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Adv. Calc. Var. [redacted] ([redacted]), 1 – 24 DOI 10.1515 / ACV.[redacted]

The Gibbs-Thomson relation for non homogeneous anisotropic phase transitions

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Communicated by [redacted]

Abstract. We prove the Gibbs-Thomson relation between the coarse grained chemical potential and the non homogeneous and anisotropic mean curvature of a phase interface within the gradient theory of phase transitions thus proving a generalization of a conjecture stated by Gurtin and proved by Luckhaus and Modica in the homogeneous and isotropic case.

Keywords. Anisotropic phase transition, Gibbs-Thomson relation for surface tension,Finsler metrics.

AMS classification. 49Q20,35B25,82B26,82B24.

1 Introduction

Interfacial models of phase transitions are a widely studied topic dating back to Lord Rayleigh and Van der Waals and can be grouped into two main classes: diffuse and sharp interface models. Several models belonging to both the classes have been successfully used to describe the formation of a (diffuse or sharp) interface between the solid and the liquid phase of a fluid undergoing a first order phase transition. Thus a very natural issue, which has proven to be challenging for both mathematicians and material scientists, has arisen: to compare the main outcomes of the two approaches. Roughly speaking, letting the thickness of the diffused interface be ε, one is lead to investigate, as ε vanishes (i.e., when the diffused interface becomes sharp), what happens to some of the relevant physical quantities described by the diffuse model and then to understand what is the relation between these coarse grained quantities and their counterparts given by the sharp interface model. In this paper we provide an answer to this type of questions finding a relation, in the case of a non homogeneous and anisotropic material, between the limit, as ε goes to zero, of the chemical potential of the system and the curvature of its sharp interface. Such a relation is commonly known as the *Gibbs-Thomson relation*. A complete answer to the same problem, in the homogeneous and isotropic case, has been provided by Luckhaus and Modica in [16].

Let $\Omega \subset \mathbb{R}^N$ be a given bounded open set representing the region occupied by the physical system and let $u : \Omega \to \mathbb{R}$ be an order parameter (it may indeed be a physical parameter such as the density of mass of the material) its values identifying the state of the system. Given $\alpha < \beta$, the sets $\Omega_\alpha = \{x \in \Omega\,;\, u(x) = \alpha\}$ and $\Omega_\beta = \{x \in \Omega\,;\, u(x) = \beta\}$ correspond to the regions where the α or the β phase is present. When a phase transition occurs, the formation of a thin interfacial layer of small ε thickness separating the two pure bulk phases Ω_α and Ω_β has been successfully described as a result of the minimization of a Ginzburg-Landau type free-energy (with such a choice the model belongs to the so called gradient theory of phase transitions). Under the hypothesis of isotropy of the physical system, for every $u \in W^{1,2}(\Omega)$, a common choice for the free-energy F_ε^{iso} of the system is

$$F_\varepsilon^{iso}(u) = \int_\Omega \varepsilon^2 |Du|^2 + W(T,u)\,dx. \tag{1.1}$$

Here $W : [0,+\infty) \times \mathbb{R} \to [0,+\infty)$, as a function of the order parameter, has a double-well shape with wells in α and β whenever T is beneath a certain critical temperature T_c. Thus, working in the range of temperature $T < T_c$, and looking at isothermal phenomena, one usually drops the dependence on the temperature replacing $W(T,u)$ by $W(u)$ in (1.1) and considers the problem of finding the equilibrium states of the system by minimizing F_ε^{iso} subject to a mean-type constraint on u that can be regarded as a constraint of constant mass if we think of u as the density of the system. For any given proportion $m \in (\alpha,\beta)$ of the mass of two phases, the coarse grained interfacial energy m_ε^{iso} of the system is then obtained by solving the following variational problem

$$m_\varepsilon^{iso} = \min \left\{ F_\varepsilon^{iso}(u);\ \int_\Omega u = m|\Omega| \right\}. \tag{1.2}$$

The issue of the convergence, as ε goes to zero, of m_ε^{iso} to what can be considered the interfacial energy of the system has been solved by Modica and Mortola in two celebrated papers ([18], [19]). In particular they proved that

$$\Gamma\text{-}\lim_{\varepsilon\to 0} \frac{F_\varepsilon^{iso}(u)}{\varepsilon} = 2\theta \mathcal{H}^{N-1}(S(u)) =: F_0^{iso}(u),$$

where $S(u)$ denotes the jump set of the function $u \in \mathrm{BV}(\Omega;\{\alpha,\beta\})$ which parametrizes the limiting interface, $\theta = \int_\alpha^\beta \sqrt{W(t)}\,dt$ is a constant representing the surface tension of the system and $\mathcal{H}^{N-1}$ denotes the $(N-1)$-dimensional (Hausdorff) surface measure. With such a result proved the authors were able to conclude that, as $\varepsilon \to 0$, $m_\varepsilon^{iso} \to m_0^{iso}$ where

$$m_0^{iso} = \min \left\{ F_0^{iso}(u);\ u \in \mathrm{BV}(\Omega;\{\alpha,\beta\}),\ \int_\Omega u = m|\Omega| \right\}$$

and that, given $u_\varepsilon \to u_0$ such that $\frac{F_\varepsilon^{iso}(u_\varepsilon)}{\varepsilon} - m_\varepsilon^{iso} \to 0$, then $F_0^{iso}(u_0) = m_0$.

After this result was proved, an interesting issue to be addressed was related to the asymptotic behavior of the chemical potential of the system in the limit as $\varepsilon \to 0$. To introduce this problem let us suppose u to be a regular function minimizing (1.2). Then u solves the Euler-Lagrange equation

$$\varepsilon^2 \Delta u - W'(u) - \lambda_\varepsilon^{iso} = 0$$

where $\lambda_\varepsilon^{iso}$ is the Lagrange multiplier due to the volume constraint. On the other hand $\lambda_\varepsilon^{iso}$ represents the chemical potential of the system under transition (see [15], [26]). Concerning the asymptotic behavior of the chemical potentials, Luckhaus and Modica in [16] gave a positive answer to a conjecture made by Gurtin in [15]. They proved the *Gibbs-Thomson relation*

$$\lim_{\varepsilon \to 0} \frac{\lambda_\varepsilon^{iso}}{\varepsilon} = 2\theta \lambda_0^{iso}$$

where λ_0^{iso} is the mean curvature of the interface.

The aim of this paper is to prove an analogous result in the case when non homogeneous anisotropic models are taken into account. Following Taylor [25] (see also [26]), the energy F_ε^{an} of such a model in the Van der Walls-Cahn-Hilliard theory is given, for any $u \in W^{1,2}(\Omega)$ by

$$F_\varepsilon^{an}(u) = \int_\Omega \varepsilon^2 f(x, Du) + W(u)\, dx, \tag{1.3}$$

the hypotheses on f and W depending on the specific physical system one wants to model. In the present paper, to exploit the standard method of Cahn-Hoffman vector fields (see [11],[26]) we restrict ourselves to the case when $f \in C^2(\Omega \times (\mathbb{R}^N); [0, +\infty))$ satisfies standard growth condition of order 2 with respect to ξ and is such that $\sqrt{f}$ is a strictly convex Finsler norm. Moreover we will suppose that $W \in C^3(\mathbb{R}; [0, +\infty))$ is a double well potential with wells in α and β and that it satisfies $p > 2$ standard growth conditions (see Remark 3.4). In this setting the equilibrium state of the system can be found by minimizing

$$m_\varepsilon^{an} := \min \left\{ F_\varepsilon^{an}(u);\ \int_\Omega u = m|\Omega| \right\}. \tag{1.4}$$

with $m \in (\alpha, \beta)$. It has been proved by Bouchitté in [9] (see also [5]) that

$$\Gamma\text{-} \lim_{\varepsilon \to 0} \frac{F_\varepsilon^{an}(u)}{\varepsilon} = \int_{S(u)} \sqrt{f(x, \nu(x))}\, d\mathcal{H}^{N-1} =: F_0^{an}(u),$$

where ν is the measure theoretic inner normal to $S(u)$. Analogously to the isotropic case, $m_\varepsilon^{an} \to m_0^{an}$ where

$$m_0^{an} = \min \left\{ F_0^{an}(u);\ u \in \mathrm{BV}(\Omega; \{\alpha, \beta\}),\ \int_\Omega u = m|\Omega| \right\}.$$

Moreover, given $u_\varepsilon \to u_0$ such that $\frac{F_\varepsilon^{an}(u_\varepsilon)}{\varepsilon} - m_\varepsilon^{an} \to 0$, then $u_0 \in \mathrm{BV}(\Omega; \{\alpha, \beta\})$, $\int_\Omega u_0 = m|\Omega|$ and $F_0^{an}(u_0) = m_0$. The Euler-Lagrange equation, for a regular u, now can be written as

$$\varepsilon^2 \operatorname{div} f_\xi(x, \nabla u) - W'(u) - \lambda_\varepsilon^{an} = 0$$

and the analog of the Gurtin's conjecture in this case can be phrased by saying that the scaled chemical potentials $\frac{\lambda_\varepsilon^{an}}{\varepsilon}$ converge, up to a multiplicative constant, to the non homogeneous and anisotropic curvature of the limit interface as ε goes to 0.

This problem has been addressed by several authors and partial results are known in the homogeneous case (see [17], [26] and references therein). Instead, we prove this generalized version of the Gurtin's conjecture in the non homogeneous and anisotropic setting. To this end we follow the main steps of the proof by Luckhaus and Modica. However we point out that, working in the framework of Finsler metrics, we cannot take advantage of the linearity of the Euler-Lagrange equation as in the isotropic case and instead our analysis relies on more abstract properties of the minimizers of the problem (1.4) as well as on some results in geometric measure theory. In particular, among the difficulties that we have to overcome, we need to generalize the statement of the Reshetnyak continuity theorem (see [22]) to the Finsler setting. We also remark that our statement is different from the statement of the main result in [16]. On one hand, we do not need to assume an L^∞ bound on the sequence of minimizers of (1.4), on the other hand we do not assume the boundedness of the sequence of chemical potentials λ_ε^{an} since, by supposing higher regularity of W (see Remark 3.4), we are able to prove it by adapting a result by X. Chen in [12].

