# Mappings of finite distortion and asymmetry of domains

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**Abstract**

We establish an anisotropic Bonnesen inequality for images of balls under homeomorphisms with exponentially integrable distortion.

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## 1 Introduction

The classical isoperimetric inequality states that if $E$ is a Borel set in $\mathbb{R}^n$, $n \geq 2$, with finite Lebesgue measure $|E|$, then the ball with the same volume has lower perimeter or, equivalently, that

$$n\omega_n^{\frac{1}{n}}|E|^{\frac{n-1}{n}} \leq P(E)\,. \tag{1.1}$$

Here $P(E)$ is the distributional perimeter of $E$, and $\omega_n$ the measure of the unit ball $B$ in $\mathbb{R}^n$. It is well known that the isoperimetric inequality holds as an equality if and only if $E$ is a ball.

In the last few years, the so-called *quantitative isoperimetric inequalities* have attracted a great interest (see for example [5, 3] and the references therein). The quantitative isoperimetric inequalities are estimates which improve the classical isoperimetric one in this sense: if (1.1) is almost an equality for a set $E$, then $E$ has to be close to a ball with respect to some geometric quantity. More precisely, Osserman ([14]) calls in this way any inequality of the form

$$\lambda(E) \leq P(E)^2 - 4\pi|E|,$$

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valid for smooth sets $E$ in the plane $\mathbb{R}^2$, where $\lambda(E)$ is a suitable non negative measure of the "asymmetry" of $E$, vanishing only if $E$ is a ball.

Inequalities of this kind are called Bonnesen style inequalities, after the results proved in the plane by Bonnesen ([2]). More precisely Bonnesen proved that

$$4\pi(R-r)^2 \leq P(E)^2 - 4\pi|E|,$$

where $R$ and $r$ are the circumradius ans the inradius of $E$, respectively.

Bonnesen's inequality does not hold for general domains in dimension greater than two. However, important quantitative isoperimetric inequalities have been obtained for general Borel sets in all dimensions (see [5, 3]). In order to describe these results let us introduce, for any Borel set $E$ in $\mathbb{R}^n$, with $0 < |E| < +\infty$, the *isoperimetric deficit of* $E$

$$\delta(E) = \frac{P(E)}{n\omega_n^{\frac{1}{n}}|E|^{\frac{n-1}{n}}} - 1 = \frac{P(E) - P(rB)}{P(rB)},$$

where $r$ is the radius of the ball having the same volume of $E$, i.e. $|E| = r^n\omega_n$.

Fuglede (see [4]) proved that if $E$ is a convex set with volume $\omega_n$, then

$$\min_{x\in\mathbb{R}^n} \operatorname{dist}_H(E, x+B) \leq C(n)\delta(E)^{\alpha(n)}$$

where $\operatorname{dist}_H(E, x+B)$ denotes the Hausdorff distance between the sets $E$ and $x+B$, and $\alpha(n)$ is a suitable exponent depending on the dimension $n$. When dealing with general non convex sets, one cannot expect the validity of inequalities like that proved by Fuglede, as can be seen by taking sets obtained by gluing thin long "tentacles" to the unit ball.

Fusco, Maggi and Pratelli in [5] gave a sharp estimate of the Fraenkel asymmetry of a set $E$ in terms of its isoperimetric deficit, proving a conjecture by Hall ([8]). Very recently, an anisotropic version of the result in [5] was established by Figalli, Maggi and Pratelli in [3]. In order to illustrate their result let us give some preliminary definitions.

Given a convex set $K$ containing the origin, and an open set $E$ with smooth boundary $\partial E$ oriented by its unit outer normal $\nu_E$, the *anisotropic perimeter of* $E$ is defined as

$$P_K(E) = \int_{\partial E} ||\nu_E(x)||_* \, d\mathcal{H}^{n-1}(x),$$

where

$$||\nu_E||_* = \sup\{y \cdot \nu_E;\ y \in K\}.$$

Note that, when $K$ is the unit ball $B$, the perimeter $P_K(E)$ coincides with the usual notion of Euclidean perimeter. It is possible to extend the definition of anisotropic perimeter also to non smooth sets, by using the notion of reduced boundary (see [1] or Section 2.1 in [3]). Then, we introduce the *isoperimetric deficit of* $E$, setting

