Homogenization of non-linear variational problems with thin low-conducting layers

Andrea BRAIDES
Dipartimento di Matematica
Università Roma 'Tor Vergata'
Via della Ricerca Scientifica
00133 Roma ITALY
braides@mat.uniroma2.it

Marc BRIANE
Centre de Mathématiques
I.N.S.A. de Rennes & I.R.M.A.R.
20 avenue des Buttes de Coësmes
35043 Rennes Cedex FRANCE
mbriane@insa-rennes.fr

Abstract

This paper deals with the homogenization of a sequence of non-linear conductivity energies in a bounded open set Ω of $\mathbb{R}^d$, for $d \geq 3$. The energy density is of the same order as $a_\varepsilon(\frac{x}{\varepsilon})\,|Du(x)|^p$, where $\varepsilon \to 0$, a_ε is periodic, u is vector-valued function in $W^{1,p}(\Omega;\mathbb{R}^m)$ and $p > 1$. The conductivity a_ε is equal to 1 in the "hard" phases composed by $N \geq 2$ two by two disjoint-closure periodic sets while a_ε tends uniformly to 0 in the "soft" phases composed by periodic thin layers which separate the hard phases. We prove that the limit energy, according to Γ-convergence, is a multi-phase functional equal to the sum of the homogenized energies (of order 1) induced by the hard phases plus an interaction energy (of order 0) due to the soft phases. The number of limit phases is less than or equal to N and is obtained by evaluating the Γ-limit of the rescaled energy of density $\varepsilon^{-p}\,a_\varepsilon(y)\,|Dv(y)|^p$ in the torus. Therefore, the homogenization result is achieved by a double Γ-convergence procedure since the cell problem depends on ε.

1 Introduction

This work is a contribution to the study of the homogenization of non-linear and non-uniformly coercive problems with complicated underlying microstructure. Problems of this type have been widely studied, particularly in the linear case, in connection with double-porosity models, where regions of low conductivity are surrounded by higher-conductivity sets. One of the first mathematical studies of the double-porosity model is due to Arbogast, Douglas and Hornung [1], subsequently revisited by Allaire [3] and extended in various ways in the linear case [5], [23], and in the non-linear case [22], [8].

If the higher-conductivity set has more than one connected component, the general form of the limit effective energy involves a multi-phase description. The mathematical approach of the double-porosity model thus belongs to the class of homogenization problems leading to homogenized vector models induced by low-conducting regions. One of the precursors of this kind of homogenization problem has been Khruslov [19], [17], [20], whose works have also been extended in different ways [11], [25], [12], [13], [4] and [14].

The usual setting for such problems is a fixed periodic microstructure, scaled by a small parameter ε, in which the coefficients of the energy are scaled differently with respect to ε in the high and low-conducting components ("hard" and "soft" phases). The effect of the low-conducting region is the appearance of an interaction energy between the hard phases, that are in this way coupled. This general feature can also be traced in other types of problems where the regions of higher conductivity have a more complex structure (see [19]).

In most of the previous works the number of interacting components is fixed by the geometry of the microstructure, and corresponds to the number of connected periodic hard phases. We address the problem of a more general setting, where also the micro-geometry is ε-dependent. In this case the number n of limit phases is itself the main unknown of the problem, and must be deduced from the asymptotic behaviour both of the geometry of the low-conducting phase and of the degenerating coefficients. In the linear setting, the determination of n can be addressed via a spectral analysis [14], that unfortunately is not easily reproducible in a non-linear framework.

Let us now state the problem. In a bounded open set of $\mathbb{R}^d$, for $d \geq 3$, we consider a sequence of energies of type

$$\mathcal{F}_\varepsilon(u) := \int_\Omega f_\varepsilon\left(\frac{x}{\varepsilon}, Du\right) dx, \quad \text{for } u \in W^{1,p}(\Omega; \mathbb{R}^m).$$

The density energy $f_\varepsilon(y, \lambda)$ is periodic in the variable y, is of the same order of the function $a_\varepsilon(y)\,|\lambda|^p$, where a_ε may be assumed to take the value 1 on N two by two disjoint periodic open connected sets E_i^ε, for $i = 1 \ldots N$. The coefficient a_ε takes values converging to 0 with ε on the complement of these sets. Contrary to the double-porosity model, this complement set has measure converging to 0.

Under some geometrical assumptions on the sets E_i^ε (and among them several weighted Poincaré-Wirtinger inequalities play an important role) we prove that the limit energy, in the sense of Γ-convergence, is a multi-phase energy of the form

$$\mathcal{F}(u_1, \ldots, u_N) = \sum_{i=1}^N \int_\Omega f_i^{\text{hom}}(Du_i)\, dx + \int_\Omega \Phi(u_1, \ldots, u_N)\, dx, \quad \text{for } u_i \in W^{1,p}(\Omega; \mathbb{R}^m).$$

The function f_i^{hom} is obtained by the usual homogenization process on the set E_i defined as the limit of E_i^ε (see [7], [10]). In the present Γ-convergence process the convergence of the sequence $u_\varepsilon \longrightarrow (u_1, \ldots, u_N)$ may be understood as

$$\lim_{\varepsilon \to 0} \sum_{i=1}^N \int_{\Omega \cap E_i^\varepsilon} |u_\varepsilon - u_i|^p\, dx = 0.$$

The interaction energy density Φ is defined through a Γ-limit procedure since the cell problem also depends on ε contrary to the analog double-porosity model dealt with in [8]. The computation of Φ is linked to the domain of the muti-phase limit energy $\mathcal{F}$, that in general is not equal to the whole $(W^{1,p}(\Omega; \mathbb{R}^m))^N$ as in [8]. Indeed, we show that the domain of the functional $\mathcal{F}$ is characterized by an equivalence relation $\mathcal{R}$ on the set $\{1, \ldots, N\}$ and the constraint that $u_i = u_j$ if $i\,\mathcal{R}\,j$. In this way we define the number n of effective phases as the number of equivalence classes modulo $\mathcal{R}$, so that, with an abuse of notation, the effective energy can be rewritten as $\mathcal{F}(u_1, \ldots, u_n)$. More precisely, the following ε-rescaled energy defined, for any $\xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^{m\times N}$, by

$$\Phi_\varepsilon^\#(\xi) := \inf\left\{F_\varepsilon^\#(v) \ :\ v \in W_\#^{1,p}(Y; \mathbb{R}^m) \text{ and } v = \xi_i \text{ in } E_i^\varepsilon\right\},$$
$$\text{where} \quad F_\varepsilon^\#(v) := \frac{1}{\varepsilon^p}\int_Y a_\varepsilon\, |Dv|^p\, dy,$$

Γ-converges (up to a subsequence) in $\mathbb{R}^{m\times N}$ to some functional $\Phi^\# : \mathbb{R}^{m\times N} \longrightarrow [0, +\infty]$. Then, the integer n is defined by

$$n = \frac{1}{m} \dim D \qquad \text{where} \qquad D := \left\{\xi \in \mathbb{R}^{m\times N} \ :\ \Phi^\#(\xi) < +\infty\right\},$$

and the energy density Φ is obtained by a double limit on the vector space D (see Proposition 2.5 and formula (2.24)).

In conclusion, the present work can be regarded both as an extension of the linear framework [17], [14] to a non-linear one, and as an extension of the recent variational approach [8] to a more intricate geometry (due to the thin layers), characterized by a double Γ-limit procedure.

The paper is organized as follows. In the first section we define precisely the geometry of the problem and we state the main results. The second section is devoted to the proof of the auxiliary results, and the third one to the proof of the Γ-convergence of the sequence $\mathcal{F}_\varepsilon$.

2 Statement of the results

Notation

- $\cdot$ denotes the scalar product and $|\cdot|$ the associated norm in any space $\mathbb{R}^M$, for $M \in \mathbb{N}^*$ ($\mathbb{N}^*$ denotes the set of strictly positive integers) ;
- 1_E denotes the *characteristic function* of the set E; if E is Lebesgue measurable then $|E|$ is its Lebesgue measure and $\displaystyle\fint_E u = \frac{1}{|E|}\int_E u\,dx$ denotes the *average* of u on E;
- d is an integer ≥ 3 and m an integer ≥ 1;
- an open subset of $\mathbb{R}^d$ is said to be *regular* if its boundary is Lipschitz; Ω denotes a bounded and regular open subset of $\mathbb{R}^d$;
- $Y := (\frac{-1}{2}, \frac{1}{2})^d$ is the unit cube in $\mathbb{R}^d$;
- $p > 1$ and $p' := \dfrac{p}{p-1}$;
- $W^{1,p}_{\#}(Y;\mathbb{R}^m)$ denotes the space of the Y-periodic vector-valued (in $\mathbb{R}^m$) functions which belong to the Sobolev space $W^{1,p}(\omega;\mathbb{R}^m)$ for any bounded open set ω of $\mathbb{R}^d$;
- for $u \in W^{1,p}(\Omega;\mathbb{R}^m)$, Du denotes the Jacobian matrix in $\mathbb{R}^{m\times d}$ defined by

$$Du := \left[\frac{\partial u_i}{\partial x_j}\right]_{\substack{1\leq i\leq m\\ 1\leq j\leq d}}.$$

2.1 Geometry of the problem

We consider $N \geq 2$ and $E_1, \ldots, E_N$, connected and regular open subsets of $\mathbb{R}^d$, $d \geq 3$, which are Y-periodic; *i.e.*, $E_i + \kappa = E_i$ for any $\kappa \in \mathbb{Z}^d$, and satisfy

$$i \neq j \implies E_i \cap E_j = \emptyset \quad \text{and} \quad \bigcup_{i=1}^{N} \overline{E_i} = \mathbb{R}^d. \tag{2.1}$$

For each $\varepsilon > 0$ and $i \in \{1, \ldots, N\}$, let $E_i^\varepsilon \subset E_i$ be a Y-periodic connected and regular open set, such that

$$j \neq i \implies \overline{E_i^\varepsilon} \cap \overline{E_j^\varepsilon} = \emptyset \qquad \text{and} \qquad \lim_{\varepsilon\to 0} |Y \cap (E_i \setminus E_i^\varepsilon)| = 0. \tag{2.2}$$

In particular, this assumption is satisfied if

$$0 < \varepsilon < \varepsilon' \implies E_i^{\varepsilon'} \subset E_i^\varepsilon \qquad \text{and} \qquad \bigcup_{\varepsilon>0} E_i^\varepsilon = E_i, \tag{2.3}$$

By (2.2), $E_1^\varepsilon, \dots, E_N^\varepsilon$ are two by two disjoint sets separated by the periodic open set

$$\omega^\varepsilon := \mathbb{R}^d \setminus \bigcup_{i=1}^{N} \overline{E_i^\varepsilon} \qquad \text{such that} \qquad \lim_{\varepsilon \to 0} |Y \cap \omega^\varepsilon| = 0, \tag{2.4}$$

which is composed of thin layers.

We consider a nonnegative Borel function $f_\varepsilon : \mathbb{R}^d \times \mathbb{R}^{m\times d} \longrightarrow \mathbb{R}_+$ depending on $\varepsilon > 0$ and which satisfies the following properties:

(*i*) $f_\varepsilon(\cdot, \lambda)$ is Y-periodic for any $\lambda \in \mathbb{R}^{m\times d}$;

(*ii*) $f_\varepsilon(y, \cdot)$ is p-homogeneous for a.e. $y \in \mathbb{R}^d$;

(*iii*) for a.e. $y \in \mathbb{R}^d \setminus \omega^\varepsilon$, $f_\varepsilon(y, \cdot) = f(y, \cdot)$ where the function $f : \mathbb{R}^d \times \mathbb{R}^{m\times d} \longrightarrow \mathbb{R}_+$ satisfies with the constants $c_0, c_1 > 0$ the p-growth condition

$$\text{a.e. } y \in \mathbb{R}^d,\ \forall\, \lambda \in \mathbb{R}^{m\times d}, \quad c_0\, |\lambda|^p \leq f(y, \lambda) \leq c_1\, |\lambda|^p; \tag{2.5}$$

(*iv*) there exists a Y-periodic positive function $a_\varepsilon : \mathbb{R}^d \longrightarrow \mathbb{R}$ such that

$$a_\varepsilon := 1 \ \text{ a.e. in } \mathbb{R}^d \setminus \omega^\varepsilon, \quad \underset{\omega^\varepsilon}{\text{ess-inf}}\ a_\varepsilon > 0 \quad \text{and} \quad \alpha_\varepsilon := \underset{\omega^\varepsilon}{\text{ess-sup}}\ a_\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} 0, \tag{2.6}$$

there exist a constant $b > 1$ such that f_ε satisfies the p-growth condition

$$\text{a.e. } y \in \mathbb{R}^d,\ \forall\, \lambda \in \mathbb{R}^{m\times d}, \quad a_\varepsilon(y)\, |\lambda|^p \leq f_\varepsilon(y, \lambda) \leq b\, a_\varepsilon(y)\, |\lambda|^p, \tag{2.7}$$

and a constant $c > 0$ such that f_ε satisfies the local Lipschitz condition

$$\begin{aligned} &\text{a.e. } y \in \omega^\varepsilon,\ \forall\, \lambda, \mu \in \mathbb{R}^{m\times d}, \\ &|f_\varepsilon(y, \lambda) - f_\varepsilon(y, \mu)| \leq c\, a_\varepsilon(y) \left(|\lambda|^{p-1} + |\mu|^{p-1}\right) |\lambda - \mu|. \end{aligned} \tag{2.8}$$

In terms of conduction the space is shared in $N \geq 2$ two by two disjoint-closure regions $E_1^\varepsilon, \dots, E_N^\varepsilon$ of conductivity given by the function f, and these regions are separated by the thin-layers region ω^ε of low conductivity given by the function a_ε.