As a final comment we remark that, as the result by Luckhaus and Modica suggests the validity of the conjecture that De Giorgi stated in [13] and that has been proved (in a modified form) in [23] in dimension $N \leq 3$ and independently in [21] in dimension $N = 2$, our main result suggests the validity of the same conjecture in the anisotropic setting. However, we would stress the fact that it is not possible to attack such a problem by exploiting the same arguments present in the previously quoted papers since some of the key ingredients (e.g., a monotonicity formula for the energy density (see [21] Theorem 3.12)) of the proofs are not yet available in the Finsler setting.

The paper is organized as follows: in Section 2 we briefly review the definition and the main properties of Finsler metrics and of anisotropic perimeters needed to set up our problem. Section 3 is then devoted to the proof of our main result.

Acknowledgements The authors wish to thank Yoshihiro Tonegawa and Giovanni Bellettini for interesting discussions on the subject of this paper. The work by Marco Cicalese was partially supported by the European Research Council under FP7, Advanced Grant n. 226234 “Analytic Techniques for Geometric and Functional Inequalities”. The work by Yuko Nagase was supported by “Progetto Mecenas, Università di Napoli Federico II and Compagnia di S. Paolo".

2 Notation and preliminaries

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary. Given $E \subset \mathbb{R}^N$, we will write $E \in \mathcal{C}^2_b(\mathbb{R}^N)$ if E is a bounded open subset of $\mathbb{R}^N$ of class C^2. Moreover we will denote by ν_E the inner unit normal vector field to its boundary ∂E. For any given $a, b \in \mathbb{R}^N$ we denote by $a \cdot b$ and $a \otimes b$ the scalar and the tensor product between a and b, respectively, and by $|a|$ the norm of a. We denote by S^{N-1} the unit sphere in $\mathbb{R}^N$. We also denote by c a positive constant which may vary from line to line.

In the following section we introduce the Finsler setting. We refer the reader to [8] and the references therein for details.

2.1 Finsler Metrics

Let $\phi : \Omega \times \mathbb{R}^N \to [0, +\infty)$ be a continuous function satisfying the following properties:

$$\phi(x, t\xi) = |t|\phi(x, \xi)\ , \qquad x \in \Omega\ ,\ \xi \in \mathbb{R}^N\ ,\ t \in \mathbb{R}\ ; \tag{2.1}$$

$$\lambda|\xi| \le \phi(x, \xi) \le \Lambda|\xi|\ , \qquad x \in \Omega\ ,\ \xi \in \mathbb{R}^N\ . \tag{2.2}$$

We say that ϕ is strictly convex if for any $x \in \Omega$ the map $\xi \mapsto \phi^2(x, \xi)$ is strictly convex on $\mathbb{R}^N$. We denote by $\phi^\circ : \Omega \times \mathbb{R}^N \to [0, +\infty)$ the dual function of ϕ defined as

$$\phi^\circ(x, \xi^*) = \sup \left\{ \frac{\xi^* \cdot \xi}{\phi(x, \xi)} \ ;\ \xi \in S^{N-1} \right\}$$

for any $x \in \Omega$ and $\xi^* \in \mathbb{R}^N$.

We say that ϕ is a *strictly convex smooth Finsler norm*, and we write $\phi \in \mathcal{N}(\Omega)$, if in addition to properties (2.1) and (2.2), ϕ and ϕ° are strictly convex and of class $C^2(\Omega \times (\mathbb{R}^N \setminus \{0\}))$. The following two sets

$$\begin{aligned} B_\phi(x) &= \{\xi \in \mathbb{R}^n |\ \phi(x, \xi) \le 1\}, \\ B_{\phi^\circ}(x) &= \{\xi^* \in \mathbb{R}^n |\ \phi^\circ(x, \xi^*) \le 1\} \end{aligned}$$

will be, as usual, referred to as *Wulff shape* and *Frank diagram*, respectively.

We recall that the ϕ*-vector* $\nu_\phi(x)$ and the *Cahn-Hoffman vector* $n_\phi(x)$, associated to a unit vector $\nu \in S^{N-1}$, are defined as

$$\nu_\phi(x) = \frac{\nu}{\phi^\circ(x, \nu)} \quad \text{and} \quad n_\phi(x) = \phi^\circ_\xi(x, \nu_\phi).$$

By the elementary properties of Finsler norms it holds that, for any $x \in \Omega$ and $\xi, \xi^* \in \mathbb{R}^N \setminus \{0\}$,

$$\phi(x, \phi^\circ_{\xi^*}(x, \xi^*)) = \phi^\circ(x, \phi_\xi(x, \xi)) = 1. \tag{2.3}$$

As a consequence of (2.3), for any $x \in \Omega$ it holds that $\nu_\phi(x) \in \partial B_\phi(x)$, $n_\phi(x) \in \partial B_{\phi^\circ}(x)$ and

$$\nu_\phi(x) \cdot n_\phi(x) = 1. \tag{2.4}$$

Let $E \in \mathcal{C}^2_b(\mathbb{R}^N)$ and, for any $x \in \partial E$, let $\nu(x)$ be the unit inner normal to ∂E at x. For a given C^1 vector field $X : \partial E \to \mathbb{R}^N$ we denote by $\operatorname{div}_\phi X$ the *ϕ-tangential divergence* of X on ∂E defined as

$$\operatorname{div}_\phi X = \operatorname{tr}\left[(\mathrm{Id} - n_\phi \otimes \nu_\phi)\nabla \widetilde{X} + \phi^\circ_x(x, \nu_\phi) \otimes \widetilde{X}\right],$$

where $\widetilde{X}$ is any smooth extension of X to a neighborhood of ∂E.

Extending to a neighborhood of ∂E the vector fields ν_ϕ and n_ϕ by regular fields while keeping the same notation, we are in a position to define the *ϕ-mean curvature* κ_ϕ of ∂E as

$$\kappa_\phi = -\operatorname{div}_\phi n_\phi \; .$$

Differentiating $\phi^\circ(x, \nu_\phi) = \phi(x, n_\phi) = 1$ with respect to x_i and exploiting (2.4) one obtains that the following relations hold on ∂E:

$$\phi^\circ_{x_i}(x, \nu_\phi) + n^j_\phi \frac{\partial \nu^j_\phi}{\partial x_i} = 0 \; , \quad i = 1, \ldots, N, \tag{2.5}$$

$$\phi_{x_i}(x, n_\phi) + \nu^j_\phi \frac{\partial n^j_\phi}{\partial x_i} = 0 \; , \quad i = 1, \ldots, N, \tag{2.6}$$

$$n^j_\phi \frac{\partial \nu^j_\phi}{\partial x_i} + \nu^j_\phi \frac{\partial n^j_\phi}{\partial x_i} = 0 \; , \quad i = 1, \ldots, N. \tag{2.7}$$

In the proof of our main result we will apply the following generalization of the divergence theorem on manifolds in the Finsler setting whose proof is a consequence of the integral representation formula for the first variation of the ϕ-anisotropic perimeter (defined in (2.8)) in terms of the ϕ-mean curvature (see (2.3) and (3.2) in [6]).

Theorem 2.1. *Let $E \in \mathcal{C}^2_b(\mathbb{R}^N)$. Let $U \subset \mathbb{R}^N$ be a neighborhood of ∂E and $g \in C^1_0(U; \mathbb{R}^N)$. Then*

$$\int_{\partial E} \kappa_\phi \nu_\phi \cdot g \, \phi^\circ(x, \nu) \, d\mathcal{H}^{N-1} = -\int_{\partial E} \operatorname{div}_\phi g \, \phi^\circ(x, \nu) \, d\mathcal{H}^{N-1}.$$

2.2 BV-functions and anisotropic perimeters

In this section we recall the basic definitions of BV functions (for more details on the subject we refer the reader to [3]) and then introduce the notion of anisotropic perimeter. We end the section by recalling some well known results in geometric measure theory that will be used in the proof of our main result.

Given a vector-valued measure μ on Ω, we denote by $|\mu|$ its total variation and we adopt the notation $\mathcal{M}(\Omega)$ for the set of all signed measures on Ω with bounded total variation. The Lebesgue measure of a set E is denoted by $|E|$. The Hausdorff $(N-1)$-dimensional measure on $\mathbb{R}^N$ is denoted by $\mathcal{H}^{N-1}$.

We recall that $u \in L^1(\Omega)$ belongs to the space $BV(\Omega)$ of functions of bounded variation if its distributional derivatives $D_i u$ belong to $\mathcal{M}(\Omega)$. We denote by Du the $\mathbb{R}^N$-valued measure whose components are $D_1 u, \ldots, D_n u$.

We say that a set E is of *finite perimeter* in Ω if its characteristic function $\chi_E \in BV(\Omega)$ and we denote by $P(E) = |D\chi_E|(\Omega)$ the *perimeter* of E in Ω. The family of sets of finite perimeter can be identified with the functions $u \in BV(\Omega; \{0,1\})$. For such functions Du can be represented as

$$Du(B) = \int_{S(u)\cap B} \nu_u \, d\mathcal{H}^{N-1}$$

for every Borel set $B \subset \Omega$, where $S(u)$ denotes the complement of the set of Lebesgue points of u and $\nu_u \in \mathbb{R}^N$ is the measure theoretic inner normal to $S(u)$. It holds that for $E = \{x\,;\ u(x) = 1\}$

$$P(E) = |Du|(\Omega) = \mathcal{H}^{N-1}(S(u) \cap \Omega).$$

The following proposition is a particular case of the chain-rule formula in $BV(\Omega)$.