$$\delta_K(E) = \frac{P_K(E)}{n|K|^{\frac{1}{n}}|E|^{\frac{1}{n'}}} - 1$$

and we give the following definition of *asymmetry.*

**Definition 1.1.** Let $U$ and $V$ be measurable sets in $\mathbb{R}^n$ with finite positive measure. We define the *relative asymmetry* $A(U,V)$,

$$A(U,V) = \inf_{b\in\mathbb{R}^n} \frac{|U \setminus b + \kappa V|}{|U|}, \quad \kappa = \Big(\frac{|U|}{|V|}\Big)^{1/n},$$

where $\kappa V = \{\kappa y :\ y \in V\}$. Notice that $A(U,V) = A(V,U)$, and $A(\lambda U, \mu V) = A(U,V)$ for every $\lambda$ and $\mu > 0$.

Then the main result of [3] is the following anisotropic quantitative isoperimetric inequality.

**Theorem 1.2.** *Let $E$ be a set of finite perimeter such that $|E| < \infty$. Then*

$$A(E,K) \leq C\sqrt{\delta_K(E)}$$

*for a constant $C$ depending only on $n$.*

The aim of this paper is to prove an anisotropic version of the Fuglede type inequality in higher dimension, restricting ourselves to the class of domains which are images of the unit ball under homeomorphisms with exponentially integrable distortion.
Before stating our main results, we recall that $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ is a *mapping of finite distortion*, if $f \in W^{1,1}_{\mathrm{loc}}(\Omega,\mathbb{R}^n)$, the Jacobian determinant $J_f \in L^1_{\mathrm{loc}}(\Omega)$, and if there is a measurable, almost everywhere finite function $K_f$ such that

$$|Df(x)|^n \leq K_f(x)J_f(x) \quad \text{for almost every } x \in \Omega.$$

If $U$ and $V$ are as in Definition 1.1, we recall that the *Hausdorff distance* between $\partial U$ and $\partial V$ is

$$\mathrm{dist}_H(\partial U, \partial V) = \max\{\sup_{x\in\partial U}\inf_{y\in\partial V}|x-y|, \sup_{y\in\partial V}\inf_{x\in\partial U}|x-y|\}.$$

We define the *relative distance* $\mathrm{dist}_R(\partial U, \partial V)$,

$$\mathrm{dist}_R(\partial U, \partial V) = \inf_{b\in\mathbb{R}^n} \frac{\mathrm{dist}_H(\partial U, b + \kappa\partial V)}{|U|^{1/n}},$$

where $\kappa$ is as above.

Our main result states that we can control the relative distance between the image of the unit sphere under a homeomorphism with exponentially integrable distortion and the boundary of a convex set with the relative asymmetry of the image of the unit ball. More precisely, we have

**Theorem 1.3.** *Let $f : B(2) \to fB(2)$ be a homeomorphism of finite distortion satisfying*

$$\int_{B(2)} \exp(\mu K_f(x))\,\mathrm{d}x \leq \mathcal{K}$$

*for some $\mu$ and $\mathcal{K} > 0$, and let $B(x,R) \subset E \subset B(x,\Lambda R)$ be a convex domain. Then*

$$\begin{aligned} \mathrm{dist}_R(fS(1), \partial E)^{n+n^2/\mu} &\leq C(n,\mu,\mathcal{K},\Lambda) A(fB(1), E) \\ &\leq C(n,\mu,\mathcal{K},\Lambda)\sqrt{\delta_E(fB(1))}. \qquad (1.2) \end{aligned}$$

Recall that the second inequality follows from Theorem 1.2. Theorem 1.3 is derived from the following more general result, which allows us to control the relative distance between the images of the unit sphere under two homeomorphisms with exponentially integrable distortion with the relative asymmetry of the images of the unit ball.