Example 2.1 In $\mathbb{R}^3$ we consider the case of the $N = 3$ Y-periodic sets $E_1^\varepsilon, E_2^\varepsilon, E_3^\varepsilon$ defined by the following intersections with Y (see Figure 1):

$$\left\{ \begin{aligned} E_1^\varepsilon \cap Y &:= \bigcup_{k=1}^{3} \left\{ y \in Y \ :\ \max_{j \neq k} |y_j| < R_1 - r_1^\varepsilon \right\} \\ E_2^\varepsilon \cap Y &:= \left[\bigcup_{k=1}^{3} \left\{ y \in Y \ :\ \max_{j \neq k} |y_j| < R_2 - r_2^\varepsilon \right\} \right] \cap \left[\bigcap_{k=1}^{3} \left\{ y \in Y \ :\ \max_{j \neq k} |y_j| > R_1 \right\} \right] \\ E_3^\varepsilon \cap Y &:= \bigcap_{k=1}^{3} \left\{ y \in Y \ :\ \max_{j \neq k} |y_j| > R_2 \right\}, \end{aligned} \right. \tag{2.9}$$

where $0 < R_1 < R_2 < \frac{1}{2}$, $r_1^\varepsilon, r_2^\varepsilon > 0$ and $r_1^\varepsilon, r_2^\varepsilon \to 0$. The sets $E_1^\varepsilon, E_2^\varepsilon$ are separated by the set ω_1^ε and $E_2^\varepsilon, E_3^\varepsilon$ by the set ω_2^ε, where

$$\left\{ \begin{aligned} \omega_1^\varepsilon \cap Y &:= \left[\bigcup_{k=1}^{3} \left\{ y \in Y \ :\ \max_{j \neq k} |y_j| < R_1 \right\} \right] \setminus \overline{E_1^\varepsilon} \\ \omega_2^\varepsilon \cap Y &:= \left[\bigcap_{k=1}^{3} \left\{ y \in Y \ :\ \max_{j \neq k} |y_j| > R_2 - r_2^\varepsilon \right\} \right] \setminus \overline{E_3^\varepsilon}. \end{aligned} \right. \tag{2.10}$$

Note that in this case the thin layer region satisfies $\omega^\varepsilon = \omega_1^\varepsilon \cup \omega_2^\varepsilon$.

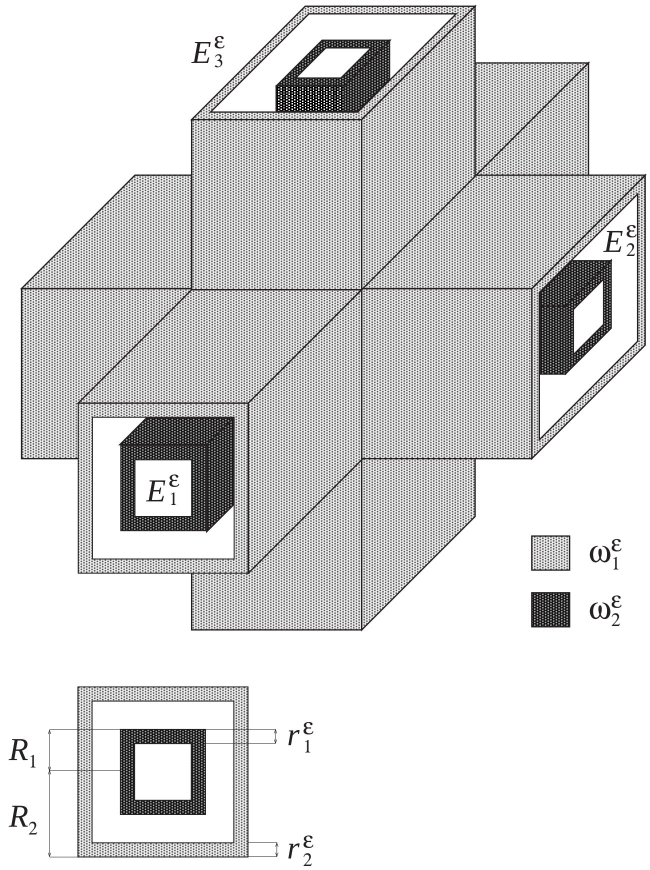

Figure 1 : *three connected regions separated by two thin layers*

Let $\alpha_1^\varepsilon, \alpha_2^\varepsilon$ be two positive sequences which converge to 0. We consider the function f_ε defined by $f_\varepsilon(y,\lambda) := a_\varepsilon(y)\,|\lambda|^p$ where

$$\begin{cases} a_\varepsilon(y) := 1 & \text{if } y \in E_1^\varepsilon \cup E_2^\varepsilon \cup E_3^\varepsilon \\ a_\varepsilon(y) := \alpha_i^\varepsilon & \text{if } y \in \omega_1^\varepsilon \cup \omega_2^\varepsilon. \end{cases} \tag{2.11}$$

2.2 Position of the problem

We are interested in the asymptotic behaviour as $\varepsilon \to 0$ of the sequence of functionals $\mathcal{F}_\varepsilon : L^p(\Omega;\mathbb{R}^m) \longrightarrow [0,+\infty]$ defined by

$$\mathcal{F}_\varepsilon(u) := \begin{cases} \displaystyle\int_\Omega f_\varepsilon\left(\frac{x}{\varepsilon}, Du\right) dx & \text{if } u \in W^{1,p}(\Omega;\mathbb{R}^m) \\ +\infty & \text{otherwise,} \end{cases} \tag{2.12}$$

where the energy density is defined in Section 2.1. As in [8] we expect the limit energy to be a multi-phases system composed by the sum of:

– the homogenized energies due to the strongly connected components $E_1^\varepsilon, \dots, E_N^\varepsilon$ (2.2),

– the interaction energy induced by the thin low-conducting layers ω^ε (2.4).

Let us make this statement precise. Under suitable assumptions (given in the next Section 2.4) we will prove that the limit energy reads as

$$\mathcal{F}(u_1,\dots,u_N) = \sum_{i=1}^N \int_\Omega f_i^{\hom}(Du_i)\,dx + \int_\Omega \Phi(u_1,\dots,u_N)\,dx, \quad \text{for } u_i \in W^{1,p}(\Omega;\mathbb{R}^m). \tag{2.13}$$

Each vector-valued function u_i, $i = 1 \ldots N$, is defined by the $L^p_{\mathrm{loc}}(\Omega; \mathbb{R}^m)$ strong limit in the component $\varepsilon E_i^\varepsilon$ of a sequence u_ε such that $\mathcal{F}_\varepsilon(u_\varepsilon)$ is bounded. So we are led to define the following sequential topology τ: for any sequence u_ε in $L^p(\Omega; \mathbb{R}^m)$ and for any $(u_1, \ldots, u_N)$ in $L^p(\Omega; \mathbb{R}^m)^N$,

$$u_\varepsilon \xrightarrow{\tau} (u_1, \ldots, u_N) \qquad \text{if} \qquad \sum_{i=1}^N \int_{\Omega \cap \varepsilon E_i^\varepsilon} |u_\varepsilon - u_i|^p \, dx \underset{\varepsilon \to 0}{\longrightarrow} 0. \tag{2.14}$$

The choice of this convergence is a consequence of the strong connectedness of the components E_i^ε (see Proposition 2.6 below). The convergence of the energy $\mathcal{F}_\varepsilon$ then has to be understood in the sense of the Γ-convergence for the topology τ. Let us recall the definition of Γ-convergence (see [6, 16]):

Definition 2.2 Let $\mathcal{F}_\varepsilon$be a sequence of functionals defined on a vector space $\mathcal{H}$ and let $\mathcal{F}$ be a functional defined on a vector space $\mathcal{K}$. Let τ be a topology defined by the convergence of sequences of $\mathcal{H}$ to vectors of $\mathcal{K}$. The sequence $\mathcal{F}_\varepsilon$ is said to Γ-converge for the topology τ if for any $u \in \mathcal{K}$,

(i) (Γ-liminf inequality) for any sequence u_ε in $\mathcal{H}$,

$$u_\varepsilon \xrightarrow{\tau} u \quad \Longrightarrow \quad \liminf_{\varepsilon \to 0} \mathcal{F}_\varepsilon(u_\varepsilon) \geq \mathcal{F}(u),$$

(ii) (Γ-limsup inequality) there exists a sequence $\overline{u}_\varepsilon$ in $\mathcal{H}$ such that

$$\overline{u}_\varepsilon \xrightarrow{\tau} u \quad \text{and} \quad \limsup_{\varepsilon \to 0} \mathcal{F}_\varepsilon(\overline{u}_\varepsilon) \leq \mathcal{F}(u).$$

Such a sequence $\overline{u}_\varepsilon$ is called a recovery sequence.

Here one has $\mathcal{H} := L^p(\Omega; \mathbb{R}^m)$, $\mathcal{K} := L^p(\Omega; \mathbb{R}^m)^N$ and τ is the topology defined by (2.14).

Contrary to [8], the dimension of the vector-valued system (2.13) is not necessarily equal to the prescribed number N of hard components since there is a reduction of the system similar to that obtained in [14] in a linear framework. Indeed, the physical characteristics (geometrical parameters and conductivity values) of the low-conducting layers may connect some of the strongly connected components between them. This is the case for example if the layers are thin enough or conducting enough. From a mathematical point of view this reduction is expressed as a restriction of the domain of the limit functional. So, if the domain of the functional $\mathcal{F}_\varepsilon$ (2.12) is equal to $W^{1,p}(\Omega; \mathbb{R}^m)$, the domain of $\mathcal{F}$ (2.13) is a subset of $W^{1,p}(\Omega; \mathbb{R}^m)^N$ in general. More precisely, the domain of $\mathcal{F}$ is associated to a subspace D of $\mathbb{R}^{m \times N}$ in such a way that

$$\mathcal{F}(u) < +\infty \text{ if and only if } u \in W^{1,p}(\Omega; \mathbb{R}^m)^N \text{ and } u \in D \text{ a.e. in } \Omega. \tag{2.15}$$

2.3 Determination of the limit energy

First of all we have to determine the subspace D which arises in the domain (2.15) of the limit energy. To this end let us consider the functional $\Phi_\varepsilon^\# : \mathbb{R}^{m \times N} \longrightarrow \mathbb{R}_+$ defined by the minimization problem

$$\begin{gathered}\Phi_\varepsilon^\#(\xi) := \inf \left\{ F_\varepsilon^\#(v) \ : \ v \in W^{1,p}_\#(Y; \mathbb{R}^m) \text{ and } v = \xi_i \text{ in } E_i^\varepsilon \right\}, \\ \text{where} \quad F_\varepsilon^\#(v) := \frac{1}{\varepsilon^p} \int_Y a_\varepsilon \, |Dv|^p \, dy,\end{gathered} \tag{2.16}$$

for $\xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^{m \times N}$ (ξ can be considered as a $(m \times N)$ matrix with its columns $\xi_1, \ldots, \xi_N$ in $\mathbb{R}^m$), where the function a_ε is defined by (2.6). Since $\mathbb{R}^{m \times N}$ is separable, there exists a subsequence, still denoted by ε, such that $\Phi^{\#}_\varepsilon$ Γ-converges in $\mathbb{R}^{m\times N}$ to a functional $\Phi^{\#} : \mathbb{R}^{m\times N} \longrightarrow [0,+\infty]$ (see *e.g.* [6] Proposition 1.42 or [16] Theorem 8.5):

$$\Phi^{\#}_\varepsilon \xrightarrow{\Gamma} \Phi^{\#} \quad \text{in } \mathbb{R}^{m\times N}. \tag{2.17}$$

The subspace D of $\mathbb{R}^{m\times N}$ is defined by

$$D := \left\{\xi \in \mathbb{R}^{m\times N} \ : \ \Phi^{\#}(\xi) < +\infty\right\}. \tag{2.18}$$

Note that the inequality

$$\forall\, v, w \in W^{1,p}_{\#}(Y;\mathbb{R}^m), \qquad \int_Y a_\varepsilon\, |Dv + Dw|^p\, dy \le 2^{p-1}\left(\int_Y a_\varepsilon\, |Dv|^p\, dy + \int_Y a_\varepsilon\, |Dw|^p\, dy\right),$$

implies that

$$\forall\, \xi, \eta \in \mathbb{R}^{m\times N}, \quad \Phi^{\#}_\varepsilon(\xi+\eta) \le 2^{p-1}\left(\Phi^{\#}_\varepsilon(\xi) + \Phi^{\#}_\varepsilon(\eta)\right), \tag{2.19}$$

which combined with the definition of the Γ-convergence of $\Phi^{\#}_\varepsilon(\xi)$, shows that D is a vector subspace of $\mathbb{R}^{m\times N}$. In fact, the set D has a very particular form:

Proposition 2.3 *There exist $n \in \{1,\dots,n\}$ and a partition $(\hat{I}_k)_{1\le k\le n}$ of the set $\{1,\dots,N\}$ such that*

$$D = \left\{\xi = (\xi_1,\dots,\xi_N) \in \mathbb{R}^{m\times N} \ : \ \forall\, k = 1\dots n, \ \forall\, i,j \in \hat{I}_k, \ \ \xi_i = \xi_j\right\}. \tag{2.20}$$

Remark 2.4 In the scalar case $m = 1$, the set D defined by (2.20) thus induces a partition of the N components $E^\varepsilon_1, \dots E^\varepsilon_N$ into $n \le N$ groups. In some sense the components of a same group $\hat{I}_k$ are glued and their union $\bigcup_{i\in\hat{I}_k} E^\varepsilon_i$ may be considered as a weakly connected domain. A similar behaviour was first shown in [14] thanks to a different approach based on a spectral analysis in the linear case.

In the sequel we will be led to use minimization functionals with non-periodic test functions. For each $K \in \mathbb{N}^*$, let us consider the functional $\Phi^K_\varepsilon : \mathbb{R}^{m\times N} \longrightarrow \mathbb{R}_+$ defined by

$$\Phi^K_\varepsilon(\xi) := \inf\left\{\frac{1}{\varepsilon^p K^d}\int_{KY} f_\varepsilon(y, Dv)\, dy : v \in W^{1,p}(KY;\mathbb{R}^m), \ v = \xi_i \text{ in } KY \cap E^\varepsilon_i\right\}. \tag{2.21}$$

Note that there exists a constant $c > 0$ (independent of K) such that $\Phi^K_\varepsilon \le c\, \Phi^{\#}_\varepsilon$ thanks to estimate (2.7) and to the fact that any function in $W^{1,p}_{\#}(Y;\mathbb{R}^m)$ belongs to $W^{1,p}(KY;\mathbb{R}^m)$. We then have the following result:

Proposition 2.5 *The functionals $\Phi^{\#}_\varepsilon$ and Φ^K_ε satisfy the following properties:*

(*i*) *For any $\xi \in D$, the sequence $\Phi^{\#}_\varepsilon(\xi)$ is bounded and more generally*

$$\forall\, \xi \in \mathbb{R}^{m\times N}, \quad \Phi^{\#}(\xi) = \lim_{\varepsilon\to 0} \Phi^{\#}_\varepsilon(\xi), \tag{2.22}$$

where $\Phi^{\#}$ is defined by (2.17).