Proposition 2.2. *Let $\alpha < \beta$, let $h : \mathbb{R} \to \mathbb{R}$ be a C^1 function and let $u \in \mathrm{BV}(\Omega; \{\alpha, \beta\})$. Then*

$$Dh(u) = (h(\beta) - h(\alpha))\nu_u \mathcal{H}^{N-1}|_{S(u)}.$$

We now recall the definitions and some properties of the anisotropic total variation for BV-functions and introduce the anisotropic perimeter (for further details we refer the reader to [2]). Let $u \in \mathrm{BV}(\Omega)$ and $\phi \in \mathcal{N}(\Omega)$. We define the *anisotropic ϕ-total variation* of Du as

$$|Du|_\phi(\Omega) = \sup\left\{ \int_\Omega u \ \operatorname{div} \sigma \, dx \ ; \ \sigma \in C_0^1(\Omega; \mathbb{R}^N) \ , \ \ \sigma(x) \in B_\phi(x) \right\}.$$

We observe that by the hypotheses on ϕ we can deduce from Theorem 5.1 in [2] that the ϕ-total variation is $L^1(\Omega)$-lower semicontinuous and admits the following integral representation

$$|Du|_\phi(\Omega) = \int_\Omega \phi^\circ(x, \nu_u) d|Du| \ \ , \quad \forall\, u \in \mathrm{BV}(\Omega).$$

Note that if $\phi(x,\xi) = |\xi|$ then the ϕ-total variation $|Du|_\phi(\Omega)$ agrees with $|Du|(\Omega)$.

We now recall the definition and some properties of anisotropic perimeters. We will follow the notation of [7]. Let $E \subset \mathbb{R}^N$ be a set of finite perimeter in Ω and let $\phi \in \mathcal{N}(\Omega)$, we define the ϕ-*anisotropic perimeter* of E in Ω as

$$P_\phi(E) = \int_{\partial^* E \cap \Omega} \phi^\circ(x, \nu(x))\, d\mathcal{H}^{N-1}, \tag{2.8}$$

where $\partial^* E$ is the reduced boundary of E and ν is the measure theoretic unit inner normal to ∂E. We observe that the ϕ-total variation of χ_E agrees with the ϕ-anisotropic perimeter of E in Ω, that is,

$$P_\phi(E) = |D\chi_E|_\phi(\Omega).$$

We warn the reader that the definition of the ϕ-anisotropic perimeter is sometimes given with ϕ in place of ϕ°.

We now state two useful propositions from geometric measure theory that we will use in the proof of our main result. Their proofs can be found in [10] and in [4], respectively. In the sequel we will denote by $\mathcal{A}(\Omega)$ the class of open subsets of Ω. Moreover we recall that a Radon space is a topological space such that every finite Borel measure is inner regular.

Proposition 2.3 (supremum of a family of measures). *Let $\mu : \mathcal{A}(\Omega) \to [0, +\infty)$ be superadditive on open sets with disjoint compact closures, let λ be a positive measure, $\{\psi_i\}_i$ be positive Borel functions such that*

$$\mu(A) \geq \int_A \psi_i \, d\lambda$$

for all open set $A \subset \Omega$.

Then for all open set $A \subset \Omega$ we have

$$\mu(A) \geq \int_A \sup_i \psi_i(x)\, d\lambda.$$

Theorem 2.4 (disintegration of a measure). *Let X, Y be Radon separable metric spaces, $\mu \in \mathcal{P}(X)$, let $\pi : X \to Y$ be a Borel map and let $\lambda = \pi_\sharp \mu \in \mathcal{P}(Y)$. Then there exist a λ-a.e. uniquely determined Borel family of probability measures $\{\mu_y\}_{y \in Y} \subset \mathcal{P}(X)$ such that*

$$\mu_y(X \setminus \pi^{-1}(y)) = 0 \;\; \text{for } \lambda\text{-a.e. } y \in Y$$

and

$$\int_X f(x)\, d\mu(x) = \int_Y \left(\int_{\pi^{-1}(y)} f(x) d\mu_y(x) \right) d\lambda(y)$$

for every Borel map $f : X \to [0, +\infty]$.

3 The Gibbs-Thomson relation

In this section we briefly review the gradient theory of non homogeneous and anisotropic phase transitions in order to present the Gibbs-Thomson relation and state our main result. We warn the reader that, in what follows, we have made the choice of not presenting some well known results, due to other authors, in their full generality. Instead, we will state them under more strong hypotheses which best fit in our setting.

In the physical literature several theories are available to model the formation of transition layers between the pure phases of a system which undergoes a phase transition. Among them the Van der Waals-Cahn-Hillard gradient theory is suitable to introduce, in a rigorous mathematical way, the so called *coarse-grained chemical potential* and to state the *Gibbs-Thomson* relation.

Let $\alpha < \beta$ and let $u_0 \in \mathrm{BV}(\Omega; \{\alpha, \beta\})$ be an order parameter of a physical system which is subject to the volume constraint $\int_\Omega u_0\, dx = m|\Omega|$ for some $m \in (\alpha, \beta)$ and that is undergoing a first order phase transition between the phases α and β. In what follows we set $\Omega_\alpha := \{x \in \Omega\,;\ u_0(x) = \alpha\}$ and $\Omega_\beta := \{x \in \Omega\,;\ u_0(x) = \beta\}$ and we agree to identify the "interface" of transition of the system with the jump set $S(u_0)$ of u_0 that is the reduced boundary of the set Ω_α.

In the Van der Waals-Cahn-Hilliard theory the sharp interface $S(u_0)$ is replaced by a diffused interface. This can be seen, for a given $0 < \sigma < \frac{\beta-\alpha}{2}$ as the set $\{x \in \Omega\,;\ u_\varepsilon(x) \in (\alpha+\sigma, \beta-\sigma)\}$, where, as ε varies, u_ε is the order parameter of a family of equilibrium states for the physical system and minimizes the following Ginzburg-Landau type energy

$$\mathcal{E}_\varepsilon(u) = \int_\Omega \varepsilon^2 f(x, Du) + W(u)\, dx,$$

subject to the volume constraint

$$(u)_\Omega = \frac{1}{|\Omega|} \int_\Omega u\, dx = m, \tag{3.1}$$

for some $f : \Omega \times \mathbb{R}^N \to [0, +\infty)$ two homogeneous in the gradient variable and some double-well potential $W : \mathbb{R} \to [0, +\infty)$ vanishing only in α and β. By assuming suitable regularity and growth hypotheses on f and W, a first goal of this theory is to prove that the thickness of the diffused interface is of order ε and that it converges to $S(u_0)$ as $\varepsilon \to 0$.

In this framework, by looking at u_ε as the density of mass of the system, the chemical potential λ_ε of the state identified by u_ε is defined as the variation of the internal energy with respect to the mass. Thus it is the Lagrange multiplier associated to the minimization of $\mathcal{E}_\varepsilon$ subject to the constraint (3.1). Supposing f, W and u_ε enough regular, one can write the Euler-Lagrange equation for such a problem to find that u_ε solves

$$\lambda_\varepsilon = \varepsilon^2 \operatorname{div} f_\xi(x, Du_\varepsilon) - W'(u_\varepsilon). \tag{3.2}$$

By recalling that the *Gibbs-Thomson* relation for a thermodynamic system with two phases states that the chemical potential is proportional to the curvature of the interface between the phases, we obtain that, in our framework, such a relation is proved if we show that, up to a multiplicative constant, $\frac{\lambda_\varepsilon}{\varepsilon}$ converges, as ε goes to 0, to the (non homogeneous and anisotropic) curvature of the limit interface. To rigorously prove such a result is the goal of the present section. We notice that the problem has been considered by several authors. In particular Luckhaus and Modica in [16] solved the problem in the homogeneous and isotropic case where $f(x,\xi) = |\xi|^2$ thus proving a conjecture by Gurtin (see [15]). Partial answers to this problem have been provided by Braun, Coriell, McFadden, Sekerka and Wheeler in [17] in the case $n \leq 3$ and f homogeneous. Even under these restrictions the authors does not provide a complete proof of the result since their argument relies on a formal asymptotic expansion of the equation (3.2).

In what follows, given $\alpha < \beta$, $f : \Omega \times \mathbb{R}^N \to [0, +\infty)$ a function of class C^2 and $W : \mathbb{R} \to [0, +\infty)$ a function of class C^0, we will consider the following set of hypotheses on f and W:

$$(H1) \qquad \begin{gathered} f(x,\cdot) \text{ is positively 2-homogeneous and strictly convex for every } x \in \Omega, \\ c_1|\xi|^2 \leq f(x,\xi) \leq c_2|\xi|^2 \text{ for every } x \in \Omega \text{ and } \xi \in \mathbb{R}^N, \end{gathered}$$

with $0 < c_1 \leq c_2$. We note that in this hypotheses $\sqrt{f} = \phi^\circ$ for some $\phi \in \mathcal{N}(\Omega)$.

$$(H2) \qquad \begin{gathered} \{t \in \mathbb{R}\ ;\ W(t) = 0\} = \{\alpha, \beta\}, \\ c_3(|t|^p - 1) \leq W(t) \leq c_4(|t|^p + 1)\ , \quad \forall t \in \mathbb{R} \end{gathered}$$

with $p \geq 2$ and $0 < c_3 \leq c_4$.

For $\varepsilon > 0$ and $m \in (\alpha, \beta)$ we introduce a conveniently scaled version of the functionals $\mathcal{E}_\varepsilon$ by defining the functionals $E_\varepsilon : L^1(\Omega) \to [0, +\infty]$ as

$$E_\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega \left(\varepsilon f(x, Du) + \frac{W(u)}{\varepsilon} \right) dx & \text{if } u \in W^{1,2}(\Omega)\,,\ (u)_\Omega = m, \\ +\infty & \text{otherwise.} \end{cases} \tag{3.3}$$

The following Γ-convergence result has been proved (under more mild hypotheses) in [9] (see also [5]). We state it in a form that is suitable for our purposes.

