A presentation of the average distance minimizing problem

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Contents

1 Introduction

The purpose of these notes is to present an overview on the fascinating "average distance problem", and more specifically about regularity results. We shall state the main known results without writing the proofs in full details but trying to explain for each of them the key ideas and giving a lot of comments.

The average distance problem has been first introduced in 2002 [4, 7] and was studied by many authors in the last 8 years [3–5, 7, 9, 12–14, 16]. However, at the moment when this note is written, the optimal regularity for the minimizers is still an open problem even in dimension 2, although it deals with elementary euclidian geometry in the plane.

During the last year, three new papers appeared more or less at the same time [3, 9, 16] which gave a new impulsion in the subject. One of those was written by the author of the present note, which takes the opportunity to make here a quick overview and try to collect in a single paper all the known results about the regularity of minimizers that was proved so far.

One objective of this note is for instance to provide enough material to anyone that would like to attack the open questions at the end of the present paper. Indeed, the problem is very elementary in nature but some questions seem quite challenging and could interpellate the curiosity of anyone.

As a matter of fact, although the present paper will contain very recent statements, there will be no real new result, except in Section 6 which contains an original but simple proposition, and except in Section 7 which annonces some new results that are done in a work in progress together with E. Mainini. However, we will try to present all the past results from a unified and maybe slightly different point of view than their original authors, which could be also interesting for specialists of the field.

Of course this paper has not the pretention of being exhaustive on the subject. Its content has been very influenced by some choices from the author, and we really encourage the reader to see the bibliography for more information and other point of views on the problem.

Let us now define the average distance problem. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, $\ell > 0$ a fixed constant, and μ a given Borel measure on Ω. In the sequel we will always assume μ to be absolutely continuous with respect to the Lebesgue measure, and with density in L^p for some $p \geq 1$. We denote d the standard euclidian distance in $\mathbb{R}^N$ and focus on the following minimization problem,

$$\min_{\Sigma \in \mathcal{A}} \mathcal{F}(\Sigma), \qquad \text{where } \mathcal{F}(\Sigma) := \int_\Omega d(x, \Sigma)\, \mathrm{d}\mu(x), \tag{1.1}$$

and where the minimum is taken over the family $\mathcal{A}$ of all the compact and connected sets $\Sigma \subset \Omega$ satisfying the length constraint $\mathcal{H}^1(\Sigma) \leq \ell$. Here $\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure. A precise definition of Hausdorff measure will be given in Section 6, but at first glance it might be enough to know that it coincides with the usual definition of length when Σ is a regular curve, say.

This problem also known as the "irrigation problem", was introduced by Buttazzo, Oudet and Stepanov in [4] and then in [7] in a more general formulation in terms of optimal mass transport problem with "free Dirichlet regions". In the sequel we will call Σ_{opt} an optimal set for the problem (1.1), but most of the time we will avoid the "*opt*" and call it simply Σ, when no confusion is possible with another competitor in $\mathcal{A}$.

An easy interpretation of the Problem (1.1) is the following. One could consider Σ as being a ressource of limited length (for instance some water in pipes) that one wants to place in the domain Ω in such a way that the average cost for people living in Ω to reach the resource Σ is minimal, according to the density of population given by the measure μ. One can find in [4, 5, 7, 12, 14] some more detailed interpretations of Problem (1.1).

We would like to mention that there exists a dual problem to Problem (1.1), which is called the "maximum distance problem" and has similar properties. We refer the reader to [12] for more details.

The existence of minimizers is rapidly established : according to the very classical Blaschke and Golab Theorems, the class $\mathcal{A}$ is compact for the Hausdorff distance and it is easy to prove that the average distance is continuous for this convergence because of the uniform convergence of $x \mapsto d(x, \Sigma)$.

Then in [7], some qualitative properties of minimizers is studied and it is proved in dimension 2 that they are topologically a tree composed by a finite union of simple curves joining by number of 3.

After [7] the question of further regularity remains difficult to solve, and it might be probable that some minimizers are not composed by C^1 curves. This is still an open problem so far, and some partial results are contained in [9, 13] and discussed in [4, 15].

