost everywhere), and then to invoke Preiss' regularity theorem [45]. Even though Preiss' theorem is notably highly involved, we choose this last alternative since it will simplify some of the subsequent arguments. Therefore, we will prove

Proposition 8. *For almost every $t > 0$,*

$$\Theta_{*,N-2}(\mu_*^t, x) = \Theta_{N-2}^*(\mu_*^t, x) \geq K\eta_3(t),$$

for $\mathcal{H}^{N-2}$ almost every $x \in \Sigma_\mu^t$. Consequently, for almost every $t > 0$ the set Σ_μ^t is (N-2)-rectifiable.

We recall that a set $\Sigma \subset \mathbb{R}^N$ is said to be (N-2)-rectifiable if $\mathcal{H}^{N-2}$ almost all of Σ can be covered by the union of countably many lipschitz images of B^{N-2}.

4 Globalizing Φ_*

In order to complete the proof of Theorem A there is still one point to clarify : the function Φ_* which appears in Theorem A is global, whereas up to now the function Φ_* constructed in Theorem 5 is only locally defined. Indeed, using Theorem 5 we may define Φ_* on every simply connected domain of $\Omega_\mu = \mathbb{R}^N \times (0, +\infty) \setminus \Sigma_\mu$. However, Ω_μ is not simply connected in general, and this raises a difficulty for defining Φ_* globally. Nevertheless its gradient $\nabla\Phi_*$ can be defined globally on Ω_μ (and verifies there the heat equation). In order to overcome this problem, we will invoke Theorem 3 of Part I.

For $m \in \mathbb{N}^*$, set $\mathcal{K}_m = B(m) \times [\frac{1}{m}, m]$, so that $\cup_m \mathcal{K}_m = \mathbb{R}^N \times (0, +\infty)$. We apply Theorem 3 to u_ε and $\mathcal{K} = \mathcal{K}_m$, so that we may write

$$u_\varepsilon = \exp(i\phi_\varepsilon^m) w_\varepsilon^m \qquad \text{on } \mathcal{K}_m, \tag{7}$$

where ϕ_ε^m solves the heat equation on $\mathcal{K}_m$,

$$\|\nabla\phi_\varepsilon^m\|_{L^\infty(\mathcal{K}_m)} \leq C(m)\sqrt{M_0|\log\varepsilon|} \tag{8}$$

and

$$\|w_\varepsilon^m\|_{L^p(\mathcal{K}_m)} \leq C(m,p) \qquad \text{for any } 1 \leq p < \frac{N+1}{N}. \tag{9}$$

Let $m \in \mathbb{N}^*$ be fixed for the moment. Extracting possibly a further subsequence of $(\varepsilon_n)_{n\in\mathbb{N}}$, we may assume without loss of generality that

$$\frac{\phi_\varepsilon^m}{\sqrt{|\log\varepsilon|}} \to \phi_*^m \qquad \text{in } \mathcal{C}^2(\mathcal{K}_{m-1}). \tag{10}$$

Moreover, passing to the limit in the equation, we infer that ϕ_*^m solves the heat equation on $\mathcal{K}_{m-1}$.

Next, let $x_0 \in \Omega_\mu$. Since Ω_μ is open, we may find a small cylindrical neighborhood Λ_{x_0} of x_0 in Ω_μ. There exist $m_0 \in \mathbb{N}$ such that for $m \geq m_0$, $\Lambda_{x_0} \subset \mathcal{K}_m$. For ε sufficiently small, we have

$$|u_\varepsilon| \geq 1 - \sigma \geq \frac{1}{2} \qquad \text{on } \Lambda_{x_0} \tag{11}$$

(where σ is the constant in Theorem 2), so that

$$u_\varepsilon = \rho_\varepsilon \exp(i\varphi_\varepsilon) \tag{12}$$

for some real-valued function φ_ε (defined up to an integer multiple of 2π). In view of (11), we may apply Theorem 2, and assert that there exists a solution Φ_ε of the heat equation on Λ_{x_0} such that

$$\|\nabla\Phi_\varepsilon - \nabla\varphi_\varepsilon\|_{L^\infty((\Lambda_{x_0})_{\frac{1}{2}})} \leq C\varepsilon^\beta. \tag{13}$$

(see Remark 5 of the introduction of Part I). On the other hand, we may write for $m \geq m_0$

$$w_\varepsilon^m = \rho_\varepsilon \exp(i\psi_\varepsilon^m) \qquad \text{on } \Lambda_{x_0} \tag{14}$$

where ψ_ε^m isa real-valued function. Combining (12), (7) and (14) we are led to

$$\nabla\varphi_\varepsilon = \nabla\phi_\varepsilon^m + \nabla\psi_\varepsilon^m, \tag{15}$$

and invoking (13), we have, **for m fixed**,

$$\left|\frac{\nabla\phi_\varepsilon^m - \nabla\Phi_\varepsilon}{\sqrt{|\log\varepsilon|}}\right| \leq \left|\frac{\nabla\psi_\varepsilon^m}{\sqrt{|\log\varepsilon|}}\right| + C\varepsilon^\beta \qquad \text{on } (\Lambda_{x_0})_{\frac{1}{2}}.$$

Using (9) we obtain

$$\left\|\frac{\nabla\phi_\varepsilon^m}{\sqrt{|\log\varepsilon|}} - \frac{\nabla\Phi_\varepsilon}{\sqrt{|\log\varepsilon|}}\right\|_{L^p((\Lambda_{x_0})_{\frac{1}{2}})} \longrightarrow \quad 0 \qquad \text{as } \varepsilon \to 0.$$

Since $\frac{\nabla\Phi_\varepsilon}{\sqrt{|\log\varepsilon|}} \to \nabla\Phi_*$ on $(\Lambda_{x_0})_{\frac{1}{2}}$, by (10) we deduce

$$\nabla\phi_*^m = \nabla\Phi_* \qquad \text{on } (\Lambda_{x_0})_{\frac{1}{2}}.$$

Since Φ_* is independent of m, changing possibly ϕ_*^m by a constant we may assume that all the ϕ_*^m coincide on $(\Lambda_{x_0})_{\frac{1}{2}}$. By analyticity, for each $n \geq m_0$ the functions $(\phi_*^m)_{m\geq n}$ coincide on $\mathcal{K}_n$. Letting n go to infinity, we define their common value ϕ_* on $\mathbb{R}^N \times (0, +\infty)$ and we set

$$\Phi_* = \phi_*. \tag{16}$$

The proof of Theorem A is now completed, combining (16), Theorem 5, Proposition 6, Corollary 2 and Proposition 8.

5 Mean curvature flows

In this section we will provide the proof of Theorem B. Since a large part of this analysis follows the lines of [4], we will only indicate the ingredients, the necessary adaptations (due to the presence of the diffuse energy) and some simplifications since rectifiability of Σ_μ^t is already available. In particular, we will avoid to refer to varifolds (or generalized varifolds) even though these important objects are hidden behind.

We first briefly recall both classical and weak notions of mean curvature flow. Then, following [30, 4] we will underline the relationship between $(\mathrm{PGL})_\varepsilon$ and this flow, leading to Theorem B.

5.1. The classical notion. Let Σ be a smooth compact manifold of dimension k, and $\gamma_0 : \Sigma \to \mathbb{R}^N$ $(N \geq k)$ a smooth embedding, so that $\Sigma^0 = \gamma_0(\Sigma)$ is a smooth k-dimensional submanifold of $\mathbb{R}^N$. The mean curvature vector at the point x of Σ^0 is the vector of the orthogonal space $(T_x\Sigma^0)^\perp$ given by

$$\vec{H}_{\Sigma^0}(x) = -\sum_{\alpha=1}^{N-k}\left(\sum_{j=1}^{k}(\tau_j\cdot\frac{\partial\nu^\alpha}{\partial\tau_j})\nu^\alpha\right) = -\sum_{\alpha=1}^{N-k}\left(\mathrm{div}_{T_x\Sigma^0}\nu^\alpha\right)\nu^\alpha, \tag{17}$$

where $(\tau_1,...,\tau_k)$ is an orthonormal moving frame on $T_x\Sigma^0$, $(\nu^1,...,\nu^{N-k})$ is an orthonormal moving frame on $(T_x\Sigma^0)^\perp$, and $\mathrm{div}_{T_x\Sigma^0}$ denotes the tangential divergence at the point x. The integral formulation of (17) is given by

$$\int_{\Sigma^0}\mathrm{div}_{T_x\Sigma^0}\vec{X}\,d\mathcal{H}^k = -\int_{\Sigma^0}\vec{H}_{\Sigma^0}\cdot\vec{X}\,d\mathcal{H}^k, \tag{18}$$

for all $\vec{X}\in\mathcal{C}_c^\infty(\mathbb{R}^N,\mathbb{R}^N)$. The vectors $\vec{H}_{\Sigma^0}(\cdot)$ are uniquely determined by (18), and in particular the definition in (17) does not depend on the choice of orthonormal frames.

Next, we introduce a time dependence, and consider a smooth family $\{\gamma_t\}_{t\in I}$ of smooth embeddings of Σ in $\mathbb{R}^N$, where I denotes some open interval containing 0. We set $\Sigma^t = \gamma_t(\Sigma)$. If χ is a smooth complactly supported function on $\mathbb{R}^N$, a standard computation shows that

$$\frac{d}{dt}\int_{\Sigma^t}\chi(x)\,d\mathcal{H}^k = \int_{\Sigma^t}\Big(-\chi(x)\vec{H}_{\Sigma^t}(x) + P(\nabla\chi(x))\Big)\cdot\vec{Y}(x)\,d\mathcal{H}^k, \tag{19}$$

where $\vec{Y}(x) = \frac{d}{ds}\gamma_s(\gamma_t^{-1}(x))$ is the velocity vector at the point x, and P denotes the orthogonal projection on $(T_x\Sigma^t)^\perp$.

The family $(\Sigma^t)_{t\in I}$ is moved by mean curvature in the classical sense if and only if

$$\frac{d}{dt}\gamma_t(m) = \vec{H}_{\Sigma^t}(\gamma_t(m)), \qquad \text{for all } m\in\Sigma \text{ and } t\in I. \tag{20}$$

In particular, if $(\Sigma^t)_{t\in I}$ is moved by mean curvature, (19) becomes

$$\frac{d}{dt}\int_{\Sigma^t}\chi(x)\,d\mathcal{H}^k = -\int_{\Sigma^t}\chi(x)|\vec{H}_{\Sigma^t}(x)|^2\,d\mathcal{H}^k + \int_{\Sigma^t}\nabla\chi(x)\cdot\vec{H}_{\Sigma^t}(x)\,d\mathcal{H}^k, \tag{21}$$

and actually (21) is equivalent to (20) if χ is taken arbitrary. Notice that the last term in the r.h.s of (21) corresponds to a transport term, whereas the first term represents a shrinking of the area. Actually, if $\chi\equiv 1$ in a neighborhood of Σ^t, then

$$\frac{d}{dt}\mathcal{H}^k(\Sigma^t) = -\int_{\Sigma^t}|\vec{H}_{\Sigma^t}(x)|^2\,d\mathcal{H}^k.$$

Finally, existence of a classical solution of (20) for small times can be established (see e.g. [52, 27]), but singularities develop in finite time.

5.2. Brakke flows. In the attempt to extend (20) or (21) to a larger class of (less regular) objects, and in particular to extend the flow past singularities, Brakke

[13] relaxed equality in (21), and considered instead sub-solutions, i.e. verifying the **inequality**

$$\frac{d}{dt}\int_{\Sigma^t}\chi(x)\,d\mathcal{H}^k \leq -\int_{\Sigma^t}\chi(x)|\vec{H}_{\Sigma^t}(x)|^2\,d\mathcal{H}^k + \int_{\Sigma^t}\nabla\chi(x)\cdot\vec{H}_{\Sigma^t}(x)\,d\mathcal{H}^k, \tag{22}$$

for all non-negative $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^N)$. Following Brakke [13], we are thus going to extend (22) to less regular objects than smooth embedded manifolds. Actually, we adopt the point of view of Ilmanen [31], which is slightly different from Brakke's original one (the difference being very tiny, to the authors understanding at least!).