(*ii*) *There exists a constant $c > 0$ such that for any $\varepsilon > 0$ and any $K \in \mathbb{N}^*$,*

$$\forall\, \xi, \eta \in D, \quad \begin{cases} \left|\Phi^K_\varepsilon(\xi)\right| \le c\, |\xi|^p \\ \left|\Phi^K_\varepsilon(\xi) - \Phi^K_\varepsilon(\eta)\right| \le c\left(|\xi|^{p-1} + |\eta|^{p-1}\right)\, |\xi - \eta|. \end{cases} \tag{2.23}$$

(*iii*) *There exists a subsequence of* ε, *still denoted by* ε, *such that for any* $K \in \mathbb{N}^*$, Φ_ε^K Γ*-converges in* $\mathbb{R}^{m\times N}$ *and pointwise converges in* D *to* $\Phi_0^K : \mathbb{R}^{m\times N} \longrightarrow [0,+\infty]$ *which satisfies estimates* (2.23).

(*iv*) *The sequence* Φ_0^K *converges in* D *as* $K \to +\infty$ *to a functional* Φ_0^∞ *which also satisfies* (2.23).

In virtue of Proposition 2.5 the energy density of the zero-order part in the limit functional (2.13) is well-defined by

$$\Phi(\xi) := \begin{cases} \lim\limits_{K\to+\infty} \lim\limits_{\varepsilon\to 0} \Phi_\varepsilon^K & \text{if } \xi \in D \\ +\infty & \text{elsewhere.} \end{cases} \tag{2.24}$$

It remains to define the "bulk" parts $f_i^{\hom}$ of the limit functional (2.13) To this end we use the following result which needs extra assumptions on the components $E_1^\varepsilon, \dots, E_N^\varepsilon$:

Proposition 2.6 *Let* E_ε, *for* $\varepsilon > 0$, *and* E *be* Y*-periodic connected and regular open subsets of* $\mathbb{R}^d$ *such that*

$$\forall\, \varepsilon > 0, \;\; E_\varepsilon \subset E \qquad \textit{and} \qquad \lim_{\varepsilon\to 0} |Y \cap (E \setminus E_\varepsilon)| = 0. \tag{2.25}$$

Assume that there exist two constants $k_0, k_1 > 0$ *such that for any* $\varepsilon > 0$, *there exists a linear extension operator* P_ε *from* $W^{1,p}(\Omega \cap \varepsilon E_\varepsilon; \mathbb{R}^m)$ *into* $W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{R}^m)$ *satisfying, for any* $u \in W^{1,p}(\Omega \cap \varepsilon E_\varepsilon; \mathbb{R}^m)$,

$$\begin{cases} P_\varepsilon u & = u & \textit{a.e. in } \Omega \cap \varepsilon E_\varepsilon \\ \displaystyle\int_{\Omega(\varepsilon k_0)} |P_\varepsilon u|^p \, dx & \le k_1 \displaystyle\int_{\Omega\cap\varepsilon E_\varepsilon} |u|^p \, dx & \\ \displaystyle\int_{\Omega(\varepsilon k_1)} |D(P_\varepsilon u)|^p \, dx & \le k_1 \displaystyle\int_{\Omega\cap\varepsilon E_\varepsilon} |Du|^p \, dx & \end{cases} \tag{2.26}$$

$$\textit{where} \quad \Omega(r) := \{x \in \Omega \;:\; \mathrm{dist}\,(x, \partial\Omega) > r\} \quad \textit{for } r > 0.$$

Let $g : \mathbb{R}^d \times \mathbb{R}^{m\times d} \longrightarrow \mathbb{R}_+$ *be a nonnegative Borel function,* Y*-periodic in the first variable and satisfying a* p*-growth condition of type* (2.5). *Let* $\mathcal{G}_\varepsilon : L^p(\Omega; \mathbb{R}^m) \longrightarrow [0,+\infty]$ *be the functional defined by*

$$\mathcal{G}_\varepsilon(v) := \begin{cases} \displaystyle\int_{\Omega\,\cap\,\varepsilon E_\varepsilon} g\left(\frac{x}{\varepsilon}, Dv\right) dx & \textit{if } v \in W^{1,p}(\Omega; \mathbb{R}^m) \\ +\infty & \textit{elsewhere.} \end{cases} \tag{2.27}$$

Then, $\mathcal{G}_\varepsilon$ Γ*-converges for the* $L^p(\Omega; \mathbb{R}^m)$*-strong convergence to the functional* $\mathcal{G}$ *defined by*

$$G(v) := \begin{cases} \displaystyle\int_\Omega g^{\hom}(Dv) \, dx & \textit{if } v \in W^{1,p}(\Omega; \mathbb{R}^m) \\ +\infty & \textit{elsewhere,} \end{cases} \tag{2.28}$$

where the energy density $g^{\hom}$ *is defined, for* $\lambda \in \mathbb{R}^{m\times d}$, *by*

$$g^{\hom}(\lambda) := \lim_{K\to+\infty} \inf \left\{ \frac{1}{K^d} \int_{KY\cap E} g(y, \lambda + Dv) \, dy \;:\; v \in W_0^{1,p}(KY; \mathbb{R}^m) \right\}, \tag{2.29}$$

where E *is the limit of* E_ε *according to* (2.25).

Remark 2.7 For a fixed $\varepsilon > 0$ the existence of a bounded extension operator satisfying (2.26) is a result of [2]. Here we need the assumption that P_ε is uniformly bounded with respect to ε.

2.4 Statement of the result

In the classical framework, namely without zero-order interaction between the different conducting phases, it is known (see *e.g.* [24], [21] and [18]) that thin lowly conducting layers may modify the homogenized conductivity in certain regimes. In the present context we want to avoid such a phenomenon, which would modify the bulk part of the limit energy, since we focus on the zero-order interaction term. Therefore, we have to assume that each conducting component E_i^ε is the only contribution to the bulk part $f_i^{\hom}$ in the limit energy (2.13) by excluding any effect of the low-conducting region. This can be done by introducing cut-off functions which separate the different components $E_1^\varepsilon, \dots, E_N^\varepsilon$. We ask to these cut-off functions to satisfy several properties in terms of Poincaré-type inequalities which may be regarded as geometrical assumptions on the thin layers:

Assume that there exist Y-periodic functions $\varphi_1^\varepsilon,\dots,\varphi_N^\varepsilon$ in $W^{1,\infty}_\#(Y)$ such that for any $i \in \{1, \dots, N\}$,

$$\begin{cases} \varphi_i^\varepsilon = 1 & \text{in } E_i^\varepsilon \\ \varphi_i^\varepsilon = 0 & \text{in } E_j^\varepsilon \quad \text{for } j \neq i, \end{cases} \tag{2.30}$$

and the following limit holds

$$\lim_{\varepsilon \to 0} \int_Y a_\varepsilon \, |\nabla \varphi_i^\varepsilon|^p \, dy = 0. \tag{2.31}$$

Assume that for any $i \in \{1, \dots, N\}$ and any $K \in \mathbb{N}^*$, there exists a positive sequence $C_{i,K}(\varepsilon)$ such that the following Poincaré-Wirtinger inequality holds in the cube KY

$$\forall\, v \in W^{1,p}(KY;\mathbb{R}^m), \quad \int_{KY} a_\varepsilon \, |\nabla \varphi_i^\varepsilon|^p \left| v - \fint_{KY \cap E_i^\varepsilon} v \right|^p dy \leq C_{i,K}(\varepsilon) \int_{KY} |Dv|^p \, dy,$$
$$\text{with} \quad \lim_{\varepsilon \to 0} C_{i,K}(\varepsilon) = 0, \tag{2.32}$$

and there exists a positive constant $C > 0$ such that the following Poincaré-Wirtinger inequality holds for periodic functions in Y

$$\forall\, v \in W^{1,p}_\#(Y;\mathbb{R}^m), \quad \int_Y a_\varepsilon \, |\nabla \varphi_i^\varepsilon|^p \left| v - \fint_{Y \cap E_i^\varepsilon} v \right|^p dy \leq C \int_Y a_\varepsilon \, |Dv|^p \, dy, \tag{2.33}$$

Also assume that there exists a constant $C > 0$ such that the following Poincaré inequality holds in the cube Y

$$\forall\, v \in W^{1,p}(Y;\mathbb{R}^m), \ v = 0 \ \text{ on } \bigcup_{j=1}^N \partial E_j^\varepsilon, \quad \int_Y a_\varepsilon \, |v|^p \, dy \leq C \int_Y a_\varepsilon \, |Dv|^p \, dy. \tag{2.34}$$

Now, we may state the main result of the paper:

Theorem 2.8 *Assume that the function Φ be defined by* (2.24) *(this is not restrictive upon extracting a subsequence of ε), assume that the sets $E_1^\varepsilon, \dots, E_N^\varepsilon$ satisfy the geometrical constraints of Section* 2.1 *and the uniform extension property* (2.26)*. Assume the existence of N cut-off functions $\varphi_1^\varepsilon,\dots,\varphi_N^\varepsilon$ satisfying conditions* (2.30)–(2.34)*. Also, assume that the energy densities f_ε satisfy the conditions of Section* 2.1*.*
Then, the energy

$$\mathcal{F}_\varepsilon(u) := \int_\Omega f_\varepsilon\left(\frac{x}{\varepsilon}, Du\right) dx, \quad \textit{for } u \in W^{1,p}(\Omega;\mathbb{R}^m),$$

Γ*-converges for the topology* τ (2.14) *to the limit energy*

$$\mathcal{F}(u_1,\dots,u_N)=\sum_{i=1}^{N}\int_\Omega f_i^{\mathrm{hom}}(Du_i)\,dx+\int_\Omega \Phi(u_1,\dots,u_N)\,dx,\quad \textit{for } u_i\in W^{1,p}(\Omega;\mathbb{R}^m),$$

where the homogenized densities f_i^{hom} *are defined by* (2.29) *with* $E:=E_i$ (2.1) *and the interaction density* Φ *is defined by* (2.24).
Moreover, the domain of the functional $\mathcal{F}$ *is characterized by*

$$(u_1,\dots,u_N)\in W^{1,p}(\Omega;\mathbb{R}^m)^N\quad \textit{and}\quad (u_1,\dots,u_N)\in D\ \textit{ a.e. in } \Omega,$$

where the set D *is defined by* (2.18).

Note that the form of function Φ may depend on the subsequence of ε we choose, being very sensitive to the geometry of the sets ω_ε, while the energy densities f_i^{hom} are independent of the subsequence.

Going back to Example 2.1 the following result allows us to illustrate Theorem 2.8:

Proposition 2.9 *Example* 2.1 *satisfies all the assumptions of Theorem 2.8 provided that*

$$\lim_{\varepsilon\to 0}\frac{\alpha_i^\varepsilon}{(r_i^\varepsilon)^{p-1}}=0\quad \textit{for } i=1,2. \tag{2.35}$$

Then, the domain (2.15) *of the limit energy* (2.13) *is characterized (up to subsequences) by*

$$\begin{cases} D=\mathbb{R}^3 & \textit{if}\ \displaystyle\lim_{\varepsilon\to 0}\frac{\alpha_i^\varepsilon}{\varepsilon^p\,(r_i^\varepsilon)^{p-1}}<+\infty \quad \textit{for } i=1,2\\ D=\{\xi\in\mathbb{R}^3:\xi_1=\xi_2\} & \textit{if}\ \displaystyle\lim_{\varepsilon\to 0}\frac{\alpha_i^\varepsilon}{\varepsilon^p\,(r_i^\varepsilon)^{p-1}}=+\infty \quad \textit{only for } i=1\\ D=\{\xi\in\mathbb{R}^3:\xi_2=\xi_3\} & \textit{if}\ \displaystyle\lim_{\varepsilon\to 0}\frac{\alpha_i^\varepsilon}{\varepsilon^p\,(r_i^\varepsilon)^{p-1}}=+\infty \quad \textit{only for } i=2\\ D=\{\xi\in\mathbb{R}^3:\xi_1=\xi_2=\xi_3\} & \textit{if}\ \displaystyle\lim_{\varepsilon\to 0}\frac{\alpha_i^\varepsilon}{\varepsilon^p\,(r_i^\varepsilon)^{p-1}}=+\infty \quad \textit{for } i=1,2. \end{cases} \tag{2.36}$$

3 Proof of the auxiliary results

3.1 Proof of Proposition 2.3

Let us start by giving a few properties of the set D defined by (2.18):

(*i*) Let $\xi=(\xi_1,\dots,\xi_N)\in D$. For any $h\in\{1,\dots,m\}$, the vector $\xi^h=(\xi_1^h,\dots,\xi_N^h)$ of $\mathbb{R}^{m\times N}$ defined by $\xi_{jh}^h:=\xi_{jh}$ and $\xi_{ji}^h:=0$ if $i\neq h$, belongs to D.

Indeed, if v^ε is a recovery sequence in $W^{1,p}_\#(Y;\mathbb{R}^m)$ such that $\Phi^\#_\varepsilon(v^\varepsilon)\to\Phi^\#(\xi)$ then the sequence w^ε, defined by $w_h^\varepsilon:=v_h^\varepsilon$ and $w_i^\varepsilon:=0$ if $i\neq h$, clearly satisfies $F^\#_\varepsilon(w^\varepsilon)\le F^\#_\varepsilon(v^\varepsilon)$ (see (2.16)), whence by the Γ-liminf inequality we obtain $\Phi^\#(\xi^h)\le\Phi^\#(\xi)<+\infty$.

(*ii*) For any $(m\times m)$ matrix of permutation P, for any $\xi=(\xi_1,\dots,\xi_N)\in D$, the vector $\xi=(P\xi_1,\dots,P\xi_N)$ belongs to D.

This is due to the fact that $F^\#_\varepsilon(v)$ only depends on the norm $|Dv|$ and hence the set D is invariant by permutation of the m coordinates.