Observe that the class of competitors $\mathcal{A}$ already has some nice rectifiable properties. Indeed, any set $\Sigma \in \mathcal{A}$ is compact, connected, and with $\mathcal{H}^1(\Sigma) < +\infty$ thus it is automatically rectifiable. A simple argument in [7] shows that any minimizer Σ is furthermore Ahlfors regular, so that any minimizer is actually uniformly rectifiable in the sense of David and Semmes [8]. This last fact is quite an exotic result since it has never been used so far to prove some further substantial regularity properties or quantitative properties on minimizers. The question then is wether one could go beyond rectifiability.

A key object in all the study of the average distance problem is the pull-back measure of μ with respect to the projection onto Σ. More precisely, if μ does not charge the Ridge set of Σ (this is the case for instance when μ is absolutely continuous with respect to the Lebesgue measure), then it is possible to chose a measurable selection of the projection multimap onto Σ, i.e. a map $\pi_\Sigma : \Omega \to \Sigma$ such that $d(x, \Sigma) = d(x, \pi_\Sigma(x))$. Then we define the measure ψ as being

$$\psi(A) := \mu(\{\pi_\Sigma^{-1}(A)\}), \text{ for any borel set } A \subset \Omega. \tag{1.2}$$

In other words $\psi = \pi_\Sigma \sharp \mu$. This measure ψ is supported on Σ and in particular depends on Σ itself but we will not make this dependance explicit to lighten the notations. The measure ψ is actually related to the curvature of Σ, and controlling ψ means controlling the regularity of Σ.

In this note, we will more or less follow the history of the problem chronologically. Before discussing the qualitative properties and regularity of minimizers, we begin with two other problems related to the average distance functional.

1.1 Link with optimal transport

Problem (1.1) can be seen as an optimal mass transport problem with a "free Dirichlet region". This was one of the main motivations of Buttazzo and Stepanov to introduce it in [7] (see also [14]). Let μ^+ and μ^- be two Borel measures on Ω. Traditionally μ^+ corresponds to a density of population representing the home-places, and μ^- is the density of workplaces. The set Σ is now representing an urban transportation network, and one wants to find an optimal network that minimizes the average cost for people moving from the home-places to the workplaces. In this model, the transportation cost is assumed to be zero whenever one moves inside the network Σ, otherwise the cost is proportional to the walking distance. Therefore, the total cost $c_\Sigma(x,y)$ to go from point x to point y in Ω is given by

$$c_\Sigma(x,y) = \min(|x-y|,\ d(x,\Sigma)+d(y,\Sigma)).$$

Now for a fixed $\Sigma \in \mathcal{A}$ one can calculate the corresponding total transportation cost by solving the Monge-Kantorovitch problem associated to c_Σ, namely

$$I(\Sigma) = \inf_{\gamma \in Adm(\mu^+,\mu^-)} \int_{\Omega\times\Omega} c_\Sigma(x,y) d\gamma(x,y)$$

where $Adm(\mu^+,\mu^-)$ is the class of all admissible γ satisfying $\pi^\pm \sharp \gamma = \mu^\pm$ with $\pi^\pm : \Omega \to \Omega$ the projections on the first and second coordinates. Then to find an optimal network it remains to minimize on Σ, and solve the problem

$$\min_{\Sigma\in\mathcal{A}} I(\Sigma). \tag{1.3}$$

The following proposition says that Problem (1.3) reduces to Problem (1.1).

Proposition 1. [14, Theorem 8.2] *For any μ^+ and μ^- there exists a measure μ such that any minimizer for Problem* (1.3) *is a minimizer for Problem* (1.1) *according to this measure μ.*

1.2 Link with q-Compliance minimizers

In this paragraph we restrict ourself to dimension $N = 2$. For any $\Sigma \in \mathcal{A}$ and $q > 0$ let us denote u_Σ the solution for the problem

$$\begin{cases} -\Delta_q u = 1 & \text{in } \Omega \setminus \Sigma \\ u = 0 & \text{on } \partial\Omega \cup \Sigma \end{cases} \tag{1.4}$$

where Δ_q denotes the q-Laplace operator. For a given Σ, the solution u_Σ for Problem (1.4) can be obtained by minimizing the energy

$$E_\Sigma(u) := \frac{1}{q}\int_{\Omega\setminus\Sigma} |\nabla u|^q \, \mathrm{d}x - \int_\Omega u \, \mathrm{d}x$$

among all the functions $u \in W_0^{1,q}(\Omega \backslash \Sigma)$.