Recall that a Radon measure ν on $\mathbb{R}^N$ is said to be k-rectifiable if there exists a k-rectifiable set Σ, and a density function $\Theta \in L^1_{\rm loc}(\mathcal{H}^k \llcorner \Sigma)$ such that

$$\nu = \Theta(\cdot)\,\mathcal{H}^k \llcorner \Sigma.$$

Since Σ is rectifiable, for $\mathcal{H}^k$-a.e. $x \in \Sigma$, there exist a unique tangent space $T_x\Sigma$ belonging to the Grassmanian $G_{N,k}$. The distributional first variation of ν is the vector-valued distribution $\delta\nu$ defined by

$$\delta\nu(\vec{X}) = \int_\Sigma \operatorname{div}_{T_x\Sigma}\vec{X}\,d\nu \qquad \text{for all } \vec{X} \in \mathcal{C}_c^\infty(\mathbb{R}^N, \mathbb{R}^N). \tag{23}$$

In case $|\delta\nu|$ is a measure absolutely continuous with respect to ν, we say that ν has a first variation and we may write

$$\delta\nu = \vec{H}\nu,$$

where $\vec{H}$ is the Radon-Nikodym derivative of $\delta\nu$ with respect to ν. In this case, formula (23) becomes

$$\int_\Sigma \operatorname{div}_{T_x\Sigma}\vec{X}\,d\nu = -\int_\Sigma \vec{H}\cdot\vec{X}\,d\nu. \tag{24}$$

Remark 3. Notice that in the smooth case, this notion coincides with the definition (17), in view of (18). Notice also that the component of $\vec{H}$ which is orthogonal to $T_x\Sigma$ is independent of the density Θ. However, if Θ is non constant, then $\vec{H}$ may have a tangential part.

We are now in position to give the precise definition of a Brakke flow. Let $(\nu_t)_{t\geq 0}$ be a family of Radon measures on $\mathbb{R}^N$. For $\chi \in \mathcal{C}_c^2(\mathbb{R}^N, \mathbb{R}^+)$, we define

$$\bar{D}_t\nu_0^t(\chi) = \limsup_{t\to t_0}\frac{\nu^t(\chi) - \nu^{t_0}(\chi)}{t - t_0}.$$

If $\nu^t \llcorner \{\chi > 0\}$ is a k-rectifiable measure which has a first variation verifying $\chi|\vec{H}|^2 \in L^1(\nu^t)$, then we set

$$\mathcal{B}(\nu^t, \chi) = -\int \chi|\vec{H}|^2\,d\nu^t + \int \nabla\chi\cdot P(\vec{H})\,d\nu^t,$$

[here P denotes $\mathcal{H}^k$-a.e. the orthogonal projection onto the tangent space to ν^t]. Otherwise, we set

$$\mathcal{B}(\nu^t, \chi) = -\infty.$$

Definition 3 (Brakke flow). *Let $(\nu_t)_{t\geq 0}$ be a family of Radon measures on $\mathbb{R}^N$. We say that $(\nu_t)_{t\geq 0}$ is a k-dimensional Brakke flow if and only if*

$$\bar{D}_t \nu^t(\chi) \leq \mathcal{B}(\nu^t, \chi), \tag{25}$$

for every $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^N, \mathbb{R}^+)$ and for all $t \geq 0$.

5.3. Relating $(\mathbf{PGL})_\varepsilon$ to mean curvature flow. The starting point of the analysis is the formal analogy of equality (21), namely

$$\frac{d}{dt}\int_{\Sigma^t} \chi(x)\, d\mathcal{H}^k = -\int_{\Sigma^t} \chi(x)|\vec{H}_{\Sigma^t}(x)|^2\, d\mathcal{H}^k + \int_{\Sigma^t} \nabla\chi(x)\cdot \vec{H}_{\Sigma^t}(x)\, d\mathcal{H}^k,$$

with the evolution of local energies for $(\mathrm{PGL})_\varepsilon$ (see (2.1) in Part I)

$$\frac{d}{dt}\int_{\mathbb{R}^N} \chi(x)\, d\mu_\varepsilon^t = -\int_{\mathbb{R}^N\times\{t\}} \chi(x)\frac{|\partial_t u_\varepsilon|^2}{|\log\varepsilon|}(x)\, dx + \int_{\mathbb{R}^N\times\{t\}} \nabla\chi(x)\cdot \frac{-\partial_t u_\varepsilon \nabla u_\varepsilon}{|\log\varepsilon|}(x)\, dx. \tag{26}$$

We already know that as $\varepsilon \to 0$, $\mu_\varepsilon^t \to \mu_*^t$. Therefore, the comparison of the two formulas suggests, at least formally, that in the limit

$$\omega_\varepsilon^t \equiv \frac{|\partial_t u_\varepsilon|^2}{|\log\varepsilon|}(x)\, dx \to |\vec{H}|^2\, d\mu_*^t, \tag{27}$$

and

$$\sigma_\varepsilon^t \equiv \frac{-\partial_t u_\varepsilon \nabla u_\varepsilon}{|\log\varepsilon|}(x)\, dx \to \vec{H}\, d\mu_*^t. \tag{28}$$

Actually, this is a little over optimistic for two reasons. First we have to deal also with the diffuse part of the energy (this will be handled thanks to Theorem A). Second, since (27) involves the quadratic term $|\vec{H}|^2$, only lower semi-continuity can be expected at first sight (this is certainly a more serious matter, and would require a much longer discussion!).

5.4. Convergence of $\boldsymbol{\sigma_\varepsilon^t}$. Consider the measure $\sigma_\varepsilon = \sigma_\varepsilon^t\, dt$ defined on $\mathbb{R}^N \times [0, +\infty)$. By Cauchy-Schwarz inequality σ_ε is uniformly bounded on $\mathbb{R}^N \times [0,T]$ for every $T > 0$, so that passing possibly to a further subsequence, we may assume that $\sigma_\varepsilon \rightharpoonup \sigma_*$ as measures. The Radon-Nikodym derivative of $|\sigma_\varepsilon|$ with respect to μ_ε verifies

$$\frac{d|\sigma_\varepsilon|}{d\mu_\varepsilon} \leq \sqrt{2}\frac{|\partial_t u_\varepsilon|}{\sqrt{e_\varepsilon(u_\varepsilon)}}.$$

On the other hand,

$$\|\frac{|\partial_t u_\varepsilon|}{\sqrt{e_\varepsilon(u_\varepsilon)}}\|_{L^2(\mathbb{R}^N\times[0,T], d\mu_\varepsilon)} \leq \int_{\mathbb{R}^N\times[0,T]} \frac{|\partial_t u_\varepsilon|^2}{|\log\varepsilon|} \leq KTM_0,$$

so that $\frac{d|\sigma_\varepsilon|}{d\mu_\varepsilon}$ is uniformly bounded in $L^2(\mathbb{R}^N \times [0,T], d\mu_\varepsilon)$. Arguing as in [4] Remark 2.2 (see also [29]) it follows that σ_* is absolutly continuous with respect to μ_*. Therefore, we may write

$$\sigma_* = \vec{\mathfrak{h}}\, \mu_*^t\, dt,$$

where $\vec{\mathfrak{h}} \in L^2(\mathbb{R}^N \times [0,T], \mu_*^t\, dt)$. In view of the decomposition in Theorem A and Part I, we infer

Lemma 4. *The measure σ_* decomposes as $\sigma_* = \sigma_*^t dt$, where for a.e. $t \geq 0$,*

$$\sigma_*^t = -\partial_t \Phi_* \cdot \nabla \Phi_*\, dx + \vec{\mathfrak{h}}\, \nu_*^t.$$

The next step will be to identify the restriction of $\vec{\mathfrak{h}}$ on Σ_μ^t with the mean curvature defined by (24). [Notice that we already know by Theorem A that ν_*^t is (N-2)-rectifiable for a.e. $t \geq 0$.]. The starting point is a classical formula involving the stress-energy tensor. Let $\vec{X} \in \mathcal{C}_c^\infty(\mathbb{R}^N, \mathbb{R}^N)$. We have, for every $t \geq 0$,

$$\begin{aligned} \frac{1}{|\log \varepsilon|} \int_{\mathbb{R}^N \times \{t\}} \left(e_\varepsilon(u_\varepsilon)\delta_{ij} - \frac{\partial u_\varepsilon}{\partial x_i}\frac{\partial u_\varepsilon}{\partial x_j} \right) \frac{\partial X^i}{\partial x_j}\, dx &= \int_{\mathbb{R}^N \times \{t\}} \vec{X} \cdot \frac{\partial_t u_\varepsilon \nabla u_\varepsilon}{|\log \varepsilon|}\, dx \\ &= - \int_{\mathbb{R}^N \times \{t\}} \vec{X} \cdot \sigma_\varepsilon^t. \end{aligned} \tag{29}$$

Formula (29) is already very close to (24), in particular the right hand side. In order to handle the diffuse energy, as well as to interpret the l.h.s as a tangential divergence, we need to analyse the weak limit of the stress-energy tensor

$$\alpha_\varepsilon^t = \left(Id - \frac{\nabla u_\varepsilon \otimes \nabla u_\varepsilon}{e_\varepsilon(u_\varepsilon)} \right) d\mu_\varepsilon^t.$$

Clearly, $|\alpha_\varepsilon^t| \leq KN\mu_\varepsilon^t$, and we may assume that

$$\alpha_\varepsilon^t \rightharpoonup \alpha_*^t \equiv A \cdot \mu_*^t,$$

where A is an $N \times N$ symmetric matrix. Since the symmetric matrix $\nabla u_\varepsilon \otimes \nabla u_\varepsilon$ is non-negative, we have

$$A \leq Id. \tag{30}$$

On the other hand,

$$Tr(e_\varepsilon(u_\varepsilon)\, Id - \nabla u_\varepsilon \otimes \nabla u_\varepsilon) = (N-2)e_\varepsilon(u_\varepsilon) + 2V_\varepsilon(u_\varepsilon).$$

Therefore, since the trace is a linear operation, passing to the limit we obtain

$$Tr(A) = (N-2) + 2\frac{dV_*}{d\mu_*}, \tag{31}$$

where the (non-negative) measure V_* is the limit (up to possibly a further subsequence) of $V_\varepsilon(u_\varepsilon)/|\log \varepsilon|$.

Going to the limit in (29), and using the decomposition in Theorem A, we obtain for a.e. $t \geq 0$,

$$\int_{\mathbb{R}^N} A^{ij} \frac{\partial X^i}{\partial x_j}\, d\nu_*^t + \int_{\mathbb{R}^N} \left(\frac{|\nabla \Phi_*|^2}{2} \delta_{ij} - \frac{\partial \Phi_*}{\partial x_i} \frac{\partial \Phi_*}{\partial x_j} \right) \frac{\partial X^i}{\partial x_j}\, dx$$
$$= - \int_{\mathbb{R}^N} \vec{X} \cdot \vec{\mathfrak{h}}\, d\nu_*^t - \int_{\mathbb{R}^N} \vec{X} \cdot \nabla \Phi_* \partial_t \Phi_*\, dx. \quad (32)$$

On the other hand, Φ_* verifies the heat equation

$$\frac{\partial \Phi_*}{\partial t} - \Delta \Phi_* = 0. \quad (33)$$

Multiplying (33) by $\vec{X} \cdot \nabla \Phi_*$, we obtain

$$\int_{\mathbb{R}^N} \left(\frac{|\nabla \Phi_*|^2}{2} \delta_{ij} - \frac{\partial \Phi_*}{\partial x_i} \frac{\partial \Phi_*}{\partial x_j} \right) \frac{\partial X^i}{\partial x_j}\, dx = - \int_{\mathbb{R}^N} \vec{X} \cdot \nabla \Phi_* \partial_t \Phi_*\, dx. \quad (34)$$

Combining (32) and (34) we have therefore proved

Lemma 5. *For a.e.* $t \geq 0$, *and for every* $\vec{X} \in \mathcal{C}_c^\infty(\mathbb{R}^N, \mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} A^{ij} \frac{\partial X^i}{\partial x_j}\, d\nu_*^t = - \int_{\mathbb{R}^N} \vec{X} \cdot \vec{\mathfrak{h}}\, d\nu_*^t. \quad (35)$$

Remark 4. The last computations are the precise mathematical expression of the fact that the linear and the topological modes do not interact.