**Theorem 1.4.** *Let $f : B(2) \to fB(2)$ and $g : B(2) \to gB(2)$ be homeomorphisms of finite distortion satisfying*

$$\int_{B(2)} \exp(\mu K_f(x)) + \exp(\mu K_g(x))\,\mathrm{d}x \leq \mathcal{K}$$

*for some $\mu$ and $\mathcal{K} > 0$. Then*

$$\mathrm{dist}_R(fS(1), gS(1))^{n+n^2/\mu} \leq C(n,\mu,\mathcal{K}) A(fB(1), gB(1)).$$

Our proofs rely on the relative isoperimetric inequality and the continuity property of the homeomorphisms with exponentially integrable distortion, as well as a chain rule and a change of variable formula.

## 2 Notations and preliminary results

We shall denote a ball in $\mathbb{R}^n$ with center $x$ and radius $r$ by $B(x,r)$, while when the ball is centered at the origin we shall omit the indication of the center, i.e. $B(r) = B(0,r)$. The corresponding notations for spheres will be $S(x,r)$ and $S(r)$.

We shall denote by $|Df|$ the operator norm of the differential matrix and by $D^\sharp f$ the adjugate of $Df$ which is defined by the formula

$$Df \cdot D^\sharp f = \mathbf{I} \cdot J_f, \qquad (2.1)$$

where, as usual, $J_f = \det Df$ and $\mathbf{I}$ is the identity matrix.

Recall that a homeomorphism $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ has finite outer distortion if its Jacobian $J_f$ is strictly positive a.e. on the set where $|Df| \neq 0$. In case $J_f(x) \geq 0$ a.e., we define its outer distortion function as

$$K_f(x) = \begin{cases} \frac{|Df(x)|^n}{J_f(x)} & \text{for } J_f(x) > 0 \\ 1 & \text{otherwise.} \end{cases} \tag{2.2}$$

We note, for a homeomorphism with finite distortion, the following relation:

$$|D^\sharp f^{-1}(y)| = K_f(f^{-1}(y)) J_{f^{-1}}(y)^{\frac{n-1}{n}} \qquad a.e.\ y \in f(\Omega) \tag{2.3}$$

(for an exhaustive treatment of the mappings with finite distortion we refer to [10]).

We shall use following result concerning the modulus of continuity of a homeomorphism with exponentially integrable distortion.

**Theorem 2.1** ([13])**.** *Let $f$ be as in Theorem 1.4. If $x$ and $y \in B(5/4)$, then*

$$|f(x) - f(y)| \leq \frac{C(n, \mu, \mathcal{K})}{\log^{\mu/n} \frac{1}{|x-y|}} |fB(1)|^{1/n}.$$

Moreover, we have the following distortion estimates

**Lemma 2.2** ([15])**.** *Let $f$ be as in Theorem 1.4. If $B(x,t) \subset B(5/4)$, then there exists a constant $C = C(n, \mu, \mathcal{K})$ such that*

$$\frac{\max_{y \in S(3/2)} |f(x) - f(y)|}{\min_{y \in S(x,t)} |f(x) - f(y)|} \leq \exp\left( C(n, \mu, \mathcal{K}) t^{-\frac{1}{n - \frac{3}{2}}} \right).$$

We will use the following consequence of the previous results.

**Lemma 2.3.** *Let $f$ be as above and assume $f(e_1) = 0$ and $|fB(1)| = \omega_n$. Then there exists $m_0 > 0$, depending only on $n$, $\mu$, and $\mathcal{K}$, such that for every $0 < t < m_0$,*

$$B(e_1, s_t) \subset f^{-1}B(t) \subset B(e_1, 1/10). \tag{2.4}$$

*Here $s_t$ satisfies*

$$t = \frac{C(n, \mu, \mathcal{K})}{\log^{\mu/n} \frac{1}{s_t}},$$

*where $C(n, \mu, \mathcal{K})$ is the constant in Theorem 2.1.*

*Proof.* The first inclusion in (2.4) follows directly from Theorem 2.1. Also, under our assumptions, there exists $b \in S(\frac{3}{2})$ such that $|f(e_1) - f(b)| = |f(b)| \geq 1$. Lemma 2.2 with $x = e_1$ yields

$$\frac{1}{\min_{y \in S(e_1, 1/10)} |f(y) - f(e_1)|} \leq \frac{\max_{b \in S(3/2)} |f(b)|}{\min_{y \in S(e_1, 1/10)} |f(y)|} \leq \varphi(n, \mu, \mathcal{K}).$$