The q-Compliance energy is then defined by

$$\mathcal{C}_q(\Sigma) := \big(1 - \frac{1}{q}\big) \int_\Omega u_\Sigma(x) \tag{1.5}$$

and one can consider the optimal set for the problem

$$\min_{\Sigma \in \mathcal{A}} \mathcal{C}_q(\Sigma). \tag{1.6}$$

In [6] it is proved that the average distance problem is equivalent to an ∞-Compliance problem as it is stated in the following proposition.

Theorem 2. [6, Theorem 3] *We endow $\mathcal{A}$ with the Hausdorff distance. Then as $q \to +\infty$ the functionals $\mathcal{C}_q$ Γ-converge to the average distance functional $\widetilde{\mathcal{F}}$ given by*

$$\widetilde{\mathcal{F}}(\Sigma) = \int_\Omega dist(x, \Sigma \cup \partial\Omega) dx.$$

The proof of Theorem 2 is not very difficult and rely on the classical fact that the L^q norm converges to the L^∞ norm. Therefore at the limit one identifies a problem of maximizing the average of some Lipschitz function u with $|\nabla u| \leq 1$, and the solution for that problem is achieved by the distance functional. We refer to [6, Section 5] for more details.

Notice that minimizers of Problem (1.6) for q fixed has never been studied even for the special case $q = 2$, although some rough arguments suggests some regularity on the optimal set Σ for this problem.

2 Topological Description

The topological description of minimizers was studied by Buttazzo and Stepanov in 2003 in [7]. All the contributions for this section are coming from [7], with some further improvements in dimension $N > 2$ due to Paolini and Stepanov [12] one year later.

The most important result proved in [7], which unfortunately holds only in dimension $N = 2$, is that Σ is a finite union of Lipschitz arcs, that meet by number of three at some finite number of triple junctions (see Proposition 3 below). It is also proved that Σ has no loops (thus it is topologically a tree) and this last fact is true for any dimension $N \geq 2$. Concerning the regularity, it is only proved in [7] that Σ is Ahlfors-Regular, but some further regularity will be the purpose of the next section.

The work in [7] is done for minimizers of Problem (1.3) i.e. optimal mass transport problem with a free Dirichlet region. But as Proposition 1 says, this problem is actually equivalent to the average distance problem, and one could rewrite all the paper [7] in this slightly easier framework without loss of generality.

For some reasons that we will try to explain below, this result holds only in dimension $N = 2$, and it is still an open problem to know wether the same description holds in higher dimension. The key point would be to extend Lemma 7 below, that is not known to hold in higher dimension. Here is a precise statement in the 2-dimensional case.

Theorem 3. [7] *Let the dimension $N = 2$ and $\ell < +\infty$. Assume that μ is a Borel measure on $\Omega \subset \mathbb{R}^2$, absolutely continuous with respect to the Lebesgue measure and with density in $L^p(\Omega, dx)$ for some $p > 4/3$. Let Σ be an optimal set for the problem* (1.1)*. Then the following hold.*
i) Σ is a finite union of simple curves.
ii) Σ contains no loop.
iii) the curves in Σ meet by number of 3.

Remark 4. Actually as was said before, *ii*) is still valid for any dimension $N > 2$. It was proved to hold for dimension $N = 2$ by Buttazzo and Stepanov [7] and then it was extended by Paolini and Stepanov for any dimension in [12].

The proof of Theorem 3 is fairly long and we will emphasis the five main steps giving only some ideas.

Step 1. Infinitesimal behavior in terms of additional length

The first step is to compute how much one can win in the average distance functional in terms of ε, attaching a little piece of set on Σ of size exactly ε. This is the content of the following Lemma.

Lemma 5. [7, 12] *For any $\Sigma \in \mathcal{A}$ there exists ε_0 such that for any $\varepsilon \leq \varepsilon_0$ one can find a connected set Σ_ε such that*

$$\mathcal{H}^1(\Sigma_\varepsilon) = \mathcal{H}^1(\Sigma) + \varepsilon$$

and

$$\mathcal{F}(\Sigma_\varepsilon) \leq \mathcal{F}(\Sigma) - \varepsilon^\alpha,$$

where $\alpha = 3/2$ if $N = 2$ and $\alpha = 2$ for $N > 2$.

Remark 6. For $N = 2$ it is a result from [7] and for $N > 2$ it holds from [12].