(*iii*) Let $D_1 \subset \mathbb{R}^N$ be the projection of D on the first coordinate. Then, there exist $n \in \{1, \dots, N\}$ and a partition $(\hat{I}_k)_{1\le k\le n}$ of $\{1, \dots, N\}$ such that $D_1 = \text{Span}\,(e^{\hat{I}_1}, \dots, e^{\hat{I}_n})$, where

$$e^I_j := \begin{cases} 1 & \text{if } j \in I \\ 0 & \text{if } j \notin I \end{cases} \qquad \text{for } I \subset \{1, \dots, N\}. \tag{3.1}$$

Indeed, for any function $T : \mathbb{R} \longrightarrow \mathbb{R}$ and any vector $\lambda = (\lambda_1, \dots, \lambda_N)$ in $\mathbb{R}^M$, set $T(\lambda) := (T(\lambda_1), \dots, T(\lambda_M))$. If T is Lipschitz then we have $F^{\#}_\varepsilon(T(v)) \le \|T'\|^p_\infty\, F^{\#}_\varepsilon(v)$ for any function $v \in W^{1,p}_{\#}(Y; \mathbb{R}^m)$, whence

$$\forall\, \xi \in D, \quad \Phi^{\#}_\varepsilon(T(\xi)) \le \|T'\|^p_\infty\, \Phi^{\#}_\varepsilon(\xi).$$

Therefore, the Γ-liminf inequality implies that $T(D) \subset D$ and hence $T(D_1) \subset D_1$. Now, by using the fact that $(1, \dots, 1) \in D_1$, property (*iii*) is a straightforward consequence of the following result:

Lemma 3.1 *Let Λ be a subspace of $\mathbb{R}^M$, $M \ge 1$, such that $(1, \dots, 1) \in \Lambda$ and for any Lipschitz function $T : \mathbb{R} \longrightarrow \mathbb{R}$, $T(\Lambda) \subset \Lambda$. Then, there exists a partition $(I_k)_{1\le k\le n}$ of the set $\{1, \dots, n\}$, where $n := \dim \Lambda$, such that $\Lambda = \text{Span}\left(e^{I_1}, \dots, e^{I_n}\right)$.*

Let us conclude the proof of Proposition 2.3. Set

$$\tilde{D} := \left\{\xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^{m\times N} \;:\; \forall\, k = 1 \dots n,\ \forall\, i, j \in \hat{I}_k,\ \ \xi_i = \xi_j\right\}.$$

Let $i, j \in \hat{I}_k$, $\xi \in D$ and $h \in \{1, \dots, m\}$. By (*i*) and (*ii*) $(\xi_{1h}, \dots, \xi_{Nh}) \in D_1$, whence by (*iii*) $\xi_{ih} = \xi_{jh}$ and thus $\xi_i = \xi_j$. Therefore, $\xi \in \tilde{D}$ and $D \subset \tilde{D}$. Inversely, let $\xi \in \tilde{D}$. For any $h \in \{1, \dots, m\}$, $(\xi_{1h}, \dots, \xi_{Nh}) \in D_1$, whence by (*i*) and (*ii*) $\xi^h \in \tilde{D}$. Therefore $\xi = \xi^1 + \dots + \xi^m \in D$ and $\tilde{D} \subset D$. We thus have the equality $\tilde{D} = D$, which establishes Proposition 2.3.

Proof of Lemma 3.1. By definition (3.1), $e^{\{1,\dots,M\}} = (1, \dots, 1) \in \Lambda$. Let $(I_1, \dots, I_q)$ be thus a maximal partition of the set $\{1, \dots, M\}$ such that the q vectors $e^{I_1}, \dots, e^{I_q}$ belong to Λ. We have $q \le n := \dim \Lambda$ since $e^{I_1}, \dots, e^{I_q}$ are clearly independent.

Assume by contradiction that $q < n$. Then, there exists $\lambda \in \Lambda \setminus \text{Span}\,(e^{I_1}, \dots, e^{I_q})$. Up to reorder and change of sign we can assume that there exists $i_1 \in I_1$ such that $\lambda_{i_1} := \max_{i\in I_1} \lambda_i > \min_{i\in I_1} \lambda_i$ and $\lambda_{i_1} \ge 0$. Let c be a positive constant large enough such that the vector $\mu := \lambda - c \sum_{i\ge 2} e^{I_i}$ satisfies $\mu_i < 0$ for any $i \notin I_1$.

Let us consider a Lipschitz function $T : \mathbb{R} \longrightarrow \mathbb{R}$ such that $T(\lambda_{i_1}) = 1$ and $T(\mu_i) = 0$ for any $\mu_i < \lambda_{i_1}$, and let us define the set $I := \{i \in I_1 \;:\; \mu_i = \lambda_{i_1}\} \ne I_1$. Then, since for any $k > 1$ and $i \in I_k$, $\mu_i < 0 \le \lambda_{i_1}$, and for any $i \in I_1 \setminus I$, $\mu_i < \lambda_{i_1}$, we have $T(\mu) = e^I \in \Lambda$. Therefore, $e^I, e^{I_1\setminus I} = e^{I_1} - e^I, \dots, e^{I_q}$ are $(q + 1)$ independent vectors in Λ, which yields the contradiction. So $q = n$ and $(e^{I_1}, \dots, e^{I_n})$ is a basis of Λ, which proves Lemma 3.1.

3.2 Proof of Proposition 2.5

(*i*) If $\xi \notin D$ by the Γ-liminf inequality $\lim\limits_{\varepsilon\to 0} \Phi^{\#}_\varepsilon(\xi) = +\infty = \Phi^{\#}(\xi)$.

Now, let $\xi \in D$. With fixed $\delta > 0$ small enough ($\delta < \frac{1}{2} \min\left\{|\xi_{ij} - \xi_{i'j'}| \;:\; \xi_{ij} \ne \xi_{i'j'}\right\}$), let $T^\delta_\xi : \mathbb{R} \longrightarrow \mathbb{R}$ be a Lipschitz function such that

$$T^\delta_\xi(t) := \xi_{ij} \ \text{ if } |t - \xi_{ij}| < \delta \qquad \text{and} \qquad \|(T^\delta_\xi)'\|_\infty \le 1 + o(1) \ \text{ as } \delta \to 0.$$

Let $\xi_\varepsilon \to \xi$ be a recovery sequence such that $\lim\limits_{\varepsilon\to 0} \Phi^{\#}_\varepsilon(\xi_\varepsilon) = \Phi^{\#}(\xi)$. Let us define $T^\delta_\xi(u) := (T^\delta_\xi(u_1), \dots, T^\delta_\xi(u_m))$ for $u = (u_1, \dots, u_m)$. Note that for ε small enough, if u is an admissible test function in $W^{1,p}_{\#}(Y;\mathbb{R}^m)$ for $\Phi^{\#}_\varepsilon(\xi_\varepsilon)$ then $T^\delta_\xi(u)$ is an admissible one for $\Phi^{\#}_\varepsilon(\xi)$. Therefore, we obtain

$$\Phi^{\#}_\varepsilon(\xi) \leq \|(T^\delta_\xi)'\|^p_\infty \, \Phi^{\#}_\varepsilon(\xi_\varepsilon) \leq \, (1+o(1))\, \Phi^{\#}_\varepsilon(\xi_\varepsilon).$$

By letting $\varepsilon \to 0$ and by the arbitrariness of δ we then get $\limsup_{\varepsilon\to 0} \Phi^{\#}_\varepsilon(\xi) \leq \Phi^{\#}(\xi)$. On the other hand, the Γ-liminf inequality yields $\Phi^{\#}(\xi) \leq \limsup_{\varepsilon\to 0} \Phi^{\#}_\varepsilon(\xi)$, and we may conclude.

(ii) Let K be a positive integer. Let $\xi,\eta \in \mathbb{R}^{m\times N}$, $\xi = (\xi_1,\dots,\xi_N)$, $\eta = (\eta_1,\dots,\eta_N)$ with $\xi_i,\eta_i \in \mathbb{R}^m$. For fixed $\delta > 0$, let $v_\varepsilon, w_\varepsilon \in W^{1,p}(KY)$ be such that $v_\varepsilon = \xi_i, w_\varepsilon = \eta_i$ in E^ε_i and

$$\Phi^K_\varepsilon(\xi) \geq \frac{1}{K^d\varepsilon^p}\int_{KY} f_\varepsilon(y,Dv_\varepsilon)\,dy - \frac{\delta}{2}, \quad \Phi^K_\varepsilon(\eta) \geq \frac{1}{K^d\varepsilon^p}\int_{KY} f_\varepsilon(y,Dw_\varepsilon)\,dy - \frac{\delta}{2}.$$

By using the inequality $(a+b)^p \leq 2^{p-1}(a^p+b^p)$ for any $a,b\in\mathbb{R}_+$, we have

$$f_\varepsilon(y,Dv_\varepsilon + Dw_\varepsilon) \leq c_p\left(f_\varepsilon(y,Dv_\varepsilon) + f_\varepsilon(y,Dw_\varepsilon)\right),$$

and integrating by parts the previous inequality over KY yields with $\delta\to 0$

$$\Phi^K_\varepsilon(\xi+\eta) \leq c_p\left(\Phi^K_\varepsilon(\xi) + \Phi^K_\varepsilon(\eta)\right). \tag{3.2}$$

Moreover, since $f_\varepsilon(y,\cdot)$ is p-homogeneous so is Φ^K_ε. Let $(\xi^1,\dots,\xi^M)$ be a basis of D $(M = mn)$ and let $\xi := \sum_{k=1}^M \alpha_k\,\xi^k$. By the additivity property (3.2), the p-homogeneity and the boundedness of $\Phi^{\#}_\varepsilon$ on D obtained in (i), we have

$$\Phi^K_\varepsilon(\xi) \leq c_p\sum_{k=1}^M |\alpha_k|^p\,\Phi^K_\varepsilon(\xi^k) \leq c_p\sum_{k=1}^M |\alpha_k|^p\,\Phi^{\#}_\varepsilon(\xi^k) \leq c\,|\xi|^p.$$

Let us prove the Lipschitz condition of (2.23). Let $\xi := \sum_{k=1}^M \alpha_k\,\xi^k$ and $\eta := \sum_{k=1}^M \beta_k\,\xi^k$ be two vectors of D. Let $\hat v^k_\varepsilon$, for $k = 1\dots M$, be a sequence in $W^{1,p}_{\#}(Y;\mathbb{R}^m)$ such that $\hat v^k_\varepsilon = \xi^k_i$ in E^ε_i and the energy $F^{\#}_\varepsilon(\hat v^k_\varepsilon)$ (see (2.16)) is bounded. For a given $\delta > 0$, let $v_\varepsilon \in W^{1,p}(KY;\mathbb{R}^m)$ be such that

$$v_\varepsilon = \xi_i \;\text{ in } E^\varepsilon_i \qquad \text{and} \qquad \frac{1}{K^d\varepsilon^p}\int_{KY} f_\varepsilon(y,Dv_\varepsilon)\,dy \leq \Phi^K_\varepsilon(\xi) + \delta.$$

Set $w_\varepsilon := v_\varepsilon + \sum_{k=1}^M(\beta_k-\alpha_k)\,\hat v^k_\varepsilon$. It is clear that $w_\varepsilon = \eta_i$ in E^ε_i and by (3.2)

$$\frac{1}{K^d\varepsilon^p}\int_{KY} f_\varepsilon(y,Dw_\varepsilon)\,dy \leq c\left(\Phi^K_\varepsilon(\xi) + \delta + |\xi-\eta|^p\right).$$

Using the properties (2.7) and (2.8) of the function f_ε, the Hölder inequality and the previous estimate yields

$$\begin{aligned}
\Phi^K_\varepsilon(\eta) - \Phi^K_\varepsilon(\xi) &\leq \frac{1}{K^d\varepsilon^p}\int_{KY}\left(f_\varepsilon(y,Dw_\varepsilon) - f_\varepsilon(y,Dv_\varepsilon)\right)\,dy + \delta \\
&\leq \left(\frac{1}{K^d\varepsilon^p}\int_{KY} a_\varepsilon\left(|Dw_\varepsilon|^p + |Dv_\varepsilon|^p\right)\,dy\right)^{\frac{1}{p'}}\left(\frac{1}{K^d\varepsilon^p}\int_{KY} a_\varepsilon\,|Dw_\varepsilon - Dv_\varepsilon|^p\,dy\right)^{\frac{1}{p}} + \delta \\
&\leq c\left(\Phi^K_\varepsilon(\xi) + \delta + |\xi-\eta|^p\right)^{\frac{1}{p'}}|\xi-\eta| + \delta \\
&\leq c\left(|\xi|^{p-1} + |\eta|^{p-1} + \delta^{\frac{1}{p'}}\right)|\xi-\eta| + \delta,
\end{aligned}$$

whence by letting $\delta \to 0$ we obtain the desired estimate.

(iii) Since $\mathbb{R}^{m\times N}$ is separable, there exists a subsequence of Φ_ε^K (not relabelled) which Γ-converges to some functional Φ_0^K in $\mathbb{R}^{m\times N}$ (see *e.g.* [6] Proposition 1.42, [16] Theorem 8.5) for any positive integer K, by using a diagonal extraction. On the other hand, the local Lipschitz condition of (2.23) implies that the sequence Φ_ε^K is equi-continuous on D with respect to ε. Therefore, the Γ-convergence in D and the pointwise convergence in D of Φ_ε^K are equivalent (see *e.g.* [16] Proposition 5.9). So Φ_ε^K pointwise converges to Φ_0^K in D. It is clear that Φ_0^K satisfies the same estimates (2.23) as Φ_ε^K. Moreover, by [8] Proposition 4.1 we have

$$\forall\, K < K' \in \mathbb{N}^*, \qquad \left(\frac{K}{K'}\right)^d \left[\frac{K'}{K+1}\right]^d \Phi_\varepsilon^K \leq \Phi_\varepsilon^{K'}, \tag{3.3}$$

where $[\cdot]$ denotes the integer part. Then, passing to the limit $\varepsilon \to 0$ in (3.3) yields the same inequality for the pointwise limit Φ_0^K in D. Again by using [8] Proposition 4.1 we obtain that Φ_0^K converges in D as $K \to +\infty$. This concludes the proof of Proposition 2.5.

