Convergence of the parabolic Ginzburg-Landau equation to motion by mean curvature

F. Bethuel, G. Orlandi and D. Smets

Abstract

For the complex parabolic Ginzburg-Landau equation, we prove that, asymptotically, vorticity evolves according to motion by mean curvature in Brakke’s weak formulation. The only assumption is a natural energy bound on the initial data. In some cases, we also prove convergence to enhanced motion in the sense of Ilmanen.

Introduction

In this paper we study the asymptotic analysis, as the parameter \( \varepsilon \) goes to zero, of the complex-valued parabolic Ginzburg-Landau equation for functions \( u_\varepsilon : \mathbb{R}^N \times \mathbb{R}^+ \rightarrow \mathbb{C} \) in space dimension \( N \geq 3 \),

\[
(PGL)_\varepsilon \quad \begin{cases} 
\frac{\partial u_\varepsilon}{\partial t} - \Delta u_\varepsilon = \frac{1}{\varepsilon^2} u_\varepsilon (1 - |u_\varepsilon|^2) & \text{on } \mathbb{R}^N \times (0, +\infty), \\
u_\varepsilon(x, 0) = u_\varepsilon^0(x) & \text{for } x \in \mathbb{R}^N.
\end{cases}
\]

It corresponds to the heat-flow for the Ginzburg-Landau energy

\[
\mathcal{E}_\varepsilon(u) = \int_{\mathbb{R}^N} e_\varepsilon(u) = \int_{\mathbb{R}^N} \frac{|
abla u|^2}{2} + V_\varepsilon(u) \quad \text{for } u : \mathbb{R}^N \rightarrow \mathbb{C},
\]

where \( V_\varepsilon \) denotes the non-convex potential

\[
V_\varepsilon(u) = \frac{(1 - |u|^2)^2}{4\varepsilon^2}.
\]

This energy plays an important role in physics, and has been studied extensively from the mathematical point of view in the last decades. It is well known that \( (PGL)_\varepsilon \) is well-posed for initial datas in \( H^1_{\text{loc}} \) with finite Ginzburg-Landau energy \( \mathcal{E}_\varepsilon(u_\varepsilon^0) \). Moreover, we have the energy identity

\[
\mathcal{E}_\varepsilon(u_\varepsilon(\cdot, T_2)) + \int_{T_1}^{T_2} \int_{\mathbb{R}^N} \left( \frac{\partial u_\varepsilon}{\partial t} \right)^2(x, t) \, dx \, dt = \mathcal{E}_\varepsilon(u_\varepsilon(\cdot, T_1)) \quad \forall 0 \leq T_1 \leq T_2,
\]

(1)
We assume that the initial condition \( u_0^\varepsilon \) verifies the bound, natural in this context,

\[
(H_0) \quad \mathcal{E}_\varepsilon(u_0^\varepsilon) \leq M_0 \log \varepsilon,
\]

where \( M_0 \) is a fixed positive constant. Therefore, in view of (I) we have

\[
\mathcal{E}_\varepsilon(u_\varepsilon(\cdot, T)) \leq \mathcal{E}_\varepsilon(u_0^\varepsilon) \leq M_0 \log \varepsilon \quad \text{for all } T \geq 0. \tag{II}
\]

The main emphasis of this paper is placed on the asymptotic limits of the Radon measures \( \mu_\varepsilon \) defined on \( \mathbb{R}^N \times [0, +\infty) \) by

\[
\mu_\varepsilon(x, t) = \frac{e_\varepsilon(u_\varepsilon(x, t))}{|\log \varepsilon|} \, dx \, dt,
\]

and of their time slices \( \mu_\varepsilon^t \) defined on \( \mathbb{R}^N \times \{t\} \) by

\[
\mu_\varepsilon^t(x) = \frac{e_\varepsilon(u_\varepsilon(x, t))}{|\log \varepsilon|} \, dx,
\]

so that \( \mu_\varepsilon = \mu_\varepsilon^t \, dt \). In view of assumption \((H_0)\) and (II), we may assume, up to a subsequence \( \varepsilon_n \to 0 \), that there exists a Radon measure \( \mu_* \) defined on \( \mathbb{R}^N \times [0, +\infty) \) such that

\[
\mu_\varepsilon \rightharpoonup \mu_* \quad \text{as measures.}
\]

Actually, passing possibly to a further subsequence, we may also assume (see Lemma 1) that

\[
\mu_\varepsilon^t \rightharpoonup \mu_*^t \quad \text{as measures on } \mathbb{R}^N \times \{t\}, \text{ for all } t \geq 0.
\]

Our main results describe the properties of the measures \( \mu_*^t \). We first have:

**Theorem A.** There exist a subset \( \Sigma_\mu \) in \( \mathbb{R}^N \times (0, +\infty) \), and a smooth real-valued function \( \Phi_* \) defined on \( \mathbb{R}^N \times (0, +\infty) \) such that the following properties hold.

i) \( \Sigma_\mu \) is closed in \( \mathbb{R}^N \times (0, +\infty) \) and for any compact subset \( \mathcal{K} \subset \mathbb{R}^N \times (0, +\infty) \setminus \Sigma_\mu \)

\[
|u_\varepsilon(x, t)| \to 1 \quad \text{uniformly on } \mathcal{K} \text{ as } \varepsilon \to 0.
\]

ii) For any \( t > 0 \), \( \Sigma_*^t \equiv \Sigma_\mu \cap \mathbb{R}^N \times \{t\} \) verifies

\[
\mathcal{H}^{N-2}(\Sigma_*^t) \leq KM_0.
\]

iii) The function \( \Phi_* \) verifies the heat equation on \( \mathbb{R}^N \times (0, +\infty) \).

iv) For each \( t > 0 \), the measure \( \mu_*^t \) can be exactly decomposed as

\[
\mu_*^t = |\nabla \Phi_*|^2 \mathcal{H}^N + \Theta_*(x, t) \mathcal{H}^{N-2} \mathbf{1}_{\Sigma_*^t}, \tag{III}
\]

where \( \Theta_*(\cdot, t) \) is a bounded function.
v) There exists a positive function \( \eta \) defined on \( \mathbb{R}^+ \) such that, for almost every \( t > 0 \), the set \( \Sigma^t_\mu \) is \((N-2)\)-rectifiable and

\[
\Theta_\ast(x, t) = \Theta_{N-2}(\mu^t_\ast, x) = \lim_{r \to 0} \frac{\mu^t_\ast(B(x, r))}{\omega_{N-2}r^{N-2}} \geq \eta(t),
\]

for \( \mathcal{H}^{N-2} \) a.e. \( x \in \Sigma^t_\mu \).

\textbf{Remark 1.} Theorem A remains valid also for \( N = 2 \). In that case \( \Sigma^t_\mu \) is therefore a finite set.

In view of the decomposition (III), \( \mu^t_\ast \) can be split into two parts. A diffuse part \( |\nabla \Phi_\ast|^2 \), and a concentrated part

\[
\nu^t_\ast = \Theta_\ast(x, t) \mathcal{H}^{N-2} \cdot \Sigma^t_\mu.
\]

By iii), the diffuse part is governed by the heat equation. Our next theorem focuses on the evolution of the concentrated part \( \nu^t_\ast \) as time varies.

\textbf{Theorem B.} The family \( (\nu^t_\ast)_{t > 0} \) is a mean curvature flow in the sense of Brakke [13].

\textbf{Comment.} We recall that there exists a classical notion of mean curvature flow for smooth compact embedded manifolds. In this case, the motion corresponds basically to the gradient flow for the area functional. It is well known that such a flow exists for small times (and is unique), but develops singularities in finite time. Asymptotic behavior (for convex bodies) and formation of singularities have been extensively studied in particular by Huisken (see [27, 28] and the references therein). Brakke [13] introduced a weak formulation which allows to encompass singularities and makes sense for (rectifiable) measures. Whereas it allows to handle a large class of objects, an important and essential flaw of Brakke’s formulation is that there is never uniqueness. Even though non-uniqueness is presumably an intrinsic property of mean curvature flow when singularities appear, a major part of non-uniqueness in Brakke’s formulation is non intrinsic, and therefore allows for weird solutions. A stronger notion of solution will be discussed in Theorem D.

More precise definitions of the above concepts will be provided in the introduction of Part II.

The proof of Theorem B relies both on the measure theoretic analysis of Ambrosio and Soner [4], and on the analysis of the structure of \( \mu_\ast \), in particular the statements in Theorem A. In [4], Ambrosio and Soner proved the result in Theorem B under the additional assumption

\[
(AS) \quad \limsup_{r \to 0} \frac{\mu^t_\ast(B(x, r))}{\omega_{N-2}r^{N-2}} \geq \eta, \quad \text{for } \mu^t_\ast\text{-a.e } x,
\]

for some constant \( \eta > 0 \). In view of the decomposition (III), assumption \((AS)\) holds if and only if \( |\nabla \Phi_\ast|^2 \) vanishes, i.e. there is no diffuse energy. If \( |\nabla \Phi_\ast|^2 \) vanishes, it
follows therefore that Theorem B can be directly deduced from [4] Theorem 5.1 and statements iv) and v) in Theorem A.

In the general case where \(|\nabla \Phi_\varepsilon|^2\) does not vanish, their argument has to be adapted, however without major changes. Indeed, one of the important consequences of our analysis is that the concentrated and diffuse energies do not interfere.

In view of the previous discussion, one may wonder if some conditions on the initial data will guarantee that there is no diffuse part. In this direction, we introduce the conditions

\[(H_1) \quad u^0_\varepsilon \equiv 1 \quad \text{in } \mathbb{R}^N \setminus B(R_1)\]

for some \(R_1 > 0\), and

\[(H_2) \quad \left\| u^0_\varepsilon \right\|_{H^{1/2}(B(R_1))} \leq M_2.\]

**Theorem C.** Assume that \(u^0_\varepsilon\) verifies \((H_0)\), \((H_1)\) and \((H_2)\). Then \(|\nabla \Phi_\varepsilon|^2 = 0\), and the family \((\mu^\varepsilon_\tau)_{\tau \geq 0}\) is a mean curvature flow in the sense of Brakke.

In stating conditions \((H_1)\) and \((H_2)\) we have not tried to be exhaustive, and there are many ways to generalize them.

We now come back to the already mentioned difficulty related to Brakke’s weak formulation, namely the strong non-uniqueness. To overcome this difficulty, Ilmanen [31] introduced the stronger notion of enhanced motion, which applies to a slightly smaller class of objects, but has much better uniqueness properties (see [31]). In this direction we prove the following.

**Theorem D.** Let \(\mathcal{M}_0\) be any given integer multiplicity \((N-2)\)-current without boundary, with bounded support and finite mass. There exists a sequence \((u^0_\varepsilon)_{\varepsilon > 0}\) and an integer multiplicity \((N-1)\)-current \(\mathcal{M}\) in \(\mathbb{R}^N \times [0, +\infty)\) such that

i) \(\partial \mathcal{M} = \mathcal{M}_0\),

ii) \(\mu^0_\varepsilon = \pi |\mathcal{M}_0|\),

and such that the pair \((\mathcal{M}, \frac{1}{\varepsilon} \mu^0_\varepsilon)\) is an enhanced motion in the sense of Ilmanen [31].

**Remark 2.** Our result is actually a little stronger than the statement of Theorem D. Indeed, we will show that any sequence \(u^0_\varepsilon\) satisfying \(Ju^0_\varepsilon \to \pi \mathcal{M}_0\) and \(\mu^0_\varepsilon = \pi |\mathcal{M}_0|\) gives rise to an Ilmanen motion. \([Ju^0_\varepsilon\) denotes the Jacobian of \(u^0_\varepsilon\) (see the introduction of Part II).]

The equation \((\text{PGL})_\varepsilon\) has already been considered in recent years. In particular, the dynamics of vortices has been described in the two dimensional case (see [32, 36]). Concerning higher dimensions \(N \geq 3\), under the assumption that the initial measure is concentrated on a smooth manifold, a conclusion similar to ours has been obtained first on a formal level by Pismen and Rubinstein [44], and then rigorously by Jerrard and
Soner [33] and Lin [37], in the time interval where the classical solution exists, that is only before the appearance of singularities. As already mentioned, a first convergence result past the singularities has been obtained by Ambrosio and Soner [4], under the crucial density assumption (AS) for the measures $\mu^\varepsilon$ discussed above. Some important asymptotic properties for solutions of $(\text{PGL})_\varepsilon$ were also considered in [40, 53, 9].

Beside these works, we had at least two important sources of inspiration in our study. The first one was the corresponding theory for the elliptic case, developed in the last decade, in particular in [7, 51, 11, 46, 38, 39, 8, 34, 12, 10]. The second one was the corresponding theory for the scalar case (i.e. the Allen-Cahn equation) developed in particular in [17, 21, 18, 22, 30, 49]. The outline of our paper bears some voluntary resemblance with the work of Ilmanen [30] (and Brakke [13]) : to stress this analogy, we will try to adopt their terminology as far as this is possible. In particular, the Clearing-Out Lemma is a step-stone in the proofs of Theorems A to D.

We divide the paper into two distinct parts. The first and longest one deals with the analysis of the functions $u_\varepsilon$, for fixed $\varepsilon$. This part involves mainly PDE techniques. The second part is devoted to the analysis of the limiting measures, and borrows some arguments of Geometric Measure Theory. The last step of the argument there will be taken directly from Ambrosio and Soner’s work [4]. The transition between the two parts is realized through delicate pointwise energy bounds which allow to translate a clearing-out lemma for functions into one for measures.

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Part I
PDE Analysis of \((PGL)_\varepsilon\).
Introduction

In this part, we derive a number of properties of solutions $u_\varepsilon$ of $(PGL)_\varepsilon$, which enter directly in the proof of the Clearing-Out Lemma (the proof of which will be completed at the beginning of Part II). We believe however that the techniques and results in this part have also an independent interest. Throughout this part, we will assume that $0 < \varepsilon < 1$. Unless explicitly stated, all the results here also hold in the two dimensional case $N = 2$. In our analysis, the sets

$$
\mathcal{V}_\varepsilon = \left\{(x,t) \in \mathbb{R}^N \times (0, +\infty), \quad |u_\varepsilon(x,t)| \leq \frac{1}{2}\right\},
$$

as well as their time slices $\mathcal{V}_\varepsilon^t = \mathcal{V}_\varepsilon \cap (\mathbb{R}^N \times \{t\})$ will play a central role. We will loosely refer to $\mathcal{V}_\varepsilon$ as the vorticity set. [In the scalar case, such a set is often referred to as the “interfaces” or “jump set”].

The two main ingredients in the proof of the Clearing-Out are a clearing-out theorem for vorticity, as well as some precise pointwise (renormalized) energy bounds.

1 Clearing-out and annihilation for vorticity

The main result here is the following.

**Theorem 1.** Let $0 < \varepsilon < 1$, $u_\varepsilon$ be a solution of $(PGL)_\varepsilon$ with $\mathcal{E}_\varepsilon(u_\varepsilon^0) < +\infty$, and $\sigma > 0$ be given. There exists $\eta_\varepsilon = \eta_\varepsilon(\sigma) > 0$ depending only on the dimension $N$ and on $\sigma$ such that if

$$
\int_{\mathbb{R}^N} e_\varepsilon(u_\varepsilon^0) \exp\left(-\frac{|x|^2}{4}\right) \leq \eta_\varepsilon |\log \varepsilon|,
$$

then

$$
|u_\varepsilon(0,1)| \geq 1 - \sigma.
$$

Note that here we do not need assumption $(H_0)$. This kind of result was obtained for $N = 3$ in [40], and for $N = 4$ in [53]. The corresponding result for the stationary case was established in [11, 51, 46, 38, 39, 8]. The restrictions on the dimension in [40, 53] seem essentially due to the fact that the term $\frac{\partial u}{\partial t}$ in $(PGL)_\varepsilon$ is treated there as a perturbation of the elliptic equation. Instead, our approach will be more parabolic in nature. Finally, let us mention that a result similar to Theorem 1 also holds in the scalar case, and enters in Ilmanen’s framework (see [30] page 436) : the proof there is fairly direct and elementary.

Our (rather lengthy) proof of Theorem 1 involves a number of tools, some of which were already used in a similar context. In particular :

- A monotonicity formula which in our case was derived first by Struwe ([51], see also [19]), in its study of the heat-flow for harmonic maps. Similar monotonicity formulas were derived by Huisken [28] for the mean curvature flow, and Ilmanen [30] for the Allen-Cahn equation.
• A localization property for the energy (see Proposition 2.4) following a result of Lin and Rivière [40] (see also [37]).
• Refined Jacobian estimates due to Jerrard and Soner [34],
and many of the techniques and ideas that were introduced for the stationary equation.

Equation (PGL)$_{\epsilon}$ has standard scaling properties. If $u_{\epsilon}$ is a solution to (PGL)$_{\epsilon}$, then for $R > 0$ the function

$$v_{\epsilon}(x, t) \equiv u_{\epsilon}(Rx, R^2t)$$

is a solution to (PGL)$_{R^{-1}\epsilon}$. We may then apply Theorem 1 to $v_{\epsilon}$. For this purpose, define, for $z_\star = (x_\star, t_\star) \in \mathbb{R}^N \times (0, +\infty)$ the scaled weighted energy, taken at time $t = t_\star$,

$$\hat{\mathcal{E}}_{w, \epsilon}(u_{\epsilon}, z_\star, R) \equiv \hat{\mathcal{E}}_{w, \epsilon}(z_\star, R) = \frac{1}{R^{N-2}} \int_{\mathbb{R}^N} e_{\epsilon}(u_{\epsilon}(x, t_\star)) \exp\left(-\frac{|x - x_\star|^2}{4R^2}\right) dx.$$ 

We have the following

**Proposition 1.** Let $T > 0$, $x_T \in \mathbb{R}^N$, and set $z_T = (x_T, T)$. Assume $u_{\epsilon}$ is a solution to (PGL)$_{\epsilon}$ on $\mathbb{R}^N \times [0, T]$ and let $R > \sqrt{2\epsilon}$. Assume moreover

$$\hat{\mathcal{E}}_{w, \epsilon}(z_T, R) \leq \eta_1(\sigma) \log \epsilon,$$  \hspace{1cm} (3)

then

$$|u_{\epsilon}(x_T, T + R^2)| \geq 1 - \sigma.$$ \hspace{1cm} (4)

The condition in (3) involves an integral on the whole of $\mathbb{R}^N$. In some situations, it will be convenient to integrate on finite domains. From this point of view, assuming (H$_0$) we have the following (in the spirit of Brakke’s original Clearing-Out [13], Lemma 6.3, but for vorticity here, not yet for the energy!).

**Proposition 2.** Let $u_{\epsilon}$ be a solution of (PGL)$_{\epsilon}$ verifying assumption (H$_0$) and $\sigma > 0$ be given. Let $x_T \in \mathbb{R}^N$, $T > 0$ and $R \geq \sqrt{2\epsilon}$. There exists a positive continuous function $\lambda$ defined on $\mathbb{R}^N_+$ such that, if

$$\tilde{\eta}(x_T, T, R) \equiv \frac{1}{R^{N-2}\log \epsilon} \int_{B(x_T, \lambda(T)R)} e_{\epsilon}(u_{\epsilon}(\cdot, T)) \leq \frac{\eta_1(\sigma)}{2}\leq \frac{\eta_1(\sigma)}{2},$$

then

$$|u_{\epsilon}(x, t)| \geq 1 - \sigma \quad \text{for } t \in [T + T_0, T + T_1] \quad \text{and } x \in B(x_T, R^2).$$

Here $T_0$ and $T_1$ are defined by

$$T_0 = \max(2\epsilon, \left(\frac{2\tilde{\eta}}{\eta_1(\sigma)}\right)^{\frac{N}{2-N}}R^2), \quad T_1 = R^2.$$
Remark 3. It follows from the proof that $\lambda(T)$ diverge as $T \to 0$. More precisely,
\[
\lambda(T) \sim \sqrt{\frac{N - 2}{2} \log T}
\quad \text{as } T \to 0,
\]
if $N \geq 3$. A slightly improved version will be proved and used in Section 4.1.

Theorem 1 and Propositions 1 and 2 have many consequences. Some are of independent interest. For instance, the simplest one is the complete annihilation of vorticity for $N \geq 3$.

Proposition 3. Assume that $N \geq 3$. Let $u_\varepsilon$ be a solution of $(PGL)_\varepsilon$ verifying assumption $(H_0)$. Then
\[
|u_\varepsilon(x, t)| \geq \frac{1}{2} \quad \text{for any } t \geq T_f \equiv \left( \frac{M_0}{\eta_{\varepsilon}} \right)^{\frac{2}{N-2}} \quad \text{and for all } x \in \mathbb{R}^N,
\]
(5)
where $\eta = \eta(\frac{1}{2})$.

In particular, there exists a function $\varphi$ defined on $\mathbb{R}^N \times [T_0, +\infty)$ such that
\[
uarepsilon = \rho \exp(i\varphi), \quad \rho = |u_\varepsilon|.
\]
The equation for the phase $\varphi$ is then the linear parabolic equation
\[
\rho \frac{\partial \varphi}{\partial t} - \text{div}(\rho^2 \nabla \varphi) = 0.
\]
(6)
From this equation (and the equation for $\rho$) one may prove that, for fixed $\varepsilon$,
\[
E_\varepsilon(u_\varepsilon(\cdot, t)) \to 0 \quad \text{as } t \to +\infty,
\]
(7)
and moreover,
\[
u_\varepsilon(\cdot, t) \to C \quad \text{as } t \to +\infty.
\]
(8)

Remark 4. The result of Proposition 3 does not hold in dimension 2. This fact is related to the so-called “slow motion of vortices” as established in [36]: vortices essentially move with a speed of order $|\log \varepsilon|^{-1}$. Therefore, a time of order $|\log \varepsilon|$ is necessary to annihilate vorticity (compared with the time $T = O(1)$ in Proposition 3). On the other hand, long-time estimates, similar to (7) and (8) have been established, for $N = 2$, in [5].
2 Improved pointwise energy bounds

Assume for a moment that \(|u_\varepsilon| = 1\) on \(\mathbb{R}^N \times [0, +\infty)\) [and in particular \(V_\varepsilon = \emptyset\)]. Then, we may write \(u_\varepsilon = \exp(i\varphi_\varepsilon)\) and \(\varphi_\varepsilon\) is determined, up to an integer multiple of \(2\pi\), by the linear parabolic problem

\[
\begin{align*}
\frac{\partial \varphi_\varepsilon}{\partial t} - \Delta \varphi &= 0 & \text{on } \mathbb{R}^N \times (0, +\infty), \\
\varphi(x, 0) &= \varphi_\varepsilon(x, 0) & \text{on } \mathbb{R}^N \times \{0\}.
\end{align*}
\] (9)

By standard regularization properties of the heat equation, we deduce that for any compact \(K \subset \mathbb{R}^N \times (0, \infty)\),

\[
|\nabla \varphi_\varepsilon|^2_{L^\infty(K)} \leq C(K) \int_{\mathbb{R}^N} \frac{|\nabla \varphi_\varepsilon|^2}{2}(x, 0) \, dx = C(K)E_\varepsilon(u_\varepsilon^0),
\]

so that

\[
\lim_{r \to 0} \frac{1}{r^N} \int_{B(x, r) \times \{t\}} \frac{|\nabla \varphi_\varepsilon|^2}{|\log \varepsilon|^2} \leq M_0 C(t), \quad \forall x \in \mathbb{R}^N, \quad \forall t > 0.
\]

In particular, going back to the discussion of the main introduction of this paper, it means that the measures \(\mu_\varepsilon^*\) are absolutely continuous with respect to the Lebesgue measure \(\mathcal{L}^N(\mathbb{R}^N)\), i.e. \(\mu_\varepsilon^* = g(x, t)\mathcal{H}^N\) for some diffuse density \(g\). Since (9) is linear, one cannot expect that \(g\) vanishes without additional assumptions, for instance compactness assumptions on the initial data \(u_\varepsilon^0\) (see [15] for related remarks in the elliptic case).

In the general situation, it is of course impossible to impose \(|u_\varepsilon| = 1\). However, on the complement of \(V_\varepsilon\), \(|u_\varepsilon| \geq \frac{1}{2}\) and the situation is quite similar. More precisely, we have

**Theorem 2.** Let \(B(x_0, R)\) be a ball in \(\mathbb{R}^N\) and \(T > 0, \Delta T > 0\) be given. Consider the cylinder

\[
\Lambda = B(x_0, R) \times [T, T + \Delta T].
\]

There exists a constant \(0 < \sigma \leq \frac{1}{2}\), and \(\beta > 0\) depending only on \(N\), such that if

\[
|u_\varepsilon| \geq 1 - \sigma \quad \text{on } \Lambda,
\] (10)

then

\[
\epsilon_\varepsilon(u_\varepsilon)(x, t) \leq C(\Lambda) \left[ \int_{\Lambda} \epsilon_\varepsilon(u_\varepsilon) + M_0 \varepsilon^\beta \right],
\] (11)

for any \((x, t) \in \Lambda_{\frac{1}{2}} = B(x_0, \frac{R}{2}) \times [T + \frac{\Delta T}{4}, T + \Delta T]\). Moreover,

\[
\epsilon_\varepsilon(u_\varepsilon) = |\nabla \Phi_\varepsilon|^2 + \kappa_\varepsilon \quad \text{in } \Lambda_{\frac{1}{2}},
\] (12)

where the functions \(\Phi_\varepsilon\) and \(\kappa_\varepsilon\) are defined on \(\Lambda_{\frac{1}{2}}\) and verify

\[
\frac{\partial \Phi_\varepsilon}{\partial t} - \Delta \Phi_\varepsilon = 0 \quad \text{in } \Lambda_{\frac{1}{2}},
\] (13)

\[
||\kappa_\varepsilon||_{L^\infty(\Lambda_{\frac{1}{2}})} \leq C(\Lambda)\varepsilon^\beta, \quad ||\nabla \Phi_\varepsilon||^2_{L^\infty(\Lambda_{\frac{1}{2}})} \leq C(\Lambda)M_0|\log \varepsilon|.
\] (14)
Remark 5. Since \( |u_\varepsilon| \geq \frac{1}{2} \) on \( \Lambda \), we may write \( u_\varepsilon = \rho_\varepsilon \exp(i\varphi_\varepsilon) \) where \( \rho_\varepsilon = |u_\varepsilon| \) and \( \varphi_\varepsilon \) is a smooth real-valued function. The proof of Theorem 2 shows actually that
\[
\|\nabla \varphi_\varepsilon - \nabla \Phi_\varepsilon\|_{L^\infty(\Lambda_2^\varepsilon)} \leq C(\Lambda)\varepsilon^\beta.
\] (15)

The result of Theorem 2 is reminiscent of a result by Chen and Struwe [19] (see also [51],[33]) developed in the context of the heat flow for harmonic maps. This technique is based on an earlier idea of Schoen [47] developed in the elliptic case. Note however that a smallness assumption on the energy is needed there. This is not the case for Theorem 2, where even a divergence of the energy (as \( |\log \varepsilon| \) is allowed. We would like also to emphasize that the proofs of Theorem 1 and 2 are completely disconnected.

Combining Theorem 1 and Theorem 2, we obtain the following immediate consequence.

Proposition 4. There exist an absolute constant \( \eta_2 > 0 \) and a positive function \( \lambda \) defined on \( \mathbb{R}^+ \) such that if, for \( x \in \mathbb{R}^N \), \( t > 0 \) and \( r > \sqrt{2\varepsilon} \), we have
\[
\int_{B(x,\lambda(t)r)} e_\varepsilon(u_\varepsilon) \leq \eta_2 r^{N-2}|\log \varepsilon|,
\]
then
\[
e_\varepsilon(u_\varepsilon) = |\nabla \Phi_\varepsilon|^2 + \kappa_\varepsilon
\]
in \( \Lambda_1(x,t,r) \equiv B(x,\frac{r}{4}) \times [t + \frac{10}{16}r^2, t + r^2] \), where \( \Phi_\varepsilon \) and \( \kappa_\varepsilon \) are as in Theorem 2. In particular,
\[
\mu_\varepsilon = \frac{e_\varepsilon(u_\varepsilon)}{|\log \varepsilon|} \leq C(t,r) \quad \text{on} \quad \Lambda_1(x,t,r).
\]

[The constant \( \eta_2 \) is actually defined as \( \eta_2 = \eta_h(\sigma) \), where \( \sigma \) is the constant in Theorem 2 and \( \eta_1 \) is the function defined in Proposition 2]

3 Identifying sources of non compactness

In the previous discussion, we already identified one possible source of non compactness, namely oscillations in the phase. However, the analysis was carried out on the complement of \( \mathcal{V}_\varepsilon \), i.e. away from vorticity. On the vorticity set on the other hand, \( u_\varepsilon \) may vanish, and this introduces some new contribution to the energy. Nevertheless, we will show that this new contribution is not a source of non compactness (at least for some weaker norm). More precisely,

Theorem 3. Let \( \mathcal{K} \subset \mathbb{R}^N \times (0, +\infty) \) be any compact set. There exist a real-valued function \( \phi_\varepsilon \) and a complex-valued function \( w_\varepsilon \), both defined on a neighborhood of \( \mathcal{K} \), such that
1. \( u_\varepsilon = w_\varepsilon \exp(i\phi_\varepsilon) \) on \( \mathcal{K} \),
2. \( \phi_\varepsilon \) verifies the heat equation on \( \mathcal{K} \),
3. \(|\nabla \phi_\varepsilon(x,t)| \leq C(\mathcal{K}) \sqrt{M_0 \log \varepsilon}\) for all \((x,t) \in \mathcal{K}\),

4. \(\|\nabla w_\varepsilon\|_{L^p(\mathcal{K})} \leq C(p,\mathcal{K})\), for any \(1 \leq p < \frac{N+1}{N}\).

Here, \(C(\mathcal{K})\) and \(C(p,\mathcal{K})\) are constants depending only on \(\mathcal{K}\), and \(\mathcal{K}\), \(p\) (and \(M_0\)) respectively.

The proof extends an argument of [9] (see also [6] for the elliptic case), and relies once more on the refined Jacobian estimates of [34].

We would like to emphasize once more that Theorem 3 provides an exact splitting of the energy in two different modes

- The topological mode, i.e. the energy related to \(w_\varepsilon\)
- The linear mode, i.e. the energy of \(\phi_\varepsilon\).

More precisely, it follows easily from Theorem 3 that for any set \(\mathcal{K}' \subset \subset \mathcal{K}\), we have

\[
\int_{\mathcal{K}'} e_\varepsilon(u_\varepsilon) = \int_{\mathcal{K}'} e_\varepsilon(w_\varepsilon) + \int_{\mathcal{K}'} |\nabla \phi_\varepsilon|^2 + O(\sqrt{\log \varepsilon}).
\]

We would also like to stress also that a new and important feature of Theorem 3 is that \(\phi_\varepsilon\) is defined and smooth even across the singular set, and verifies globally (on \(\mathcal{K}\)) the heat flow. Going back to Theorem A, this fact will be determinant to define the function \(\Phi\), globally. For Theorem B, it will allow us to prove that the linear mode does not perturb the topological mode, which undergoes its own (Brakke) motion.

One possible way to remove the linear mode is to impose additional compactness on the initial data. We will not try to find the most general assumptions in that direction, but instead give simple conditions which keep however the essential features of the problem. Assume next that \(u_0^0\) verifies the additional conditions

\[(H_1) \quad u_\varepsilon^0 \equiv 1 \quad \text{in} \ \mathbb{R}^N \setminus B(R_1)\]

for some \(R_1 > 0\), and

\[(H_2) \quad \|u_\varepsilon^0\|_{H^\frac{N}{N+1}(B(R_1))} \leq M_2.\]

Then a stronger conclusion holds.

**Theorem 4.** Assume that \(u_\varepsilon^0\) verifies \((H_0)\), \((H_1)\) and \((H_2)\). Then for any \(1 \leq p < \frac{N+1}{N}\) and any compact set \(\mathcal{K} \subset \mathbb{R}^N \times (0, +\infty)\), we have

\[\|\nabla u_\varepsilon\|_{L^p(\mathcal{K})} \leq C(p, \mathcal{K}),\]

where \(C(p, \mathcal{K})\) is a constant depending only on \(p\), \(\mathcal{K}\), \(M_0\) and \(M_2\).
Theorem 4 is of course of particular interest if one is interested in the asymptotic behavior of the function $u_\varepsilon$ itself. We will not carry out this analysis here (see [9] for a related discussion for boundary value problems on compact domains).

Combining Theorem 1, Theorem 2 and Theorem 4 we finally derive the following, in the same spirit as Proposition 4.

**Proposition 5.** Assume that $(H_0), (H_1)$ and $(H_2)$ hold. There exist an absolute constant $\eta_2 > 0$ and a positive function $\lambda$ defined on $\mathbb{R}^+_\varepsilon$ such that if, for $x \in \mathbb{R}^N$, $t > 0$ and $r > \sqrt{\varepsilon}$, we have

\[
\int_{B(x, \varepsilon^2 t)} e_\varepsilon(u_\varepsilon) \leq \eta_2 r^{N-2} \log \varepsilon, \tag{16}
\]

then

\[
e_\varepsilon(u_\varepsilon) \leq C(M_0, M_2) r^{-2}
\]

in $\Lambda_{\frac{1}{2}}(x, t, r) \equiv B(x, \frac{\varepsilon^2}{2}) \times \left[t + \frac{64}{63} r^2, t + r^2\right]$.

[Here $\eta_2 = \eta(\sigma)$ is the same constant as in Proposition 4].

## 1 Pointwise estimates

In this section we recall (standard) pointwise parabolic estimates. Although these estimates are presumably well known to the experts, we are not aware of precise statements in the (Ginzburg-Landau) literature. For the reader’s convenience, we therefore provide complete proofs.

**Proposition 1.1.** Let $u_\varepsilon$ be a solution of \((PGL)_\varepsilon\) with $E_\varepsilon(u_\varepsilon^0) < +\infty$. Then there exists a constant $K > 0$ depending only on $N$ such that, for $t \geq \varepsilon^2$ and $x \in \mathbb{R}^N$, we have

\[
|u_\varepsilon(x, t)| \leq 3, \quad |\nabla u_\varepsilon(x, t)| \leq \frac{K}{\varepsilon}, \quad |\frac{\partial u_\varepsilon}{\partial t}(x, t)| \leq \frac{K}{\varepsilon^2}.
\]

[Note in particular that $K$ is independent of the initial data].

**Proof.** It is convenient to make the following change of variable, setting

\[
U(x, t) = u_\varepsilon(x, \varepsilon^2 t),
\]

so that the function $U$ verifies

\[
\frac{\partial U}{\partial t} - \Delta U = U(1 - |U|^2) \quad \text{on} \mathbb{R}^N \times [0, +\infty). \tag{1.1}
\]

It is therefore sufficient to prove that for $t \geq 1$ and $x \in \mathbb{R}^N$,

\[
|U(x, t)| \leq K, \quad |\nabla U(x, t)| \leq K, \quad |\frac{\partial U}{\partial t}(x, t)| \leq K.
\]
We begin with the $L^\infty$ estimate for $U$. Set
\[ \sigma(x, t) := |U(x, t)|^2 - 1. \]
Multiplying equation (1.1) by $U$ we are led to the equation for $\sigma$,
\[ \frac{\partial \sigma}{\partial t} - \Delta \sigma + 2|\nabla U|^2 + 2(\sigma + 1)\sigma = 0. \tag{1.2} \]
Consider next the ODE
\[ y'(t) + 2(y(t) + 1)y(t) = 0, \tag{1.3} \]
and notice that (1.3) possesses an explicit solution $y_0$ which blows-up as $t$ tends to zero, namely
\[ y_0(t) := \frac{\exp(-t/2)}{1 - \exp(-t/2)} \quad \text{for } t > 0. \]
We claim that
\[ \sigma(t, x) \leq y_0(t), \quad \text{for all } t > 0 \text{ and } x \in \mathbb{R}^N, \tag{1.4} \]
so that, in particular,
\[ |U(x, t)|^2 = \sigma(x, t) + 1 \leq 9 \quad \text{for all } t \geq \frac{1}{4} \text{ and } x \in \mathbb{R}^N. \]
Indeed, set $\tilde{\sigma}(x, t) = y_0(t)$. Then,
\[ \frac{\partial \tilde{\sigma}}{\partial t} - \Delta \tilde{\sigma} + 2(\tilde{\sigma} + 1)\tilde{\sigma} = 0, \]
and therefore by (1.2),
\[ \frac{\partial}{\partial t}(\tilde{\sigma} - \sigma) - \Delta(\tilde{\sigma} - \sigma) + 2(\tilde{\sigma} - \sigma)(1 + \tilde{\sigma} + \sigma) \geq 0. \]
Note that $1 + \sigma + \tilde{\sigma} = |U|^2 + \tilde{\sigma} \geq 0$. The maximum principle implies that
\[ \tilde{\sigma}(x, t) - \sigma(x, t) \geq 0 \quad \text{for all } t > 0 \text{ and } x \in \mathbb{R}^N, \]
which proves the claim (1.4).

We next turn to the space and time derivatives. Since $|U(x, t)| \leq 3$ for $t \geq 1/4$, we have
\[ \left| U(1 - |U|^2) \right| \leq 24 \quad \text{for } t \geq \frac{1}{4}. \]
Let $p > N + 1$ be fixed. It follows from the standard regularity theory for the linear heat equation (see e.g. [35]) that for each compact set $\mathcal{F} \subset \mathbb{R}^N \times [1/4, +\infty)$ we have
\[ \|\partial_t U\|_{L^p(\mathcal{F})} \leq K(\mathcal{F}) \quad \text{and} \quad \|D^2 U\|_{L^p(\mathcal{F})} \leq K(\mathcal{F}). \]
In particular, by the Sobolev embedding and the $L^\infty$ bound for $U$ we obtain
\[
\|U\|_{C^{0,\alpha}(\mathbb{R}^N \times [1/2, +\infty))} \leq K, \tag{1.5}
\]
where $\alpha = (1 - N/p)/2$. It follows from (1.5) that moreover
\[
\|U(1 - |U|^2)\|_{C^{0,\alpha}(\mathbb{R}^N \times [1/2, +\infty))} \leq K.
\]
Invoking the $C^{0,\alpha}$ regularity theory (see e.g. [24]), we obtain
\[
\|U\|_{C^{1,\alpha/2}(\mathbb{R}^N \times [1, +\infty))} \leq K,
\]
and the proof is complete. \hfill \Box

**Remark 1.1.** It follows from the proof of Proposition 1.1 that the bound
\[
|u_\varepsilon(x, t)|^2 \leq 1 + C \exp\left(-\frac{t}{2\varepsilon^2}\right)
\]
holds for $t \geq \varepsilon^2$.

We have the following variant of Proposition 1.1.

**Proposition 1.2.** Assume $u_\varepsilon$ is a solution of $(PGL)_\varepsilon$ such that $\mathcal{E}_\varepsilon(u_\varepsilon^0) < +\infty$, and that for some constants $C_0 \geq 1$, $C_1 \geq 0$ and $C_2 \geq 0$,
\[
|u_\varepsilon^0(x)| \leq C_0, \quad |\nabla u_\varepsilon^0(x)| \leq \frac{C_1}{\varepsilon}, \quad |D^2 u_\varepsilon^0(x)| \leq \frac{C_2}{\varepsilon^2}, \quad \forall x \in \mathbb{R}^N.
\]
Then, for any $x \in \mathbb{R}^N$ and any $t > 0$,
\[
|u_\varepsilon(x, t)| \leq C_0, \quad |\nabla u_\varepsilon(x, t)| \leq \frac{K}{\varepsilon}, \quad \left| \frac{\partial u_\varepsilon}{\partial t}(x, t) \right| \leq \frac{K}{\varepsilon^2},
\]
where $K$ depends only on $C_0$, $C_1$ and $C_2$.

**Proof.** As in the proof of Proposition 1.1, we work with the rescaled function $U$. It follows from (1.2) and the maximum principle that
\[
|U(x, t)| \leq \sup_{x \in \mathbb{R}^N} |U(0, x)| \leq C_0.
\]
It remains to prove the bounds on the space and time derivatives. Since these estimates are already known for $t \geq 1$ by Proposition 1.1, we only need to consider the case $t \in (0, 1]$. For the space derivative, we use the following Bochner type inequality
\[
\frac{\partial}{\partial t} (|\nabla U|^2) - \Delta (|\nabla U|^2) \leq K|\nabla U|^2, \tag{1.6}
\]
so that
\[
\frac{\partial}{\partial t} (\exp(Kt)|\nabla U|^2) - \Delta (\exp(Kt)|\nabla U|^2) \leq 0.
\]
The conclusion then follows from the maximum principle. For the time derivative, one argues similarly, using the inequality
\[
\frac{\partial}{\partial t}(\frac{|\partial U}{\partial t}|^2) - \Delta(\frac{|\partial U}{\partial t}|^2) \leq K|\frac{\partial U}{\partial t}|^2
\]
and the fact that, for \( t = 0 \), we have by assumption
\[
|\frac{\partial U}{\partial t}|^2 = |\Delta U + U(1 - |U|^2)|^2 \leq K.
\]
\[\square\]

Proposition 1.1 above provides an upper bound for \( |u_\varepsilon| \). Our next lemma provides a local lower bound on \( |u_\varepsilon| \), when we know it is away from zero on some region.

Since we have to deal with parabolic problems, it is natural to consider parabolic cylinders of the type
\[
\Lambda_\alpha(x_0, T, R, \Delta T) = B(x_0, \alpha R) \times [T + (1 - \alpha^2)\Delta T, T + \Delta T].
\]
Sometimes, it will be convenient to choose \( \Delta T = R \) and write \( \Lambda_\alpha(x_0, T, R) \). Finally if this is not misleading we will simply write \( \Lambda_\alpha \), and \( \Lambda \) if \( \alpha = 1 \).

**Lemma 1.1.** Let \( u_\varepsilon \) be a solution of \((PGL)_\varepsilon\), verifying \( \mathcal{E}_\varepsilon(u_\varepsilon^0) < +\infty \). Let \( x_0 \in \mathbb{R}^N \), \( R > 0 \), \( T \geq 0 \) and \( \Delta T > 0 \) be given. Assume that
\[
|u_\varepsilon| \geq \frac{1}{2} \quad \text{on} \quad \Lambda(x_0, T, R, \Delta T),
\]
then
\[
1 - |u_\varepsilon| \leq C(\alpha, \Lambda)\varepsilon^2 \left( \|\nabla \varphi_\varepsilon\|_{L^2(\Lambda)} + \log \varepsilon \right) \quad \text{on} \quad \Lambda_\alpha,
\]
where \( \varphi_\varepsilon \) is defined on \( \Lambda \), up to a multiple of \( 2\pi \), by \( u_\varepsilon = |u_\varepsilon| \exp(i\varphi_\varepsilon) \).

**Proof.** We may always assume that \( T \geq \varepsilon \), otherwise we consider a smaller cylinder. Set \( \rho = |u_\varepsilon| \) and \( \theta = 1 - \rho \). The function \( \theta \) verifies the equation
\[
\frac{\partial \theta}{\partial t} - \Delta \theta + \frac{\theta}{\varepsilon^2} = (1 - \theta)|\nabla \varphi_\varepsilon|^2 - \frac{1}{\varepsilon^2}\theta(\theta - 1)^2.
\]
On the other hand, by Proposition 1.1, we already know that \( \theta \geq -\exp(-\frac{1}{\varepsilon}) \), so that
\[
\frac{\partial \theta}{\partial t} - \Delta \theta + \frac{\theta}{\varepsilon^2} \leq 2\|\nabla \varphi_\varepsilon\|^2 + C\varepsilon^{-2}\exp(-\frac{1}{\varepsilon}). \tag{1.7}
\]
We next construct an upper solution for (1.7). Let \( \chi \) be a smooth cut-off function defined on \( \mathbb{R}^N \) such that \( 0 \leq \chi \leq 1 \) and
\[
\chi \equiv 1 \quad \text{on} \quad B(x_0, \alpha R), \quad \chi \equiv 0 \quad \text{on} \quad \mathbb{R}^N \setminus B(x_0, \frac{1 + \alpha}{2} R).
\]
Consider the function $\tau$ defined on $[T, T + \Delta T]$ by
\[
\tau(t) = \frac{1}{2} - \frac{1}{2} \exp \left( \frac{t - T}{(1 - \alpha^2) \Delta T} \log \varepsilon^2 \right),
\]
and set
\[
\sigma_0(x, t) = \frac{1}{2} - \tau(t) \chi(x).
\]
We have $\sigma_0 \geq 0$ and
\[
|\partial_{t} \sigma_0| = |\tau'(t)| \chi(x) \leq \frac{1}{(1 - \alpha^2) \Delta T} \log \varepsilon, \quad |\Delta \sigma_0| \leq \tau(t) |\Delta \chi(x)| \leq C(\Lambda),
\]
so that
\[
\frac{\partial \sigma_0}{\partial t} - \Delta \sigma_0 + \frac{\sigma_0}{\varepsilon^2} \geq -C(\Lambda) \log \varepsilon.
\]
Finally, set
\[
\sigma = \sigma_0 + 2 \varepsilon^2 \left( \|\nabla \varphi_\varepsilon\|_{L^\infty(\Lambda)}^2 + C(\Lambda) \log \varepsilon \right).
\]
By construction,
\[
\frac{\partial \sigma}{\partial t} - \Delta \sigma + \frac{\sigma}{\varepsilon^2} \geq 2 \|\nabla \varphi_\varepsilon\|_{L^\infty(\Lambda)}^2 + C(\Lambda) \log \varepsilon \geq \frac{\partial \theta}{\partial t} - \Delta \theta + \frac{\theta}{\varepsilon^2}.
\]
On the other hand,
\[
\sigma \geq \frac{1}{2} \geq \theta \quad \text{on} \quad B(x_0, R) \times \{T\} \cup \partial B(x_0, R) \times [T, T + \Delta T],
\]
so that by the maximum principle $\theta \leq \sigma$ on $\Lambda$. Since $\chi \equiv 1$ on $B(x_0, \alpha R)$, we have on $\Lambda_\alpha$
\[
\sigma(x, t) = \frac{1}{2} \exp \left( \frac{t - T}{(1 - \alpha^2) \Delta T} \log \varepsilon^2 \right) + 2 \varepsilon^2 \left( \|\nabla \varphi_\varepsilon\|_{L^\infty(\Lambda)}^2 + C(\Lambda) \log \varepsilon \right)
\leq \frac{1}{2} \varepsilon^2 \left( \|\nabla \varphi_\varepsilon\|_{L^\infty(\Lambda)}^2 + C(\Lambda) \log \varepsilon \right)
\]
and the proof is complete. \hfill \Box

2 Toolbox

The purpose of this section is to present a number of tools, which will enter directly in the proof of Theorem 1. As mentioned earlier, some of them are already available in the literature. We will adapt their statements to our needs. Note that all the results in this section remain valid for vector-valued maps $u_\varepsilon : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}^k$, for every $k \geq 1$, $u_\varepsilon$ solution to (PGL)$_\varepsilon$.  

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2.1 Evolution of localized energies

Identity (1) of the introduction states a global decrease in time of the energy. In this section, we recall some classical results, describing the behavior of localized integrals of energy.

Lemma 2.1. Let $\chi$ be a bounded lipschitz function on $\mathbb{R}^N$. Then, for any $T \geq 0$ we have

$$
\frac{d}{dt} \int_{\mathbb{R}^N \times \{T\}} e_\varepsilon(u_\varepsilon) \chi(x) \, dx = -\int_{\mathbb{R}^N \times \{T\}} |\partial_t u_\varepsilon|^2 \chi(x) \, dx - \int_{\mathbb{R}^N \times \{T\}} \partial_t u_\varepsilon \nabla u_\varepsilon \cdot \nabla \chi \, dx. \quad (2.1)
$$

In particular, for any $0 \leq T_1 \leq T_2$ we have

$$
\int_{\mathbb{R}^N \times \{T_2\}} e_\varepsilon(u_\varepsilon) \chi(x) \, dx - \int_{\mathbb{R}^N \times \{T_1\}} e_\varepsilon(u_\varepsilon) \chi(x) \, dx =
- \int_{\mathbb{R}^N \times [T_1, T_2]} |\partial_t u_\varepsilon|^2 \chi(x) \, dx \, dt - \int_{\mathbb{R}^N \times [T_1, T_2]} \partial_t u_\varepsilon \nabla u_\varepsilon \cdot \nabla \chi \, dx \, dt. \quad (2.2)
$$

Proof. We have

$$
\frac{d}{dt} \left( \frac{|\nabla u_\varepsilon|^2}{2} + V_\varepsilon(u_\varepsilon) \chi \right) = \nabla u_\varepsilon \cdot \nabla(\partial_t u_\varepsilon) \chi + V'_\varepsilon(u_\varepsilon) \partial_t u_\varepsilon \chi.
$$

Integrating by parts on $\mathbb{R}^N \times \{T\}$ we thus have

$$
\frac{d}{dt} \int_{\mathbb{R}^N \times \{T\}} e_\varepsilon(u_\varepsilon) \chi(x) \, dx = -\int_{\mathbb{R}^N \times \{T\}} (-\Delta u_\varepsilon + V'_\varepsilon(u_\varepsilon)) \partial_t u_\varepsilon \chi(x) \, dx
- \int_{\mathbb{R}^N \times \{T\}} \partial_t u_\varepsilon \nabla u_\varepsilon \cdot \nabla \chi \, dx
$$

and the conclusion follows since $u_\varepsilon$ verifies $(PGL)_\varepsilon$. \hfill \Box

As a straightforward consequence we obtain the following semi-decreasing property.

Corollary 2.1. Let $\chi$ be as above, then

$$
\frac{d}{dt} \int_{\mathbb{R}^N \times \{T\}} e_\varepsilon(u_\varepsilon) \chi^2(x) \, dx \leq 2 \| \nabla \chi \|^2_L = \mathcal{E}(u_\varepsilon^0).
$$

2.2 The monotonicity formula

Let $u \equiv u_\varepsilon$ be a solution to $(PGL)_\varepsilon$ verifying $(H_1)$. For simplicity, we will drop the subscripts $\varepsilon$ when this is not misleading. For $(x_*, t_*) \in \mathbb{R}^N \times \mathbb{R}^+$ we set

$$z_* = (x_*, t_*) .
$$

For $0 < R \leq \sqrt{t_*}$ we define the weighted energy

$$
E_w(z_*, R) \equiv E_w(u; z_*, R) \equiv \mathcal{E}_{w}(u, x_*, t_* - R^2, R) , \quad (2.3)
$$
that is,

$$E_w(z_*, R) = \int_{\mathbb{R}^N} e_x(u(x, t_*) - R^2)) \exp(-\frac{|x - x_*|^2}{4R^2}) dx,$$

and the corresponding scaled energy

$$\tilde{E}_w(z_*, R) = \frac{1}{R^{N-2}} E_w(z_*, R) = \frac{1}{R^{N-2}} \int_{\mathbb{R}^N} e_x(u(x, t_*) - R^2)) \exp(-\frac{|x - x_*|^2}{4R^2}) dx.$$

We emphasize the fact that the above integral is computed at the time $t = t_* - R^2$, and not at time $t = t_*$, as it is the case for $E_x$, i.e. a shift in time $\delta t = -R^2$ has been introduced. Note also that in (2.4) and (2.5) the weight becomes small outside the ball $B(x_*, R)$. Moreover, the following inequality holds

$$\exp\left(\frac{1}{4}\right) \tilde{E}_w(z_*, R) \geq \frac{1}{R^{N-2}} \int_{B(x_*, R)} e_x(u(x, t_*) - R^2)) dx.$$

The right-hand side of (2.6) arises naturally in the stationary equation, where its monotonicity properties (with respect to the radius $R$) play an important role. In our parabolic setting, we recall once more that the time $t$ at which $E_w$ and $\tilde{E}_w$ are computed is related to $R$ by

$$t = t_* - R^2.$$

This is consistent with the usual parabolic scaling (for $\lambda > 0$)

$$\begin{aligned}
\{ x & \rightarrow \lambda x \\
\lambda t & \rightarrow \lambda^2 t
\end{aligned},$$

which leaves the linear heat equation invariant.