Proof. We will sketch the proof of Lemma 5 contained in [12], which gives the weaker exponent $\alpha = 2$ in dimension 2 but which is much more easier to prove and which will be enough for the next steps. Indeed, for any ε one can discretize a bit and cover Σ with a finite union of balls $B(x_i, \varepsilon)$, in such a way that the $B(x_i, \varepsilon/2)$ are disjoint and each of them contains a piece of Σ of length at least $\varepsilon/2$. Then we select a ball $B(x_{i_0}, \varepsilon)$ such that $\psi(B(x_{i_0}, \varepsilon))$ is maximum among all those balls. For that ball it is easy to see that

$$\psi(B(x_{i_0}, \varepsilon)) \geq \varepsilon \frac{\psi(\Sigma)}{2\mathcal{H}^1(\Sigma)}.$$

Then at this point x_{i_0} one attaches a star-like set K_ε of total length ε. This set K_ε is of the form

$$K_\varepsilon := \bigcup_{k=1}^{N} \{te_k;\ t \in [-\varepsilon/2N, \varepsilon/2N]\},$$

where the e_k form an orthonormal basis of $\mathbb{R}^N$. This set K_ε permits us to win $C\varepsilon$ in terms of distance, is each direction of the euclidian space $\mathbb{R}^N$, thus for all the points x that was projected onto $B(x_{i_0}, \varepsilon)$ (see [12, Lemma 3.2.]). Therefore the total gain in the average distance functional is

$$C\varepsilon\psi(B(x_{i_0}, \varepsilon)) \geq C\varepsilon^2,$$

and so follows the Lemma. □

Step 2. Absence of Loop

The second step is to prove that Σ has no loop (i.e. a image of S^1 by a homeomorphism). The general idea is that if Σ had a loop, then one could cut a piece somewhere in this loop and Σ would stay connected, thus be an admissible competitor. Then we have right to add somewhere this winning length using Step 1 in order to decrease the functional. The most difficult part is to prove that one can cut in the loop by paying only something like $c\varepsilon^2$ in the average distance, with c small as we want. Then thank to Step 1 we can win $C\varepsilon^2$ attaching the star-like set somewhere else and get the desired contradiction because $C > c$. A way to cut judiciously in the loop is contained in [12, Lemma 5.4.]. The idea is to first find a point $\mathcal{H}^1$-a.e. in the loop such that Σ admits a tangent line at this point, and then for ε small enough one cut the ball $B(x, \varepsilon)$ and at the same time attach some little pieces of set of size $\varepsilon/10$ "orthogonal" to the tangent line and fixed at some remaining boundary points lying on $\Sigma \cap \partial B(x, \varepsilon)$. By this way all the points in $B(x, 10\varepsilon)$ have lost $C\varepsilon$ (but there are only of total mass $\mu(B(x, \varepsilon) = o(\varepsilon))$ and the points far away didn't lost much thanks to the little orthogonal pieces of segment that was attached for this purpose.

Observe that $\Sigma \setminus B(x, \varepsilon)$ does not remain connected in general, even if x lies inside a loop of Σ. Therefore one has to find a suitable set $D_\varepsilon \subset \Sigma \cap B(x, \varepsilon)$ containing x, of diameter of order ε and that has this property. This is always possible using a topological argument, but one has to prove it.

Step 3. Existence of atom

The next step is to prove the following "key Lemma".

Lemma 7. [7] *Let $N = 2$, and let Σ be an optimal set for the Problem* (1.1)*. Then there exists $x \in \Sigma$ such that*

$$\mu\{y \in \Omega; d(y, \Sigma) = d(y, x)\} > 0.$$

In other words, x is an atom for the measure ψ.

It is very natural to think that any endpoint x of Σ would be an atom for ψ. This has been proven in dimension $N = 2$. Notice that the existence of endpoint is given by Step 2, since we know now that Σ is a tree thus contains at least 2 endpoints. Curiously enough, this Lemma is the one that is not known to hold in higher dimension $N > 2$, even if the existence of endpoint is always true without any restriction on the dimension N. It is worth mentioning that thank to some work from Stepanov [14], all the topological description in dimension $N > 2$ would directly follow from an analogue of Lemma 7 in higher dimension.

The proof of Lemma 7 in dimension 2 is technical, and we refer the reader to [7] for more details.