Recall that we already know that Σ_μ^t is rectifiable for a.e. $t \geq 0$. Comparing (35) with (24) in order to identify $\vec{\mathfrak{h}}$ with the mean curvature of ν^t, we merely have to prove that the matrix A corresponds to the orthogonal projection P onto the tangent space $T_x\Sigma_\mu^t$. We follow closely the argument of [4] : however, our presentation is more direct, since rectifiability is already established. We first have

Lemma 6. *For a.e.* $t \geq 0$,

$$A(x) \left[\int_{T_x \Sigma_\mu^t} \nabla \chi(y)\, d\mathcal{H}^{N-2}(y) \right] = 0 \qquad \textit{for } \mathcal{H}^{N-2}\textit{-a.e. } x \in \Sigma_\mu^t, \quad (36)$$

and for all $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^N, \mathbb{R})$.

Sketch of the proof. Let $x \in \Sigma_\mu^t$ be such that $T_x\Sigma_\mu^t$ exists and such that x is a Lebesgue point for Θ_* (with respect to $\mathcal{H}^{N-2}$) and for A (with respect to ν_*^t). For $r > 0$, consider the vector field $\vec{X}_{r,l}(y) = \chi(\frac{x-y}{r}) e_l$. Inserting $\vec{X}_{r,l}$ into (35) and letting $r \to 0$, we obtain, by difference of homogeneity, that the r.h.s is neglectible with respect to the l.h.s., and the conclusion follows. ☐

A straightforward consequence is

Corollary 3. *For t and x as in Lemma 6,*

$$\left(T_x\Sigma^t_\mu\right)^\perp \subseteq \operatorname{Ker} A(x).$$

With a little more elementary linear algebra, we further deduce

Corollary 4. *For t and x as in Lemma 6, $A = P$ is the orthogonal projection onto the tangent space $T_x\Sigma^t_\mu$.*

Proof. By (30), $A \leq Id$, and therefore all the eigenvalues $\lambda_1, ..., \lambda_N$ of A are less or equal to 1. By (31), $\operatorname{Tr}(A) \geq N-2$, so that $\sum_{i=1}^N \lambda_i \geq N-2$. On the other hand, by Corollary 3, A has at least two eigenvalues, say λ_1 and λ_2, equal to zero. Therefore, $\lambda_i = 1$ for $i = 3, ..., N$. In particular A is an orthogonal projection on an (N-2)-dimensional space. Since $\operatorname{Ker} A(x) \supseteq (T_x\Sigma^t_\mu)^\perp$, and since $\dim(T_x\Sigma^t_\mu) = N-2$, the conclusion follows. □

Remark 5. Corollary 3 and 4 have many important consequences.
i) Using (31), we deduce that $\frac{dV_*}{d\mu_*} = 0$, i.e. there is only kinetic energy in the limit.
ii) Let $(\tau_1, \ldots, \tau_N)$ be an orthonormal frame such that $T_x\Sigma^t_\mu$ is spanned by $(\tau_3, \ldots, \tau_N)$. In view of the expression of the stress-energy tensor in these coordinates, we infer that the energy concentrates in the (τ_1, τ_2) plane (i.e. $(T_x\Sigma^t_\mu)^\perp$) and uniformly with respect to the direction. In particular, since σ^t_ε is colinear to ∇u_ε, this suggests strongly that $\vec{\mathfrak{h}}$ is perpendicular to $T_x\Sigma_\mu$. Such an argument is made rigorous in [4] (Proposition 6.2). This remark has presumably many other important consequences, but we will not discuss them here.

Combining the previous arguments, we have finally proved

Proposition 9. *For a.e. $t \geq 0$, ν^t_* has a first variation and*

$$\delta\nu^t_* = \vec{\mathfrak{h}}\,\nu^t_*,$$

i.e. $\vec{\mathfrak{h}}$ is the mean curvature of ν^t_*.

Semi-continuity of $\boldsymbol{\omega^t_\varepsilon}$. The purpose of this subsection is to show that for a.e. $t \geq 0$,

$$\liminf_{\varepsilon\to 0} \int_{\mathbb{R}^N\times\{t\}} \chi \frac{|\partial_t u_\varepsilon|^2}{|\log\varepsilon|} \geq \int_{\mathbb{R}^N\times\{t\}} \chi\, |\vec{\mathfrak{h}}|^2 d\nu^t_* + \int_{\mathbb{R}^N\times\{t\}} \chi\, |\partial_t \Phi_*|^2\, dx.$$

It is tempting to write on Σ^t_μ

$$\frac{|\partial_t u_\varepsilon|^2}{|\log\varepsilon|} \geq \frac{|\partial_t u_\varepsilon \nabla u_\varepsilon|^2}{|\log\varepsilon||\nabla u_\varepsilon|^2} \geq \frac{1}{2}\frac{|\partial_t u_\varepsilon \nabla u_\varepsilon|^2}{e_\varepsilon(u_\varepsilon)}\,\mu^t_\varepsilon \gtrsim \frac{1}{2}|\vec{H}|^2\mu^t_*.$$

These formal (but essentially correct) inequalities do not allow to conclude, in view of the extra factor $\frac{1}{2}$. Fortunately, the last inequality is far from being optimal. Indeed, weak convergence does not imply convergence of the squared quantities!

Remark 6. In the scalar case, i.e. for the Allen-Cahn equation, this difficulty does not arises since $\frac{|\nabla u_\varepsilon|^2}{2} \simeq V_\varepsilon(u_\varepsilon)$ there, so that $|\nabla u_\varepsilon|^2 \simeq e_\varepsilon(u_\varepsilon)$. The difficulty there however was to establish the balance between the kinetic and potential terms (see [30] Section 8.1).

In order to handle the factor $\frac{1}{2}$, a determinant idea of [4] in this context was to recast the problem in the framework of Young measures, which turn out to be an appropriate concept to analyse the energies of the oscillations. In this direction, set $p_\varepsilon = \frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|}$, and consider the measure (defined on $\mathbb{R}^N \times \mathbb{R}^{2N}$)

$$\tilde{\omega}_\varepsilon^t = \delta_{p_\varepsilon(x)} \frac{|\partial_t u_\varepsilon \cdot p_\varepsilon|^2}{|\log \varepsilon|} \, dx.$$

Extracting possibly a further subsequence, we may assume that $\tilde{\omega}_\varepsilon^t \, dt \to \tilde{\omega}_*$ as measures. We deduce from the analysis of Part I and Theorem A,

Lemma 7. *The measure $\tilde{\omega}_*$ decomposes as $\tilde{\omega}_* = \tilde{\omega}_*^t \, dt$, and for a.e. $t \geq 0$*

$$\tilde{\omega}_*^t = \Pi_{*,x}^t(p) \, |\partial_t \Phi_*|^2 \, dx + \mathfrak{W}_*^t,$$

where $\Pi_{*,x}^t$ is a measure on $\mathbb{R}^{2N}$ (with support on the unit ball) and $\mathfrak{W}_*^t = \tilde{\omega}_*^t \llcorner \Sigma_\mu^t$. Moreover, $\Pi_{*,x}^t(\mathbb{R}^{2N}) = 1$.

[Notice that the measure $\Pi_{*,x}^t$ arises from the possible oscillations of the phase Φ_ε of u_ε, but is not disturbing since it acts linearly].

The main ingredient that we will borrow directly from the analysis of [4] Section 6 can be formulated as the following.

Proposition 10 (Ambrosio and Soner). *For a.e. $t \geq 0$, and every $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^N)$,*

$$\int_{\mathbb{R}^N \times \mathbb{R}^{2N}} \chi(x) \, \mathfrak{W}_*^t(x,p) \geq \int_{\mathbb{R}^N} \chi \, |\vec{\mathfrak{h}}|^2 \, d\nu_*^t.$$

At this stage, we are (finally!) in position to complete the proof of Theorem B.

Proof of Theorem B. In view of Theorem 4.4 in [4], it suffices to establish the integral version of (25). Let $0 < T_0 < T_1$. We integrate (26) on $[T_0, T_1]$ and let ε go to zero. Combining the results of Lemma 4, Proposition 9, Lemma 7, Remark 5, Proposition 10 and Theorem A, we obtain

$$\begin{aligned} \nu_*^{T_1}(\chi) - \nu_*^{T_0}(\chi) + \int_{\mathbb{R}^N \times \{T_1\}} \chi \, |\nabla \Phi_*|^2 \, dx - \int_{\mathbb{R}^N \times \{T_0\}} \chi \, |\nabla \Phi_*|^2 \, dx \\ \leq - \int_{\mathbb{R}^N \times [T_0,T_1]} \chi \, |\vec{\mathfrak{h}}|^2 \, d\nu_* + \int_{\mathbb{R}^N \times [T_0,T_1]} \nabla \chi P(\vec{\mathfrak{h}}) \, d\nu_* \\ - \int_{\mathbb{R}^N \times [T_0,T_1]} \chi |\partial_t \Phi_*|^2 \, dx \, dt + \int_{\mathbb{R}^N \times [T_0,T_1]} \nabla \chi \nabla \Phi_* \partial_t \Phi_*. \quad (37) \end{aligned}$$

Since Φ_* verifies the heat equation, we have the identity

$$\int_{\mathbb{R}^N\times\{T_1\}} \chi\,|\nabla\Phi_*|^2\,dx - \int_{\mathbb{R}^N\times\{T_0\}} \chi\,|\nabla\Phi_*|^2\,dx = \int_{\mathbb{R}^N\times[T_0,T_1]} \chi|\partial_t\Phi_*|^2\,dx\,dt + \int_{\mathbb{R}^N\times[T_0,T_1]} \nabla\chi\nabla\Phi_*\partial_t\Phi_*. \quad (38)$$

Combining (37) and (38) we obtain

$$\nu_*^{T_1}(\chi) - \nu_*^{T_0}(\chi) \le -\int_{\mathbb{R}^N\times[T_0,T_1]} \chi\,|\vec{\mathfrak{h}}|^2\,d\nu_* + \int_{\mathbb{R}^N\times[T_0,T_1]} \nabla\chi P(\vec{\mathfrak{h}})\,d\nu_*.$$

As mentioned above, this integral formulation implies (25), under suitable assumptions which are fullfilled here, namely rectifiability of Σ_μ^t, lower bounds on the density Θ_*, and orthogonality of the mean curvature $\vec{\mathfrak{h}}$ with $(T_x\Sigma_\mu^t)^\perp$.

The proof of Theorem B is complete. □

6 Ilmanen enhanced motion

The notion of motion by mean curvature in the sense of Brakke has many interesting properties, in particular the fact that the area functional decreases along the flow, as expected from the classical motion. Unfortunately, as already mentioned in the main introduction, this notion strongly lacks of uniqueness. Indeed, if $(\mu^t)_{t\ge 0}$ is a Brakke flow, so is also $(g(t)\mu^t)_{t\ge 0}$, where g is an arbitrary non increasing function on $\mathbb{R}^+$. In particular, the trivial solution given by $\nu^0 = \mu^0$ and $\nu^t \equiv 0$ for $t > 0$ is not excluded a priori. Actually, for $(\mathrm{PGL})_\varepsilon$ such a situation may occur (as in the Allen-Cahn equation), at least in three distinct cases :

- Concentrated phase : the initial data is of the form $u_\varepsilon^0 = \exp(i\varphi_\varepsilon^0\sqrt{|\log\varepsilon|})$, where $|\nabla\varphi_\varepsilon^0|^2$ is bounded in L^1 and concentrates on a (N-2)-dimensional set Σ_0.

- Low density : we present an example is dimension 3. In the plane (x_1, x_3), consider a standard dipole (i.e. with "least" energy) of two vortices away from the origin and separated by a length ε^η (where $0 < \eta < 1$ is fixed), so that the energy in the plane is of order $\pi\eta|\log\varepsilon|$. Rotate the dipole along the x_3 axis so that $e_\varepsilon(u_\varepsilon^0)$ concentrates on a circle with a 1-density proportionnal to η. If η is chosen sufficiently small, then $\mu_*^t \equiv 0$ for $t > 0$ by the clearing-out lemma.

- Hidden mean curvature : consider in the (x_1, x_2) plane the standard circle S^1. Approximate it, weakly in the sense of measures, by a collection $\mathcal{B}_i$ of small circles centered on S^1 and of radii $\sim \frac{1}{i}$. By Theorem D, for each $i \in \mathbb{N}_*$ there exist initial data $(u_\varepsilon^{0,i})$ such that the limiting measures $\mu_*^{t,i}$ evolves according to the classical motion of the small circles, whose lifetime is of the order of i^{-2}. By a diagonal argument, it is therefore possible to construct a sequence u_ε^0 such that $\mu_*^0 = S^1$ but $\mu_*^t \equiv 0$ for $t > 0$.