In this context, the following monotonicity formula for $\tilde{E}_w$ was derived first by Struwe [50] for the heat-flow of harmonic maps (see also [19, 28]). In a different context Giga and Kohn [26] used related ideas.

**Proposition 2.1.** We have

$$\frac{d\tilde{E}_w(z_*, R)}{dR} \bigg|_{R=r} = \frac{1}{r^{N-1}} \int_{\mathbb{R}^N \times \{t_* - r^2\}} \frac{1}{2r^2} (|x - x_*| \cdot \nabla u + 2r^2 \partial_t u)^2 \exp(-\frac{|x - x_*|^2}{4r^2}) dx$$

$$+ \frac{1}{r^{N-1}} \int_{\mathbb{R}^N \times \{t_* - r^2\}} 2V_x(u) \exp(-\frac{|x - x_*|^2}{4r^2}) dx$$

$$= \frac{1}{2r} \int_{\mathbb{R}^N \times \{t_* - r^2\}} (|x - x_*| \cdot \nabla u + 2(t - t_*) \partial_t u)^2 G(x - x_*, t - t_*) dx$$

$$+ r \int_{\mathbb{R}^N \times \{t_* - r^2\}} 2V_x(u(x, t)) G(x - x_*, t - t_*) dx,$$

where $G(x, t)$ denotes, up to a multiplicative factor $\pi^{-N/2}$, the heat kernel

$$\begin{aligned}
\{ G(x, t) &= \frac{1}{(4\pi t)^N} \exp\left(-\frac{|x|^2}{4t}\right) \text{ for } t > 0 \\
G(x, t) &= 0 \text{ for } t \leq 0
\end{aligned}.$$
In particular,
\[
\frac{d\tilde{E}_w}{dR}(z_*, R) \geq 0,
\]
i.e. \(\tilde{E}_w(z_*, R)\) is a non-decreasing function of \(R\).

Proof. Set \(\tilde{E}_w(R) \equiv \tilde{E}_w(z_*, R)\). Due to translation invariance, it is sufficient to consider the case \(z_* = (x_*, t_*) = (0, 0)\), so that \(u\) is defined on \(\mathbb{R}^N \times [-t_*, +\infty)\). In order to keep the integration domain fixed with respect to \(R\), we consider the following change of variables, for \(z = (x, y) \in \mathbb{R}^N \times [-t_*, +\infty)\):
\[
z = (x, t) = (Ry, R^2 \tau) = \Phi_R(y, \tau) = \Phi_R(z').
\]
Set \(u_R(z') = u \circ \Phi_R(z') = u(z)\), i.e. \(u_R(y, \tau) = u(Ry, R^2 \tau) = u(x, t)\), so that in particular
\[
\nabla u_R(z') = R \nabla u_R(z), \quad \frac{\partial u_R}{\partial \tau}(z') = R^2 \frac{\partial u}{\partial t}(z), \quad \Delta u_R(z') = R^2 \Delta u(z).
\]
It follows that
\[
\frac{\partial u_R}{\partial \tau} - \Delta u_R = \frac{R^2}{\varepsilon^2} u_R(1 - |u_R|^2) = -R^2 V_e'(u_R).
\]
Moreover,
\[
\frac{d}{dR} u_R(z') = \frac{d}{dR} u(Ry, R^2 \tau) = y \cdot \nabla u(z) + 2R \frac{\partial u}{\partial t}(z).
\]
From (2.12) and (2.10) we deduce the formula
\[
R \frac{du_R}{dt}(z') = x \cdot \nabla u(z) + 2t \frac{\partial u}{\partial t}(z) = y \cdot \nabla u_R(z') + 2 \frac{\partial u_R}{\partial \tau}(z').
\]
The scaled energy \(\tilde{E}_w(R)\) (defined by formula (2.5)) can be then expressed as follows
\[
\tilde{E}_w(R) = \int_{\mathbb{R}^N \setminus \{-1\}} \left[ \frac{\nabla u_R(y, -1)^2}{2} + R^2 V_e(u_R(y, -1)) \right] \exp\left(-\frac{|y|^2}{4}\right) dy.
\]
Taking into account (2.9), (2.11) and (2.13), we compute
\[
\frac{d\tilde{E}_w}{dR} = \int_{\mathbb{R}^N \times \{-1\}} \left[ \nabla u_r \cdot \nabla \left( \frac{du_R}{dR} \right) + r^2 V_e'(u_r) \cdot \frac{du_R}{dR} + 2r V_e(u_r) \right] \exp\left(-\frac{|y|^2}{4}\right) dy
\]
\[
= \int_{\mathbb{R}^N \times \{-1\}} \left[ (-\Delta u_r + \frac{y}{2} \cdot \nabla u_r + r^2 V_e'(u_r)) \cdot \frac{du_R}{dR} + 2r V_e(u_r) \right] \exp\left(-\frac{|y|^2}{4}\right) dy
\]
\[
= \int_{\mathbb{R}^N \times \{-1\}} \left[ \frac{y}{2} \cdot \nabla u_r - \frac{\partial u_r}{\partial \tau} \right] \left[ \frac{1}{r} (y \cdot \nabla u_r - 2 \frac{\partial u_r}{\partial \tau}) + 2r V_e(u_r) \right] \exp\left(-\frac{|y|^2}{4}\right) dy
\]
\[
= \int_{\mathbb{R}^N \times \{-1\}} \left[ \frac{1}{2r} (y \cdot \nabla u_r - 2 \frac{\partial u_r}{\partial \tau})^2 + 2r V_e(u_r) \right] \exp\left(-\frac{|y|^2}{4}\right) dy
\]
\[
= \int_{\mathbb{R}^N \times \{-r^2\}} \left[ \frac{1}{2r} (x \cdot \nabla u - 2 \frac{\partial u}{\partial t})^2 + 2r V_e(u) \right] r^{-N} \exp\left(-\frac{|x|^2}{4r^2}\right) dx
\]
\[
= \int_{\mathbb{R}^N \times \{-r^2\}} \left[ \frac{1}{2r} (x \cdot \nabla u + 2t \frac{\partial u}{\partial t})^2 + 2r V_e(u) \right] G(x, t) dx.
\]
The last formula in the above computation gives precisely (2.7) in the particular case $z_0 = (x_*, t_*) = (0, 0)$. \hfill \Box

### 2.3 Space-time estimates and auxiliary functions

Let $u \equiv u_\varepsilon$ be a solution to (PGL)$_\varepsilon$ verifying $E_\varepsilon(u_\varepsilon^0) < +\infty$.

**Lemma 2.2.** For any $z_0 = (x_*, t_*) \in \mathbb{R}^N \times \mathbb{R}^+$, the following equality holds, for $R_* = \sqrt{t_*}$:

$$
\int_{\mathbb{R}^N \times [0, t_*)} (V_\varepsilon(u) + \Xi(u, z_0)) G(x - x_*, t - t_*) dx dt = \frac{1}{t_*} \int_{\mathbb{R}^N \times \{0\}} e_\varepsilon(u(\cdot, 0)) \exp\left(-\frac{|x - x_*|^2}{4t_*}\right) dx = \tilde{E}_w(z_0, R_*) ,
$$

(2.16)

where we have set

$$
\Xi(u, z_0)(x, t) = \frac{1}{4\sqrt{|t - t_*|}}[(x - x_*) \cdot \nabla u + 2(t - t_*) \partial_t u]^2.
$$

(2.17)

**Proof.** Integrating equality (2.7) from zero to $R_*$ we obtain

$$
\tilde{E}_w(z_0, R_*) - \tilde{E}_w(z_0, 0) = \int_0^{R_*} 2\pi r dr \int_{\mathbb{R}^N \times \{t_* - r^2\}} V_\varepsilon(u(x, t)) G(x - x_*, t - t_*) dx \nonumber
$$

$$
+ \int_0^{R_*} 2\pi r dr \int_{\mathbb{R}^N \times \{t_* - r^2\}} \frac{1}{4r^2}[(x - x_*) \cdot \nabla u - 2r^2 \partial_t u]^2 G(x - x_*, t - t_*) dx .
$$

(2.18)

Expressing the integral on the right-hand side of (2.18) in the variable $t \equiv t_* - r^2$ (so that $dt = -2\pi dr$) yields

$$
\tilde{E}_w(z_0, R_*) - \tilde{E}_w(z_0, 0) = - \int_{t_*}^0 dt \int_{\mathbb{R}^N \times \{t\}} V_\varepsilon(u(x, t)) G(x - x_*, t - t_*) dx
$$

$$
- \int_{t_*}^0 dt \int_{\mathbb{R}^N \times \{t\}} \frac{1}{4\sqrt{|t - t_*|}}[(x - x_*) \cdot \nabla u - 2r^2 \partial_t u]^2 G(x - x_*, t - t_*) dx .
$$

(2.19)

Finally, since $u$ is smooth on $\mathbb{R}^N \times (0, +\infty)$ and with finite energy on each time slice, we obtain

$$
\tilde{E}_w(z_0, 0) = 0 ,
$$

so that the proof is complete. \hfill \Box

The following elementary lemma will be useful for further purposes.

**Lemma 2.3.** Let $0 < t_* < T$, and $z_0 = (x_*, t_*)$. We have

$$
\tilde{E}_w,c(z_0, \sqrt{t_*}) \leq \left( \frac{T}{t_*} \right)^{\frac{N}{2}} \exp\left(\frac{|x_T - x_*|^2}{4(T - t_*)}\right) \tilde{E}_w,c((x_T, 0), \sqrt{T}) , \quad \forall x_T \in \mathbb{R}^N .
$$

(2.20)
Proof. By definition of $\tilde{E}_w$, we have
\[
\tilde{E}_w(z_*, \sqrt{t_*}) = \frac{1}{t_*^{N/2}} \int_{\mathbb{R}^N} e_{\varepsilon}(u(x, 0)) \exp(-\frac{|x - x_*|^2}{4t_*})dx \\
= \left(\frac{T}{t_*}\right)^{N/2} \frac{1}{T^{N/2}} \int_{\mathbb{R}^N} e_{\varepsilon}(u(x, 0)) \exp(-\frac{|x - x_T|^2}{4T})Q(x)dx,
\]
where the function $Q$ is defined on $\mathbb{R}^N$ as
\[
Q(x) = \exp\left(\frac{|x - x_T|^2}{4T} - \frac{|x - x_*|^2}{4t_*}\right) \quad \forall x \in \mathbb{R}^N.
\] (2.22)
Clearly $Q$ is positive and bounded on $\mathbb{R}^N$. Its maximum is achieved at a point $x_0 \in \mathbb{R}^N$ such that
\[
\frac{(x_0 - x_T)}{T} = \frac{(x_0 - x_*)}{t_*},
\]
so that
\[
x_0 - x_* = \frac{(x_* - x_T)}{T - t_*}t_*, \quad x_0 - x_T = \frac{(x_* - x_T)}{T - t_*}T.
\] (2.23)
Inserting (2.23) in (2.22), we are led to
\[
\sup_{x \in \mathbb{R}^N} Q(x) = Q(x_0) = \exp\left(\frac{|x_* - x_T|^2}{4(T - t_*)}\right).
\] (2.24)
Hence, combining (2.24) with (2.21) we obtain
\[
\tilde{E}_w(z_*, \sqrt{t_*}) \leq \left(\frac{T}{t_*}\right)^{N/2} \exp\left(\frac{|x_* - x_T|^2}{4(T - t_*)}\right) \int_{\mathbb{R}^N} e_{\varepsilon}(u(x, 0)) \exp(-\frac{|x - x_T|^2}{4T})dx,
\] (2.25)
and (2.20) follows. \qed

Next, let $T > 0$ be given and let $f \in L^\infty(\mathbb{R}^N \times [0, T])$ be such that
\[
|f(z)| \leq V_{\varepsilon}(|u(z)|), \quad \text{for any } z = (x, t) \in \mathbb{R}^N \times [0, T].
\] (2.26)
We consider the solution $\omega$ of the heat equation with source term $f$, i.e. $\omega$ solves
\[
\begin{cases}
\frac{\partial \omega}{\partial t} - \Delta \omega = f & \text{on } \mathbb{R}^N \times [0, T], \\
\omega(x, 0) = 0 & \text{for } x \in \mathbb{R}^N.
\end{cases}
\] (2.27)
The following $L^\infty$-estimate, which already played a key role in the elliptic setting (see [8]), will enter similarly in the proof of Theorem 1.

Lemma 2.4. For any $z_* = (x_*, t_*) \in \mathbb{R}^N \times [0, T]$, we have
\[
|\omega(z_*)| \leq \pi^{-N/2} \tilde{E}_w(z_*, \sqrt{t_*}).
\] (2.28)
Proof. The function \( \omega \) is given explicitly by Duhamel’s formula
\[
\omega(z_\ast) = \frac{\pi^{-N/2}}{\sqrt{\pi}} \left( \frac{t}{4(t-t_\ast)} \right)^{\frac{N-1}{2}} \int_{\mathbb{R}^N \times [0,t_\ast]} f(x,t) G(x-x_\ast, t-t_\ast) \, dx \, dt,
\]
so that, by (2.26),
\[
|\omega(z_\ast)| \leq \frac{\pi^{-N/2}}{\sqrt{\pi}} \left( \frac{t}{4(t-t_\ast)} \right)^{\frac{N-1}{2}} \left| V_\epsilon(u(x,t)) G(x-x_\ast, t-t_\ast) \right| \, dx \, dt,
\]
and the conclusion follows from (2.16).

Combining Lemma 2.3 and Lemma 2.4 we obtain

**Proposition 2.2.** Let \( T > 0 \), \( x_T \in \mathbb{R}^N \). For any \( z = (x,t) \in \mathbb{R}^N \times [0,T] \), the following estimate holds
\[
|\omega(z)| \leq \left( \frac{T}{t} \right)^{\frac{\gamma}{2}} \exp \left( \frac{|x_T - x|^2}{4(T-t)} \right) \mathcal{E}_{\omega,\epsilon}(x_T, 0, \sqrt{T}), \quad \forall x_T \in \mathbb{R}^N. \tag{2.29}
\]

### 2.4 Bounds for the scaled weighted energy \( \mathcal{E}_{w,\epsilon} \)

Our next lemma provides an upper bound for \( \mathcal{E}_{w,\epsilon}(z, R) \) in terms of \( \mathcal{E}_{w,\epsilon}(x_T, 0, \sqrt{T}) \) provided \( z < T \) and \( R \) is sufficiently small. More precisely, we have

**Lemma 2.5.** Let \( T > 0 \), and \( z = (x,t) \in \mathbb{R}^N \times [0,T] \). We have the inequality
\[
\mathcal{E}_{w,\epsilon}(z, R) \leq \left( \frac{T}{t+R^2} \right)^{\frac{\gamma}{2}} \exp \left( \frac{|x_T - x|^2}{4(T-t-R^2)} \right) \mathcal{E}_{w,\epsilon}(x_T, 0, \sqrt{T}), \tag{2.30}
\]
for any \( x_T \in \mathbb{R}^N \), and for \( 0 < R < \sqrt{T-t} \).

**Proof.** In view of the monotonicity formula (2.8), we have the inequality
\[
\mathcal{E}_{w,\epsilon}(z, R) = \tilde{\mathcal{E}}_w((x,t+R^2), R) \leq \tilde{\mathcal{E}}_w((x,t+R^2), \sqrt{t+R^2}). \tag{2.31}
\]
By Lemma 2.3 applied to \( z_\ast = (x,t+R^2) \), we obtain
\[
\tilde{\mathcal{E}}_w((x,t+R^2), \sqrt{t+R^2}) \leq \left( \frac{T}{t+R^2} \right)^{\frac{\gamma}{2}} \exp \left( \frac{|x_T - x|^2}{4(T-t-R^2)} \right) \mathcal{E}_{w,\epsilon}(x_T, 0, \sqrt{T}), \tag{2.32}
\]
for any \( x_T \in \mathbb{R}^N \). Combining (2.32) with (2.31) the conclusion follows.

**Comment.** Note that (2.30) holds in particular for small \( R \). It can therefore be understood as a regularizing property of \( (PGL)_\epsilon \). Indeed, starting with an arbitrary initial condition, the gradient of the solution at time \( t \) remains bounded in the Morrey space \( L^{2, N-2} \) (so that the solution itself remains bounded in BMO, locally).
2.5 Localizing the energy

In some of the proofs of the main results, it will be convenient to work on bounded domains for fixed time slices (in particular in view of the elliptic estimates needed there). On the other hand, the definition of $\mathcal{E}_w$ and $\tilde{E}_w$ involves integration on the whole space (even though the weight has an extremely fast decay at infinity). In order to overcome this difficulty, we will make use of two kinds of localization methods. The first one is a fairly elementary consequence of the monotonicity formula and can be stated as follows.

**Proposition 2.3.** Let $T > 0$, $x_T \in \mathbb{R}^N$ and $R > \sqrt{2\varepsilon}$. Assume $u_\varepsilon$ is a solution to $(PGL)_\varepsilon$ verifying $(H_0)$. Then the following inequality holds, for any $\lambda > 0$,

$$
\int_{\mathbb{R}^N \times \{T\}} e_\varepsilon(u_\varepsilon) e^{-\frac{|x-x_T|^2}{4R^2}} dx \leq \int_{B(x_T, \lambda R) \times \{T\}} e_\varepsilon(u_\varepsilon) + \left(\frac{\sqrt{2}R}{\sqrt{T+2R^2}}\right)^{N-2} M_0 \exp(-\frac{\lambda^2}{8}) |\log \varepsilon|.
$$

**Proof.** It suffices obviously to prove that

$$
\int_{\{|x-x_T| \geq \lambda R\} \times \{T\}} e_\varepsilon(u_\varepsilon) e^{-\frac{|x-x_T|^2}{4R^2}} dx \leq \left(\frac{\sqrt{2}R}{\sqrt{T+2R^2}}\right)^{N-2} M_0 \exp(-\frac{\lambda^2}{8}) |\log \varepsilon|. \tag{2.33}
$$

First, we write $\exp(-\frac{|x-x_T|^2}{4R^2}) = |\exp(-\frac{|x-x_T|^2}{4R^2})|^2$, so that on $\mathbb{R}^N \setminus B(x_T, \lambda R)$ we have

$$
\exp(-\frac{|x-x_T|^2}{4R^2}) \leq \exp(-\frac{\lambda^2}{8}) \exp(-\frac{|x-x_T|^2}{8R^2}). \tag{2.34}
$$

On the other hand, applying the monotonicity formula at the point $(x_T, T+2R^2)$, we obtain

$$
\frac{1}{(\sqrt{2}R)^{N-2}} \int_{\mathbb{R}^N \times \{T\}} e_\varepsilon(u_\varepsilon) e^{-\frac{|x-x_T|^2}{4R^2}} dx \leq \frac{1}{(\sqrt{T+2R^2})^{N-2}} \int_{\mathbb{R}^N \times \{0\}} e_\varepsilon(u_\varepsilon) e^{-\frac{|x-x_T|^2}{T+2R^2}} dx
$$

$$
\leq \frac{1}{(\sqrt{T+2R^2})^{N-2}} M_0 |\log \varepsilon|. \tag{2.35}
$$

Combining (2.33),(2.34) and (2.35) the conclusion follows. \hfill \Box

The idea of the second localization method originated in [40] and is based on a Pohozaev type inequality.

**Proposition 2.4.** Let $0 < t < T$. The following inequality holds, for any $x_T \in \mathbb{R}^N$,

$$
\int_{\mathbb{R}^N \times \{t\}} e_\varepsilon(u) \frac{|x-x_T|^2}{4(T-t)} \exp(-\frac{|x-x_T|^2}{4(T-t)}) dx \leq \frac{N}{2} \int_{\mathbb{R}^N \times \{t\}} e_\varepsilon(u) \exp(-\frac{|x-x_T|^2}{4(T-t)}) dx
$$

$$
+ \int_{\mathbb{R}^N \times \{t\}} [V_\varepsilon(u) + 3 \Xi(u, x_T)] \exp(-\frac{|x-x_T|^2}{4(T-t)}) dx. \tag{2.36}
$$

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As a consequence,

\[
\int_{\mathbb{R}^N \times \{t\}} e_\varepsilon(u) \exp\left(-\frac{|x - x_T|^2}{4(T - t)}\right) \, dx \leq \int_{B(x_T, r_T) \times \{t\}} e_\varepsilon(u) \exp\left(-\frac{|x - x_T|^2}{4(T - t)}\right) \, dx
\]

\[
+ \frac{2}{N} \int_{\mathbb{R}^N \times \{t\}} [V_\varepsilon(u) + 3 \Xi(u, z_T)] \exp\left(-\frac{|x - x_T|^2}{4(T - t)}\right) \, dx,
\]

where \( r_T = 2\sqrt{N(T - t)} \).

Note that the radius \( r_T \) of the ball \( B(x_T, r_T) \) where the first integral of the r.h.s. of (2.37) is computed is proportional to \( \sqrt{T - t} \), which is the width of the parabolic cone with vertex \( z_T = (x_T, T) \).

The proof of Proposition 2.4 relies on the following inequality.

**Lemma 2.6.** Let \( 0 < T_1 \leq T_2 < T, \ x_T \in \mathbb{R}^N, \ z_T = (x_T, T) \). We have

\[
\int_{\mathbb{R}^N \times [T_1, T_2]} e_\varepsilon(u) \frac{|x - x_T|^2}{4(T - t)} \exp\left(-\frac{|x - x_T|^2}{4(T - t)}\right) \, dx \, dt
\]

\[
\leq (T - T_1)^{N/2} \tilde{E}_\omega(z_T, \sqrt{T - T_1}) - (T - T_2)^{N/2} \tilde{E}_\omega(z_T, \sqrt{T - T_2})
\]

\[
+ \int_{\mathbb{R}^N \times [T_1, T_2]} \frac{1}{2(T - t)} [(x - x_T) \cdot \nabla u - 2(T - t) \partial_t u]^2 \exp\left(-\frac{|x - x_T|^2}{4(T - t)}\right) \, dx \, dt.
\]

**Proof.** The idea is to multiply \((\text{PGL})_\varepsilon\) by the multiplier \(2(T - t) \partial_t u \exp\left(-\frac{|x - x_T|^2}{4(T - t)}\right)\) and integrate on \( \mathbb{R}^N \times [T_1, T_2] \). One obtains, after integration by parts in the space variable,

\[
\int_{T_1}^{T_2} \int_{\mathbb{R}^N} 2(T - t) \partial_t u^2 \exp\left(-\frac{|x_T - x|^2}{4(T - t)}\right) \, dx \, dt
\]

\[
= \int_{T_1}^{T_2} \int_{\mathbb{R}^N} 2(T - t) \Delta u \partial_t u \exp\left(-\frac{|x_T - x|^2}{4(T - t)}\right) \, dx \, dt
\]

\[
- \int_{T_1}^{T_2} \int_{\mathbb{R}^N} 2(T - t) \frac{\partial}{\partial t} [V_\varepsilon(u)] \exp\left(-\frac{|x_T - x|^2}{4(T - t)}\right) \, dx \, dt
\]

\[
= - \int_{T_1}^{T_2} \int_{\mathbb{R}^N} \nabla u \cdot \left( 2(T - t) \frac{\partial}{\partial t} \nabla u - (T - t) \partial_t u \frac{x - x_T}{T - t} \right) \exp\left(-\frac{|x_T - x|^2}{4(T - t)}\right) \, dx \, dt
\]

\[
- \int_{T_1}^{T_2} \int_{\mathbb{R}^N} 2(T - t) \frac{\partial}{\partial t} [V_\varepsilon(u)] \exp\left(-\frac{|x_T - x|^2}{4(T - t)}\right) \, dx \, dt
\]

\[
= - \int_{T_1}^{T_2} \int_{\mathbb{R}^N} (T - t) \frac{\partial}{\partial t} \left[ 2V_\varepsilon(u) + 2V_\varepsilon(u) \right] \exp\left(-\frac{|x_T - x|^2}{4(T - t)}\right) \, dx \, dt
\]

\[
+ \int_{T_1}^{T_2} \int_{\mathbb{R}^N} ((x - x_T) \cdot \nabla u) \frac{\partial u}{\partial t} \exp\left(-\frac{|x_T - x|^2}{4(T - t)}\right) \, dx \, dt.
\]
Integration by parts in the time variable now yields

$$
\int_{T_1}^{T_2} \int_{\mathbb{R}^N} 2(T - t)|\partial_t u|^2 \exp\left(-\frac{|x_T - x|^2}{4(T - t)}\right) dx \, dt
$$

$$
= - \int_{T_1}^{T_2} \int_{\mathbb{R}^N} (T - t) \frac{\partial}{\partial t} \left[ |\nabla u|^2 + 2V_\varepsilon(u)\right] \exp\left(-\frac{|x_T - x|^2}{4(T - t)}\right) dx \, dt
$$

$$
- \int_{T_1}^{T_2} \int_{\mathbb{R}^N} \frac{|x_T - x|^2}{4(T - t)} \left[ |\nabla u|^2 + 2V_\varepsilon(u)\right] \exp\left(-\frac{|x_T - x|^2}{4(T - t)}\right) dx \, dt
$$

$$
+ \int_{\mathbb{R}^N \times \{T_1\}} (T - T_1) \left[ |\nabla u|^2 + 2V_\varepsilon(u)\right] \exp\left(-\frac{|x_T - x|^2}{4(T - t)}\right) dx
$$

$$
- \int_{\mathbb{R}^N \times \{T_2\}} (T - T_2) \left[ |\nabla u|^2 + 2V_\varepsilon(u)\right] \exp\left(-\frac{|x_T - x|^2}{4(T - t)}\right) dx
$$

$$
+ \int_{T_1}^{T_2} \int_{\mathbb{R}^N} (x - x_T) \cdot \nabla u \frac{\partial u}{\partial t} \exp\left(-\frac{|x_T - x|^2}{4(T - t)}\right) dx \, dt.
$$

Adding the integral

$$
\int_{T_1}^{T_2} \int_{\mathbb{R}^N} \frac{1}{2(T - t)} |(x - x_T) \cdot \nabla u|^2 \exp\left(-\frac{|x_T - x|^2}{4(T - t)}\right) dx \, dt
$$

to equation (2.40) we finally obtain

$$
\int_{T_1}^{T_2} \int_{\mathbb{R}^N} \frac{1}{2(T - t)} |(x - x_T) \cdot \nabla u - 2(T - t) \partial_t u|^2 \exp\left(-\frac{|x_T - x|^2}{4(T - t)}\right) dx \, dt
$$

$$
+ \int_{T_1}^{T_2} \int_{\mathbb{R}^N} \left( 1 + \frac{|x_T - x|^2}{4(T - t)} \right) \left[ |\nabla u|^2 + 2V_\varepsilon(u)\right] \exp\left(-\frac{|x_T - x|^2}{4(T - t)}\right) dx \, dt
$$

$$
= + \int_{\mathbb{R}^N \times \{T_1\}} (T - T_1) \left[ |\nabla u|^2 + 2V_\varepsilon(u)\right] \exp\left(-\frac{|x_T - x|^2}{4(T - t)}\right) dx
$$

$$
- \int_{\mathbb{R}^N \times \{T_2\}} (T - T_2) \left[ |\nabla u|^2 + 2V_\varepsilon(u)\right] \exp\left(-\frac{|x_T - x|^2}{4(T - t)}\right) dx
$$

$$
+ \int_{T_1}^{T_2} \int_{\mathbb{R}^N} \frac{(x - x_T) \cdot \nabla u}{2(T - t)} [(x - x_T) \cdot \nabla u] \exp\left(-\frac{|x_T - x|^2}{4(T - t)}\right) dx \, dt.
$$

We bound the last term in (2.41), using the inequality $ab \leq \frac{a^2}{4} + b^2$, with the choice

$$
a = \frac{(x - x_T) \cdot \nabla u}{\sqrt{2(T - t)}} \exp\left(-\frac{|x_T - x|^2}{8(T - t)}\right)
$$

and

$$
b = \frac{[(x - x_T) \cdot \nabla u - 2(T - t) \partial_t u]}{\sqrt{2(T - t)}} \exp\left(-\frac{|x_T - x|^2}{8(T - t)}\right),
$$

and the desired conclusion follows. \qed
Proof of Proposition 2.4. Let $0 < t < T$ be given and fixed and apply Lemma 2.6 with $T_1 = t$, $T_2 = t + \Delta t$, for $\Delta t > 0$. We divide by $\Delta t$ and let $\Delta t$ tend to zero in (2.38). This yields

\[
\int_{\mathbb{R}^N \times \{t\}} e_{\varepsilon}(u) \frac{|x - x_T|^2}{4(T-t)} \exp\left(-\frac{|x - x_T|^2}{4(T-t)}\right) dx \\
\leq \int_{\mathbb{R}^N \times \{t\}} \frac{1}{2(T-t)} [(x - x_T) \cdot \nabla u - 2(T-t) \partial_t u]^2 \exp\left(-\frac{|x - x_T|^2}{4(T-t)}\right) dx \\
+ g'(T-t),
\]

where we have set
\[
g(s) := s^{N/2} \tilde{E}_w(z_T, \sqrt{s}). \tag{2.43}
\]

Since
\[
g'(T-t) = \frac{N}{2} (T-t)^{N-2} \tilde{E}_w(z_T, \sqrt{T-t}) + \frac{(T-t)^{\frac{N-1}{2}}}{2} \frac{d}{dR} \tilde{E}_w(\sqrt{T-t}),
\]
we obtain, using the monotonicity formula in Proposition 2.1 and (2.42),

\[
\int_{\mathbb{R}^N \times \{t\}} e_{\varepsilon}(u) \frac{|x - x_T|^2}{4(T-t)} \exp\left(-\frac{|x - x_T|^2}{4(T-t)}\right) dx \leq \frac{N}{2} \int_{\mathbb{R}^N \times \{t\}} e_{\varepsilon}(u) \exp\left(-\frac{|x - x_T|^2}{4(T-t)}\right) dx \\
+ \int_{\mathbb{R}^N \times \{t\}} [V_{\varepsilon}(u) + 3 \Xi(u, z_T)] \exp\left(-\frac{|x - x_T|^2}{4(T-t)}\right) dx, \tag{2.44}
\]

which proves (2.36). For (2.37), consider the region

\[
A := \left\{ x \in \mathbb{R}^N \text{ s.t. } \frac{|x - x_T|^2}{8(T-t)} \leq \frac{N}{2} \right\}.
\]

We deduce from (2.44) that

\[
\int_{\mathbb{R}^N \times \{t\}} e_{\varepsilon}(u) \frac{|x - x_T|^2}{4(T-t)} \exp\left(-\frac{|x - x_T|^2}{4(T-t)}\right) dx \leq \frac{N}{2} \int_{A \times \{t\}} e_{\varepsilon}(u) \exp\left(-\frac{|x - x_T|^2}{4(T-t)}\right) dx \\
+ \int_{(\mathbb{R}^N \setminus A) \times \{t\}} e_{\varepsilon}(u) \frac{|x - x_T|^2}{8(T-t)} \exp\left(-\frac{|x - x_T|^2}{4(T-t)}\right) dx \\
+ \int_{\mathbb{R}^N \times \{t\}} [V_{\varepsilon}(u) + 3 \Xi(u, z_T)] \exp\left(-\frac{|x - x_T|^2}{4(T-t)}\right) dx, \tag{2.45}
\]

so that

\[
\int_{(\mathbb{R}^N \setminus A) \times \{t\}} \frac{N}{2} e_{\varepsilon}(u) \exp\left(-\frac{|x - x_T|^2}{4(T-t)}\right) dx \leq \frac{N}{2} \int_{A \times \{t\}} e_{\varepsilon}(u) \exp\left(-\frac{|x - x_T|^2}{4(T-t)}\right) dx \\
+ \int_{\mathbb{R}^N \times \{t\}} [V_{\varepsilon}(u) + 3 \Xi(u, z_T)] \exp\left(-\frac{|x - x_T|^2}{4(T-t)}\right) dx, \tag{2.46}
\]

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and finally
\[
\int_{\mathbb{R}^N \times \{t\}} e_{\epsilon}(u) \exp\left(-\frac{|x-x_T|^2}{4(T-t)}\right) dx \leq 2 \int_{A \times \{t\}} e_{\epsilon}(u) \exp\left(-\frac{|x-x_T|^2}{4(T-t)}\right) dx \\
+ \frac{2}{N} \int_{\mathbb{R}^N \times \{t\}} [V_{\epsilon}(u) + 3 \Xi(u, z_T)] \exp\left(-\frac{|x-x_T|^2}{4(T-t)}\right) dx. \tag{2.47}
\]
This completes the proof. \[\Box\]

2.6 Choice of an appropriate scaling

Let \( z_T = (x_T, T) \) as above, and set
\[
E_{w, \epsilon}(R) \equiv E_{w, \epsilon}(z_T, R) \equiv E_{w, \epsilon}(u_\epsilon; z_T, R),
\]
and accordingly
\[
\tilde{E}_{w, \epsilon}(R) \equiv \tilde{E}_{w, \epsilon}(z_T, R) \equiv \tilde{E}_{w, \epsilon}(u_\epsilon; z_T, R).
\]
Let \( 0 < \delta < 1/16 \) be fixed. We have

**Proposition 2.5.** There exists a constant \( \varepsilon_1 > 0 \) depending only on \( T \) and \( \delta \), such that, for \( \varepsilon \leq \varepsilon_1 \), there exists \( R_1 > 0 \), with \( R_1 \in (\varepsilon^{1/2}, \sqrt{T}) \) such that
\[
0 \leq \tilde{E}_{w, \epsilon}(R_1) - \tilde{E}_{w, \epsilon}(\delta R_1) \leq 4\log\delta \left| \frac{\tilde{E}_{w, \epsilon}(\sqrt{T})}{\log\varepsilon} \right|,
\tag{2.48}
\]
and therefore
\[
\int_{T-R_1^2}^{T-\delta^2 R_1^2} \int_{\mathbb{R}^N} (V_{\epsilon}(u) + \Xi(u, z_T)) \exp\left(-\frac{|x-x_T|^2}{4\delta^2 R_1^2}\right) dx dt \leq 4\log\delta \left| \frac{\tilde{E}_{w, \epsilon}(\sqrt{T})}{\log\varepsilon} \right| \tag{2.49}
\]

**Proof.** Set \( R = \sqrt{T} \), and for \( n \in \mathbb{N} \), \( R_n = \delta^n R \). Let \( k_0 \) be the largest integer such that
\[
\delta^{k_0-1} R \geq \varepsilon 1/2.
\]
We have
\[
k_0 = \left\lceil \frac{(\log \varepsilon)/2 - \log R}{\log \delta} \right\rceil,
\]
where, for \( \alpha \in \mathbb{R} \), \( \lceil \alpha \rceil \) denotes the largest integer less or equal to \( \alpha \), so that, if \( \varepsilon \leq R^4 \delta^{-8} \) then \( k_0 \) verifies
\[
k_0 - 2 \leq \frac{|\log \varepsilon|}{|\log \delta|} \tag{2.50}
\]
On the other hand, we have the equality
\[
\tilde{E}_w(\delta R) - \tilde{E}_w(\delta^{k_0} R) = \sum_{j=2}^{k_0} \left( \tilde{E}_w(\delta^{j-1} R) - \tilde{E}_w(\delta^j R) \right),
\]

and all the terms of the sum of the right hand side of the equality are non negative. Therefore, there exists \( k_1 \in \{2, \ldots, k_0 \} \) such that
\[
\tilde{E}_w(\delta^{k_1-1} R) - \tilde{E}_w(\delta^{k_1} R) \leq \frac{\tilde{E}_w(R)}{k_0 - 2} \leq 4\log \delta \frac{\tilde{E}_w(R)}{|\log \varepsilon|},
\]
where we have used (2.50) for the last inequality. We therefore set \( R_1 = \delta^{k_1-1} R \).
Inequality (2.49) is a direct consequence of the monotonicity formula. \( \square \)

**Blowing-up.** In view of Proposition 2.5 we perform the following change of variables
\[
\tilde{x} = \frac{x - x_T}{R_1}, \quad \tilde{t} = \frac{t - T}{R_1^2} + 1
\]
so that \((x_T, T)\) becomes in new variables \((0, 1)\), and \((x_T, T - R_1^2)\) becomes \((0, 0)\). Set
\[
\varepsilon = \frac{\varepsilon}{R_1}
\]
and define the map \( v_\varepsilon : \mathbb{R}^N \times (0, +\infty) \to \mathbb{C} \) by
\[
v_\varepsilon(\tilde{x}, \tilde{t}) = u_\varepsilon(x, t),
\]
so that \( v_\varepsilon \) verifies the equation
\[
\frac{\partial v_\varepsilon}{\partial \bar{t}} - \Delta v_\varepsilon = \frac{1}{\varepsilon^2} v_\varepsilon(1 - |v_\varepsilon|^2) \quad \text{on } \mathbb{R}^N \times (0, +\infty),
\]
(2.51)
i.e. \( v_\varepsilon \) is a solution to \((\text{PGL})_\varepsilon\). Note that
\[
\frac{\varepsilon}{T} \leq \varepsilon \leq \varepsilon^{1/2},
\]
(2.52)
therefore \( \varepsilon \to 0 \) as \( \varepsilon \to 0 \), \( |\log \varepsilon| \geq |\log \varepsilon|/2 \) and the asymptotic analysis for \((\text{PGL})_\varepsilon\) is also valid for \((\text{PGL})_\varepsilon\). In the sequel we skip the tildes on the new variables for simplicity.

**Lemma 2.7.** We have,
\[
|v_\varepsilon(x)| \leq 3, \quad |\nabla v_\varepsilon(x)| \leq \frac{K}{\varepsilon}, \quad |\partial_t v_\varepsilon(x)| \leq \frac{K}{\varepsilon^2}
\]
(2.53)
for any \((x, t) \in \mathbb{R}^N \times (0, +\infty)\). Moreover,
\[
\tilde{E}_{w, \varepsilon}(v_\varepsilon, (0,1), 1) = \tilde{E}_{w, \varepsilon}(u_\varepsilon, z_T, R_1),
\]
(2.54)
\[
\tilde{E}_{w, \varepsilon}((0,1), 1) - \tilde{E}_{w, \varepsilon}((0,1), \delta) \leq 4\log \delta \frac{\tilde{E}_{w, \varepsilon}(z_T, \sqrt{T})}{|\log \varepsilon|},
\]
(2.55)
and
\[
\int_{\mathbb{R}^N \times [0,1-\delta^2]} [V_\varepsilon(v_\varepsilon) + \Xi(v_\varepsilon, (0,1))] G(x, t - 1) \, dx \, dt \leq 4\log \delta \frac{\tilde{E}_{w, \varepsilon}(z_T, \sqrt{T})}{|\log \varepsilon|}.
\]
(2.56)

**Proof.** This is a direct consequence of the scaling invariance of each term. \( \square \)
3 Proof of Theorem 1

3.1 Change of scale and improved energy decay

Let \( u_\varepsilon \) be a solution of \((\text{PGL})_\varepsilon \) as in Theorem 1, i.e. satisfying the bounds

\[
\mathcal{E}_\varepsilon(u_\varepsilon^0) \leq M_0 |\log \varepsilon| \quad \text{(3.1)}
\]

\[
\int_{\mathbb{R}^N} e_\varepsilon(u_\varepsilon^0) \exp(-|x|^2/4) \leq \eta |\log \varepsilon|. \quad \text{(3.2)}
\]

Let \( 0 < \delta < \frac{1}{16} \) be fixed, but to be determined later at the very end of the proof. Let also \( T = 1 \), and \( \varepsilon_T = (0,1) \). Recall that in section 2.6 we have constructed a rescaled map \( v_\varepsilon \) defined by

\[
v_\varepsilon(x,t) = u_\varepsilon(R_1x, R_1^2(t - 1) + 1), \quad \varepsilon = \frac{\varepsilon_T}{R_1}, \quad \varepsilon \leq \varepsilon \leq \varepsilon^{1/2},
\]

for some appropriate choice of \( R_1 \). In particular, the function \( v_\varepsilon \) is a solution of \((\text{PGL})_\varepsilon \) and it follows from the monotonicity formula that

\[
\bar{E}_{\varepsilon}(v_\varepsilon, (0,1), 1) = \bar{E}_{\varepsilon}(u_\varepsilon, (0,1), 1) = \eta |\log \varepsilon| \leq \eta |\log \varepsilon| \leq 2\eta |\log \varepsilon|, \quad \text{(3.3)}
\]

where we have set

\[
\bar{\eta}(v_\varepsilon) = \frac{\bar{E}_{\varepsilon}(v_\varepsilon)}{|\log \varepsilon|}.
\]

In view of Lemma 2.7, we have the estimates

\[
|v_\varepsilon| \leq 3 \quad \text{on} \ \mathbb{R}^N \times [0, +\infty), \quad \text{(3.4)}
\]

\[
|\nabla v_\varepsilon| \leq \frac{K}{\varepsilon}, \quad |\partial_t v_\varepsilon| \leq \frac{K}{\varepsilon^2} \quad \text{on} \ \mathbb{R}^N \times [0, +\infty), \quad \text{(3.5)}
\]

\[
\frac{1}{\varepsilon} \exp(-\frac{|x|^2}{4}) \leq \exp(-\frac{|x|^2}{4}) \leq 2 \exp(-\frac{|x|^2}{4}), \quad \text{(3.6)}
\]

\[
\int_{\mathbb{R}^N \times [0,1-\delta^2]} V_\varepsilon(v_\varepsilon) \frac{1}{(1-t)^{N/2}} \exp(-\frac{|x|^2}{4(1-t)}) \, dx \, dt \leq 4 |\log \delta| \bar{\eta} \leq 8 |\log \delta| \eta, \quad \text{(3.7)}
\]

\[
\int_{\mathbb{R}^N \times [0,1-\delta^2]} \Xi(v_\varepsilon, (0,1)) \frac{1}{(1-t)^{N/2}} \exp(-\frac{|x|^2}{4(1-t)}) \, dx \, dt \leq 4 |\log \delta| \bar{\eta} \leq 8 |\log \delta| \eta. \quad \text{(3.8)}
\]

Note that \( v_\varepsilon(0,1) = u_\varepsilon(0,1) \). Thus, in order to prove Theorem 1 it suffices to establish that \( v_\varepsilon \) verifies

\[
|v_\varepsilon(0,1)| \geq 1 - \sigma. \quad \text{(3.9)}
\]

Throughout this section, we will work with \( v_\varepsilon \) instead of \( u_\varepsilon \). The main advantage to do so is that we have the additional estimates (3.4,3.6,3.7,3.8) which provide uniform bounds which are independent of \( \varepsilon \). In the definition of \( \bar{E}_{\varepsilon}, \mathcal{E}_{\varepsilon} \), and the various quantities involved in the proof, we will thus skip the reference to \( v_\varepsilon \) or even \( \varepsilon \) if this is not misleading.

The main ingredient in the proof of (3.9), i.e. Theorem 1, is the following \( \delta \)-energy decay estimate.
Proposition 3.1. There exists constants $0 < \delta_0 < \frac{1}{50}$, $0 < \epsilon_0 < \frac{1}{7}$, and $\eta_0 > 0$ such that for $0 < \eta \leq \eta_0$ the following inequality holds

$$\tilde{E}_{w,\epsilon}(v_{\epsilon}, (0, 1), \delta_0) \leq \frac{1}{2} \tilde{E}_{w,\epsilon}(v_{\epsilon}, (0, 1), 1) + \mathcal{R}(\eta),$$  \hspace{1cm} (3.10)

where $\mathcal{R}(\eta)$ tends to zero as $\eta \to 0$.

We postpone the proof of Proposition 3.1 and show first how it implies Theorem 1.

3.2 Proposition 3.1 implies Theorem 1

Assume $0 < \eta \leq \eta_0$ and set $\lambda(\sigma) = \frac{\sqrt{r_e}}{2K}$, where $\sigma$ is the constant appearing in the statement of Theorem 1, whereas $K$ is the constant appearing in (3.5). Set $r_e = \min(1, \lambda(\sigma)\epsilon)$ and $T_e = \max(0, 1 - \lambda^2(\sigma)\epsilon^2) = 1 - r_e^2$. We claim that

$$\frac{1}{\epsilon N} \int_{B(e)} (1 - |v_{\epsilon}(x, T_e)|^2)^2 \leq \mathcal{R}_1(\eta),$$  \hspace{1cm} (3.11)

where $\mathcal{R}_1(\eta) \to 0$ as $\eta \to 0$.

**Proof of the claim.** Combining (3.6) and (3.10) we are led to

$$\tilde{E}_{w,\epsilon}(v_{\epsilon}, (0, 1), 1) \leq 8\log\delta|\eta| + 2\mathcal{R}(\eta).$$  \hspace{1cm} (3.12)

Assume first that $\lambda(\sigma)\epsilon \leq 1$, so that $T_e = 1 - \lambda^2(\sigma)\epsilon^2$. We deduce from the monotonicity formula that

$$\tilde{E}_{w,\epsilon}(v_{\epsilon}, (0, 1), \lambda(\sigma)\epsilon) \leq \tilde{E}_{w,\epsilon}(v_{\epsilon}, (0, 1), 1)$$  \hspace{1cm} (3.13)

so that, combining (3.12) and (3.13) we obtain

$$\tilde{E}_{w,\epsilon}(v_{\epsilon}, (0, 1), \lambda(\sigma)\epsilon) \leq 8\log\delta|\eta| + 2\mathcal{R}(\eta).$$

If $\lambda(\sigma)\epsilon \geq 1$, then $r_e = 1, T_e = 0$ so that

$$\tilde{E}_{w,\epsilon}(v_{\epsilon}, (0, 1), r_e) \leq \eta|\log\epsilon| \leq \eta|\log\lambda(\sigma)|.$$  

In both cases the claim (3.11) follows from the inequality

$$\frac{1}{\epsilon N} \int_{B(e)} (1 - |v_{\epsilon}(x, T_e)|^2)^2 \leq \frac{C(\sigma)}{r_e^{N-2}} \int_{B(e)} \frac{(1 - |v_{\epsilon}(x, T_e)|^2)^2}{\epsilon^2} \exp\left(-\frac{|x|^2}{r_e^2}\right)dx \leq C(\sigma) \tilde{E}_{w,\epsilon}(v_{\epsilon}, (0, 1), r_e),$$  \hspace{1cm} (3.14)

valid for some constant $C(\sigma)$ depending only on $\sigma$ and $N$.

Arguing as in the proof of Lemma III.2 in [8], we are led to

$$1 - |v_{\epsilon}(0, T_e)| \leq C \left(\frac{1}{\epsilon N} \int_{B(e)} (1 - |v_{\epsilon}(x, T_e)|^2)^2\right)^\frac{1}{1+2} \leq C\mathcal{R}_1(\eta)^{\frac{1}{1+2}}.$$  \hspace{1cm} (3.15)
On the other hand, by (3.5),
\[ |v_e(0,T_e) - v_e(0,1)| \leq \frac{K}{\epsilon^2}(T_e - 1) \leq \frac{\sigma}{2}. \]  \hspace{1cm} (3.16)

Combining (3.15) and (3.16), we obtain
\[ |1 - |v_e(0,1)|| \leq \frac{\sigma}{2} + C\mathcal{R}_1(\eta) \frac{1}{\epsilon^2} \]
so that the conclusion follows if \( \eta_0 \) is chosen sufficiently small, since \( \mathcal{R}_1(\eta) \to 0 \) as \( \eta \to 0 \).

### 3.3 Paving the way to Proposition 3.1

As in [8], let us first consider the ideal situation where
\[ |v_e| \equiv 1 \quad \text{on} \quad \mathbb{R}^N \times [0,1]. \]

Then, we may write \( v_e = \exp(\mathbb{i}\varphi) \) where the phase \( \varphi : \mathbb{R}^N \times [0,1] \to \mathbb{R} \) is uniquely defined, up to a constant multiple of \( 2\pi \). The equation for the phase \( \varphi \) is then the linear heat equation
\[ \frac{\partial \varphi}{\partial t} - \Delta \varphi = 0 \quad \text{on} \quad \mathbb{R}^N \times (0,1). \]

Notice that in that situation, \( |\nabla v_e| = |\nabla \varphi| \) so that \( e_e(v_e) = |\nabla \varphi|^2/2 \) and \( \partial_t v_e = |\partial \varphi| \). Moreover, \( |\nabla \varphi|^2 \) verifies the equation
\[ \frac{\partial |\nabla \varphi|^2}{\partial t} - \Delta (|\nabla \varphi|^2) = -2|\nabla \varphi|^2 \leq 0 \]
so that for any \( 0 < \delta < 1 \), and any \( x_s \in \mathbb{R}^N \),
\[ |\nabla \varphi(x_s,1-\delta^2)|^2 \leq \int_{\mathbb{R}^N} \frac{1}{\pi N^2(1-\delta^2)^{N/2}} e_e(v_e(x,0)) \exp\left(-\frac{|x-x_s|^2}{4(1-\delta^2)}\right) dx. \]  \hspace{1cm} (3.17)

For \( \mu \in \mathbb{R}^N \) and \( \sigma > 0 \), consider the Gaussian \( N(\mu,\sigma^2) = \frac{1}{(4\pi)^{N/2}\delta^N} \exp(-\frac{|x-\mu|^2}{4\sigma^2}) \). We deduce form (3.17) that
\[ \frac{1}{(4\pi)^{N/2}\delta^N} \int_{\mathbb{R}^N} |\nabla \varphi(x_s,1-\delta^2)|^2 \exp\left(-\frac{|x_s|^2}{4\delta^2}\right) dx_s \]
\[ \leq \int_{\mathbb{R}^N \times \mathbb{R}^N} N(0,\delta^2)(x_s) N(x,1-\delta^2)(x_s) e_e(v_e(x,0)) dx_s dx \]
\[ \leq \int_{\mathbb{R}^N} \left[N(0,\delta^2) * N(0,1-\delta^2)\right](x) e_e(v_e(x,0)) dx \]
\[ = \int_{\mathbb{R}^N} N(0,1) e_e(v_e(x,0)) dx, \]  \hspace{1cm} (3.18)
i.e.,
\[
\tilde{E}_{w,e}(v_e, (0, 1), \delta) \leq \delta^2 \tilde{E}_{w,e}(v_e, (0, 1), 1)
\]
so that (3.10) is verified for \(\delta \leq \sqrt{1/2}\).

In the general case \(v_e\) may vanish, so that it is not possible to find a phase \(\varphi\) which is globally defined. However, if locally we may write \(v_e = \rho \exp(i\varphi)\), then

\[
v_e \times \nabla v_e = \rho^2 \nabla \varphi
\]

so that when \(\rho\) is close to 1, \(v_e \times \nabla v_e\) represents essentially the gradient of the phase. The quantity \(v_e \times \nabla v_e\) is always globally defined, in contrast with the phase. The following decomposition formula is then the starting point of the analysis of \(|\nabla v_e|^2\)

\[
4|v_e|^2|\nabla v_e|^2 = 4|v_e \times \nabla v_e|^2 + |\nabla|v_e|^2| = 4|v_e \times \nabla v_e|^2 + 4\rho^2|\nabla \rho|^2, \tag{3.19}
\]

where \(\rho = |v_e|\) is the modulus.