Step 4. Finite number of curves

The next step is to prove that the value of the atom $\psi(\{x\})$ for any endpoint $x \in \Sigma$ actually does not really depend on x. Namely, there exists a constant $C_0 > 0$ such that $C_0^{-1} \leq \psi(\{x\}) \leq C_0$ for any endpoint $x \in \Sigma$. This is not very difficult to prove. Indeed, if x and y are two endpoints it suffice to compare Σ with the set made by cutting a piece of Σ near y of size r and adding a circle of length exactly r around x. By this way we loose approximatively r times $\psi(\{y\})$ in the average distance functional but up to a constant π we win r in each direction for the points that are projected onto x. We get the desired estimate passing to the limit $r \to 0$ and then exchanging the role of x and y.

From this fact we obtain that the number of endpoints must be finite, and since Σ is a tree we deduce that the number of curves and junctions points must also be finite, so follows $i)$.

Step 5. Only triple junctions

The final step is to prove that only 3 curves can meet at a junction point. This is done arguing by contradiction and comparing with a Steiner connexion. This operation obviously provide some gain of length, and the distance functional is not much affected because there is almost no points that are projected near the junction point. We refer to [7, Theorem 8.1.] for a detailed proof.

3 Some further regularity in dimension 2

3.1 Ahlfors regularity

A closed set Σ is said to be Ahlfors regular if there exists some constants $C_1, C_2 > 0$ and a radius $r_0 > 0$ such that

$$C_1 r \leq \mathcal{H}^1(\Sigma \cap B(x,r)) \leq C_2 r \qquad \forall x \in \Sigma, \forall r \leq r_0.$$

Firstly notice that in our case since Σ is connected, then the lower bound is trivial with $C_1 = 1$ provided $r \leq diam(\Sigma)$. Subsequently, to prove the Ahlfors regularity of minimizers it is enough to prove the upper bound. This is not very difficult to prove in dimension 2 with $C_2 = 3\pi$, and this is done in [7]. Indeed assume by contradiction that $\mathcal{H}^1(\Sigma \cap B(x,r)) \geq 3\pi r$ for some ball $B(x,r)$ centered on Σ. Then it suffice to compare the optimal set Σ with $\Sigma' := (\Sigma \setminus B(x,r)) \cup \partial B(x,r)$. By this way Σ' is still a competitor, the average distance is improved or at most stays unchanged (except for the points inside $B(x,r)$ of total mass $o(r)$), and it remains some little length, namely $\mathcal{H}^1(\Sigma \cap B(x,r)) - 2\pi r \geq \pi r$, to add somewhere in order to decrease the average distance (provided $\mu \in L^p$ with some $p > 4/3$) and get a contradiction when r is small enough.

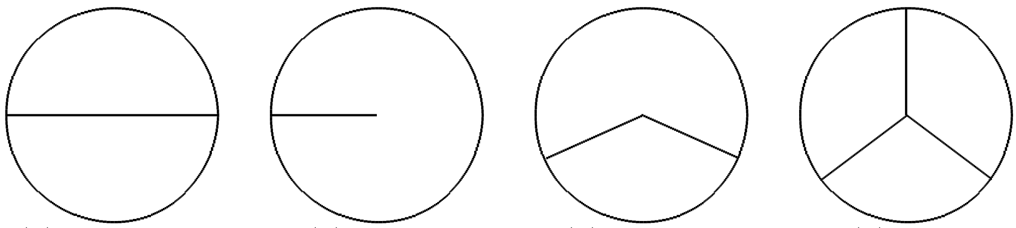
The Ahlfors regularity in dimension $N > 2$ is more delicate but remains true. This was proved by Paolini and Stepanov in [12]. Observe that the competitor $(\Sigma \setminus B(x,r)) \cup \partial B(x,r)$ of Buttazzo and Stepanov does not work anymore in higher dimension and one has to argue with differently.

Theorem 8. [12] *Let Σ be an optimal set for the Problem* (1.1) *in dimension $N \geq 2$, with $\mu \in L^p$ for some $p \geq N/(N-1)$. Then Σ is Ahlfors regular.*

Remark 9. The lower bound $p > 4/3$ of Buttazzo and Stepanov is slightly better than the one of Theorem 8 for the particular case $N = 2$.

3.2 Blow up limits

Thank to Santambrogio and Tilli [13], the blow up limit of the minimal set Σ at point $x \in \Sigma$ exists for any x, and belongs to a short list. Indeed, they prove that any blow up sequence $\Sigma_r := \frac{1}{r}(\Sigma \cap B(x,r) - x)$ with $x \in \Sigma$, converges in $B(0,1)$ (for the Hausdorff distance) to some limit $\Sigma_0(x)$ when $r \to 0$, and the limit is one of the following below, up to a rotation.