Theorem 3.1. *Let f and W satisfy $(H1)$ and $(H2)$ respectively and let E_ε be as in* (3.3)*. Then E_ε Γ-converges, with respect to the $L^1(\Omega)$-topology, to the functional $E_0 : L^1(\Omega) \to [0, +\infty]$, defined as*

$$E_0(u) = \begin{cases} 2\theta P_\phi(\{x \in \Omega :\ u(x) = \alpha\}) & \textit{if } u \in \mathrm{BV}(\Omega; \{\alpha, \beta\})\,,\ (u)_\Omega = m \\ +\infty & \textit{otherwise,} \end{cases}$$

where $\theta = \int_\alpha^\beta \sqrt{W(t)}\,dt$*, with* $\phi^\circ = \sqrt{f}$*.*

Corollary 3.2. *Let* $\{\varepsilon_n\}$ *be a sequence of positive real numbers converging to zero. For any* $m \in (\alpha, \beta)$ *and* $p > 2$ *let* $u_{\varepsilon_n} \in W^{1,2}(\Omega)$ *be a solution of the problem*

$$m_{\varepsilon_n} = \min\left\{ \int_\Omega \left(\varepsilon_n^2 f(x, Du) + W(u)\right)\, dx \; ; \; \int_\Omega u\, dx = m|\Omega| \right\}. \tag{3.4}$$

Then upon extracting a subsequence (not relabelled), $u_{\varepsilon_n} \to u_0 \in \mathrm{BV}(\Omega; \{\alpha, \beta\})$ *in* $L^1(\Omega)$*. Moreover* Ω_α *is a solution of*

$$m_0 = \min\left\{ P_\phi(E) \; ; \; |E| = \frac{\beta - m}{\beta - \alpha}|\Omega| \right\},$$

with $\phi^\circ = \sqrt{f}$*.*

We now state our main result.

Theorem 3.3. *Let* f *satisfy hypotheses* $(H1)$ *and* $W : \mathbb{R} \to [0, \infty)$ *be of class* C^3 *satisfying* $(H2)$ *with* $W''(\alpha), W''(\beta) \geq c_0 > 0$*. Let* $\{u_{\varepsilon_n}\}$ *and* u_0 *be as in Corollary 3.2 and* λ_{ε_n} *be as in* (3.2)*. Suppose that* $\Omega_\alpha \in \mathcal{C}_b^2(\mathbb{R}^N)$*, then, up to subsequences (not relabeled),*

$$\lim_{n\to\infty} \frac{\lambda_{\varepsilon_n}}{\varepsilon_n} = \frac{2\theta}{\beta - \alpha}\lambda_0$$

where $\theta = \int_\alpha^\beta \sqrt{W(t)}\,dt$ *and* λ_0 *is the* ϕ*-anisotropic curvature of* $\Omega \cap \partial^*\Omega_\alpha$ *being* $\sqrt{f} = \phi^\circ$*.*

Before proving our result we make some comments on our hypotheses.

Remark 3.4 (regularity hypotheses on f and W). We observe that the case of a less regular f, for instance such that $\sqrt{f}$ is a crystalline norm, would be of great interest. Unfortunately in this case the approach due to Cahn and Hoffmann (see [11]), as we use in the present paper, cannot be exploited (the Cahn-Hoffmann vector field is not even uniquely defined in the crystalline case). We also observe that all the assumptions on W but (H1) are required only in order to prove the boundedness of the sequence $\{\frac{\lambda_\varepsilon}{\varepsilon}\}$. For this reason they could be removed if the previous sequence would be assumed to be a priori bounded as in [16].

Remark 3.5 (regularity hypotheses on Ω_α). The hypothesis on the regularity of Ω_α is needed to apply Theorem 2.1. We remark that this hypothesis can be dropped in the homogeneous case in dimension $n \leq 7$. Indeed in this case it is possible to exploit the standard regularity theory for Almgren's elliptic parametric integrals (see [1], [24] and also [20] Chapter 12).

Proof of Theorem 3.3 Without loss of generality we take $\alpha = -1$ and $\beta = 1$. Moreover, for simplicity of notation we drop the dependence on n in the sequences and we set

$$l_\varepsilon = -\frac{\lambda_\varepsilon}{\varepsilon}\,, \quad e_+^\varepsilon(x) = \varepsilon f(x, Du_\varepsilon(x)) + \frac{W(u_\varepsilon(x))}{\varepsilon}\,,$$
$$e_-^\varepsilon(x) = \varepsilon f(x, Du_\varepsilon(x)) - \frac{W(u_\varepsilon(x))}{\varepsilon}.$$

We also note that, by the hypotheses on f, there exists $\phi \in \mathcal{N}(\Omega)$ such that $\sqrt{f} = \phi^\circ$, moreover since $\{u_\varepsilon\}$ is a minimizing sequence, there exists a constant M such that for any $\varepsilon \in (0,1)$

$$\int_\Omega e_+^\varepsilon(x)\,dx \leq M. \tag{3.5}$$

The proof is divided in four steps.

Step 1 (discrepancy and curvature terms) By Lemma 3.6, upon extracting a subsequence (not relabeled) to $\{l_\varepsilon\}$, there exists $l_0 = \lim_{\varepsilon\to 0} l_\varepsilon$. Our next aim is to identify l_0. By adding and subtracting $\int_\Omega \varepsilon f(x, Du_\varepsilon)\,dx$ in equation (3.17), we get that, for any $g \in C_0^1(\Omega;\mathbb{R}^N)$,

$$\begin{aligned} -\int_\Omega l_\varepsilon u_\varepsilon \ \operatorname{div} g\ dx = &-\underbrace{\int_\Omega \left(\varepsilon f(x, Du_\varepsilon) - \frac{W(u_\varepsilon)}{\varepsilon}\right) \operatorname{div} g\ dx}_{A_\varepsilon} \\ &+ \underbrace{\int_\Omega 2\varepsilon f(x, Du_\varepsilon)\ \operatorname{div} g - \varepsilon f_{x_j}(x, Du_\varepsilon) g_j + \varepsilon f_{\xi_i}(x, Du_\varepsilon) D_j u_\varepsilon D_i g_j\ dx}_{B_\varepsilon}. \end{aligned} \tag{3.6}$$

The term $A_\varepsilon = \int_\Omega e_-^\varepsilon(x)\,dx$ is usually referred to as the *discrepancy term* while the term B_ε is referred to as the *curvature term*.

We now recall that $u_\varepsilon \to u_0$ in $L^1(\Omega)$ and that $u_0 \in \mathrm{BV}(\Omega;\{-1,1\})$ by Corollary 3.2. Thus taking the limit as $\varepsilon \to 0$ in the left hand side of (3.6), we get

$$\begin{aligned} \lim_{\varepsilon\to 0}\left(l_\varepsilon \int_\Omega u_\varepsilon \operatorname{div} g\ dx\right) &= l_0 \int_\Omega u_0 \operatorname{div} g\,dx = 2l_0 \int_{\Omega\cap\partial\Omega_\alpha} \nu\cdot g\, d\mathcal{H}^{N-1} \\ &= 2l_0 \int_{\Omega\cap\partial\Omega_\alpha} \nu_\phi \cdot g\, \phi^\circ(x,\nu) d\mathcal{H}^{N-1}, \end{aligned}$$

where $\nu(x) : \partial\Omega_\alpha \to S^{N-1}$ is the outer unit normal to $\partial\Omega_\alpha$. It remains to study the limit as $\varepsilon \to 0$ of A_ε and B_ε. This will be our purpose in the next steps of the proof.

Step 2 (re-parametrization of u_ε) In this step we use a standard strategy (see [16]) to re-parametrize u_ε using a primitive of $\sqrt{W}$ and we then prove a key convergence result for the ϕ°-total variations of the re-parametrizations. We set

$$\varphi(s) = \int_{-1}^{s} \sqrt{W(t)}\,dt$$

and we define $v_\varepsilon(x) = \varphi(u_\varepsilon(x))$. It holds that

$$Dv_\varepsilon = \varphi'(u_\varepsilon)Du_\varepsilon = \sqrt{W(u_\varepsilon)}Du_\varepsilon.$$

The family $\{v_\varepsilon\}$ is uniformly bounded in $\mathrm{BV}(\Omega)$. Indeed, by exploiting the classical Modica-Mortola's trick we have that

$$\int_\Omega |Dv_\varepsilon|\,dx \le \frac{1}{2}\int_\Omega \frac{\varepsilon|Du_\varepsilon|^2}{2} + \frac{W(u_\varepsilon)}{\varepsilon}\,dx \le c\int_\Omega e_+^\varepsilon\,dx \le cM. \tag{3.7}$$

By the compactness theorem in BV (see [3]), there exists $v_0 \in \mathrm{BV}(\Omega)$ such that, up to subsequences,

$$\lim_{\varepsilon\to 0}\int_\Omega |v_\varepsilon - v_0|\,dx = 0 \quad \text{and} \quad \int_\Omega |Dv_0| \le \liminf_{\varepsilon\to 0}\int_\Omega |Dv_\varepsilon|\,.$$

Let us denote by φ^{-1} the inverse function of φ. By the previous convergence results we easily deduce that

$$u_0 = \varphi^{-1}(v_0).$$

Thanks to the chain rule formula for BV- functions (see Proposition 2.2) we can also explicitly compute Dv_0.