Remark 7. The two first cases are related to specific properties of $(\mathrm{PGL})_\varepsilon$, whereas the third is intrinsically related to motion by mean curvature.

The three cases have a common feature : the Jacobians of u_ε^0 converge to zero as ε tends to 0, at least in the sense of distributions. We consider next the space-time Jacobian of u_ε,

$$\mathcal{J} u_\varepsilon = \sum_{0\le i<j\le N} (\partial_{x_i} u_\varepsilon \times \partial_{x_j} u_\varepsilon)\, dx_i \wedge dx_j,$$

with the convention that $x_0 \equiv t$. In view of the space-times bounds on the Ginzburg-Landau energy, we may invoke the work of Jerrard and Soner [34] (see also [1]), to assert that

$$\mathcal{J} u_\varepsilon \rightharpoonup \mathcal{J}_* \qquad \text{in } \left(\mathcal{C}_c^{0,\alpha}(\mathbb{R}^N \times \mathbb{R}^+)\right)^*,$$

where $\mathcal{J}_*$ is an (N-1)-rectifiable vector-valued measure. Moreover, it is shown in [34, 1], that $\frac{1}{\pi}\mathcal{J}_*$ can be identified with an integer multiplicity (N-1)-current, whose boundary is exactly $\mathcal{J}_*^0$ (the slice at time zero), and

$$\frac{1}{\pi}|\mathcal{J}_*^t| \le \mu_*^t, \qquad \text{for } t \ge 0.$$

Here, $\mathcal{J}_*^t$ denotes the slice of the current $\mathcal{J}_*$ on $\mathbb{R}^N \times \{t\}$ (which we will prove to be well defined), and coincides with the limit in the sense of currents of the space Jacobians $Ju_\varepsilon(\cdot, t)$. The (N-1)-rectifiable set $\Sigma_{\mathcal{J}}$ supporting $\mathcal{J}_*$ represents the concentration set of vorticity, and has therefore a great significance (presumably for the applications, more than the energy). Obviously, $\Sigma_{\mathcal{J}} \subseteq \Sigma_\mu$, and it is rather easy to construct examples where they are different (think of two approaching circles with opposite orientations). Notice also that $\mathcal{J}_*$, a priori, has more structure than μ_*, since it has an orientation and integer multiplicity (modulo π).

The previous discussion naturally leads to Ilmanen's notion of enhanced (mean curvature) motion, which we recall now.

Let $\mathcal{M}_0$ be an integer multiplicity (N-2)-current in $\mathbb{R}^N$, without boundary. We assume for simplicity that $\mathcal{M}_0$ is bounded and of finite mass. Let $\mathcal{M}$ be an integer multiplicity (N-1)-current in $\mathbb{R}^N \times [0, +\infty)$, and $\{\mu^t\}_{t\ge 0}$ a familly of non-negative Radon measures on $\mathbb{R}^N$.

Definition 4 (Enhanced motion). *The pair $\{\mathcal{M}, \{\mu^t\}_{t\ge0}\}$ is called an enhanced motion with initial condition $\mathcal{M}_0$ if and only if*

i) $\partial\mathcal{M} = \mathcal{M}_0$.

ii) $\mu^0 = |\mathcal{M}_0|$.

iii) The measure defined on $\mathbb{R}^+$ by $\mathcal{T}(B) = |\mathcal{M}|(\mathbb{R}^N \times B)$, for any Borel set B, is absolutly continuous with respect to the Lebesgue measure on $\mathbb{R}^+$.

iv) For all $t \ge 0$,

$$\mu^t \ge |\mathcal{M}_t|,$$

where $\mathcal{M}_t$ denotes the slice of $\mathcal{M}$ at time t.

v) $\{\mu^t\}_{t\geq 0}$ is a Brakke flow.

Remark 8. Notice in particular that conditions *i)* and *iii)* are closer to what one actually would normally expect from a Cauchy problem. In Ilmanen's terminology, $\mathcal{M}$ is called the under-current, and provides, in view of iv), a lower bound, which rules out sudden shrinking.

In [31], Ilmanen established the existence of an enhanced motion, for any initial data as above (actually in any codimension). Theorem D provides an alternative construction in codimension 2. The two solutions may differ, since there remains still some possible non-uniqueness for an enhanced motion (see the discussion on "matching motion" in [31]). Moreover, in the smooth case, there is uniqueness for an enhanced motion (before singularities appear) and it coincides with the classical notion.

We are now in position to present the proof of Theorem D.

Proof of Theorem D. The proof essentially relies on a combination of results proven in [1, 34].

Construction of u_ε^0. The existence of a family $(u_\varepsilon^0)_{\varepsilon>0}$ satisfying (H_0) and assumption ii) follows directly from [1], Theorem 1.1, ii). More precisely, the family $(u_\varepsilon^0)_{\varepsilon>0}$ constructed there verifies

$$\frac{1}{\pi} J u_\varepsilon^0 \to \mathcal{M}_0 \qquad \text{in } [\mathcal{C}_c^{0,\alpha}(\mathbb{R}^N)]^*, \tag{39}$$

$$\frac{1}{\pi}\mu_*^0 = |\mathcal{M}_0| \equiv M_0\,, \tag{40}$$

and the additional compactness conditions (H_1), (H_2), as well as the bound $|u_\varepsilon^0| \leq 1$.

Construction of $\mathcal{M}$. We next consider the solution u_ε of $(\mathrm{PGL})_\varepsilon$ with initial datum u_ε^0, verifying (39) and (40) above. In view of (H_1) and (H_2), we may apply Theorem C to μ_*, so that μ_* has no diffuse part, i.e. $\mu_* = \nu_*$. In particular, by Theorem B, $\{\mu_*^t\}_{t\geq 0}$ is a Brakke flow, and property v) of Definition 4 is established.

In view of (I), the space-time Ginzburg-Landau energy is bounded in $\mathbb{R}^N \times [0,T]$ for every $T > 0$, more precisely

$$\int_{\mathbb{R}^N\times[0,T]} \frac{1}{2}|\nabla_{x,t} u_\varepsilon|^2 + V_\varepsilon(u_\varepsilon) \leq M_0(T+1)|\log\varepsilon|.$$

We deduce from [34, 1] that

$$\mathcal{J} u_\varepsilon \to \mathcal{J}_* \qquad \text{in } [\mathcal{C}_c^{0,\alpha}(\mathbb{R}^N\times\mathbb{R}^+)]^*,$$

where $\mathcal{J}_*$ is a (N-1) integer multiplicity current. Notice that $\mathcal{J}_*$ has compact support in $\mathbb{R}^N \times [0, T_f+1]$ (see Proposition 3 for the definition of T_f). We set

$$\mathcal{M} = \frac{1}{\pi}\mathcal{J}_*\,.$$

We claim that

$$\partial \mathcal{M} = \mathcal{M}_0 \,, \tag{41}$$

i.e., ii) is verified.

Indeed, for every test form $\chi \in \mathcal{C}^\infty(\mathbb{R}^N \times \mathbb{R}^+)$ we have, by Stokes theorem,

$$\begin{aligned}
\int_{\mathbb{R}^N \times \mathbb{R}^+} \mathcal{J} u_\varepsilon \wedge \delta\chi &= \frac{1}{2}\int_{\mathbb{R}^N \times \mathbb{R}^+} \delta(u_\varepsilon \times \delta u_\varepsilon) \wedge \delta\chi \\
= \frac{1}{2}\int_{\mathbb{R}^N \times \mathbb{R}^+} \delta(u_\varepsilon \times \delta u_\varepsilon \wedge \delta\chi) &= -\frac{1}{2}\int_{\mathbb{R}^N \times \{0\}} (u_\varepsilon^0 \times du_\varepsilon^0) \wedge d\chi \\
= \frac{1}{2}\int_{\mathbb{R}^N \times \{0\}} d(u_\varepsilon^0 \times du_\varepsilon^0) \wedge \chi &= \int_{\mathbb{R}^N \times \{0\}} J u_\varepsilon^0 \wedge \chi \,.
\end{aligned} \tag{42}$$

In view of the compactness results in Theorem 4, and (H_1), (H_2), we may pass to the limit as $\varepsilon \to 0$, so that

$$\mathcal{M}(\delta\chi) = \mathcal{M}_0(\chi) \,,$$

which establishes the claim.

At this stage, we have shown that the pair $\{\mathcal{M}, \{\mu_*^t\}_{t\geq 0}\}$ verifies i), ii), v) of Definition 4.

Proof of iv). By definition of slicing, and arguing as in (42), we have

$$\mathcal{M}_t = \lim_{\varepsilon\to 0} J u_\varepsilon(\cdot, t) \,.$$

Therefore iv) follows from [34, 1].

Proof of iii). Let I=[a,b] be a bounded interval in $\mathbb{R}$. We claim that

$$|\mathcal{M}|(\mathbb{R}^N \times [a,b]) \leq C(M_0)|b-a|^{1/2} \,, \tag{43}$$

which clearly implies iii). Recall that

$$|\mathcal{M}|(\mathbb{R}^N \times [a,b]) = \sup\{\mathcal{M}(\chi),\ |\chi| \leq 1, \quad \text{supp}\chi \subset \mathbb{R}^N \times [a,b]\} \,.$$

In order to prove (43), we need first to go back to the level ε. Let $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^N \times [a,b])$, we have

$$\mathcal{M}(\chi) = \lim_{\varepsilon\to 0} \int J u_\varepsilon \wedge \chi \,.$$

In order to estimate the integral on the r.h.s, we distinguish the purely spatial components of the Jacobian, and the space-time components. For the spatial components, we have by Lemma 3.13 (see also [1]),

$$\begin{aligned}
\left| \int_{\mathbb{R}^N \times [a,b]} (*\chi)_{ij} \frac{\partial u_\varepsilon}{\partial x_i} \times \frac{\partial u_\varepsilon}{\partial x_j} \right| &\leq \frac{K}{|\log\varepsilon|} \|\chi\|_\infty \int_{\mathbb{R}^N \times [a,b]} e_\varepsilon(u_\varepsilon) \\
&\quad + K\varepsilon^\beta \|\chi\|_{C^1} \left(1 + \int_{\mathbb{R}^N \times [a,b]} e_\varepsilon(u_\varepsilon)\right) \\
&\leq K M_0 |b-a| \left(\|\chi\|_\infty + \varepsilon^\beta(|\log\varepsilon| + 1)\|\chi\|_{C^1}\right) .
\end{aligned} \tag{44}$$

In order to handle the space-time component, we rescale the function u_ε with respect to the time variable. Consider the interval $I' = [a, b']$, where $b' = a + |b-a|^{1/2}$, and the function w_ε defined on $\mathbb{R}^N \times I'$, by

$$w_\varepsilon(\cdot, s) = u_\varepsilon(\cdot, (s-a)|b-a|^{1/2} + a)\,, \qquad\qquad s \in I'\,,$$

so that

$$\frac{\partial w_\varepsilon}{\partial s}(\cdot, s) = |b-a|^{1/2} \frac{\partial u_\varepsilon}{\partial t}(\cdot, (s-a)|b-a|^{1/2} + a)$$

and

$$\int_{\mathbb{R}^N \times I'} \left|\frac{\partial w_\varepsilon}{\partial s}\right|^2 dxds = |b-a|^{1/2} \int_{\mathbb{R}^N \times [a,b]} \left|\frac{\partial u_\varepsilon}{\partial t}\right|^2 dxdt \leq K|b-a|^{1/2} M_0 |\log\varepsilon|\,.$$

On the other hand, by the energy inequality (I),

$$\int_{\mathbb{R}^N \times I'} \frac{|\nabla w_\varepsilon|^2}{2} + V_\varepsilon(w_\varepsilon) \leq |b-a|^{1/2} M_0 |\log\varepsilon|\,,$$

so that

$$\int_{\mathbb{R}^N \times I'} \frac{|\nabla_{x,s} w_\varepsilon|^2}{2} + V_\varepsilon(w_\varepsilon) \leq K|b-a|^{1/2} M_0 |\log\varepsilon|\,. \tag{45}$$

We apply the estimate of Lemma 3.13 to the function w_ε in $\mathbb{R}^N \times I'$. This yields, in view of (45),

$$\left|\int_{\mathbb{R}^N \times I'} (*\tilde{\chi})_{0j} \frac{\partial w_\varepsilon}{\partial s} \times \frac{\partial w_\varepsilon}{\partial x_j}\right| \leq KM_0|b-a|^{1/2}(\|\chi\|_\infty + \varepsilon^\beta(|\log\varepsilon| + 1)\|\chi\|_{C^1}). \tag{46}$$

[Here we set $\tilde{\chi}(\cdot, s) = \chi(\cdot, (s-a)|b-a|^{1/2} + a)$]

Finally, we have

$$\int_{\mathbb{R}^N \times [a,b]} (*\chi)_{0j} \frac{\partial u_\varepsilon}{\partial t} \times \frac{\partial u_\varepsilon}{\partial x_j} = \int_{\mathbb{R}^N \times I'} (*\tilde{\chi})_{0j} \frac{\partial w_\varepsilon}{\partial s} \times \frac{\partial w_\varepsilon}{\partial x_j}. \tag{47}$$

Combining (44), (46) and (47), we are led to

$$\left|\int_{\mathbb{R}^N \times [a,b]} \mathcal{J}u_\varepsilon \wedge \chi\right| \leq KM_0(|b-a|^{1/2} + |b-a|)\|\chi\|_\infty + o(1)\,. \tag{48}$$

Passing to the limit in (48) as $\varepsilon \to 0$, we derive (43). □

6 Properties of Σ_μ

The purpose of this Section is to provide detailed proofs of some technical statements, concerning Σ_μ, in the introduction to Part II. More precisely, we will prove (3), Lemma 3, Theorem 6 and Propositions 6, 7 and 8. We begin with a few elementary observations which we will use later in the proofs.