$\Sigma_0(x)$ is a diameter	$\Sigma_0(x)$ is a radius	$\Sigma_0(x)$ is a corner	$\Sigma_0(x)$ is a tripod
x is a regular point	x is an endpoint	x is a corner point	x is a junction point
$\psi(\{x\}) = 0$	$\psi(\{x\}) > 0$	$\psi(\{x\}) > 0$	$\psi(\{x\}) = 0$

The existence of blow up limits is proven by Γ-convergence. For any $r > 0$, Santambrogio and Tilli identify a minimisation problem that Σ_r satisfies in the unit ball and they are able to compute the Γ-limit of this problem. At the limit they obtain that $\Sigma_0(x)$ itself solves a certain minimisation problem in the unit ball that somehow inherit the properties of the original average distance minimisation problem. This is enough to classify the possible limiting sets, and the only possibilities are given by the list above, up to rotations. They use some first order calculus and in particular they need the "key Lemma" (Lemma 7). Consequently, the blow up limits are characterized only in dimension $N = 2$.

It should be pointed out that no quantitative estimate can be derived from their method. In other words, the existence of a blow up limit is established without knowing any speed of convergence of Σ_r to its "tangent set" Σ_0. Therefore, no regularity result follows directly from this blow up procedure, unfortunately !

On the other hand the Γ-convergence argument provides some more information about the tangent set $\Sigma_0(x)$, as for instance its direction, involving the measure μ and the set of projected points $\pi_\Sigma^{-1}(\{x\})$. More precisely, for any atom $x \in \Sigma$ for the measure ψ (i.e. x is either a corner point or an endpoint) let us define ν_x the image measure of $\mu \llcorner \pi_\Sigma^{-1}(x)$ by the application $y \mapsto \frac{y-x}{\|y-x\|}$. Then we define the vector

$$\bar{v}(x) := \int_{S^1+x} (v - x)\, \mathrm{d}\nu_x(v).$$

It follows from [13] that the vector $\bar{v}(x)$ is oriented in the direction of the blow up limit $\Sigma_0(x)$, i.e. is in the direction or the radius if x is an endpoint, and in the direction of the bisectrix of the angle if x is a corner point. Let us now denote

$$\lambda(x) := \|\bar{v}(x)\|.$$

Then it can be proved (see e.g. Remark 38 of [9]) that there exists a constant λ_0 such that $\lambda(x) = \lambda_0$ for any endpoint x. Now if x is a corner point, the constant λ_0 is linked to the aperture of the corner by the following very nice identity coming from [13, Theorem 3.7.],

$$\lambda_0 \cos\big(\frac{\theta(x)}{2}\big) = \lambda(x)$$

where $\theta(x)$ the smallest angle between the two rays of the bow up limit at point x.

Notice that we always have $\lambda(x) \leq \psi(\{x\})$ so that for any corner point x the value of $\psi(\{x\})$ bounds its aperture. It is not very difficult to exclude all the corners of aperture greater than $\frac{2\pi}{3}$ (see [9, Proposition 40]). On the other hand there is no lower bound on the aperture of corners, this would actually be a great advance in the problem.

3.3 $C^{1,1}$-regularity near triple points

The second nice contribution of Santambrogio and Tilli [13] is a sufficient condition to obtain $C^{1,1}$ regularity in a neighborhood of a point $x \in \Sigma$, involving the diameter of the set of points that are projected on $\Sigma \cap B(x, r)$. Since this condition is satisfied in a small enough neighborhood of any triple point, they obtain that any triple point admits a small neighborhood in which the three pieces of curve of Σ are $C^{1,1}$.

Theorem 10. [13, Theorem 4.1] *Let Σ be an optimal set for the Problem* (1.1) *in dimension $N = 2$, with some measure $\mu \in L^\infty$. Then there exists a constant $C > 0$ such that any curve $\gamma \subset \Sigma$ satisfying*

$$\sup_{x\in\gamma} diam(\pi_\Sigma^{-1}(\{x\})) \leq C, \tag{3.1}$$

is a $C^{1,1}$ regular curve.

Remark 11. In [13] one can also find a bound on the curvature of γ depending on the best constant in (3.1).

Remark 12. Since $\pi_\Sigma^{-1}(\{x\}) = \emptyset$ for any triple point x, by semicontinuity of the diameter we deduce from Theorem 10 that any triple point of Σ admits a small neighborhood for which the three pieces of curve are $C^{1,1}$.