In order to establish (3.10), it suffices to prove a similar inequality when \(\delta_0\) is replaced by some \(\delta \in [\delta_0, 2\delta_0]\). That is, we will show that there exist \(\delta \in [\delta_0, 2\delta_0]\) such that

\[
\tilde{E}_{w,e}(v_e, (0, 1), \delta) \leq \frac{1}{2} \tilde{E}_{w,e}(v_e, (0, 1), 1) + \mathcal{R}(\eta). \tag{3.20}
\]

We will determine \(\delta\) using averaging arguments, for quantities which will be integrated on constant time slices [and bounded thanks to (3.6,3.7,3.8)]. For that purpose, we introduce first some notation. Set, for \(t \in [0, 1],\)

\[
A(t) = \frac{1}{(1-t)^{N/2}} \int_{\mathbb{R}^N} V_e(v_e) \exp(-\frac{|x|^2}{4(1-t)}) \, dx,
\]

\[
B(t) = \frac{1}{(1-t)^{N/2}} \int_{\mathbb{R}^N} \Xi(v_e, (0, 1), (x, t)) \exp(-\frac{|x|^2}{4(1-t)}) \, dx.
\]

By (3.7) and (3.8) we have therefore

\[
\int_0^{1-\delta^2} A(t) \, dt \leq 4|\log \delta_0|\eta \tag{3.21}
\]

and

\[
\int_0^{1-\delta^2} B(t) \, dt \leq 4|\log \delta_0|\eta. \tag{3.22}
\]

We first observe that the left hand side of (3.20), i.e. \(\tilde{E}_{w,e}(v_e, (0, 1), \delta)\), involves an integral on the whole \(\mathbb{R}^N\). However, for “many” choices of \(\delta\), we may localize this integral.
3.4 Localizing the energy on appropriate time slices

Consider the set $\Theta_1$ defined by

$$\Theta_1 = \left\{ t \in [1 - 4 \delta_0^2, 1 - \delta_0^2] \mid A(t) + B(t) \leq \frac{32|\log \delta_0| \eta}{\delta_0^2} \right\}. \quad (3.23)$$

**Lemma 3.1.** We have

$$\text{meas}(\Theta_1) \geq \frac{3}{4} \text{meas} \left( [1 - 4 \delta_0^2, 1 - \delta_0^2] \right).$$

**Proof.** The proof is an easy consequence of (3.21) and (3.22). \qed

**Lemma 3.2.** The following inequality holds for any $t \in \Theta_1$:

$$\bar{E}_{w,c}(\nu_\epsilon, (0,1), \delta) \leq \frac{1}{\delta^{N-2}} \int_{B(2\sqrt{\delta} \times \{t\})} \epsilon c(\nu_\epsilon) + K|\log \delta| \eta,$$

where $\delta = \sqrt{1 - t}$.

**Proof.** The proof is an immediate consequence of Proposition 2.4 and the definition of $\Theta_1$. \qed

3.5 Improved energy decay estimate for the modulus

Set $\sigma_\epsilon = 1 - |\nu_\epsilon|^2$. Recall that $\nu_\epsilon$ verifies the equation

$$\partial_t \sigma_\epsilon - \Delta \sigma_\epsilon = 2|\nabla \nu_\epsilon|^2 - \frac{2}{\epsilon^2} \sigma_\epsilon (1 - \sigma_\epsilon) \quad \text{on } \mathbb{R}^N \times (0, +\infty). \quad (3.24)$$

Let $\delta \in [\delta_0, 2\delta_0]$ be given. Our first aim is to bound $\int_{B(1) \times \{t\}} |\nabla \sigma_\epsilon|^2$, where $t = 1 - \delta^2$.

**Lemma 3.3.** The following inequality holds

$$\int_{B(1) \times \{t\}} |\nabla \sigma_\epsilon|^2 \leq C(\delta_0) \left( \int_{\mathbb{R}^N \times \{t\}} V_\epsilon(\nu_\epsilon) \exp\left(-\frac{|x|^2}{4\delta^2}\right) \right)^{1/2}$$

$$\cdot \left( \int_{\mathbb{R}^N \times \{t\}} \left( |\nabla \nu_\epsilon|^2 + \frac{|x|}{2\delta^2} \cdot \nabla \nu_\epsilon - \partial_t \nu_\epsilon \right) \exp\left(-\frac{|x|^2}{4\delta^2}\right) \right)^{1/2}, \quad (3.25)$$

where $C(\delta_0) = K\delta_0^{-2} \exp(\frac{1}{4\delta_0})$.

**Proof.** Let $r_1 \in [1, 2]$, multiplying (3.24) by $\sigma_\epsilon$ and integrating by parts on $B(r_1)$ we obtain

$$\int_{B(r_1)} |\nabla \nu_\epsilon|^2 = 2\int_{B(r_1)} |\nabla \nu_\epsilon|^2 \sigma_\epsilon - \int_{B(r_1)} \partial_t \sigma_\epsilon \cdot \sigma_\epsilon$$

$$- \int_{\partial B(r_1)} \partial_r \sigma_\epsilon \cdot \sigma_\epsilon - \frac{2}{\epsilon^2} \int_{B(r_1)} \sigma_\epsilon^2 (1 - \sigma_\epsilon) \quad (3.26)$$

$$\leq 2\int_{B(r_1)} |\nabla \nu_\epsilon|^2 + \int_{B(r_1)} |\partial_t \sigma_\epsilon \cdot \sigma_\epsilon| + \int_{\partial B(r_1)} |\partial_r \sigma_\epsilon \cdot \sigma_\epsilon|. \quad (3.27)$$

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Here we have used for the last inequality the fact that \((1 - \sigma_e) \sigma_e^2 \geq 0\). In order to bound the last term of the r.h.s. of the previous inequality, we choose \(r_1 \in [1, 2]\) so that
\[
\int_{\partial B(r_1) \times \{t\}} |\nabla v_e|^2 \exp\left(-\frac{|x|^2}{4\delta^2}\right) \leq \int_{\mathbb{R}^N \times \{t\}} |\nabla v_e|^2 \exp\left(-\frac{|x|^2}{4\delta^2}\right)
\]
and
\[
\int_{\partial B(r_1) \times \{t\}} V_e(v_e) \exp\left(-\frac{|x|^2}{4\delta^2}\right) \leq \int_{\mathbb{R}^N \times \{t\}} V_e(v_e) \exp\left(-\frac{|x|^2}{4\delta^2}\right).
\]
For this choice of \(r_1\), we have therefore
\[
\int_{\partial B(r_1) \times \{t\}} |\partial_v \sigma_e \cdot \sigma_e| \leq K \epsilon \left( \int_{\partial B(r_1) \times \{t\}} |\nabla v_e| \cdot \left| \frac{1 - |v_e|^2}{\epsilon} \right| \right)
\]
\[
\leq K \epsilon \exp\left(\frac{1}{\delta_0^2}\right) \left( \int_{\partial B(r_1) \times \{t\}} |\nabla v_e|^2 \exp\left(-\frac{|x|^2}{4\delta^2}\right) \right)^{1/2} \left( \int_{\partial B(r_1) \times \{t\}} V_e(v_e) \exp\left(-\frac{|x|^2}{4\delta^2}\right) \right)^{1/2}
\]
\[
\leq K \epsilon \exp\left(\frac{1}{\delta_0^2}\right) \left( \int_{\mathbb{R}^N \times \{t\}} |\nabla v_e|^2 \exp\left(-\frac{|x|^2}{4\delta^2}\right) \right)^{1/2} \left( \int_{\mathbb{R}^N \times \{t\}} V_e(v_e) \exp\left(-\frac{|x|^2}{4\delta^2}\right) \right)^{1/2}.
\]
(3.27)

Finally, we estimate the remaining two terms on the r.h.s. of (3.26). First, we have by (3.5)
\[
\int_{B(r_1) \times \{t\}} |\nabla v_e|^2 \sigma_e \leq K \int_{B(r_1) \times \{t\}} |\nabla v_e| \cdot \left| \frac{1 - |v_e|^2}{\epsilon} \right|
\]
\[
\leq K \epsilon \exp\left(\frac{1}{\delta_0^2}\right) \left( \int_{\mathbb{R}^N \times \{t\}} |\nabla v_e|^2 \exp\left(-\frac{|x|^2}{4\delta^2}\right) \right)^{1/2} \left( \int_{\mathbb{R}^N \times \{t\}} V_e(v_e) \exp\left(-\frac{|x|^2}{4\delta^2}\right) \right)^{1/2}.
\]
(3.28)

Similarly,
\[
\int_{B(r_1) \times \{t\}} |\partial_v \sigma_e \cdot \sigma_e| \leq K \epsilon \int_{B(r_1) \times \{t\}} |\partial_v | \cdot \left| \frac{1 - |v_e|^2}{\epsilon} \right|
\]
\[
\leq K \epsilon \exp\left(\frac{1}{\delta_0^2}\right) \left( \int_{B(r_1) \times \{t\}} |\partial_v v_e|^2 \exp\left(-\frac{|x|^2}{4\delta^2}\right) \right)^{1/2} \left( \int_{B(r_1) \times \{t\}} V_e(v_e) \exp\left(-\frac{|x|^2}{4\delta^2}\right) \right)^{1/2}
\]
\[
\leq K \epsilon \exp\left(\frac{1}{\delta_0^2}\right) \left( \int_{B(r_1) \times \{t\}} V_e(v_e) \exp\left(-\frac{|x|^2}{4\delta^2}\right) \right)^{1/2} \left( \int_{B(r_1) \times \{t\}} \left| \frac{x}{2 \delta^2} \cdot \nabla v_e \right|^2 + |\frac{x}{2 \delta^2} \cdot \nabla v_e - \partial_v v_e|^2 \exp\left(-\frac{|x|^2}{4\delta^2}\right) \right)^{1/2}.
\]
(3.29)

Combining (3.27), (3.28) and (3.29) we derive the conclusion.

The previous lemma allows us to estimate the contribution of the modulus to the energy on appropriate time slices. More precisely,
Proposition 3.2. For any \( t \in \Theta_1 \) we have
\[
\int_{B(1) \times \{ t \}} \frac{1}{2} |\nabla v|_e^2 + \frac{(1 - |v|_e^2)}{4\epsilon^2} \leq C_1(\delta_0) \left[ \eta^{1/2}(E_{w,e}(v_e, (0, 1), 1) + 1) \right],
\]
where \( C_1(\delta_0) = K\delta_0^{N-4} \exp \left( \frac{1}{\delta_0} \right) \log \delta_0 \).

Proof. By (3.25), we have
\[
\int_{B(1) \times \{ t \}} \frac{1}{2} |\nabla v|_e^2 \leq C(\delta_0) |\delta^N A(t)|^{1/2} \left[ \delta^{(N-2)/2} E_{w,e}(v_e, (0, 1), \delta) + \delta^{(N-2)/2} B(t)^{1/2} \right] \leq C(\delta_0) \delta^{N-1} A(t)^{1/2} \left[ \tilde{E}_{w,e}(v_e, (0, 1), 1) + B(t)^{1/2} \right],
\]
and we have made use of the monotonicity formula for the last inequality.

For \( t \in \Theta_1 \), \( A(t) + B(t) \leq 32 |\log \delta_0| \delta_0^{-2} \eta \), so that
\[
\int_{B(1) \times \{ t \}} \frac{1}{2} |\nabla v|_e^2 \leq K C(\delta_0) \delta_0^{-N-2} |\log \delta_0| \eta^{1/2} \left( \tilde{E}_{w,e}(v_e, (0, 1), 1) + \eta^{1/2} \right) \leq K C(\delta_0) \delta_0^{-N-2} |\log \delta_0| \left[ \eta^{1/2}(\tilde{E}_{w,e}(v_e, (0, 1), 1) + 1) \right].
\]

Finally, we have for the potential and for \( t \in \Theta_1 \),
\[
\frac{1}{4\epsilon^2} \int_{B(1) \times \{ t \}} (1 - |v|_e^2) \leq \frac{(1 - |v|_e^2)}{4\epsilon^2} A(t) \leq K\delta_0^{N-2} |\log \delta_0| \exp \left( \frac{1}{\delta_0} \right) \eta
\]
and the conclusion follows. \( \Box \)

3.6 Hodge - de Rham decomposition of \( v_e \times dv_e \)

In view of (3.19) and the previous subsection, it remains to provide an improved decay estimate for \( |v_e \times dv_e|^2 \). For that purpose, we will introduce as for the elliptic case an appropriate Hodge - de Rham decomposition of \( v_e \times dv_e \). We would like to emphasize the fact that the estimates obtained so far work equally well if we consider instead vector-valued maps \( u_e : \mathbb{R}^N \times \mathbb{R}^+ \rightarrow \mathbb{R}^k, k \geq 1 \). The techniques of the present subsection however heavily rely on the fact that \( k = 2 \), i.e. \( u_e \) is complex-valued.

Let \( \chi \in C^\infty_c(\mathbb{R}^N) \) be such that \( 0 \leq \chi \leq 1, \chi \equiv 1 \) on \( B(2) \) and \( \chi \equiv 0 \) on \( \mathbb{R}^N \setminus B(4) \). We assume moreover that \( \| \nabla \chi \|_\infty \leq 1 \). Consider for \( t > 0 \) the two-form \( \psi_t \) defined on \( \mathbb{R}^N \times \{ t \} \) by
\[
\psi_t = -G_N * d(v_e \times dv_e) \chi \quad \text{on } \mathbb{R}^N \times \{ t \}
\]
where \( G_N \) denotes the Green’s function of the Laplace operator in dimension \( N \),
\[
G_N(x) = -\frac{\omega_{N-1}}{|x|^{N-2}} \quad \text{for } N > 2 \quad \text{and} \quad G_2(x) = \frac{1}{2\pi} \log |x|.
\]
Note in particular that
\[
-\Delta \psi_t = d(v_e \times dv_e) \chi \quad \text{on } \mathbb{R}^N \times \{ t \}
\]
and that, for \( N \geq 3 \),
\[
|\psi|(x) \to 0 \quad \text{as} \quad |x| \to +\infty.
\]
Since \( -\Delta = dd^* + d^*d \) and since \( \chi \equiv 1 \) on \( B(2) \) it follows that
\[
d(v_e \times dv_e - d^*\psi_t) = d^*d\psi_t \equiv \zeta_t \quad \text{on} \quad B(2) \times \{t\}.
\] (3.32)
We observe that
\[
\Delta(d\psi_t) = 0 \quad \text{on} \quad B(2) \times \{t\}.
\] (3.33)
Indeed, we have
\[
\Delta(d\psi_t) = d(\Delta \psi_t) = d(d(v_e \times dv_e)) = 0.
\]
It follows that the two-form \( \zeta_t = d^*d\psi_t \) is closed, since
\[
d\zeta_t = d(d^*d\psi_t) = dd^*(d\psi_t) = -\Delta(d\psi_t) - d^*d(d\psi_t) = -\Delta(d\psi_t) = 0.
\] (3.34)
By Poincaré Lemma, there exists therefore a 1-form \( \xi_t \) defined on \( B(3/2) \times \{t\} \) such that
\[
\begin{cases}
    d\xi_t = \zeta_t & \text{on} \quad B(3/2) \times \{t\} \\
    d^*\xi_t = 0 & \text{on} \quad B(3/2) \times \{t\}
\end{cases}
\] (3.35)
and
\[
\|\xi_t\|_{L^2(B(3/2) \times \{t\})} \leq K\|\zeta_t\|_{L^2(B(3/2) \times \{t\})}.
\] (3.36)
[Note that such a form \( \xi_t \) is not uniquely defined]

Going back to (3.32), we may write
\[
d(v_e \times dv_e - d^*\psi_t - \xi_t) = 0 \quad \text{on} \quad B(3/2) \times \{t\}.
\]
Invoking once more the Poincaré Lemma, we deduce that there exists some function \( \varphi_t \) uniquely determined on \( B(3/2) \times \{t\} \) (up to an additive constant) such that
\[
v_e \times dv_e = d\varphi_t + d^*\psi_t + \xi_t \quad \text{on} \quad B(3/2) \times \{t\}.
\] (3.37)
This is precisely the Hodge - de Rham decomposition of \( v_e \times dv_e \) which best fits our needs. We are going to estimate the \( L^2 \) norm of each of the three terms on the r.h.s. of (3.37) successively. As we will see, the most delicate estimate is for \( \psi_t \). Although it will enter in the final estimates for \( \xi_t \) and \( \varphi_t \), we will treat these last two terms first.

### 3.7 Estimate for \( \xi_t \)

Since \( d\psi_t \) is harmonic on \( B(2) \) by (3.32), we have for any \( k \in \mathbb{N} \),
\[
\|d\psi_t\|_{C^k(B(3/2) \times \{t\})} \leq K_k\|d\psi_t\|_{L^2(B(2) \times \{t\})} \leq K_k\|\nabla\psi_t\|_{L^2(B(2) \times \{t\})}.
\] (3.38)
On the other hand, since \( \zeta_t = d^*d\psi_t \), it follows that
\[
\|\zeta_t\|_{C^k(B(3/4) \times \{t\})} \leq K_k\|\nabla\psi_t\|_{L^2(B(2) \times \{t\})}
\]
and going back to (3.36) we obtain the estimate :

**Lemma 3.4.** We have,
\[
\|\xi_t\|_{L^2(B(3/2) \times \{t\})} \leq K\|\nabla\psi_t\|_{L^2(B(2) \times \{t\})}.
\]
3.8 Estimate for $\varphi_t$

The first step is to derive an elliptic equation for $\varphi_t$. This equation involves a linear elliptic operator (with a first order term) which appears naturally in the context of parabolic equations (see [26]). In a second step we provide some simple linear estimates for this operator. We finally use them to complete the estimates for $\varphi_t$.

The equation for $\varphi_t$. Taking the external product of $({\mathcal PGL})_x$ for $\nu_t$ with $\nu_e$, we obtain

$$v_e \times \partial_{\nu} v_e + d^*(v_e \times dv_e) = 0 \quad \text{on } \mathbb{R}^N \times (0, +\infty).$$

[Note that if $v_e = \rho \exp (i \phi)$ then (3.39) is equivalent to (6)] The term $d^*(v_e \times dv_e)$ can be computed using the Hodge - de Rham decomposition (3.37). We have, since $d^* \xi_t = 0$,

$$d^*(v_e \times dv_e) = -\Delta \varphi_t \quad \text{on } B(3/2) \times \{t\}.$$ 

On the other hand, we may write

$$v_e \times \partial_{\nu} v_e = -v_e \times \left( \frac{x}{2 \delta^2} \cdot \nabla v_e - \partial_{\nu} v_e \right) + \frac{x}{2 \delta^2} \cdot (v_e \times \nabla v_e)$$

and

$$\frac{x}{2 \delta^2} \cdot (v_e \times \nabla v_e) = \frac{x}{2 \delta^2} (\nabla \varphi_t + d^* \psi_t + \xi_t).$$

Going back to (3.39) we thus obtain

$$-\Delta \varphi_t + \frac{x}{2 \delta^2} \cdot \nabla \varphi_t = v_e \times \left( \frac{x}{2 \delta^2} \cdot \nabla v_e - \partial_{\nu} v_e \right) - (d^* \psi_t + \xi_t) \cdot \frac{x}{2 \delta^2} \quad \text{on } B(3/2) \times \{t\}. (3.40)$$

In view of (3.40), we are led to consider the linear elliptic operator

$$L_\delta \equiv -\Delta + \frac{x}{2 \delta^2} \cdot \nabla = -\exp \left( \frac{|x|^2}{4 \delta^2} \right) \text{div} \left( \exp \left( -\frac{|x|^2}{4 \delta^2} \right) \nabla \right).$$

Linear estimates for $L_\delta$. Let $r > 0$ and consider functions $v$ and $f$ on $B(r)$ such that

$$L_\delta v = f \quad \text{on } B(r).$$

(3.41)

The next lemma corresponds to the of Pohozaev’s identity for the operator $L_\delta$.

Lemma 3.5. Let $v$ and $f$ satisfy (3.41), then the following equality holds

$$\int_{B(r)} \left[ \left( \frac{N - 2}{2} - \frac{|x|^2}{4 \delta^2} \right) |\nabla v|^2 \exp \left( -\frac{|x|^2}{4 \delta^2} \right) \right] + \int_{\partial B(r)} x \cdot \nabla v f \exp \left( -\frac{|x|^2}{4 \delta^2} \right)$$

$$= -\frac{r}{2} \int_{\partial B(r)} |\nabla v|^2 \exp \left( -\frac{|x|^2}{4 \delta^2} \right) - r \int_{\partial B(r)} |\partial_{\nu} v|^2 \exp \left( -\frac{|x|^2}{4 \delta^2} \right).$$

(3.42)
Proof. We multiply $\Delta v$ by $x \cdot \nabla v \exp(-\frac{|x|^2}{4\delta^2})$ and integrate by parts on $B(r)$. This yields,

$$
\int_{B(r)} \Delta v \left( \sum_i x_i \partial_i v \right) \exp(-\frac{|x|^2}{4\delta^2})
= -\int_{B(r)} \nabla v \cdot \nabla \left( \sum_i x_i \partial_i v \right) \exp(-\frac{|x|^2}{4\delta^2}) + \int_{B(r)} \frac{x}{2\delta^2} \cdot \nabla v \left( \sum_i x_i \partial_i v \right) \exp(-\frac{|x|^2}{4\delta^2})
+ \int_{\partial B(r)} \partial_r v \left( \sum_i x_i \partial_i v \right) \exp(-\frac{|x|^2}{4\delta^2})
$$

$$
= -\int_{B(r)} \sum_j |\partial_j v|^2 \exp(-\frac{|x|^2}{4\delta^2}) - \int_{B(r)} \sum_i \frac{x_i}{2} \partial_i (|\partial_j v|^2) \exp(-\frac{|x|^2}{4\delta^2})
+ \int_{B(r)} \frac{x}{2\delta^2} \cdot \nabla v \left( \sum_i x_i \partial_i v \right) \exp(-\frac{|x|^2}{4\delta^2})
+ \int_{\partial B(r)} \partial_r v \left( \sum_i x_i \partial_i v \right) \exp(-\frac{|x|^2}{4\delta^2})
$$

(3.43)

$$
= -\int_{B(r)} \sum_j |\partial_j v|^2 \exp(-\frac{|x|^2}{4\delta^2}) + \int_{B(r)} \sum_i \frac{1}{2} |\partial_j v|^2 \exp(-\frac{|x|^2}{4\delta^2})
- \int_{B(r)} \frac{|x|^2}{4\delta^2} |\nabla v|^2 \exp(-\frac{|x|^2}{4\delta^2}) + \int_{B(r)} \frac{x}{2\delta^2} \cdot \nabla v \left( \sum_i x_i \partial_i v \right) \exp(-\frac{|x|^2}{4\delta^2})
- \int_{\partial B(r)} \sum_j r \frac{1}{2} |\partial_j v|^2 \exp(-\frac{|x|^2}{4\delta^2}) + \int_{\partial B(r)} \partial_r v \left( \sum_i x_i \partial_i v \right) \exp(-\frac{|x|^2}{4\delta^2})
$$

$$
= \int_{B(r)} \frac{N - 2}{2} \left| \nabla v \right|^2 \exp(-\frac{|x|^2}{4\delta^2}) + \int_{B(r)} \frac{x}{2\delta^2} \cdot \nabla v \left( \sum_i x_i \partial_i v \right) \exp(-\frac{|x|^2}{4\delta^2})
- \frac{r}{2} \int_{\partial B(r)} \left| \nabla v \right|^2 \exp(-\frac{|x|^2}{4\delta^2}) + r \int_{\partial B(r)} |\partial_r v|^2 \exp(-\frac{|x|^2}{4\delta^2})
$$

The conclusion then follows from (3.41).

\[ \blacksquare \]

**Corollary 3.1.** We have, if $v$ and $f$ satisfy (3.41), the inequality

$$
\int_{\partial B(r)} \left| \nabla v \right|^2 \exp(-\frac{|x|^2}{4\delta^2}) \leq \frac{N - 2}{r} \int_{B(r)} \left| \nabla v \right|^2 \exp(-\frac{|x|^2}{4\delta^2}) + \int_{\partial B(r)} \left| \partial_r v \right|^2 \exp(-\frac{|x|^2}{4\delta^2})
$$

(3.44)
Proof. It suffices to note that
\[
\int_{B(r)} x \cdot \nabla v f \exp(-\frac{|x|^2}{4\delta^2}) - \int_{B(r)} \frac{|x|^2}{4\delta^2} |\nabla v|^2 \exp(-\frac{|x|^2}{4\delta^2})
\leq \left( \int_{B(r)} |x|^2 |\nabla v|^2 \exp(-\frac{|x|^2}{4\delta^2}) \right)^{1/2} \left( \int_{B(r)} f^2 \exp(-\frac{|x|^2}{4\delta^2}) \right)^{1/2} - \int_{B(r)} \frac{|x|^2}{4\delta^2} |\nabla v|^2 \exp(-\frac{|x|^2}{4\delta^2})
\leq \frac{1}{2} \int_{B(r)} f^2 \exp(-\frac{|x|^2}{4\delta^2}),
\]
since $4\delta^2 \leq 1$. \hfill \qed

Next, we consider the boundary value problem
\[
\begin{align*}
L_\delta v &= f & \text{on } B(r) \\
\frac{\partial v}{\partial r} &= g & \text{on } \partial B(r).
\end{align*}
\] (3.45)

Lemma 3.6. There exists some constants $C(\delta, r)$ depending only and continuously on $\delta$ and $r$, such that if $v, f, g$ verify (3.45) then
\[
\int_{B(r)} |\nabla v|^2 \exp(-\frac{|x|^2}{4\delta^2}) \leq C(\delta, r) \left[ \int_{B(r)} f^2 \exp(-\frac{|x|^2}{4\delta^2}) + \left( \int_{\partial B(r)} g^2 \exp(-\frac{|x|^2}{4\delta^2}) \right)^{1/2} \right] + K\tau \int_{\partial B(r)} g^2 \exp(-\frac{|x|^2}{4\delta^2}),
\] (3.46)
where $K$ depends only possibly on $N$ but not on $\delta$ or $r$.

Proof. Note that (3.46) involves only the gradient of $v$, whereas if $v$ is a solution to (3.45) so is $v + c$ for every $c \in \mathbb{R}$. Therefore we may assume that
\[
\int_{\partial B(r)} v = 0.
\] (3.47)
It is convenient to use the divergence form of the equation, namely
\[
-\text{div} \left( \exp(-\frac{|x|^2}{4\delta^2}) \nabla v \right) = \exp(-\frac{|x|^2}{4\delta^2}) f.
\] (3.48)
We multiply (3.48) by $v$ and integrate by parts on $B(r)$ to obtain
\[
\int_{B(r)} |\nabla v|^2 \exp(-\frac{|x|^2}{4\delta^2}) = \int_{B(r)} f v \exp(-\frac{|x|^2}{4\delta^2}) - \int_{B(r)} \partial_r v \cdot v \exp(-\frac{|x|^2}{4\delta^2})
\leq \left( \int_{B(r)} f^2 \exp(-\frac{|x|^2}{4\delta^2}) \right)^{1/2} \left( \int_{B(r)} v^2 \exp(-\frac{|x|^2}{4\delta^2}) \right)^{1/2} + \left( \int_{\partial B(r)} g^2 \exp(-\frac{|x|^2}{4\delta^2}) \right)^{1/2} \left( \int_{\partial B(r)} v^2 \exp(-\frac{|x|^2}{4\delta^2}) \right)^{1/2}.
\] (3.49)
In view of (3.47), we have by the Poincaré-Wirtinger inequality
\[
\int_{\partial B(r)} v^2 \leq \frac{r^2}{N-1} \int_{\partial B(r)} |\nabla v|^2. \tag{3.50}
\]

By (3.44) we thus have
\[
\int_{\partial B(r)} v^2 \exp\left(-\frac{|x|^2}{4\delta^2}\right) \leq r \int_{B(r)} |\nabla v|^2 \exp\left(-\frac{|x|^2}{4\delta^2}\right) + \frac{r}{N-1} \int_{B(r)} f^2 \exp\left(-\frac{|x|^2}{4\delta^2}\right) + \frac{r^2}{N-1} \int_{\partial B(r)} g^2 \exp\left(-\frac{|x|^2}{4\delta^2}\right). \tag{3.51}
\]

On the other hand, standard elliptic estimates yield
\[
\int_{B(r)} v^2 \exp\left(-\frac{|x|^2}{4\delta^2}\right) \leq C(\delta, r) \left[ \int_{B(r)} f^2 \exp\left(-\frac{|x|^2}{4\delta^2}\right) + \int_{\partial B(r)} g^2 \exp\left(-\frac{|x|^2}{4\delta^2}\right) \right] \tag{3.52}
\]
where the constant \( C(\delta, r) \) may depend (strongly) on \( \delta \) and \( r \). Going back to (3.49) we bound the second term on the r.h.s. by
\[
\left(\int_{\partial B(r)} g^2 \exp\left(-\frac{|x|^2}{4\delta^2}\right)\right)^{1/2} \leq r \left(\int_{\partial B(r)} v^2 \exp\left(-\frac{|x|^2}{4\delta^2}\right)\right)^{1/2} \leq \frac{r}{2} \int_{\partial B(r)} g^2 \exp\left(-\frac{|x|^2}{4\delta^2}\right)
\]
\[
\leq r \int_{\partial B(r)} g^2 \exp\left(-\frac{|x|^2}{4\delta^2}\right) + \frac{1}{2} \int_{B(r)} |\nabla v|^2 \exp\left(-\frac{|x|^2}{4\delta^2}\right) + \frac{1}{2} \int_{B(r)} f^2 \exp\left(-\frac{|x|^2}{4\delta^2}\right).
\]

The first term on the r.h.s. of (3.49) is estimated as follows
\[
\left(\int_{B(r)} f^2 \exp\left(-\frac{|x|^2}{4\delta^2}\right)\right)^{1/2} \left(\int_{B(r)} v^2 \exp\left(-\frac{|x|^2}{4\delta^2}\right)\right)^{1/2} \leq C(\delta, r) \left[ \int_{B(r)} f^2 \exp\left(-\frac{|x|^2}{4\delta^2}\right) + \int_{B(r)} f^2 \exp\left(-\frac{|x|^2}{4\delta^2}\right) \right]
\]
and the conclusion follows.

We are now in position to complete the estimates for \( \varphi_t \).

**Estimates for \( \varphi_t \).** Recall that for every \( 0 < r < 3/2 \), \( \varphi_t \) verifies the equation
\[
\begin{aligned}
L_\xi \varphi_t &= f & \text{on } B(r) \times \{t\} \\
\frac{\partial \varphi_t}{\partial r} &= g & \text{on } \partial B(r) \times \{t\}
\end{aligned}
\]
where \( f \) and \( g \) are defined by
\[
f = v_e \times \left( \frac{x}{2\delta^2} \cdot \nabla v_e - \partial_t v_e \right) - (d^* \psi_t + \xi_t) \cdot \frac{x}{2\delta^2} & \quad \text{on } B(3/2) \times \{t\} \tag{3.53}
\]
and
\[ g = \nu_e \times \frac{\partial \nu_e}{\partial r} - (d^* \psi_t + \xi_t)_N \quad \text{on } \partial B(r) \times \{t\}. \tag{3.54} \]

In view of Lemma 3.6 we choose \( r \in [1, 3/2] \) such that
\[
\int_{\partial B(r) \times \{t\}} |\nabla \nu_e|^2 \exp\left(-\frac{|x|^2}{4\delta^2}\right) \leq 12 \int_{(B(3/2) \setminus B(1)) \times \{t\}} |\nabla \nu_e|^2
\]
\[
\int_{\partial B(r) \times \{t\}} |\nabla \psi_t|^2 \exp\left(-\frac{|x|^2}{4\delta^2}\right) \leq 12 \int_{(B(3/2) \setminus B(1)) \times \{t\}} |\nabla \psi_t|^2
\]
\[
\int_{\partial B(r) \times \{t\}} |\xi_t|^2 \exp\left(-\frac{|x|^2}{4\delta^2}\right) \leq 12 \int_{(B(3/2) \setminus B(1)) \times \{t\}} |\xi_t|^2
\]
so that
\[
\int_{\partial B(r)} g^2 \exp\left(-\frac{|x|^2}{4\delta^2}\right) \leq K \int_{(B(3/2) \setminus B(1)) \times \{t\}} \left(|\nabla \nu_e|^2 + |\nabla \psi_t|^2 + |\xi_t|^2\right) \exp\left(-\frac{|x|^2}{4\delta^2}\right) \tag{3.58}
\]

Our main estimate for \( \varphi_t \) is the following proposition.

**Proposition 3.3.** We have
\[
\int_{B(1)} |\nabla \varphi_t|^2 \exp\left(-\frac{|x|^2}{4\delta^2}\right) \leq K \delta^N \tilde{E}_w,\varepsilon(v_\varepsilon, (0, 1), \delta) + C(\delta_0) \left[R(t) + R(t)^{\frac{1}{2}} \tilde{E}_w,\varepsilon(v_\varepsilon, (0, 1), \delta)^{\frac{1}{2}}\right], \tag{3.59}
\]

where \( C(\delta_0) \) is a constant depending only on \( \delta_0 \), and \( R(t) \) is defined as
\[
R(t) = \int_{\mathbb{R}^N \times \{t\}} \left[\Xi(v_\varepsilon, (0, 1)) + V_\varepsilon(v_\varepsilon) + (|\nabla \psi_t|^2 + |\xi_t|^2) 1_{B(3/2)}\right] \exp\left(-\frac{|x|^2}{4\delta^2}\right) dx.
\]

**Proof.** We apply Lemma 3.6 to \( \varphi_t \). Clearly, in view of the definition (3.53) of \( f \),
\[
\int_{B(r)} f^2 \exp\left(-\frac{|x|^2}{4\delta^2}\right) \leq C(\delta) R(t). \tag{3.60}
\]

On the other hand, by (3.58),
\[
\int_{B(r)} g^2 \exp\left(-\frac{|x|^2}{4\delta^2}\right) \leq K \int_{(B(3/2) \setminus B(1)) \times \{t\}} |\nabla \nu_e|^2 \exp\left(-\frac{|x|^2}{4\delta^2}\right) + K R(t). \tag{3.61}
\]

The important observation is that
\[
\int_{(B(3/2) \setminus B(1)) \times \{t\}} |\nabla \nu_e|^2 \exp\left(-\frac{|x|^2}{4\delta^2}\right) \leq 4\delta^2 \int_{(B(3/2) \setminus B(1)) \times \{t\}} \frac{|x|^2}{4\delta^2} |\nabla \nu_e|^2 \exp\left(-\frac{|x|^2}{4\delta^2}\right)
\]
\[
\leq 2N \delta^N \tilde{E}_w,\varepsilon(v_\varepsilon, (0, 1), \delta) + 3R(t),
\]
where we have used (2.44) for the last inequality. The conclusion then follows from Lemma 3.6. \( \square \)
The following is a direct consequence of Proposition 3.3 and the definition of $\Theta_1$.

**Corollary 3.2.** For $t \in \Theta_1$, we have

$$\int_{B(2\sqrt{\delta_0})} |\nabla \varphi_\delta|^2 \leq K\delta_0^N E_{w,\varepsilon}(\varphi_\delta, (0,1), 1) + C(\delta_0) \left[ \eta + \eta^{1/2} E_{w,\varepsilon}(\varphi_\delta, (0,1), 1) \right],$$

where

$$R_2(t) = C(\delta_0) \left[ \int_{B(3/2)} \left( |\nabla \psi_t|^2 + |\xi_t|^2 \right) + \left( \int_{B(3/2)} |\nabla \psi_t|^2 + |\xi_t|^2 \right)^{1/2} E_{w,\varepsilon}(\varphi_\delta, (0,1), 1) \right],$$

and $C(\delta_0) = K \exp(-\frac{4}{\delta_0})$.

### 3.9 Splitting $\psi_t$

We turn next to the estimate for $\psi_t$. As already announced, this is the key part, and our main contribution in the proof of Theorem 1.

Recall that $\psi_t$ verifies the equation

$$-\Delta \psi_t = 2d(\varphi_t \times d\varphi_t) \chi \quad \text{on } \mathbb{R}^N \times \{t\}, \quad (3.62)$$

where $t = 1 - \delta^2$, $\delta \in [\delta_0, 2\delta_0]$ is fixed but to be determined later, and the cut-off function $\chi$ depends only on $x$, verifies $0 \leq \chi \leq 1$, $\chi \equiv 1$ on $B(2)$, $\chi \equiv 0$ on $\mathbb{R}^N \setminus B(4)$, and $|\nabla \chi| \leq 1$.

First, as in [8], we define a reprojecton of $\varphi_t$ in the following way. Let $\tau$ be the rel-valued function defined on $\mathbb{R}^N \times (0, +\infty)$ by

$$\tau(x, t) = p(|\varphi_t(x, t)|)$$

where $p(\cdot)$ is a function : $[0, 3] \to [\frac{1}{4}, 2]$ verifying the properties

$$\begin{cases} p(s) = \frac{1}{4} & \text{if } \frac{1}{4} \leq s \\ p(s) = 1 & \text{if } 0 \leq s \leq \frac{1}{4} \\ |p'(s)| \leq 4 & \text{for all } s. \end{cases} \quad (3.63)$$

By construction, $\tau$ verifies the inequality

$$\left| 1 - \tau^2(x) \right| \leq K \left| 1 - |\varphi_t(x)|^2 \right|. \quad (3.64)$$

Set $\tilde{\varphi}_t = \tau \varphi_t$, so that

$$\begin{cases} \tilde{\varphi}_t = \varphi_t & \text{if } |\varphi_t| \leq \frac{1}{4} \\ |\tilde{\varphi}_t| = 1 & \text{if } |\varphi_t| \geq \frac{1}{4}. \end{cases}$$

The main motivation for the previous construction is the following observation.
Lemma 3.7. We have
\[ d(\tilde{v}_e \times d\tilde{v}_e)(x) = 2 \sum_{i<j} (\partial_i \tilde{v}_e \times \partial_j \tilde{v}_e) \, dx_i \wedge dx_j. \] (3.65)

In particular,
\[ d(\tilde{v}_e \times d\tilde{v}_e)(x) = 0 \quad \text{if} \quad |v_e(x)| \geq \frac{1}{2} \] (3.66)
and therefore
\[ |d(\tilde{v}_e \times d\tilde{v}_e)| \leq K \frac{(1 - |v_e|^2)^2}{4e^2} = K V_e(v_e) \quad \text{on} \quad \mathbb{R}^N \times (0, +\infty). \] (3.67)

Proof. The identity (3.65) follows easily from the definition of \( d \) and the identity \( d^2 \equiv 0 \).
For (3.66), we notice that if \( |v_e(x)| \geq \frac{1}{2} \) then \( |\tilde{v}_e(x)| = 1 \) so that \( \partial_i \tilde{v}_e \) and \( \partial_j \tilde{v}_e \) are collinear on the set
\[ \mathcal{O} = \left\{ x \in \mathbb{R}^N \quad \text{s.t.} \quad |v_e(x)| \geq \frac{1}{2} \right\}. \]
Finally (3.67) follows from (3.66) and the bound (3.5).

We decompose \( \psi_t \) as
\[ \psi_t = \psi_{1,t} + \psi_{2,t} \quad \text{on} \quad \mathbb{R}^N \times \{t\} \]
where
\[ \begin{align*}
\psi_{1,t} &= -G_N * d(\tilde{v}_e \times d\tilde{v}_e) \chi \quad \text{on} \quad \mathbb{R}^N \times \{t\} \\
\psi_{2,t} &= -G_N * d((1 - \tau^2) v_e \times dv_e) \chi \quad \text{on} \quad \mathbb{R}^N \times \{t\}
\end{align*} \]
so that
\[ \begin{align*}
-\Delta \psi_{1,t} &= d(\tilde{v}_e \times d\tilde{v}_e) \chi \quad \text{on} \quad \mathbb{R}^N \times \{t\} \\
-\Delta \psi_{2,t} &= d((1 - \tau^2) v_e \times dv_e) \chi \quad \text{on} \quad \mathbb{R}^N \times \{t\}.
\end{align*} \] (3.68)

In view of its definition, \( \psi_{2,t} \) is an error term arising from the projection \( \tilde{v}_e \) of \( v_e \). This term can be handled easily as we see next.

3.10 \( L^2 \) estimate for \( \nabla \psi_{2,t} \)

The following inequality holds for \( \psi_{2,t} \).

Lemma 3.8. We have
\[ \int_{\mathbb{R}^N \times \{t\}} |\nabla \psi_{2,t}|^2 \leq C(\delta_0) \int_{\mathbb{R}^N \times \{t\}} V_e(v_e) \exp(-\frac{|x|^2}{4\delta_0^2}) \, dx. \] (3.69)

where \( C(\delta_0) = K \exp(4/\delta_0^2) \) and \( K \) is a constant depending only on \( N \).
Proof. We multiply the second equation of (3.68) by $\psi_{2,t}$ and integrate by parts on $\mathbb{R}^N$. This yields
\[
\int_{\mathbb{R}^N \times \{t\}} |\nabla \psi_{2,t}|^2 \leq \int_{\mathbb{R}^N \times \{t\}} |1 - \tau^2| |v_e \times dv_e| (\chi |\nabla \psi_{2,t}| + |\nabla \chi| |\psi_{2,t}|)
\]
\[
\leq K \left( \int_{B(4) \times \{t\}} V_e(v_e) \right) \frac{1}{2} \left[ \left( \int_{\mathbb{R}^N \times \{t\}} |\nabla \psi_{2,t}|^2 \right)^{\frac{1}{2}} + \left( \int_{\mathbb{R}^N \times \{t\}} |\nabla \chi| |\psi_{2,t}|^2 \right)^{\frac{1}{2}} \right]
\]
where we have used (3.5), (3.64) and the Sobolev inequality. It follows that
\[
\int_{\mathbb{R}^N \times \{t\}} |\nabla \psi_{2,t}|^2 \leq K \exp\left( \frac{4}{\delta_0^2} \right) \int_{\mathbb{R}^N \times \{t\}} V_e(v_e) \exp(-\frac{|x|^2}{4\delta_0^2}) \, dx
\]
and the proof is complete. \qed

We next turn to the estimate for $\psi_{1,t}$. We will first present a simple proof in dimension two, and then give the proof for $N \geq 3$. Although this proof might be adapted for the case $N = 2$, we believe that the simple arguments in case $N = 2$ will shed some insight for the general case.

### 3.11 $L^2$ estimate for $\nabla \psi_{1,t}$ when $N = 2$

The following estimate holds.

**Lemma 3.9.** For every $t \in [1 - 4\delta_0^2, 1 - \delta_0^2]$ we have
\[
\int_{B(2) \times \{t\}} |\nabla \psi_{1,t}|^2 \leq C(\delta_0) \left[ \int_{\mathbb{R}^2 \times \{t\}} V_e(v_e) \exp(-\frac{|x|^2}{4\delta_0^2}) \, dx \right]^2 \log |\epsilon|.
\]

**Proof.** In view of Lemma 3.7 we have
\[
\|\Delta \psi_{1,t}\|_{L^1(\mathbb{R}^2)} \leq K \int_{B(4) \times \{t\}} V_e(v_e)
\]
and by standard elliptic estimates
\[
\|\psi_{1,t}\|_{W^{1,p}(B(4))} \leq K_p \int_{B(4) \times \{t\}} V_e(v_e)
\]
for any $1 \leq p < 2$. On the other hand, we have
\[
\Delta (\psi_{1,t} \chi) = (\Delta \psi_{1,t}) \chi + 2 \nabla \psi_{1,t} \nabla \chi + \psi_{1,t} \Delta \chi
\]
so that
\[
\|\Delta (\psi_{1,t} \chi)\|_{L^1(\mathbb{R}^2)} \leq K \int_{B(4) \times \{t\}} V_e(v_e).
\]
To complete the proof, we present an unpublished argument of a preliminary version of [8], which relies on the following inequality, due to [14] (see also [16] and [54]).
Lemma 3.10. We have, for any $u \in H^2(\mathbb{R}^2)$,
\[
\|u\|_{L^\infty(\mathbb{R}^2)} \leq K \|u\|_{H^1(\mathbb{R}^2)} \left[ 1 + \log^4 \left( 1 + \|u\|_{H^2(\mathbb{R}^2)} \right) \right].
\]

We apply Lemma 3.10 to $\psi_{1,t}\chi$. Since
\[
\|\Delta(\psi_{1,t}\chi)\|_{L^\infty(\mathbb{R}^2)} \leq \frac{K}{\epsilon^2}
\]
by (3.5), we deduce that
\[
\|\psi_{1,t}\chi\|_{H^2(\mathbb{R}^2)} \leq \frac{K}{\epsilon^2}
\]
and therefore
\[
\|\psi_{1,t}\chi\|_{L^\infty(\mathbb{R}^2)} \leq K \|\psi_{1,t}\chi\|_{H^1(\mathbb{R}^2)} \log \epsilon^{\frac{1}{2}}. \tag{3.71}
\]
On the other hand, form standard elliptic estimates and using (3.71) we obtain
\[
\|\psi_{1,t}\chi\|_{H^1(\mathbb{R}^2)}^2 \leq K \|\Delta(\psi_{1,t}\chi)\|_{L^1(\mathbb{R}^2)} \|\psi_{1,t}\chi\|_{L^\infty(\mathbb{R}^2)}
\leq K \|\Delta(\psi_{1,t}\chi)\|_{L^1(\mathbb{R}^2)} \|\psi_{1,t}\chi\|_{H^1(\mathbb{R}^2)} \log \epsilon^{\frac{1}{2}}. \tag{3.72}
\]
The conclusion then follows from (3.70).

Remark 3.1. i) The main point here is the $L^\infty$ estimate for $\psi_{1,t}$. The only property of the equation which is used is the pointwise $L^\infty$ bound on $\nabla v_\epsilon$ in (3.5). A similar type of $L^\infty$ estimate is also used in an essential way for the elliptic case in [8]. The proof there uses, besides (3.5), the monotonicity formula.

ii) Recall that $H^s(\mathbb{R}^2) \hookrightarrow L^\infty_{\text{loc}}(\mathbb{R}^2)$ for $s > 1$. This is however not true for $s = 1$, which is therefore critical for the previous embedding. Lemma 3.10 can thus be interpreted as an interpolation inequality in the critical dimension. There are generalizations of Lemma 3.10 for higher dimension (see [14, 16, 54]), nevertheless they involve critical Sobolev spaces for the corresponding dimension, which require more regularity than $H^1$.

iii) The proof of Lemma 3.10 can be obtained in the Fourier variable by a decomposition in high and low frequencies. This idea will be used also in our estimate of $\psi_{1,t}$ in the next section, however we have to use additional ingredients related to the nonlinear parabolic nature of (PGL)_\epsilon.

3.12 $L^2$ estimate for $\psi_{1,t}$ when $N \geq 3$

The analog of Lemma 3.9 in higher dimension is the following.

Proposition 3.4. There exists a subset $\Theta_2 \subseteq [1 - 4\delta_0^2, 1 - \delta_0^2]$ such that
\[
\text{meas} \left( \Theta_2 \right) \geq \frac{3}{4} \text{meas} \left( [1 - 4\delta_0^2, 1 - \delta_0^2] \right) \tag{3.73}
\]
and for each $t \in \Theta_2$,

$$
\int_{\mathbb{R}^N \setminus \{t\}} |\nabla \psi_{1,t}|^2 \leq C(\delta_0) e^{b \tilde{E}_{w,\epsilon}(v_e, (0, 1), 1)} + C(\delta_0) \int_{\mathbb{R}^N \setminus \{t\}} V_e(v_e) \exp(-\frac{|x|^2}{4\delta^2}) \, dx \left( \tilde{E}_{w,\epsilon}(v_e, (0, 1), 1) + 1 \right), \quad (3.74)
$$

where $C(\delta_0) = K \exp\left(\frac{3\epsilon^2}{\delta}\right)$ and $K$ is a constant depending only on $N$.

**Comment.** In contrast with Lemma 3.9, we are only able to establish inequality (3.74) for appropriate time slices.

The proof of Proposition 3.4 is rather involved. We divide it therefore in several steps.

**Step 1 : Splitting $\psi_t$.**

In view of the proof for the elliptic case in [8], as well as in view of Lemma 3.9, it is tempting to believe that a similar $L^\infty$ bound for $\psi_{1,t}$ can be derived for $N \geq 3$.

Nevertheless, this may not be true (see however [53] for $N = 4$). To overcome this difficulty, we perform a splitting of $\psi_{1,t}$ in high and low frequencies,

$$
\psi_{1,t} = \psi_{1,t}^h + \psi_{1,t}^l \quad \text{on } \mathbb{R}^N \times \{t\}, \quad (3.75)
$$

We will derive an $L^\infty$ estimate for the low frequency part $\psi_{1,t}^l$ and a smallness property for the (weaker) $L^2$ norm of $\psi_{1,t}^h$. For the sake of simplicity, we write $\psi_1$ instead of $\psi_{1,t}$ and similarly $\psi_1^h$ and $\psi_1^l$, whenever this does not lead to a confusion. The high frequencies are essentially contained in $\psi_1^h$, whereas $\psi_1^l$ stands for the low frequency range. Since

$$
\psi_1 = G_N * d(v_e \times d\nu_e) \chi,
$$

we define the splitting (3.75) introducing an appropriate splitting of the kernel $G_N$. More precisely, we write

$$
G_N = G_N^h + G_N^l \equiv m(|x|) G_N + (1 - m(|x|)) G_N
$$

where $m$ is some non negative function with compact support which we will define now. Choose $\alpha \in (\frac{2}{3}, \frac{3}{4})$ and consider the non negative function $l$ defined on $\mathbb{R}^+$ by

$$
l(s) = \begin{cases} 
0 & \text{if } s \leq \epsilon^\alpha \\
((\frac{s}{\epsilon^\alpha})^{N-1} - 1)(2^{N-1} - 1)^{-1} & \text{if } \epsilon^\alpha \leq s \leq 2\epsilon^\alpha \\
1 & \text{if } 2\epsilon^\alpha \leq s \leq 16 \\
(2^{N-1} - (\frac{s}{16})^{N-1})(2^{N-1} - 1)^{-1} & \text{if } 16 \leq s \leq 32 \\
0 & \text{if } s \geq 32.
\end{cases}
$$

We set

$$
m(s) = \begin{cases} 
1 - l(s) & \text{if } 0 \leq s \leq 16 \\
0 & \text{if } s \geq 16.
\end{cases}
$$

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In particular, $m$ is lipschitz with compact support, and
\[
\begin{cases}
m(s) \equiv 1 & \text{for } s \in (0, \epsilon^a) \\
m(s) \equiv 0 & \text{for } s \in (2\epsilon^a, +\infty) \\
|\dot{m}(s)| \leq K\epsilon^{-a}.
\end{cases}
\]

Finally, we define
\[
\psi_i^\epsilon = G_N^i \ast d(\tilde{v}_\epsilon \times d\tilde{v}_\epsilon) \chi \quad \text{on } \mathbb{R}^N \times \{t\},
\]
and
\[
\psi_r^\epsilon = G_N^r \ast d(\tilde{v}_\epsilon \times d\tilde{v}_\epsilon) \chi \quad \text{on } \mathbb{R}^N \times \{t\}.
\]

The following properties of the kernel $G_N^i$ will be useful. The proofs are elementary and left to the reader.

**Lemma 3.11.** We have
\[
\|\nabla G_N^i\|_{L^1(\mathbb{R}^N)} \leq K\epsilon^a,
\]
and
\[
\|\Delta G_N^i\|_{\mathcal{M}(\mathbb{R}^N)} \leq K,
\]
where $\mathcal{M}$ denotes the set of finite Radon measures on $\mathbb{R}^N$ and $K$ is a constant depending only on $N$.

We first begin with the $L^\infty$ estimate for $\psi_i^\epsilon$.