Lemma 6.1. *Let* $(x,t) \in \Sigma_\mu$ *and* $0 < r < \sqrt{t}$. *Then, we have*

$$r^{2-N}\mu_*^{t-r^2}(B(x,\lambda(t-r^2)r)) > \eta_2,$$

where η_2 *is the constant in Theorem 5.*

Proof. Indeed, assume by contradiction that

$$r^{2-N}\mu_*^{t-r^2}(B(x,\lambda(t-r^2)r)) \le \eta_2.$$

Then, by Theorem 5, for every $\tau \in [t - \frac{1}{16}r^2, t]$

$$\mu_*^\tau = |\nabla\Phi_*|^2\,dx \qquad \text{on } B(x, \frac{1}{4}r),$$

where Φ_* is smooth. We are going to show that

$$s^{2-N}\int_{\mathbb{R}^N} \exp(-\frac{|x-y|^2}{4s^2})\,d\mu_*^{t-s^2} \to 0, \tag{6.1}$$

as $s \to 0$. Indeed, we write

$$\begin{aligned} s^{2-N}\int_{B(x,\frac{1}{8}r)} \exp(-\frac{|x-y|^2}{4s^2})\,d\mu_*^{t-s^2} &\le s^{2-N}\|\nabla\Phi_*\|^2_{L^\infty(B(x,\frac{1}{8}r))}\int_{\mathbb{R}^N}\exp(-\frac{|x-y|^2}{4s^2})\,dx \\ &\le K\|\nabla\Phi_*\|^2_{L^\infty(B(x,\frac{1}{8}r))} s^2 \to 0, \end{aligned} \tag{6.2}$$

as $s \to 0$. On the other hand,

$$s^{2-N}\int_{\mathbb{R}^N\setminus B(x,\frac{1}{8}r)} \exp(-\frac{|x-y|^2}{4s^2})\,d\mu_*^{t-s^2} \le s^{2-N}\exp(-\frac{r^2}{256s^2})M_0 \to 0, \tag{6.3}$$

as $\varepsilon \to 0$. Combining (6.2) and (6.3), (6.1) follows and hence $\Theta^P_{N-2}(\mu_*, (x,t)) = 0$, i.e.

$$(x,t) \notin \Sigma_\mu,$$

a contradiction. ☐

Lemma 6.2. *The function* $(x,t) \mapsto \Theta^P_{N-2}(\mu_*, (x,t))$ *is upper semi-continuous on the set* $\mathbb{R}^N \times (0, +\infty)$.

Proof. Let $(x,t) \in \mathbb{R}^N \times (0,+\infty)$, and let $(x_n,t_n)_{n\in\mathbb{N}}$ be a sequence such that $(x_n,t_n) \to (x,t)$. We are going to show that

$$\limsup_{n\to+\infty} \Theta^P_{N-2}(\mu_*, (x_n,t_n)) \le \Theta^P_{N-2}(\mu_*, (x,t)). \tag{6.4}$$

Let $0 < r < \frac{1}{2}\sqrt{t}$ be fixed for the moment. For n sufficiently large, set $r_n = \sqrt{r^2 + t_n - t}$, so that $t - r^2 = t_n - r_n^2$. By monotonicity we have

$$\begin{aligned}\Theta^P_{N-2}(\mu_*, (x_n, t_n)) &\leq \frac{1}{r_n^{N-2}} \int_{\mathbb{R}^N} \exp(-\frac{|y - x_n|^2}{4r^2})\, d\mu_*^{t_n - r_n^2}(y) \\ &= \frac{1}{r_n^{N-2}} \int_{\mathbb{R}^N} \exp(-\frac{|y - x_n|^2}{4r^2})\, d\mu_*^{t - r^2}(y).\end{aligned}$$

Letting n tends to $+\infty$, we obtain

$$\limsup_{n\to+\infty} \Theta^P_{N-2}(\mu_*, (x_n, t_n)) \leq \frac{1}{r^{N-2}} \int_{\mathbb{R}^N} \exp(-\frac{|x - y|^2}{4r^2})\, d\mu_*^{t-r^2}(y).$$

Next, we let $r \to 0$, and (6.4) follows. □

6.1 Proof of Lemma 3

Let $x \in \mathbb{R}^N$ and $t > 0$. We have, for every $0 < r < \frac{1}{2}\sqrt{t}$,

$$\begin{aligned}\frac{\mu_*^t(B(x, r))}{\omega_{N-2} r^{N-2}} &\leq \exp(\frac{1}{4}) \frac{1}{\omega_{N-2} r^{N-2}} \int_{\mathbb{R}^N} \exp(-\frac{|x - y|^2}{4r^2})\, d\mu_*^t(y) \\ &\leq \exp(\frac{1}{4}) \frac{1}{\omega_{N-2}(t + r^2)^{\frac{N-2}{2}}} \int_{\mathbb{R}^N} \exp(-\frac{|x - y|^2}{4r^2})\, d\mu_*^0(y) \\ &\leq \exp(\frac{1}{4}) \frac{M_0}{\omega_{N-2}} t^{\frac{2-N}{2}},\end{aligned}$$

where we have used the monotonicity formula (Lemma 2) for the second inequality.

6.2 Proof of inequality (3)

Let $(x, t) \in \mathbb{R}^N \times (0, +\infty)$ be given. Let $0 < r < t$ be fixed for the moment. We write, for every $0 < s < \sqrt{t}$,

$$\begin{aligned}\frac{1}{r^{N-2}} \mu_*^t(B(x, r)) &\leq \frac{1}{r^{N-2}} \exp(\frac{1}{4}) \int_{\mathbb{R}^N} \exp(-\frac{|x - y|^2}{4r^2})\, d\mu_*^t(y) \\ &\leq \frac{1}{(r^2 + s^2)^{\frac{N-2}{2}}} \exp(\frac{1}{4}) \int_{\mathbb{R}^N} \exp(-\frac{|x - y|^2}{4(r^2 + s^2)})\, d\mu_*^{t-s^2}(y),\end{aligned}$$

where we have used the monotonicity (at the point $(x, t + r^2)$) for the last inequality. Next, we choose $s = \sqrt{r}$. This yields

$$\frac{1}{r^{N-2}} \mu_*^t(B(x, r)) \leq \frac{1}{(r^2 + r)^{\frac{N-2}{2}}} \exp(\frac{1}{4}) \int_{\mathbb{R}^N} \exp(-\frac{|x - y|^2}{4(r^2 + r)})\, d\mu_*^{t-r}(y). \tag{6.5}$$

In the last integral, we decompose

$$\mathbb{R}^N = B(x,1) \cup (\mathbb{R}^N \setminus B(x,1)).$$

On $B(x,1)$, observe that

$$\exp(-\frac{|x-y|^2}{4(r^2+r)}) \leq K \exp(-\frac{|x-y|^2}{4r}),$$

for some absolute constant K. On the other hand, on $\mathbb{R}^N \setminus B(x,1)$, we have

$$\int_{\mathbb{R}^N \setminus B(x,1)} \exp(-\frac{|x-y|^2}{4(r^2+r)})\, d\mu_*^{t-r}(y) \leq \exp(-\frac{1}{4(r^2+r)}) M_0.$$

Going back to (6.5), we are led to

$$\frac{1}{r^{N-2}} \mu_*^t(B(x,r)) \leq \frac{K}{r^{\frac{N-2}{2}}} \int_{\mathbb{R}^N} \exp(-\frac{|x-y|^2}{4r})\, d\mu_*^{t-r} + \frac{1}{r^{\frac{N-2}{2}}} \exp(-\frac{1}{4(r^2+r)}) M_0.$$

Letting r go to zero the conclusion follows.

6.3 Proof of Theorem 6

Let $(x,t) \in \mathbb{R}^N \times (0,+\infty)$ and $0 < r < \sqrt{t}$. We have

$$r^{2-N} \mu_*^{t-r^2}(B(x, \lambda(t-r^2)r)) \leq \exp(\frac{\lambda^2(t-r^2)}{4})\, \mathfrak{E}_\mu((x,t),r). \tag{6.6}$$

Consider therefore the function

$$\eta_3(s) = \exp(-\lambda^2(s)/4)\eta_2,$$

and assume next that, for some $0 < r < \sqrt{t}$,

$$\mathfrak{E}_\mu((x,t),r) \leq \eta_3(t-r^2).$$

Then, by (6.6),

$$\mu_*^{t-r^2}(B(x,\lambda(t-r^2)r)) \leq \eta_2$$

and the conclusion follows by Lemma 6.1. □

6.4 Proof of Proposition 6

Proof of i). In view of Corollary 1, we have

$$\Sigma_\mu = \left\{(x,t) \in \mathbb{R}^N \times (0,+\infty),\ \Theta^P_{N-2}(\mu_*, (x,t)) \geq \eta_3(t)\right\}.$$

Since $\eta_3(\cdot)$ is continuous, and since $\Theta^P_{N-2}(\mu_*, \cdot)$ is upper semi-continuous by Lemma 6.2, so is $\Theta^P_{N-2}(\mu_*, \cdot) - \eta_3(\cdot)$ on $\mathbb{R}^N \times (0, +\infty)$. The conclusion follows.

Proof of ii). We proceed in two steps. Firstly, we establish the estimate for $t = 1$ and secondly we argue by scaling.

Step 1 : The case $t = 1$. Let $0 < \delta < \frac{1}{4}$. Consider a standard (say parallepipedic) covering of $\mathbb{R}^N$ such that

$$\mathbb{R}^N \subseteq \cup_{j \in I} B(x_j, \delta), \qquad \text{and} \qquad B(x_i, \frac{\delta}{2}) \cap B(x_j, \frac{\delta}{2}) = \emptyset \text{ for } i \neq j.$$

Set

$$I_\delta = \left\{ i \text{ s.t. } B(x_i, \delta) \cap \Sigma^1_\mu \neq \emptyset \right\}.$$

For $i \in I_\delta$, there exists some $y_i \in \Sigma^1_\mu \cap B(x_i, \delta)$. Hence, by Lemma 6.1,

$$\mu_*^{1-\delta^2}(B(y_i, \lambda(1-\delta^2)\delta)) > \eta_2 \delta^{2-N},$$

and in particular

$$\mu_*^{1-\delta^2}(B(x_i, (\lambda(1-\delta^2)+1)\delta)) > \eta_2 \delta^{2-N}. \tag{6.7}$$

On the other hand, since the balls $B(x_i, \frac{\delta}{2})$ are disjoints, the balls $B(x_i, (\lambda(1-\delta^2)+1)\delta)$ cover at most K times $\mathbb{R}^N$, where K is a constant depending only on N, for $\delta < \frac{1}{4}$. Therefore,

$$\sum_{i \in I_\delta} \mu_*^{1-\delta^2}(B(x_i, (\lambda(1-\delta^2)+1)\delta)) \leq K M_0. \tag{6.8}$$

Combining (6.7) and (6.8) we obtain

$$\sharp I_\delta \leq K M_0 \delta^{2-N}.$$

Since by definition, $\mathcal{H}^{N-2}(\Sigma^1_\mu) \leq K \limsup_{\delta \to 0} (\sharp I_\delta) \delta^{N-2}$, the conclusion follows.