$$\begin{aligned} Dv_0 &= D(\varphi(u_0)) = (\varphi(1) - \varphi(-1))\nu_{u_0}\mathcal{H}^{N-1}|_{S(u_0)} \\ &= \int_{-1}^{1}\sqrt{W(s)}\,ds\,\nu_{u_0}\mathcal{H}^{N-1}|_{S(u_0)} = \theta\,\nu_{u_0}\mathcal{H}^{N-1}|_{S(u_0)}. \end{aligned} \tag{3.8}$$

We claim that the ϕ-total variation of Dv_ε converges as $\varepsilon \to 0$ to the ϕ-anisotropic perimeter of the jump set of u_0, that is

$$\lim_{\varepsilon\to 0}\int_\Omega \sqrt{f(x, Dv_\varepsilon)}\,dx = \int_{\Omega\cap\partial\Omega_\alpha}\sqrt{f(x,\nu_{u_0})}\,d\mathcal{H}^{N-1} = \theta P_\phi(\Omega_\alpha). \tag{3.9}$$

The claim is a consequence of Theorem 3.1 and of a density argument. Indeed, first we note that by the homogeneity assumptions on f and by Theorem 3.1 we have

$$\begin{aligned} \limsup_{\varepsilon\to 0}\int_\Omega \sqrt{f(x, Dv_\varepsilon)}\,dx &= \limsup_{\varepsilon\to 0}\int_\Omega \sqrt{f(x, Du_\varepsilon)}\sqrt{W(u_\varepsilon)}\,dx \\ &\le \frac{1}{2}\limsup_{\varepsilon\to 0}\int_\Omega \varepsilon f(x, Du_\varepsilon) + \frac{W(u_\varepsilon)}{\varepsilon}\,dx \\ &= \frac{1}{2}\limsup_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) = \theta P_\phi(\Omega_\alpha). \end{aligned} \tag{3.10}$$

In order to obtain the $\liminf$ inequality we start by observing that, since $v_\varepsilon \to v_0$ in $\mathrm{BV}(\Omega)$, by Reshetnyak lower semicontinuity theorem (see for example Theorem 2.38 in [3]), we have

$$\int_\Omega |Dv_0|_\phi = \int_\Omega \sqrt{f\left(x, \frac{Dv_0}{|Dv_0|}\right)}\,d|Dv_0| \le \liminf_{\varepsilon\to 0}\int_\Omega \sqrt{f(x, Dv_\varepsilon)}\,dx. \tag{3.11}$$

Moreover, for any $\zeta \in S^{N-1}$ by the defnition of dual norm and by (3.8) (again using the notation $\phi^\circ = \sqrt{f}$), we can write

$$\int_\Omega \phi^\circ \left(x, \frac{Dv_0}{|Dv_0|} \right) d|Dv_0| \geq \int_\Omega \frac{1}{\phi(x,\zeta)} \left| \frac{Dv_0}{|Dv_0|} \cdot \zeta \right| d|Dv_0| \\ = \theta \int_{\Omega \cap \partial\Omega_\alpha} \frac{1}{\phi(x,\zeta)} |\nu_{u_0} \cdot \zeta| \, d\mathcal{H}^{N-1}. \tag{3.12}$$

Let now $\{\zeta_i\}_{i\in\mathbb{N}}$ be a dense subset of S^{N-1}. From (3.12), applying Proposition 2.3 with $\lambda = \mathcal{H}^{N-1}|_{\partial\Omega_\alpha}$, $\mu(A) = \int_\Omega \phi^\circ \left(x, \frac{Dv_0}{|Dv_0|} \right) d|Dv_0|$ and $\psi_i = \theta \, \chi_{\partial\Omega_\alpha} \frac{1}{\phi(x,\zeta)} |\nu_{u_0} \cdot \zeta_i|$, we obtain

$$\int_\Omega \phi^\circ \left(x, \frac{Dv_0}{|Dv_0|} \right) d|Dv_0| \geq \theta \int_\Omega \sup_{i\in\mathbb{N}} \frac{1}{\phi(x,\zeta_i)} |\nu_{u_0} \cdot \zeta_i| \, d\mathcal{H}^{N-1} \\ = \theta \int_{\Omega\cap\partial\Omega_\alpha} \phi^\circ(x, \nu_{u_0}) \, d\mathcal{H}^{N-1}.$$

The previous inequality, togheter with (3.10) and (3.11) prove the claim.

Step 3 (negligibility of the discrepancy term) We now prove that the discrepancy term vanishes as ε goes to 0. By using a simple algebraic argument (see Lemma 1 in [16]) the claim will follow directly by formula (3.9).

Set $a_\varepsilon = \sqrt{\varepsilon f(x, Du_\varepsilon)}$ and $b_\varepsilon = \sqrt{\frac{W(u_\varepsilon)}{\varepsilon}}$. Formula (3.9) reads as

$$\lim_{\varepsilon\to 0} \int_\Omega a_\varepsilon b_\varepsilon \, dx = \theta P_\phi(\Omega_\alpha).$$

Moreover, again by Theorem 3.1, we also have that

$$\lim_{\varepsilon\to 0} \int_\Omega a_\varepsilon^2 + b_\varepsilon^2 \, dx = \lim_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) = 2\theta P_\phi(\Omega_\alpha).$$

By the previous relations we get the claim noticing that

$$\lim_{\varepsilon\to 0} \int_\Omega (a_\varepsilon - b_\varepsilon)^2 \, dx = \lim_{\varepsilon\to 0} \left(\int_\Omega a_\varepsilon^2 + b_\varepsilon^2 \, dx - 2 \int_\Omega a_\varepsilon b_\varepsilon \right) dx = 0$$

which turns out to imply, by the energy estimate,

$$0 \leq \lim_{\varepsilon\to 0} \int_\Omega |a_\varepsilon^2 - b_\varepsilon^2| \, dx = \lim_{\varepsilon\to 0} \int_\Omega |(a_\varepsilon + b_\varepsilon)(a_\varepsilon - b_\varepsilon)| \, dx \\ \leq \lim_{\varepsilon\to 0} \left(\int_\Omega |a_\varepsilon^2 - a_\varepsilon b_\varepsilon| \, dx + \int_\Omega |a_\varepsilon b_\varepsilon - b_\varepsilon^2| \, dx \right) \\ \leq cM \lim_{\varepsilon\to 0} \int_\Omega (a_\varepsilon - b_\varepsilon)^2 = 0\,.$$

Step 4 (convergence of the curvature term) In this last step we analyze the limit behaviour of B_ε as ε goes to 0. We claim that

$$\lim_{\varepsilon\to 0} B_\varepsilon = -2\theta \int_{\Omega\cap\partial\Omega_\alpha} \nu\cdot g\ \kappa_\phi\ d\mathcal{H}^{N-1},$$

where we have denoted by where $\nu(x) : \partial\Omega_\alpha \to S^{N-1}$ is the outer unit normal to $\partial\Omega_\alpha$. To prove the claim we start by noticing that from the energy estimate (3.5), we can deduce that

$$\begin{aligned} 0 &= \lim_{\varepsilon\to 0}\int_\Omega |\varepsilon f(x,Du_\varepsilon) - \sqrt{f(x,Du_\varepsilon)}\sqrt{W(u_\varepsilon)}|\,dx \\ &= \lim_{\varepsilon\to 0}\int_\Omega |\varepsilon f(x,Du_\varepsilon) - \sqrt{f(x,Dv_\varepsilon)}|\,dx. \end{aligned} \tag{3.13}$$

We now use the homogeneity of f to rewrite B_ε as

$$B_\varepsilon = \int_\Omega 2\varepsilon f(x,Du_\varepsilon) H(x,Du_\varepsilon)\,dx,$$

where we have set

$$H(x,\xi) = \mathrm{tr}\left[(\mathrm{Id} - \frac{1}{2}\frac{f_\xi(x,\xi)}{\sqrt{f(x,\xi)}} \otimes \frac{\xi}{\sqrt{f(x,\xi)}})\ Dg + \frac{f_x(x,\xi)}{2f(x,\xi)}\otimes g\right].$$

Let $\Omega_\varepsilon := \{x\in\Omega\ :\ |Dv_\varepsilon|_\phi \neq 0\}$. By the boundedness of the function H, the equality $\frac{Du_\varepsilon}{|Du_\varepsilon|_\phi} = \frac{Dv_\varepsilon}{|Dv_\varepsilon|_\phi}$ on Ω_ε and by using (3.13) we have

$$\begin{aligned} \lim_{\varepsilon\to 0}\frac{B_\varepsilon}{2} &= \lim_{\varepsilon\to 0}\int_\Omega \varepsilon f(x,Du_\varepsilon)H(x,Du_\varepsilon)\,dx = \lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon}\sqrt{f(x,Dv_\varepsilon)}H(x,Dv_\varepsilon)\,dx \\ &= \lim_{\varepsilon\to 0}\int_\Omega \mathrm{tr}\left[(\mathrm{Id} - \phi^\circ_\xi\left(x,\frac{Dv_\varepsilon}{|Dv_\varepsilon|_\phi}\right)\otimes\frac{Dv_\varepsilon}{|Dv_\varepsilon|_\phi})Dg + \phi^\circ_x\left(x,\frac{Dv_\varepsilon}{|Dv_\varepsilon|_\phi}\right)\otimes g\right] d|Dv_\varepsilon|_\phi \end{aligned}$$

where we recall that $\phi^\circ = \sqrt{f}$. We are now in a position to apply Lemma 3.7, (3.8) and Theorem 2.1 to conclude that

$$\begin{aligned} \lim_{\varepsilon\to 0} B_\varepsilon &= 2\int_{\Omega\cap\partial\Omega_\alpha} \mathrm{tr}\left[(\mathrm{Id} - \phi^\circ_\xi(x,\nu_\phi)\otimes\nu_\phi)Dg + \phi^\circ_x(x,\nu_\phi)\otimes g\right] d|Dv_0|_\phi \\ &= 2\theta\int_{\Omega\cap\partial\Omega_\alpha} \mathrm{tr}\left[(\mathrm{Id} - n_\phi\otimes\nu_\phi)Dg + \phi^\circ_x(x,\nu_\phi)\otimes g\right]\phi^\circ(x,\nu)\,d\mathcal{H}^{N-1} \\ &= 2\theta\int_{\Omega\cap\partial\Omega_\alpha} \mathrm{div}_\phi\, g\,\phi^\circ(x,\nu)\,d\mathcal{H}^{N-1} = -2\theta\int_{\Omega\cap\partial\Omega_\alpha}\nu\cdot g\ \kappa_\phi\ d\mathcal{H}^{N-1}. \end{aligned}$$

□

The following Lemma is obtained adapting an argument by X. Chen in [12] to the Finsler setting. Note that, unlike Chen's result, here we have to deal with Dirichlet instead of Neumann boundary conditions.