The second inclusion follows once $m_0$ is chosen to be small enough depending only on $n$, $\mu$, and $\mathcal{K}$. $\square$

**Corollary 2.4.** *Let $0 < t < m_0$, where $m_0$ is as in Lemma 2.3. Then*

$$|f^{-1}(B(t)) \cap B(1)|^{(n-1)/n} \leq C(n) \mathcal{H}^{n-1}(f^{-1}(S(t)) \cap B(1)). \tag{2.5}$$

*Proof.* The second inclusion in (2.4) guarantees that

$$|f^{-1}(B(t)) \cap B(1)| \leq |B(1) \setminus f^{-1}B(t)|.$$

Therefore, the claim follows by the relative isoperimetric inequality in a ball (see formula (3.43) in [1]) . $\square$

# 3 Proof of Theorem 1.4

We assume that $A(fB(1), gB(1)) > 0$. Since the distortions $K_f$ and $K_g$ are not affected by postcompositions with affine maps, we may assume that $|fB(1)| = \omega_n$ and

$$A(fB(1), gB(1)) = \omega_n^{-1} |fB(1) \setminus gB(1)| = \omega_n^{-1} |gB(1) \setminus fB(1)|.$$

We denote $m = \operatorname{dist}_H(fS(1), gS(1))$. In order to prove the theorem, we need to show that

$$m^{n + n^2/\mu} \leq C(n, \mu, \mathcal{K}) |fB(1) \setminus gB(1)|.$$

Now either there exists a point $y_0 \in fS(1)$ such that

$$\operatorname{dist}_H(y_0, gS(1)) = m, \tag{3.1}$$

or a point $z_0 \in gS(1)$ such that

$$\operatorname{dist}_H(z_0, fS(1)) = m. \tag{3.2}$$

From now on we assume that (3.1) holds, otherwise we change the roles of $f$ and $g$. By precomposing $f$ with a rotation and postcomposing with a translation, if necessary, we may assume that $y_0 = 0$ and $f^{-1}(y_0) = e_1$.

Since $\operatorname{dist}_H(0, gS(1)) = m$, we have that either

$$B(m) \cap fB(1) \subset \mathbb{R}^n \setminus gB(1),$$

or

$$B(m) \setminus fB(1) \subset gB(1),$$

depending on whether or not $0 \in \mathbb{R}^n \setminus gB(1)$. In the first case we have

$$|fB(1) \setminus gB(1)| \geq |B(m) \cap fB(1) \setminus gB(1)| = |B(m) \cap fB(1)|,$$

so in order to prove the theorem it suffices to show that

$$m^{n+n^2/\mu} \leq C(n, \mu, \mathcal{K})|B(m) \cap fB(1)|. \tag{3.3}$$

In the second case

$$|gB(1) \setminus fB(1)| \geq |B(m) \setminus fB(1)|,$$

so in this case the theorem follows if

$$m^{n+n^2/\mu} \leq C(n, \mu, \mathcal{K})|B(m) \setminus fB(1)|. \tag{3.4}$$

The proofs of estimates (3.3) and (3.4) are very similar, and therefore we only give the proof of (3.3).

**Proposition 3.1.** *Let $B(m) \cap fB(1) \subset \mathbb{R}^n \setminus gB(1)$. Then* (3.3) *holds.*

*Proof of Proposition 3.1.* We first assume that $m \leq m_0$, where $m_0$ is as in Lemma 2.3. We choose $t_0 = m/2$ and an increasing sequence $(t_j)_{j=1}^k$ of radii inductively such that if

$$|f^{-1}(B(m)) \cap B(1)| > 2|f^{-1}(B(t_{j-1})) \cap B(1)|,$$

then we choose $m/2 < t_j < m$ such that

$$|f^{-1}(B(t_j)) \cap B(1)| = 2|f^{-1}(B(t_{j-1})) \cap B(1)|, \tag{3.5}$$

otherwise $j = k$ and we choose $t_k = m$. We claim that

$$k \leq C(n) \log_2 \frac{1}{s_{\frac{m}{2}}} \leq C(n, \mu, \mathcal{K}) m^{-n/\mu}. \tag{3.6}$$