Step 2 : Invariance by scaling. For $t_0 > 0$ fixed, consider the function

$$v_\epsilon(x,t) = u_\varepsilon(\sqrt{t_0}\, x, t_0 t)$$

where $\epsilon = \frac{\varepsilon}{\sqrt{t_0}}$, so that

$$v_\epsilon(x, 1) = u_\varepsilon(\sqrt{t_0}\, x, t_0),$$

v_ϵ verifies $(\mathrm{PGL})_\epsilon$, and $E_\epsilon(v^0_\epsilon) = t_0^{\frac{2-N}{2}} E_\varepsilon(u^0_\varepsilon)$. Letting $\varepsilon_n \to 0$, so does $\epsilon_n = \frac{\varepsilon_n}{\sqrt{t_0}}$, and

$$\Sigma^t_\mu(u) = t_0^{\frac{1}{2}} \Sigma_\mu^{\frac{t}{t_0}}(v)$$

(with obvious notations), in particular

$$\Sigma^{t_0}_\mu(u) = t_0^{\frac{1}{2}} \Sigma^1_\mu(v),$$

so that

$$\mathcal{H}^{N-2}(\Sigma^{t_0}_\mu(u)) = t_0^{\frac{N-2}{2}} \mathcal{H}^{N-2}(\Sigma^1_\mu(v)).$$

By Step 1 applied to v_ϵ and the corresponding measure $\Sigma_\mu(v)$, we obtain

$$\mathcal{H}^{N-2}(\Sigma_\mu^1(v)) \leq K \sup_{n\in\mathbb{N}} \left(E_{\epsilon_n}(v^0_{\epsilon_n})\right) \leq K t_0^{\frac{2-N}{2}} M_0.$$

Therefore,

$$\mathcal{H}^{N-2}(\Sigma_\mu^{t_0}(u)) \leq K M_0,$$

and the conclusion ii) follows.

Proof of iii). By i), we know that Σ_μ^t is closed, and hence measurable. Therefore,

$$\mu_*^t = \mu_*^t \llcorner (\mathbb{R}^N \setminus \Sigma_\mu^t) + \mu_*^t \llcorner \Sigma_\mu^t. \tag{6.9}$$

We claim that there exists a smooth function g defined on the open set $\mathbb{R}^N \times (0, +\infty) \setminus \Sigma_\mu$ such that

$$\mu_*^t \llcorner (\mathbb{R}^N \setminus \Sigma_\mu^t) = g \cdot \mathcal{H}^N.$$

Indeed, let $x \in \mathbb{R}^N \setminus \Sigma_\mu^t$. Then by definition

$$\lim_{r\to 0} \mathfrak{E}_\mu((x,t), r) = 0,$$

so that for some r_0 sufficiently small

$$\mathfrak{E}_\mu((x,t), r_0) \leq \eta_3(t - r_0^2).$$

Therefore, by (6.6),

$$\mu_*^{t-r_0^2}(B(x, \lambda(t - r_0^2) r_0) \leq \eta_2 r_0^{N-2},$$

and we infer from Theorem 5 that for all $s \in [t - \frac{1}{16} r_0^2, t]$,

$$\mu_*^s \equiv g(\cdot, s)\, \mathcal{H}^N \qquad \text{on } B(x, \frac{1}{4} r_0),$$

for some smooth function g. Notice that at this point, we may only locally write $g = |\nabla \Phi_*|^2$, for some smooth Φ_*. We will see later that Φ_* is global, whereas g is obviously already globally defined on $\mathbb{R}^N \times (0, +\infty) \setminus \Sigma_\mu$.

Since $\mathcal{H}^{N-2}(\Sigma_\mu^t) < +\infty$ we have $\mathcal{H}^N(\Sigma_\mu^t) = 0$, and hence

$$\mu_*^t \llcorner (\mathbb{R}^N \setminus \Sigma_\mu^t) = g(\cdot, t)\, \mathcal{H}^N, \tag{6.10}$$

which establishes the claim.

Next, we deduce from Lemma 3 that $\mu_*^t \llcorner \Sigma_\mu^t$ is absolutely continuous with respect to the measure $\mathcal{H}^{N-2}$, and by ii) that the measure $\mathcal{H}^{N-2} \llcorner \Sigma_\mu^t$ is finite. We may therefore apply the Radon-Nikodym Theorem, which yields

$$\mu_*^t \llcorner \Sigma_\mu^t = \Theta_*(x,t) \mathcal{H}^{N-2} \llcorner \Sigma_\mu^t, \tag{6.11}$$

where Θ_* is the Radon-Nikodym derivative. By Lemma 3, it verifies the bound

$$\Theta_*(x,t) \leq K M_0 t^{\frac{2-N}{2}}.$$

Combining (6.9), (6.10) and (6.11) conclusion iii) follows and the proof is complete. □

6.5 Proof of Proposition 7

In this section, we shall use some very basic estimate for the time derivative $\partial_t u_\varepsilon$, namely

$$\frac{1}{|\log \varepsilon|} \int_{\mathbb{R}^N \times (0,T]} \left| \frac{\partial u_\varepsilon}{\partial t} \right|^2 \leq M_0 T, \qquad \text{for every } T > 0.$$

Therefore, extracting possibly a further subsequence, we may assume that there exists some non negative Radon measure ω_* defined on $\mathbb{R}^N \times (0, +\infty)$ such that

$$\frac{1}{|\log \varepsilon|} \left| \frac{\partial u_\varepsilon}{\partial t} \right|^2 \to \omega_* \qquad \text{as measures,}$$

so that

$$\omega_*(\mathbb{R}^N \times (0, T]) \leq M_0 T.$$

Since we already know that $\Sigma_\mu \subset \mathbb{R}^N \times (0, T_f + 1)$, where T_f is the constant in Proposition 3 (after which vorticity has been wiped out), we restrict our attention to this portion of space-time. Next, we introduce some subsets of $\mathbb{R}^N \times (0, T_f + 1)$, which are concentration sets for the time derivative. Set, for $l \in \mathbb{N}_*$ and $q \in \mathbb{R}+_*$ to be fixed later,

$$A_l(\omega_*) = \left\{ (x,t) \in \mathbb{R}^N \times (0, T_f + 1), \ \limsup_{r \to 0} \frac{1}{r^q} \int_{B(x,lr) \times [t-r^2, t]} \omega_* \geq 1 \right\}.$$

Concentration sets for bounded measures are classical in the litterature, see e.g. [54]. In a context similar to ours, they have been used in [41] in a related way. The following shows that $A_l(\omega_*)$ is small in some appropriate sense.

Lemma 6.3. *For each $l \in \mathbb{N}_*$,*

$$\mathcal{H}_P^q(A_l(\omega_*)) < +\infty,$$

where $\mathcal{H}_P^q$ denotes the q-dimensional Hausdorff measure with respect to the parabolic distance $d_P((x,t),(x',t')) = \max(|x - x'|, |t - t'|^{\frac{1}{2}})$.

Proof. Let $\delta > 0$ be given, and fixed for the moment. For $(x,t) \in A_l(\omega_*)$, there exist $r = r(x,t) < \delta$ such that

$$\int_{B(x,lr) \times [t-r^2,t]} \omega_* \geq r^q. \tag{6.12}$$

Clearly, $\cup_{(x,t) \in A_l(\omega_*)} \Gamma_l^P(x,t,r(x,t))$ covers $A_l(\omega_*)$, where we have set

$$\Gamma_l^P(x,t,r(x,t)) = B(x, lr(x,t)) \times (t - r(x,t)^2, t).$$

Notice that $\operatorname{diam}(\Gamma_l^P) \leq lr$. By [23] 2.8.9, we may apply the Besicovitch covering theorem. In particular, there exists an integer $m(l,N)$ depending only on N and l, and there exists a sub-covering of the form

$$A_l(\omega_*) \subset \bigcup_{i=1}^{m(l,N)} \left(\bigcup_{j \in J_i^\delta} \Gamma_l^P(x_j, t_j, r(x_j, t_j)) \right),$$

where for fixed i, the sets $\Gamma_j = \Gamma_l^P(x_j, t_j, r(x_j, t_j))$ are disjoints. Consequently, it follows from (6.12) that for each $i = 1, ..., m(l, N)$,

$$\sum_{j\in J_i^\delta} r(x_j, t_j)^q \le \sum_{j\in J_i^\delta} \int_{\Gamma_j} \omega_* \le \int_{\mathbb{R}^N\times(0,T_f+1]} \omega_* \le C(M_0).$$

Therefore,

$$\sum_{i=1}^{m(l,N)} \sum_{j\in J_i^\delta} \operatorname{diam}(\Gamma_j)^q \le m(l, N) l^q C(M_0).$$

Note that the constant on the r.h.s is independent of δ. Hence, letting $\delta \to 0$, we obtain

$$\mathcal{H}_P^q(A_l(\omega_*)) \le \limsup_{\delta\to 0} \left(\sum_{i=1}^{m(l,N)} \sum_{j\in J_i^\delta} \operatorname{diam}(\Gamma_j)^q \right) \le m(l, N) l^q C(M_0),$$

and the proof is complete. □

We fix $q = N - \frac{3}{2}$. This choice has no specific geometrical meaning, but is convenient as the following shows.

Corollary 6.1. *We have,*

$$\mathcal{H}^{N-1}(\cup_{l\in\mathbb{N}_*} A_l(\omega_*)) = 0. \tag{6.13}$$

Hence, for almost every $t > 0$

$$\mathcal{H}^{N-2}(\cup_{l\in\mathbb{N}_*} A_l^t(\omega_*)) = 0, \tag{6.14}$$

where $A_l^t(\omega_*) = A_l(\omega_*) \cap \mathbb{R}^N \times \{t\}$.

Proof. Since, by the previous lemma, $\mathcal{H}_P^{N-\frac{3}{2}}(A_l(\omega_*)) < +\infty$, if follows that

$$\mathcal{H}_P^{N-1}(A_l(\omega_*)) = 0.$$

On the other hand, parabolic balls are smaller than euclidian balls of the same radius, so that the parabolic Hausdorff measure dominates the euclidian Hausdorff measure. It follows that

$$\mathcal{H}^{N-1}(\cup_{l\in\mathbb{N}_*} A_l(\omega_*)) = 0,$$

and the proof is complete. □

Next, we introduce the set

$$\Omega_\omega = \left(\mathbb{R}^N \times (0, T_f + 1)\right) \setminus \bigcup_{n\in\mathbb{N}_*} A_l(\omega_*).$$

Lemma 6.4. *Let $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$. Then, for $(x, t) \in \Omega_\omega$,*

$$\lim_{r\to 0} \left(\frac{1}{r^{N-2}} \int_{\mathbb{R}^N} \chi(\frac{y-x}{r})\, d\mu_*^t(y) - \frac{1}{r^{N-2}} \int_{\mathbb{R}^N} \chi(\frac{y-x}{r})\, d\mu_*^{t-r^2}(y) \right) = 0.$$

Proof. We need to go back first to the level of the functions u_ε. For $0 < r < \sqrt{t}$, by Lemma 2.1 we have

$$\int_{\mathbb{R}^N\times\{t\}} \frac{e_\varepsilon(u_\varepsilon)}{|\log\varepsilon|}\chi(\frac{y-x}{r})\,dx - \int_{\mathbb{R}^N\times\{t-r^2\}} \frac{e_\varepsilon(u_\varepsilon)}{|\log\varepsilon|}\chi(\frac{y-x}{r})\,dx$$
$$= -\int_{\mathbb{R}^N\times[t-r^2,t]} \frac{|\partial_t u_\varepsilon|^2}{|\log\varepsilon|}\chi(\frac{y-x}{r})\,dxdt - \frac{1}{r|\log\varepsilon|}\int_{\mathbb{R}^N\times[t-r^2,t]} \partial_t u_\varepsilon \nabla u_\varepsilon\cdot\nabla\chi(\frac{y-x}{r})\,dxdt.$$