3.3 Proof of Proposition 2.6

Since $L^p(\Omega;\mathbb{R}^m)$ is separable, we can assume that the functional $\mathcal{G}_\varepsilon$ defined by (2.27) Γ-converges to some functional $\mathcal{G}_0$ for the strong topology of $L^p(\Omega;\mathbb{R}^m)$. The inclusion $E_\varepsilon \subset E$ of (2.25) clearly implies that $\mathcal{G}_0 \leq \mathcal{G}$ defined by (2.28). Note that $\mathcal{G}$ is the Γ-limit of the functional $\mathcal{G}_\varepsilon^E$ defined by (2.27) with the fixed periodic set E, according to [10] (see also [9] Chapter 20).

Let us prove the inverse inequality $\mathcal{G} \leq \mathcal{G}_0$. Let $v \in L^p(\Omega;\mathbb{R}^m)$ and let $v_\varepsilon \in L^p(\Omega;\mathbb{R}^m)$ be a recovering sequence which strongly converges to v and such that $\mathcal{G}_\varepsilon(v_\varepsilon) \to \mathcal{G}_0(v)$. If $\mathcal{G}_0(v) = +\infty$ then $\mathcal{G}(v) \leq \mathcal{G}_0(v)$. Now, assume that $\mathcal{G}_0(v) < +\infty$. Then, $\mathcal{G}_\varepsilon(v_\varepsilon)$ is bounded, whence by the p-growth condition on g combined with the uniform extension property (2.26) we deduce that $P_\varepsilon v_\varepsilon$ is bounded in $W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$ and $P_\varepsilon v_\varepsilon$ weakly converges to v in $W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$ (up to a subsequence).

Let ω be an open subset of $\mathbb{R}^d$ such that $\omega \Subset \Omega$. We have

$$\liminf_{\varepsilon\to 0} \int_{\Omega\cap\varepsilon E_\varepsilon} g\left(\frac{x}{\varepsilon}, Dv_\varepsilon\right) dx \geq \liminf_{\varepsilon\to 0} \int_{\omega\cap\varepsilon E_\varepsilon} g\left(\frac{x}{\varepsilon}, D(P_\varepsilon v_\varepsilon)\right) dx. \tag{3.4}$$

By a result due to Fonseca et al. [15] there exists a bounded subsequence (not relabelled) w_ε in $W^{1,p}(\omega;\mathbb{R}^m)$ such that

$$\left|\,\{w_\varepsilon \neq P_\varepsilon v_\varepsilon\} \cup \{Dw_\varepsilon \neq D(P_\varepsilon v_\varepsilon)\}\,\right| \underset{\varepsilon\to 0}{\longrightarrow} 0 \quad \text{and} \quad |Dw_\varepsilon|^p \text{ equi-integrable.}$$

Using the p-growth of g yields

$$\begin{aligned}
&\int_{\omega\cap\varepsilon E_\varepsilon} g\left(\frac{x}{\varepsilon}, D(P_\varepsilon v_\varepsilon)\right) dx \geq \int_{\omega\cap\varepsilon E_\varepsilon\cap\{Dw_\varepsilon = D(P_\varepsilon v_\varepsilon)\}} g\left(\frac{x}{\varepsilon}, Dw_\varepsilon\right) dx \\
&\geq \int_{\omega\cap\varepsilon E_\varepsilon} g\left(\frac{x}{\varepsilon}, Dw_\varepsilon\right) dx - c\int_{\omega\cap\varepsilon E_\varepsilon\cap\{Dw_\varepsilon \neq D(P_\varepsilon v_\varepsilon)\}} |Dw_\varepsilon|^p\, dx,
\end{aligned}$$

whence by (3.4) and the equi-integrability of $|Dw_\varepsilon|^p$

$$\liminf_{\varepsilon\to 0} \int_{\omega\cap\varepsilon E_\varepsilon} g\left(\frac{x}{\varepsilon}, Dv_\varepsilon\right) dx \geq \liminf_{\varepsilon\to 0} \int_{\omega\cap\varepsilon E_\varepsilon} g\left(\frac{x}{\varepsilon}, Dw_\varepsilon\right) dx. \tag{3.5}$$

Moreover, we have

$$\int_{\omega\cap\varepsilon E_\varepsilon} g\left(\frac{x}{\varepsilon}, Dw_\varepsilon\right) dx = \int_{\omega\cap\varepsilon E} g\left(\frac{x}{\varepsilon}, Dw_\varepsilon\right) dx - \int_{\omega\cap\varepsilon(E\setminus E_\varepsilon)} g\left(\frac{x}{\varepsilon}, Dw_\varepsilon\right) dx, \tag{3.6}$$

in which the last term tends to 0 again by the equi-integrability of $|Dw_\varepsilon|^p$ combined with (2.25). Then (3.5) combined with (3.6) and the Γ-liminf inequality satisfied by $\mathcal{G}^E_\varepsilon$ imply that

$$\liminf_{\varepsilon\to 0}\int_{\omega\cap\varepsilon E_\varepsilon} g\left(\frac{x}{\varepsilon}, Dv_\varepsilon\right)dx \geq \liminf_{\varepsilon\to 0}\int_{\omega\cap\varepsilon E} g\left(\frac{x}{\varepsilon}, Dw_\varepsilon\right)dx \geq \int_\omega g^{\mathrm{hom}}(Dv)\,dx.$$

Therefore, we obtain

$$\mathcal{G}_0(v) = \lim_{\varepsilon\to 0}\ \mathcal{G}_\varepsilon(v_\varepsilon) \geq \liminf_{\varepsilon\to 0}\int_{\omega\cap\varepsilon E_\varepsilon} g\left(\frac{x}{\varepsilon}, Dv_\varepsilon\right)dx \geq \int_\omega g^{\mathrm{hom}}(Dv)\,dx,$$

whence $\mathcal{G}_0(v) \geq \mathcal{G}(v)$ by the arbitrariness of ω. The proof is thus established.

4 Proof of Theorem 2.8

The proof of Theorem 2.8 is divided in three sections. In the first section we determine the domain of the Γ-limit of the energy $\mathcal{F}_\varepsilon$ defined by (2.12). The second section is devoted to the Γ-liminf inequality, and the third one to the Γ-limsup inequality according to Definition 2.2.

4.1 Determination of the domain of the limit energy

We proceed in two steps. In the first step we prove an inequality which is an auxiliary result for the second step. In the second step we prove that any multi-phase limit $(u_1,\dots,u_N)$, according to the topology (2.14), of a sequence u_ε with bounded energy, $\mathcal{F}_\varepsilon(u_\varepsilon) \leq c$, belongs to $W^{1,p}(\Omega;\mathbb{R}^m)^N$ and a.e. to the set D defined by (2.20).

First step : A preliminary inequality.
Let $k \in \{1,\dots,n\}$, let $i,j \in \hat{I}_k$ where the set $\hat{I}_k$ is defined in Proposition 2.3 and denote $Y^\varepsilon_h := E^\varepsilon_h \cap Y$ for $h = 1\dots N$. Let us prove that the optimal constant $C(\varepsilon)$ of the inequality

$$\forall\, v \in W^{1,p}_\#(Y;\mathbb{R}^m),\qquad \left|\fint_{Y_i^\varepsilon} v - \fint_{Y_j^\varepsilon} v\right| \leq C(\varepsilon)\left(\frac{1}{\varepsilon^p}\int_Y a_\varepsilon\,|Dv|^p\,dy + \int_Y |v|^p\,dy\right)^{\frac{1}{p}}, \qquad \text{satisfies}\quad \lim_{\varepsilon\to 0} C(\varepsilon) = 0, \tag{4.1}$$

To this end, let us consider a function v_ε in $W^{1,p}_\#(Y;\mathbb{R}^m)$ such that

$$G^\#_\varepsilon(v_\varepsilon) := \frac{1}{\varepsilon^p}\int_Y a_\varepsilon\,|Dv_\varepsilon|^p\,dy + \int_Y |v_\varepsilon|^p\,dy = 1 \quad\text{and}\quad C(\varepsilon) = \left|\fint_{Y_i^\varepsilon} v_\varepsilon - \fint_{Y_j^\varepsilon} v_\varepsilon\right|. \tag{4.2}$$

Since v_ε is bounded in $L^p(Y;\mathbb{R}^m)$, we have (up to a subsequence)

$$\xi^\varepsilon_h := \fint_{Y^\varepsilon_h} v_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} \xi_h \quad \text{for } h = 1\dots N.$$

Let us consider the function

$$\tilde{v}_\varepsilon := v_\varepsilon - \sum_{h=1}^N \varphi^\varepsilon_h\left(v_\varepsilon - \fint_{Y^\varepsilon_h} v_\varepsilon\right),$$

where φ^ε_k are the cut-off functions defined by (2.30). The energy estimate (4.2) satisfied by the sequence v_ε combined with the Poincaré-Wirtinger inequality (2.33) implies that

$$F^\#_\varepsilon(\tilde{v}_\varepsilon) = \frac{1}{\varepsilon^p}\int_Y a_\varepsilon\,|D\tilde{v}_\varepsilon|^p\,dy \leq \frac{c}{\varepsilon^p}\int_Y a_\varepsilon\,|Dv_\varepsilon|^p\,dy \leq c\,G^\#_\varepsilon(v_\varepsilon) \leq c.$$

Moreover, we have $\tilde{v}_\varepsilon = \xi_h^\varepsilon$ in E_h^ε for $h = 1 \dots N$. Then $\tilde{v}_\varepsilon$ is a suitable test function for the minimization problem (2.16), and the Γ-liminf inequality of (2.17) yields

$$\Phi^\#(\xi_1, \dots, \xi_N) \le \liminf_{\varepsilon \to 0} \ \Phi_\varepsilon^\#(\xi_1^\varepsilon, \dots, \xi_N^\varepsilon) \le \liminf_{\varepsilon \to 0} \ F_\varepsilon^\#(\tilde{v}_\varepsilon) \le c,$$

whence by definitions (2.18) and (2.20) $(\xi_1, \dots, \xi_N) \in D$, and in particular $\xi_i = \xi_j$. Therefore, by (4.2) $C(\varepsilon) \to 0$.

Second step : Property of the multi-phase limit of a sequence with bounded energy.

Let u_ε be a sequence in $W^{1,p}(\Omega;\mathbb{R}^m)$ with bounded energy, *i.e.*, $\mathcal{F}_\varepsilon(u_\varepsilon) \le c$, which converges to a multi-phase function $(u_1, \dots, u_N) \in (L^p(\Omega;\mathbb{R}^m))^N$ for the topology (2.14). Thanks to the uniform extension property (2.26) satisfied by each component E_i^ε for $i = 1 \dots N$, the sequence u_ε in $\Omega \cap \varepsilon E_i^\varepsilon$ can be extended to a bounded sequence in $W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$ which weakly converges (up to a subsequence) to u_i in $W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$. The function u_i actually belongs to $W^{1,p}(\Omega;\mathbb{R}^m)$. Indeed, using the semi-lower continuity of the L^2-norm of the gradient and the second inequality of (2.26) yields for any open set $\omega \Subset \Omega$,

$$\begin{aligned} \int_\omega |\nabla u_i|^2 \, dx &\le \liminf_{\varepsilon \to 0} \int_\omega |\nabla P_\varepsilon (u_{\varepsilon | \Omega \cap \varepsilon E_i^\varepsilon})|^2 \, dx \\ &\le k_1 \liminf_{\varepsilon \to 0} \int_{\Omega \cap \varepsilon E_i^\varepsilon} |\nabla u_\varepsilon|^2 \, dx \le k_1 \liminf_{\varepsilon \to 0} \ \mathcal{F}_\varepsilon(u_\varepsilon) \le k_1 c, \end{aligned}$$

where the constant $k_1 c$ is independent of the open set ω.

It remains to prove that $(u_1, \dots, u_N)$ also belongs a.e. to the set D defined by (2.20). To this end, let us consider, for fixed $k \in \{1, \dots, n\}$, $i, j \in \hat{I}_k$ and $h \in \{1, \dots, m\}$, the solution w_ε in $W^{1,p}_\#(Y;\mathbb{R}^m)$ of the problem

$$\begin{aligned} \forall v \in W^{1,p}_\#(Y;\mathbb{R}^m), \quad & \frac{1}{\varepsilon^p} \int_Y a_\varepsilon |Dw_\varepsilon|^{p-2} Dw_\varepsilon : Dv \, dy + \int_Y |w_\varepsilon|^{p-2} w_\varepsilon \cdot v \, dy \\ & = \fint_{Y_i^\varepsilon} v_h \, dy - \fint_{Y_j^\varepsilon} v_h \, dy, \end{aligned} \tag{4.3}$$

or equivalently, in the whole space $\mathbb{R}^d$,

$$-\frac{1}{\varepsilon^p} \operatorname{Div} \left(a_\varepsilon |Dw_\varepsilon|^{p-2} Dw_\varepsilon \right) + |w_\varepsilon|^{p-2} w_\varepsilon = \frac{1_{E_i^\varepsilon}}{|Y_i^\varepsilon|} \, e_h - \frac{1_{E_j^\varepsilon}}{|Y_j^\varepsilon|} \, e_h \quad \text{in } \mathcal{D}'(\mathbb{R}^d), \tag{4.4}$$

where $\operatorname{Div} W$ denotes the $\mathbb{R}^m$-distribution obtained by taking the divergence of the lines of the $\mathbb{R}^{m \times d}$-valued function W, and $(e_1, \dots, e_m)$ the canonical basis of $\mathbb{R}^m$. Putting the function w_ε as test function in (4.3) and using inequality (4.1) yield

$$\frac{1}{\varepsilon^p} \int_Y a_\varepsilon |Dw_\varepsilon|^p \, dy + \int_Y |w_\varepsilon|^p \, dy \le c \, C(\varepsilon)^{p'} \underset{\varepsilon \to 0}{\longrightarrow} 0. \tag{4.5}$$