**Step 2 : $L^\infty$ estimate for $\psi_i^\epsilon$.**

First, notice that $G_N^r(x) = G_N(x)$ for $|x| \geq 2\epsilon^a$. In particular, since $\chi$ has compact support in $B(4)$, it follows that
\[
\psi_i^\epsilon = G_N \ast d(\tilde{v}_\epsilon \times d\tilde{v}_\epsilon) \chi \quad \text{on } (\mathbb{R}^N \setminus B(4 + 2\epsilon^a)) \times \{t\}. \tag{3.76}
\]

Therefore, $\psi_i^\epsilon$ is harmonic on $(\mathbb{R}^N \setminus B(5))$ (provided $\epsilon$ is sufficiently small). Hence, by the maximum principle,
\[
\|\psi_i^\epsilon\|_{L^\infty(\mathbb{R}^N \times \{t\})} \leq \|\psi_r^\epsilon\|_{L^\infty(B(5) \times \{t\})}. \tag{3.77}
\]

On the other hand, on the larger ball $B(12)$, one has by the definition of $m$, and in view of the support of $\chi$,
\[
\psi_i^\epsilon = l(|x|)G_N \ast d(\tilde{v}_\epsilon \times d\tilde{v}_\epsilon) \chi \quad \text{on } B(12) \times \{t\}. \tag{3.78}
\]

Recall also that $\text{supp}(l) \subseteq B(32)$ so that $l(|x|)G_N \ast d(\tilde{v}_\epsilon \times d\tilde{v}_\epsilon) \chi$ has compact support in $B(36)$. Combining (3.77) and (3.78), we obtain
\[
\|\psi_i^\epsilon\|_{L^\infty(\mathbb{R}^N \times \{t\})} \leq \|l(|x|)G_N \ast d(\tilde{v}_\epsilon \times d\tilde{v}_\epsilon) \chi\|_{L^\infty(\mathbb{R}^N \times \{t\})}. \tag{3.79}
\]

In order to estimate the r.h.s. of (3.79), we invoke the following lemma, which motivated the precise definition of $l$. A similar construction was already used in our previous work on the NLS equation [10].
Lemma 3.12. Let \( f \in L^1(\mathbb{R}^N) \). The following equality holds for any \( y \in \mathbb{R}^N \).

\[
(l(|x|)G_N * f)(y) = \int_{\alpha}^{\infty} r^{-1} \left\{ \frac{1}{2^{N-2}} \int_{B(y,2r) \times \{t\}} f(x) h(|x-y|, r) \, dx \right\} \, dr
\]

\[
+ \left[ \frac{r^{2-N}}{N-2} \int_{B(y,2r) \times \{t\}} f(x) h(|x-y|, r) \, dx \right]_{\alpha}^{16}, \tag{3.80}
\]

where the lipschitz cut-off function \( h \) is defined on \( \mathbb{R}^+ \times \mathbb{R}^+ \) by

\[
h(s, r) = \omega_{N-1} \frac{(N-1)(N-2)}{2^{N-1}} \begin{cases} \frac{1}{2r-s} & \text{if } 0 \leq s \leq r, \\ \frac{1}{2r} & \text{if } r \leq s \leq 2r, \\ 0 & \text{if } s \geq 2r. \end{cases}
\]

Proof. We start with the r.h.s. of the equality. Integrating by parts in the variable \( r \), we obtain

\[
\int_{\alpha}^{16} r^{-1} \left\{ \frac{1}{2^{N-2}} \int_{B(y,2r) \times \{t\}} f(x) h(|x-y|, r) \, dx \right\} \, dr
\]

\[
+ \left[ \frac{r^{2-N}}{N-2} \int_{B(y,2r) \times \{t\}} f(x) h(|x-y|, r) \, dx \right]_{\alpha}^{16}
\]

\[
= \int_{\alpha}^{16} \frac{r^{2-N}}{N-2} \int_{B(y,2r) \times \{t\}} f(x) \left( \frac{\partial h}{\partial r} \right) (|x-y|, r) \, dx \, dr. \tag{3.81}
\]

Here, we have used the fact that \( h(2r, r) = 0 \) for each \( r > 0 \). Notice that

\[
\frac{\partial h}{\partial r}(|x-y|, r) = \omega_{N-1} \frac{(N-1)(N-2)}{2^{N-1}} \frac{|x-y|}{r^2}
\]

for \( x \in B(y,2r) \setminus B(y,r) \)

and is equal to zero elsewhere. The last term in (3.81) can thus be rewritten as

\[
\int_{\alpha}^{16} r^{-N} \omega_{N-1} \frac{N-1}{2^{N-1}} \int_{(B(y,2r) \setminus B(y,r)) \times \{t\}} f(x) |x-y| \, dx \, dr,
\tag{3.82}
\]

and therefore also as

\[
\omega_{N-1} \frac{N-1}{2^{N-1}} \int_{\alpha}^{16} r^{-N} \int_r^{2r} s \left[ \int_{\partial B(y,s) \times \{t\}} f(x) \, dx \right] ds \, dr \equiv \mathcal{I}(y).
\tag{3.83}
\]

Using Fubini’s theorem, we obtain

\[
\mathcal{I}(y) = \omega_{N-1} \frac{N-1}{2^{N-1}} \int_{\alpha}^{32} s \left[ \int_{\partial B(y,s) \times \{t\}} f(x) \, dx \right] \int_{\min(s/2, \alpha)}^{\max(s,16)} r^{-N} \, dr \, ds. \tag{3.84}
\]

Note that by construction, \( l \) verifies

\[
\frac{N-1}{2^{N-1}} \int_{\min(s/2, \alpha)}^{\max(s,16)} r^{-N} \, dr \equiv l(s) s^{1-N}.
\]
Therefore, since \( \text{supp}(l) \subseteq [e^\alpha, 32] \) we can rewrite

\[
I(y) = \int_{e^\alpha}^{32} \omega_{N-1} s^{2-N} \left[ \int_{B(y,s)} f(x) \, dx \right] l(s) \, ds
\]

\[
= \int_{\mathbb{R}^N \times \{t\}} \frac{\omega_{N-1}}{|y - x|^{N-2}} f(|y - x|) f(x) \, dx,
\]

and the proof is complete. \(\square\)

In view of the previous lemma, we have, for any \( y \in \mathbb{R}^N \),

\[
\left( l(|x|) G_N \ast d(\tilde{v}_e \times d\tilde{v}_e) \chi \right)(y) = \int_{e^\alpha}^{16} \frac{1}{r} J_r(y) \, dr + \frac{1}{N-2} \left[ J_{16}(y) - J_{e^\alpha}(y) \right],
\]

where we set

\[
J_r(y) = \frac{1}{r^{N-2}} \int_{B(y,2r) \times \{t\}} d(\tilde{v}_e \times d\tilde{v}_e) h(|x - y|, r) \chi(x) \, dx.
\]

In particular, for any \( y \in \mathbb{R}^N \)

\[
\left| \left( l(|x|) G_N \ast d(\tilde{v}_e \times d\tilde{v}_e) \chi \right)(y) \right| \leq K \sup_{r \in [e^\alpha, 16]} \left[ \int_{e^\alpha}^{16} \frac{dr}{r} + \frac{2}{N-2} \right]
\]

\[
\leq K \sup_{r \in [e^\alpha, 16]} \left| J_r(y) \right| \left| \|\log \epsilon\| + 1 \right|. \tag{3.85}
\]

In view of (3.65) and the monotonicity formula, one may derive the bound

\[
\left| J_r(y) \right| \leq KE_w e^\alpha(0, (0, 1), 1).
\]

This bound however is far from being satisfactory for our purposes. To proceed further, we argue as in [10], and use a refined estimate due to Jerrard and Soner [34] which relies on the special structure of the Jacobian

\[
J\tilde{v}_e = \frac{1}{2} d(\tilde{v}_e \times d\tilde{v}_e).
\]

More precisely, we have

**Lemma 3.13 (Jerrard & Soner).** Let \( w \in H^1_{\text{loc}}(\mathbb{R}^N) \), \( \varphi \in \mathcal{C}_c^{0,1}(\mathbb{R}^N, \Lambda^2 \mathbb{R}^N) \) and set \( K = \text{supp}(\varphi) \). Then there exist some constants \( K > 0 \) and \( 0 < \beta < 1 \) depending only on \( N \) such that

\[
\left| \int_{\mathbb{R}^N} \langle Jw, \varphi \rangle \right| \leq \frac{K}{|\log \epsilon|} \|\varphi\|_{L^\infty} \int_K \epsilon_r(w) + K \epsilon^\beta \|\varphi\|_{C^1_\epsilon}(1 + \int_K \epsilon_r(w))(1 + |K|^2). \tag{3.86}
\]

With the help of the previous lemma and of the analysis in Section 2.4, we obtain the following.
Lemma 3.14. Let $\beta > 0$ given in Lemma 3.13. We have, for any $y \in B(36),$

$$\sup_{r \in [e^\alpha, 16]} |\mathcal{J}_r(y)| \leq K \exp\left(\frac{34^2}{\delta_0^2} \left( \frac{\tilde{E}_{w,\epsilon}(v_\epsilon, (0,1), 1)}{|\log \epsilon|} + \epsilon^4 \right) \right).$$

Proof. Define the rescaled functions $\tilde{v}_{e, y, r}$ and $\chi_{y, r}$ by

$$\tilde{v}_{e, y, r}(x) = \tilde{v}_e(rx + y) \quad \text{and} \quad \chi_{y, r} = \chi(rx + y).$$

Define also $\epsilon_r = \frac{\epsilon}{r},$ and notice that for $r \in [e^\alpha, 16]$ we have

$$|\log \epsilon_r| \geq (1 - \alpha)|\log \epsilon| \geq \frac{1}{4}|\log \epsilon|.$$  \hspace{1cm} (3.87)

By scaling and the definition of $h,$ we obtain

$$\mathcal{J}_r(y) \equiv \frac{1}{r^{N-2}} \int_{B(y,2r) \times \{t\}} d(\tilde{v}_e \times d\tilde{v}_e) h(|x - y|, r) \chi(x) \, dx$$

$$= \int_{B(2) \times \{t\}} 2J\tilde{v}_{e, y, r}(x) h(x, 1) \chi_{y, r}(x) \, dx. \hspace{1cm} (3.88)$$

Note that since $\chi$ has compact support and $r \leq 16,$

$$\|2h(\cdot, 1)\chi_{y, r}(\cdot)\|_{C_{0,1}(B(2))} \leq K$$

where the constant $K$ depends only on $N.$ We apply Jerrard-Soner estimate (3.86) to $w = \tilde{v}_{e, y, r}$ and $\varphi = 2h(\cdot, 1)\chi_{y, r}.$ In view of (3.87), this yields

$$\left| \int_{B(2) \times \{t\}} 2J\tilde{v}_{e, y, r}(x) h(x, 1) \chi_{y, r}(x) \, dx \right| \leq K \left( \frac{\int_{B(2) \times \{t\}} e_\epsilon(\tilde{v}_{e, y, r}) + \epsilon_r^\beta}{|\log \epsilon_r|} \right)$$

$$\leq K \left( \frac{\int_{B(2) \times \{t\}} e_\epsilon(\tilde{v}_{e, y, r}) + \epsilon_r^\beta}{|\log \epsilon|} \right). \hspace{1cm} (3.89)$$

On the one hand, for $e^\alpha \leq r \leq \frac{\delta}{2},$

$$\int_{B(2) \times \{t\}} e_\epsilon(\tilde{v}_{e, y, r}) = \frac{1}{r^{N-2}} \int_{B(y,2r) \times \{t\}} e_\epsilon(\tilde{v}_e)$$

$$\leq K \frac{1}{r^{N-2}} \int_{\mathbb{R}^N \times \{t\}} e_\epsilon(\tilde{v}_e) \exp\left(-\frac{|x - y|^2}{4r^2}\right) \, dx$$

$$= K \tilde{E}_{w, \epsilon}(v_\epsilon, (y, 1 - \delta^2), r) \leq K \exp\left(\frac{1}{12}\tilde{E}_{w, \epsilon}(v_\epsilon, (0,1), 1)\right), \hspace{1cm} (3.90)$$

where we have used Lemma 2.5 for the last inequality.

On the other hand, for $\frac{\delta}{2} \leq r \leq 16,$

$$\int_{B(2) \times \{t\}} e_\epsilon(\tilde{v}_{e, y, r}) = \frac{1}{r^{N-2}} \int_{B(y,2r) \times \{t\}} e_\epsilon(\tilde{v}_e)$$

$$\leq K \exp\left(\frac{|y| + 32)^2}{4\delta_0^2}\right) \frac{1}{\delta^{N-2}} \int_{\mathbb{R}^N \times \{t\}} e_\epsilon(\tilde{v}_e) \exp\left(-\frac{|x|^2}{4\delta^2}\right) \, dx$$

$$\leq K \exp\left(\frac{34^2}{\delta_0^2}\right) \tilde{E}_{w, \epsilon}(v_\epsilon, (0,1), 1), \hspace{1cm} (3.91)$$

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where we have used the monotonicity formula for the last inequality. The conclusion then follows from (3.88), (3.89), (3.90) and (3.91). 

We are now in position to derive our $L^\infty$ estimate for $\psi_i^\varepsilon$.

**Lemma 3.15.** There exists a constant $K$ depending only on $N$ such that

$$\|\psi_i^\varepsilon\|_{L^\infty(\mathbb{R}^N \cup \{t\})} \leq C(\delta_0) \left( \hat{E}_{w,e}(v_e, (0,1), 1) + 1 \right),$$  

(3.92)

where $C(\delta_0) = K \exp(\frac{3\delta^2}{\delta_0})$.

**Proof.** Recall that by (3.79) we have

$$\|\psi_i^\varepsilon\|_{L^\infty(\mathbb{R}^N \cup \{t\})} \leq \|l(|x|)G_N \ast d(\tilde{v}_e \times d\tilde{v}_e)\chi\|_{L^\infty(\mathbb{R}^N \cup \{t\})}.$$  

Since $\text{supp}(l) \subseteq B(32)$ and $\text{supp}(\chi) \subseteq B(4)$, we also have

$$\left(l(|x|)G_N \ast d(\tilde{v}_e \times d\tilde{v}_e)\chi\right)(y) = 0 \quad \text{for } y \in \mathbb{R}^N \setminus B(36).$$

Therefore we only need to consider the case $y \in B(36)$, and the conclusion follows using (3.85) and Lemma 3.14. 

We next turn to the estimates for the high frequency part of $\psi_1$, namely $\psi_i^1$.

**Step 3 : $L^2$ estimate for $\psi_i^1$.**

Since $\psi_i^1 = G_N^1 \ast d(\tilde{v}_e \times d\tilde{v}_e)\chi$ and since $\|\nabla G_N^1\|_{L^1} \leq K\varepsilon^\alpha$, a few computations yield the following lemma.

**Lemma 3.16.** There exists a constant $K$ depending only on $N$ such that

$$\int_{\mathbb{R}^N \setminus \{t\}} |\psi_i^1|^2 \leq C(\delta_0)\varepsilon^{2\alpha} \hat{E}_{w,e}(v_e, (0,1), 1),$$  

(3.93)

where $C(\delta_0) = K \exp(\frac{\delta_0}{\delta_0})$.

**Proof.** We have

$$\psi_i^1 = m(|x|) \frac{\omega_{N-1}}{|x|^{N-2}} \ast d(\tilde{v}_e \times d\tilde{v}_e)\chi$$

$$= m(|x|) \frac{\omega_{N-1}}{|x|^{N-2}} \ast d(\tilde{v}_e \times d\tilde{v}_e)\chi - m(|x|) \frac{\omega_{N-1}}{|x|^{N-2}} \ast (\tilde{v}_e \times d\tilde{v}_e \cdot d\chi)$$

$$= d\left(m(|x|) \frac{\omega_{N-1}}{|x|^{N-2}} \right) \ast (\tilde{v}_e \times d\tilde{v}_e)\chi - m(|x|) \frac{\omega_{N-1}}{|x|^{N-2}} \ast (\tilde{v}_e \times d\tilde{v}_e \cdot d\chi).$$  

(3.94)

Note that by Lemma 3.11,

$$\left\|d\left(m(|x|) \frac{\omega_{N-1}}{|x|^{N-2}} \right)\right\|_{L^1(\mathbb{R}^N)} \leq K\varepsilon^\alpha,$$  

(3.95)

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and that
\[ \left\| m(|x|) \frac{\omega^{N-1}}{|x|^{N-2}} \right\|_{L^1(\mathbb{R}^N)} \leq K \epsilon^a \leq K \epsilon^a. \] (3.96)

From (3.94) we thus infer that
\[ \left\| \psi_1 \right\|_{L^2(\mathbb{R}^N \times \{t\})} \leq K \epsilon^a \left( \int_{B(1) \times \{t\}} |\nabla \psi_1|^2 \right)^{1/2} \leq K \exp \left( \frac{2}{\delta_0^2} \right) \epsilon^a \bar{E}_{w,e}(v_\epsilon, (0,1), \delta)^{1/2} \] (3.97)
and the conclusion follows using the monotonicity formula. □

**Step 4: Introducing an auxiliary parabolic problem.**

Recall that
\[ -\Delta \psi_1 = d(\tilde{v}_e \times d\tilde{v}_e) \chi \quad \text{on} \ \mathbb{R}^N \times \{t\}. \] (3.98)

In view of the result of Section 2.3, it is tempting to compare \( \psi_1 \) with the solution \( \psi_1^* \) of the parabolic problem
\[ \begin{cases} \partial_t \psi_1^* - \Delta \psi_1^* = d(\tilde{v}_e \times d\tilde{v}_e) \chi & \text{on} \ \mathbb{R}^N \times [0, +\infty) \\ \psi_1^*(.,0) = 0 & \text{on} \ \mathbb{R}^N \times \{0\}. \end{cases} \] (3.99)

In view of Lemma 3.7, we have
\[ |d(\tilde{v}_e \times d\tilde{v}_e) \chi| \leq K \left( \frac{1 - |v_\epsilon|^2}{4\epsilon^2} \right) \chi \quad \text{on} \ \mathbb{R}^N \times [0, \infty), \] (3.100)
where the constant \( K \) depends only on \( N \), and the results of Section 2.3 apply directly to \( \psi_1^* \). This yields

**Lemma 3.17.** We have, for any \( \delta \in [1 - 4\delta_0^2, 1 - \delta^2] \),
\[ \left\| \psi_1^*(., 1 - \delta^2) \right\|_{L^\infty(\mathbb{R}^N)} \leq C(\delta_0) \bar{E}_{w,e}(v_\epsilon, (0,1), 1), \] (3.101)

and
\[ \int_{\mathbb{R}^N \times [0,1 - \delta^2]} |\nabla \psi_1^*|^2 \leq C(\delta_0) \bar{E}_{w,e}(v_\epsilon, (0,1), 1), \] (3.102)
where \( C(\delta_0) = K \exp(4/\delta_0^2) \) and the constant \( K \) depends only on \( N \).

**Proof.** For (3.101), consider the function \( f \) defined by
\[ f(x,t) = \frac{(1 - |v_\epsilon(x,t)|^2)^2}{4\epsilon^2} \chi(x) \quad \text{on} \ \mathbb{R}^N \times [0, \infty) \]
and let \( \omega \) be the solution of
\[ \begin{cases} \frac{\partial \omega}{\partial t} - \Delta \omega = f & \text{on} \ \mathbb{R}^N \times [0, \infty), \\ \omega(x,0) = 0 & \text{for} \ x \in \mathbb{R}^N. \end{cases} \]
It follows from the maximum principle and (3.100) that
\[ |\psi_1^*(x, 1 - \delta^2)| \leq K \omega(x, 1 - \delta^2) \quad \forall \ x \in \mathbb{R}^N. \tag{3.103} \]
We deduce from Proposition 2.2 with \( T = 1 \) and \( \epsilon_T = (0, 1) \) that
\[ \omega(x, 1 - \delta^2) \leq K \left( \frac{1}{1 - \delta^2} \right)^{N/2} \exp \left( \frac{|x|^2}{4\delta^2} \right) \tilde{E}_{\omega,e}(v_{\epsilon}, (0, 1), 1). \]
On the other hand, since \( \chi \) is supported in \( B(4) \), we deduce from Duhamel’s representation formula for \( \omega \) that
\[ \sup_{x \in \mathbb{R}^N} \omega(x, 1 - \delta^2) = \sup_{x \in B(1)} \omega(x, 1 - \delta^2), \]
and therefore
\[ \sup_{x \in \mathbb{R}^N} \omega(x, 1 - \delta^2) \leq K \exp \left( \frac{4}{\delta_0^2} \right) \tilde{E}_{\omega,e}(v_{\epsilon}, (0, 1), 1). \]
The estimate (3.101) then follows from (3.103).

We next turn to (3.102). Multiplying (3.99) by \( \psi_1^* \) and integrating by parts we obtain
\[ \frac{1}{2} \int_{\mathbb{R}^N \times [1 - \delta_0^2]} |\psi_1^*|^2 + \int_{\mathbb{R}^N \times [1 - \delta_0^2]} |\nabla \psi_1^*|^2 = \int_{\mathbb{R}^N \times [0, 1 - \delta_0^2]} \langle d(\bar{v} \times d\bar{v}) \chi, \psi_1^* \rangle. \tag{3.104} \]
Therefore,
\[ \int_{\mathbb{R}^N \times [0, 1 - \delta_0^2]} |\nabla \psi_1^*|^2 \leq \int_{\mathbb{R}^N \times [0, 1 - \delta_0^2]} (|\bar{v} \times d\bar{v}| \cdot |\psi_1^*| + \chi |\nabla \psi_1^*|) \]
\[ \leq K \exp \left( \frac{2}{\delta_0^2} \right) \left( \int_{\mathbb{R}^N \times [0, 1 - \delta_0^2]} |\nabla \psi_1^*|^2 \right) \frac{1}{2} \tilde{E}_{\omega,e}(v_{\epsilon}, (0, 1), 1) \frac{1}{2} \tag{3.105} \]
\[ \leq \frac{1}{2} \int_{\mathbb{R}^N \times [0, 1 - \delta_0^2]} |\nabla \psi_1^*|^2 + K \exp \left( \frac{4}{\delta_0^2} \right) \tilde{E}_{\omega,e}(v_{\epsilon}, (0, 1), 1). \]
Here, we have used the Sobolev inequality and the monotonicity of \( \tilde{E}_{\omega,e}(v_{\epsilon}, (0, 1), \cdot) \). Estimate (3.102) follows and the proof is complete. \( \square \)

**Comment**: Estimate (3.102) seems a little disappointing, since it does not offer any improvement for the energy (in the spirit of Proposition 3.4). However, a little more computations show, using (3.101), that
\[ \int_{\mathbb{R}^N} \int_{1 - \delta_0^2} |\nabla \psi_1^*|^2 \leq C(\delta_0) \left( \int_{\mathbb{R}^N} \int_{1 - \delta_0^2} V_{\epsilon}(v_{\epsilon}) \exp \left( -\frac{|x|^2}{4(1 - t)} \right) dx \right) \tilde{E}_{\omega,e}(v_{\epsilon}, (0, 1), 1) + 1. \]
Notice that this inequality involves only integration on \( \mathbb{R}^N \times [1 - \delta_0^2, 1 - \delta_0^2] \) whereas (3.102) involves integration on \( \mathbb{R}^N \times [0, 1 - \delta_0^2] \). We will not make use of the previous bound.

**Step 5: \( L^2 \) estimate for \( \partial_t \psi_1^* \) on appropriate time slices.**
In order to compare \( \psi_1^* \) with \( \psi_1 \), it seems natural to try to derive some bound on the time derivative \( \partial_t \psi_1^* \). In this direction, we have the following estimate.
Lemma 3.18. We have,

\[
\int_{\mathbb{R}^N \times [0, 1 - \delta_0^2]} |\partial_t \psi^*_1|^2 \leq C(\delta_0) \epsilon^{-1} \tilde{E}_{w, \epsilon}(v_\epsilon, (0, 1), 1),
\]

where \( C(\delta_0) = K \delta_0^{-1} \exp \left( \frac{4}{\delta_0^2} \right) \) and \( K \) is a constant depending only on \( N \).

A straightforward corollary is the following.

Corollary 3.3. There exists a set \( \Theta_2 \subseteq [1 - 4 \delta_0^2, 1 - \delta_0^2] \) such that

\[
\text{meas}(\Theta_2) \geq \frac{3}{4} \text{meas} \left( [1 - 4 \delta_0^2, 1 - \delta_0^2] \right)
\]

and for each \( t \in \Theta_2 \),

\[
\int_{\mathbb{R}^N \times \{t\}} |\partial_t \psi^*_1|^2 \leq C(\delta_0) \epsilon^{-1} \tilde{E}_{w, \epsilon}(v_\epsilon, (0, 1), 1),
\]

where \( C(\delta_0) = K \delta_0^{-3} \exp \left( \frac{4}{\delta_0^2} \right) \) and \( K \) is a constant depending only on \( N \).

Comment. At first sight, this estimate seems rather poor, since the r.h.s. diverges as \(|\log \epsilon| \epsilon^{-1}\), whereas for \( v_\epsilon \) we already know that

\[
\int_{\mathbb{R}^N \times [0, 1 - \delta_0^2]} |\partial_t v_\epsilon|^2 \leq \int_{\mathbb{R}^N \times \{0\}} e_\epsilon(v_\epsilon).
\]

If one assumes \((H_1)\) then the r.h.s. of the previous inequality behaves as \(|\log \epsilon| \epsilon^{-1}\). However, estimate (3.109) is deeply related to the fact that \((\text{PGL})_\epsilon\) is the heat flow for the Ginzburg-Landau energy. Linear estimates based on the pointwise bound \(|\nabla \bar{v}_\epsilon| \leq K \epsilon^{-1}\) would lead only to estimates of order \(\epsilon^{-2}\). In this respect, (3.108) presents a substantial improvement which is again related to the divergence structure of the term \(d(\bar{v}_\epsilon \times d\bar{v}_\epsilon)\). This improvement will be crucial for estimate (3.134).

In order to prove Lemma 3.18, we begin with the following estimate for the time derivative \(\partial_t v_\epsilon\).

Lemma 3.19. We have

\[
\int_{\mathbb{R}^N \times [0, 1 - \delta_0^2]} |\partial_t v_\epsilon|^2 \exp \left( -\frac{|x|^2}{4(1 - t)} \right) dx dt \leq K \delta_0^{-2} \tilde{E}_{w, \epsilon}(v_\epsilon, (0, 1), 1).
\]

Proof. By definition of \(\Xi\), we have

\[
\int_{\mathbb{R}^N \times [0, 1 - \delta_0^2]} |\partial_t v_\epsilon|^2 \exp \left( -\frac{|x|^2}{4(1 - t)} \right) dx dt
\]

\[
\leq \int_{\mathbb{R}^N \times [0, 1 - \delta_0^2]} \frac{1}{1 - t} \left( \Xi(v_\epsilon, (0, 1)) + \frac{|x|^2}{4(1 - t)} |\nabla v_\epsilon|^2 \right) \exp \left( -\frac{|x|^2}{4(1 - t)} \right) dx dt.
\]

The conclusion follows, using (2.36), (3.7), (3.8) and the monotonicity formula. \(\square\)
Proof of Lemma 3.18. We multiply the equation for \( \psi_1^* \), namely

\[
\partial_t \psi_1^* - \Delta \psi_1^* = d(\tilde{\nu}_e \times d\tilde{\nu}_e) \chi \quad \text{on } \mathbb{R}^N \times [0, \infty),
\]

by \( \partial_t \psi_1^* \) and integrate by parts on \( \mathbb{R}^N \times [0, 1 - \delta_0^2] \). We obtain

\[
\begin{aligned}
\int_{\mathbb{R}^N \times [0, 1 - \delta_0^2]} |\partial_t \psi_1^*|^2 \leq & -\frac{1}{2} \int_0^{1 - \delta_0^2} \left( \int_{\mathbb{R}^N} |\nabla \psi_1^*|^2 \, dx \right) dt + \int_{\mathbb{R}^N \times [0, 1 - \delta_0^2]} \langle d(\tilde{\nu}_e \times d\tilde{\nu}_e) \chi, \partial_t \psi_1^* \rangle \\
= & -\frac{1}{2} \int_{\mathbb{R}^N \times \{1 - \delta_0^2\}} |\nabla \psi_1^*|^2 + \int_{\mathbb{R}^N \times [0, 1 - \delta_0^2]} \langle d(\tilde{\nu}_e \times d\tilde{\nu}_e) \chi, \partial_t \psi_1^* \rangle.
\end{aligned}
\]

(3.112)

Since the first term in the r.h.s. of (3.112) is non positive, we only need to concentrate on the term

\[
\int_{\mathbb{R}^N \times [0, 1 - \delta_0^2]} \langle d(\tilde{\nu}_e \times d\tilde{\nu}_e) \chi, \partial_t \psi_1^* \rangle.
\]

The main idea is to exchange space and time derivatives of \( \tilde{\nu}_e \) and \( \psi_1^* \), and for that purpose we proceed by two successive integrations by parts. Set \( \mathcal{U} = \mathbb{R}^N \times [0, 1 - \delta_0^2] \).

We first have

\[
\int_{\mathcal{U}} \langle d(\tilde{\nu}_e \times d\tilde{\nu}_e) \chi, \partial_t \psi_1^* \rangle = \sum_{i < j} \int_{\mathcal{U}} \left( \partial_i (\tilde{\nu}_e \times \partial_j \tilde{\nu}_e) - \partial_j (\tilde{\nu}_e \times \partial_i \tilde{\nu}_e) \right) \chi \partial_t \psi_{1,ij}^*
\]

\[
= \sum_{i < j} \int_{\mathcal{U}} -\tilde{\nu}_e \times \partial_j \tilde{\nu}_e \left( \partial_i (\partial_t \psi_{1,ij}^*) \chi + \partial_i \chi \partial_t \psi_{1,ij}^* \right) + \tilde{\nu}_e \times \partial_i \tilde{\nu}_e \left( \partial_j (\partial_t \psi_{1,ij}^*) \chi + \partial_j \chi \partial_t \psi_{1,ij}^* \right)
\]

\[
= \sum_{i < j} \int_{\mathcal{U}} \partial_i (\tilde{\nu}_e \times \partial_j \tilde{\nu}_e) \chi \partial_t \psi_{1,ij}^* - \partial_i (\tilde{\nu}_e \times \partial_i \tilde{\nu}_e) \chi \partial_t \psi_{1,ij}^*
\]

\[
+ \sum_{i < j} \int_{\mathcal{U}} (\tilde{\nu}_e \times \partial_j \tilde{\nu}_e) \partial_i \chi \partial_t \psi_{1,ij}^* + (\tilde{\nu}_e \times \partial_i \tilde{\nu}_e) \partial_j \chi \partial_t \psi_{1,ij}^*
\]

\[
- \sum_{i < j} \int_{\mathbb{R}^N \times \{1 - \delta_0^2\}} (\tilde{\nu}_e \times \partial_j \tilde{\nu}_e) \partial_t \psi_{1,ij}^* \chi - (\tilde{\nu}_e \times \partial_i \tilde{\nu}_e) \partial_t \psi_{1,ij}^* \chi.
\]

(3.113)

[Here, we write \( \psi_1^* = \sum_{i < j} \psi_{1,ij}^* \, dx_i \wedge dx_j \)] Notice that

\[
\int_{\mathcal{U}} \partial_j (\tilde{\nu}_e \times \partial_i \tilde{\nu}_e) \partial_t \psi_{1,ij}^* \chi
\]

\[
= -\int_{\mathcal{U}} (\tilde{\nu}_e \times \partial_i \tilde{\nu}_e) \partial_j \psi_{1,ij}^* \chi + \int_{\mathcal{U}} (\tilde{\nu}_e \times \partial_i \tilde{\nu}_e) \partial_t \psi_{1,ij}^* \partial_j \chi
\]

\[
= \int_{\mathcal{U}} \partial_i (\tilde{\nu}_e \times \partial_i \tilde{\nu}_e) \partial_j \psi_{1,ij}^* \chi + \int_{\mathcal{U}} (\tilde{\nu}_e \times \partial_i \tilde{\nu}_e) \left( \partial_j \psi_{1,ij}^* \partial \chi - \partial \psi_{1,ij}^* \partial_j \chi \right).
\]

(3.114)

Combining (3.113) and (3.114) we obtain, after some easy algebra,

\[
\int_{\mathcal{U}} \langle d(\tilde{\nu}_e \times d\tilde{\nu}_e) \chi, \partial_t \psi_1^* \rangle = T_1 + T_2 + T_3 + T_4,
\]

(3.115)

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where
\[ T_1 = \sum_{i<j} \int_{\mathcal{U}} (\partial_i \tilde{v}_e \times \partial_j \tilde{v}_e) \partial_i \psi^*_{1,ij} \chi + (\partial_i \tilde{v}_e \times \partial_i \tilde{v}_e) \partial_j \psi^*_{1,ij} \chi, \]
\[ T_2 = \sum_{i<j} \int_{\mathcal{U}} (\tilde{v}_e \times \partial_i \tilde{v}_e) \partial_j \psi^*_{1,ij} \partial_i \chi - (\tilde{v}_e \times \partial_i \tilde{v}_e) \partial_j \psi^*_{1,ij} \partial_j \chi, \]
\[ T_3 = \sum_{i<j} \int_{\mathcal{U}} -(\tilde{v}_e \times \partial_j \tilde{v}_e) \partial_i \psi^*_{1,ij} \partial_i \chi + (\tilde{v}_e \times \partial_i \tilde{v}_e) \partial_j \psi^*_{1,ij} \partial_j \chi, \]

and
\[ T_4 = -\sum_{i<j} \int_{\mathbb{R}^N \times (1-\delta_0)} (\tilde{v}_e \times \partial_j \tilde{v}_e) \partial_i \psi^*_{1,ij} \chi - (\tilde{v}_e \times \partial_i \tilde{v}_e) \partial_j \psi^*_{1,ij} \chi. \]

We first estimate \( T_1 \). By (3.5) we obtain,
\[ T_1 \leq \frac{K}{\epsilon} \left( \int_{\mathcal{U}} |\partial_e \tilde{v}_e|^2 \chi^2 \right)^{\frac{1}{2}} \left( \int_{\mathcal{U}} |\nabla \psi_{1}^*|^2 \right)^{\frac{1}{2}}. \quad (3.116) \]

It follows from (3.110) and the definition of \( \tilde{v}_e \) that
\[ \int_{\mathcal{U}} |\partial_e \tilde{v}_e|^2 \chi^2 \leq K \exp\left( \frac{4}{\delta_0^2} \right) \int_{\mathcal{U}} |\partial_v v_e|^2 \exp\left( -\frac{|x|^2}{4(1-t)} \right) \leq K \delta_0^{-2} \exp\left( \frac{4}{\delta_0^2} \right) \tilde{E}_{w,e}(v_e, (0, 1), 1), \quad (3.117) \]

and from (3.102) that
\[ \int_{\mathbb{R}^N \times [0, 1-\delta_0]} |\nabla \psi_{1}^*|^2 \leq K \exp\left( \frac{4}{\delta_0^2} \right) \tilde{E}_{w,e}(v_e, (0, 1), 1). \quad (3.118) \]

Combining (3.116), (3.117) and (3.118), the estimate for \( T_1 \) can be completed as
\[ T_1 \leq K \delta_0^{-1} \exp\left( \frac{4}{\delta_0^2} \right) \epsilon^{-1} \tilde{E}_{w,e}(v_e, (0, 1), 1). \quad (3.119) \]

We turn next to \( T_2 \), which is estimated exactly as \( T_1 \) except that we don’t need to invoke estimate (3.5). This yields
\[ T_2 \leq K \left( \int_{\mathcal{U}} |\partial_e \tilde{v}_e| |\nabla \chi|^2 \right)^{\frac{1}{2}} \left( \int_{\mathcal{U}} |\nabla \psi_{1}^*|^2 \right)^{\frac{1}{2}} \leq K \delta_0^{-1} \exp\left( \frac{4}{\delta_0^2} \right) \tilde{E}_{w,e}(v_e, (0, 1), 1). \quad (3.120) \]

For \( T_3 \) we obtain, using the monotonicity formula,
\[ T_3 \leq K \left( \int_{\mathcal{U}} |\nabla \tilde{v}_e|^2 |\nabla \chi|^2 \right)^{\frac{1}{2}} \left( \int_{\mathcal{U}} |\partial \psi_{1}^*|^2 \right)^{\frac{1}{2}} \leq K \exp\left( \frac{4}{\delta_0^2} \right) \tilde{E}_{w,e}(v_e, (0, 1), 1) + \frac{1}{2} \int_{\mathcal{U}} |\partial \psi_{1}^*|^2. \quad (3.121) \]
[Notice that the factor 1/2 in front of the last term on the r.h.s. of (3.121) will allow us to absorb it in the l.h.s. of (3.112)]. Finally, for $T_i$, we obtain, using once more the monotonicity formula,

$$
T_i \leq K \left( \int_{\mathbb{R}^N \times \{-\delta_0^i\}} |\nabla \psi_i|^2 |\chi|^2 \right)^\frac{1}{2} \left( \int_{\mathbb{R}^N \times \{-\delta_0^i\}} |\nabla \psi_i^*|^2 \right)^\frac{1}{2}
$$

$$
\leq K \exp \left( \frac{4}{\delta_0^i} \right) \tilde{E}_{w, e}(v, (0, 1), 1) + \frac{1}{4} \int_{\mathbb{R}^N \times \{-\delta_0^i\}} |\nabla \psi_i^*|^2.
$$

(3.122)

[Here again, the presence of the factor 1/4 in front of the last term in (3.122), will allow us to absorb it in the l.h.s. of (3.112)].

Combining (3.112) and (3.115) with the estimates (3.119), (3.120), (3.121) and (3.122), we finally obtain

$$
\int_{\mathbb{R}^N \times [0,1-\delta_0^i]} \left| \partial_t \psi_i^* \right|^2 \leq K \delta_0^{-1} \exp \left( \frac{4}{\delta_0^i} \right) e^{-1} \tilde{E}_{w, e}(v, (0, 1), 1),
$$

(3.123)

and the proof is completed.

**Step 6 : Proof of Proposition 3.4 completed.** Let us recall the estimates that we have obtained so far for $\psi_i = \psi_i^t + \psi_i^* + \psi_i^t$.

For $t \in \Theta_2$ ($\Theta_2$ given by Lemma 3.3), we have

$$
\|\psi_i^t\|_{L^\infty(\mathbb{R}^N \times \{t\})} \leq C(\delta_0) \left( \tilde{E}_{w, e}(v, (0, 1), 1) + 1 \right),
$$

(3.124)

$$
\int_{\mathbb{R}^N \times \{t\}} |\psi_i^t|^2 \leq C(\delta_0) \epsilon^{2\alpha} \tilde{E}_{w, e}(v, (0, 1), 1),
$$

(3.125)

$$
\|\psi_i^t(., t)\|_{L^\infty(\mathbb{R}^N)} \leq C(\delta_0) \tilde{E}_{w, e}(v, (0, 1), 1),
$$

(3.126)

$$
\int_{\mathbb{R}^N \times \{t\}} |\partial_t \psi_i^*|^2 \leq C(\delta_0) \epsilon^{-1} \tilde{E}_{w, e}(v, (0, 1), 1),
$$

(3.127)

and

$$
|d(\tilde{v}_c \times d\tilde{v}_c) \chi| \leq K \frac{(1 - |v_c|^2)^2}{4\epsilon^2} \chi \quad \text{on} \ \mathbb{R}^N \times [0, \infty),
$$

(3.128)

where $C(\delta_0) \leq K \exp \left( \frac{3\epsilon^2}{\delta_0^2} \right)$ and $K$ is a constant depending only on $N$. We also recall that $\psi_1$ and $\psi_1^*$ verify the equations

$$
-\Delta \psi_1 = d(\tilde{v}_c \times d\tilde{v}_c) \chi \quad \text{on} \ \mathbb{R}^N \times \{t\},
$$

(3.129)

$$
\partial_t \psi_1^* - \Delta \psi_1^* = d(\tilde{v}_c \times d\tilde{v}_c) \chi \quad \text{on} \ \mathbb{R}^N \times [0, \infty).
$$

(3.130)

In order to complete the estimate for $|\nabla \psi_i|^2$, we write

$$
|\nabla \psi_i|^2 = \nabla \psi_i \cdot \nabla \psi_i^t + \nabla \psi_i \cdot \nabla \psi_i^t.
$$
and integrate each of the terms of the r.h.s separately. Multiplying (3.129) by $\psi_1^*$ we obtain :

$$
\left| \int_{\mathbb{R}^N \times \{t\}} \nabla \psi_1 \cdot \nabla \psi_1^* \right| \leq C(\delta_0) \int_{\mathbb{R}^N \times \{t\}} V_e(v_e) \exp\left( -\frac{|x|^2}{4\delta^2} \right) dx 
$$

$$
\left( \tilde{E}_{w,e}(v_e, (0,1), 1) + 1 \right). \quad (3.31)
$$

where we have used the $L^\infty$ estimate (3.124) and the $L^1$ estimate (3.128). Similarly, multiplying (3.129) by $\psi_1^t$ we are led to

$$
\left| \int_{\mathbb{R}^N \times \{t\}} \nabla \psi_1 \cdot \nabla \psi_1^t \right| \leq K \left| \int_{\mathbb{R}^N \times \{t\}} \langle d(\tilde{v}_e \times d\tilde{v}_e) \chi, \psi_1^t \rangle \right|. \quad (3.32)
$$

We bound the r.h.s. of (3.32) using the equation for $\psi_1^t$. We obtain, multiplying (3.130) by $\psi_1^t$ and integrating by parts on $\mathbb{R}^N \times \{t\}$, the equality

$$
\int_{\mathbb{R}^N \times \{t\}} \langle d(\tilde{v}_e \times d\tilde{v}_e) \chi, \psi_1^t \rangle = \int_{\mathbb{R}^N \times \{t\}} \partial_t \psi_1^* \cdot \psi_1^t + \int_{\mathbb{R}^N \times \{t\}} \nabla \psi_1^* \cdot \psi_1^t. \quad (3.33)
$$

For the first term of (3.33) we invoke Lemma 3.18 (i.e. estimate (3.127)) and Lemma 3.16 (i.e. estimate (3.125)). By the Cauchy-Schwarz inequality, we therefore obtain

$$
\left| \int_{\mathbb{R}^N \times \{t\}} \langle \partial_t \psi_1^*, \psi_1^t \rangle \right| \leq C(\delta_0) e^{\alpha \frac{s}{2}} \tilde{E}_{w,e}(v_e, (0,1), 1) \leq C(\delta_0) e^{\frac{s}{2}} \tilde{E}_{w,e}(v_e, (0,1), 1). \quad (3.34)
$$

Finally, we turn to the last term in (3.33), that is

$$
\int_{\mathbb{R}^N \times \{t\}} \nabla \psi_1^* \cdot \nabla \psi_1^t = - \int_{\mathbb{R}^N \times \{t\}} \Delta \psi_1^* \cdot \psi_1^t. \quad (3.35)
$$

Notice that

$$
-\Delta \psi_1^* = \Delta G_N^i \ast d(\tilde{v}_e \times d\tilde{v}_e) \chi.
$$

By standard estimates for convolutions we have

$$
\left\| \Delta G_N^i \ast d(\tilde{v}_e \times d\tilde{v}_e) \chi \right\|_{L^1(\mathbb{R}^N \times \{t\})} \leq \left\| \Delta G_N^i \right\|_{\mathcal{M}(\mathbb{R}^N)} \cdot \left\| d(\tilde{v}_e \times d\tilde{v}_e) \chi \right\|_{L^1(\mathbb{R}^N \times \{t\})}
$$

$$
\leq C(\delta_0) \int_{\mathbb{R}^N \times \{t\}} V_e(v_e) \exp\left( -\frac{|x|^2}{4\delta^2} \right) dx,
$$

where we have used Lemma 3.11 and (3.129). Going back to (3.35) we obtain, by (3.126),

$$
\left| \int_{\mathbb{R}^N \times \{t\}} \nabla \psi_1^* \cdot \nabla \psi_1^t \right| \leq C(\delta_0) \int_{\mathbb{R}^N \times \{t\}} V_e(v_e) \exp\left( -\frac{|x|^2}{4\delta^2} \right) dx \tilde{E}_{w,e}(v_e, (0,1), 1). \quad (3.36)
$$
Combining (3.132), (3.133), (3.134) and (3.136) we obtain

\[
\left| \int_{\mathbb{R}^N \times \{t\}} \nabla \psi_1 \cdot \nabla \psi_1^t \right| \leq C(\delta_0) \epsilon^{b} \bar{E}_{w,c}(v_e, (0, 1), 1) \\
+ C(\delta_0) \int_{\mathbb{R}^N \times \{t\}} V_e(v_e) \exp\left(-\frac{|x|^2}{4\delta^2}\right) dx \bar{E}_{w,c}(v_e, (0, 1), 1). \tag{3.137}
\]

Finally, adding (3.131) to (3.137) we obtain the estimate for \( \nabla \psi_1 \),

\[
\int_{\mathbb{R}^N \times \{t\}} |\nabla \psi_1|^2 \leq C(\delta_0) \epsilon^{b} \bar{E}_{w,c}(v_e, (0, 1), 1) \\
+ C(\delta_0) \int_{\mathbb{R}^N \times \{t\}} V_e(v_e) \exp\left(-\frac{|x|^2}{4\delta^2}\right) dx \left( \bar{E}_{w,c}(v_e, (0, 1), 1) + 1 \right),
\]

which ends the proof.

\[ \square \]

### 3.13 Proof of Proposition 3.1 completed

Recall that

\[ v_e \times dv_e = d\varphi_t + d\psi_{1,t} + d\psi_{2,t} + \xi_t \quad \text{on} \quad B(1) \times \{t\}, \]

and that by (3.19),

\[ 4|v_e|^2|\nabla v_e|^2 = 4|v_e \times \nabla v_e|^2 + |\nabla |v_e||^2 = 4|v_e \times \nabla v_e|^2 + 4\rho^2|\nabla \rho|^2, \]

where \( \rho = |v_e| \) denotes the modulus. Using the fact that

\[ 4(1 - |v_e|^2) \cdot |\nabla v_e|^2 \leq K \frac{|1 - |v_e||^2}{\epsilon} |\nabla v_e|^2 \leq 2|\nabla v_e|^2 + KV_e(v_e), \]

we therefore obtain, using the Hodge - de Rham decomposition (3.37),

\[ e_e(v_e) \leq K \left[ |\nabla \varphi|^2 + |\nabla \rho|^2 + |\nabla \psi_{1,t}|^2 + |\nabla \psi_{2,t}|^2 + |\xi_t|^2 + V_e(v_e) \right] \quad \text{on} \quad B(1) \times \{t\}. \tag{3.138} \]

On the other hand, we have by Lemma 3.2, for \( t \in \Theta_1 \),

\[ \bar{E}_{w,c}(v_e, (0, 1), \delta) \leq \frac{1}{\delta^{N-2}} \int_{B(2\sqrt{N} \delta) \times \{t\}} e_e(v_e) + K |\log \delta| \eta, \tag{3.139} \]

where \( \delta = \sqrt{1 - t} \in [\delta_0, 2\delta_0] \).

We emphasize the fact that at this stage \( \delta_0 \) has not been determined yet. In order to use (3.138), we first impose the condition

\[ 4\sqrt{N} \delta_0 \leq 1, \tag{3.140} \]

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so that, if (3.140) is verified, we have, for \( t \in \Theta_1 \),
\[
\int_{B(2\sqrt{N}\delta) \times \{t\}} e_c(v_c) \leq K \int_{B(2\sqrt{N}\delta) \times \{t\}} |\nabla \varphi_t|^2 + |\nabla \rho|^2 + |\nabla \psi_{1,t}|^2 + |\nabla \psi_{2,t}|^2 + |\xi_t|^2 + V_c(v_c).
\] (3.141)
For each of the terms on the r.h.s. of (3.141), we may safely replace the small ball \( B(2\sqrt{N}\delta) \) by the larger ball \( B(1) \), except for the term involving \( \varphi_t \) for which it is crucial to integrate on a ball of radius of order \( \delta \) (see Corollary 3.2).
Notice that \( \Theta_1 \cap \Theta_2 \neq \emptyset \). Indeed
\[
\text{meas}(\Theta_1 \cap \Theta_2) \geq \frac{3}{2} \delta_0^2,
\]
by Lemma 3.1 and Corollary 3.3. Therefore, combining the estimates in Proposition 3.2, Lemma 3.4, Corollary 3.2, Lemma 3.8 and Proposition 3.4, we obtain, for \( t \in \Theta_1 \cap \Theta_2 \),
\[
\int_{B(2\sqrt{N}\delta) \times \{t\}} e_c(v_c) \leq K(\delta_0^N + C(\delta_0)\eta^\frac{1}{\gamma}) \tilde{E}_{w,c}(v_c, (0,1), 1) + C(\delta_0)\eta^\frac{1}{\gamma}.
\]
Hence,
\[
\frac{1}{\delta^{N-2}} \int_{B(2\sqrt{N}\delta) \times \{t\}} e_c(v_c) \leq K(\delta_0^2 + C(\delta_0)\eta^\frac{1}{\gamma}) \tilde{E}_{w,c}(v_c, (0,1), 1) + C(\delta_0)\eta^\frac{1}{\gamma},
\] (3.142)
where \( C(\delta_0) \) depends only on \( \delta_0 \) and \( K \) depends only on \( N \).

We fix \( \delta_0 \) such that (3.140) holds and such that
\[
K \delta_0^2 \leq \frac{1}{4}.
\]
From now on, \( \delta_0 \) is completely determined. So is \( C(\delta_0) \) in (3.142). Therefore, choosing \( \eta_0 \) such that
\[
C(\delta_0)\eta_0^\frac{1}{\gamma} \leq \frac{1}{4},
\]
we have for \( 0 \leq \eta \leq \eta_0 \), combining (3.142) with (3.139),
\[
\tilde{E}_{w,c}(v_c, (0,1), \delta) \leq \frac{1}{2} \tilde{E}_{w,c}(v_c, (0,1), 1) + \mathcal{R}(\eta),
\]
where
\[
\mathcal{R}(\eta) = C(\delta_0)\eta^\frac{1}{\gamma} + K|\log \delta_0|\eta.
\]
This finishes the proof of Proposition 3.1, and hence the proof of Theorem 1 is completed. \( \square \)

4 Consequences of Theorem 1

In this section, we prove some consequences of Theorem 1 which were announced in the introduction. Proposition 1 is immediate and we leave the proof to the reader. We present the proofs of Proposition 2 and 3, and we add another consequence, which allows to localize vorticity under some additional compactness properties of the initial data \( v_c^0 \).
4.1 Proof of Proposition 2

Let \( x_0 \) be any given point in \( B(x_T, \frac{R}{2}) \). We claim that we can find \( 0 < \lambda(T) \) such that

\[
\mathcal{E}_{\tilde{w}, \varepsilon}((x_0, T), r) \leq \eta_1 |\log \varepsilon|, \quad \text{for every } \sqrt{T_0} < r < \sqrt{T_1} = R,
\]

provided \( \eta \leq \frac{\eta_1}{2} \) [recall that \( \lambda \) enters in the definition of \( \eta \)].