Remark 13. The constant C in the statement of Theorem 4.1. could be fairly small, and depends on all the parameters of the problem, i.e. Ω, μ, and could even depend on Σ itself.

Proof of Theorem 10. Let us explain now why the regularity result in Theorem 10 holds and how the condition (3.1) appears. The first step is to use the "key Lemma" (Lemma 7) in order to obtain a refinement of Lemma 5. More precisely, using the existence of atom for ψ at an endpoint it can be shown that Lemma 5 still holds with $\alpha = 1$ (instead of 4/3). The idea is to find a ball contained in $\pi_\Sigma^{-1}(x_0)$, for x_0 any endpoint, and add a segment of size ε in the direction of this ball to win $C\varepsilon$ in the average distance functional (instead of $\varepsilon^{4/3}$ as in the original statement of Lemma 5).

Now to find some regularity the natural idea is to compare $\Sigma \cap B(x,r)$ with a diameter, where x is a point of Σ. Assume for simplicity that $\Sigma \cap \partial B(x,r)$ consists in exactly two points z and y which are approximatively diametrally opposed (this is always possible for r small enough by a technical Lemma contained in [13]). Then we compare Σ with the set $\Sigma \setminus B(x,r) \cup [z,y]$. Denoting D_H the Hausdorff distance between $\Sigma \cap B(x,r)$ and the segment $[z,y]$, we obtain that all the points of $\pi_\Sigma^{-1}(B(x,r))$ have lost at most D_H in terms of distance, whereas by the refined Lemme 5 we know that we can win C times the difference of length. Therefore we obtain that

$$C(\mathcal{H}^1(\Sigma \cap B(x,r)) - \mathcal{H}^1([z,y])) \leq D_H \psi(B(x,r)).$$

On the other hand simply using Pythagoras we can prove that (see [13, (2.2)])

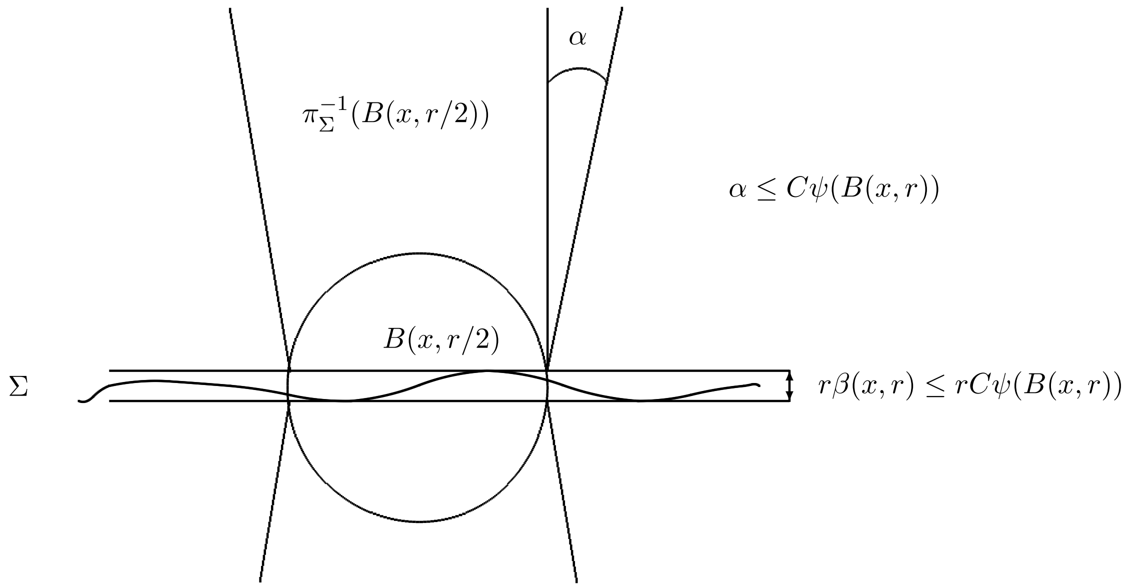
$$\mathcal{H}^1(\Sigma \cap B(x,r))^2 - \mathcal{H}^1([z,y])^2 \geq D_H^2.$$

Therefore we obtain the following crucial estimate on the flatness of Σ in the ball $B(x,r)$,

$$\frac{D_H}{r} \leq C\psi(B(x,r)) \tag{3.2}$$

The quantity $\frac{D_H}{r}$ measures how flat is the set Σ is in the ball $B(x,r)$. It is comparable to the classical β-number of Peter Jones, and we will denote $\beta(x,r) := \frac{D_H}{r}$. A consequence of (3.2) is that any decay on $\psi(B(x,r))$ like a power of r leads to some $C^{1,\alpha}$ regularity on Σ (see also Proposition 22 related to that fact).