Lemma 3.6. *Let f satisfy hypotheses $(H1)$ and $W:\mathbb{R}\to[0,\infty)$ be of class C^3 satisfying $(H2)$ with $W''(\alpha)$, $W''(\beta)\geq c_0>0$. Let $\{u_{\varepsilon_n}\}$ be as in Corollary 3.2 and λ_{ε_n} be as in* (3.2). *Then the sequence $\left\{\frac{\lambda_{\varepsilon_n}}{\varepsilon_n}\right\}$ is bounded.*

Proof. For simplicity of notation, we drop the dependence on n in the sequences and we set $l_\varepsilon=-\frac{\lambda_\varepsilon}{\varepsilon}$. We observe that, since u_ε is a minimizer for (3.4), it satisfies the elliptic equation

$$l_\varepsilon=-\varepsilon\operatorname{div} f_\xi(x,Du_\varepsilon)+\frac{W'(u_\varepsilon)}{\varepsilon}, \tag{3.14}$$

and it has constant mean value in Ω

$$(u_\varepsilon)_\Omega=\frac{1}{|\Omega|}\int_\Omega u_\varepsilon\,dx=m\in(-1,1). \tag{3.15}$$

We observe that (3.5) implies that $\{u_\varepsilon\}$ is uniformly bounded in $L^2(\Omega)$ and moreover, by the hypotheses on W, we can easily deduce that

$$\int_\Omega(|u_\varepsilon|-1)^2\,dx\leq cM\varepsilon \tag{3.16}$$

Following an argument by Chen (see the proof of Lemma 3.4 in [12]), using a smoothing procedure and the elliptic regularity theory, one can infer from (3.14), (3.15) and (3.5) the boundedness of $\{l_\varepsilon\}_{\varepsilon\in(0,1)}$. To this aim we write the weak form of (3.14) using as a test function $\varphi=g\cdot Du_\varepsilon$, where $g\in C_0^1(\Omega;\mathbb{R}^N)$. Integrating by parts we have

$$\begin{aligned}-\int_\Omega l_\varepsilon u_\varepsilon\operatorname{div} g\,dx=&\varepsilon\int_\Omega f_\xi(x,Du_\varepsilon)\cdot D\varphi+\frac{W'(u_\varepsilon)}{\varepsilon}\varphi\,dx\\ &=-\int_\Omega e_+^\varepsilon(x)\operatorname{div} g\,dx-\int_\Omega\varepsilon f_x(x,Du_\varepsilon)\cdot g\,dx\\ &\quad+\int_\Omega\operatorname{tr}[(\varepsilon f_\xi(x,Du_\varepsilon)\otimes Du_\varepsilon)Dg]\,dx.\end{aligned} \tag{3.17}$$

By choosing $g=D\psi$ for a suitable $\psi\in C^2(\Omega)$ in (3.17) we get

$$\begin{aligned}l_\varepsilon\int_\Omega\Delta\psi\,u_\varepsilon\,dx=&\int_\Omega\operatorname{tr}[D^2\psi\left(e_+^\varepsilon(u_\varepsilon)\operatorname{Id}-\varepsilon f_\xi(x,Du_\varepsilon)\otimes Du_\varepsilon\right)]\\ &+\varepsilon f_x(x,Du_\varepsilon)\cdot D\psi\,dx.\end{aligned} \tag{3.18}$$

Setting

$$I'_\varepsilon=\int_\Omega\Delta\psi\,u_\varepsilon\,dx$$

and

$$I''_\varepsilon=\int_\Omega\operatorname{tr}[D^2\psi\left(\operatorname{Id}e_+^\varepsilon(x)-\varepsilon f_\xi(x,Du_\varepsilon)\otimes Du_\varepsilon\right)]+\varepsilon f_x(x,Du_\varepsilon)\cdot D\psi\,dx\,,$$

we can write

$$l_\varepsilon = \frac{I''_\varepsilon}{I'_\varepsilon}$$

and the proof will follow once we find an upper-bound for I''_ε and a lower-bound for I'_ε uniformly w.r.t. ε. The desired bounds will be obtained choosing carefully the function ψ. Let ρ_η be a standard family of mollifiers and consider, for $\eta < \eta_0$, $u^\eta_\varepsilon = \rho_\eta * u_\varepsilon \in C^\infty(\Omega)$ the η-regularization of u_ε. Here we have assumed that u_ε has been extended by reflection to the η_0-neighborhood of Ω

$$\{x \in \mathbb{R}^N;\ \mathrm{dist}(x, \Omega) \le \eta_0\}.$$

Let $\widetilde{\Omega} \subset\subset \Omega$ and let $\widetilde{\psi}$ be the unique solution of the Dirichlet problem

$$\begin{cases} -\Delta\widetilde{\psi} = u^\eta_\varepsilon - (u^\eta_\varepsilon)_{\widetilde{\Omega}} & \text{in } \tilde{\Omega}, \\ \widetilde{\psi} = 0 & \text{on } \partial\tilde{\Omega}. \end{cases}$$

By classical regularity results (cfr.[14] Theorem 6.14) $\widetilde{\psi} \in C^{2,\alpha}(\overline{\widetilde{\Omega}})$. Let ψ be a smooth extension of $\widetilde{\psi}$ in $\mathbb{R}^N$ with compact support in Ω' with $\widetilde{\Omega} \subset \Omega' \subset \Omega$ (see [14] Lemma 6.37). We observe that, by the definition of u^η_ε and by (3.16), the following estimate holds:

$$\begin{aligned} \|u^\eta_\varepsilon\|_{C(\Omega)} &\le \left\| \int_{B_1} \rho_1(y) u_\varepsilon(x - \eta y)\, dy \right\|_{C(\Omega)} \\ &\le 1 + \sup_{x\in\Omega} \int_{B_1} \rho_1(y)\, \big| |u_\varepsilon(x-\eta y)| - 1 \big|\ dy \\ &\le 1 + c\eta^{-\frac{N}{2}} \big\| |u_\varepsilon| - 1 \big\|_{L^2(\Omega)} \le 1 + c\varepsilon^{\frac12}\eta^{-\frac{N}{2}}. \end{aligned}$$

A similar computation gives

$$\|u^\eta_\varepsilon\|_{C^1(\Omega)} \le c\eta^{-1}(1 + \varepsilon^{\frac12}\eta^{-\frac{N}{2}}).$$

By classical elliptic estimates we also have that

$$\|\psi\|_{C^2(\Omega)} \le c\|u^\eta_\varepsilon\|_{C^1(\Omega)} \le c\eta^{-1}(1 + \varepsilon^{\frac12}\eta^{-\frac{N}{2}}).$$

The previous estimates together with assumption (H1) and (3.5) imply the following estimate for I''_ε:

$$\begin{aligned} |I''_\varepsilon| \le & c\|\psi\|_{C^2(\Omega)} \int_\Omega e^\varepsilon_+(x) + \varepsilon|Du_\varepsilon|^2 |\,\mathrm{tr}[f_\xi\left(x, \frac{Du_\varepsilon}{|Du_\varepsilon|}\right) \otimes \frac{Du_\varepsilon}{|Du_\varepsilon|}]|\, dx \\ & + c\|\psi\|_{C^1(\Omega)} \int_\Omega \varepsilon|Du_\varepsilon|^2 \left| f_x\left(x, \frac{Du_\varepsilon}{|Du_\varepsilon|}\right)\right| dx \qquad (3.19) \\ \le & cK\|\psi\|_{C^2(\Omega)} \int_\Omega e^\varepsilon_+(x)\, dx \le cKM\eta^{-1}(1 + \varepsilon^{\frac12}\eta^{-\frac{N}{2}}) \end{aligned}$$

where

$$K=\sup_{(x,\xi)\in\Omega\times S^{N-1}}\{|f_\xi\,(x,\xi)\,|+|f_x\,(x,\xi)\,|\}.$$

Concerning I'_ε, we first observe that it can be conveniently rewritten as

$$\begin{aligned}
I'_\varepsilon &= \int_{\widetilde\Omega}(u^\eta_\varepsilon-(u^\eta_\varepsilon)_{\widetilde\Omega})u_\varepsilon\,dx+\int_{\Omega'\setminus\widetilde\Omega}\Delta\psi\;u_\varepsilon\,dx\\
&=\int_{\widetilde\Omega}(u^\eta_\varepsilon-u_\varepsilon)u_\varepsilon\,dx+\int_{\widetilde\Omega}(u^2_\varepsilon-1)\,dx+|\widetilde\Omega|\left(1-(u_\varepsilon)^2_{\widetilde\Omega}\right)\\
&\quad+|\widetilde\Omega|(u_\varepsilon)_{\widetilde\Omega}((u_\varepsilon)_{\widetilde\Omega}-(u^\eta_\varepsilon)_{\widetilde\Omega})+\int_{\Omega'\setminus\widetilde\Omega}\Delta\psi\;u_\varepsilon\,dx.
\end{aligned}\tag{3.20}$$