**Proof of the claim.** We invoke Proposition 2.3. Let \( \lambda > 0 \) and \( \sqrt{T_0} < r < \sqrt{T_1} = R \), we have

\[
\mathcal{E}_{\tilde{w}, \varepsilon}((x_0, T), r) \leq \frac{1}{r^{N-2}} \int_{B(x_0, \lambda T) \times \{T\}} e_{\varepsilon}(u_{\varepsilon}) + \left( \frac{\sqrt{2}}{\sqrt{T} + 2r^2} \right)^{N-2} M_0 \exp\left( -\frac{\lambda^2}{8} \right) |\log \varepsilon|. \tag{4.2}
\]

First we choose \( \lambda_0(T) \) such that

\[
\left( \frac{2}{T} \right)^{\frac{N-2}{2}} M_0 \exp\left( -\frac{\lambda^2}{8} \right) \leq \frac{\eta_1}{2}. \tag{4.3}
\]

Set

\[ \lambda(T) = \max(2, 2\lambda_0(T)). \]

Since \( x_0 \) belongs to \( B(x_T, \frac{R}{2}) \) and \( r < R \), it follows that

\[ B(x_0, \lambda_0(T) r) \subset B(x_T, \lambda(T) R). \]

Therefore,

\[
\frac{1}{r^{N-2}} \int_{B(x_0, \lambda_0(T) r) \times \{T\}} e_{\varepsilon}(u_{\varepsilon}) \leq \left( \frac{R}{r} \right)^{N-2} \frac{1}{r^{N-2}} \int_{B(x_T, \lambda(T) R) \times \{T\}} e_{\varepsilon}(u_{\varepsilon}) = \left( \frac{R}{r} \right)^{N-2} \eta |\log \varepsilon| \leq \left( \frac{R}{\sqrt{T_0}} \right)^{N-2} \eta |\log \varepsilon|.
\]

Choosing \( T_0 \) of the form \( T_0 = K \eta^{\frac{2}{N-2}} R^2 \), we obtain

\[
\frac{1}{r^{N-2}} \int_{B(x_0, \lambda_0(T) r) \times \{T\}} e_{\varepsilon}(u_{\varepsilon}) \leq K^{-\frac{N-2}{2}} |\log \varepsilon|. \tag{4.4}
\]

It suffices then to fix the constant \( K \) as

\[ K = \left( \frac{\eta_1}{2} \right)^{\frac{N-2}{2}}, \]

so that combining (4.2),(4.3) and (4.4) we obtain (4.1) and the claim is proved. The conclusion then follows from Proposition 1. \( \square \)

In the next section, we will make use of the following easy variant of Proposition 2.
Proposition 4.1. Let $u_\varepsilon$ be a solution of $(PGL)_\varepsilon$ verifying assumption $(H_0)$. Let $x_T \in \mathbb{R}^N$, $T \geq 0$ and $R \geq \sqrt{2\varepsilon}$. There exists a positive continuous function $\lambda$ defined on $(\mathbb{R}^+)^2$ such that, if
\[
\tilde{\eta}(x_T, T, R) \equiv \frac{1}{R^{N-2}} |\log \varepsilon| \int_{B(x_T, \lambda(T, R) R)} e_\varepsilon(u_\varepsilon(\cdot, T)) \leq \frac{\eta_1}{2}
\]
then
\[
|u_\varepsilon(x, t)| \geq \frac{1}{2} \quad \text{for } t \in [T + T_0, T + T_1] \quad \text{and } x \in B(x_T, \frac{R}{2}).
\]
The function $\lambda(T, R)$ verifies
\[
\lambda(T, R) \sim \sqrt{\frac{N-2}{2} |\log(T + R^2)|}, \quad \text{for } (T, R) \to (0, 0),
\]
and in particular $\lambda(T, R) R$ remains bounded as $R \to 0$, for any $T$.

4.2 Proof of Proposition 3
We have, for any $x_0 \in \mathbb{R}^N$ and $t \geq T_f$,
\[
\tilde{\mathcal{E}}_{w, \varepsilon}(x_0, 0, 0) \frac{1}{\sqrt{t}^{N-2}} \int_{\mathbb{R}^N} e_\varepsilon(u_0) \exp(-\frac{|y-x|^2}{4t}) dy \\
\leq \frac{1}{\sqrt{t}^{N-2}} M_0 |\log \varepsilon| \leq T_f \frac{N-2}{2} M_0 |\log \varepsilon| \\
\leq \eta_1 |\log \varepsilon|,
\]
in view of the definition of $T_f$. The conclusion follows from Proposition 1. \hfill \Box

4.3 Localizing vorticity
In this section, we assume that $u_\varepsilon^0$ is localized in some large ball $B(R_1)$. More precisely, we will assume that there exists $R_1 > 0$ such that
\[
(H_1) \quad u_\varepsilon^0 \equiv 1 \quad \text{on } \mathbb{R}^N \setminus B(R_1).
\]
In particular, there is no vorticity outside $B(R_1)$ at time zero. In this situation, we will show that $\mathcal{V}_\varepsilon \cap \{t \geq 2\varepsilon\}$ remains confined in a bounded region of $\mathbb{R}^N \times (0, +\infty)$. In view of Proposition 3, we already know that
\[
\mathcal{V}_\varepsilon \subset \mathbb{R}^N \times [0, T_f], \quad \text{where } T_f = \left( \frac{M_0}{\eta_1} \right)^{\frac{2}{N-2}}.
\]
We thus need to prove that, under assumption $(H_1)$, horizontal spreading is excluded. More precisely, we have
Proposition 4.2. Assume \( u^\theta_\varepsilon \) verifies \((H_0)\) and \((H_1)\). Then there exists \( \bar{R} > 0 \) depending on \( M_0 \) and \( R_1 \), but not on \( \varepsilon \), such that

\[
|u_\varepsilon(x,t)| \geq \frac{1}{2} \quad \text{for all } x \in \mathbb{R}^N \setminus B(\bar{R}) \text{ and } t \geq 2\varepsilon.
\]  

(4.5)

Proof. In view of Proposition 3, (4.5) is already established for \( T \geq T_f \). We therefore assume \( t \leq T_f \). Set

\[
\tau = \max_{0 < R \leq \sqrt{T_f}} \lambda(0, R) R,
\]

where \( \lambda \) is the function defined in Proposition 4.1. Note that \( \tau \) is finite in view of the last remark in Proposition 4.1. Let \( x_0 \in \mathbb{R}^N \setminus B(R_1 + \tau) \), and \( \sqrt{2\varepsilon} < R < \sqrt{T_f} \). We have

\[
\eta(x_0, 0, R) \leq \frac{1}{RN-2|\log \varepsilon|} \int_{B(x_0, \tau)} \epsilon_\varepsilon(u_\varepsilon^0) = 0,
\]

where we have used \((H_1)\). Applying Proposition 4.1 for \( T = 0 \), \( x_T = x_0 \) and \( R \) we obtain the desired conclusion setting \( \bar{R} = R_1 + \tau \).

\[\Box\]

5 Improved pointwise bounds and compactness

The aim of this section is to provide proofs to Theorems 2, 3 and 4.

5.1 Proof of Theorem 2

Since by assumption (10), \( |u_\varepsilon| \geq 1 - \sigma \geq \frac{1}{\varepsilon} \) on \( \Lambda \), there is some real-value function \( \varphi_\varepsilon \) defined on \( \Lambda \) such that

\[
u_\varepsilon = \rho_\varepsilon \exp(i\varphi_\varepsilon) \quad \text{in } \Lambda,
\]

(5.1)

where \( \rho_\varepsilon = |u_\varepsilon| \). Changing \( u_\varepsilon \) possibly by a constant phase, we may impose the additional condition

\[
\frac{1}{|\Lambda|} \int_{\Lambda} \varphi_\varepsilon = 0.
\]

(5.2)

We split as previously the estimates for the phase \( \varphi_\varepsilon \) and for the modulus \( \rho_\varepsilon \), and we begin with the phase. Inserting (5.1) into (PGL)_\varepsilon we are led to the parabolic equation

\[
\rho^2_\varepsilon \frac{\partial \varphi_\varepsilon}{\partial t} - \text{div}(\rho_\varepsilon \nabla \varphi_\varepsilon) = 0 \quad \text{in } \Lambda.
\]

(5.3)

In contrast with the equation for the modulus, (5.3) has the advantage that the explicit dependence on \( \varepsilon \) has been removed. We will handle (5.3) as a linear equation for the function \( \varphi_\varepsilon \), \( \rho_\varepsilon \) being considered as a coefficient. In the sequel, we write \( \varphi = \varphi_\varepsilon \) and \( \rho = \rho_\varepsilon \) when this is not misleading. In order to work on a finite domain, we consider the truncated function \( \tilde{\varphi} \) defined on \( \mathbb{R}^N \times [T, T + \Delta T] \) by

\[
\tilde{\varphi}(x,t) = \varphi(x,t) \chi(x),
\]
where $\chi$ is a smooth cut-off function such that

$$
\chi \equiv 1 \text{ on } B\left(\frac{4}{5}R\right) \quad \text{and} \quad \chi \equiv 0 \text{ on } \mathbb{R}^N \setminus B\left(\frac{5}{6}R\right).
$$

The function $\tilde{\varphi}$ then verifies the equation

$$
\rho^2 \frac{\partial \tilde{\varphi}}{\partial t} - \text{div}(\rho \nabla \tilde{\varphi}) = \text{div}(\rho^2 \varphi \nabla \chi) + \rho^2 \nabla \chi \cdot \nabla \varphi \quad \text{in } \Lambda.
$$

Moreover, by construction

$$
\text{supp}(\tilde{\varphi}) \subset B\left(\frac{4}{5}R\right) \times [T, T + \Delta T],
$$

and in particular $\tilde{\varphi} = 0$ on the vertical part of the boundary of $\Lambda$. By a mean value argument, we may choose some $t_0 \in [T + T + \frac{\Delta T}{4}]$ such that

$$
\int_{B(R) \times \{t_0\}} e_\varepsilon(u_\varepsilon) \leq \frac{4}{\Delta T} \int_\Lambda e_\varepsilon(u_\varepsilon)
$$

and we set

$$
\Lambda_0 = B(R) \times [t_0, T + \Delta T] \supset \Lambda_0^2 = B\left(\frac{3}{4}R\right) \times [T + \frac{\Delta T}{4}, T + \Delta T].
$$

Since by assumption $\rho$ is close to 1, it is natural to treat the l.h.s. of (5.4) as a perturbation of a heat operator, and to rewrite (5.4) as follows

$$
\frac{\partial \tilde{\varphi}}{\partial t} - \Delta \tilde{\varphi} = \text{div}(\rho^2 - 1) \nabla \tilde{\varphi}) + (1 - \rho^2) \frac{\partial \tilde{\varphi}}{\partial t} + \text{div}(\rho^2 \varphi \nabla \chi) + \rho^2 \nabla \chi \cdot \nabla \varphi \quad \text{in } \Lambda.
$$

We introduce the function $\varphi_0$ defined on $\Lambda_0$ as the solution of

$$
\begin{cases}
\frac{\partial \varphi_0}{\partial t} - \Delta \varphi_0 = \text{div}(\rho^2 \varphi \nabla \chi) + \rho^2 \nabla \chi \cdot \nabla \varphi & \text{in } \Lambda_0, \\
\varphi_0(x, t_0) = \tilde{\varphi}(x, t_0) & \text{on } B(R) \times \{t_0\}, \\
\varphi_0(x, t) = 0 & \forall x \in \partial B(R), \forall t \geq t_0.
\end{cases}
$$

In particular, since $\chi \equiv 1$ on $B\left(\frac{4}{5}R\right)$, we have

$$
\frac{\partial \varphi_0}{\partial t} - \Delta \varphi_0 = 0 \quad \text{in } B\left(\frac{4}{5}R\right) \times [t_0, T + \Delta T].
$$

We set $\varphi_1 = \tilde{\varphi} - \varphi_0$, i.e.

$$
\tilde{\varphi} = \varphi_0 + \varphi_1.
$$

We will show that $\varphi_1$ is essentially a perturbation term.

At this stage, we divide the estimates into several steps. We start with linear estimates for $\varphi_0$. 

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Step 1: Estimates for $\varphi_0$. We claim that

$$\|\nabla \varphi_0\|_{L^2 L^2(\Lambda_0)} \leq C_1(\Lambda) \left[ \int_{\Lambda} e_\varepsilon(u_\varepsilon) \right]$$

(5.7)

and

$$\|\nabla \varphi_0\|_{L^\infty(\Lambda_0)} \leq C_2(\Lambda) \left[ \int_{\Lambda} e_\varepsilon(u_\varepsilon) \right],$$

(5.8)

where $2^* = \frac{2N}{N-2}$ is the Sobolev exponent in dimension $N$, and, for $1 < p, q < +\infty$,

$$L^p L^q(\Lambda_0) = \{ f \text{ measurable on } \Lambda_0 \text{ s.t. } \int_{\Lambda_0} \left( \int_{B(R)} |f|^q \right)^{\frac{p}{q}} < +\infty \}$$

[We recall the obvious identity $L^p L^p(\Lambda_0) = L^p(\Lambda_0)$].

Proof. We write $\varphi_0 = \varphi_0^0 + \varphi_0^1$, where $\varphi_0^0$ is defined by

$$\begin{cases}
\frac{\partial \varphi_0^0}{\partial t} - \Delta \varphi_0^0 = 0 & \text{in } \Lambda, \\
\varphi_0^0(x, t_0) = \tilde{\varphi}(x, t_0) & \text{on } B(R) \times \{t_0\}, \\
\varphi_0^0(x, t) = 0 & \forall x \in \partial B(R), \forall t \geq t_0.
\end{cases}$$

By standard estimates for the heat equation, we have

$$\|\nabla^2 \varphi_0^0\|_{L^2(\Lambda_0)} \leq C(\Lambda) \|\nabla \tilde{\varphi}\|_{L^2(B(R) \times \{t_0\})} \leq C(\Lambda) \|e_\varepsilon(u_\varepsilon)\|_{L^1(\Lambda)},$$

and therefore by Sobolev embedding

$$\|\nabla \varphi_0^0\|_{L^2 L^2(\Lambda_0)} \leq C(\Lambda) \|e_\varepsilon(u_\varepsilon)\|_{L^1(\Lambda)}.$$  

(5.9)

We turn next to $\varphi_0^1$. Let $T$ be the linear mapping which, to any function $f$ defined on $\Lambda_0$, associates the unique solution $v = T f$ of the problem

$$\begin{cases}
\frac{\partial v}{\partial t} - \Delta v = f & \text{in } \Lambda, \\
v = 0 & \text{on } B(R) \times \{t_0\}, \\
v = 0 & \forall x \in \partial B(R), \forall t \geq t_0.
\end{cases}$$

It is well known that the operators $f \mapsto \nabla^2(T f), f \mapsto \frac{\partial}{\partial t}(T f)$ and $g \mapsto \nabla(T(\text{div } g))$ are linear continuous on $L^p L^q(\Lambda_0)$ (see e.g. [35]). With these notations, we may write

$$\varphi_0^1 = T f + T(\text{div } g)$$

where

$$f = \rho^2 \nabla \chi \cdot \nabla \varphi, \quad g = \rho^2 \varphi \nabla \chi.$$  

We have the easy estimate

$$\|f\|_{L^2(\Lambda_0)} \leq C(\Lambda) \|\nabla \varphi\|_{L^2(\Lambda_0)} \leq C(\Lambda) \|e_\varepsilon(u_\varepsilon)\|_{L^1(\Lambda)},$$

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and in view of (5.2) and Sobolev embedding,
\[ \|g\|_{L^2 L^{2^*}(\Lambda_0)} \leq C\|\varphi\|_{L^2 L^{2^*}(\Lambda_0)} \leq C(\Lambda)\|\nabla \varphi\|_{L^2 L^{2^*}(\Lambda_0)} \leq C(\Lambda)\|e_\varepsilon(u_\varepsilon)\|_{L^1(\Lambda)} . \]

Therefore, by the linear theory for \( T \) mentioned above,
\[ \begin{align*}
\|\nabla \varphi_0\|_{L^2 L^{2^*}(\Lambda_0)} &\leq \|\nabla (Tf)\|_{L^2 L^{2^*}(\Lambda_0)} + \|\nabla (T(\operatorname{div} g))\|_{L^2 L^{2^*}(\Lambda_0)} \\
&\leq C(\Lambda) \left[ \|f\|_{L^2 L^{2^*}(\Lambda_0)} + \|g\|_{L^2 L^{2^*}(\Lambda_0)} \right] \\
&\leq C(\Lambda) \|e_\varepsilon(u_\varepsilon)\|_{L^1(\Lambda)} .
\end{align*} \] (5.10)

Combining (5.9) and (5.10) we obtain (5.7). Finally, (5.8) follows from (5.6), (5.7) and standard estimates for the homogeneous heat equation.

**Step 2 : The equation for \( \varphi_1 \).** The function \( \varphi_1 \) verifies the evolution problem
\[ \begin{align*}
\frac{\partial \varphi_1}{\partial t} - \Delta \varphi_1 &= \operatorname{div}(\nabla (\varphi + (1 - \varphi^2) \frac{\partial \varphi}{\partial t})) + \varphi_0 + \varphi_1 \\
\varphi_1(x, t_0) &= 0 & \text{in } \Lambda, \\
\varphi_1(x, t) &= 0 & \text{on } B(R) \times \{t_0\}, \\
\varphi_1(x, t) &= 0 & \forall x \in \partial B(R), \forall t \geq t_0. \tag{5.11}
\end{align*} \]

It is convenient to rewrite equation (5.11) as
\[ \frac{\partial \varphi_1}{\partial t} - \Delta \varphi_1 = \operatorname{div}(\nabla (\varphi + (1 - \varphi^2) \varphi_0)) + f_0 + \varphi_1, \tag{5.12} \]
where we have set
\[ f_0 = (1 - \varphi^2) \frac{\partial \varphi}{\partial t} \quad \text{and} \quad g_0 = (\varphi + (1 - \varphi^2) \varphi_0). \]

We have, for any 1 \( \leq p < 2, \)
\[ \|f_0\|^p_{L^2 L^p(\Lambda_0)} \leq C(\Lambda) M_0 \varepsilon^{2-p} \|\log \varepsilon\|. \tag{5.13} \]

Indeed, for any \( t \in [t_0, T + \Delta T] \)
\[ \int_{B(R) \times \{t\}} |f_0|^p = \int_{B(R) \times \{t\}} |1 - \varphi^2| \frac{\partial \varphi}{\partial t} \leq \left( \int_{B(R) \times \{t\}} \left| \frac{\partial \varphi}{\partial t} \right|^2 \right)^{\frac{p}{2}} \left( \int_{B(R) \times \{t\}} (1 - \varphi^2)^{\frac{2}{2-p}} \right)^{\frac{2-p}{2}} \leq C(\Lambda) \left( \int_{B(R) \times \{t\}} \left| \frac{\partial \varphi}{\partial t} \right|^2 \right)^{\frac{p}{2}} \varepsilon^{2-p} \|M_0 \| \|\log \varepsilon\|^{\frac{2-p}{2}}. \]

Hence,
\[ \int_{t_0}^{T + \Delta T} \left( \int_{B(R) \times \{t\}} |h_0|^2 \right)^{\frac{p}{2}} \leq C(\Lambda) \left[ \int_{\Lambda_0} \left| \frac{\partial \varphi}{\partial t} \right|^2 \right] \varepsilon^{\frac{p}{2}(2-p)} \|M_0 \| \|\log \varepsilon\|^{\frac{2-p}{2}} \leq C(\Lambda) \varepsilon^{\frac{p}{2}(2-p)} \|M_0 \| \|\log \varepsilon\|^{\frac{2-p}{2}}, \]

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and (5.13) follows. Similarly using (5.7), we obtain, for any $2 \leq q < 2^*$, the estimate for $g_0$

$$
\|g_0\|_{L^2(L^q(\Lambda_0))} \leq C(\Lambda) M_0 \varepsilon^{2^* - q} \|\log \varepsilon\|.
$$

(5.14)

We now estimate $\varphi_1$ from (5.12) through a fixed point argument.

**Step 3: The fixed point argument.** Equation (5.12) may be rewritten as

$$
\varphi_1 = \mathcal{T}(\text{div}(\rho^2 - 1) \nabla \varphi_1)) + \mathcal{T}(f_0 + \text{div} g_0),
$$

which is of the form

$$(\text{Id} - A)\varphi_1 = b$$

where $A$ is the linear operator $v \mapsto \mathcal{T}(\text{div}(\rho^2 - 1) \nabla v))$ and $b = \mathcal{T}(f_0 + \text{div} g_0)$. To go further we need to specify the function space on which we consider this operator. Set $I = [t_0, T + \Delta T]$. Fix $p$ and $q$ such that they verify the conditions

$$
1 < p < 2, \quad q = p^* = \frac{Np}{N - p}, \quad \text{and} \quad 2 < q < 2^*
$$

[Although the choice of possible $p$ and $q$ verifying the previous conditions shrinks as $N$ increases, it never becomes void!]. Consider the Banach space

$$
X_q = \left\{ v \in W^{1,2}(I, W^{-1,q}(B(R))) \cap L^2(I, W^{1,q}(B(R))) \text{ s.t. } v(0) = 0 \right\}.
$$

It follows from the linear theory for $\mathcal{T}$ mentioned earlier that $A : X_q \to X_q$ is linear continuous and that

$$
\|A\|_{L^q(X_q)} \leq C(q) \|1 - \rho\|_{L^\infty(\Lambda_0)}.
$$

In particular, we may fix $\sigma > 0$ such that

$$
C(q) \|1 - \rho\|_{L^\infty(\Lambda_0)} \leq C(q) \sigma < \frac{1}{2}.
$$

With this choice of $\sigma$, we deduce that $I - A$ is invertible on $X_q$ and

$$
\|\varphi_1\|_{X_q} \leq C\|b\|_{X_q}.
$$

(5.15)

Finally, by (5.13), (5.14) and Sobolev embedding we obtain

$$
\|b\|_{X_q} \leq \|\mathcal{T}f_0\|_{X_q} + \|\mathcal{T}(\text{div} g_0)\|_{X_q}
\leq \|\nabla\mathcal{T}f_0\|_{L^{2,L^{q'}}} + \|\partial_t\mathcal{T}f_0\|_{L^{2,W^{-1,q}}} + \|g_0\|_{L^2L^q}
\leq C(\Lambda) \left\| \left( \|
abla^2 + \text{Id}\right)\mathcal{T}f_0\|_{L^{2,L^p}} + \|\partial_t\mathcal{T}f_0\|_{L^{2,L^p}} + \|g_0\|_{L^2L^q} \right\|
\leq C(\Lambda) \left\| f_0\|_{L^{2,L^p}} + \|g_0\|_{L^2L^q} \right\|
\leq C(\Lambda) (\varepsilon^2 - \varepsilon^{2^*} - \varepsilon^{2^* - q}) (M_0 + 1) |\log \varepsilon|.
$$

For the third inequality, we have used the fact that $L^p \hookrightarrow W^{-1,q}$ (recall our choice $q = p^*$). This, in turn, is a consequence of the Sobolev embedding $W^{1,q'} \hookrightarrow L^{p'}$ which
follows from the identity \((q')^* = ((p^*)')^* = p'\), where stars and primes refer to Hölder and Sobolev conjugates in dimension \(N\).

The following estimate for \(\varphi_1\) then follows from (5.15)

\[
\|\nabla \varphi_1\|_{L^2 L^\infty(\Lambda_0)} \leq C(\Lambda) (\varepsilon^{\frac{n-2}{p}} + \varepsilon^{\frac{n-2}{2-p}}) (M_0 + 1)|\log \varepsilon|.
\]  

(5.16)

We now combine the estimates for \(\varphi_0\) and \(\varphi_1\).

**Step 4 : Improved integrability of \(\nabla \tilde{\varphi}\).** Combining (5.7) and (5.16) we obtain

\[
\|\nabla \tilde{\varphi}\|_{L^2 L^\infty(\Lambda_0)} \leq C(\Lambda) (M_0 + 1)|\log \varepsilon|.
\]  

(5.17)

**Comment.** Since \(q > 2\), the previous estimate presents a substantial improvement over the corresponding inequality with \(q\) replaced by 2, which follows directly from \((H_0)\). This improvement is crucial in order to prove the smallness of both the modulus and potential terms in the energy, which we derive now.

**Step 5 : Estimates for the modulus and potential terms.**

The function \(\rho\) satisfies the equation

\[
\frac{\partial \rho}{\partial t} - \Delta \rho + \rho|\nabla \varphi|^2 = \rho \frac{(1 - \rho^2)}{\varepsilon^2}.
\]  

(5.18)

Since \(\chi \equiv 1\) on \(B(\frac{1}{5}R)\), we have \(\varphi = \tilde{\varphi}\) on \(B(\frac{1}{5}R)\). Let \(\xi\) be a non-negative cut-off function such that \(\xi \equiv 1\) on \(B(\frac{1}{3}R)\) and \(\xi \equiv 0\) outside \(B(\frac{1}{3}R)\). Multiplying (5.18) by \((1 - \rho^2)\xi\) and integrating by parts we obtain

\[
\int_{\Lambda_0} 2\rho|\nabla \rho|^2 \xi + \int_{\Lambda_0} \rho \frac{(1 - \rho^2)^2}{\varepsilon^2} = \int_{\Lambda_0} \frac{\partial \rho}{\partial t} (1 - \rho^2)\xi
\]

\[
+ \int_{\Lambda_0} \nabla \rho \cdot \nabla \xi (1 - \rho^2) + \int_{\Lambda_0} \rho (1 - \rho^2) |\nabla \tilde{\varphi}|^2 \xi.
\]

Hence, since \(\rho \geq \frac{1}{2}\) on \(\Lambda\) we obtain

\[
\int_{\Lambda_0} \frac{1}{2} |\nabla \rho|^2 + V_\varepsilon (u_\varepsilon) \leq K\varepsilon \left( \int_{\Lambda_0} |\nabla \rho|^2 \right)^{\frac{1}{2}} \left( \int_{\Lambda_0} V_\varepsilon (u_\varepsilon) \right)^{\frac{1}{2}}
\]

\[
+ K \int_0^{T + \Delta T} \left( \int_{B(\frac{4}{5}R)} |\nabla \tilde{\varphi}|^2 \right)^{\frac{1}{2}} \left( \int_{B(\frac{4}{5}R)} (1 - \rho^2) \right)^{\frac{n-2}{2}} dt
\]

so that using (5.17) we finally infer that

\[
\int_{\Lambda_0} \left[ |\nabla u_\varepsilon|^2 + V_\varepsilon (u_\varepsilon) \right] \leq C(\Lambda) (M_0 + 1) (\varepsilon^{\frac{2}{q-2}} + \varepsilon)|\log \varepsilon|^2.
\]  

(5.19)

To summarize, we have proved at this stage that
\[ e_\varepsilon(u_\varepsilon) \leq |\nabla \varphi_0|^2 + r_\varepsilon, \] (5.20)

for some \( r_\varepsilon \geq 0 \) which verifies

\[ \int_{\Lambda^\alpha_F} r_\varepsilon \leq C(A)M_0\varepsilon^\alpha, \] (5.21)

for some small \( \alpha > 0 \) depending only on \( N \). Therefore, we set

\[ \Phi_\varepsilon = \varphi_0. \]

**Step 6 : Proof of the \( L^\infty \) bound (11) for the energy.**

This step relies on a result by Chen and Struwe [19] (see also [51] and [47]), which provides an \( L^\infty \) bound for the Ginzburg Landau energy on a cylinder, provided the \( L^1 \) norm of the energy on a larger cylinder is small. More precisely we have

**Proposition 5.1.** (see [19]) Let \( 0 < \varepsilon < 1 \) and let \( v_\varepsilon \) be a solution of \( (PGL)_\varepsilon \) on the cylinder \( \Lambda^0_R = B(R) \times [0, R^2] \) for some \( R > 0 \). Then there exists a constant \( \gamma_0 > 0 \), depending only on \( N \) such that if \( R > \sqrt{\varepsilon} \) and

\[ \frac{1}{RN} \int_{\Lambda_R} e_\varepsilon(v_\varepsilon) \leq \gamma_0 \] (5.22)

then

\[ e_\varepsilon(v_\varepsilon)(x, t) \leq K \frac{1}{RN^{\gamma_0}} \int_{\Lambda_R} e_\varepsilon(v_\varepsilon) \] (5.23)

for any \( (x, t) \in B(\frac{R}{2}) \times [\frac{3}{4}R, R] \).

In our situation, (5.22) is not met, in general, for the function \( u_\varepsilon \) itself. However, we will use Proposition 5.2 for a suitably scaled version of \( u_\varepsilon \), for which (5.22) applies.

Let \( \sqrt{\varepsilon} < r_0 < \frac{1}{8}R \), to be determined later, set \( \varepsilon = \frac{r_0^2}{R^2} \) and let \( (x_0, t_0) \in \Lambda^\alpha_F \) be fixed. Consider the map \( v_\varepsilon \) defined on \( \Lambda^0_1 = B(1) \times [0, 1] \) by

\[ v_\varepsilon(x, t) = u_\varepsilon \left( \frac{x - x_0}{r_0}, \frac{(t - t_0) + r_0^2}{r_0^2} \right) \]

so that

\[ u_\varepsilon(x_0, t_0) = v_\varepsilon(0, 1). \]

By scaling, we have

\[ \int_{\Lambda^0_1} e_\varepsilon(v_\varepsilon) = \frac{1}{r_0^N} \int_{\Lambda_0(x_0, t_0)} e_\varepsilon(u_\varepsilon) \] (5.24)

where we set \( \Lambda_0(x_0, t_0) = B(x_0, r_0) \times [t_0 - r_0^2, t_0] \). Note in particular, since \( r_0 < \frac{1}{8}R \), that

\[ \Lambda_0(x_0, t_0) \subset \Lambda^\alpha_F. \]
and we may apply the decomposition (5.20) to assert that

\[
\int_{\Lambda_0(x_0,t_0)} e_\varepsilon(u_\varepsilon) \leq \text{meas}(\Lambda_0(x_0,t_0)) \cdot \|\nabla \Phi_\varepsilon\|_{L^\infty}^2 + \int_{\Lambda_0^T} r_\varepsilon
\]

\[
\leq \omega_N C_2(\Lambda) r_0^{N+2} \|e_\varepsilon(u_\varepsilon)\|_{L^1(\Lambda)} + C(\Lambda) M_0 \varepsilon^\alpha.
\]

Hence, going back to (5.24)

\[
\int_{\Lambda_0^T} e_\varepsilon(v_\varepsilon) \leq \omega_N C_2(\Lambda) r_0^2 \|e_\varepsilon(u_\varepsilon)\|_{L^1(\Lambda)} + C(\Lambda) M_0 r_0^{-N} \varepsilon^\alpha.
\]  \hfill (5.25)

Therefore, we choose

\[
r_0 = \inf \left\{ \frac{1}{8} R, \left( \frac{2 \omega_N C_2(\Lambda) \|e_\varepsilon(u_\varepsilon)\|_{L^1(\Lambda)}}{\gamma_0} \right)^{-\frac{1}{2}} \right\}.
\]

Note in particular that \( r_0^{-N} \) diverges at most as \( |\log \varepsilon|^\frac{1}{2} \). Hence, for \( \varepsilon \) sufficiently small,

\[
C(\Lambda) M_0 r_0^{-N} \varepsilon^\alpha \leq \frac{\gamma_0}{2}.
\]

On the other hand, by construction,

\[
\omega_N C_2(\Lambda) r_0^2 \|e_\varepsilon(u_\varepsilon)\|_{L^1(\Lambda)} \leq \frac{\gamma_0}{2}.
\]

Applying Proposition 5.2 to \( v_\varepsilon \), together with \( R = 1 \), we therefore deduce

\[
r_0^2 e_\varepsilon(u_\varepsilon)(x_0, t_0) = e_\varepsilon(v_\varepsilon)(0, 1) \leq K \int_{\Lambda_0^T} e_\varepsilon(v_\varepsilon)
\]

\[
\leq K \omega_N C_2(\Lambda) r_0^2 \|e_\varepsilon(u_\varepsilon)\|_{L^1(\Lambda)} + C(\Lambda) M_0 r_0^{-N} \varepsilon^\alpha,
\]

which leads to

\[
e_\varepsilon(u_\varepsilon)(x_0, t_0) \leq C(\Lambda) \int_{\Lambda} e_\varepsilon(u_\varepsilon) + C(\Lambda) M_0 \varepsilon^\beta,
\]

for some constant \( 0 < \beta < \alpha \). This proves (11), for every \( (x_0, t_0) \in \Lambda_{\frac{3}{8}} \).

The remainder of the proof is devoted to the \( L^\infty \) estimates for \( \kappa_\varepsilon \). We start with the modulus and potential terms.

**Step 7 : Improved estimates for \( \nabla \rho \) and \( V_\varepsilon(u_\varepsilon) \).** Set \( \theta = 1 - \rho \). Applying Lemma 1.1 to the cylinder \( \Lambda_{\frac{3}{8}} \), we obtain

\[
|\theta| \leq C(\Lambda) \varepsilon^2 \left( \|\nabla \varphi\|_{L^\infty(\Lambda_{\frac{3}{8}})} + |\log \varepsilon| \right) \leq C(\Lambda) \varepsilon^2 |\log \varepsilon| \quad \text{on } \Lambda_{\frac{3}{16}},
\]  \hfill (5.26)

where we invoke (11) for the last inequality. Going back to (5.18) and using once more (11), we infer that

\[
|\partial_t \theta - \Delta \theta| \leq C(\Lambda) |\log \varepsilon| \quad \text{on } \Lambda_{\frac{3}{16}}.
\]  \hfill (5.27)
Since \((5.27)\) is an \(L^\infty\) bound, we deduce by standard linear theory that, for every \(1 < q_1 < +\infty\) and \(1 < q_2 < +\infty\),

\[
\|\theta\|_{W^{1,q_1}(L^{q_2}(B))} \leq C(\Lambda) |\log \varepsilon|, \quad \|\theta\|_{L^{q_1}(L^{2,q_2}(B))} \leq C(\Lambda) |\log \varepsilon|,
\]

where \(I = [T + \frac{1}{2}\Delta T, T + \Delta T]\) and \(B = B(x_0, \frac{1}{2}R)\). By interpolation (see e.g. \([42, 35]\)), we obtain

\[
\|\theta\|_{W^{1,q_1}(L^{2,q_2}(B))} \leq C(\Lambda) |\log \varepsilon|.
\]

Choosing \(q_1\) and \(q_2\) sufficiently large (in particular \(q_1 > 3, q_2 > 3N\)), we obtain that for every \(0 < \gamma < 1\),

\[
\|\theta\|_{C^{0,\frac{\gamma}{2}}(I;C^{1,\gamma}(B))} \leq C(\gamma, \Lambda) |\log \varepsilon|.
\]

On the other hand, from \((5.26)\) we have

\[
\|\theta\|_{L^\infty(I;L^\infty(B))} \leq C(\Lambda) \varepsilon^2 |\log \varepsilon|,
\]

and therefore by interpolation again

\[
\|\theta\|_{C^{0,\frac{\gamma}{2}}(I;C^{1,\gamma}(B))} \leq C(\Lambda) \varepsilon^\alpha \tag{5.28}
\]

for some (small) \(\alpha > 0\). In particular, we have

\[
|\nabla \rho|_{L^\infty(A_\frac{1}{2})} = |\nabla \theta|_{L^\infty(A_\frac{1}{2})} \leq C(\Lambda) \varepsilon^\alpha. \tag{5.29}
\]

Finally, in view of \((5.26)\) once more, we obtain

\[
V_\varepsilon(u_\varepsilon) \leq K \frac{\theta^2}{\varepsilon^2} \leq C(\Lambda) \varepsilon^2 |\log \varepsilon|^2 \quad \text{on } A_{\frac{1}{2}}
\]

so that

\[
|\nabla \rho| + V_\varepsilon(u_\varepsilon) \leq C(\Lambda) \varepsilon^\alpha \quad \text{on } A_{\frac{1}{2}}. \tag{5.30}
\]

**Step 8 : Improved \(L^\infty\) estimates for \(\nabla \varphi\).** Going back to \((5.12)\), we estimate again \(f_0\) and \(g_0\), but now with the help of the improved estimates for \(\rho\). First, for \(g_0\), we have by \((5.28)\)

\[
\|g_0\|_{C^{0,\frac{\gamma}{2}}(I;C^{1,\gamma}(B))} \leq C(\Lambda) \varepsilon^\alpha |\log \varepsilon|.
\]

For \(f_0\), first notice that since we work on \(A_{\frac{1}{2}}, \varphi = \tilde{\varphi}\) and therefore \(f_0 = (1 - \rho^2) \partial_t \varphi\). From the equation for \(\varphi\),

\[
\rho^2 \frac{\partial \varphi}{\partial t} - \text{div}(\rho^2 \nabla \varphi) = 0,
\]

and from the \(\alpha\)-Hölder regularity bound for \(\rho\), we infer \(\alpha\)-Hölder regularity bounds for \(\partial_t \varphi\), of the order \(|\log \varepsilon|\). Since on the other hand \((5.28)\) holds, we deduce that

\[
\|f_0\|_{C^{0,\alpha}(A_{\frac{1}{2}})} \leq C(\Lambda) \varepsilon^\alpha |\log \varepsilon|^2.
\]
Going back to (5.12) and invoking Schauder theory, we obtain
\[ \| |\nabla \varphi_1| |_{C^0_c(\Lambda^+)} \leq C(\Lambda) \varepsilon^\beta, \] (5.31)
for some \( \beta > 0. \)

**Step 9: Estimate (14) completed.** We write on \( \Lambda^+ \)
\[
e_\varepsilon(u_\varepsilon) = |\nabla u_\varepsilon|^2 + V_\varepsilon(u_\varepsilon)
= |\nabla \rho|^2 + \rho^2 |\nabla \varphi|^2 + V_\varepsilon(u_\varepsilon)
\equiv |\nabla \Phi_\varepsilon|^2 + \kappa_\varepsilon,
\]
and the conclusion follows directly from our previous estimates. The proof of Theorem 2 is thus completed.

In order to prove Theorem 3, we turn next to a new Hodge–de Rham decomposition which is specially tailored for situations where wild oscillations in the phase are present. This decomposition will later help us proving that the linear and topological modes do not interact.

### 5.2 Hodge–de Rham decomposition without compactness

Let \( k \in \mathbb{N}, \, k \geq 3, \) and consider a smooth bounded domain \( \Omega \) in \( \mathbb{R}^k, \) such that \( \pi_1(\partial \Omega) = 0. \) [since \( k \geq 3, \) this is the case for instance if \( \Omega \) is topologically a ball]. Let \( \delta \) and \( \delta^* \) denote respectively the exterior differentiation operator for differential forms on \( \mathbb{R}^k, \) and its formal adjoint [since in the sequel we will take \( k = N + 1, \) we do not use the notations \( d \) and \( d^*, \) which we restrict to \( \mathbb{R}^N \) for the ease of reading]. Let \( v_\varepsilon \) be a smooth complex-valued function defined on \( \Omega. \) We assume that, for some constant \( M_2 > 0, \) \( v_\varepsilon \) verifies the bounds
\[
\int_{\Omega} e_\varepsilon(v_\varepsilon) \leq M_2 |\log \varepsilon|, \tag{5.32}
\]
\[
\int_{\partial \Omega} e_\varepsilon(v_\varepsilon) \leq M_2 |\log \varepsilon|, \tag{5.33}
\]
and
\[
|v_\varepsilon| \leq 3. \tag{5.34}
\]
Then we have

**Proposition 5.2.** Assume that \( v_\varepsilon \) verifies (5.32), (5.33), (5.34). Then there exists a smooth function \( \Phi, \) a smooth 1-form \( \zeta, \) and a smooth 2-form \( \Psi \) defined on \( \Omega, \) such that
\[
v_\varepsilon \times \delta v_\varepsilon = \delta \Phi + \delta^* \Psi + \zeta, \quad \delta \Psi = 0 \text{ in } \Omega, \quad \Psi_T = 0 \text{ on } \partial \Omega, \tag{5.35}
\]
and
\[
\|\nabla \Phi\|_{L^2(\Omega)} + \|\nabla \Psi\|_{L^2(\Omega)} \leq C(\Omega) M_2 |\log \varepsilon|. \tag{5.36}
\]
Moreover, for any $1 \leq p < \frac{k}{k-1}$ we have

\[
\begin{align*}
\|\nabla \Psi\|_{L^p(\Omega)} &\leq C(p, \Omega) M_2, \\
\|\zeta\|_{L^p(\Omega)} &\leq C(p, \Omega) M_2 \varepsilon^\frac{1}{2},
\end{align*}
\] (5.37)

where $C(p, \Omega)$ is a constant depending only on $p$ and $\Omega$.

**Comment.** The terms $\Psi$ and $\zeta$ in the decomposition (5.35) are bounded in suitable norms. Notice however that it is not possible to find a uniform bound on $\Phi$ in any reasonable norm. In vague terms, one might say that the possible lack of compactness of $v_\varepsilon \times \delta v_\varepsilon$ has been completely “locked” into $\Phi$.

**Proof.** We split the proof into two steps. In the first step, we take care of the boundary $\Sigma = \partial \Omega$ (which is by assumption a smooth (k-1)-dimensional manifold), and of the Hodge - de Rham decomposition of the restriction $(v_\varepsilon \times \delta v_\varepsilon)_\Sigma$ to $\Sigma$. Then, we “gauge away” the possible lack of compactness.

**Step 1 : HdR decomposition on $\Sigma$.** Since by assumption $\Sigma$ is simply connected, we may write

\[
(v_\varepsilon \times \delta v_\varepsilon)_\Sigma = v_\varepsilon \times d_{\Sigma} v_\varepsilon = d_{\Sigma} \Phi_\varepsilon^\Sigma + d_{\Sigma}^a \Psi_\varepsilon^\Sigma \quad \text{on } \Sigma, \quad \text{with } d_{\Sigma} \Psi_\varepsilon^\Sigma = 0 \quad \text{on } \Sigma,
\] (5.38)

where $d_{\Sigma}$ denotes the exterior derivative for forms on $\Sigma$, and $d_{\Sigma}^a$ its formal adjoint. Moreover, by orthogonality, we have

\[
\|\nabla_{\Sigma} \Phi_\varepsilon^\Sigma\|_{L^2}^2 + \|\nabla_{\Sigma} \Psi_\varepsilon^\Sigma\|_{L^2}^2 \leq K M_2 |\log \varepsilon|.
\] (5.39)

On the other hand, we claim that for $1 \leq p < \frac{k}{k-1}$,

\[
\|\nabla_{\Sigma} \Psi_\varepsilon^\Sigma\|_{L^p} \leq C(p, \Omega) M_2.
\] (5.40)

Indeed, applying $d_{\Sigma}$ to (5.38), we obtain

\[
-\Delta_{\Sigma} \Psi_\varepsilon^\Sigma = d_{\Sigma} (v_\varepsilon \times d_{\Sigma} v_\varepsilon) = 2J_{\Sigma} v_\varepsilon \quad \text{on } \Sigma.
\] (5.41)

By the Jerrard-Soner estimate [34], we know that, for any $0 < \alpha < 1$,

\[
\|J_{\Sigma} v_\varepsilon\|_{[C^{\alpha,0}], \Sigma} \leq C(\alpha, \Omega) \frac{1}{|\log \varepsilon|} \int_{\partial \Omega} e_\varepsilon(v_\varepsilon) \leq C(\alpha, \Omega) M_2.
\] (5.42)

By the Sobolev embedding, if $q > k - 1$ we have $W^{1,q}(\Sigma) \hookrightarrow C^{0,\alpha}(\Sigma)$ for $\alpha = 1 - \frac{k-1}{q}$, so that by duality $[C^{0,\alpha}(\Sigma)]^* \hookrightarrow [W^{1,q}(\Sigma)]^* = W^{-1,q}(\Sigma)$ where $\frac{1}{p} + \frac{1}{q} = 1$. By elliptic regularity theory, we therefore deduce from (5.41) and (5.42) that, for $1 \leq p < \frac{k}{k-1}$,

\[
\|\Psi_\varepsilon^\Sigma\|_{W^{1,p}} \leq C(p, \Omega) \|J_{\Sigma} v_\varepsilon\|_{W^{-1,p}} \leq C(p, \Omega) \|J_{\Sigma} v_\varepsilon\|_{[C^{0,\alpha}(\Sigma)]^*} \leq C(p, \Omega) M_2.
\] (5.43)
We consider next the harmonic extension $\Phi_\varepsilon^0$ of $\Phi_\varepsilon^\Sigma$ to $\Omega$, i.e.
\[
\begin{cases}
\Delta \Phi_\varepsilon^0 = 0 & \text{in } \Omega, \\
\Phi_\varepsilon^0 = \Phi_\varepsilon^\Sigma & \text{on } \Sigma.
\end{cases}
\]
In view of (5.39), we have
\[
\|\nabla \Phi_\varepsilon^0\|_{L^2(\Omega)}^2 \leq KM_2|\log \varepsilon|.
\] (5.44)

**Step 2**: “Gauge transformation” of $v_\varepsilon$. On $\Omega$ we consider the map $w_\varepsilon$ defined by
\[
w_\varepsilon = v_\varepsilon \exp(-i \Phi_\varepsilon^0) \quad \text{in } \Omega.
\]
Notice that $|w_\varepsilon| = |v_\varepsilon|$. Moreover, a simple computation shows that
\[
w_\varepsilon \times \delta w_\varepsilon = v_\varepsilon \times \delta v_\varepsilon - |v_\varepsilon|^2 \delta \Phi_\varepsilon^0 = v_\varepsilon \times \delta v_\varepsilon - \delta \Phi_\varepsilon^0 + (1 - |v_\varepsilon|^2) \delta \Phi_\varepsilon^0.
\] (5.45)
Since by assumption (5.34) $|v_\varepsilon| \leq 3$, we have
\[
|\nabla w_\varepsilon| \leq |\nabla v_\varepsilon| + 3|\nabla \Phi_\varepsilon^0|
\] (5.46)
and hence
\[
\|\nabla w_\varepsilon\|_{L^2(\Omega)}^2 + \varepsilon^{-2} \|(1 - |v_\varepsilon|^2)\|_{L^2(\Omega)}^2 \leq KM_2|\log \varepsilon|.
\] (5.47)
By Hölder inequality, we have for $1 \leq p < 2$,
\[
\|(1 - |v_\varepsilon|^2)\|_{L^p(\Omega)}^p \leq K M \varepsilon^{2-p} |\log \varepsilon|.
\] (5.48)
and similarly
\[
\|(1 - |v_\varepsilon|^2) d_\Sigma \Phi_\varepsilon^\Sigma\|_{L^p(\Sigma)}^p \leq K M \varepsilon^{2-p} |\log \varepsilon|.
\] (5.49)
Next, we apply the Hodge - de Rham decomposition to $w_\varepsilon \times \delta w_\varepsilon$ on $\Omega$ so that
\[
\begin{cases}
w_\varepsilon \times \delta w_\varepsilon = \delta \Phi_\varepsilon^1 + \delta^* \Psi_\varepsilon & \text{in } \Omega, \\
\delta^* \Psi_\varepsilon = 0 & \text{in } \Omega, \\
\Phi_\varepsilon^1 = 0, \ (\Psi_\varepsilon)_\tau = 0 & \text{on } \Sigma.
\end{cases}
\]
By orthogonality, we have
\[
\|\nabla \Phi_\varepsilon^1\|_{L^2(\Omega)} \leq K M |\log \varepsilon|.
\] (5.50)
Arguing as above, we are led to the elliptic problem
\[
\begin{cases}
-\Delta \Psi_\varepsilon = \omega_\varepsilon \equiv 2Jw_\varepsilon & \text{in } \Omega \\
(\Psi_\varepsilon)_\tau = 0 & \text{on } \Sigma, \\
(\delta^* \Psi_\varepsilon)_\tau = A_\varepsilon \equiv (w_\varepsilon \times \delta w_\varepsilon)_\tau = d_\Sigma^* \Psi_\varepsilon^\Sigma + (1 - |v_\varepsilon|^2) d_\Sigma \Phi_\varepsilon^\Sigma & \text{on } \Sigma.
\end{cases}
\] (5.51)
In view of (5.43) and (5.49) we have, for any $1 \leq p < \frac{k}{k-1}$,
\[
\|A_\varepsilon\|_{L^p(\Sigma)} \leq C(p, \Omega) M_2.
\]
On the other hand, by (5.33) we may invoke Proposition II.1 case ii) in [9] to conclude that for any $0 < \alpha < 1$,
\[
\|\omega_\varepsilon\|_{C^{0,\alpha}(\Omega)} \leq C(\alpha, \Omega) M_2.
\]
[The previous inequality does not follow immediately from Jerrard-Soner’s work [34] since $\omega_\varepsilon$ does not have compact support in $\Omega$.]

In order to conclude, we first invoke the following linear estimate [for a proof see e.g. [8] and the references therein].

**Lemma 5.1.** Let $1 < p < +\infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let $l \in \mathbb{N}, 1 \leq l \leq k$. Let $\Psi$ and $\omega$ be $l$-forms on $\Omega$ and $A$ be an $(l-1)$-form on $\Sigma = \partial \Omega$. Assume that
\[
\begin{cases}
-\Delta \Psi = \omega & \text{in } \Omega \\
\Psi_\tau = 0, & (\delta^* \Psi)_\tau = A & \text{on } \Sigma.
\end{cases}
\]

There exists some constant $C(p, \Omega)$, depending only on $p$ and $\Omega$ such that
\[
\|\Psi\|_{W^{1,p}(\Omega)} \leq C(p, \Omega) \left[ \|\omega\|_{W^{1,q}(\Omega)} + \|A\|_{W^{1,q}(\Sigma)} \right].
\]

**Proof of Proposition 5.2 completed.** For any $1 \leq p < \frac{k}{k-1}$ and $\frac{1}{p} + \frac{1}{q} = 1$ we have,
\[
\| A_\varepsilon \|_{W^{1,\frac{q}{l+p}(\Omega)}} \leq \| A_\varepsilon \|_{L^q(\Omega)} = \| A_\varepsilon \|_{L^p(\Omega)} \leq C(p, \Omega) M_2.
\]

Arguing as for (5.43), we obtain
\[
\| \omega_\varepsilon \|_{W^{1,\frac{q}{l+p}(\Omega)}} \leq C(p, \Omega) \| \omega_\varepsilon \|_{C^{0,\alpha}(\Omega)} \leq C(\alpha, \Omega) M_2.
\]

Therefore, we deduce from Lemma 5.1 that
\[
\| \Psi_\varepsilon \|_{W^{1,p}} \leq C(p, \Omega) M_2. \tag{5.52}
\]

Set
\[
\Psi = \Psi_\varepsilon, \quad \Phi = \Phi_\varepsilon^0 + \Phi_\varepsilon^1, \quad \text{and} \quad \zeta(\|v_\varepsilon\|^2 - 1) \delta \Phi_\varepsilon^0.
\]

Then,
\[
v_\varepsilon \times \delta v_\varepsilon = w_\varepsilon \times \delta w_\varepsilon + |v_\varepsilon|^2 \delta \Phi_\varepsilon^0 = \delta \Phi_\varepsilon^1 + \delta^* \Psi_\varepsilon + \delta \Phi_\varepsilon^0 - (1 - |v_\varepsilon|^2) \delta \Phi_\varepsilon^0 = \delta \tilde{\Phi} + \delta^* \Psi + \zeta,
\]

and the conclusion follows from (5.44), (5.48), (5.50) and (5.52). \hfill $\square$

### 5.3 Evolution of the phase

Let $u_\varepsilon$ be a solution of (PGL)$_\varepsilon$ verifying $(H_0)$. Let $K$ be any compact subset of $\mathbb{R}^N \times (0, +\infty)$. We first choose a parabolic cylinder $\Lambda$ which contains $K$
\[
K \subset \Lambda \equiv B \times (T_0, T_1) \subset \mathbb{R}^N \times (0, +\infty).
\]

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Here $B$ is some open ball in $\mathbb{R}^N$ and $0 < T_0 < T_1$. Next, let $\Omega$ be a smooth bounded domain with simply connected boundary, such that

$$K \subset \Lambda \subset \Omega \subset \mathbb{R}^N \times (0, +\infty).$$

Without loss of generality, we may assume that for $\varepsilon$ sufficiently small

$$\int_\Omega e_\varepsilon(u_\varepsilon) \leq M_2|\log \varepsilon|, \quad \int_{\partial \Omega} e_\varepsilon(u_\varepsilon) \leq M_2|\log \varepsilon|, \quad \text{and} \quad |u_\varepsilon| \leq 3,$$

where $M_2 = C(K)M_0$ and $C(K)$ depends only on $K$. We apply Proposition 5.2 to $u_\varepsilon$. This yields

$$u_\varepsilon \times \delta u_\varepsilon = \delta \Phi + \delta^* \Psi + \zeta, \quad \delta \Psi = 0 \text{ in } \Omega, \quad \Psi_T = 0 \text{ on } \partial \Omega, \quad (5.53)$$

where $\Phi$, $\Psi$ and $\zeta$ verify the bounds (5.37). In view of (5.37), we have already obtained good estimates for $\Psi$ and $\zeta$. In order to handle $\Phi$, we first prove that it solves an evolution equation.