On the other hand, for a given $\varphi \in C_c^\infty(\Omega)$, putting the function φu_ε as test function in the ε-rescaled equation (4.4), and using successively the Hölder inequality, estimates (4.5) and (2.7), yield

$$
\begin{aligned}
&\left| \frac{1}{|Y_i^\varepsilon|} \int_{\Omega\cap\varepsilon E_i^\varepsilon} (\varphi u_\varepsilon)_h \, dx - \frac{1}{|Y_j^\varepsilon|} \int_{\Omega\cap\varepsilon E_j^\varepsilon} (\varphi u_\varepsilon)_h \, dx \right| \\
&= \left| \frac{1}{\varepsilon^{p-1}} \int_\Omega \left(a_\varepsilon \, |Dw_\varepsilon|^{p-2} Dw_\varepsilon\right) \left(\frac{x}{\varepsilon}\right) : D(\varphi u_\varepsilon) \, dx + \int_\Omega \left(|w_\varepsilon|^{p-2} w_\varepsilon\right) \left(\frac{x}{\varepsilon}\right) \cdot (\varphi u_\varepsilon) \, dx \right| \\
&\le \frac{c}{\varepsilon^{p-1}} \left(\int_Y a_\varepsilon \, |Dw_\varepsilon|^p \, dy \right)^{\frac{1}{p'}} \left(\int_\Omega a_\varepsilon \left(\frac{x}{\varepsilon}\right) |D(\varphi u_\varepsilon)|^p \, dx \right)^{\frac{1}{p}} \\
&+ c \left(\int_Y |w_\varepsilon|^p \, dy \right)^{\frac{1}{p'}} \left(\int_\Omega |\varphi u_\varepsilon|^p \, dx \right)^{\frac{1}{p}} \\
&\le c\, C(\varepsilon) \left(\int_\Omega f_\varepsilon \left(\frac{x}{\varepsilon}, D(\varphi u_\varepsilon)\right) dx \right)^{\frac{1}{p}} + c\, C(\varepsilon) \left(\int_\Omega |\varphi u_\varepsilon|^p \, dx \right)^{\frac{1}{p}}
\end{aligned}
$$

which implies the estimate

$$
\begin{aligned}
&\left| \frac{1}{|Y_i^\varepsilon|} \int_{\Omega\cap\varepsilon E_i^\varepsilon} (\varphi u_\varepsilon)_h \, dx - \frac{1}{|Y_j^\varepsilon|} \int_{\Omega\cap\varepsilon E_j^\varepsilon} (\varphi u_\varepsilon)_h \, dx \right| \\
&\le c\, C(\varepsilon) \left((\mathcal{F}_\varepsilon(\varphi u_\varepsilon))^{\frac{1}{p}} + \|\varphi u_\varepsilon\|_{L^\infty(\Omega)} \right).
\end{aligned}
\tag{4.6}
$$

First case : If u_ε is uniformly bounded in Ω, the energy $\mathcal{F}_\varepsilon(\varphi u_\varepsilon)$ is bounded, whence the right-hand side of (4.6) tend to 0. By passing to the limit in (4.6) we thus obtain

$$
\forall\, \varphi \in C_c^\infty(\Omega), \qquad \int_\Omega \varphi(u_i - u_j)_h \, dx = 0,
$$

which implies $u_i = u_j$ a.e. in Ω for any $i, j \in \hat{I}_k$. Therefore, $(u_1, \ldots, u_N) \in D$ a.e. in Ω by the definition (2.20) of D.
Second case : Let T_K, $K \in \mathbb{N}^*$, be the function defined by $T_K(t) := \min(K, \max(-K, t))$, $t \in \mathbb{R}$. The sequence $T_K(u_\varepsilon)$ is bounded by K and has a bounded energy since the condition (2.7) satisfied by f_ε combined with the 1-Lipschitz property of T_K implies that $\mathcal{F}_\varepsilon(T_K(u_\varepsilon)) \le c\, \mathcal{F}_\varepsilon(u_\varepsilon)$. By the Lipschitz property of T_K the sequence $T_K(u_\varepsilon)$ τ-converges to the multi-phase function $(T_K(u_1), \ldots, T_K(u_N))$. Then, by the first case we obtain $(T_K(u_1), \ldots, T_K(u_N)) \in D$ a.e. in Ω. Since $T_K(u_i)$ strongly converges to u_i in $L^p(\Omega; \mathbb{R}^m)$ and hence a.e. in Ω (up to a subsequence), passing to the limit $K \to +\infty$ finally yields $(u_1, \ldots, u_N) \in D$ a.e. in Ω.

4.2 Proof of the lower-bound inequality

The proof follows closely that in [8] Section 5, to which we refer for the details that remain unchanged.

Let u_ε be a sequence converging to $u = (u_1, \ldots, u_N)$ such that $\sup_\varepsilon \mathcal{F}_\varepsilon(u_\varepsilon) < +\infty$. Note that by the previous section we have $u(x) \in D$ for a.e. $x \in \Omega$. By Proposition 2.6 we have

$$
\begin{aligned}
\liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon) \;&\ge\; \sum_{i=1}^N \liminf_{\varepsilon\to 0} \int_{\Omega\cap\varepsilon E_i^\varepsilon} f_\varepsilon\Big(\frac{x}{\varepsilon}, Du_\varepsilon\Big) \, dx + \liminf_{\varepsilon\to 0} \int_{\Omega\cap\omega_\varepsilon} f\Big(\frac{x}{\varepsilon}, Du_\varepsilon\Big) \, dx \\
&\ge\; \sum_{i=1}^N \liminf_{\varepsilon\to 0} \int_\Omega f_i^{\hom}(Du_i) \, dx + \liminf_{\varepsilon\to 0} \int_{\Omega\cap\varepsilon\omega_\varepsilon} f\Big(\frac{x}{\varepsilon}, Du_\varepsilon\Big) \, dx.
\end{aligned}
$$

It then remains to prove that

$$
\liminf_{\varepsilon\to 0} \int_{\Omega\cap\varepsilon\omega_\varepsilon} f\Big(\frac{x}{\varepsilon}, Du_\varepsilon\Big) \, dx \ge \int_\Omega \Phi(u_1, \ldots, u_N) \, dx\,. \tag{4.7}
$$

With fixed $K \in \mathbb{N}^*$ for all $j \in \mathbb{Z}^d$ such that

$$Q_K^j := Kj + KY \subset \frac{1}{\varepsilon}\Omega, \tag{4.8}$$

we define

$$\overline{u}_{i,\varepsilon}^j := \fint_{\varepsilon(Q_K^j \cap E_i^\varepsilon)} u_\varepsilon \, dx, \qquad w_{i,\varepsilon}^j(x) = u_\varepsilon(x) - \overline{u}_{i,\varepsilon}^j \quad \text{for } x \in \varepsilon(Q_K^j \cap E_i^\varepsilon), \tag{4.9}$$

and, with a slight abuse of notation, again by $w_{i,\varepsilon}^j$ the extension of the latter function given by (2.26). Upon replacing Q_K^j by $Kj + (K - k_0)Y$, we may then assume that

$$\begin{aligned} \int_{\varepsilon Q_K^j} |w_{i,\varepsilon}^j|^p \, dx &\leq c_K \int_{\varepsilon(Q_K^j \cap E_i^\varepsilon)} |u_\varepsilon - \overline{u}_{i,\varepsilon}^j|^p \, dx && (4.10) \\ \int_{\varepsilon Q_K^j} |Dw_{i,\varepsilon}^j|^p \, dx &\leq c_K \int_{\varepsilon(Q_K^j \cap E_i^\varepsilon)} |Du_\varepsilon|^p \, dx, && (4.11) \end{aligned}$$

and hence, after applying the Poincaré-Wirtinger inequality on $\varepsilon(Q_K^j \cap E_i^\varepsilon)$, that

$$\int_{\varepsilon Q_K^j} |w_{i,\varepsilon}^j|^p \, dx \leq \varepsilon^p c_K \int_{\varepsilon(Q_K^j \cap E_i^\varepsilon)} |Du_\varepsilon|^p \, dx, \tag{4.12}$$

for some constant c_K depending on K only.

Define now the functions

$$w_\varepsilon^j(x) = \sum_{i=1}^N \varphi_i^\varepsilon\Big(\frac{x}{\varepsilon}\Big) w_{i,\varepsilon}^j(x),$$

where φ_i^ε are as in (2.30)–(2.34). Note that, by (4.12)

$$\int_{\varepsilon Q_K^j} |w_\varepsilon^j|^p \, dx \leq \varepsilon^p c_K \sum_{i=1}^N \int_{\varepsilon(Q_K^j \cap E_i^\varepsilon)} |Du_\varepsilon|^p \, dx. \tag{4.13}$$

Let I_ε^K be the set of all j such that (4.8) holds. We have by (2.8), Hölder's inequality, (2.7), and (2.32)

$$\begin{aligned} &\Big| \sum_{j \in I_\varepsilon^K} \int_{\varepsilon(Q_K^j \cap \omega_\varepsilon)} \Big(f_\varepsilon\Big(\frac{x}{\varepsilon}, Du_\varepsilon - Dw_\varepsilon^j\Big) - f_\varepsilon\Big(\frac{x}{\varepsilon}, Du_\varepsilon\Big)\Big) \, dx \Big| \\ \leq\ & c\Big(\sum_{j \in I_\varepsilon^K} \int_{\varepsilon(Q_K^j \cap \omega_\varepsilon)} a_\varepsilon\Big(\frac{x}{\varepsilon}\Big)(|Du_\varepsilon|^p + |Dw_\varepsilon^j|^p) \, dx \Big)^{1/p'} \Big(\sum_{j \in I_\varepsilon^K} \int_{\varepsilon(Q_K^j \cap \omega_\varepsilon)} a_\varepsilon\Big(\frac{x}{\varepsilon}\Big) |Dw_\varepsilon^j|^p \, dx \Big)^{1/p} \\ \leq\ & c\Big(\sum_{j \in I_\varepsilon^K} \sum_{i=1}^N \int_{\varepsilon(Q_K^j \cap \omega_\varepsilon)} \Big(\frac{1}{\varepsilon^p} a_\varepsilon\Big(\frac{x}{\varepsilon}\Big) \Big| D\varphi_i^\varepsilon\Big(\frac{x}{\varepsilon}\Big) w_{i,\varepsilon}^j(x) \Big|^p + a_\varepsilon\Big(\frac{x}{\varepsilon}\Big) |Dw_{i,\varepsilon}^j(x)|^p \Big) \, dx \Big)^{1/p} \\ \leq\ & c\Big(\sum_{j \in I_\varepsilon^K} \sum_{i=1}^N \Big(C_{i,K}(\varepsilon) \int_{\varepsilon Q_K^j} |Dw_{i,\varepsilon}^j|^p \, dx + \alpha_\varepsilon \int_{\varepsilon Q_K^j} |Dw_{i,\varepsilon}^j|^p \, dx \Big) \Big)^{1/p}. \end{aligned}$$

By the limit conditions in (2.32) and (2.6) we then obtain

$$\lim_{\varepsilon \to 0} \sum_{j \in I_\varepsilon^K} \int_{\varepsilon(Q_K^j \cap \omega_\varepsilon)} \Big(f_\varepsilon\Big(\frac{x}{\varepsilon}, Du_\varepsilon - Dw_\varepsilon^j\Big) - f_\varepsilon\Big(\frac{x}{\varepsilon}, Du_\varepsilon\Big)\Big) \, dx = 0,$$

that yields

$$\liminf_{\varepsilon\to 0}\int_{\Omega\cap\varepsilon\omega_\varepsilon} f_\varepsilon\Big(\frac{x}{\varepsilon}, Du_\varepsilon\Big)\,dx \geq \liminf_{\varepsilon\to 0}\sum_{j\in I_\varepsilon^K}\int_{\varepsilon(Q_K^j\cap\omega_\varepsilon)} f_\varepsilon\Big(\frac{x}{\varepsilon}, Du_\varepsilon - Dw_\varepsilon^j\Big)\,dx.$$

On the other hand, the functions w_ε^j are constructed in such a way that $u_\varepsilon - w_\varepsilon^j$ is an admissible test function for $\Phi_\varepsilon^K(\xi)$ in (2.21) with $\xi_i = \overline{u}_{i,\varepsilon}^j$, so that we get

$$\liminf_{\varepsilon\to 0}\int_{\Omega\cap\varepsilon\omega_\varepsilon} f_\varepsilon\Big(\frac{x}{\varepsilon}, Du_\varepsilon\Big)\,dx \geq \liminf_{\varepsilon\to 0}\sum_{j\in I_\varepsilon^K}\varepsilon^p K^p \Phi_\varepsilon^K(\overline{u}_{1,\varepsilon}^j,\ldots,\overline{u}_{1,\varepsilon}^j). \tag{4.14}$$

Upon defining $\overline{u}_\varepsilon$ by setting $(\overline{u}_\varepsilon)_i = \sum_{j\in I_\varepsilon^K} 1_{\varepsilon Q_K^j}\overline{u}_{i,\varepsilon}^j$, we can rewrite

$$\sum_{j\in I_\varepsilon^K}\varepsilon^p K^p \Phi_\varepsilon^K(\overline{u}_{1,\varepsilon}^j,\ldots,\overline{u}_{1,\varepsilon}^j) = \int_\Omega \Phi_\varepsilon^K(\overline{u}_\varepsilon)\,dx.$$

Since $\overline{u}_\varepsilon$ strongly converges to u in $L^p_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$ and, upon extracting a subsequence, a.e. in Ω, by Fatou's lemma and Proposition 2.5 (iii) we have

$$\begin{aligned}\liminf_{\varepsilon\to 0}\int_{\Omega\cap\varepsilon\omega_\varepsilon} f_\varepsilon\Big(\frac{x}{\varepsilon}, Du_\varepsilon\Big)\,dx &\geq \liminf_{\varepsilon\to 0}\int_\Omega \Phi_\varepsilon^K(\overline{u}_\varepsilon)\,dx\\ &\geq \int_\Omega \liminf_{\varepsilon\to 0}\Phi_\varepsilon^K(\overline{u}_\varepsilon)\,dx \geq \int_\Omega \Phi_0^K(u)\,dx.\end{aligned}$$

It suffices eventually to apply Proposition 2.5 (iv) to obtain (4.7) and conclude the proof.

4.3 Proof of the upper-bound inequality

We now have to construct a recovery sequence for each function $u = (u_1,\ldots,u_N)$ such that $\mathcal{F}(u) < +\infty$; i.e., for all functions $u \in (W^{1,p}(\Omega;\mathbb{R}^m))^N$ such that $u(x)\in D$ a.e. By a density argument it suffices to deal with piecewise-affine u. We will only give a proof for u linear, since the extension to u piecewise affine follows standard arguments, as in [8].