The second inequality in (3.6) follows from Lemma 2.3. In order to prove the first inequality we use the definition of $(t_j)$ to write (note that $t_k = m$)

$$2^{k-1} \leq \prod_{j=1}^{k} \frac{|f^{-1}(B(t_j)) \cap B(1)|}{|f^{-1}(B(t_{j-1})) \cap B(1)|} = \frac{|f^{-1}(B(m)) \cap B(1)|}{|f^{-1}(B(\frac{m}{2})) \cap B(1)|}. \tag{3.7}$$

Since $B(e_1, s_{\frac{m}{2}}) \subset f^{-1}(B(\frac{m}{2}))$ and $f^{-1}(B(m)) \subset B(e_1, \frac{1}{10})$, by Lemma 2.3, the right term in (3.7) can be estimated by $C(n)s_{\frac{m}{2}}^{-n}$. Taking logarithms gives the first inequality in (3.7).

Next, by [9] (see also [12]), the restriction of $f^{-1}$ to $F_t := S(t) \cap fB(1)$ belongs to $W^{1,n}$ for almost every $m/2 < t < m$. We denote $E_t = f^{-1}F_t$. By [11],

$$\mathcal{H}^{n-1}(E_t) \leq \int_{F_t} |D^\sharp f^{-1}(y)| \, \mathrm{d}\mathcal{H}^{n-1}(y) \tag{3.8}$$

for such radii $t$. Let $V_t = f^{-1}(B(t)) \cap B(1)$ and $V_j = V_{t_j}$. We integrate both sides of (3.8) over the interval $(t_{j-1}, t_j)$, $j = 1, \ldots, k$. By the relative isoperimetric inequality (2.5), and our choice (3.5) of the radii $t_j$, the left integral is estimated from below by

$$C(n)(t_j - t_{j-1})|V_j|^{(n-1)/n}.$$

Using the formula (2.3), Hölder's inequality, and change of variables, the right integral is estimated from above as follows:

$$\begin{aligned}
&\int_{fB(1)\cap B(t_j)\setminus B(t_{j-1})} |D^\sharp f^{-1}(y)| \, \mathrm{d}y \\
= \; &\int_{fB(1)\cap B(t_j)\setminus B(t_{j-1})} K_f(f^{-1}(y))^{1/n} J_{f^{-1}}(y)^{(n-1)/n} \, \mathrm{d}y \\
\leq \; &|fB(1) \cap B(t_j) \setminus B(t_{j-1})|^{1/n} \Big( \int_{V_j} K_f(x)^{1/(n-1)} \, \mathrm{d}x \Big)^{(n-1)/n}.
\end{aligned}$$

We denote $|fB(1) \cap B(t_j) \setminus B(t_{j-1})|$ by $q_j$. Combining the estimates and taking the measure of $V_j$ to the right yields

$$(t_j - t_{j-1}) \leq C q_j^{1/n} \Big( |V_j|^{-1} \int_{V_j} K_f(x)^{1/(n-1)} \, \mathrm{d}x \Big)^{(n-1)/n}, \tag{3.9}$$

where $C$ depends only on $n$. Applying Jensen's inequality to the convex function $t \mapsto \exp(\mu t^{n-1})$, and using the integrability assumption on $K_f$, gives

$$\begin{aligned}
&|V_j|^{-1} \int_{V_j} K_f(x)^{1/(n-1)} \, \mathrm{d}x \\
\leq \; &\mu^{-1} \Big( \log \big( |V_j|^{-1} \int_{V_j} \exp(\mu K_f(x)) \, \mathrm{d}x \big) \Big)^{1/(n-1)} \\
\leq \; &\mu^{-1} \Big( \log \Big( \mathcal{K} |V_j|^{-1} \Big) \Big)^{1/(n-1)}.
\end{aligned} \tag{3.10}$$

Since $|V_j| \geq C(n) s_m^n$, where $s_m$ is as in Lemma 2.3, the second inequality in (3.6) yields

$$\Big( \log |V_j|^{-1} \Big)^{1/(n-1)} \leq C(n, \mu, \mathcal{K}) m^{-n/(\mu(n-1))}.$$