**Lemma 5.2.** The function $\Phi$ in (5.53) verifies the equation

$$\frac{\partial \Phi}{\partial t} - \Delta \Phi = d^*(\delta^* \Psi + \zeta - P_t(\delta^* \Psi + \zeta)dt) - P_t(\delta^* \Psi + \zeta) \text{ in } \Omega. \quad (5.54)$$

Here, for a 1-form $\omega$ on $\Lambda$, we denote by $P_t(\omega)$ its $dt$ component.

**Proof.** Taking the exterior product of $(PGL)_\varepsilon$ with $u_\varepsilon$, we are led to

$$u_\varepsilon \times \frac{\partial u_\varepsilon}{\partial t} - \text{div}(u_\varepsilon \times \nabla u_\varepsilon) = 0 \quad \text{in } \Lambda. \quad (5.55)$$

On the other hand, in view of the decomposition (5.53),

$$\begin{cases}
    u_\varepsilon \times \frac{\partial u_\varepsilon}{\partial t} = \delta \Phi + \delta^* \Psi + \zeta - P_t(\delta^* \Psi + \zeta)dt, \\
    u_\varepsilon \times \frac{\partial u_\varepsilon}{\partial t} = \frac{\partial \Phi}{\partial t} + P_t(\delta^* \Psi + \zeta). \quad (5.56)
\end{cases}$$

Combining (5.55) and (5.56) leads to the conclusion. $\square$

### 5.4 Proof of Theorem 3

Let $u_\varepsilon$ be a solution of $(PGL)_\varepsilon$ verifying $(H_0)$, and let $\Lambda$, $K$, $\Psi$, $\Phi$ and $\zeta$ be as in Section 5.3. Without loss of generality, we may assume that

$$\int_{\partial \Lambda} |\nabla_{x,t} \Phi|^2 \leq C(K)M_0|\log \varepsilon|, \quad (5.57)$$

where $\nabla_{x,t}$ denotes the gradient with respect to both space and time coordinates. Indeed, since by Proposition 5.2

$$\|\nabla_{x,t} \Phi\|_{L^2(\Omega)} \leq C(\Omega)M_2|\log \varepsilon|,$$
if (5.57) were not verified for our original $\Lambda$, we could shrink it to a smaller cylinder, still containing $K$ and verifying (5.57). We decompose next the proof in two steps.

**Step 1 : Defining $\varphi_\varepsilon$.** We set $\partial \Lambda = \mathcal{O}_0 \cup \mathcal{O}_1$, where

$$\mathcal{O}_0 = (B \times \{T_0\}) \cup (\partial B \times [T_0, T_1]) \quad \text{and} \quad \mathcal{O}_1 = B \times \{T_1\}.$$ 

Let $\Phi_1$ be the unique solution of the parabolic problem

$$\begin{cases} \frac{\partial \Phi_1}{\partial t} - \Delta \Phi_1 = d^\ast \delta^\ast \Psi + \zeta - P_t(\delta^\ast \Psi + \zeta) dt \quad \text{in} \quad \Lambda, \\ \Phi_1 = 0 \quad \text{on} \quad \mathcal{O}_0. \end{cases}$$ (5.58)

Since by (5.37) we have

$$\|\delta^\ast \Psi + \zeta - P_t(\delta^\ast \Psi + \zeta) dt\|_{L^p(\Lambda)} + \|P_t(\delta^\ast \Psi + \zeta)\|_{L^p(\Lambda)} \leq C(p, K)M_0,$$

it follows from standard estimates for the non-homogeneous heat equation that

$$\|\nabla \Phi_1\|_{L^p(\Lambda)} \leq C(p, K)M_0. \quad (5.59)$$

Finally, we set

$$\varphi_\varepsilon = \Phi - \Phi_1$$

and

$$w_\varepsilon = u_\varepsilon \exp(-i \varphi_\varepsilon).$$

By construction, $\varphi_\varepsilon$ verifies the homogeneous heat equation

$$\begin{cases} \frac{\partial \varphi_\varepsilon}{\partial t} - \Delta \varphi_\varepsilon = 0 \quad \text{in} \quad \Lambda, \\ \varphi_\varepsilon = \Phi \quad \text{on} \quad \mathcal{O}_1. \end{cases}$$ (5.60)

From standard regularity theory for the heat equation, we have

$$\|\nabla \varphi_\varepsilon\|^2_{L^\infty(K)} \leq C(K) \|\nabla_x \Phi\|^2_{L^2(\mathcal{O}_0)} \leq C(K)M_0|\log \varepsilon|. \quad (5.61)$$

This establishes the third statement of Theorem 3. We next turn to the fourth and last one.

**Step 2 : $W^{1,p}$ estimates for $w_\varepsilon$.** First notice that

$$|w_\varepsilon|^2 |\nabla w_\varepsilon|^2 = |w_\varepsilon|^2 |\nabla |w_\varepsilon||^2 + |w_\varepsilon \times \nabla w_\varepsilon|^2,$$

and hence, since $|w_\varepsilon| = |u_\varepsilon|$,

$$\int_{K \cap \{ |u_\varepsilon| \geq \frac{1}{2} \}} |\nabla w_\varepsilon|^p \leq C(p) \left[ \int_K |w_\varepsilon \times dw_\varepsilon|^p + \int_K |\nabla |w_\varepsilon||^p \right]. \quad (5.62)$$

On the other hand, since $|\nabla w_\varepsilon| \leq |\nabla u_\varepsilon| + 3|\nabla \varphi_\varepsilon| \leq C(K)M_0\varepsilon^{-1}$, we have

$$\int_{K \cap \{ |u_\varepsilon| \leq \frac{1}{2} \}} |\nabla w_\varepsilon|^p \leq C(K)M_0\varepsilon^{2-p} \int_K \frac{(1 - |u_\varepsilon|^2)^2}{4\varepsilon^2} \leq C(p, K)M_0. \quad (5.63)$$
By construction, we have
\[ w_\varepsilon \times \delta w_\varepsilon = u_\varepsilon \times \delta u_\varepsilon - |u_\varepsilon|^2 \delta \varphi_\varepsilon = \delta^* \Psi + \delta \Phi_1 + (1 - |u_\varepsilon|^2) \delta \varphi_\varepsilon. \] (5.64)

Since by the Hölder and Cauchy-Schwarz inequalities
\[ \|(1 - |u_\varepsilon|^2) \delta \varphi_\varepsilon\|_{L^p(K)} \leq C(p, K) \|(1 - |u_\varepsilon|^2)\|_{L^2(K)}^{1 - \frac{1}{p}} \|\delta \varphi_\varepsilon\|_{L^2(K)} \leq C(p, K) \varepsilon^{2 - \frac{1}{p}} |\log \varepsilon| M_0, \]

it follows from (5.64), (5.37) and (5.59) that
\[ \int_K |w_\varepsilon \times \delta w_\varepsilon|^p \leq C(p, K) (M_0 + 1). \] (5.65)

It remains to bound the $L^p$ norm of the gradient of the modulus. For that purpose, we use the following lemma.

**Lemma 5.3.** Set $\rho = |u_\varepsilon|$. The following bound holds, for any compact subset $K \subset \mathbb{R}^N \times (0, +\infty)$, and any $1 \leq p < 2$,
\[ \int_K |\nabla |u_\varepsilon||^p \leq C(K) (M_0 + 1) \varepsilon^{1 - \frac{p}{2}} |\log \varepsilon|, \]

where the constant $C(K)$ depends only on $K$.

**Proof.** The function $\rho$ satisfies the equation
\[ \frac{\partial \rho^2}{\partial t} - \Delta \rho^2 + 2 |\nabla u_\varepsilon|^2 = \frac{2}{\varepsilon^2} \rho^2 (1 - \rho^2). \] (5.66)

Let us introduce the set
\[ A = \{(x, t) \in \Omega, \; \rho(x, t) > 1 - \varepsilon^{1/2}\} \]
and the function
\[ \bar{\rho} = \max\{\rho, 1 - \varepsilon^{1/2}\}, \]
so that $\rho = \rho$ on $A$ and $0 \leq 1 - \bar{\rho} \leq \varepsilon^{1/2}$ in $\Omega$.

Next, let $\chi_K$ be a cut-off function in $\mathcal{D}(\Omega)$ such that $0 \leq \chi_K \leq 1$ on $\Omega$, $\chi_K \equiv 1$ on $K$, and $|\nabla \chi_K| \leq C(K)$.

Multiplying equation (5.66) by $\chi_K(\bar{\rho}^2 - 1)$ (which is compactly supported in $\Omega$), and integrating over $\Omega$ we obtain
\[ \int_\Omega \nabla \rho^2 \nabla \bar{\rho}^2 \chi_K + \int_\Omega \frac{2\rho(1 - \rho^2)(1 - \bar{\rho}^2)}{\varepsilon^2} \chi_K = \int_\Omega (1 - \bar{\rho}^2) |\nabla u_\varepsilon|^2 + \int_\Omega \nabla \rho^2 \nabla \chi_K (1 - \bar{\rho}^2) - \int_\Omega \frac{\partial \rho^2}{\partial t} (\bar{\rho}^2 - 1) \chi_K. \]
It follows that on the set \( A_K = A \cap K \) we have
\[
\int_{A_K} |\nabla \rho|^2 = \int_{A_K} \nabla \rho \cdot \nabla \rho \\
\leq 2\varepsilon^{1/2} \int_{\Omega} |\nabla u_\varepsilon|^2 + C(K) \int_{\Omega} |\nabla \rho| |1 - \rho|^2 + C(K) M_0 \varepsilon^{1/2} |\log \varepsilon| \\
\leq 2\varepsilon^{1/2} \int_{\Omega} |\nabla u_\varepsilon|^2 + C(K) \varepsilon \left[ \int_{\Omega} |\nabla \rho|^2 + \frac{1}{4} \frac{(1-\rho)^2}{\varepsilon^2} \right] + C(K) M_0 \varepsilon^{1/4},
\]

hence, since \( \rho \geq 1 - \varepsilon^{1/2} \) on \( A_K \), we have, for \( \varepsilon \leq 1/4 \),
\[
\int_{A_K} |\nabla \rho|^2 \leq 4 \int_{A_K} |\nabla \rho|^2 \leq C(K) M_0 \varepsilon^{1/4}. 	ag{5.67}
\]

On the other hand, on \( B_K = K \setminus A \) we have \( \int_{B_K} (1 - \rho)^2 \leq C(K) M_0 \varepsilon^2 |\log \varepsilon| \) and hence, since \( (1 - \rho) \geq \varepsilon^{1/2} \) on \( B_K \), it follows that \( |B_K| \leq C(K) M_0 \varepsilon |\log \varepsilon| \). Thus
\[
\int_{B_K} |\nabla \rho|^p \leq \left( \int_{\Omega} |\nabla \rho|^2 \right)^{p/2} |B_K|^{1-p/2} \leq C(K) (M_0 + 1) \varepsilon^{1-p/2} |\log \varepsilon|. 	ag{5.68}
\]

Adding (5.67) and (5.68) we complete the estimate.

Combining (5.62), (5.63), (5.65) and Lemma 5.3, the proof of Theorem 3 is complete.

\[ \square \]

5.5 Hodge - de Rham decomposition with compactness

In this section, we adapt the strategy of the proof of Section 5.2 assuming the compactness assumptions \((H_1)\) and \((H_2)\). We are going to work in the domain
\[
\Omega_\varepsilon = \mathbb{R}^N \times (2\varepsilon, +\infty).
\]

Notice that in contrast to the results of Section 5.2 the domain here is not compact (but the initial data possesses compactness!). Notice also that for technical reasons (in particular in view of Proposition 1.1 and Proposition 4.2) that we have removed the boundary layer present in \( \mathbb{R}^N \times [0, 2\varepsilon] \). This however introduces a new difficulty, namely we have to keep track of the compactness across this boundary layer.

As in the proof of Theorem 1, we begin with a reprojection of \( u_\varepsilon \). Let \( p : \mathbb{R}^+ \to \mathbb{R}^+ \) verify (3.63), let \( \tau = p(|u_\varepsilon|) \) and set
\[
\tilde{u}_\varepsilon = \tau u_\varepsilon,
\]
so that \( |\tilde{u}_\varepsilon| \leq 1 \) on \( \mathbb{R}^N \times (0, +\infty) \),
\[
|\tilde{u}_\varepsilon| = u_\varepsilon \text{ if } |u_\varepsilon| \leq \frac{1}{4}, \quad |\tilde{u}_\varepsilon| = 1 \text{ if } |u_\varepsilon| \geq \frac{1}{2},
\]

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and
\[ J\tilde{u}_\varepsilon = 0 \text{ on } \mathcal{V}_\varepsilon = \{(x,t), \ |u_\varepsilon(x,t)| \geq \frac{1}{2}\}. \]

In particular,
\[ \text{supp} \ (J\tilde{u}_\varepsilon) \subset B(R_1) \times [0,T_f]. \]

We have

**Proposition 5.3.** Assume that \( u_\varepsilon \) verifies \((PGL)_\varepsilon, (H_0), (H_1) \) and \((H_2)\). Then there exists a smooth function \( \Phi \) and a smooth 2-form \( \Psi \) defined on \( \tilde{\Omega}_\varepsilon \) such that

\[
\begin{align*}
\tilde{u}_\varepsilon \times \delta\tilde{u}_\varepsilon &= \delta\Phi + \delta^s \Psi & \text{in } \Omega_\varepsilon, \\
\delta\Psi &= 0 & \text{in } \Omega_\varepsilon, \\
\Phi &= 0, \ \Psi_\top = 0 & \text{on } \partial\Omega_\varepsilon = \mathbb{R}^N \times \{2\varepsilon\}. 
\end{align*}
\]

Moreover,
\[ \|\nabla_{x,t}\Phi\|^2_{L^2(\mathbb{R}^N \times [0,T])} + \|
abla_{x,t}\Psi\|^2_{L^2(\mathbb{R}^N \times [0,T])} \leq CM_0 \log \varepsilon |T|, \quad \text{for all } T \geq 0, \quad (5.70) \]

and for any \( 1 \leq p < \frac{N+1}{N} \) and any compact subset \( K \subset \mathbb{R}^N \times (2\varepsilon, +\infty) \) we have
\[ \|\nabla_{x,t}\Phi\|_{L^p(K)} + \|
abla_{x,t}\Psi\|_{L^p(K)} \leq C(p, K, M_0). \quad (5.71) \]

**Proof.** The Hodge - de Rham decomposition of the 1-form \( \tilde{u}_\varepsilon \times \delta\tilde{u}_\varepsilon \) on \( \Omega_\varepsilon \) leads directly to \((5.69)\) and \((5.70)\). Moreover, applying the \( \delta^s \) operator to \((5.69)\) on \( \Omega_\varepsilon \) we are led to the elliptic problem

\[
\begin{cases}
-\Delta_{x,t}\Psi = 2J_{x,t}\tilde{u}_\varepsilon & \text{in } \Omega_\varepsilon \\
\Psi_\top = 0, \ (\delta^s \Psi)_\top = \tilde{u}_\varepsilon \times \delta\tilde{u}_\varepsilon & \text{on } \partial\Omega_\varepsilon. 
\end{cases}
\]

(5.72)

In the sequel we write simply \( \Delta \) instead of \( \Delta_{x,t} \) and similarly for \( J \), when this is not misleading. Since \( \Psi \) is a 2-form it has \( \frac{N(N+1)}{2} \) different scalar components

\[ \Psi = \sum_{1 \leq i \leq j \leq N} \Psi_{i,j} dx_i \wedge dx_j + \sum_{1 \leq j \leq N} \Psi_j dt \wedge dx_j. \]

Going back to \((5.72)\), we see that the boundary condition on \( \partial\Omega_\varepsilon \) decouple into Neumann conditions for the functions \( \Psi_j \), namely

\[ \frac{\partial \Psi_j}{\partial x_N} = \tilde{u}_\varepsilon \times \frac{\partial u_\varepsilon}{\partial x_j} \quad \text{on } \partial\Omega_\varepsilon, \]

whereas for the functions \( \Psi_{i,j} \) we have homogeneous Dirichlet conditions

\[ \Psi_{i,j} = 0 \quad \text{on } \partial\Omega_\varepsilon. \]

We divide the proof of \((5.71)\) in several steps.
Step 1: $L^p$ estimate for $\nabla \Psi_{i,j}$. We introduce the reflection operator $P_\varepsilon$, which, to any function $f$ defined on $\Omega_\varepsilon$, associates its reflected function $P_\varepsilon f$ defined on $\mathbb{R}^N \times (-\infty, 2\varepsilon)$ by

$$P_\varepsilon f(x, t) = f(x, -t + 4\varepsilon) \quad \forall x \in \mathbb{R}^N, \ t < 2\varepsilon.$$ 

We extend $\Psi_{i,j}$ to $\mathbb{R}^{N+1}$ by setting

$$\Psi_{i,j}(x, t) = -P_\varepsilon \Psi_{i,j}(x, t)$$

so that

$$-\Delta \Psi_{i,j} = 2 \tilde{a}_{x_i} \wedge \tilde{a}_{x_j} - P_\varepsilon (2 \tilde{a}_{x_i} \wedge \tilde{a}_{x_j}) \quad \text{in } \mathbb{R}^{N+1}. \quad (5.73)$$

Invoking Proposition II.1, case ii) of [9], we deduce that the right-hand side of (5.73) is bounded in $[C^{0, a}(\mathbb{R}^{N+1})]^*$, and arguing as in the proof of Proposition 5.2 we deduce that

$$\|\nabla \Psi_{i,j}\|_{L^p(K)} \leq C(p, K, M_0). \quad (5.74)$$

The corresponding estimate for $\Psi_j$ is less direct. The compactness assumption on the initial data will be determinant in the computation. We are going to use the following.

Step 2: Compactness at the initial time. Let $\chi$ be any function in $C^1_c(\mathbb{R}^N)$. We have, for any $q > N$,

$$\left| \int_{\mathbb{R}^N} u_\varepsilon^0 \times \left( \frac{\partial u_\varepsilon^0}{\partial x_j} \right) \chi \right| \leq C(q) \left[ \|u_\varepsilon^0\|_{H^{\frac{1}{2}}(\mathbb{R}^N)}^2 + 1 + M_0\varepsilon \|\log \varepsilon\| \right] \|\chi\|_{W^{1, \frac{1}{q}}(\mathbb{R}^N)}. \quad (5.75)$$

Proof. First notice that since $u_\varepsilon^0$ is constant outside $B(R_1)$, we only need to consider the case $\chi \in C^1_c(B(2R_1))$. For the same reason, we also have

$$\|u_\varepsilon^0\|_{H^{\frac{1}{2}}(B(2R_1))} \leq C \left( \|u_\varepsilon^0\|_{H^{\frac{1}{2}}(\mathbb{R}^N)} + 1 \right).$$

Consider the function $\tilde{u}_\varepsilon^0$ defined on $\mathbb{R}^N$ by $\tilde{u}_\varepsilon^0 = u_\varepsilon^0$ if $|u_\varepsilon^0| \leq 1$, $\tilde{u}_\varepsilon^0 = u_\varepsilon^0/|u_\varepsilon^0|$ otherwise. We have $\tilde{u}_\varepsilon^0 \times d\tilde{u}_\varepsilon = u_\varepsilon^0 \times du_\varepsilon$ if $|u_\varepsilon^0| \leq 1$, and $\tilde{u}_\varepsilon^0 \times d\tilde{u}_\varepsilon = \frac{1}{|u_\varepsilon^0|} u_\varepsilon^0 \times du_\varepsilon$, if $|u_\varepsilon^0| \geq 1$. Next, we use the embedding $W^{1, \frac{1}{q}} \hookrightarrow C^0 \cap H^{\frac{1}{q}}$, and the fact that $H^{\frac{1}{2}} \cap L^\infty$ is an algebra. Since $|\tilde{u}_\varepsilon^0| \leq 1$, $\chi \tilde{u}_\varepsilon^0$ belongs to $H^{\frac{1}{2}} \cap L^\infty$ and therefore

$$\|\chi \tilde{u}_\varepsilon^0\|_{H^{\frac{1}{2}}(B(2R_1))} \leq C \left( \|\chi\|_{L^\infty(B(2R_1))} \|\tilde{u}_\varepsilon^0\|_{H^{\frac{1}{2}}(B(2R_1))} + \|\tilde{u}_\varepsilon^0\|_{L^\infty(B(2R_1))} \|\chi\|_{H^{\frac{1}{2}}(B(2R_1))} \right)$$

$$\leq C(q) \left( \|\chi\|_{W^{1, \frac{1}{q}}(B(2R_1))} \|\tilde{u}_\varepsilon^0\|_{H^{\frac{1}{2}}(B(2R_1))} + \|\tilde{u}_\varepsilon^0\|_{H^{\frac{1}{2}}(B(2R_1))} \right)$$

$$\leq C(q) \left( 1 + \|\tilde{u}_\varepsilon^0\|_{H^{\frac{1}{2}}(B(2R_1))} \right) \|\chi\|_{W^{1, \frac{1}{q}}(B(2R_1))}.$$ 

Hence, we obtain

$$\left| \int_{B(2R_1)} \chi(\tilde{a}_\varepsilon^0 \times d\tilde{a}_\varepsilon^0) \right| \leq \|d\tilde{a}_\varepsilon^0\|_{H^{-\frac{1}{2}}(B(2R_1))} \|\chi\|_{W^{1, \frac{1}{q}}(B(2R_1))} \left( 1 + \|\tilde{u}_\varepsilon^0\|_{H^{\frac{1}{2}}(B(2R_1))} \right). \quad (5.76)$$
On the other hand, by construction
\[ |u^0 \times du^0 - \tilde{u}^0 \times d\tilde{u}^0| \leq \left| |u^0|^2 - 1 \right| \cdot |du^0| \leq \sqrt{2} \varepsilon \left( \frac{(1 - |u^0|^2)^2}{4\varepsilon^2} + \frac{|\nabla u^0|^2}{2} \right). \]
so that
\[ \int_{\mathbb{R}^N} \chi(\tilde{u}^0 \times d\tilde{u}^0 - u^0 \times du^0) \leq C(q) M_0 \varepsilon \log \varepsilon \| \chi \|_{W^{1,\frac{1}{2}}(\mathbb{R}^N)}. \] (5.77)
Combining (5.76) with (5.77) we derive the conclusion. \hfill \Box

**Step 3 : Propagating compactness.** We claim that
\[ \| u_\varepsilon(\cdot, 2\varepsilon) - u^0(\cdot) \|_{L^2(\mathbb{R}^N)} \leq C M_0^{\frac{1}{2}} \varepsilon^{\frac{1}{2}} \log \varepsilon^{\frac{1}{2}}, \] (5.78)
and that for any \( \chi \in C^1_c(\mathbb{R}^N) \),
\[ \left| \int_{\mathbb{R}^N \times (2\varepsilon)} (u^0 \times \frac{\partial u^0}{\partial x_j}) \chi - \int_{\mathbb{R}^N} (u^0 \times \frac{\partial u^0}{\partial x_j}) \chi \right| \leq C M_0 \varepsilon^{\frac{1}{2}} |\log \varepsilon| \left( \| \chi \|_{W^{1,\infty}} + |\text{supp}(\chi)|^{\frac{1}{2}} \right). \] (5.79)

**Proof.** Define on \( \mathbb{R}^N \) the function \( u_i^f(x) = u_\varepsilon(x, 2\varepsilon) \). We have, for \( x \in \mathbb{R}^N \),
\[ |u_i^f(x) - u_i^0(x)|^2 \leq \left( \int_0^{2\varepsilon} \frac{\partial u^f}{\partial t}(x, s) ds \right)^2 \leq 2\varepsilon \int_0^{2\varepsilon} \left( \frac{\partial u^f}{\partial t}(x, s) \right)^2 ds. \]
Hence, by integration,
\[ \| u_i^f - u_i^0 \|_{L^2(\mathbb{R}^N)} \leq 2\varepsilon \int_{\mathbb{R}^N \times (0, 2\varepsilon)} \left( \frac{\partial u^f}{\partial t} \right)^2 \leq 2 M_0 \varepsilon |\log \varepsilon|, \]
and (5.78) follows. For (5.79) we write
\[ u_i^f \times \frac{\partial u_i^f}{\partial x_j} - u_i^0 \times \frac{\partial u_i^0}{\partial x_j} = (u_i^f - u_i^0) \times \frac{\partial u_i^0}{\partial x_j} + u_i^0 \times \frac{\partial}{\partial x_j}(u_i^f - u_i^0). \]
For the first term on the r.h.s., we obtain by Cauchy-Schwarz inequality
\[ \left| \int_{\mathbb{R}^N} (u_i^f - u_i^0) \times \frac{\partial u_i^0}{\partial x_j} \chi \right| \leq \| u_i^f - u_i^0 \|_{L^2(\mathbb{R}^N)} \| \frac{\partial u_i^0}{\partial x_j} \|_{L^2(\mathbb{R}^N)} \| \chi \|_{L^{\infty}} \leq C \| \chi \|_{L^{\infty}} M_0 \varepsilon^{\frac{1}{2}} |\log \varepsilon|. \] (5.80)
For the second term, we integrate by parts
\[ \left| \int_{\mathbb{R}^N} u_i^0 \times \frac{\partial}{\partial x_j}(u_i^f - u_i^0) \chi \right| \leq \int_{\mathbb{R}^N} |u_i^f - u_i^0| \left| \frac{\partial u_i^0}{\partial x_j} \right| |\nabla \chi| + \int_{\mathbb{R}^N} \left| \frac{\partial u_i^0}{\partial x_j}(u_i^f - u_i^0) \chi \right| \leq \| u_i^f - u_i^0 \|_{L^2(\mathbb{R}^N)} \| u_i^0 \|_{L^2(\text{supp}(\chi))} \| \nabla \chi \|_{L^{\infty}} \] \[ + \| \nabla u_i^0 \|_{L^2(\mathbb{R}^N)} \| u_i^f - u_i^0 \|_{L^2(\mathbb{R}^N)} \| \chi \|_{L^{\infty}} \leq C M_0 \varepsilon^{\frac{1}{2}} |\log \varepsilon| \left( \| \chi \|_{W^{1,\infty}} + |\text{supp}(\chi)|^{\frac{1}{2}} \right), \] (5.81)
where we have used the bound
\[ \|u^0_\varepsilon\|_{L^2(\text{supp}(\chi))} \leq C \|1 - |u^0_\varepsilon|^2\|_{L^2(\mathbb{R}^N)} + C |\text{supp}(\chi)|^{1/2} \leq C \varepsilon E_\varepsilon(u^0_\varepsilon)^{1/2} + C |\text{supp}(\chi)|^{1/2}. \]
Combining (5.80) and (5.81) we deduce (5.79). \hfill \Box

Combining (5.79) and (5.75) we are led to
\[ \int_{\mathbb{R}^N \times \{2\varepsilon\}} u_\varepsilon \times (\frac{\partial u_\varepsilon}{\partial x_j}) \chi \leq C \left[ \|u^0_\varepsilon\|_{H^{1/2}(\mathbb{R}^N)}^2 + 1 + M_0 \varepsilon |\log \varepsilon| \right] (\|\chi\|_{W^{1,\infty}} + |\text{supp}(\chi)|^{1/2}), \]
and arguing as in Step 2 we conclude that
\[ \int_{\mathbb{R}^N \times \{2\varepsilon\}} \tilde{u}_\varepsilon \times (\frac{\partial \tilde{u}_\varepsilon}{\partial x_j}) \chi \leq C \left[ \|u^0_\varepsilon\|_{H^{1/2}(\mathbb{R}^N)}^2 + 1 + M_0 \varepsilon |\log \varepsilon| \right] (\|\chi\|_{W^{1,\infty}} + |\text{supp}(\chi)|^{1/2}). \] \hfill (5.82)

**Step 4 : Improving (5.82).** Let \( \chi \) be in \( C^1(\mathbb{R}^N) \) such that
\[ \|\nabla \chi\|_{L^\infty(\mathbb{R}^N)} + \|\chi\|_{L^2(\mathbb{R}^N)} < +\infty. \]
Then we claim that
\[ \int_{\mathbb{R}^N \times \{2\varepsilon\}} \tilde{u}_\varepsilon \times (\frac{\partial \tilde{u}_\varepsilon}{\partial x_j}) \chi \leq C \left[ \|u^0_\varepsilon\|_{H^{1/2}(\mathbb{R}^N)}^2 + 1 + M_0 \varepsilon |\log \varepsilon|^{1/2} \right] (\|\nabla \chi\|_{L^\infty} + \|\chi\|_{L^2}). \] \hfill (5.83)

**Proof.** Let \( \xi \) be a smooth non negative cut-off function such that \( \xi \equiv 1 \) on \( B(R_1) \) and \( \xi \equiv 0 \) outside \( B(R_2) \). We write \( \chi = \chi \xi + \chi(1 - \xi) \). By (5.82) we have
\[ \int_{\mathbb{R}^N \times \{2\varepsilon\}} \tilde{u}_\varepsilon \times (\frac{\partial \tilde{u}_\varepsilon}{\partial x_j}) \chi \xi \leq C \left[ \|u^0_\varepsilon\|_{H^{1/2}(\mathbb{R}^N)}^2 + 1 + M_0 \varepsilon |\log \varepsilon| \right] (\|\nabla \chi\|_{L^\infty} + \|\chi\|_{L^2} + 1). \]
On the other hand, we have
\[ \int_{\mathbb{R}^N \times \{2\varepsilon\}} \tilde{u}_\varepsilon \times (\frac{\partial \tilde{u}_\varepsilon}{\partial x_j}) \chi (1 - \xi) \leq C \left[ \int_{\mathbb{R}^N \times \{2\varepsilon\}} e_\varepsilon(u_\varepsilon)(1 - \xi)^2 \right]^{1/2} \|\chi\|_{L^2(\mathbb{R}^N)}. \]
By Corollary 2.1 and assumption \((H_1)\),
\[ \int_{\mathbb{R}^N \times \{2\varepsilon\}} e_\varepsilon(u_\varepsilon)(1 - \xi)^2 \leq \int_{\mathbb{R}^N \times \{0\}} e_\varepsilon(u_\varepsilon)(1 - \xi)^2 + 4\varepsilon |\log \varepsilon| M_0 \|\nabla \xi\|_{L^\infty}^2 \leq 4\varepsilon |\log \varepsilon| M_0 \|\nabla \xi\|_{L^\infty}^2, \]
so that the conclusion follows. \hfill \Box

**Step 5 : \( L^p \) estimate for \( \nabla \Psi_j \).** Let \( K \) be any compact set in \( \mathbb{R}^N \times (2\varepsilon, +\infty) \). Then, for any \( 1 \leq p < \frac{N+1}{N} \),
\[ \|\nabla \Psi_j\|_{L^p(K)} \leq C(p, K) M_0. \] \hfill (5.84)
Proof. We argue by duality. Let \( \frac{1}{q} + \frac{1}{p} = 1 \), so that in particular \( q > N + 1 \), and let \( h \) be any vector field in \( L^q(\mathbb{R}^N) \) with compact support in \( K \). We introduce the solution \( \zeta \) of the dual problem

\[
\begin{aligned}
-\Delta \zeta &= \text{div } h \quad \text{in } \mathbb{R}^N \times \{ 2\varepsilon, +\infty \}, \\
& \frac{\partial \zeta}{\partial \nu} = 0 \quad \text{on } \mathbb{R}^N \times \{ 2\varepsilon \} 
\end{aligned}
\]  

(5.85)

(Actually, \( \zeta \) is uniquely defined up to a constant). Extending \( \zeta \) by reflection on the whole of \( \mathbb{R}^N \), i.e. setting

\[
\zeta = P_\varepsilon(\zeta) \quad \text{on } \mathbb{R}^N \times (-\infty, 2\varepsilon),
\]

then \( \zeta \) solves the equation

\[
-\Delta \zeta = \text{div } h + \text{div}(P_\varepsilon h) \quad \text{on } \mathbb{R}^{N+1}.
\]

It follows by standard elliptic estimates that

\[
\| \nabla \zeta \|_{L^q(\mathbb{R}^N)} \leq C\| h \|_{L^p(\mathbb{R}^N)},
\]

and

\[
\zeta \in C^\infty(\mathbb{R}^{N+1} \setminus (K \cup P_\varepsilon K)),
\]

where \( P_\varepsilon K \) obviously denotes the reflection of \( K \). Moreover, since \( \zeta \) is defined up to an additive constant, we may also assume that

\[
|\zeta(x, t)| \leq C(K) \frac{\| h \|_{L^p}}{\text{dist}((x, t), K \cup P_\varepsilon K)^{N-1}}
\]

and that

\[
|\nabla \zeta(x, t)| \leq C(K) \frac{\| h \|_{L^p}}{\text{dist}((x, t), K \cup P_\varepsilon K)^N}.
\]

We turn back to the system (5.72). Multiplying the equation of (5.72) corresponding to \( \Psi_j \) by \( \zeta \) and integrating by parts on \( \Omega_\varepsilon \) we obtain

\[
\int_{\Omega_\varepsilon} \nabla \Psi_j \nabla \zeta = \int_{\Omega_\varepsilon} 2(\tilde{u}_t \times \tilde{u}_x) \zeta + \int_{\mathbb{R}^N \times \{ 2\varepsilon \}} \tilde{u}_x \times \frac{\partial \tilde{u}_\varepsilon}{\partial x_j} \zeta.
\]

On the other hand, multiplying (5.70) by \( \Psi_j \) and integrating by parts, we obtain similarly

\[
\int_{\Omega_\varepsilon} \nabla \Psi_j \nabla \zeta = \int_{\Omega_\varepsilon} h \cdot \nabla \Psi_j
\]

Hence, combining the previous relations,

\[
\left| \int_{\Omega_\varepsilon} h \cdot \nabla \Psi_j \right| \leq \left| \int_{\Omega_\varepsilon} 2(\tilde{u}_t \times \tilde{u}_x) \zeta \right| + \left| \int_{\mathbb{R}^N \times \{ 2\varepsilon \}} \tilde{u}_x \times \frac{\partial \tilde{u}_\varepsilon}{\partial x_j} \zeta \right|.
\]  

(5.86)

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Arguing as in Step 1, we are led to the inequality
\[ \left| \int_{\Omega_t} 2(\tilde{u}_t \times \tilde{u}_x) \zeta \right| \leq CM_0 \| \nabla \zeta \|^2_{L^2(\mathbb{R}^N)} \leq CM_0 \| h \|_{L^8(\mathbb{R}^N)}. \]

For the second term on the r.h.s. of (5.86) we invoke Step 4. This yields
\[ \left| \int_{\mathbb{R}^N \times \{2\varepsilon\}} \tilde{u}_\varepsilon \times \frac{\partial \tilde{u}_\varepsilon}{\partial x_j} \zeta \right| \leq C \left( 1 + M_2 + M_0 \| \nabla \zeta \|_{L^\infty} + \| \zeta \|_{L^2} \right) \leq C(K) (1 + M_0 + M_2) \| h \|_{L^8}. \]

Going back to (5.86) we thus obtain
\[ \left| \int_{\Omega_t} h \cdot \nabla \Psi_j \right| \leq C(K) (1 + M_0 + M_2) \| h \|_{L^6(\Omega_t)}, \]
and since \( h \) was arbitrary the conclusion follows. \( \square \)

**Step 6 : Estimate for \( \nabla \Phi \).** As in Section 5.3 we derive a parabolic equation for the phase, using (5.55) once more. We have, recalling that \( \tau = p(|u_\varepsilon|) \),
\[ u_\varepsilon \times \delta u_\varepsilon = \tau^{-2} \tilde{u}_\varepsilon \times \delta \tilde{u}_\varepsilon = \tau^{-2} \delta \Phi + \tau^{-2} \delta^s \Psi. \]

Hence,
\[ u_\varepsilon \times du_\varepsilon = \tau^{-2} d\Phi + \tau^{-2} (\delta^s \Psi - P_t(\delta^s \Psi) dt) \]
\[ u_\varepsilon \times \frac{\partial u_\varepsilon}{\partial t} = \tau^{-2} \frac{\partial \Phi}{\partial t} + \tau^{-2} P_t(\delta^s \Psi), \]
and (5.55) leads to
\[ \tau^{-2} \frac{\partial \Phi}{\partial t} - \text{div}(\tau^{-2} \nabla \Phi) = \tau^{-2} \left( \delta \Phi - P_t(\delta^s \Psi) dt \right) - \tau^{-2} P_t(\delta^s \Psi). \]

We obtain therefore
\[ \frac{\partial \Phi}{\partial t} - \Delta \Phi = \tau^s \left( \tau^{-2} \delta \Phi - P_t(\delta^s \Psi) dt \right) - \tau^{-2} P_t(\delta^s \Psi) + (1 - \tau^{-2}) \frac{\partial \Phi}{\partial t} + \text{div}(1 - \tau^{-2}) \nabla \Phi. \]

By (3.64), \( |1 - \tau^{-2}| \leq C|1 - |u_\varepsilon||^2 \). Given the \( L^p \) estimate for \( \nabla \Psi \) obtained in Steps 1 and 5, and arguing as in Step 1 of the proof of Theorem 3, we finally derive the bound
\[ \| \nabla \Phi \|_{L^p(K)} \leq C(p, K) \]
for any \( 1 \leq p < \frac{N+1}{N} \), and the proof of Proposition 5.3 is complete. \( \square \)
5.6 Proof of Theorem 4

First notice that the same way we obtained (5.62) and (5.63) we have here

\[ \int_K |\nabla u_\varepsilon|^p \leq C(p) \left[ \int_K |u_\varepsilon \times du_\varepsilon|^p + \int_K |\nabla u_\varepsilon||^p \right] + C(p, K) M_0. \]

By Lemma 5.3 we have

\[ \int_K |\nabla u_\varepsilon||^p \leq C(K) (M_0 + 1) \varepsilon^{1-\frac{p}{2}} |\log \varepsilon|. \]

On the other hand,

\[ |u_\varepsilon \times du_\varepsilon|^p = \tau^{-2p} |\tilde{u}_\varepsilon \times d\tilde{u}_\varepsilon|^p \leq C(p) (|\nabla \Phi|^p + |\nabla \Psi|^p). \]

Hence, by Proposition 5.3

\[ \int_K |u_\varepsilon \times du_\varepsilon|^p \leq C(p, K) \]

and the conclusion follows.  \( \square \)

5.7 Proof of Proposition 5

Since we assume \((H_1)\) and \((H_2)\), we may apply Theorem 4 (with \(p = 1\)), so that

\[ \int_{\Lambda_1^+} |\nabla u_\varepsilon| \leq C; \quad (5.87) \]

where \(\Lambda_1 = \Lambda_1^+(x, t, r)\). Since \(u_\varepsilon\) verifies (16), we may also apply Proposition 4 so that

\[ e_\varepsilon(u_\varepsilon) = |\nabla \Phi_\varepsilon|^2 + \kappa_\varepsilon \quad \text{on } \Lambda_1^+, \quad (5.88) \]

where \(\kappa_\varepsilon\) is bounded in \(L^\infty\) and \(\Phi_\varepsilon\) verifies the heat flow on \(\Lambda_1^-.\) Recall that \(\Phi_\varepsilon\) was constructed in the proof of Theorem 2 and verifies (5.5). Notice that on \(B(r) \times \{t_0\}\) we may impose the additional condition

\[ \int_{B(r) \times \{t_0\}} |\nabla u_\varepsilon| \leq C. \quad (5.89) \]

Going back to (5.5), we verify that all the terms on the r.h.s. are bounded in some suitable norm, say \(L^1\). On the other hand, the initial value is also bounded by (5.89). Since \(\varphi_0 = \Phi_\varepsilon\) solves the heat equation on \(\Lambda_1^+,\) we therefore deduce that

\[ |\nabla \Phi_\varepsilon| \leq C \quad \text{on } \Lambda_1^+ \]

and the conclusion follows from (5.88).
Part II

Analysis of the measures $\mu_*^t$
Introduction

As mentioned in our main Introduction, the focus of this paper is on the asymptotic limits, as \( \varepsilon \to 0 \), of the Radon measures \( \mu_\varepsilon \) defined on \( \mathbb{R}^N \times [0, +\infty) \) by

\[
\mu_\varepsilon(x, t) = \frac{e_\varepsilon(u_\varepsilon(x, t))}{|\log \varepsilon|} \, dx \, dt,
\]

where for \( 0 < \varepsilon < 1 \), the functions \( u_\varepsilon \) are solutions of (PGL)\( _\varepsilon \) satisfying assumption \( (H_0) \). We are specially interested in the properties of the time slices \( \mu_\varepsilon^t \) defined by

\[
\mu_\varepsilon^t(x) = \frac{e_\varepsilon(u_\varepsilon(x, t))}{|\log \varepsilon|} \, dx.
\]

In view of assumption \( (H_0) \) and inequality (II), we may assume that for a subsequence \( \varepsilon_n \to 0 \), there exists a Radon measure \( \mu_* \) defined on \( \mathbb{R}^N \times [0, +\infty) \) such that

\[
\mu_{\varepsilon_n} \rightharpoonup \mu_* \quad \text{as measures.} 
\]

(1)

Following Brakke [13], we may also assume weak convergence of \( \mu_{\varepsilon_n}^t \) for all \( t > 0 \), in the sense of measures.

**Lemma 1.** There exist a subsequence of \( \varepsilon_n \) (still denoted \( \varepsilon_n \)) such that

\[
\mu_{\varepsilon_n}^t \rightharpoonup \mu_*^t \quad \text{for all } t \geq 0,
\]

where \( \mu_*^t \) is a finite Radon measure on \( \mathbb{R}^N \) for all \( t \geq 0 \). Moreover, \( \mu_* = \mu_*^t \, dt \).

The proof in [31] carries over word for word. We fix such a sequence \( \varepsilon_n \), and we will therefore write \( \varepsilon \) instead of \( \varepsilon_n \) in the sequel, when this is not misleading. We also identify in some places the measure \( \mu_*^t \) with a measure on \( \mathbb{R}^N \times \{t\} \), and we will even sometimes identify \( \mathbb{R}^N \) and \( \mathbb{R}^N \times \{t\} \).

Some properties of the functions \( u_* \) can be translated directly in the language of the measure \( \mu_* \). Firstly, an easy consequence of the monotonicity formula (for \( u_* \)) is,

**Lemma 2.** For each \( t > 0 \) and \( x \in \mathbb{R}^N \), the function \( \mathcal{E}_\mu((x, t), \cdot) \) defined on \( \mathbb{R}^+ \) by

\[
r \mapsto \mathcal{E}_\mu((x, t), r) = \frac{1}{r^{N-2}} \int_{\mathbb{R}^N} \exp(-\frac{|x - y|^2}{4r^2}) \, d\mu_*^{t-r^2}(y)
\]

is non-decreasing for \( 0 < r < \sqrt{t} \).

Secondly, important consequences of the analysis developed in Part I are given by the following.
Theorem 5. i) There exist an absolute constant $\eta_2 > 0$, and a positive continuous function $\lambda$ defined on $\mathbb{R}^+_s$ such that if, for $x \in \mathbb{R}^N$, $t > 0$ and $r > 0$ we have
\[
\mu_*^t(B(x, \lambda(t)r)) < \eta_2 r^{N-2},
\]
then for every $s \in [t + \frac{15}{16}r^2, t + r^2]$, $\mu_*^s$ is absolutely continuous with respect to the Lebesgue measure on the ball $B(x, \frac{1}{4}r)$. More precisely,
\[
\mu_*^s = |\nabla \Phi_*|^2 \, dx \quad \text{on } B(x, \frac{1}{4}R),
\]
where $\Phi_*$ satisfies the heat equation in $\Lambda_{\frac{1}{4}} = B(x_0, \frac{1}{4}r) \times \left[t + \frac{15}{16}r^2, t + r^2\right]$.

ii) If $u_\varepsilon$ verifies the conditions (H$_1$) and (H$_2$) in addition to (2), then for every $s \in [t + \frac{15}{16}r^2, t + r^2]$,
\[
\mu_*^s \equiv 0 \quad \text{on } B(x, \frac{1}{4}R).
\]

Remark 1. Note that the constant $\eta_2$ and the function $\lambda$ are the same as in Proposition 4 of Part I. Notice also that $\mu_* = |\nabla \Phi_*|^2 \, dx \, dt$ on $\Lambda_{\frac{1}{4}}$, and that $|\nabla \Phi_*|^2$ is a smooth function.

We briefly sketch the proof of Theorem 5, which is a rather direct consequence of Theorems 1, 2 and 4 of Part I. We begin with case i). If (2) is verified, then for $\varepsilon = \varepsilon_n$ small enough
\[
\int_{B(x, \lambda(t)r)} e_\varepsilon(u_\varepsilon) \leq \eta_2 r^{N-2}|\log \varepsilon|,
\]
so that we may invoke Proposition 4. This yields
\[
e_\varepsilon(u_\varepsilon) = |\nabla \Phi_\varepsilon|^2 + \kappa_\varepsilon \quad \text{in } \Lambda_{\frac{1}{8}},
\]
where $\Phi_\varepsilon$ verifies the heat equation in $\Lambda_{\frac{1}{8}}$ and
\[
|\nabla \Phi_\varepsilon|^2 \leq C(\Lambda)|\log \varepsilon|, \quad |\kappa_\varepsilon| \leq C(\Lambda)\varepsilon^{\beta} \quad \text{in } \Lambda_{\frac{1}{4}}.
\]
Extracting possibly a further subsequence we may assume that
\[
\frac{\Phi_\varepsilon}{|\log \varepsilon|} \to \Phi_* \quad \text{uniformly on } \Lambda_{\frac{1}{16}}.
\]
Since $\Phi_\varepsilon$ verifies the heat equation, it follows that for every $k \in \mathbb{N}
\frac{\Phi_\varepsilon}{|\log \varepsilon|} \to \Phi_* \quad \text{in } C^k(\Lambda_{\frac{1}{4}}),
\]
and $\Phi_*$ verifies the heat equation on $\Lambda_{\frac{1}{4}}$. On the other hand,
\[
\kappa_\varepsilon \to 0 \quad \text{uniformly on } \Lambda_{\frac{1}{4}},
\]
so that
\[
e_\varepsilon(u_\varepsilon) \to |\nabla \Phi_*|^2 \quad \text{uniformly on } \Lambda_{\frac{1}{16}}.
\]
For case ii), we argue similarly, invoking Proposition 5.
1 Densities and concentration set

In order to analyse geometric properties of the measures $\mu_*$ and $\mu^t$, an important concept is that of densities. For a given Radon measure $\nu$ on $\mathbb{R}^N$, we have the classical definition.

**Definition 1.** For $m \in \mathbb{N}$, the $m$-dimensional lower density of $\nu$ at the point $x$ is defined by

$$\Theta_m(\nu, x) = \liminf_{r \to 0} \frac{\nu(B(x, r))}{\omega_m r^m},$$

where $\omega_m$ denotes the volume of the unit ball $B^m$. Similarly, the $m$-dimensional upper density $\Theta^*_m(\nu^t, x)$ is given by

$$\Theta^*_m(\nu, x) = \limsup_{r \to 0} \frac{\nu(B(x, r))}{\omega_m r^m}.$$

When both quantities coincide, $\nu$ admits a $m$-dimensional density $\Theta_m(\nu, x)$ at the point $x$, defined as the common value.

Since the energy measure is expected to concentrate on $(N-2)$-dimensional objects, our main efforts will be devoted to the study of the density $\Theta_m(\mu^t_\cdot, \cdot)$. Invoking the monotonicity formula once more, we have

**Lemma 3.** For all $x \in \mathbb{R}^N$ and for all $t > 0$,

$$\Theta_m(\mu^t_\cdot, x) \leq \Theta^*_m(\mu^t_\cdot, x) \leq KM_0 t^{\frac{N-2}{2}} < +\infty.$$

The previous provides an upper-bound. For regularity properties (of the concentration set) it is well known that lower bounds play a key role. However, there are some conceptual difficulties to attack $\Theta_m(\mu^t_\cdot, \cdot)$ directly (since the equation depends on time). Instead, we will first work on the space-time measure $\mu_*$, and recall the notion of parabolic density, which is natural in view of Lemma 2.

**Definition 2.** Let $\nu$ be a Radon measure on $\mathbb{R}^N \times [0, +\infty)$ such that $\nu = \nu^t dt$. For $t > 0$ and $m \in \mathbb{N}$, the parabolic $m$-dimensional lower density of $\nu$ at the point $(x, t)$ is defined by

$$\Theta^*_{m}(\nu, (x, t)) = \liminf_{r \to 0} \frac{1}{r^m} \int_{\mathbb{R}^N} \exp\left(-\frac{|x-y|^2}{4r^2}\right) d\nu^{t-r^2}(y).$$

The parabolic upper density and parabolic density are defined accordingly, and denoted respectively by $\Theta^*_m$ and $\Theta^P_m$.

**Remark 2.** Notice that $\Theta^P_m$ is not the classical density, in the spirit of Definition 1, for the parabolic metric defined by $d_P((x, t), (x', t')) = \max(|x - x'|, |t - t'|^{\frac{1}{2}})$.
It clearly follows from monotonicity that the limit in Definition 2 is decreasing, so that $\Theta_{N-2}^p(\mu^*, (x, t))$ exists everywhere in $\mathbb{R}^N \times (0, +\infty)$. Another consequence, which we will prove later (see Section 6.2) is that

$$\Theta_{N-2}^p(\mu^*, (x, t)) \geq K \Theta_{*N-2}(\mu^t_*, x)$$

(3)

for some explicit constant $K$. Motivated by this inequality, we set

$$\Sigma = \{(x, t) \in \mathbb{R}^N \times (0, +\infty) \text{ s.t. } \Theta_{N-2}^p(\mu^*, (x, t)) > 0\},$$

and for $t > 0$,

$$\Sigma_t^t = \Sigma \cap (\mathbb{R}^N \times \{t\}).$$

An obvious consequence of (3) is that

$$\Theta_{*N-2}(\mu^t_*, x) = 0 \quad \text{on } \mathbb{R}^N \setminus \Sigma^t_\mu.$$  

(4)

2 First properties of $\Sigma_\mu$

As in Brakle's and Ilmanen's works ([13, 30]) the main tool in the study of geometric properties of $\Sigma_\mu$ is the following.