Let $u(x) := (\lambda_1 x,\ldots,\lambda_N x)$, with $\lambda_i\in\mathbb{R}^{m\times d}$. Note that the constraint $u(x)\in D$ a.e. implies that $\lambda_i = \lambda_j$ if the two indices belong to the same element of the partition defined in Proposition 2.3.

With fixed $\delta > 0$, for all $i\in\{1,\ldots,N\}$ we choose $K\in\mathbb{N}^*$ and $u_i^K\in W_0^{1,p}(KY;\mathbb{R}^m)$ such that

$$\int_{KY\cap E_i^\varepsilon} f_\varepsilon(y, Du_i^K + \lambda_i)\,dy \leq K^d(f_{\mathrm{hom}}^i(\lambda_i) + \delta), \tag{4.15}$$

and for all $j\in\mathbb{Z}^d$ we choose $v_\varepsilon^j\in W^{1,p}(KY;\mathbb{R}^m)$ such that

$$\frac{1}{\varepsilon^p K^d}\int_{KY} f_\varepsilon(y, Dv_\varepsilon^j)\,dy \leq \Phi_\varepsilon^K(\lambda_1 x_\varepsilon^j,\ldots,\lambda_N x_\varepsilon^j) + \delta, \tag{4.16}$$

where

$$x_\varepsilon^j := \varepsilon K j \qquad\text{and}\qquad v_\varepsilon^j := \lambda_i x_\varepsilon^j \text{ on } KY\cap E_i^\varepsilon. \tag{4.17}$$

Moreover, we also choose $\widehat{v}_\varepsilon^h\in W_\#^{1,p}(Y;\mathbb{R}^m)$ such that $\widehat{v}_\varepsilon^h = \lambda_i^h$ on $Y\cap E_i^\varepsilon$ and

$$\frac{1}{\varepsilon^p}\int_Y a_\varepsilon(y)|D\widehat{v}_\varepsilon^h|^p\,dy \leq \Phi_\varepsilon^\#(\lambda_1^h,\ldots,\lambda_N^h) + \frac{\delta}{N}.$$

The function $\widehat{v}_\varepsilon \in W^{1,p}_{\#}(Y;\mathbb{R}^{m\times N})$ defined by $\widehat{v}_\varepsilon := (\widehat{v}^1_\varepsilon,\dots,\widehat{v}^N_\varepsilon) \in (\mathbb{R}^m)^N$ satisfies

$$\begin{aligned} \frac{1}{\varepsilon^p}\int_Y a_\varepsilon(y)|D\widehat{v}_\varepsilon|^p\,dy &\leq \sum_{h=1}^N \Phi^{\#}_\varepsilon(\lambda^h_1,\dots,\lambda^h_N)+\delta, \\ \widehat{v}_\varepsilon = \lambda_i \quad &\text{in} \quad Y\cap E^\varepsilon_i. \end{aligned} \tag{4.18}$$

Let $\psi_K \in W^{1,\infty}_{\#}(KY)$ be such that

$$\psi_K(x) = 1 \text{ in } (K-1)Y, \qquad \psi_K(x) = 0 \text{ on } \partial(KY). \tag{4.19}$$

If $x \in \varepsilon(Kj+KY)$, $j\in\mathbb{Z}^d$, we set

$$w_\varepsilon(x) := \sum_{i=1}^N \varepsilon\varphi^\varepsilon_i\Big(\frac{x}{\varepsilon}\Big)u^K_i\Big(\frac{x-x^j_\varepsilon}{\varepsilon}\Big)+v^j_\varepsilon\Big(\frac{x-x^j_\varepsilon}{\varepsilon}\Big)+\widehat{v}_\varepsilon\Big(\frac{x}{\varepsilon}\Big)(x-x^j_\varepsilon) \tag{4.20}$$

and finally

$$u_\varepsilon(x) := \psi_K\Big(\frac{x}{\varepsilon}\Big)w_\varepsilon(x)+\Big(1-\psi_K\Big(\frac{x}{\varepsilon}\Big)\Big)\,\widehat{v}_\varepsilon\Big(\frac{x}{\varepsilon}\Big)x \tag{4.21}$$

The functions u_ε constructed in this way satisfy

$$u_\varepsilon(x) = \widehat{v}_\varepsilon\Big(\frac{x}{\varepsilon}\Big)x \qquad \text{on} \quad \partial(x^j_\varepsilon+\varepsilon KY), \tag{4.22}$$

$$u_\varepsilon(x) = \varepsilon u^K_i\Big(\frac{x-x^j_\varepsilon}{\varepsilon}\Big)+\lambda_i x \qquad \text{in} \quad (x^j_\varepsilon+\varepsilon(K-1)Y)\cap\varepsilon E^\varepsilon_i \tag{4.23}$$

and in particular u_ε uniformly converges to $(\lambda_1 x,\dots,\lambda_N x)$.

Let J^ε_K be the set of all $j\in\mathbb{Z}^d$ such that

$$\varepsilon(Kj+KY)\cap\Omega \neq \emptyset. \tag{4.24}$$

By reasoning similarly as in [8] Section 6, we may then estimate

$$\begin{aligned} &\limsup_{\varepsilon\to0}\int_\Omega f_\varepsilon\Big(\frac{x}{\varepsilon},Du_\varepsilon\Big)\,dx \\ \leq\ &\limsup_{\varepsilon\to0}\sum_{j\in J^\varepsilon_K}\int_{x^j_\varepsilon+\varepsilon KY} f_\varepsilon\Big(\frac{x}{\varepsilon},Du_\varepsilon\Big)\,dx \\ \leq\ &\limsup_{\varepsilon\to0}\sum_{j\in J^\varepsilon_K}\int_{x^j_\varepsilon+\varepsilon(K-1)Y} f_\varepsilon\Big(\frac{x}{\varepsilon},Du_\varepsilon\Big)\,dx+o(1). \end{aligned}$$

The $o(1)$ remainder in this formula comes from the contribution of the term with $\nabla\psi_K$. Applying the Poincaré inequality (2.34) to the function $v^j_\varepsilon - \widehat{v}_\varepsilon x^j_\varepsilon$ (which are equal to 0 on ∂E^ε_i) leads to the energy of v^j_ε over the set $KY\setminus(K-1)Y$ (which contains the support of $\nabla\psi_K$). Proceeding as in Proposition 4.1 of [8] one can check that this energy is bounded by the values of the function $\Phi^K_\varepsilon - \big(\frac{K-2}{K}\big)^d\,\Phi^{K-2}_\varepsilon$ at the points $(\lambda_1 x^j_\varepsilon,\dots,\lambda_N x^j_\varepsilon)$. However, thanks to the Lipschitz condition (2.23) satisfied by Φ^K_ε combined with the Ascoli theorem, the sequence Φ^K_ε uniformly converges to Φ as $\varepsilon\to0$ and $K\to+\infty$. Therefore, the sequence $\Phi^K_\varepsilon - \big(\frac{K-2}{K}\big)^d\,\Phi^{K-2}_\varepsilon$ uniformly converges to 0, which implies the desired $o(1)$. We can then proceed in our estimate obtaining

$$
\begin{aligned}
&\limsup_{\varepsilon\to 0}\int_\Omega f_\varepsilon\Big(\frac{x}{\varepsilon}, Du_\varepsilon\Big)\,dx\\
\le\ &\limsup_{\varepsilon\to 0}\sum_{j\in J_K^\varepsilon}\int_{x_\varepsilon^j+\varepsilon KY} f_\varepsilon\Big(\frac{x}{\varepsilon}, Dw_\varepsilon\Big)\,dx + o(1)\\
\le\ &\sum_{i=1}^N \limsup_{\varepsilon\to 0}\sum_{j\in J_K^\varepsilon}\int_{(x_\varepsilon^j+\varepsilon KY)\cap\varepsilon E_i^\varepsilon} f_\varepsilon\Big(\frac{x}{\varepsilon}, Du_i^K\Big(\frac{x-x_\varepsilon^j}{\varepsilon}\Big)+\lambda_i\Big)\,dx\\
&+\limsup_{\varepsilon\to 0}\sum_{j\in J_K^\varepsilon}\frac{1}{\varepsilon^p}\int_{(x_\varepsilon^j+\varepsilon KY)\cap\varepsilon\omega_\varepsilon} f_\varepsilon\Bigg(\frac{x}{\varepsilon}, \sum_{i=1}^N \varepsilon\Bigg(\nabla\varphi_i^\varepsilon\Big(\frac{x}{\varepsilon}\Big)u_i^K\Big(\frac{x-x_\varepsilon^j}{\varepsilon}\Big)\\
&\qquad +\varphi_i^\varepsilon\Big(\frac{x}{\varepsilon}\Big)Du_i^K\Big(\frac{x-x_\varepsilon^j}{\varepsilon}\Big)\Bigg) + Dv_\varepsilon^j\Big(\frac{x-x_\varepsilon^j}{\varepsilon}\Big) + D\widehat{v}_\varepsilon\Big(\frac{x}{\varepsilon}\Big)(x-x_\varepsilon^j) + \widehat{v}_\varepsilon\Big(\frac{x}{\varepsilon}\Big)\Bigg)\,dx + o(1)\\
\le\ &\sum_{i=1}^N \limsup_{\varepsilon\to 0}\sum_{j\in J_K^\varepsilon}\varepsilon^d\int_{KY\cap E_i^\varepsilon} f_\varepsilon(y, Du_i^K+\lambda_i)\,dy\\
&+\limsup_{\varepsilon\to 0}\sum_{j\in J_K^\varepsilon}\frac{\varepsilon^d}{\varepsilon^p}\int_{KY\cap\omega_\varepsilon} f_\varepsilon(y, Dv_\varepsilon^j)\,dx + o(1).
\end{aligned}
$$

The $o(1)$ in the last line of the latter formula holds true for the following reasons. Firstly, the energy term with u_i^K in the fourth line of the formula is small by combining the Poincaré-Wirtinger inequality (2.32), limit (2.31) and estimate (4.15), so is the term with Du_i^K thanks to the factor ε. Secondly, the term with $D\widehat{v}_\varepsilon$ in the fifth line of the formula is small thanks to estimate (4.18) and to $|x-x_\varepsilon^j|\le K\varepsilon$. Thirdly, the last term with $\widehat{v}_\varepsilon$ in the fifth line of the formula is controlled by considering the difference $\widehat{v}_\varepsilon - \sum_{h=1}^N \varphi_h^\varepsilon$ and by using the Poincaré inequality (2.34), estimate (4.18) and limit (2.31).

Using (4.15) and (4.16) we then obtain

$$
\begin{aligned}
\limsup_{\varepsilon\to 0}\mathcal{F}(u_\varepsilon) \ \le\ & \sum_{i=1}^N \limsup_{\varepsilon\to 0}\sum_{j\in J_K^\varepsilon}\varepsilon^d K^d(f_{\rm hom}^i(\lambda_i)+\delta)\\
&+\limsup_{\varepsilon\to 0}\sum_{j\in J_K^\varepsilon}\varepsilon^d K^d(\Phi_\varepsilon^K(\lambda_1 x_\varepsilon^j,\dots,\lambda_N x_\varepsilon^j)+\delta)+o(1)\\
\le\ & \sum_{i=1}^N |\Omega| f_{\rm hom}^i(\lambda_i) + \limsup_{\varepsilon\to 0}\int_\Omega \Phi_\varepsilon^K(\overline{u}_\varepsilon) + 2\delta|\Omega| + o(1),
\end{aligned}
$$

where $\overline{u}_\varepsilon(x) = (\lambda_1 x_\varepsilon^j,\dots,\lambda_N x_\varepsilon^j)$ if $x\in x_\varepsilon^j+\varepsilon KY$. Taking into account the uniform convergence of $\overline{u}_\varepsilon$ to u and that of Φ_K^ε to Φ (which is due to the Lipschitz condition (2.23) combined with the Ascoli theorem), and letting $\varepsilon\to 0$ and then $K\to+\infty$, and by the arbitrariness of δ, we eventually obtain

$$
\limsup_{\varepsilon\to 0}\mathcal{F}(u_\varepsilon) \le \sum_{i=1}^N \int_\Omega f_{\rm hom}^i(Du_i)\,dx + \int_\Omega \Phi(u_1,\dots,u_N)\,dx
$$

as desired.

4.4 An example

This section contains the proof of Proposition 2.9, illustrating Example 2.1. The proof is divided in three steps. In the first step we define three cut-off functions which satisfy conditions (2.30) and (2.31). In the second step we prove that these functions satisfy the Poincaré type inequalities (2.32), (2.33) and (2.34). In the third step we determine the domain D (2.20).

First step : Definition of the cut-off functions.

Let $0 < R < R'$ and let $\psi_{R,R'}$ be the function defined on $[R, R']$ by

$$\psi_{R,R'}(r) := \frac{R' - r}{R' - R} \quad \text{for } r \in [R, R'].$$

Let $k \in \{1, 2, 3\}$ and set $r := \max\limits_{j \neq k} |y_j|$. Let $\psi_1^{k,\varepsilon}$ and $\psi_3^{k,\varepsilon}$ be the periodic functions in $W^{1,\infty}_{\#}(Y)$ defined on Y by

$$\psi_1^{k,\varepsilon}(y) := \begin{cases} 0 & \text{if } r > R_1 \\ 1 & \text{if } r < R_1 - r_1^\varepsilon \\ \psi_{R_1 - r_1^\varepsilon, R_1}(r) & \text{if } R_1 - r_1^\varepsilon \leq r \leq R_1, \end{cases}$$

$$\psi_3^{k,\varepsilon}(y) := \begin{cases} 1 & \text{if } r > R_2 \\ 0 & \text{if } r < R_2 - r_2^\varepsilon \\ 1 - \psi_{R_2 - r_2^\varepsilon, R_2}(r) & \text{if } R_2 - r_2^\varepsilon \leq r \leq R_2. \end{cases}$$

Then, we define the cut-off functions $\varphi_1^\varepsilon, \varphi_2^\varepsilon, \varphi_3^\varepsilon$ by

$$\begin{cases} \varphi_1^\varepsilon := 1 - (1 - \psi_1^{1,\varepsilon})\,(1 - \psi_1^{2,\varepsilon})\,(1 - \psi_1^{3,\varepsilon}) \\ \varphi_3^\varepsilon := \psi_3^{1,\varepsilon}\,\psi_3^{2,\varepsilon}\,\psi_3^{3,\varepsilon} \\ \varphi_2^\varepsilon := (1 - \varphi_1^\varepsilon)\,(1 - \varphi_3^\varepsilon). \end{cases} \tag{4.25}$$

By taking into account the conditions $\alpha_i^\varepsilon\,(r_i^\varepsilon)^{1-p} \to 0$ for $i = 1, 2$, it is clear that $\varphi_1^\varepsilon, \varphi_2^\varepsilon, \varphi_3^\varepsilon$ satisfy (2.30) and (2.31).