Combining with (3.9) and (3.10) gives

$$t_j - t_{j-1} \leq C q_j^{1/n} m^{-1/\mu}.$$

Finally, we add over $j$ and use the Cauchy-Schwarz inequality:

$$\begin{aligned} \frac{m}{2} &= \sum_{j=1}^{k} t_j - t_{j-1} \leq C m^{-1/\mu} \sum_{j=1}^{k} q_j^{1/n} \\ &\leq C m^{-1/\mu} k^{(n-1)/n} \Big( \sum_{j=1}^{k} q_j \Big)^{1/n}. \end{aligned} \tag{3.11}$$

Since

$$\sum_{j=1}^{k} q_j \leq |B(m) \cap fB(1)|,$$

using estimates (3.11) and (3.6), we get

$$m \leq C(n, \mu, \mathcal{K}) |B(m) \cap fB(1)|^{1/n} m^{-n/\mu},$$

which implies (3.3).

We next assume that $m \geq m_0$. By applying the previous argument with $m = m_0$, we see that

$$|B(m) \cap fB(1)| \geq |B(m_0) \cap fB(1)| \geq C_1(n, \mu, \mathcal{K}). \tag{3.12}$$

On the other hand, Lemma 2.2 shows that we always have

$$m \leq \operatorname{diam} fS(1) \leq C_2(n, \mu, \mathcal{K}) |fB(1)| = C_3(n, \mu, \mathcal{K}). \tag{3.13}$$

Combining (3.12) and (3.13) gives (3.3). The proof is complete. □

# 4 Proof of Theorem 1.3

We use the following result of Gehring and Väisälä [7, Theorem 5.2] (they only consider the case $n = 3$, but the proof extends to all dimensions $n \geq 2$).

**Theorem 4.1.** *Let $B(x, R) \subset E \subset B(x, \Lambda R)$ be a convex domain. Then there exists a $K$-quasiconformal mapping $g : \mathbb{R}^n \to \mathbb{R}^n$, where $K$ depends only on $n$ and $\Lambda$, such that $gB(1) = E$.*

We now prove Theorem 1.3, using Theorems 1.4 and 4.1. As in the proof of Theorem 1.4, we may assume that $|fB(1)| = |E| = \omega_n$ and

$$A(fB(1), E) = \omega_n^{-1} |fB(1) \setminus E|.$$

By Lemma 2.2, there exist $x \in \mathbb{R}^n$, and $t > 1$, depending only on $n$, $\mu$, and $\mathcal{K}$, such that

$$B(x, t^{-1}) \subset fB(1) \subset B(x, t).$$

We may assume that $\Lambda$ is the smallest constant for which there exist $x \in \mathbb{R}^n$ and $R > 0$ such that

$$B(x, R) \subset E \subset B(x, \Lambda R).$$

Then, if $\Lambda$ is large enough depending on $t$, the convexity of $E$ implies that $A(fB(1), E) \geq 1/100$. We have $R < 1$ because $|E| = \omega_n$, and so

$$\mathrm{dist}_R(fS(1), \partial E) \leq \mathrm{diam}\, fS(1) + \mathrm{diam}\, \partial E \leq C(n, \mu, \mathcal{K}) + \Lambda.$$

Therefore,

$$\mathrm{dist}_R(fS(1), \partial E) \leq C(n, \mu, \mathcal{K}, \Lambda) A(fB(1), E)$$

when $\Lambda \geq \Lambda_0(n, \mu, \mathcal{K})$.

Now let $\Lambda \leq \Lambda_0(n, \mu, \mathcal{K})$. An application of Theorem 4.1 gives a $K(n, \mu, \mathcal{K})$-quasiconformal homeomorphism $g : B(2) \to gB(2)$ such that $gB(1) = E$. We can now apply Theorem 1.4 to $f$ and $g$, since

$$\int_{B(2)} \exp(\mu K_g(x))\, \mathrm{d}x \leq C(n, \mu, \mathcal{K}).$$

This gives the first inequality at (1.2). The second inequality follows from Theorem 1.2.

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