**Theorem 6 (Clearing-Out).** There exist a positive continuous function $\eta_3$ defined on $\mathbb{R}^*_+$, such that for any $(x, t) \in \mathbb{R}^N \times (0, +\infty)$ and any $0 < r < \sqrt{t}$, if

$$\mathcal{C}_\mu((x, t), r) \equiv \frac{1}{r^{N-2}} \int_{\mathbb{R}^N} \exp\left(-\frac{|x - y|^2}{4r^2}\right) d\mu^t_*(y) \leq \eta_3(t - r^2)$$

then

$$(x, t) \notin \Sigma_\mu.$$  

Theorem 6 is a consequence of Theorem 5. An immediate corollary is

**Corollary 1.** For any $(x, t) \in \Sigma_\mu$, we have

$$\Theta_{N-2}^p(\mu^*, (x, t)) \geq \eta_3(t).$$

At this stage, we are in position to derive the following, without invoking any further property of the equation (PGL)$_\varepsilon$.

**Proposition 6.**

(i) The set $\Sigma_\mu$ is closed in $\mathbb{R}^N \times (0, +\infty)$.

(ii) For any $t > 0$,

$$\mathcal{H}^{N-2}(\Sigma^t_\mu) \leq K M_0 < +\infty.$$  

(iii) For any $t > 0$, the measure $\mu^t_*$ can be decomposed as

$$\mu^t_* = g(x, t)\mathcal{H}^N + \Theta_*(x, t)\mathcal{H}^{N-2} \setminus \Sigma^t_\mu,$$

where $g$ is some smooth function defined on $\mathbb{R}^N \times (0, +\infty) \setminus \Sigma_\mu$ and $\Theta_*$ verifies the bound $\Theta_*(x, t) \leq K M_0 I_{\Sigma^t_\mu}$.  

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Comment. a) The function $\Theta_\ast$ in decomposition $\text{iii}$ is the Radon-Nikodym derivative of $\mu_\ast^t \ll \Sigma^t_\mu$ with respect to $\mathcal{H}^{N-2}$; at this stage we may just assert that it lies between the lower and upper densities.

b) Concerning $g$, it can be locally defined as $|\nabla \Phi_\ast|^2$ for some smooth $\Phi_\ast$ verifying the heat equation. The function $\Phi_\ast$ however is not yet globally defined.

3 Regularity of $\Sigma^t_\mu$

As already mentioned, lower bounds for $\Theta_{\ast, N-2}$ will play an important role for regularity issues; however, up to now we have only lower bounds for $\Theta_{\ast, N-2}'$ (see Corollary 1). The next result provides the reverse inequality to (3).

Proposition 7. For almost every $t > 0$, the following inequality holds

$$\Theta_{\ast, N-2}(\mu^t_\ast, x) \geq K \theta_{N-2}^p(\mu^t_\ast, (x, t))$$

for $\mathcal{H}^{N-2}$ almost every $x \in \mathbb{R}^N$.

Combining Corollary 1 and Proposition 7 we are led to

Corollary 2. For almost every $t \geq 0$,

$$\Theta_{\ast, N-2}(\mu^t_\ast, x) \geq K \eta_\beta(t), \quad \text{for } \mathcal{H}^{N-2}-a.e. \ x \in \Sigma^t_\mu. \quad (6)$$

At this point, combining Theorem 5 with Corollary 2 and Ambrosio-Soner's work [4], the proof of Theorem C is complete. Indeed, since $\nabla \Phi_\ast = 0$, there is no diffuse part and (AS) holds.

To proceed further towards the proofs of Theorem A and Theorem B, we have to deal with the diffuse part, and different kinds of arguments could then lead to regularity for $\Sigma^t_\mu$. One way is to follow the arguments of [4] (as above for Theorem C), which rely on a curvature equation for $\mu^t_\ast$ and Allard's first rectifiability theorem (see [48]). Another possible way is to prove the existence of the density $\Theta_{N-2}$ ($\mathcal{H}^{N-2}$ almost everywhere), and then to invoke Preiss' regularity theorem [45]. Even though Preiss' theorem is notably highly involved, we choose this last alternative since it will simplify some of the subsequent arguments. Therefore, we will prove

Proposition 8. For almost every $t > 0$,

$$\Theta_{\ast, N-2}(\mu^t_\ast, x) = \Theta_{N-2}^*(\mu^t_\ast, x) \geq K \eta_\beta(t),$$

for $\mathcal{H}^{N-2}$ almost every $x \in \Sigma^t_\mu$. Consequently, for almost every $t > 0$ the set $\Sigma^t_\mu$ is $(N-2)$-rectifiable.

We recall that a set $\Sigma \subset \mathbb{R}^N$ is said to be $(N-2)$-rectifiable if $\mathcal{H}^{N-2}$ almost all of $\Sigma$ can be covered by the union of countably many Lipschitz images of $B^{N-2}$. 

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4 Globalizing $\Phi_*$

In order to complete the proof of Theorem A there is still one point to clarify: the function $\Phi_*$ which appears in Theorem A is global, whereas up to now the function $\Phi_*$ constructed in Theorem 5 is only locally defined. Indeed, using Theorem 5 we may define $\Phi_*$ on every simply connected domain of $\Omega_\mu = \mathbb{R}^N \times (0, +\infty) \setminus \Sigma_\mu$. However, $\Omega_\mu$ is not simply connected in general, and this raises a difficulty for defining $\Phi_*$ globally. Nevertheless its gradient $\nabla \Phi_*$ can be defined globally on $\Omega_\mu$ (and verifies there the heat equation). In order to overcome this problem, we will invoke Theorem 3 of Part I.

For $m \in \mathbb{N}^*$, set $\mathcal{K}_m = B(m) \times [\frac{1}{m}, m]$, so that $\bigcup_{m} \mathcal{K}_m = \mathbb{R}^N \times (0, +\infty)$. We apply Theorem 3 to $u_\varepsilon$ and $\mathcal{K} = \mathcal{K}_m$, so that we may write

$$u_\varepsilon = \exp(i\phi_\varepsilon^m)w_\varepsilon^m \quad \text{on } \mathcal{K}_m,$$

where $\phi_\varepsilon^m$ solves the heat equation on $\mathcal{K}_m$,

$$\|\nabla \phi_\varepsilon^m\|_{L^\infty(\mathcal{K}_m)} \leq C(m)\sqrt{M_0 \|\log \varepsilon\|}$$

and

$$\|w_\varepsilon^m\|_{L^p(\mathcal{K}_m)} \leq C(m, p) \quad \text{for any } 1 \leq p < \frac{N + 1}{N}.$$  \hspace{1cm} (9)

Let $m \in \mathbb{N}^*$ be fixed for the moment. Extracting possibly a further subsequence of $(\varepsilon_n)_{n \in \mathbb{N}}$, we may assume without loss of generality that

$$\frac{\phi_\varepsilon^m}{\sqrt{\|\log \varepsilon\|}} \rightarrow \phi_*^m \quad \text{in } C^2(\mathcal{K}_{m-1}).$$

Moreover, passing to the limit in the equation, we infer that $\phi_*^m$ solves the heat equation on $\mathcal{K}_{m-1}$.

Next, let $x_0 \in \Omega_\mu$. Since $\Omega_\mu$ is open, we may find a small cylindrical neighborhood $\Lambda_{x_0}$ of $x_0$ in $\Omega_\mu$. There exist $m_0 \in \mathbb{N}$ such that for $m \geq m_0$, $\Lambda_{x_0} \subset \mathcal{K}_m$. For $\varepsilon$ sufficiently small, we have

$$|u_\varepsilon| \geq 1 - \sigma \geq \frac{1}{2} \quad \text{on } \Lambda_{x_0}$$

(where $\sigma$ is the constant in Theorem 2), so that

$$u_\varepsilon = \rho_\varepsilon \exp(i\varphi_\varepsilon)$$

for some real-valued function $\varphi_\varepsilon$ (defined up to an integer multiple of $2\pi$). In view of (11), we may apply Theorem 2, and assert that there exists a solution $\Phi_\varepsilon$ of the heat equation on $\Lambda_{x_0}$ such that

$$\|\nabla \Phi_\varepsilon - \nabla \varphi_\varepsilon\|_{L^\infty(\Lambda_{x_0})} \leq C\varepsilon^\beta.$$  \hspace{1cm} (13)
(see Remark 5 of the introduction of Part I). On the other hand, we may write for \( m \geq m_0 \)

\[
    w^m_\varepsilon = \rho_\varepsilon \exp(i\psi^m_\varepsilon) \quad \text{on } \Lambda_{x_0} \tag{14}
\]

where \( \psi^m_\varepsilon \) is a real-valued function. Combining (12), (7) and (14) we are led to

\[
    \nabla \varphi_\varepsilon = \nabla \phi^m_\varepsilon + \nabla \psi^m_\varepsilon, \tag{15}
\]

and invoking (13), we have, for m fixed,

\[
    \left| \frac{\nabla \phi^m_\varepsilon - \nabla \Phi_*}{\sqrt{\log \varepsilon}} \right| \leq \left| \frac{\nabla \psi^m_\varepsilon}{\sqrt{\log \varepsilon}} \right| + C \varepsilon^\beta \quad \text{on } (\Lambda_{x_0})_\frac{1}{2}.
\]

Using (9) we obtain

\[
    \left( \frac{\nabla \phi^m_\varepsilon - \nabla \Phi_*}{\sqrt{\log \varepsilon}} \right)_{L^p((\Lambda_{x_0})_\frac{1}{2})} \to 0 \quad \text{as } \varepsilon \to 0.
\]

Since \( \frac{\nabla \phi^m_\varepsilon}{\sqrt{\log \varepsilon}} \to \nabla \Phi_* \) on \( (\Lambda_{x_0})_\frac{1}{2} \), by (10) we deduce

\[
    \nabla \phi^m_\varepsilon = \nabla \Phi_* \quad \text{on } (\Lambda_{x_0})_\frac{1}{2}.
\]

Since \( \Phi_* \) is independent of \( m \), changing possibly \( \phi^m_* \) by a constant we may assume that all the \( \phi^m_* \) coincide on \( (\Lambda_{x_0})_\frac{1}{2} \). By analyticity, for each \( n \geq m_0 \) the functions \( (\phi^m_*)_m \geq n \) coincide on \( \mathcal{K}_n \). Letting \( n \) go to infinity, we define their common value \( \phi_* \) on \( \mathbb{R}^N \times (0, +\infty) \) and we set

\[
    \phi_* = \phi_* . \tag{16}
\]

The proof of Theorem A is now completed, combining (16), Theorem 5, Proposition 6, Corollary 2 and Proposition 8.

5 Mean curvature flows

In this section we will provide the proof of Theorem B. Since a large part of this analysis follows the lines of [4], we will only indicate the ingredients, the necessary adaptations (due to the presence of the diffuse energy) and some simplifications since rectifiability of \( \Sigma^t_\mu \) is already available. In particular, we will avoid to refer to varifolds (or generalized varifolds) even though these important objects are hidden behind.

We first briefly recall both classical and weak notions of mean curvature flow. Then, following [30, 4] we will underline the relationship between \((\PGL)_\varepsilon\) and this flow, leading to Theorem B.

5.1 The classical notion. Let \( \Sigma \) be a smooth compact manifold of dimension \( k \), and \( \gamma_0 : \Sigma \to \mathbb{R}^N \) \( (N \geq k) \) a smooth embedding, so that \( \Sigma^t = \gamma_0(\Sigma) \) is a smooth
k-dimensional submanifold of \( \mathbb{R}^N \). The mean curvature vector at the point \( x \) of \( \Sigma^0 \) is the vector of the orthogonal space \( (T_x \Sigma^0)^\perp \) given by

\[
\bar{H}_{\Sigma^0}(x) = -\sum_{a=1}^{N-k} \left( \sum_{j=1}^{k} (\tau_j \cdot \frac{\partial h^a}{\partial \tau_j}) \nu^a \right) = -\sum_{a=1}^{N-k} \left( \text{div}_{T_x \Sigma^0} \nu^a \right) \nu^a,
\]

where \((\tau_1, \ldots, \tau_k)\) is an orthonormal moving frame on \( T_x \Sigma^0 \), \((\nu^1, \ldots, \nu^{N-k})\) is an orthonormal moving frame on \((T_x \Sigma^0)^\perp\), and \( \text{div}_{T_x \Sigma^0} \) denotes the tangential divergence at the point \( x \). The integral formulation of (17) is given by

\[
\int_{\Sigma^0} \text{div}_{T_x \Sigma^0} \bar{X} \, d\mathcal{H}^k = -\int_{\Sigma^0} \bar{H}_{\Sigma^0} \cdot \bar{X} \, d\mathcal{H}^k,
\]

for all \( \bar{X} \in \mathcal{C}^\infty(\mathbb{R}^N, \mathbb{R}^N) \). The vectors \( \bar{H}_{\Sigma^0} \) are uniquely determined by (18), and in particular the definition in (17) does not depend on the choice of orthonormal frames.

Next, we introduce a time dependence, and consider a smooth family \( \{\gamma_t\}_{t \in I} \) of smooth embeddings of \( \Sigma \) in \( \mathbb{R}^N \), where \( I \) denotes some open interval containing 0. We set \( \Sigma^t = \gamma_t(\Sigma) \). If \( \chi \) is a smooth compactly supported function on \( \mathbb{R}^N \), a standard computation shows that

\[
\frac{d}{dt} \int_{\Sigma^t} \chi(x) \, d\mathcal{H}^k = \int_{\Sigma^t} \left( -\chi(x) \ddot{H}_{\Sigma^t}(x) + P(\nabla \chi(x)) \right) \cdot \dot{Y} (x) \, d\mathcal{H}^k,
\]

where \( \dot{Y}(x) = \frac{d}{dt} \gamma_t(\gamma_t^{-1}(x)) \) is the velocity vector at the point \( x \), and \( P \) denotes the orthogonal projection on \((T_x \Sigma^t)^\perp\).

The family \( (\Sigma^t)_{t \in I} \) is moved by mean curvature in the classical sense if and only if

\[
\frac{d}{dt} \gamma_t(m) = \bar{H}_{\Sigma^t}(\gamma_t(m)), \quad \text{for all } m \in \Sigma \text{ and } t \in I.
\]

In particular, if \( (\Sigma^t)_{t \in I} \) is moved by mean curvature, (19) becomes

\[
\frac{d}{dt} \int_{\Sigma^t} \chi(x) \, d\mathcal{H}^k = -\int_{\Sigma^t} \chi(x) |\bar{H}_{\Sigma^t}(x)|^2 \, d\mathcal{H}^k + \int_{\Sigma^t} \nabla \chi(x) \cdot \bar{H}_{\Sigma^t}(x) \, d\mathcal{H}^k,
\]

and actually (21) is equivalent to (20) if \( \chi \) is taken arbitrary. Notice that the last term in the r.h.s of (21) corresponds to a transport term, whereas the first term represents a shrinking of the area. Actually, if \( \chi \equiv 1 \) in a neighborhood of \( \Sigma^t \), then

\[
\frac{d}{dt} \mathcal{H}^k(\Sigma^t) = -\int_{\Sigma^t} |\bar{H}_{\Sigma^t}(x)|^2 \, d\mathcal{H}^k.
\]

Finally, existence of a classical solution of (20) for small times can be established (see e.g. [52, 27]), but singularities develop in finite time.

5.2. Brakke flows. In the attempt to extend (20) or (21) to a larger class of (less regular) objects, and in particular to extend the flow past singularities, Brakke
relaxed equality in (21), and considered instead sub-solutions, i.e. verifying the
inequality
\[
\frac{d}{dt} \int_\Sigma \chi(x) \, dH^k \leq - \int_\Sigma \chi(x) |\tilde{H}_\Sigma(x)|^2 \, dH^k + \int_\Sigma \nabla \chi(x) \cdot \tilde{H}_\Sigma(x) \, dH^k,
\]
for all non-negative \( \chi \in C_\infty_c(\mathbb{R}^N) \). Following Brakke [13], we are thus going to extend (22) to less regular objects than smooth embedded manifolds. Actually, we adopt the point of view of Ilmanen [31], which is slightly different from Brakke’s original one (the difference being very tiny, to the authors understanding at least!).

Recall that a Radon measure \( \nu \) on \( \mathbb{R}^N \) is said to be k-rectifiable if there exists a k-rectifiable set \( \Sigma \), and a density function \( \Theta \in L^1_{\text{loc}}(H^k \llcorner \Sigma) \) such that
\[
\nu = \Theta(\cdot) \, H^k \llcorner \Sigma.
\]
Since \( \Sigma \) is rectifiable, for \( H^k \)-a.e. \( x \in \Sigma \), there exist a unique tangent space \( T_x \Sigma \) belonging to the Grassmanian \( G_{N,k} \). The distributional first variation of \( \nu \) is the vector-valued distribution \( \delta \nu \) defined by
\[
\delta \nu(\vec{X}) = \int_\Sigma \text{div}_{T_x \Sigma} \vec{X} \, d\nu \quad \text{for all } \vec{X} \in C_\infty_c(\mathbb{R}^N, \mathbb{R}^N).
\]
In case \( |\delta \nu| \) is a measure absolutely continuous with respect to \( \nu \), we say that \( \nu \) has a first variation and we may write
\[
\delta \nu = \tilde{H} \, \nu,
\]
where \( \tilde{H} \) is the Radon-Nikodym derivative of \( \delta \nu \) with respect to \( \nu \). In this case, formula (23) becomes
\[
\int_\Sigma \text{div}_{T_x \Sigma} \vec{X} \, d\nu = - \int_\Sigma \tilde{H} \cdot \vec{X} \, d\nu.
\]
**Remark 3.** Notice that in the smooth case, this notion coincides with the definition (17), in view of (18). Notice also that the component of \( \tilde{H} \) which is orthogonal to \( T_x \Sigma \) is independent of the density \( \Theta \). However, if \( \Theta \) is non constant, then \( \tilde{H} \) may have a tangential part.

We are now in position to give the precise definition of a Brakke flow. Let \( (\nu_t)_{t \geq 0} \) be a family of Radon measures on \( \mathbb{R}^N \). For \( \chi \in C_\infty_c(\mathbb{R}^N, \mathbb{R}^+) \), we define
\[
\tilde{D}_t \nu_t(\chi) = \limsup_{t \to t_0} \frac{\nu_t^t(\chi) - \nu_t^{t_0}(\chi)}{t - t_0}.
\]
If \( \nu_t^t \ll \{ \chi > 0 \} \) is a k-rectifiable measure which has a first variation verifying \( \chi |\tilde{H}|^2 \in L^1(\nu^t) \), then we set
\[
\mathcal{B}(\nu^t, \chi) = - \int \chi |\tilde{H}|^2 \, d\nu^t + \int \nabla \chi \cdot P(\tilde{H}) \, d\nu^t,
\]
[here \( P \) denotes \( H^k \)-a.e. the orthogonal projection onto the tangent space to \( \nu^t \)]. Otherwise, we set
\[
\mathcal{B}(\nu^t, \chi) = -\infty.
\]
Definition 3 (Brakke flow). Let $(\nu_t)_{t \geq 0}$ be a family of Radon measures on $\mathbb{R}^N$. We say that $(\nu_t)_{t \geq 0}$ is a $k$-dimensional Brakke flow if and only if

$$D_t \nu^t(\chi) \leq \mathcal{B}(\nu^t, \chi),$$

for every $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^N, \mathbb{R}^+)$ and for all $t \geq 0$.

5.3. Relating $(\text{PGL})_\varepsilon$ to mean curvature flow. The starting point of the analysis is the formal analogy of equality (21), namely

$$\frac{d}{dt} \int_{\Sigma^t} \chi(x) \, d\mathcal{H}^k = -\int_{\Sigma^t} \chi(x) |\vec{H}_{\Sigma^t}(x)|^2 \, d\mathcal{H}^k + \int_{\Sigma^t} \nabla \chi(x) \cdot \vec{H}_{\Sigma^t}(x) \, d\mathcal{H}^k,$$

with the evolution of local energies for $(\text{PGL})_\varepsilon$ (see (2.1) in Part I)

$$\frac{d}{dt} \int_{\mathbb{R}^N} \chi(x) \, d\mu^t_\varepsilon = -\int_{\mathbb{R}^N} \chi(x) \frac{|\partial_t u_\varepsilon|^2}{|\log \varepsilon|} \, dx + \int_{\mathbb{R}^N} \chi(x) \frac{-\partial_t u_\varepsilon \nabla u_\varepsilon}{|\log \varepsilon|} \, dx. \quad (26)$$

We already know that as $\varepsilon \to 0$, $\mu^t_\varepsilon \to \mu^t_*$. Therefore, the comparison of the two formulas suggests, at least formally, that in the limit

$$\omega^t_\varepsilon \equiv \frac{|\partial_t u_\varepsilon|^2}{|\log \varepsilon|} \, dx \to |\vec{H}|^2 \, d\mu^t_*, \quad (27)$$

and

$$\sigma^t_\varepsilon \equiv \frac{-\partial_t u_\varepsilon \nabla u_\varepsilon}{|\log \varepsilon|} \, dx \to \vec{H} \, d\mu^t_* \quad (28)$$

Actually, this is a little over optimistic for two reasons. First we have to deal also with the diffuse part of the energy (this will be handled thanks to Theorem A). Second, since (27) involves the quadratic term $|\vec{H}|^2$, only lower semi-continuity can be expected at first sight (this is certainly a more serious matter, and would require a much longer discussion).

5.4. Convergence of $\sigma^t_\varepsilon$. Consider the measure $\sigma_\varepsilon = \sigma^t_\varepsilon \, dt$ defined on $\mathbb{R}^N \times [0, +\infty)$. By Cauchy-Schwarz inequality $\sigma_\varepsilon$ is uniformly bounded on $\mathbb{R}^N \times [0, T]$ for every $T > 0$, so that passing possibly to a further subsequence, we may assume that $\sigma_\varepsilon \to \sigma_*$ as measures. The Radon-Nikodym derivative of $|\sigma_\varepsilon|$ with respect to $\mu_\varepsilon$ verifies

$$\frac{d|\sigma_\varepsilon|}{d\mu_\varepsilon} \leq \sqrt{2} \frac{|\partial_t u_\varepsilon|}{e_\varepsilon(u_\varepsilon)}.$$

On the other hand,

$$\left\| \frac{|\partial_t u_\varepsilon|}{\sqrt{e_\varepsilon(u_\varepsilon)}} \right\|_{L^2(\mathbb{R}^N \times [0, T], d\mu_\varepsilon)} \leq \int_{\mathbb{R}^N \times [0, T]} \frac{|\partial_t u_\varepsilon|^2}{|\log \varepsilon|} \, \leq KT M_0,$$
so that \( \frac{d\sigma}{dt} \) is uniformly bounded in \( L^2(\mathbb{R}^N \times [0, T], d\mu_t) \). Arguing as in [4] Remark 2.2 (see also [29]) it follows that \( \sigma_* \) is absolutely continuous with respect to \( \mu_* \). Therefore, we may write

\[
\sigma_* = \tilde{h} \mu_*^t dt,
\]

where \( \tilde{h} \in L^2(\mathbb{R}^N \times [0, T], \mu_*^t dt) \). In view of the decomposition in Theorem A and Part I, we infer

**Lemma 4.** The measure \( \sigma_* \) decomposes as \( \sigma_* = \sigma_*^t dt \), where for a.e. \( t \geq 0 \),

\[
\sigma_*^t = -\partial \Phi_* \cdot \nabla \Phi_* \, dx + \tilde{h} \nu_*^t.
\]

The next step will be to identify the restriction of \( \tilde{h} \) on \( \Sigma^t \) with the mean curvature defined by (24). [Notice that we already know by Theorem A that \( \nu_*^t \) is (N-2)-rectifiable for a.e. \( t \geq 0 \).] The starting point is a classical formula involving the stress-energy tensor. Let \( \tilde{X} \in C^\infty(\mathbb{R}^N, \mathbb{R}^N) \). We have, for every \( t \geq 0 \),

\[
\frac{1}{|\log \varepsilon|} \int_{\mathbb{R}^N \times \{t\}} \left( e_\varepsilon(u_\varepsilon) \delta_{ij} - \frac{\partial u_\varepsilon}{\partial x_i} \frac{\partial u_\varepsilon}{\partial x_j} \right) \frac{\partial X^i}{\partial x_j} \, dx = \int_{\mathbb{R}^N \times \{t\}} \tilde{X} \cdot \frac{\partial u_\varepsilon}{|\log \varepsilon|} \nabla u_\varepsilon \, dx
\]

(29)

Formula (29) is already very close to (24), in particular the right hand side. In order to handle the diffuse energy, as well as to interpret the l.h.s as a tangential divergence, we need to analyse the weak limit of the stress-energy tensor

\[
\alpha^t_\varepsilon = \left( Id - \frac{\nabla u_\varepsilon \otimes \nabla u_\varepsilon}{e_\varepsilon(u_\varepsilon)} \right) d\mu^t_\varepsilon.
\]

Clearly, \( |\alpha^t_\varepsilon| \leq KN\mu^t_\varepsilon \), and we may assume that

\[
\alpha^t_\varepsilon \rightharpoonup \alpha_*^t \equiv A \cdot \mu_*^t,
\]

where \( A \) is an \( N \times N \) symmetric matrix. Since the symmetric matrix \( \nabla u_\varepsilon \otimes \nabla u_\varepsilon \) is non-negative, we have

\[
A \leq Id.
\]

(30)

On the other hand,

\[
Tr(e_\varepsilon(u_\varepsilon) \, Id - \nabla u_\varepsilon \otimes \nabla u_\varepsilon) = (N - 2)e_\varepsilon(u_\varepsilon) + 2V_\varepsilon(u_\varepsilon).
\]

Therefore, since the trace is a linear operation, passing to the limit we obtain

\[
Tr(A) = (N - 2) + 2\frac{dV_*}{d\mu_*},
\]

(31)

where the (non-negative) measure \( V_* \) is the limit (up to possibly a further subsequence) of \( V_\varepsilon(u_\varepsilon)/|\log \varepsilon| \).
Going to the limit in (20), and using the decomposition in Theorem A, we obtain for a.e. \( t \geq 0 \),

\[
\int_{\mathbb{R}^N} A^{ij} \frac{\partial X_i}{\partial x_j} \, dv^\mu + \int_{\mathbb{R}^N} \left( \frac{\nabla \Phi \cdot \nabla \Phi}{2} - \frac{\partial \Phi}{\partial x_i} \frac{\partial \Phi}{\partial x_j} \right) \frac{\partial X_i}{\partial x_j} \, dx = - \int_{\mathbb{R}^N} \vec{X} \cdot \vec{b} \, dv^\mu - \int_{\mathbb{R}^N} \vec{X} \cdot \nabla \Phi \, d\mu, \tag{32}
\]

On the other hand, \( \Phi \) verifies the heat equation

\[
\frac{\partial \Phi}{\partial t} - \Delta \Phi = 0. \tag{33}
\]

Multiplying (33) by \( \vec{X} \cdot \nabla \Phi \), we obtain

\[
\int_{\mathbb{R}^N} \left( \frac{\nabla \Phi \cdot \nabla \Phi}{2} - \frac{\partial \Phi}{\partial x_i} \frac{\partial \Phi}{\partial x_j} \right) \frac{\partial X_i}{\partial x_j} \, dx = - \int_{\mathbb{R}^N} \vec{X} \cdot \nabla \Phi \, d\mu. \tag{34}
\]

Combining (32) and (34) we have therefore proved

**Lemma 5.** For a.e. \( t \geq 0 \), and for every \( \vec{X} \in C^\infty_c(\mathbb{R}^N, \mathbb{R}^N) \),

\[
\int_{\mathbb{R}^N} A^{ij} \frac{\partial X_i}{\partial x_j} \, dv^\mu = - \int_{\mathbb{R}^N} \vec{X} \cdot \vec{b} \, dv^\mu. \tag{35}
\]

**Remark 4.** The last computations are the precise mathematical expression of the fact that the linear and the topological modes do not interact.

Recall that we already know that \( \Sigma^t_\mu \) is rectifiable for a.e. \( t \geq 0 \). Comparing (35) with (24) in order to identify \( \vec{b} \) with the mean curvature of \( \nu^t \), we merely have to prove that the matrix \( A \) corresponds to the orthogonal projection \( P \) onto the tangent space \( T_x \Sigma^t_\mu \). We follow closely the argument of [4] : however, our presentation is more direct, since rectifiability is already established. We first have

**Lemma 6.** For a.e. \( t \geq 0 \),

\[
A(x) \left[ \int_{T_x \Sigma^t_\mu} \nabla \chi(y) \, d\mathcal{H}^{N-2}(y) \right] = 0 \quad \text{for } \mathcal{H}^{N-2}-\text{a.e. } x \in \Sigma^t_\mu, \tag{36}
\]

and for all \( \chi \in C^\infty_c(\mathbb{R}^N, \mathbb{R}) \).

**Sketch of the proof.** Let \( x \in \Sigma^t_\mu \) be such that \( T_x \Sigma^t_\mu \) exists and such that \( x \) is a Lebesgue point for \( \Theta_\mu \) (with respect to \( \mathcal{H}^{N-2} \)) and for \( A \) (with respect to \( \nu^t_\mu \)). For \( r > 0 \), consider the vector field \( \vec{X}_{\tau r}(y) = \chi\left( \frac{y-x}{r} \right) \epsilon_i \). Inserting \( \vec{X}_{\tau r} \) into (35) and letting \( r \to 0 \), we obtain, by difference of homogeneity, that the r.h.s is negligible with respect to the l.h.s., and the conclusion follows. \( \square \)

A straightforward consequence is
Corollary 3. For $t$ and $x$ as in Lemma 6,

$$\left(T_x \Sigma^t_{\mu}\right) \subseteq \text{Ker } A(x).$$

With a little more elementary linear algebra, we further deduce

Corollary 4. For $t$ and $x$ as in Lemma 6, $A = P$ is the orthogonal projection onto the tangent space $T_x \Sigma^t_{\mu}$.

Proof. By (30), $A \leq Id$, and therefore all the eigenvalues $\lambda_1, ..., \lambda_N$ of $A$ are less or equal to 1. By (31), $\text{Tr}(A) \geq N - 2$, so that $\sum_{i=1}^{N} \lambda_i \geq N - 2$. On the other hand, by Corollary 3, $A$ has at least two eigenvalues, say $\lambda_1$ and $\lambda_2$, equal to zero. Therefore, $\lambda_i = 1$ for $i = 3, ..., N$. In particular $A$ is an orthogonal projection on an $(N-2)$-dimensional space. Since $\text{Ker } A(x) \supseteq \left(T_x \Sigma^t_{\mu}\right) \perp$, and since $\dim \left(T_x \Sigma^t_{\mu}\right) = N - 2$, the conclusion follows. □

Remark 5. Corollary 3 and 4 have many important consequences.

i) Using (31), we deduce that $\frac{d}{dt} \mu_t = 0$, i.e. there is only kinetic energy in the limit.

ii) Let $(\tau_1, ..., \tau_N)$ be an orthonormal frame such that $T_x \Sigma^t_{\mu}$ is spanned by $(\tau_3, ..., \tau_N)$. In view of the expression of the stress-energy tensor in these coordinates, we infer that the energy concentrates in the $(\tau_1, \tau_2)$ plane (i.e. $\left(T_x \Sigma^t_{\mu}\right) \perp$) and uniformly with respect to the direction. In particular, since $\sigma^e_t$ is colinear to $\nabla u_e$, this suggests strongly that $\bar{H}$ is perpendicular to $T_x \Sigma^t_{\mu}$. Such an argument is made rigorous in [4] (Proposition 6.2). This remark has presumably many other important consequences, but we will not discuss them here.

Combining the previous arguments, we have finally proved

Proposition 9. For a.e. $t \geq 0$, $\nu^t$ has a first variation and

$$\delta \nu^t = \bar{H} \nu^t,$$

i.e. $\bar{H}$ is the mean curvature of $\nu^t$.

Semi-continuity of $\omega^t$. The purpose of this subsection is to show that for a.e. $t \geq 0$,

$$\liminf_{\varepsilon \to 0} \int_{\mathbb{R}^N \times \{t\}} \chi \frac{\left|\partial_t u_e\right|^2}{\log \varepsilon} \geq \int_{\mathbb{R}^N \times \{t\}} \chi \left|\bar{H}\right|^2 d\nu^t + \int_{\mathbb{R}^N \times \{t\}} \chi \left|\partial_t \Phi^t\right|^2 dx.$$

It is tempting to write on $\Sigma^t_{\mu}$

$$\frac{\left|\partial_t u_e\right|^2}{\log \varepsilon} \geq \frac{\left|\partial_t u_e \nabla u_e\right|^2}{\log \varepsilon \left|\nabla u_e\right|^2} \geq \frac{1}{2} \frac{\left|\partial_t u_e \nabla u_e\right|^2}{e^t(u_e)} \mu^t_{e^t} \geq \frac{1}{2} \left|\bar{H}\right|^2 \mu^t_{e^t}.$$

These formal (but essentially correct) inequalities do not allow to conclude, in view of the extra factor $\frac{1}{2}$. Fortunately, the last inequality is far from being optimal. Indeed, weak convergence does not imply convergence of the squared quantities!
Remark 6. In the scalar case, i.e. for the Allen-Cahn equation, this difficulty does not arise since \[ \frac{|\nabla u_\varepsilon|^2}{2} \simeq V_\varepsilon(u_\varepsilon) \] there, so that \[ |\nabla u_\varepsilon|^2 \simeq e_\varepsilon(u_\varepsilon). \] The difficulty there however was to establish the balance between the kinetic and potential terms (see [30] Section 8.1).

In order to handle the factor \( \frac{1}{2} \), a determinant idea of [4] in this context was to recast the problem in the framework of Young measures, which turn out to be an appropriate concept to analyse the energies of the oscillations. In this direction, set \( p_\varepsilon = \frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|^2} \) and consider the measure (defined on \( \mathbb{R}^N \times \mathbb{R}^{2N} \))

\[
\tilde{\omega}_\varepsilon^t = \delta_{p_t(x)} \frac{|\partial_1 \omega_\varepsilon \cdot p_\varepsilon|^2}{|\log \varepsilon|} dx.
\]

Extracting possibly a further subsequence, we may assume that \( \tilde{\omega}_\varepsilon^t dt \to \tilde{\omega}_* \) as measures. We deduce from the analysis of Part I and Theorem A,

**Lemma 7.** The measure \( \tilde{\omega}_* \) decomposes as \( \tilde{\omega}_* = \tilde{\omega}_*^t dt \), and for a.e. \( t \geq 0 \)

\[
\tilde{\omega}_*^t = \Pi_*^t(x) \frac{|\partial \Phi_\varepsilon|^2}{|\log \varepsilon|} dx + \mathcal{M}_*^t,
\]

where \( \Pi_*^t \) is a measure on \( \mathbb{R}^{2N} \) (with support on the unit ball) and \( \mathcal{M}_*^t = \tilde{\omega}_*^t \cdot \Sigma_*^t \). Moreover, \( \Pi_*^t(x) \mathbb{R}^{2N} = 1 \).

[Notice that the measure \( \Pi_*^t \) arises from the possible oscillations of the phase \( \Phi_\varepsilon \) of \( u_\varepsilon \), but is not disturbing since it acts linearly].

The main ingredient that we will borrow directly from the analysis of [4] Section 6 can be formulated as the following.

**Proposition 10 (Ambrosio and Soner).** For a.e. \( t \geq 0 \), and every \( \chi \in C_c^\infty(\mathbb{R}^N) \),

\[
\int_{\mathbb{R}^N \times \mathbb{R}^{2N}} \chi(x) \mathcal{M}_*^t(x, p) \geq \int_{\mathbb{R}^N} \chi(\tilde{\Phi})^2 d\nu_*^t.
\]

At this stage, we are (finally!) in position to complete the proof of Theorem B.

**Proof of Theorem B.** In view of Theorem 4.4 in [4], it suffices to establish the integral version of (25). Let \( 0 < T_0 < T_1 \). We integrate (26) on \( [T_0, T_1] \) and let \( \varepsilon \) go to zero. Combining the results of Lemma 4, Proposition 9, Lemma 7, Remark 5, Proposition 10 and Theorem A, we obtain

\[
\nu_*^{T_1}(\chi) - \nu_*^{T_0}(\chi) + \int_{\mathbb{R}^N \times [T_1]} \chi |\nabla \Phi_*|^2 dx - \int_{\mathbb{R}^N \times [T_0]} \chi |\nabla \Phi_*|^2 dx
\]

\[
\leq \int_{\mathbb{R}^N \times [T_0, T_1]} \chi |\tilde{\Phi}|^2 d\nu_* + \int_{\mathbb{R}^N \times [T_0, T_1]} \nabla \chi P(\tilde{\Phi}) d\nu_*
\]

\[
- \int_{\mathbb{R}^N \times [T_0, T_1]} \chi |\partial_1 \Phi_*|^2 dx dt + \int_{\mathbb{R}^N \times [T_0, T_1]} \nabla \chi \nabla \Phi_* \partial_1 \Phi_* \quad (37)
\]

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Since $\Phi_*$ verifies the heat equation, we have the identity
\[
\int_{\mathbb{R}^N \times \{T_1\}} \chi |\nabla \Phi_*|^2 \, dx - \int_{\mathbb{R}^N \times \{T_0\}} \chi |\nabla \Phi_*|^2 \, dx
= \int_{\mathbb{R}^N \times [T_0,T_1]} \chi |\partial_t \Phi_*|^2 \, dx \, dt + \int_{\mathbb{R}^N \times [T_0,T_1]} \nabla \chi \nabla \Phi_* \partial_t \Phi_* \quad (38)
\]
Combining (37) and (38) we obtain
\[
\nu_*^{T_1}(\chi) - \nu_*^{T_0}(\chi) \leq - \int_{\mathbb{R}^N \times [T_0,T_1]} \chi |\tilde{\eta}|^2 \, d\nu_* + \int_{\mathbb{R}^N \times [T_0,T_1]} \nabla \chi P(\tilde{\eta}) \, d\nu_*.
\]
As mentioned above, this integral formulation implies (25), under suitable assumptions which are fulfilled here, namely rectifiability of $\Sigma^t_\mu$, lower bounds on the density $\Theta_*$, and orthogonality of the mean curvature $\tilde{\eta}$ with $(T_\mu \Sigma^t_\mu)^\perp$.

The proof of Theorem B is complete. \hfill \Box

6 Ilmanen enhanced motion

The notion of motion by mean curvature in the sense of Brakke has many interesting properties, in particular the fact that the area functional decreases along the flow, as expected from the classical motion. Unfortunately, as already mentioned in the main introduction, this notion strongly lacks of uniqueness. Indeed, if $(\mu^t)_{t \geq 0}$ is a Brakke flow, so is also $(g(t)\mu^t)_{t \geq 0}$, where $g$ is an arbitrary non increasing function on $\mathbb{R}^+$. In particular, the trivial solution given by $\nu^0 = \mu^0$ and $\nu^t \equiv 0$ for $t > 0$ is not excluded a priori. Actually, for $(PGL)_\varepsilon$ such a situation may occur (as in the Allen-Cahn equation), at least in three distinct cases:

- **Concentrated phase**: the initial data is of the form $u^0_\varepsilon = \exp(i\varphi^0_\varepsilon \sqrt{\log \varepsilon})$, where $|\nabla \varphi^0_\varepsilon|^2$ is bounded in $L^1$ and concentrates on a $(N-2)$-dimensional set $\Sigma_0$.

- **Low density**: we present an example is dimension 3. In the plane $(x_1,x_2)$, consider a standard dipole (i.e. with “least” energy) of two vortices away from the origin and separated by a length $\varepsilon^n$ (where $0 < n < 1$ is fixed), so that the energy in the plane is of order $\pi n \log \varepsilon$. Rotate the dipole along the $x_3$ axis so that $c_\varepsilon(u^0_\varepsilon)$ concentrates on a circle with a 1-density proportional to $\eta$. If $\eta$ is chosen sufficiently small, then $\mu^t_\varepsilon \equiv 0$ for $t > 0$ by the clearing-out lemma.

- **Hidden mean curvature**: consider in the $(x_1,x_2)$ plane the standard circle $S^1$. Approximate it, weakly in the sense of measures, by a collection $\mathcal{B}_i$ of small circles centered on $S^1$ and of radii $\sim \frac{1}{i}$. By Theorem D, for each $i \in \mathbb{N}$, there exist initial data $(u_{\varepsilon,i}^0)$ such that the limiting measures $\mu_{\varepsilon,i}^t$ evolves according to the classical motion of the small circles, whose lifetime is of the order of $i^{-2}$. By a diagonal argument, it is therefore possible to construct a sequence $u^0_\varepsilon$ such that $\mu^0_\varepsilon = S^1$ but $\mu^t_\varepsilon \equiv 0$ for $t > 0$.

**Remark 7.** The two first cases are related to specific properties of $(PGL)_\varepsilon$, whereas the third is intrinsically related to motion by mean curvature.
The three cases have a common feature: the Jacobians of \( u^\varepsilon \) converge to zero as \( \varepsilon \) tends to 0, at least in the sense of distributions. We consider next the space-time Jacobian of \( u^\varepsilon \),

\[
\mathcal{J} u^\varepsilon = \sum_{0 \leq i < j \leq N} (\partial x_i u^\varepsilon \times \partial x_j u^\varepsilon) \, dx_i \wedge dx_j,
\]

with the convention that \( x_0 \equiv t \). In view of the space-times bounds on the Ginzburg-Landau energy, we may invoke the work of Jerrard and Soner [34] (see also [1]), to assert that

\[
\mathcal{J} u^\varepsilon \rightharpoonup \mathcal{J}_* \quad \text{in} \quad (C^0_{c, \alpha}(\mathbb{R}^N \times \mathbb{R}^+))^*,
\]

where \( \mathcal{J}_* \) is an \((N-1)\)-rectifiable vector-valued measure. Moreover, it is shown in [34, 1], that \( \frac{1}{\pi} \mathcal{J}_* \) can be identified with an integer multiplicity \((N-1)\)-current, whose boundary is exactly \( \mathcal{J}^0 \) (the slice at time zero), and

\[
\frac{1}{\pi} |\mathcal{J}_*^t| \leq \mu_*^t, \quad \text{for} \, t \geq 0.
\]

Here, \( \mathcal{J}_*^t \) denotes the slice of the current \( \mathcal{J}_* \) on \( \mathbb{R}^N \times \{t\} \) (which we will prove to be well defined), and coincides with the limit in the sense of currents of the space Jacobians \( J u^\varepsilon(\cdot, t) \). The \((N-1)\)-rectifiable set \( \Sigma_{\mathcal{J}} \) supporting \( \mathcal{J}_* \) represents the concentration set of vorticity, and has therefore a great significance (presumably for the applications, more than the energy). Obviously, \( \Sigma_{\mathcal{J}} \subseteq \Sigma_{\mu} \), and it is rather easy to construct examples where they are different (think of two approaching circles with opposite orientations). Notice also that \( \mathcal{J}_* \), a priori, has more structure than \( \mu_* \), since it has an orientation and integer multiplicity (modulo \( \pi \)).

The previous discussion naturally leads to Ilmanen’s notion of enhanced (mean curvature) motion, which we recall now.

Let \( \mathcal{M}_0 \) be an integer multiplicity \((N-2)\)-current in \( \mathbb{R}^N \), without boundary. We assume for simplicity that \( \mathcal{M}_0 \) is bounded and of finite mass. Let \( \mathcal{M} \) be an integer multiplicity \((N-1)\)-current in \( \mathbb{R}^N \times [0, +\infty) \), and \( \{\mu^t\}_{t \geq 0} \) a family of non-negative Radon measures on \( \mathbb{R}^N \).

**Definition 4 (Enhanced motion).** The pair \( \{\mathcal{M}, \{\mu^t\}_{t \geq 0}\} \) is called an enhanced motion with initial condition \( \mathcal{M}_0 \) if and only if

\begin{enumerate}
  \item[i)] \( \partial \mathcal{M} = \mathcal{M}_0 \).
  \item[ii)] \( \mu^0 = |\mathcal{M}_0| \).
  \item[iii)] The measure defined on \( \mathbb{R}^+ \) by \( \mathcal{T}(B) = |\mathcal{M}|(\mathbb{R}^N \times B) \), for any Borel set \( B \), is absolutely continuous with respect to the Lebesgue measure on \( \mathbb{R}^+ \).
  \item[iv)] For all \( t \geq 0 \),
  \[
  \mu^t \geq |\mathcal{M}_t|,
  \]
  where \( \mathcal{M}_t \) denotes the slice of \( \mathcal{M} \) at time \( t \).
\end{enumerate}
v) \( \{ \mu_t^I \}_{t \geq 0} \) is a Brakke flow.

**Remark 8.** Notice in particular that conditions i) and iii) are closer to what one actually would normally expect from a Cauchy problem. In Ilmanen’s terminology, \( \mathcal{M} \) is called the under-current, and provides, in view of iv), a lower bound, which rules out sudden shrinking.

In [31], Ilmanen established the existence of an enhanced motion, for any initial data as above (actually in any codimension). Theorem D provides an alternative construction in codimension 2. The two solutions may differ, since there remains still some possible non-uniqueness for an enhanced motion (see the discussion on “matching motion” in [31]). Moreover, in the smooth case, there is uniqueness for an enhanced motion (before singularities appear) and it coincides with the classical notion.

We are now in position to present the proof of Theorem D.

**Proof of Theorem D.** The proof essentially relies on a combination of results proven in [1, 34].

**Construction of \( u^0_\varepsilon \).** The existence of a family \( (u^0_\varepsilon)_{\varepsilon > 0} \) satisfying \((H_0)\) and assumption ii) follows directly from [1, Theorem 1.1, ii). More precisely, the family \( (u^0_\varepsilon)_{\varepsilon > 0} \) constructed there verifies

\[
\frac{1}{\pi} J u^0_\varepsilon \to \mathcal{M}_0 \quad \text{in } [\mathcal{C}_c^{0, \alpha}(\mathbb{R}^N)]^*, \tag{39}
\]

and the additional compactness conditions \((H_1), (H_2)\), as well as the bound \( |u^0_\varepsilon| \leq 1 \).

**Construction of \( \mathcal{M} \).** We next consider the solution \( u_\varepsilon \) of (PGL)\( _\varepsilon \) with initial datum \( u^0_\varepsilon \), verifying (39) and (40) above. In view of \((H_1)\) and \((H_2)\), we may apply Theorem C to \( \mu_* \), so that \( \mu_* \) has no diffuse part, i.e. \( \mu_* = \nu_* \). In particular, by Theorem B, \( \{ \mu^I_t \}_{t \geq 0} \) is a Brakke flow, and property v) of Definition 4 is established.

In view of (i), the space-time Ginzburg-Landau energy is bounded in \( \mathbb{R}^N \times [0, T] \) for every \( T > 0 \), more precisely

\[
\int_{\mathbb{R}^N \times [0, T]} \frac{1}{2} |\nabla_x u_\varepsilon|^2 + V_\varepsilon(u_\varepsilon) \leq M_0(T + 1) |\log \varepsilon|. 
\]

We deduce from [34, 1] that

\[
\mathcal{J} u_\varepsilon \to \mathcal{J}_* \quad \text{in } [\mathcal{C}_c^{0, \alpha}(\mathbb{R}^N \times \mathbb{R}^+)]^*,
\]

where \( \mathcal{J}_* \) is a \((N-1)\) integer multiplicity current. Notice that \( \mathcal{J}_* \) has compact support in \( \mathbb{R}^N \times [0, T_f + 1] \) (see Proposition 3 for the definition of \( T_f \)). We set

\[
\mathcal{M} = \frac{1}{\pi} \mathcal{J}_*.
\]

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We claim that
\[ \partial \mathcal{M} = \mathcal{M}_0, \] (41)
i.e., ii) is verified.
Indeed, for every test form \( \chi \in C^\infty(\mathbb{R}^N \times \mathbb{R}^+) \) we have, by Stokes theorem,
\[
\begin{align*}
\int_{\mathbb{R}^N \times \mathbb{R}^+} \mathcal{J} u_\varepsilon \wedge \delta \chi &= \frac{1}{2} \int_{\mathbb{R}^N \times \mathbb{R}^+} \delta (u_\varepsilon \times \mathring{u}_\varepsilon) \wedge \delta \chi \\
&= \frac{1}{2} \int_{\mathbb{R}^N \times \mathbb{R}^+} \delta (u_\varepsilon \times \delta u_\varepsilon \wedge \delta \chi) = \frac{1}{2} \int_{\mathbb{R}^N \times \{0\}} (u_\varepsilon^0 \times du_\varepsilon^0) \wedge d\chi \\
&= \frac{1}{2} \int_{\mathbb{R}^N \times \{0\}} d(u_\varepsilon^0 \times du_\varepsilon^0) \wedge \chi = \int_{\mathbb{R}^N \times \{0\}} \mathcal{J} u_\varepsilon^0 \wedge \chi.
\end{align*}
\] (42)
In view of the compactness results in Theorem 4, and \((H_1), (H_2)\), we may pass to the limit as \( \varepsilon \to 0 \), so that
\[ \mathcal{M}(\delta \chi) = \mathcal{M}_0(\chi), \]
which establishes the claim.
At this stage, we have shown that the pair \( \{ \mathcal{M}, \{ \mathcal{M}_\varepsilon \}_{\varepsilon \geq 0} \} \) verifies i), ii), v) of Definition 4.

**Proof of iv).** By definition of slicing, and arguing as in (42), we have
\[ \mathcal{M}_t = \lim_{\varepsilon \to 0} J u_\varepsilon(\cdot, t). \]
Therefore iv) follows from [34, 1].