Let $l \in \mathbb{N}_*$ such that $\mathrm{supp}(\chi) \subset B(l)$. We set $\Lambda = B(x, lr) \times [t - r^2, t]$, and estimate the last term in the previous identity by the Cauchy-Schwarz inequality,

$$\frac{1}{r|\log\varepsilon|}\left|\int_\Lambda \partial_t u_\varepsilon\nabla u_\varepsilon\cdot\nabla\chi(\frac{y-x}{r})\right| \le \left(\int_\Lambda \frac{|\partial_t u_\varepsilon|^2}{|\log\varepsilon|}\right)^{\frac12}\left(\int_\Lambda \frac{|\nabla u_\varepsilon|^2}{r^2|\log\varepsilon|}\right)^{\frac12}\|\nabla\chi\|_\infty.$$

We now let ε go to zero, therefore obtaining the inequality for measures

$$\left|\frac{1}{r^{N-2}}\int_{\mathbb{R}^N}\chi(\frac{y-x}{r})\,(d\mu_*^t - d\mu_*^{t-r^2})(y)\right|$$
$$\le [\frac{1}{r^{N-2}}\int_\Lambda \omega_* + \left(\frac{1}{r^{N-2}}\int_\Lambda\omega_*\right)^{\frac12}\left(\frac{1}{r^N}\int_\Lambda d\mu_*\right)^{\frac12}]\,\|\chi\|_\infty. \quad (6.15)$$

Obviously, we have

$$\frac{1}{r^{N-2}}\int_\Lambda \omega_* \le r^{\frac12}\left(\frac{1}{r^{N-\frac32}}\int_\Lambda\omega_*\right).$$

On the other hand, it follows from the monotonicity that

$$\frac{1}{r^N}\int_\Lambda d\mu_* \le C(l)t^{\frac{2-N}{2}}M_0.$$

Therefore, the right hand side of (6.15) can be bounded by

$$\mathcal{R}(r) = C(t,l,M_0)\|\chi\|_{C^1}r^{\frac14}\left[1 + \frac{1}{r^{N-\frac32}}\int_\Lambda\omega_*\right].$$

Since by assumption $(x,t) \in \Omega_\omega$, letting r go to zero, we obtain

$$\lim_{r\to0}\mathcal{R}(r) \le 2C(t,l,M_0)\|\chi\|_{C^1}\lim_{r\to0}r^{\frac14} = 0,$$

and the proof is complete. □

In Lemma 6.4, we have assumed that χ has compact support. The following shows that the result still holds for $\chi = \exp(-\frac{|x|^2}{4})$, which is of special interest in view of the monotonicity!

Corollary 6.2. *We have, for $(x,t)\in\Omega_\omega$,*

$$\lim_{r\to 0}\left(\frac{1}{r^{N-2}}\int_{\mathbb{R}^N}\exp(-\frac{|x-y|^2}{4r^2})\,d\mu_*^t(y)-\frac{1}{r^{N-2}}\int_{\mathbb{R}^N}\exp(-\frac{|x-y|^2}{4r^2})\,d\mu_*^{t-r^2}(y)\right)=0. \tag{6.16}$$

In particular, for $(x,t)\in\Sigma_\mu\cap\Omega_\omega$, the following limit exists and verifies the inequality

$$\lim_{r\to 0}\frac{1}{r^{N-2}}\int_{\mathbb{R}^N}\exp(-\frac{|x-y|^2}{4r^2})\,d\mu_*^t(y)\geq \eta_3(t). \tag{6.17}$$

Proof. Let ζ be a smooth cut-off function such that $0\leq\zeta\leq 1$, $\zeta\equiv 1$ on $B(1)$ and $\zeta\equiv 0$ outside $B(2)$. For $l\in\mathbb{N}_*$, consider the function ζ_l defined by $\zeta_l(y)=\zeta(\frac{y}{l})$, and set

$$\chi_l(y)=\exp(-\frac{|y|^2}{4})\zeta_l(y)\qquad\text{for } y\in\mathbb{R}^N.$$

We apply Lemma 6.4 to χ_l, so that

$$\lim_{r\to 0}\left(\frac{1}{r^{N-2}}\int_{\mathbb{R}^N}\chi_l(\frac{y-x}{r})\,d\mu_*^t(y)-\frac{1}{r^{N-2}}\int_{\mathbb{R}^N}\chi_l(\frac{y-x}{r})\,d\mu_*^{t-r^2}(y)\right)=0. \tag{6.18}$$

On the other hand, we claim that, for every $s\in[t-r^2,t]$,

$$\frac{1}{r^{N-2}}\int_{\mathbb{R}^N}\left[\exp(-\frac{|x-y|^2}{4r^2})-\chi_l(\frac{y-x}{r})\right]\,d\mu_*^s(y)\leq Ks^{\frac{2-N}{2}}M_0\exp(-\frac{l^2}{8}). \tag{6.19}$$

Indeed, notice first that

$$\begin{aligned}\exp(-\frac{|x-y|^2}{4r^2})-\chi_l(\frac{y-x}{r})\exp(-\frac{|x-y|^2}{4r^2})(1-\zeta_l(\frac{y-x}{r}))\\ \leq\exp(-\frac{|x-y|^2}{8r^2})\exp(-\frac{l^2}{8}).\end{aligned}$$

Secondly, by the monotonicity formula

$$\frac{1}{(\sqrt{2}r)^{N-2}}\int_{\mathbb{R}^N}\exp(-\frac{|x-y|^2}{8r^2})\,d\mu_*^s(y)\leq\frac{1}{(s+2r^2)^{\frac{N-2}{2}}}M_0,$$

and the claim follows.

Note the the r.h.s. of (6.19) does not depend on r, for $r<\frac{1}{2}\sqrt{t}$. Combining (6.18) and (6.17) we are led to

$$\limsup_{r\to 0}\left|\frac{1}{r^{N-2}}\int_{\mathbb{R}^N}\exp(-\frac{|x-y|^2}{4r^2})\,(d\mu_*^t-d\mu_*^{t-r^2})(y)\right|\leq Kt^{\frac{2-N}{2}}M_0\exp(-\frac{l^2}{8}).$$

Since l was arbitrary the conclusion follows. ☐

We are now in position to present the proof of Proposition 7.

Proof of Proposition 7. For $(x,t) \in \Omega_\omega$, set

$$\tilde{\Theta}_{N-2}(\mu_*^t, x) = \lim_{r\to 0} \frac{1}{r^{N-2}} \int_{\mathbb{R}^N} \exp(-\frac{|x-y|^2}{4r^2})\, d\mu_*^t(y).$$

In view of Corollary 6.2, $\tilde{\Theta}_{N-2}(\mu_*^t, x)$ exists on Ω_ω and

$$\tilde{\Theta}(\mu_*^t, x) = \Theta_{N-2}^P(\mu_*, (x,t)). \tag{6.20}$$

If $(x,t) \notin \Sigma_\mu$, then $\Theta_{N-2}^P(\mu_*, (x,t)) = 0$ so that (5) is obviously verified. Therefore, we assume in the sequel that $(x,t) \in \Sigma_\mu \cap \Omega_\omega$. Arguing as for the claim in Corollary 6.2, we obtain

$$\frac{1}{(lr)^{N-2}} \int_{B(x,rl)} d\mu_*^t \geq \frac{K}{l^{N-2}} \frac{1}{r^{N-2}} \int_{\mathbb{R}^N} \exp(-\frac{|x-y|^2}{4r^2})\, d\mu_*^t - K \exp(-\frac{l^2}{8}) t^{\frac{2-N}{2}} M_0.$$

Hence, letting r go to zero, and by (6.20),

$$\Theta_{*,N-2}(\mu_*^t, x) \geq \frac{K}{l^{N-2}} \left(\Theta_{N-2}^P(\mu_*, (x,t)) - K l^{N-2} \exp(-\frac{l^2}{8}) t^{\frac{2-N}{2}} M_0 \right). \tag{6.21}$$

In order to obtain (5), we invoke the fact that on Σ_μ, $\Theta_{N-2}^P \geq \eta_3(t)$, and therefore we choose l sufficiently large so that

$$K l^{N-2} \exp(-\frac{l^2}{8}) t^{\frac{2-N}{2}} M_0 \leq \frac{1}{2}\eta_3(t) \leq \frac{1}{2}\Theta_{N-2}^P(\mu_*, (x,t)).$$

Going back to (6.21), with this choice of l, we obtain

$$\Theta_{*,N-2}(\mu_*^t, x) \geq \frac{K}{2l^{N-2}} \Theta_{N-2}^P(\mu_*, (x,t)),$$

and the proof is complete. □

6.6 Proof of Proposition 8

We turn finally to the proof of Proposition 8. Once more, the starting point is Corollary 6.2. Let $(x,t) \in \Omega_\omega$ be given and fixed throughout. We consider the vector-space

$$F = \left\{ g \in L^\infty(\mathbb{R}^+, \mathbb{R}) \text{ s.t. } I(g) = \lim_{r\to 0} I_r(g) \text{ exists and is finite} \right\},$$

where for $r > 0$, $I_r(g) = \frac{1}{r^{N-2}} \int_{\mathbb{R}^N} g(\frac{|x-y|}{r})\, d\mu_*^t(y)$. Notice that I_r and I are linear forms on F. With this notation, the statement of Proposition 8 is precisely that the caracteristic function $1_{[0,1]}$ of the interval $[0,1]$ belongs to F. In order to establish that fact, we derive first some basic properties of F.

Lemma 6.5. *i) For every $s > 0$, the function e_s defined on $\mathbb{R}^+$ by $e_s(l) = \exp(-l^2 s)$ belongs to F.*

ii) Set $A(s) = I(e_s)$, then we have the identity

$$A(s) = A(1)s^{\frac{2-N}{2}}. \tag{6.22}$$

Proof. The case $s = \frac{1}{4}$ follows immediatly from Corollary 6.2. For the general case, we argue by scaling. Indeed, we have for any $s > 0$,

$$\begin{aligned} I(e_{\frac{1}{4}}) &= \lim_{r\to 0} \frac{1}{r^{N-2}} \int_{\mathbb{R}^N} \exp(-\frac{|x-y|^2}{4r^2})\, d\mu_*^t(y) \\ &= \lim_{r\to 0} (\frac{2\sqrt{s}}{r})^{N-2} \int_{\mathbb{R}^N} \exp(-\frac{|x-y|^2}{r^2}s)\, d\mu_*^t(y) \\ &= (4s)^{\frac{N-2}{2}} \lim_{r\to 0} \frac{1}{r^{N-2}} \int_{\mathbb{R}^N} e_s(\frac{|x-y|}{r})\, d\mu_*^t(y) \\ &= (4s)^{\frac{N-2}{2}} I(e_s), \end{aligned}$$

so that $I(e_s)$ exists and $I(e_s) = (4s)^{\frac{2-N}{2}} I(e_{\frac{1}{4}})$.

Statement ii) then follows from the previous relation. □

Remark 6.1. The argument above shows more generally that if g belongs to F, the the scaled function g_s defined by $g_s(l) = g(\sqrt{s}\, l)$ belongs also to F.

Lemma 6.6. *For every $k \in \mathbb{N}$, the function $l \mapsto l^{2k} \exp(-l^2)$ belongs to F.*

Proof. The case $k = 0$ follows from Lemma 6.5, with $s = 1$. We provide first a detailed proof for the case $k = 1$. First note that by (6.22) A is smooth on $\mathbb{R}_*^+$. We are going to prove that for $s > 0$,

$$A'(s) = \lim_{r\to 0} \frac{1}{r^{N-2}} \int_{\mathbb{R}^N} \frac{\partial}{\partial s} e_s(l) d\mu_*^t(y) = \lim_{r\to 0} -\frac{1}{r^{N-2}} \int_{\mathbb{R}^N} \frac{|x-y|^2}{r^2} \exp(-\frac{|x-y|^2}{r^2}s) d\mu_*^t(y), \tag{6.23}$$

and in particular that the limit in the r.h.s do exists.