In order to estimate the different terms in (3.20) we set

$$\tilde\phi(s):=\int_{-1}^{s}\sqrt{\tilde W(t)}\,dt\quad\text{with } \tilde W(t)=\min\{W(t),1+|t|^2\}$$

and we note that, by the assumptions on W, there exists a constants $c>0$ such that

$$c|s_1-s_2|^2\le|\tilde\phi(s_1)-\tilde\phi(s_2)|\quad,\qquad\forall\,s_1,s_2\in\mathbb{R}.\tag{3.21}$$

Moreover, for $\tilde v_\varepsilon=\tilde\phi(u_\varepsilon)$ we have, as in (3.7), that

$$\int_\Omega|\nabla\tilde v_\varepsilon|dx\le cM.$$

From the previous inequality, the definition of u^η_ε and (3.21) we infer that

$$\|\nabla u^\eta_\varepsilon\|_{L^2(\Omega)}\le c\eta^{-1}\|u_\varepsilon\|_{L^2(\Omega)}\le c\eta^{-1}$$

and that

$$\begin{aligned}
\int_\Omega|u^\eta_\varepsilon-u_\varepsilon|^2dx &\le\int_\Omega\int_{B_1}\rho_1(y)|u_\varepsilon(x-\eta y)-u_\varepsilon(x)|^2\,dy\,dx\\
&\le c\int_\Omega\int_{B_1}\rho_1(y)|\tilde v_\varepsilon(x-\eta y)-\tilde v_\varepsilon(x)|\,dy\,dx\\
&\le c\eta\|\nabla\tilde v_\varepsilon\|_{L^1(\Omega)}\le cM\eta.
\end{aligned}\tag{3.22}$$

We can now deduce from (3.22) that

$$\begin{aligned}
\int_{\widetilde\Omega}(u^\eta_\varepsilon-u_\varepsilon)u_\varepsilon\,dx &\ge-\int_{\widetilde\Omega}|u^\eta_\varepsilon-u_\varepsilon|\,|u_\varepsilon|\,dx\\
&\ge-\|u^\eta_\varepsilon-u_\varepsilon\|_{L^2(\Omega)}\|u_\varepsilon\|_{L^2(\Omega)}\ge-c\eta^{\frac12}
\end{aligned}\tag{3.23}$$

and, by (3.16), that

$$\int_{\widetilde{\Omega}} |u_\varepsilon^2 - 1|\,dx = \int_\Omega (u_\varepsilon - 1)(u_\varepsilon + 1)\,dx \\ \geq -c\|u_\varepsilon - 1\|_{L^2(\Omega)} \left(\|u_\varepsilon\|_{L^2(\Omega)} + 1\right) \geq -c\varepsilon^{\frac{1}{2}}. \tag{3.24}$$

We now choose $\widetilde{\Omega}$ such that $|(u_\varepsilon)_{\widetilde{\Omega}}| \leq c < 1$ which turns out to imply, for ε small enough, that

$$|\widetilde{\Omega}|(u_\varepsilon)_{\widetilde{\Omega}}((u_\varepsilon)_{\widetilde{\Omega}} - (u_\varepsilon^\eta)_{\widetilde{\Omega}}) \geq -c|\widetilde{\Omega}|\,\|u_\varepsilon - u_\varepsilon^\eta\|_{L^2(\Omega)} \geq -c\eta^{\frac{1}{2}} \tag{3.25}$$

and that

$$|\widetilde{\Omega}|\left(1 - (u_\varepsilon)^2_{\widetilde{\Omega}}\right) \geq c|\widetilde{\Omega}|. \tag{3.26}$$

Indeed, by Rellich Theorem, $\{u_\varepsilon\}$ is precompact in $L^{2^*}(\Omega)$ and then, up to extracting a subsequence, we have that

$$(u_\varepsilon)_{\widetilde{\Omega}} = \frac{|\Omega|}{|\widetilde{\Omega}|}(u_\varepsilon)_\Omega - \frac{1}{|\widetilde{\Omega}|}\int_{\Omega\setminus\widetilde{\Omega}} u_\varepsilon dx = \frac{|\Omega|}{|\widetilde{\Omega}|}m - \frac{1}{|\widetilde{\Omega}|}\int_{\Omega\setminus\widetilde{\Omega}} u_\varepsilon dx.$$

Thus, since $m < 1$, we may choose $\widetilde{\Omega}$ sufficiently big in order to have that

$$|(u_\varepsilon)_{\widetilde{\Omega}}| \leq \frac{|\Omega|}{|\widetilde{\Omega}|}m + \frac{1}{|\widetilde{\Omega}|}\left(\|u_\varepsilon\|_{2^*}|\Omega\setminus\widetilde{\Omega}|^{\frac{1}{(2^*)'}}\right) \\ \leq \frac{|\Omega|}{|\widetilde{\Omega}|}m + \frac{1}{|\widetilde{\Omega}|}c|\Omega\setminus\widetilde{\Omega}|^{\frac{1}{(2^*)'}} \leq c < 1,$$

that, in turn, implies (3.25) and (3.26). It remains to observe that, choosing Ω' such that $|\Omega'\setminus\widetilde{\Omega}|^{\frac{1}{2}} \leq \eta^2$, we have that

$$\left|\int_{\Omega'\setminus\widetilde{\Omega}} \Delta\psi\, u_\varepsilon\, dx\right| \leq \int_{\Omega'\setminus\widetilde{\Omega}} \|\psi\|_{C^2(\Omega)}|u_\varepsilon|\,dx \leq c\|\psi\|_{C^2(\Omega)}\|u_\varepsilon\|_{2,\Omega}|\Omega'\setminus\widetilde{\Omega}|^{\frac{1}{2}} \\ \leq c\eta(1 + \varepsilon^{\frac{1}{2}}\eta^{-\frac{N}{2}})$$

and we can conclude, combining (3.20),(3.23),(3.24),(3.25) and (3.26), that

$$I'_\varepsilon \geq |\widetilde{\Omega}|(1 - m^2) - c(\varepsilon^{\frac{1}{2}} + \eta^{\frac{1}{2}}) - c\eta^{-1}(1 + \varepsilon^{\frac{1}{2}}\eta^{-\frac{N}{2}}). \tag{3.27}$$

Gathering together (3.27) and (3.19) it follows that

$$|l_\varepsilon| \leq \frac{cKM\eta^{-1}(1 + \varepsilon^{\frac{1}{2}}\eta^{-\frac{N}{2}})}{|\widetilde{\Omega}|c - c(\varepsilon^{\frac{1}{2}} + \eta^{\frac{1}{2}}) - c\eta(1 + \varepsilon^{\frac{1}{2}}\eta^{-\frac{N}{2}})},$$

which implies the desired estimate for a sufficiently small η independent of ε. □

The following result is a version of the Reshetnyak continuity theorem (see [22] and also [16]) in the Finsler setting.

Lemma 3.7. *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded open set. Let* ϕ *be a strictly convex Finsler norm and* $\{v_\varepsilon\}$ *be a family of functions of class* $C^1(\Omega)$ *and let* $v_0 \in BV(\Omega)$. *Assume that* v_ε *converges to* v_0 *in* $\mathrm{BV}(\Omega)$ *and that*

$$\lim_{\varepsilon \to 0} \int_\Omega \phi(x, Dv_\varepsilon)\, dx = |Dv_0|_\phi(\Omega). \tag{3.28}$$

Then for any function $F(x,p) \in C(\Omega \times \mathbb{R}^n)$ *satisfying*

$$F(x,tp) = tF(x,p) \quad for \quad x \in \Omega, \quad p \in \mathbb{R}^n, \;\; t \geq 0 \tag{3.29}$$

and

$$F(x,p) = 0 \quad for \quad x \notin K, \;\; p \in \mathbb{R}^n \tag{3.30}$$

with K *is a fixed compact subset of* Ω, *we have*

$$\lim_{\varepsilon \to 0} \int_\Omega F(x, Dv_\varepsilon)\, dx = \int_\Omega F\left(x, \frac{\nu_{v_0}}{\phi(x,\nu_{v_0})}\right)\, d|Dv_0|_\phi.$$

Proof. Let $\theta_\varepsilon : \Omega \to \{(x,p)\,;\, x \in \Omega\,,\; p \in \partial B_\phi(x)\}$ be defined by

$$\theta_\varepsilon(x) = \left(x, \frac{Dv_\varepsilon(x)}{\phi(x, Dv_\varepsilon(x))}\right)$$

and let us consider the family of measures $\mu_\varepsilon = (\theta_\varepsilon)_\sharp |Dv_\varepsilon|_\phi$. By the definition of μ_ε, for any $f \in C_c(\Omega \times \mathbb{R}^N)$

$$\int_{\Omega \times \mathbb{R}^N} f(x,p)\, d\mu_\varepsilon = \int_\Omega f\left(x, \frac{Dv_\varepsilon}{\phi(x, Dv_\varepsilon(x))}\right) \phi(x, Dv_\varepsilon(x))\, dx. \tag{3.31}$$

Analogously let us consider the function $\theta_0 : \Omega \to \{(x,p)\,;\, x \in \Omega\,,\; p \in \partial B_\phi(x)\}$ defined as

$$\theta_0(x) = \left(x, \frac{\nu_{v_0}}{\phi(x, \nu_{v_0})}\right)$$

and the measure $\mu_0 = (\theta_0)_\sharp |Dv_0|_\phi$. We therefore have that for any $f \in C_c(\Omega \times \mathbb{R}^N)$

$$\int_{\Omega \times \mathbb{R}^N} f(x,p)\, d\mu_0 = \int_\Omega f\left(x, \nu_{v_0}\right)\, d|Dv_0|. \tag{3.32}$$

By the regularity of F and by (3.30) we have that (3.31) and (3.32) hold for $f = F$. Thus, taking into account (3.29), the claim follows by proving that $\mu_\varepsilon \overset{*}{\rightharpoonup} \mu_0$.