Second step : Proof of the Poincaré type inequalities.

The proof of the weighted Poincaré-Wirtinger inequality (2.32) is based on the following one:

Let Q_ε be the thick cylinder of axis Oy_3 and of (outer and inner) radii $(R - r_\varepsilon, R)$ with $0 < R < \frac{1}{2}$ and $r_\varepsilon \to 0$, defined by

$$Q_\varepsilon := \{y \in Y \ : \ R - r_\varepsilon < \max\left(|y_1|, |y_2|\right) < R\},$$

and let Q be the thick cylinder of axis Oy_3 and of radii $(R - r_\varepsilon, R')$ or $(R + R', R)$ with $0 < R' < \min\left(R, \frac{1}{2} - R\right)$. Then, there exists a constant $C > 0$ independent of ε such that

$$\forall\, v \in W^{1,p}(Y), \qquad \int_{Q_\varepsilon} \left| v - \fint_Q v \right|^p dy \leq C \left((r_\varepsilon)^p \int_{Q_\varepsilon} |\nabla v|^p\, dy + r_\varepsilon \int_Q |\nabla v|^p\, dy \right). \tag{4.26}$$

Let us now prove (4.26) when Q is the cylinder of radii $(R + R', R)$ for example. We can assume that Q and Q_ε have a circular section thanks to a change of variables. Let $\Gamma \subset Y$ be the cylinder of height 1 and of radius R between Q and Q_ε, let $v \in C^1(\overline{Y})$ and denote by $v_{|\Gamma}$ the trace of v on Γ. We have the following inequality

$$\begin{aligned} &\int_{Q_\varepsilon} \left| v - \fint_Q v \right|^p dy \\ \leq\ & c_p \left(\int_{Q_\varepsilon} |\, v - v_{|\Gamma}\,|^p\, dy + \int_{Q_\varepsilon} \left| v_{|\Gamma} - \fint_\Gamma v \right|^p dy + \int_{Q_\varepsilon} \left| \fint_\Gamma v - \fint_Q v \right|^p dy \right). \end{aligned} \tag{4.27}$$

By a Poincaré-Wirtinger type inequality we have

$$\int_{Q_\varepsilon} \left| \fint_\Gamma v - \fint_Q v \right|^p dy = |Q_\varepsilon| \left| \fint_\Gamma v - \fint_Q v \right|^p dy \le c\, r_\varepsilon \int_Q |\nabla v|^p \, dy.$$

We pass in polar coordinates for evaluating the two first terms of the right-hand side of (4.27). Since $v_{|\Gamma}$ does not depend on the radial coordinate, using successively the embedding of $W^{1,p}(Q)$ into $L^p(\Gamma)$ and a Poincaré-Wirtinger type inequality in Q yields

$$\int_{Q_\varepsilon} \left| v_{|\Gamma} - \fint_\Gamma v \right|^p dy \le c\, r_\varepsilon \int_\Gamma \left| v_{|\Gamma} - \fint_\Gamma v \right|^p dy \le c\, r_\varepsilon \left\| v - \fint_\Gamma v \right\|^p_{W^{1,p}(Q)} \le c\, r_\varepsilon \int_Q |\nabla v|^p \, dy.$$

Moreover, we have for any $y \in Q_\varepsilon$,

$$v(y) - v_{|\Gamma}(y) = v(r, \theta, y_3) - v(R, \theta, y_3) = \int_R^r \frac{\partial v}{\partial \rho}(r, \theta, y_3)\, d\rho,$$

whence by using the Hölder inequality

$$\begin{aligned}
\int_{Q_\varepsilon} \left| v - v_{|\Gamma} \right|^p dy &\le \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_0^{2\pi} \int_{R - r_\varepsilon}^R \left(\int_r^R \rho^{-\frac{p'}{p}} \, d\rho \right)^{\frac{p}{p'}} \left(\int_r^R \left| \frac{\partial v}{\partial \rho}(r, \theta, y_3) \right|^p \rho d\rho \right) r dr d\theta dy_3 \\
&\le c \left(\int_{R - r_\varepsilon}^R (R - r)^{p-1} \, dr \right) \int_{Q_\varepsilon} |\nabla v|^p \, dy \\
&\le c\, (r_\varepsilon)^p \int_{Q_\varepsilon} |\nabla v|^p \, dy.
\end{aligned}$$

Putting the three previous inequalities in (4.27) yields the desired inequality (4.26). Similarly, we can prove the following Poincaré type inequality

$$\forall\, v \in W^{1,p}(Y), \ v = 0 \text{ on } \partial Q \cap \partial Q_\varepsilon, \quad \int_{Q_\varepsilon} |v|^p \, dy \le C\, (r_\varepsilon)^p \int_{Q_\varepsilon} |\nabla v|^p \, dy. \tag{4.28}$$

Let us prove inequality (2.32) for $i = 1$, the cases $i = 2, 3$ being quite similar. Let $v \in W^{1,p}(Y)$. For each $k \in \{1, 2, 3\}$, let $Q_1^{k,\varepsilon}$ and $Q_\varepsilon^{k,\varepsilon}$ be the thick cylinders defined by

$$\begin{cases}
Q_1^{k,\varepsilon} & := \left\{ y \in Y \ : \ \max\limits_{j \ne k} |y_j| \le R_1 - r_1^\varepsilon \right\} \qquad = \left\{ y \in Y \ : \ \psi_1^{k,\varepsilon}(y) = 1 \right\} \\
Q_\varepsilon^{k,\varepsilon} & := \left\{ y \in Y \ : \ R_1 - r_1^\varepsilon < \max\limits_{j \ne k} |y_j| < R_1 \right\} \ = \left\{ y \in Y \ : \ 0 < \psi_1^{k,\varepsilon}(y) < 1 \right\}.
\end{cases}$$

By using the definition (4.25) of the function φ_1^ε in terms of the functions $\psi_1^{k,\varepsilon}$ and the fact that $|\nabla \psi_1^{k,\varepsilon}| \le c\, (r_1^\varepsilon)^{-1}$, we have

$$\begin{aligned}
&\int_Y a_\varepsilon \, |\nabla \varphi_1^\varepsilon|^p \left| v - \fint_{Y \cap E_1^\varepsilon} v \right|^p dy \\
&\le c\, \frac{\alpha_1^\varepsilon}{(r_1^\varepsilon)^p} \sum_{k=1}^3 \left(\int_{Q_\varepsilon^{k,\varepsilon}} \left| v - \fint_{Q_1^{k,\varepsilon}} v \right|^p dy + \int_{Q_\varepsilon^{k,\varepsilon}} \left| \fint_{Q_1^{k,\varepsilon}} v - \fint_{Y \cap E_1^\varepsilon} v \right|^p dy \right) \\
&\le c\, \frac{\alpha_1^\varepsilon}{(r_1^\varepsilon)^p} \sum_{k=1}^3 \left(\int_{Q_\varepsilon^{k,\varepsilon}} \left| v - \fint_{Q_1^{k,\varepsilon}} v \right|^p dy + r_1^\varepsilon \left| \fint_{Q_1^{k,\varepsilon}} v - \fint_{Y \cap E_1^\varepsilon} v \right|^p dy \right).
\end{aligned}$$

On the one hand, the Poincaré-Wirtinger inequality (4.26) implies that

$$\int_{Q_\varepsilon^{k,\varepsilon}} \left| v - \fint_{Q_1^{k,\varepsilon}} v \right|^p dy \le c \left((r_1^\varepsilon)^p \int_{Q_\varepsilon^{k,\varepsilon}} |\nabla v|^p dy + r_1^\varepsilon \int_{Q_1^{k,\varepsilon}} |\nabla v|^p dy \right).$$

On the other hand, a Poincaré-Wirtinger type inequality yields

$$\left| \fint_{Q_1^{k,\varepsilon}} v - \fint_{Y\cap E_1^\varepsilon} v \right|^p \le c \int_{Y\cap E_1^\varepsilon} |\nabla v|^p \, dy.$$

By combining the three previous estimates and by the definition (2.10) of ω_1^ε we thus obtain the inequality

$$\int_Y a_\varepsilon \, |\nabla \varphi_1^\varepsilon|^p \left| v - \fint_{Y\cap E_1^\varepsilon} v \right|^p dy \le c \left(\alpha_1^\varepsilon \int_{Y\cap\omega_1^\varepsilon} |\nabla v|^p \, dy + \frac{\alpha_1^\varepsilon}{(r_1^\varepsilon)^{p-1}} \int_{Y\cap E_1^\varepsilon} |\nabla v|^p \, dy \right),$$

which yields inequalities (2.32) and (2.33) for $i = 1$ since α_1^ε and $\alpha_1^\varepsilon \, (r_1^\varepsilon)^{1-p} \to 0$.

Similarly, we deduce from inequality (4.28) the following Poincaré type inequality

$$\forall\, v \in W^{1,p}(Y),\ v = 0 \text{ on } Y \cap \partial E_i^\varepsilon, \quad \int_{Y\cap\omega_\varepsilon} a_\varepsilon \, |v|^p \, dy \le c\, (r_1^\varepsilon + r_2^\varepsilon)^p \int_{Y\cap\omega_\varepsilon} a_\varepsilon \, |\nabla v|^p \, dy,$$

which implies (2.34).

Third step : Determination of the set D.

One the one hand, putting the test function $\varphi_1^\varepsilon \lambda_1 + \varphi_2^\varepsilon \lambda_2 + \varphi_3^\varepsilon \lambda_3$, for a fixed $\lambda \in \mathbb{R}^3$, in the minimization problem (2.16) yields with the definition (4.25) of the functions φ_i^ε and the definition (2.10) of the thin layers ω_j^ε,

$$\begin{aligned}\Phi_\varepsilon^\#(\xi) \quad &\le \frac{\alpha_1^\varepsilon}{\varepsilon^p} \int_{\omega_1^\varepsilon\cap Y} |\, \nabla \left(\varphi_1^\varepsilon \xi_1 + (1-\varphi_1^\varepsilon)\xi_2\right)|^p \, dy + \frac{\alpha_2^\varepsilon}{\varepsilon^p} \int_{\omega_2^\varepsilon\cap Y} |\, \nabla \left((1-\varphi_3^\varepsilon)\xi_2 + \varphi_3^\varepsilon \xi_3\right)|^p \, dy \\ &\le c \left(\frac{\alpha_1^\varepsilon}{\varepsilon^p \, (r_1^\varepsilon)^{p-1}} \, |\xi_1 - \xi_2|^p + \frac{\alpha_2^\varepsilon}{\varepsilon^p \, (r_2^\varepsilon)^{p-1}} \, |\xi_2 - \xi_3|^p \right)\end{aligned} \tag{4.29}$$

On the other hand, let Γ_1 and Γ_1^ε be the cylinder of axis Oy_1 defined by

$$\begin{cases} \Gamma_1 & := \left\{ y \in Y \ : \ \max\left(|y_2|, |y_3|\right) = R_1 \ \text{ and } \ R_2 < y_3 < \frac{1}{2} \right\} \\ \Gamma_1^\varepsilon & := \left\{ y \in Y \ : \ \max\left(|y_2|, |y_3|\right) = R_1 - r_1^\varepsilon \ \text{ and } \ R_2 < y_3 < \frac{1}{2} \right\}. \end{cases}$$

Proceeding as in the proof of inequality (4.26) by passing into polar coordinates and using the Hölder inequality, we obtain that there exists a positive constant C such that

$$\forall\, v \in W^{1,p}(Y), \quad \left| \fint_{\Gamma_1} v - \fint_{\Gamma_1^\varepsilon} v \right|^p \le C\, (r_1^\varepsilon)^{p-1} \int_{\omega_1^\varepsilon\cap Y} |\nabla v|^p \, dy \le C \, \frac{(r_1^\varepsilon)^{p-1}}{\alpha_1^\varepsilon} \int_Y a_\varepsilon \, |\nabla v|^p \, dy.$$

Then, putting the minimizer $v_\varepsilon \in W^{1,p}_\#(Y)$ such that

$$\Phi_\varepsilon^\#(\xi) = \frac{1}{\varepsilon^p} \int_Y a_\varepsilon \, |\nabla v_\varepsilon|^p \, dy \quad \text{with} \quad v_\varepsilon = \xi_i \text{ in } E_i^\varepsilon,$$

in the previous inequality and noting that $v_\varepsilon = \xi_1$ on Γ_1^ε and $v_\varepsilon = \xi_2$ on Γ_1, yield

$$\Phi_\varepsilon^\#(\xi) \ge C^{-1} \, \frac{\alpha_1^\varepsilon}{\varepsilon^p \, (r_1^\varepsilon)^{p-1}} \, |\xi_1 - \xi_2|^p.$$

Similarly, there exists a positive constant c such that

$$\Phi_\varepsilon^\#(\xi) \ge c \, \frac{\alpha_1^\varepsilon}{\varepsilon^p \, (r_1^\varepsilon)^{p-1}} \, |\xi_2 - \xi_3|^p.$$

The two previous estimates combined with (4.29) imply that there exist a constant $c > 1$ such that

$$c^{-1}\,\Phi^{\#}_{\varepsilon}(\xi) \le \frac{\alpha_1^{\varepsilon}}{\varepsilon^p\,(r_1^{\varepsilon})^{p-1}}\,|\xi_1-\xi_2|^p + \frac{\alpha_2^{\varepsilon}}{\varepsilon^p\,(r_2^{\varepsilon})^{p-1}}\,|\xi_2-\xi_3|^p \le c\,\Phi^{\#}_{\varepsilon}(\xi),$$

which combined with the definition (2.20) of D implies (2.36).

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