Let $s > 0$ and $\Delta s \in \mathbb{R}$ so that $s + \Delta s > 0$. We have, for $l \in \mathbb{R}^+$,

$$e_{s+\Delta s}(l) - e_s(l) = -\exp(-l^2 s)(1 - \exp(-l^2 \Delta s)),$$

and by Taylor expansion, for any $k \in \mathbb{R}^+$,

$$|1 - \exp(-k) - k| \leq \frac{k^2}{2}.$$

Hence, for any $y \in \mathbb{R}^N$, we have (choosing $k = \frac{|x-y|^2}{r^2}\Delta s$)

$$\begin{aligned} \left| (e_{s+\Delta s} - e_s)(\frac{|x-y|}{r}) + \exp(-\frac{|x-y|^2}{r^2}s)\frac{|x-y|^2}{r^2}\Delta s \right| &\leq \exp(-\frac{|x-y|^2}{r^2}s)\frac{|x-y|^4}{r^4}(\Delta s)^2 \\ &\leq C(s)\exp(-\frac{|x-y|^2}{2r^2}s)(\Delta s)^2. \end{aligned}$$

Integrating against the measure μ_*^t on $\mathbb{R}^N$, we are led to

$$\frac{1}{r^{N-2}}\int_{\mathbb{R}^N}\left[\frac{e_{s+\Delta s}-e_s}{\Delta s}(\frac{|x-y|}{r})+\exp(-\frac{|x-y|^2}{r^2}s)\frac{|x-y|^2}{r^2}\right]d\mu_*^t$$
$$\leq K\frac{\Delta s}{r^{N-2}}\int_{\mathbb{R}^N}\exp(-\frac{|x-y|^2}{2r^2}s)d\mu_*^t\leq C(s)M_0\Delta s.$$

Note that the r.h.s side does not depend on r, therefore letting $\Delta s\to 0$ identity (6.23) follows. Applying (6.23) with $s=1$, we deduce that the function $l\mapsto l^2\exp(-l^2)$ belongs to F. A similar computation shows that

$$\frac{d^k}{ds^k}A(s)==\lim_{r\to 0}\frac{(-1)^k}{r^{N-2}}\int_{\mathbb{R}^N}\frac{|x-y|^{2k}}{r^{2k}}\exp(-\frac{|x-y|^2}{r^2}s)d\mu_*^t(y),\tag{6.24}$$

so that the function $l\mapsto l^{2k}\exp(-l^2)$ belongs to F. □

Lemma 6.7. *The set*

$$W=\left\{g\in\mathcal{C}_c^2(\mathbb{R}^+)\ s.t.\ g'(0)=0\right\}$$

is included in F.

Proof. For a function g defined on $\mathbb{R}^+$, we consider its extension $\tilde g$ to $\mathbb{R}$ so that $\tilde g$ is even. In particular, g belongs to W if and only if $\tilde g$ belongs to $\mathcal{C}_c^2(\mathbb{R})$.

Next, for $k\in\mathbb{N}$, we consider the subset V_k of $L^2(\mathbb{R})$ defined by

$$V_k=\text{Vect}\left\{l\mapsto l^{2j}\exp(-l^2),\ j\in\{0,...,k\}\right\}.$$

In view of Lemma 6.6 the restriction of elements of V_k to $\mathbb{R}^+$ belongs to F. We are going to show that elements of W can be suitably approximated by elements of V_k (as $k\to+\infty$), so that the conclusion will follow. For that purpose, we recall some well-known facts concerning Hermite polynomials, and which enter directly in our argument.

Hermite polynomials For $n\in\mathbb{N}$, the Hermite polynomials H_n can be expressed by Rodrigues' formula

$$H_m(l)=(-1)^m\exp(l^2)\frac{d^m}{dl^m}\exp(-l^2).$$

The degree of H_m is exactly m, and H_m is even if m is even, odd if m is odd. Set, for $l\in\mathbb{R}$,

$$\psi_m(l)=c_mH_m(l)\exp(-\frac{l^2}{2}),\qquad\text{where } c_m=(\sqrt{\pi}2^mn!)^{-\frac{1}{2}}.$$

The function ψ_m verify the first order differential relations

$$(l-\frac{d}{dl})\psi_m=\sqrt{2(m+1)}\psi_{m+1},\qquad(l+\frac{d}{dl})\psi_m=\sqrt{2m}\psi_{m-1},$$

so that for $m \geq 0$,

$$\sqrt{2}\frac{d}{dl}\psi_m = \sqrt{m}\psi_{m-1} - \sqrt{m+1}\psi_{m+1}, \tag{6.25}$$

and for $m \geq 0$,

$$-\frac{d^2}{dl^2}\psi_m + l^2\psi_m = 2(m+1)\psi_m \tag{6.26}$$

(i.e. the ψ_m are eigenfunctions of the harmonic oscillator). Moreover, the family $\{\psi_m\}_{m\in\mathbb{N}}$ is a Hilbert basis of $L^2(\mathbb{R})$. For $f \in L^2(\mathbb{R})$, set $c_m(f) = \langle f, \psi_m\rangle_{L^2(\mathbb{R})}$. If f belongs to $\mathcal{C}^2_c(\mathbb{R})$, then we have, by (6.26),

$$c_m(f) = \frac{1}{2(m+1)}\langle -\frac{d^2}{dl^2}\psi_m + l^2\psi_m, f\rangle = \frac{1}{2(m+1)}\langle \psi_m, -\frac{d^2}{dl^2}f + l^2 f\rangle, \tag{6.27}$$

and by (6.25), for $m \geq 1$

$$c_m(\frac{df}{dl}) = \langle \psi_m, \frac{df}{dl}\rangle = \langle -\frac{d\psi_m}{dl}, f\rangle = \sqrt{\frac{m+1}{2}}c_{m+1}(f) - \sqrt{\frac{m}{2}}c_{m-1}(f). \tag{6.28}$$

Let P_m be the orthogonal projection (for the L^2-scalar product) onto the space $W_m = \mathrm{vect}_{0\leq j\leq m}\{\psi_m\}$. For $f \in \mathcal{C}^2_c(\mathbb{R})$, we have by the Bessel-Parseval identity and (6.27)

$$\begin{aligned} \|f - P_m f\|^2_{L^2} = \sum_{j\geq m+1} c_j^2(f) &\leq \frac{1}{4(m+1)^2}\sum_{j\geq m+1} c_j^2(-f'' + l^2 f) \\ &\leq \frac{1}{4(m+1)^2}\left[\|f''\|^2_{L^2} + \|l^2 f\|^2_{L^2}\right]. \end{aligned} \tag{6.29}$$

Since, by (6.25), we have

$$\begin{aligned} \frac{d}{dl}(f - P_m(f)) &= \sum_{j\geq m+1} c_j(f)\frac{d\psi_m}{dl}\sqrt{2}\sum_{j\geq m+1} c_j(f)\left[\sqrt{j}\psi_{j-1} - \sqrt{j+1}\psi_{j+1}\right] \\ &= \sqrt{2}\sum_{j\geq m}\left[\sqrt{j+1}c_{j+1}(f) - \sqrt{j}c_{j-1}(f)\right]\psi_j - \sqrt{m}c_{m-1}(f)\psi_m. \end{aligned}$$

We deduce similarily that

$$\|\frac{d}{dl}(f - P_m(f))\|^2_{L^2} \leq K\sum_{j\geq m} jc_j^2(f) \leq \frac{K}{m}\left[\|f''\|_{L^2} + \|l^2 f\|_{L^2}\right], \tag{6.30}$$

and finally combining (6.29) and (6.30)

$$\|f - P_m(f)\|_\infty \leq K\|f - P_m(f)\|_{H^1} \leq \frac{K}{\sqrt{m}}\left[\|f''\|_{L^2} + \|l^2 f\|_{L^2}\right]. \tag{6.31}$$

Proof of Lemma 6.7 completed. Let $g \in W$ be given, and consider the function f defined for $l \in \mathbb{R}$ by $f(l) = \tilde{g}(l)\exp(\frac{l^2}{2})$, so that $f \in \mathcal{C}^2_c(\mathbb{R})$ and is even. For $m \in \mathbb{N}_*$, set

$h_m = f - P_m(f)$ and $g_m = P_m(f) e_{\frac{1}{2}} = P_m(f) \exp(-\frac{l^2}{2})$. Since f is even, $P_m(f)$ is even also, and consequently g_n is even, of the form $g_m(l) = Q_m(l) \exp(-l^2)$, where Q_m is an even polynomial of degree less or equal to m. In view of Lemma 6.6, the restriction of g_m to $\mathbb{R}^+$ belongs to F. Since $g = g_m + h_m e_{\frac{1}{2}}$, we may write for $0 < r < 1$

$$I_r(g) = I_r(g_m) + I_r(h_m e_{\frac{1}{2}}). \tag{6.32}$$

By (6.31), we have

$$\|h_m\|_\infty \leq \frac{C(g)}{\sqrt{m}},$$

where $C(g)$ is independent of m, so that for $0 < r < 1$

$$\left| I_r(h_m e_{\frac{1}{2}}) \right| \leq \frac{C(g)}{\sqrt{m}}. \tag{6.33}$$

On the other hand, since g_m belongs to F, $I_r(g_m) \to I(g_m)$, for all $m \in \mathbb{N}$. We claim that the sequence $\{I(g_m)\}_{m \in \mathbb{N}}$ converges as $m \to +\infty$. Indeed, for $k \geq m$, we have by (6.32) and (6.33), for $0 < r < 1$,

$$|I_r(g_k) - I_r(g_m)| \leq \frac{C(g)}{\sqrt{m}}.$$

Letting $r \to 0$, we deduce that

$$|I(g_k) - I(g_m)| \leq \frac{C(g)}{\sqrt{m}}$$

so that $(I(g_m))_{m \in \mathbb{N}}$ is a Cauchy sequence and hence converges to a limit L. We finally prove that $I_r(g) \to L$ as $r \to 0$. Indeed, let $\delta > 0$ be given. In view of (6.33) we may choose m_0 such that $|I_r(h_{m_0} e_{\frac{1}{2}})| \leq \frac{\delta}{4}$ for $0 < r < 1$, and $|I(g_{m_0}) - L| \leq \frac{\delta}{4}$. Going back to (6.32) we have therefore, for $0 < r < 1$,

$$|I_r(g) - L| \leq |I_r(g_{m_0}) - I(g_{m_0})| + \frac{\delta}{2}.$$

Choosing $r_0 > 0$ such that for $0 < r < r_0$, $|I_r(g_{m_0}) - I(g_{m_0})| \leq \frac{\delta}{2}$, we deduce that for $0 < r < r_0$,

$$|I_r(g) - L| \leq \frac{\delta}{2}.$$

Since δ was arbitrary, it follows that $I_r(g)$ converges to L as $r \to 0$, and hence g belongs to F. $\square$

Proof of Proposition 8. In view of the above discussion, we only need to prove that the caracteristic function $1_{[0,1]}$ of the interval $[0,1]$ belongs to F. Let $(g_n)_{n \in \mathbb{N}}$ be an increasing sequence of functions defined on $\mathbb{R}^+$ verifying

$$g_n \in \mathcal{C}_c^2(\mathbb{R}^+), \qquad g_n'(0) = 0, \qquad g_n \leq 1_{[0,1]}, \quad \text{and} \quad \check{g}_n \geq 1_{[0,1]},$$

where $\breve{g}_n(l) = \frac{n+1}{n} g_n(\frac{nl}{n+1})$. Note that by Lemma 6.7, $g_n \in F$ and $\breve{g}_n \in F$ for all $n \in \mathbb{N}$. Let

$$L = \lim_{n \to +\infty} I(g_n) = \sup_{n \in \mathbb{N}} I(g_n). \tag{6.34}$$

By Remark 6.1, we also have

$$\lim_{n \to +\infty} I(\breve{g}_n) = \lim_{n \to +\infty} \frac{n+1}{n} (\frac{n}{n+1})^{\frac{2-N}{2}} I(g_n) = L. \tag{6.35}$$

Finally, since $g_n \leq 1_{[0,1]} \leq \breve{g}_n$, for each $0 < r < 1$ and $n \in \mathbb{N}$ we have

$$I_r(g_n) \leq I_r(1_{[0,1]}) \leq I_r(\breve{g}_n). \tag{6.36}$$

Combining (6.34), (6.35) and (6.36) we obtain $L = \lim_{r \to 0} I_r(1_{[0,1]})$, and the proof is complete. □

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