Since

$$\mu_\varepsilon(\Omega \times \mathbb{R}^N) = \int_\Omega \phi(x, Dv_\varepsilon(x))\, dx\,,$$

μ_ε is uniformly bounded by (3.28). Then, up to subsequences, it weakly star converges to a Radon measure μ. It remains to prove that $\mu = \mu_0$.

Applying Theorem 2.4 with $Y = \Omega$, $X = \bigcup_{x\in\Omega}(\{x\} \times \partial B_\phi(x))$, $\pi(x, y) = x$, for every $x \in \Omega$ there exist a probability measure λ_x supported on $\{x\} \times \partial B_\phi(x)$, (that we will simply identify by $\partial B_\phi(x)$) such that, for every Borel function $f : X \to [0, +\infty]$,

$$\int_{\Omega\times\mathbb{R}^N} f(x, p)\, d\mu = \int_\Omega \int_{\partial B_\phi(x)} f(x, p)\, d\lambda_x(p)\, d\omega(x) \tag{3.33}$$

where $\omega = \pi_\sharp\mu$. Thus it is left to prove that $\omega = |Dv_0|_\phi$ and that $\lambda_x = \delta\left(\frac{\nu_{v_0}(x)}{\phi(x,\nu_{v_0}(x))}\right)$. Let us consider $h \in C_c(\Omega; \mathbb{R}^N)$. Using $f(x, p) = h(x) \cdot p$ in (3.33), we have

$$\int_{\Omega\times\mathbb{R}^N} h(x) \cdot p\, d\mu = \int_\Omega h(x) \cdot \int_{\partial B_\phi(x)} p\, d\lambda_x(p)\; d\omega(x). \tag{3.34}$$

By weak star convergence and by the hypotheses we get

$$\begin{aligned}\int_{\Omega\times\mathbb{R}^N} h(x) \cdot p\, d\mu &= \lim_{\varepsilon\to 0} \int_{\Omega\times\mathbb{R}^N} h(x) \cdot p\, d\mu_\varepsilon \\ &= \lim_{\varepsilon\to 0} \int_\Omega h(x) \cdot \frac{Dv_\varepsilon(x)}{\phi(x, Dv_\varepsilon(x))}\, d|Dv_\varepsilon|_\phi \\ &= \lim_{\varepsilon\to 0} \int_\Omega h(x) \cdot Dv_\varepsilon(x)\, dx = \int_\Omega h(x) \cdot d\, Dv_0(x)\end{aligned} \tag{3.35}$$

By (3.34) and (3.35) we deduce that

$$\left(\int_{\partial B_\phi(x)} p d\lambda_x(p)\right) d\omega(x) = dDv_0(x)$$

which gives $|Dv_0|_\phi << \omega$. Thus there exists a ω-measurable function $\gamma : \Omega \to \mathbb{R}^+$ such that $|Dv_0|_\phi = \gamma\omega$ and that, for ω-a.e. $x \in \Omega$, it holds

$$\int_{\partial B_\phi(x)} p\, d\lambda_x(p) = \frac{\nu_{v_0}(x)}{\phi(x, \nu_{v_0}(x))}\gamma(x)$$

and then

$$\phi\left(x, \int_{\partial B_\phi(x)} p d\lambda_x(p)\right) = \gamma(x). \tag{3.36}$$

By (3.28) we finally have

$$\begin{aligned}\int_\Omega \Big(\int_{\partial B_\phi(x)} \phi(x, p) d\lambda_x(p)\Big)\, d\omega(x) &= \mu(\Omega \times \mathbb{R}^N) = \lim_{\varepsilon\to 0} \mu_\varepsilon(\Omega \times \mathbb{R}^N) \\ &= \lim_{\varepsilon\to 0} \int_\Omega d|Dv_\varepsilon|_\phi = \int_\Omega d|Dv_0|_\phi \\ &= \int_\Omega \gamma(x)\, d\omega(x) = \int_\Omega \phi\left(x, \int_{\partial B_\phi(x)} p d\lambda_x(p)\right) d\omega(x).\end{aligned} \tag{3.37}$$

By gathering together (3.36) and (3.37) we have

$$\phi\left(x, \int_{\partial B_\phi(x)} p\, d\lambda_x(p)\right) = \int_{\partial B_\phi(x)} \phi(x,p) d\lambda_x(p)$$

which, by the strict convexity of ϕ, implies $\lambda_x = \delta\left(\frac{\nu_{v_0}(x)}{\phi(x,\nu_{v_0}(x))}\right)$.

□

References

[1] F.J.jun. Almgren. Existence and regularity almost everywhere of solutions to elliptic variational problems among surfaces of varying topological type and singularity structure. *Ann. of Math.*, 87:321–391, 1968.

[2] M. Amar and G. Bellettini. A notion of total variation depending on a metric with discontinuous coefficients. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 11(1):91–133, 1994.

[3] Luigi Ambrosio, Nicola Fusco, and Diego Pallara. *Functions of bounded variation and free discontinuity problems.* Oxford Mathematical Monographs. Oxford: Clarendon Press. xviii, 434 p., 2000.

[4] Luigi Ambrosio, Nicola Gigli and Giuseppe Savaré. *Gradient flows in metric spaces and in the space of probability measures.* Lectures in Mathematics, ETH Zürich. Basel: Birkhäuser. vii, 333 p. EUR 34.24 , 2005.

[5] Nadia Ansini, Andrea Braides and Valeria Chiadò Piat. Gradient theory of phase transitions in composite media. *Proc. R. Soc. Edinb., Sect. A, Math.*, 133(2):265–296, 2003.

[6] G. Bellettini and I. Fragalà. Elliptic approximations of prescribed mean curvature surfaces in Finsler geometry. *Asymptotic Anal.*, 22(2):87–111, 2000.

[7] G. Bellettini and L. Mugnai. Anisotropic geometric functionals and gradient flows. *Banach Center Publications*, 86:21–43, 2009.

[8] G. Bellettini and M. Paolini. Anisotropic motion by mean curvature in the context of Finsler geometry. *Hokkaido Math. J.*, 25(3):537–566, 1996.

[9] Guy Bouchitté. Singular perturbations of variational problems arising from a two-phase transition model. *Appl. Math. Optim.*, 21(3):289–314, 1990.

[10] A. Braides. Free discontinuity problems and their non-local approximation. Buttazzo, G. (ed.) et al., Calculus of variations and partial differential equations. Topics on geometrical evolution problems and degree theory. Based on a summer school, Pisa, Italy, September 1996. Berlin: Springer. 171-180, 327-337, 2000.

[11] J. W. Cahn and D. W. Hoffman. A vector thermodynamics for anisotropic surfaces. ii. curved and faceted surfaces. *Acta Metall. Mater.*, 22:1205–1214, 1974.

[12] Xinfu Chen. Global asymptotic limit of solutions of the Cahn-Hilliard equation. *J. Differ. Geom.*, 44(2):262–311, 1996.

[13] Ennio De Giorgi. Some remarks on Γ-convergence and least squares method. In: Dal Maso,G., Dell' Antonio, G.F. (eds), Composite media and homogenization theory (Trieste, 1990), 135–142, Progr. Nonlinear Differential Equations Appl., 5, BirkhŁuser Boston, Boston, MA, 1991.

[14] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order. 2nd ed.* Grundlehren der Mathematischen Wissenschaften, 224. Berlin etc.: Springer-Verlag. XIII, 513 p. DM 128.00; $ 47.80 , 1983.

[15] Morton E. Gurtin. Some results and conjectures in the gradient theory of phase transitions. Metastability and incompletely posed problems, Proc. Workshop, Minneapolis/Minn. 1984/85, IMA Vol. Math. Appl. 3, 135-146, 1987.

[16] Stephan Luckhaus and Luciano Modica. The Gibbs-Thompson relation within the gradient theory of phase transitions. *Arch. Ration. Mech. Anal.*, 107(1):71–83, 1989.

[17] G. B. McFadden, A. A. Wheeler, R. J. Braun, S. R. Coriell, and R. F. Sekerka. Phase field models for anisotropic interfacses. *Phys. Rev. E*, 48(3):2016–2024, 1993.

[18] Luciano Modica. The gradient theory of phase transitions and the minimal interface criterion. *Arch. Ration. Mech. Anal.*, 98:123–142, 1987.

[19] Luciano Modica and Stefano Mortola. Un esempio di Γ^--convergenza. *Boll. Unione Mat. Ital., V. Ser., B*, 14:285–299, 1977.

[20] Morgan, Frank. Geometric measure theory. A beginner's guide. 4th ed. *San Diego, CA: Academic Press. viii, 249 p. 2009*.

[21] Yuko Nagase and Yoshihiro Tonegawa. A singular perturbation problem with integral curvature bound. *Hiroshima Math. J.* 37(3):455–489, 2007.

[22] Yu.G. Reshetnyak. Weak convergence of completely additive vector functions on a set. *Sib. Math. J.*, 9:1039–1045, 1968.

[23] Röger Matthias and Schätzle Reiner. On a modified conjecture of De Giorgi. *Math. Z.* 254(4): 675–714, 2006.

[24] Richard Schoen and Leon Simon. A new proof of the regularity theorem for rectifiable currents which minimize parametric elliptic functionals. *Indiana Univ. Math. J.*, 31:415–434, 1982.

[25] J. E. Taylor. Mean curvature and weighted mean curvature. *Acta Metall. Mater.*, 40:1475–1485, 1992.

[26] A.A. Wheeler. Cahn-Hoffman ξ-vector and its relation to diffuse interface models of phase transitions. *J. Stat. Phys.*, 95(5-6):1245–1280, 1999.

Received ■■■

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