**Proof of iii).** Let \( I = [a, b] \) be a bounded interval in \( \mathbb{R} \). We claim that
\[ |\mathcal{M}|([\mathbb{R}^N \times [a, b]] \leq C(M_0) |b - a|^{1/2}, \] (43)
which clearly implies iii). Recall that
\[ |\mathcal{M}|([\mathbb{R}^N \times [a, b]] = \sup \{ \mathcal{M}(\chi), |\chi| \leq 1, \text{supp} \chi \subset \mathbb{R}^N \times [a, b] \}. \]
In order to prove (43), we need first to go back to the level \( \varepsilon \). Let \( \chi \in C^\infty_c(\mathbb{R}^N \times [a, b]) \), we have
\[ \mathcal{M}(\chi) = \lim_{\varepsilon \to 0} \int J u_\varepsilon \wedge \chi. \]
In order to estimate the integral on the r.h.s, we distinguish the purely spatial components of the Jacobian, and the space-time components. For the spatial components, we have by Lemma 3.13 (see also [1]),
\[
\left| \int_{\mathbb{R}^N \times [a, b]} (\ast \chi)_ij \frac{\partial u_\varepsilon}{\partial x_i} \times \frac{\partial u_\varepsilon}{\partial x_j} \right| \leq \frac{K}{|\log \varepsilon|} \|\chi\|_{\infty} \int_{\mathbb{R}^N \times [a, b]} e_\varepsilon(u_\varepsilon) + K\varepsilon^\beta \|\chi\|_{C^1} \left( 1 + \int_{\mathbb{R}^N \times [a, b]} e_\varepsilon(u_\varepsilon) \right) \] (44)
\[ \leq KM_0 |b - a| \left( \|\chi\|_{\infty} + \varepsilon^\beta (|\log \varepsilon| + 1) \|\chi\|_{C^1} \right). \]
In order to handle the space-time component, we rescale the function \( u_\varepsilon \) with respect to the time variable. Consider the interval \( I' = [a, b'] \), where \( b' = a + |b - a|^{1/2} \), and the function \( w_\varepsilon \) defined on \( \mathbb{R}^N \times I' \), by

\[
w_\varepsilon(\cdot, s) = u_\varepsilon(\cdot, (s - a) |b - a|^{1/2} + a), \quad s \in I',
\]

so that

\[
\frac{\partial w_\varepsilon}{\partial s}(\cdot, s) = |b - a|^{1/2} \frac{\partial u_\varepsilon}{\partial t}(\cdot, (s - a) |b - a|^{1/2} + a)
\]

and

\[
\int_{\mathbb{R}^N \times I'} \left| \frac{\partial w_\varepsilon}{\partial s} \right|^2 dx ds = |b - a|^{1/2} \int_{\mathbb{R}^N \times [a,b]} \left| \frac{\partial u_\varepsilon}{\partial t} \right|^2 dx dt \leq K |b - a|^{1/2} M_0 |\log \varepsilon|.
\]

On the other hand, by the energy inequality (I),

\[
\int_{\mathbb{R}^N \times I'} \frac{|\nabla w_\varepsilon|^2}{2} + V_\varepsilon(w_\varepsilon) \leq |b - a|^{1/2} M_0 |\log \varepsilon|,
\]

so that

\[
\int_{\mathbb{R}^N \times I'} \frac{|\nabla w_\varepsilon|^2}{2} + V_\varepsilon(w_\varepsilon) \leq K |b - a|^{1/2} M_0 |\log \varepsilon|.
\]

(45)

We apply the estimate of Lemma 3.13 to the function \( w_\varepsilon \) in \( \mathbb{R}^N \times I' \). This yields, in view of (45),

\[
\left| \int_{\mathbb{R}^N \times I'} (\ast \bar{\chi})_0 j \frac{\partial w_\varepsilon}{\partial s} \times \frac{\partial w_\varepsilon}{\partial x_j} \right| \leq K M_0 |b - a|^{1/2} (|\chi|_\infty + \varepsilon^\beta (|\log \varepsilon| + 1) |\chi|_{C^1} ).
\]

(46)

[Here we set \( \bar{\chi}(\cdot, s) = \chi(\cdot, (s - a) |b - a|^{1/2} + a) \)]

Finally, we have

\[
\int_{\mathbb{R}^N \times [a,b]} (\ast \bar{\chi})_0 j \frac{\partial u_\varepsilon}{\partial t} \times \frac{\partial u_\varepsilon}{\partial x_j} = \int_{\mathbb{R}^N \times I'} (\ast \bar{\chi})_0 j \frac{\partial w_\varepsilon}{\partial s} \times \frac{\partial w_\varepsilon}{\partial x_j}.
\]

(47)

Combining (44), (46) and (47), we are led to

\[
\left| \int_{\mathbb{R}^N \times [a,b]} \mathcal{J} u_\varepsilon \wedge \chi \right| \leq K M_0 (|b - a|^{1/2} + |b - a| |\chi|_\infty + o(1)).
\]

(48)

Passing to the limit in (48) as \( \varepsilon \to 0 \), we derive (43). \( \square \)

6 Properties of \( \Sigma_\mu \)

The purpose of this Section is to provide detailed proofs of some technical statements, concerning \( \Sigma_\mu \), in the introduction to Part II. More precisely, we will prove (3), Lemma 3, Theorem 6 and Propositions 6, 7 and 8. We begin with a few elementary observations which we will use later in the proofs.
Lemma 6.1. Let \((x, t) \in \Sigma_\mu\) and \(0 < r < \sqrt{t}\). Then, we have
\[
r^{2-N} \mu_*^{t-r^2}(B(x, \lambda(t - r^2)r)) > \eta_2,
\]
where \(\eta_2\) is the constant in Theorem 5.

Proof. Indeed, assume by contradiction that
\[
r^{2-N} \mu_*^{t-r^2}(B(x, \lambda(t - r^2)r)) \leq \eta_2.
\]
Then, by Theorem 5, for every \(\tau \in [t - \frac{1}{16}r^2, t]\)
\[
\mu_*^{\tau} = |\nabla \Phi_*|^2 dx \quad \text{on } B(x, \frac{1}{4}r),
\]
where \(\Phi_*\) is smooth. We are going to show that
\[
s^{2-N} \int_{\mathbb{R}^N} \exp\left(-\frac{|x-y|^2}{4s^2}\right) d\mu_*^{t-s^2} \to 0, \quad (6.1)
\]
as \(s \to 0\). Indeed, we write
\[
s^{2-N} \int_{B(x, \frac{1}{8}r)} \exp\left(-\frac{|x-y|^2}{4s^2}\right) d\mu_*^{t-s^2} \leq s^{2-N} \|\nabla \Phi_*\|^2_{L^\infty(B(x, \frac{1}{8}r))} \int_{\mathbb{R}^N} \exp\left(-\frac{|x-y|^2}{4s^2}\right) dx
\]
\[
\leq K \|\nabla \Phi_*\|^2_{L^\infty(B(x, \frac{1}{8}r))} s^2 \to 0, \quad (6.2)
\]
as \(s \to 0\). On the other hand,
\[
s^{2-N} \int_{\mathbb{R}^N \setminus B(x, \frac{1}{8}r)} \exp\left(-\frac{|x-y|^2}{4s^2}\right) d\mu_*^{t-s^2} \leq s^{2-N} \exp\left(-\frac{r^2}{256s^2}\right)M_0 \to 0, \quad (6.3)
\]
as \(s \to 0\). Combining (6.2) and (6.3), (6.1) follows and hence \(\Theta_{N-2}^r(\mu_*, (x, t)) = 0\), i.e.
\[
(x, t) \notin \Sigma_\mu,
\]
a contradiction. \(\square\)

Lemma 6.2. The function \((x, t) \mapsto \Theta_{N-2}^r(\mu_*, (x, t))\) is upper semi-continuous on the set \(\mathbb{R}^N \times (0, +\infty)\).

Proof. Let \((x, t) \in \mathbb{R}^N \times (0, +\infty)\), and let \((x_n, t_n)_{n \in \mathbb{N}}\) be a sequence such that \((x_n, t_n) \to (x, t)\). We are going to show that
\[
\limsup_{n \to +\infty} \Theta_{N-2}^r(\mu_*, (x_n, t_n)) \leq \Theta_{N-2}^r(\mu_*, (x, t)). \quad (6.4)
\]
Let $0 < r < \frac{1}{2}\sqrt{t}$ be fixed for the moment. For $n$ sufficiently large, set $r_n = \sqrt{t} + t_n - t$, so that $t - r^2 = t_n - r_n^2$. By monotonicity we have

$$
\Theta_{N-2}^r(\mu_*, (x_n, t_n)) \leq \frac{1}{r_{N-2}^N} \int_{\mathbb{R}^N} \exp(-\frac{|y - x_n|^2}{4r^2}) \, d\mu_{r_n-r^2}(y) \\
= \frac{1}{r_{N-2}^N} \int_{\mathbb{R}^N} \exp(-\frac{|y - x_n|^2}{4r^2}) \, d\mu_{r-r^2}(y).
$$

Letting $n$ tends to $+\infty$, we obtain

$$
\limsup_{n \to +\infty} \Theta_{N-2}^r(\mu_*, (x_n, t_n)) \leq \frac{1}{r_{N-2}^N} \int_{\mathbb{R}^N} \exp(-\frac{|x - y|^2}{4r^2}) \, d\mu_{r-r^2}(y).
$$

Next, we let $r \to 0$, and (6.4) follows.

### 6.1 Proof of Lemma 3

Let $x \in \mathbb{R}^N$ and $t > 0$. We have, for every $0 < r < \frac{1}{2}\sqrt{t}$,

$$
\frac{\mu_r(B(x, r))}{\omega_{N-2} r^{N-2}} \leq \exp\left(\frac{1}{4}\right) \frac{1}{\omega_{N-2} r^{N-2}} \int_{\mathbb{R}^N} \exp(-\frac{|x - y|^2}{4r^2}) \, d\mu_r(y) \\
\leq \exp\left(\frac{1}{4}\right) \frac{1}{\omega_{N-2} (t + r^2)^{\frac{N-2}{2}}} \int_{\mathbb{R}^N} \exp(-\frac{|x - y|^2}{4r^2}) \, d\mu_r(y) \\
\leq \exp\left(\frac{1}{4}\right) \frac{M_0}{\omega_{N-2} t^{\frac{N-2}{2}}},
$$

where we have used the monotonicity formula (Lemma 2) for the second inequality.

### 6.2 Proof of inequality (3)

Let $(x, t) \in \mathbb{R}^N \times (0, +\infty)$ be given. Let $0 < r < t$ be fixed for the moment. We write, for every $0 < s < \sqrt{t}$,

$$
\frac{1}{r^{N-2}} \mu_s(B(x, r)) \leq \frac{1}{r^{N-2}} \exp\left(\frac{1}{4}\right) \int_{\mathbb{R}^N} \exp(-\frac{|x - y|^2}{4r^2}) \, d\mu_s(y) \\
\leq \frac{1}{(r^2 + s^2)^{\frac{N-2}{2}}} \exp\left(\frac{1}{4}\right) \int_{\mathbb{R}^N} \exp(-\frac{|x - y|^2}{4(r^2 + s^2)}) \, d\mu_{r^2-r^2}(y),
$$

where we have used the monotonicity (at the point $(x, t + r^2)$) for the last inequality. Next, we choose $s = \sqrt{r}$. This yields

$$
\frac{1}{r^{N-2}} \mu_s(B(x, r)) \leq \frac{1}{(r^2 + r^2)^{\frac{N-2}{2}}} \exp\left(\frac{1}{4}\right) \int_{\mathbb{R}^N} \exp(-\frac{|x - y|^2}{4(r^2 + r^2)}) \, d\mu_{r^2-r^2}(y).
$$

(6.5)
In the last integral, we decompose
\[ \mathbb{R}^N = B(x, 1) \cup (\mathbb{R}^N \setminus B(x, 1)). \]
On \( B(x, 1) \), observe that
\[ \exp\left(-\frac{|x - y|^2}{4(r^2 + r)}\right) \leq K \exp\left(-\frac{|x - y|^2}{4r}\right), \]
for some absolute constant \( K \). On the other hand, on \( \mathbb{R}^N \setminus B(x, 1) \), we have
\[ \int_{\mathbb{R}^N \setminus B(x, 1)} \exp\left(-\frac{|x - y|^2}{4(r^2 + r)}\right) d\mu_*^t(y) \leq \exp\left(-\frac{1}{4(r^2 + r)}\right) M_0. \]
Going back to (6.5), we are led to
\[ \frac{1}{r^{N-2}} \mu_*^t(B(x, r)) \leq \frac{K}{r^{\frac{N-2}{2}}} \int_{\mathbb{R}^N} \exp\left(-\frac{|x - y|^2}{4r}\right) d\mu_*^t(y) + \frac{1}{r^{\frac{N-2}{2}}} \exp\left(-\frac{1}{4(r^2 + r)}\right) M_0. \]
Letting \( r \) go to zero the conclusion follows.

### 6.3 Proof of Theorem 6

Let \((x, t) \in \mathbb{R}^N \times (0, +\infty)\) and \(0 < r < \sqrt{t}\). We have
\[ r^{2-N} \mu_*^{t-r^2}(B(x, \lambda(t-r^2)r)) \leq \exp\left(\frac{\lambda^2(t-r^2)}{4}\right) \mathcal{E}_\mu((x, t), r). \quad (6.6) \]
Consider therefore the function
\[ \eta_3(s) = \exp(-\lambda^2(s)/4) \eta_2, \]
and assume next that, for some \(0 < r < \sqrt{t}\),
\[ \mathcal{E}_\mu((x, t), r) \leq \eta_3(t - r^2). \]
Then, by (6.6),
\[ \mu_*^{t-r^2}(B(x, \lambda(t-r^2)r)) \leq \eta_2 \]
and the conclusion follows by Lemma 6.1.

### 6.4 Proof of Proposition 6

**Proof of i).** In view of Corollary 1, we have
\[ \Sigma_\mu = \left\{ (x, t) \in \mathbb{R}^N \times (0, +\infty), \Theta^\nu_{N-2}(\mu_*, (x, t)) \geq \eta_3(t) \right\}. \]
Since $\eta_{\delta}(\cdot)$ is continuous, and since $\Theta_{N-2}^{\mu}(\mu_{\ast}, \cdot)$ is upper semi-continuous by Lemma 6.2, so is $\Theta_{N-2}^{\mu}(\mu_{\ast}, \cdot) - \eta_{\delta}(\cdot)$ on $\mathbb{R}^N \times (0, +\infty)$. The conclusion follows.

**Proof of ii).** We proceed in two steps. Firstly, we establish the estimate for $t = 1$ and secondly we argue by scaling.

**Step 1 : The case $t = 1$.** Let $0 < \delta < \frac{1}{4}$. Consider a standard (say parallelepiped) covering of $\mathbb{R}^N$ such that

$$\mathbb{R}^N \subseteq \bigcup_{x \in I} B(x, \delta), \quad \text{and} \quad B(x_i, \frac{\delta}{2}) \cap B(x_j, \frac{\delta}{2}) = \emptyset \text{ for } i \neq j.$$

Set

$$I_\delta = \left\{ i \text{ s.t. } B(x_i, \delta) \cap \Sigma_{\mu}^1 \neq \emptyset \right\}.$$

For $i \in I_\delta$, there exists some $y_i \in \Sigma_{\mu}^1 \cap B(x_i, \delta)$. Hence, by Lemma 6.1,

$$\mu_{\ast}^{1-\delta^2}(B(y_i, \lambda(1-\delta^2)\delta)) > \eta_2 \delta^{2-N},$$

and in particular

$$\mu_{\ast}^{1-\delta^2}(B(x_i, (\lambda(1-\delta^2) + 1)\delta)) > \eta_2 \delta^{2-N}. \quad (6.7)$$

On the other hand, since the balls $B(x_i, \frac{\delta}{2})$ are disjoint, the balls $B(x_i, (\lambda(1-\delta^2) + 1)\delta)$ cover at most $K$ times $\mathbb{R}^N$, where $K$ is a constant depending only on $N$, for $\delta < \frac{1}{4}$. Therefore,

$$\sum_{i \in I_\delta} \mu_{\ast}^{1-\delta^2}(B(x_i, (\lambda(1-\delta^2) + 1)\delta)) \leq KM_0. \quad (6.8)$$

Combining (6.7) and (6.8) we obtain

$$\sharp I_\delta \leq KM_0 \delta^{2-N}.$$

Since by definition, $\mathcal{H}^{N-2}(\Sigma_{\mu}^1) \leq K \limsup_{\delta \to 0}(\sharp I_\delta)\delta^{N-2}$, the conclusion follows.

**Step 2 : Invariance by scaling.** For $t_0 > 0$ fixed, consider the function

$$v_\epsilon(x, t) = u_\epsilon(\sqrt{t_0} x, t_0 t)$$

where $\epsilon = \frac{\delta}{\sqrt{t_0}}$, so that

$$v_\epsilon(x, 1) = u_\epsilon(\sqrt{t_0} x, t_0),$$

$v_\epsilon$ verifies (PGL)$_\epsilon$, and $E_\epsilon(v_\epsilon^0) = t_0^{2-\delta} E_\epsilon(v_0^0)$. Letting $\epsilon_n \to 0$, so does $\epsilon_n = \frac{\epsilon_n}{\sqrt{t_0}}$, and

$$\Sigma_{\mu}^1(u) = t_0^{\frac{1}{2}} \Sigma_{\mu_0}^1(v)$$

(with obvious notations), in particular

$$\Sigma_{\mu}^{10}(u) = t_0^{\frac{1}{2}} \Sigma_{\mu}^1(v),$$

so that

$$\mathcal{H}^{N-2}(\Sigma_{\mu}^{10}(u)) = t_0^{\frac{N-2}{2}} \mathcal{H}^{N-2}(\Sigma_{\mu}^1(v)).$$
By Step 1 applied to \( v_e \) and the corresponding measure \( \Sigma_\mu(v) \), we obtain

\[
\mathcal{H}^{N-2}(\Sigma_\mu^1(v)) \leq K \sup_{n \in \mathbb{N}} \left( E_{e_n}(v_{e_n}) \right) \leq K t_0^{\frac{2-N}{2}} M_0.
\]

Therefore,

\[
\mathcal{H}^{N-2}(\Sigma_\mu(u)) \leq K M_0,
\]

and the conclusion ii) follows.

**Proof of iii).** By i), we know that \( \Sigma_\mu^t \) is closed, and hence measurable. Therefore,

\[
\mu^t_\star = \mu^t_\star \llcorner (\mathbb{R}^N \setminus \Sigma_\mu^t) + \mu^t_\star \llcorner \Sigma_\mu^t. \tag{6.9}
\]

We claim that there exists a smooth function \( g \) defined on the open set \( \mathbb{R}^N \times (0, +\infty) \setminus \Sigma_\mu \) such that

\[
\mu^t_\star \llcorner (\mathbb{R}^N \setminus \Sigma_\mu^t) = g \cdot \mathcal{H}^N.
\]

Indeed, let \( x \in \mathbb{R}^N \setminus \Sigma_\mu^t \). Then by definition

\[
\lim_{r \to 0} \mathcal{E}_\mu((x,t),r) = 0,
\]

so that for some \( r_0 \) sufficiently small

\[
\mathcal{E}_\mu((x,t),r_0) \leq \eta_3 (t - r_0^2).
\]

Therefore, by (6.6),

\[
\mu^t_\star \llcorner (\mathbb{R}^N \setminus \Sigma_\mu^t) = g \cdot \mathcal{H}^N \quad \text{on} \quad B(x, \frac{1}{4} r_0),
\]

for some smooth function \( g \). Notice that at this point, we may only locally write \( g = |\nabla \Phi_*|^2 \), for some smooth \( \Phi_* \). We will see later that \( \Phi_* \) is global, whereas \( g \) is obviously already globally defined on \( \mathbb{R}^N \times (0, +\infty) \setminus \Sigma_\mu \).

Since \( \mathcal{H}^{N-2}(\Sigma_\mu^t) < +\infty \) we have \( \mathcal{H}^N(\Sigma_\mu^t) = 0 \), and hence

\[
\mu^t_\star \llcorner (\mathbb{R}^N \setminus \Sigma_\mu^t) = g \cdot \mathcal{H}^N, \tag{6.10}
\]

which establishes the claim.

Next, we deduce from Lemma 3 that \( \mu^t_\star \llcorner \Sigma_\mu^t \) is absolutely continuous with respect to the measure \( \mathcal{H}^{N-2} \), and by ii) that the measure \( \mathcal{H}^{N-2} \llcorner \Sigma_\mu^t \) is finite. We may therefore apply the Radon-Nikodym Theorem, which yields

\[
\mu^t_\star \llcorner \Sigma_\mu^t = \Theta_\star(x,t) \mathcal{H}^{N-2} \llcorner \Sigma_\mu^t, \tag{6.11}
\]

where \( \Theta_\star \) is the Radon-Nikodym derivative. By Lemma 3, it verifies the bound

\[
\Theta_\star(x,t) \leq K M_0 t^{\frac{2-N}{2}}.
\]

Combining (6.9), (6.10) and (6.11) conclusion iii) follows and the proof is complete.
6.5 Proof of Proposition 7

In this section, we shall use some very basic estimate for the time derivative \( \partial_t u_\varepsilon \), namely
\[
\frac{1}{|\log \varepsilon|} \int_{\mathbb{R}^N \times (0,T]} \left| \frac{\partial u_\varepsilon}{\partial t} \right|^2 \leq M_0 T, \quad \text{for every } T > 0.
\]

Therefore, extracting possibly a further subsequence, we may assume that there exists some non-negative Radon measure \( \omega_\ast \) defined on \( \mathbb{R}^N \times (0, +\infty) \) such that
\[
\frac{1}{|\log \varepsilon|} \left| \frac{\partial u_\varepsilon}{\partial t} \right|^2 \to \omega_\ast \quad \text{as measures},
\]
so that
\[
\omega_\ast(\mathbb{R}^N \times (0,T]) \leq M_0 T.
\]

Since we already know that \( \Sigma_\mu \subset \mathbb{R}^N \times (0, T_f + 1) \), where \( T_f \) is the constant in Proposition 3 (after which vorticity has been wiped out), we restrict our attention to this portion of space-time. Next, we introduce some subsets of \( \mathbb{R}^N \times (0, T_f + 1) \), which are concentration sets for the time derivative. Set, for \( l \in \mathbb{N}_\ast \) and \( q \in \mathbb{R}_+ \), to be fixed later,
\[
A_l(\omega_\ast) = \left\{ (x, t) \in \mathbb{R}^N \times (0, T_f + 1), \limsup_{r \to 0} \frac{1}{r^d} \int_{B(x, lr) \times [t-r^2, t]} \omega_\ast \geq 1 \right\}.
\]

Concentration sets for bounded measures are classical in the literature, see e.g. [54]. In a context similar to ours, they have been used in [41] in a related way. The following shows that \( A_l(\omega_\ast) \) is small in some appropriate sense.

**Lemma 6.3.** For each \( l \in \mathbb{N}_\ast \),
\[
\mathcal{H}_p^q(A_l(\omega_\ast)) < +\infty,
\]
where \( \mathcal{H}_p^q \) denotes the \( q \)-dimensional Hausdorff measure with respect to the parabolic distance \( dp((x,t),(x',t')) = \max\{|x-x'|,|t-t'|^{1/2} \} \).

**Proof.** Let \( \delta > 0 \) be given, and fixed for the moment. For \( (x,t) \in A_l(\omega_\ast) \), there exist \( r = r(x,t) < \delta \) such that
\[
\int_{B(x,lr) \times [t-r^2, t]} \omega_\ast \geq r^d. \tag{6.12}
\]
Clearly, \( \bigcup_{(x,t) \in A_l(\omega_\ast)} \Gamma^P_t(x,t,r(x,t)) \) covers \( A_l(\omega_\ast) \), where we have set
\[
\Gamma^P_t(x,t,r(x,t)) = B(x, lr(x,t)) \times (t - r(x,t)^2, t).
\]
Notice that \( \text{diam}(\Gamma^P_t) \leq lr \). By [23] 2.8.9, we may apply the Besicovitch covering theorem. In particular, there exists an integer \( m(l,N) \) depending only on \( N \) and \( l \), and there exists a sub-covering of the form
\[
A_l(\omega_\ast) \subset \bigcup_{i=1}^{m(l,N)} \left( \bigcup_{j \in i^l} \Gamma^P_t(x_j,t_j,r(x_j,t_j)) \right),
\]

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where for fixed $i$, the sets $\Gamma_j = \Gamma^i_j(x_j, t_j, r(x_j, t_j))$ are disjoint. Consequently, it follows from (6.12) that for each $i = 1, ..., m(l, N)$,

$$\sum_{j \in \mathcal{J}^i} r(x_j, t_j)^q \leq \sum_{j \in \mathcal{J}^i} \int_{\Gamma_j} \omega_* \leq \int_{\mathbb{R}^N \times [0, T_f + 1]} \omega_* \leq C(M_0).$$

Therefore,

$$m(l, N) \sum_{i=1}^{m(l, N)} \sum_{j \in \mathcal{J}^i} \text{diam}(\Gamma_j)^q \leq m(l, N) l^q C(M_0).$$

Note that the constant on the r.h.s is independent of $\delta$. Hence, letting $\delta \to 0$, we obtain

$$\mathcal{H}_p^q(A_l(\omega_*)) \leq \limsup_{\delta \to 0} \left( m(l, N) \sum_{i=1}^{m(l, N)} \sum_{j \in \mathcal{J}^i} \text{diam}(\Gamma_j)^q \right) \leq m(l, N) l^q C(M_0),$$

and the proof is complete. \hfill \Box

We fix $q = N - \frac{3}{2}$. This choice has no specific geometrical meaning, but is convenient as the following shows.

**Corollary 6.1.** We have,

$$\mathcal{H}^{N-1}(\bigcup_{l \in \mathcal{L}_t} A_l(\omega_*)) = 0. \quad (6.13)$$

Hence, for almost every $t > 0$

$$\mathcal{H}^{N-2}(\bigcup_{l \in \mathcal{L}_t} A_l^i(\omega_*)) = 0, \quad (6.14)$$

where $A_l^i(\omega_*) = A_l(\omega_*) \cap \mathbb{R}^N \times \{t\}$.

**Proof.** Since, by the previous lemma, $\mathcal{H}_p^{N-\frac{3}{2}}(A_l(\omega_*)) < +\infty$, it follows that

$$\mathcal{H}_p^{N-1}(A_l(\omega_*)) = 0.$$

On the other hand, parabolic balls are smaller than euclidian balls of the same radius, so that the parabolic Hausdorff measure dominates the euclidian Hausdorff measure. It follows that

$$\mathcal{H}^{N-1}(\bigcup_{l \in \mathcal{L}_t} A_l(\omega_*)) = 0,$$

and the proof is complete. \hfill \Box

Next, we introduce the set

$$\Omega_\omega = \left( \mathbb{R}^N \times (0, T_f + 1) \right) \setminus \bigcup_{n \in \mathbb{N}} A_t(\omega_*).$$

**Lemma 6.4.** Let $\chi \in C_0^\infty(\mathbb{R}^N)$. Then, for $(x, t) \in \Omega_\omega$,

$$\lim_{r \to 0} \left( \frac{1}{r^{N-2}} \int_{\mathbb{R}^N} \chi\left( \frac{y-x}{r} \right) \mu_*^t(y) - \frac{1}{r^{N-2}} \int_{\mathbb{R}^N} \chi\left( \frac{y-x}{r} \right) \mu_*^{t-r^2}(y) \right) = 0.$$
Proof. We need to go back first to the level of the functions $u_\varepsilon$. For $0 < r < \sqrt{t}$, by Lemma 2.1 we have

$$
\int_{\mathbb{R}^N \times \{t\}} e_\varepsilon(u_\varepsilon) \left( \frac{y - x}{r} \right) dx - \int_{\mathbb{R}^N \times \{t - r^2\}} e_\varepsilon(u_\varepsilon) \left( \frac{y - x}{r} \right) dx
$$

$$
= - \int_{\mathbb{R}^N \times \{t - r^2\}} \frac{\partial_t u_\varepsilon}{|\log \varepsilon|} \chi \left( \frac{y - x}{r} \right) dx dt - \frac{1}{r |\log \varepsilon|} \int_{\mathbb{R}^N \times \{t - r^2\}} \partial_t u_\varepsilon \nabla u_\varepsilon \cdot \nabla \chi \left( \frac{y - x}{r} \right) dx dt.
$$

Let $l \in \mathbb{N}$, such that supp$(\chi) \subset B(l)$. We set $\Lambda = B(x, lr) \times \{t - r^2, t\}$, and estimate the last term in the previous identity by the Cauchy-Schwarz inequality,

$$
\frac{1}{r |\log \varepsilon|} \left| \int_{\Lambda} \partial_t u_\varepsilon \nabla u_\varepsilon \cdot \nabla \chi \left( \frac{y - x}{r} \right) \right| \leq \left( \int_{\Lambda} \frac{\partial_t u_\varepsilon}{|\log \varepsilon|} \right)^{\frac{1}{2}} \left( \int_{\Lambda} \frac{|\nabla u_\varepsilon|^2}{|\log \varepsilon|} \right)^{\frac{1}{2}} \|\nabla \chi\|_\infty.
$$

We now let $\varepsilon$ go to zero, therefore obtaining the inequality for measures

$$
\frac{1}{r^{N-2}} \int_{\mathbb{R}^N} \chi \left( \frac{y - x}{r} \right) \left( d\mu^l - d\mu_* - r^2 \right)(y) 
\leq \left[ \frac{1}{r^{N-2}} \int_{\Lambda} \omega_* + \left( \frac{1}{r^{N-2}} \int_{\Lambda} \omega_* \right)^{\frac{1}{2}} \left( \frac{1}{r^N} \int_{\Lambda} d\mu_* \right)^{\frac{1}{2}} \right] \|\chi\|_\infty. \quad (6.15)
$$

Obviously, we have

$$
\frac{1}{r^{N-2}} \int_{\Lambda} \omega_* \leq r^\frac{1}{2} \left( \frac{1}{r^{N-\frac{3}{2}}} \int_{\Lambda} \omega_* \right).
$$

On the other hand, it follows from the monotonicity that

$$
\frac{1}{r^N} \int_{\Lambda} d\mu_* \leq C(t) t^{\frac{2-N}{2}} M_0.
$$

Therefore, the right hand side of (6.15) can be bounded by

$$
\mathcal{R}(r) = C(t, l, M_0) \|\chi\|_{C^1} r^\frac{1}{2} \left[ 1 + \frac{1}{r^{N-\frac{3}{2}}} \int_{\Lambda} \omega_* \right].
$$

Since by assumption $(x, t) \in \Omega_\varepsilon$, letting $r$ go to zero, we obtain

$$
\lim_{r \to 0} \mathcal{R}(r) \leq 2C(t, l, M_0) \|\chi\|_{C^1} \lim_{r \to 0} r^\frac{1}{2} = 0,
$$

and the proof is complete. \qed

In Lemma 6.4, we have assumed that $\chi$ has compact support. The following shows that the result still holds for $\chi = \exp(-\frac{|x|^2}{4})$, which is of special interest in view of the monotonicity!
Corollary 6.2. We have, for \((x, t) \in \Omega_\omega\),

\[
\lim_{r \to 0} \left( \frac{1}{r^{N-2}} \int_{\mathbb{R}^N} \exp\left(-\frac{|x-y|^2}{4r^2}\right) \, d\mu^t_s(y) - \frac{1}{r^{N-2}} \int_{\mathbb{R}^N} \exp\left(-\frac{|x-y|^2}{4r^2}\right) \, d\mu^{t-r^2}_s(y) \right) = 0.
\] (6.16)

In particular, for \((x, t) \in \Sigma_\mu \cap \Omega_\omega\), the following limit exists and verifies the inequality

\[
\lim_{r \to 0} \frac{1}{r^{N-2}} \int_{\mathbb{R}^N} \exp\left(-\frac{|x-y|^2}{4r^2}\right) \, d\mu^t_s(y) \geq \eta_3(t).
\] (6.17)

Proof. Let \(\zeta\) be a smooth cut-off function such that \(0 \leq \zeta \leq 1\), \(\zeta \equiv 1\) on \(B(1)\) and \(\zeta \equiv 0\) outside \(B(2)\). For \(l \in \mathbb{N}\), consider the function \(\zeta_l\) defined by \(\zeta_l(y) = \zeta\left(\frac{y}{l}\right)\), and set

\[
\chi_l(y) = \exp\left(-\frac{|y|^2}{4}\right)\zeta_l(y) \quad \text{for } y \in \mathbb{R}^N.
\]

We apply Lemma 6.4 to \(\chi_0\), so that

\[
\lim_{r \to 0} \left( \frac{1}{r^{N-2}} \int_{\mathbb{R}^N} \chi_l\left(\frac{x-y}{r}\right) \, d\mu^t_s(y) - \frac{1}{r^{N-2}} \int_{\mathbb{R}^N} \chi_l\left(\frac{x-y}{r}\right) \, d\mu^{t-r^2}_s(y) \right) = 0. \tag{6.18}
\]

On the other hand, we claim that, for every \(s \in [t-r^2, t]\),

\[
\frac{1}{r^{N-2}} \int_{\mathbb{R}^N} \left[ \exp\left(-\frac{|x-y|^2}{4r^2}\right) - \chi_l\left(\frac{x-y}{r}\right) \right] \, d\mu^t_s(y) \leq K s^{\frac{2-N}{2}} M_0 \exp\left(-\frac{l^2}{8}\right). \tag{6.19}
\]

Indeed, notice first that

\[
\exp\left(-\frac{|x-y|^2}{4r^2}\right) - \chi_l\left(\frac{x-y}{r}\right) \exp\left(-\frac{|x-y|^2}{4r^2}\right) (1 - \zeta_l\left(\frac{x-y}{r}\right))
\]

\[
\leq \exp\left(-\frac{|x-y|^2}{8r^2}\right) \exp\left(-\frac{l^2}{8}\right).
\]

Secondly, by the monotonicity formula

\[
\frac{1}{(\sqrt{2r})^{N-2}} \int_{\mathbb{R}^N} \exp\left(-\frac{|x-y|^2}{8r^2}\right) \, d\mu^t_s(y) \leq \frac{1}{(s+2r^2)^{\frac{N-2}{2}}} M_0,
\]

and the claim follows.

Note the the r.h.s. of (6.19) does not depend on \(r\), for \(r < \frac{1}{2}\sqrt{t}\). Combining (6.18) and (6.17) we are led to

\[
\lim\sup_{r \to 0} \left| \frac{1}{r^{N-2}} \int_{\mathbb{R}^N} \exp\left(-\frac{|x-y|^2}{4r^2}\right) \left( d\mu^t_s - d\mu^{t-r^2}_s \right)(y) \right| \leq K t^{\frac{2-N}{2}} M_0 \exp\left(-\frac{l^2}{8}\right).
\]

Since \(l\) was arbitrary the conclusion follows. \(\Box\)
We are now in position to present the proof of Proposition 7.

**Proof of Proposition 7.** For \((x, t) \in \Omega_o\), set

\[
\tilde{\Theta}_{N-2}(\mu_*, x) = \lim_{r \to 0} \frac{1}{r^{N-2}} \int_{\mathbb{R}^N} \exp\left(-\frac{|x-y|^2}{4r^2}\right) \, d\mu'_t(y).
\]

In view of Corollary 6.2, \(\tilde{\Theta}_{N-2}(\mu_*, x)\) exists on \(\Omega_o\) and

\[
\tilde{\Theta}(\mu'_t, x) = \Theta_{N-2}(\mu_*, (x, t)).
\]

(6.20)

If \((x, t) \notin \Sigma_\mu\), then \(\Theta_{N-2}^{P}(\mu_*, (x, t)) = 0\) so that (5) is obviously verified. Therefore, we assume in the sequel that \((x, t) \in \Sigma_\mu \cap \Omega_o\). Arguing as for the claim in Corollary 6.2, we obtain

\[
\frac{1}{(lr)^{N-2}} \int_{B(x, t)} d\mu'_t \geq \frac{K}{l^{N-2}} \frac{1}{r^{N-2}} \int_{\mathbb{R}^N} \exp\left(-\frac{|x-y|^2}{4r^2}\right) \, d\mu'_t - K \exp\left(-\frac{t^2}{8}\right) \frac{2^{2N}}{l^{2N}} M_0.
\]

Hence, letting \(r\) go to zero, and by (6.20),

\[
\Theta_{*, N-2}(\mu'_t, x) \geq \frac{K}{l^{N-2}} \left( \Theta_{N-2}^{P}(\mu_*, (x, t)) - K l^{N-2} \exp\left(-\frac{t^2}{8}\right) \frac{2^{2N}}{l^{2N}} M_0 \right).
\]

(6.21)

In order to obtain (5), we invoke the fact that on \(\Sigma_\mu\), \(\Theta_{N-2}^{P} \geq \eta_b(t)\), and therefore we choose \(l\) sufficiently large so that

\[
K l^{N-2} \exp\left(-\frac{t^2}{8}\right) \frac{2^{2N}}{l^{2N}} M_0 \leq \frac{1}{2} \eta_b(t) \leq \frac{1}{2} \Theta_{N-2}^{P}(\mu_*, (x, t)).
\]

Going back to (6.21), with this choice of \(l\), we obtain

\[
\Theta_{*, N-2}(\mu'_t, x) \geq \frac{K}{2l^{N-2}} \Theta_{N-2}^{P}(\mu_*, (x, t)),
\]

and the proof is complete.

\(\square\)

### 6.6 Proof of Proposition 8

We turn finally to the proof of Proposition 8. Once more, the starting point is Corollary 6.2. Let \((x, t) \in \Omega_o\) be given and fixed throughout. We consider the vector-space

\[
F = \left\{ g \in L^\infty(\mathbb{R}^+, \mathbb{R}) \text{ s.t. } I(g) = \lim_{r \to 0} I_r(g) \text{ exists and is finite} \right\},
\]

where for \(r > 0\), 

\[
I_r(g) = \frac{1}{r^{N-2}} \int_{\mathbb{R}^N} g\frac{|x-y|}{r} \, d\mu'_t(y).
\]

Notice that \(I_r\) and \(I\) are linear forms on \(F\). With this notation, the statement of Proposition 8 is precisely that the characteristic function \(1_{[0,1]}\) of the interval \([0,1]\) belongs to \(F\). In order to establish that fact, we derive first some basic properties of \(F\).
Lemma 6.5. i) For every $s > 0$, the function $e_s$ defined on $\mathbb{R}^+$ by $e_s(l) = \exp(-l^2s)$ belongs to $F$.

ii) Set $A(s) = I(e_s)$, then we have the identity

$$A(s) = A(1)s^{\frac{2-N}{2}}. \tag{6.22}$$

Proof. The case $s = \frac{1}{t}$ follows immediately from Corollary 6.2. For the general case, we argue by scaling. Indeed, we have for any $s > 0$,

$$I(e_1) = \lim_{r \to 0} \frac{1}{r^{N-2}} \int_{\mathbb{R}^N} \exp(-\frac{|x-y|^2}{4r^2}) \, d\mu_s(y)$$

$$= \lim_{r \to 0} (\frac{2 \sqrt{s}}{r})^{N-2} \int_{\mathbb{R}^N} \exp(-\frac{|x-y|^2}{r^2}) \, d\mu_s(y)$$

$$= (4s)^{\frac{N-2}{2}} \lim_{r \to 0} \frac{1}{r^{N-2}} \int_{\mathbb{R}^N} e_s(\frac{|x-y|}{r}) \, d\mu_s(y)$$

$$= (4s)^{\frac{N-2}{2}} I(e_s),$$

so that $I(e_s)$ exists and $I(e_s) = (4s)^{\frac{2-N}{2}} I(e_1)$.

Statement ii) then follows from the previous relation. \hfill \Box

Remark 6.1. The argument above shows more generally that if $g$ belongs to $F$, the the scaled function $g_s$ defined by $g_s(l) = g(\sqrt{s}l)$ belongs also to $F$.

Lemma 6.6. For every $k \in \mathbb{N}$, the function $l \mapsto l^{2k} \exp(-l^2)$ belongs to $F$.

Proof. The case $k = 0$ follows from Lemma 6.5, with $s = 1$. We provide first a detailed proof for the case $k = 1$. First note that by (6.22) $A$ is smooth on $\mathbb{R}^+$. We are going to prove that for $s > 0$,

$$A'(s) = \lim_{r \to 0} \frac{1}{r^{N-2}} \int_{\mathbb{R}^N} \frac{\partial}{\partial s} e_s(l) d\mu_s(y) = \lim_{r \to 0} -\frac{1}{r^{N-2}} \int_{\mathbb{R}^N} \frac{|x-y|^2}{r^2} \exp(-\frac{|x-y|^2}{r^2}) d\mu_s(y), \tag{6.23}$$

and in particular that the limit in the r.h.s do exists.

Let $s > 0$ and $\Delta s \in \mathbb{R}$ so that $s + \Delta s > 0$. We have, for $l \in \mathbb{R}^+$,

$$e_{s+\Delta s}(l) - e_s(l) = -\exp(-l^2s)(1 - \exp(-l^2\Delta s)),$$

and by Taylor expansion, for any $k \in \mathbb{R}^+$,

$$|1 - \exp(-k) - k| \leq \frac{k^2}{2}.$$

Hence, for any $y \in \mathbb{R}^N$, we have (choosing $k = \frac{\Delta s}{r^2}$)

$$\left|(e_{s+\Delta s} - e_s)(\frac{|x-y|}{r}) + \exp(-\frac{|x-y|^2}{r^2}) \frac{|x-y|^2}{r^2} \Delta s \right| \leq \exp(-\frac{|x-y|^2}{r^2} - s) \frac{|x-y|^4}{r^4}(\Delta s)^2$$

$$\leq C(s) \exp(-\frac{|x-y|^2}{2r^2} s)(\Delta s)^2.$$
Integrating against the measure \( \mu^t_s \) on \( \mathbb{R}^N \), we are led to

\[
\frac{1}{r^{N-2}} \int_{\mathbb{R}^N} \left[ \frac{e_{s+\Delta s} - e_s}{\Delta s} \left( \frac{|x-y|}{r} \right) + \exp\left( -\frac{|x-y|^2}{2r^2} \right) \right] d\mu^t_s \leq K \frac{\Delta s}{r^{N-2}} \int_{\mathbb{R}^N} \exp\left( -\frac{|x-y|^2}{2r^2} \right) d\mu^t_s \leq C(s)M_0 \Delta s.
\]

Note that the r.h.s side does not depend on \( r \), therefore letting \( \Delta s \to 0 \) identity (6.23) follows. Applying (6.23) with \( s = 1 \), we deduce that the function \( l \mapsto l^2 \exp(-l^2) \) belongs to \( F \). A similar computation shows that

\[
\frac{d^k}{ds^k} A(s) = \lim_{t \to 0} \frac{(-1)^k}{t^{N-2}} \int_{\mathbb{R}^N} \frac{|x-y|^{2k}}{y^{2k}} \exp\left( -\frac{|x-y|^2}{2r^2} \right) d\mu^t_s(y), \tag{6.24}
\]

so that the function \( l \mapsto l^{2k} \exp(-l^2) \) belongs to \( F \).

\[\square\]

**Lemma 6.7.** The set

\[ W = \left\{ g \in C^2_0(\mathbb{R}^+) \text{ s.t. } g(0) = 0 \right\} \]

is included in \( F \).

**Proof.** For a function \( g \) defined on \( \mathbb{R}^+ \), we consider its extension \( \tilde{g} \) to \( \mathbb{R} \) so that \( \tilde{g} \) is even. In particular, \( g \) belongs to \( W \) if and only if \( \tilde{g} \) belongs to \( C^2_0(\mathbb{R}) \).

Next, for \( k \in \mathbb{N} \), we consider the subset \( V_k \) of \( L^2(\mathbb{R}) \) defined by

\[ V_k = \text{Vect} \left\{ l \mapsto l^{2j} \exp(-l^2), \; j \in \{0,...,k\} \right\} \]

In view of Lemma 6.6 the restriction of elements of \( V_k \) to \( \mathbb{R}^+ \) belongs to \( F \). We are going to show that elements of \( W \) can be suitably approximated by elements of \( V_k \) (as \( k \to +\infty \)), so that the conclusion will follow. For that purpose, we recall some well-known facts concerning Hermite polynomials, and which enter directly in our argument.

**Hermite polynomials** For \( n \in \mathbb{N} \), the Hermite polynomials \( H_n \) can be expressed by Rodrigues’ formula

\[ H_n(l) = (-1)^m \exp(l^2) \frac{d^m}{dl^m} \exp(-l^2). \]

The degree of \( H_n \) is exactly \( m \), and \( H_n \) is even if \( m \) is even, odd if \( m \) is odd. Set, for \( l \in \mathbb{R} \),

\[ \psi_n(l) = c_n H_n(l) \exp(-\frac{l^2}{2}), \quad \text{where } c_n = (\sqrt{\pi}2^nn!)^{-\frac{1}{4}}. \]

The function \( \psi_n \) verify the first order differential relations

\[ (l - \frac{d}{dl}) \psi_n = \sqrt{2(m+1)} \psi_{n+1}, \quad (l + \frac{d}{dl}) \psi_n = \sqrt{2m} \psi_{n-1}, \]

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so that for $m \geq 0$,
\[
\sqrt{2} \frac{d}{dl} \psi_m = \sqrt{m} \psi_{m-1} - \sqrt{m+1} \psi_{m+1},
\] (6.25)
and for $m \geq 0$,
\[
- \frac{d^2}{dl^2} \psi_m + l^2 \psi_m = 2(m + 1) \psi_m
\] (6.26)
(i.e. the $\psi_m$ are eigenfunctions of the harmonic oscillator). Moreover, the family 
\(\{\psi_m\}_{m \in \mathbb{N}}\) is a Hilbert basis of $L^2(\mathbb{R})$. For $f \in L^2(\mathbb{R})$, set $c_m(f) = \langle f, \psi_m \rangle_{L^2(\mathbb{R})}$. If $f$ belongs to $C^2_c(\mathbb{R})$, then we have, by (6.26),
\[
c_m(f) = \frac{1}{2(m + 1)} \left( - \frac{d^2}{dl^2} \psi_m + l^2 \psi_m, f \right) = \frac{1}{2(m + 1)} \left( \psi_m, - \frac{d^2}{dl^2} f + l^2 f \right),
\] (6.27)
and by (6.25), for $m \geq 1$
\[
c_m \left( \frac{df}{dl} \right) = \left( \psi_m, \frac{df}{dl} \right) = \left( - \frac{d\psi_m}{dl}, f \right) = \sqrt{\frac{m+1}{2}} c_{m+1}(f) - \sqrt{\frac{m}{2}} c_{m-1}(f).
\] (6.28)
Let $P_m$ be the orthogonal projection (for the $L^2$-scalar product) onto the space $W_m = \text{vect}_{0 \leq j \leq m} \{\psi_j\}$. For $f \in C^2_c(\mathbb{R})$, we have by the Bessel-Parseval identity and (6.27)
\[
\|f - P_m f\|_{L^2}^2 = \sum_{j \geq m+1} c_j^2(f) \leq \frac{1}{4(m + 1)^2} \sum_{j \geq m+1} c_j^2(-f'' + l^2 f)
\] (6.29)
\[
\leq \frac{1}{4(m + 1)^2} \left[ \|f''\|_{L^2}^2 + \|l^2 f\|_{L^2}^2 \right].
\]
Since, by (6.25), we have
\[
\frac{d}{dl} (f - P_m f) = \sum_{j \geq m+1} c_j(f) \frac{d\psi_m}{dl} \sqrt{2} \sum_{j \geq m+1} c_j(f) \left[ \sqrt{j} \psi_{j-1} - \sqrt{j + 1} \psi_{j+1} \right]
\] (6.30)
\[
= \sqrt{2} \sum_{j \geq m} \left[ \sqrt{j + 1} c_{j+1}(f) - \sqrt{j} c_{j-1}(f) \right] \psi_j - \sqrt{m} c_{m-1}(f) \psi_m.
\]
We deduce similarly that
\[
\|\frac{d}{dl} (f - P_m(f))\|_{L^2}^2 \leq K \sum_{j \geq m} j c_j^2(f) \leq \frac{K}{m} \left[ \|f''\|_{L^2}^2 + \|l^2 f\|_{L^2}^2 \right],
\] (6.31)
and finally combining (6.29) and (6.30)
\[
\|f - P_m (f)\|_{\infty} \leq K \|f - P_m (f)\|_{H^1} \leq \frac{K}{\sqrt{m}} \left[ \|f''\|_{L^2}^2 + \|l^2 f\|_{L^2}^2 \right].
\]

**Proof of Lemma 6.7 completed.** Let $g \in W$ be given, and consider the function $f$ defined for $l \in \mathbb{R}$ by $f(l) = \tilde{g}(l) \exp(i \frac{l^2}{2})$, so that $f \in C^2_c(\mathbb{R})$ and is even. For $m \in \mathbb{N}_*$, set
\[ h_m = f - P_m(f) \text{ and } g_m = P_m(f)e^{-\frac{t}{m}} = P_m(f) \exp\left(-\frac{t^2}{m^2}\right). \] Since \( f \) is even, \( P_m(f) \) is even also, and consequently \( g_m \) is even, of the form \( g_m(t) = Q_m(t) \exp(-t^2) \), where \( Q_m \) is an even polynomial of degree less or equal to \( m \). In view of Lemma 6.6, the restriction of \( g_m \) to \( \mathbb{R}^+ \) belongs to \( F \). Since \( g = g_m + h_m e^{-\frac{t}{m}} \), we may write for \( 0 < r < 1 \)

\[ I_r(g) = I_r(g_m) + I_r(h_m e^{-\frac{t}{m}}). \] (6.32)

By (6.31), we have

\[ \|h_m\| \leq \frac{C(g)}{\sqrt{m}}, \]

where \( C(g) \) is independent of \( m \), so that for \( 0 < r < 1 \)

\[ |I_r(h_m e^{-\frac{t}{m}})| \leq \frac{C(g)}{\sqrt{m}}. \] (6.33)

On the other hand, since \( g_m \) belongs to \( F \), \( I_r(g_m) \to I(g_m) \), for all \( m \in \mathbb{N} \). We claim that the sequence \( \{I(g_m)\}_{m \in \mathbb{N}} \) converges as \( m \to +\infty \). Indeed, for \( k \geq m \), we have by (6.32) and (6.33), for \( 0 < r < 1 \),

\[ |I_r(g_k) - I_r(g_m)| \leq \frac{C(g)}{\sqrt{m}}. \]

Letting \( r \to 0 \), we deduce that

\[ |I(g_k) - I(g_m)| \leq \frac{C(g)}{\sqrt{m}} \]

so that \( (I(g_m))_{m \in \mathbb{N}} \) is a Cauchy sequence and hence converges to a limit \( L \). We finally prove that \( I_r(g) \to L \) as \( r \to 0 \). Indeed, let \( \delta > 0 \) be given. In view of (6.33) we may choose \( m_0 \) such that \( |I_r(h_m e^{-\frac{t}{m}})| \leq \frac{\delta}{4} \) for \( 0 < r < 1 \), and \( |I(g_m) - L| \leq \frac{\delta}{4} \). Going back to (6.32) we have therefore, for \( 0 < r < 1 \),

\[ |I_r(g) - L| \leq |I_r(g_{m_0}) - I(g_{m_0})| + \frac{\delta}{2}. \]

Choosing \( r_0 > 0 \) such that for \( 0 < r < r_0 \), \( |I_r(g_{m_0}) - I(g_{m_0})| \leq \frac{\delta}{2} \), we deduce that for \( 0 < r < r_0 \),

\[ |I_r(g) - L| \leq \frac{\delta}{2}. \]

Since \( \delta \) was arbitrary, it follows that \( I_r(g) \) converges to \( L \) as \( r \to 0 \), and hence \( g \) belongs to \( F \). \[ \square \]

**Proof of Proposition 8.** In view of the above discussion, we only need to prove that the characteristic function \( 1_{[0,1]} \) of the interval \( [0,1] \) belongs to \( F \). Let \( (g_n)_{n \in \mathbb{N}} \) be an increasing sequence of functions defined on \( \mathbb{R}^+ \) verifying

\[ g_n \in \mathcal{C}^2(\mathbb{R}^+), \quad g_n(0) = 0, \quad g_n \leq 1_{[0,1]}, \quad \text{and} \quad \bar{g}_n \geq 1_{[0,1]}, \]

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where $\tilde{g}_n(l) = \frac{n+1}{n} g_n\left(\frac{n}{n+1}\right)$. Note that by Lemma 6.7, $g_n \in F$ and $\tilde{g}_n \in F$ for all $n \in \mathbb{N}$. Let

$$L = \lim_{n \to +\infty} I(g_n) = \sup_{n \in \mathbb{N}} I(g_n).$$

(6.34)

By Remark 6.1, we also have

$$\lim_{n \to +\infty} I(\tilde{g}_n) = \lim_{n \to +\infty} \frac{n+1}{n} \left(\frac{n}{n+1}\right)^{2n/3} I(g_n) = L.$$  

(6.35)

Finally, since $g_n \leq 1_{[0,1]} \leq \tilde{g}_n$, for each $0 < r < 1$ and $n \in \mathbb{N}$ we have

$$I_r(g_n) \leq I_r(1_{[0,1]}) \leq I_r(\tilde{g}_n).$$

(6.36)

Combining (6.34), (6.35) and (6.36) we obtain $L = \lim_{r \to 0} I_r(1_{[0,1]})$, and the proof is complete. }

\[\square\]
Bibliography


Addresses

**Fabrice Bethuel**, Laboratoire Jacques-Louis Lions, Université de Paris 6, 4 place Jussieu BC 187, 75252 Paris, France & Institut Universitaire de France. E-mail: bethuel@ann.jussieu.fr

**Giandomenico Orlandi**, Dipartimento di Informatica, Università di Verona, Strada le Grazie, 37134 Verona, Italy. E-mail: orlandi@sci.univr.it

**Didier Smets**, Laboratoire Jacques-Louis Lions, Université de Paris 6, 4 place Jussieu BC 187, 75252 Paris, France. E-mail: smets@ann.jussieu.fr