Subsequently, Theorem 10 follows from a decay estimate like $\psi(B(x,r)) \leq Cr$, that will hold provided that the diameter of $\pi_\Sigma^{-1}(\{x\})$ is small enough. This decay estimate is found by a nice geometrical argument. Indeed, up to remove a finite set of atoms we may assume that $\psi(B(x,r))$ is fairly small, say less than 1/100 so that Σ is already 1/100 flat in the ball $B(x,r)$. Then we use the following intuitive argument : If $\Sigma \cap B(x,r)$ is completely flat (i.e. is a segment), then the transportation rays ending in $\Sigma \cap B(x,r)$ (i.e. segments of the form $[y, \pi_\Sigma(y)]$ with $\pi_\Sigma(y) \in \Sigma \cap B(x,r)$) are exactly orthogonal to Σ. Now if $\Sigma \cap B(x,r)$ is "almost flat", namely $\beta(x,r) \leq \delta$, then the angle between the transportation rays ending in $\Sigma \cap B(x,r/2)$ which are large enough (i.e. such that y is outside $B(x,10r)$) is less than $C\delta$. And since $\beta(x,r)$ is always less than $\psi(B(x,r))$ (inequality (3.2)), this angle is actually less than $C\psi(B(x,r))$. Therefore we can localize the set $\pi_\Sigma^{-1}(B(x,r/2))$ in an "almost" rectangular domain with borders making an angle comparable to $\psi(x,r)$ like in the following picture.



Now for some fixed r_0 let us denote $D := diam(\pi_\Sigma^{-1}(B(x,r_0))$. From the above geometrical facts we get the following important estimate for any $r < r_0$,

$$\psi(B(x,r/2)) = \mu(\pi_\Sigma^{-1}(B(x,r/2))) \leq CD(\psi(B(x,r)+r) + Cr^2. \tag{3.3}$$

Therefore, if $CD < 1$ one can bootstrap the argument, iterate (3.3) and sum the estimates to get

$$\psi(B(x,r)) \leq Cr$$

as desired. This is how one obtains some $C^{1,1}$ regularity provided D is small enough, and concludes the proof of the Theorem. □

3.4 Weak regularity outside triple points

Away from triple points we are not able to prove $C^{1,1}$ regularity, but it has been proved in [9] that Σ is locally at least as regular as the graph of a convex fonction, namely that the Right and Left tangent maps admit some Right and Left limits at every point and are semicontinuous. More precisely, for a given parametrization γ of an injective Lipschitz arc $\Gamma \subset \Sigma$, by existence of blow up limits one can define the Left and Right tangent half-lines at every point $x \in \Gamma$ by

$$T_R(x) := x + \mathbb{R}^+. \lim_{h\to 0^+} \frac{\gamma(t_0+h)-\gamma(t_0)}{h}$$
$$T_L(x) := x + \mathbb{R}^+. \lim_{h\to 0^+} \frac{\gamma(t_0-h)-\gamma(t_0)}{h}$$

Then we have the following.

Theorem 14. [9] *Let $\Gamma \subset \Sigma$ be an open injective Lipschitz arc. Then the Right and Left tangent maps $x \mapsto T_R(x)$ and $x \mapsto T_L(x)$ are semicontinuous, i.e. for every $y_0 \in \Gamma$,*

$$\lim_{\substack{y\to y_0\\ y<_\gamma y_0}} T_L(y) = T_L(y_0) \quad \textit{and} \quad \lim_{\substack{y\to y_0\\ y>_\gamma y_0}} T_R(y) = T_R(y_0).$$

In addition the limit from the other side exists and we have

$$\lim_{\substack{y\to y_0 \\ y>_\gamma y_0}} T_L(y) = T_R(y_0) \quad \textit{and} \quad \lim_{\substack{y\to y_0 \\ y<_\gamma y_0}} T_R(y) = T_L(y_0).$$

An interesting and immediate consequence is the following result.

Corollary 15. [9] *Assume that $\Gamma \subset \Sigma$ is a relatively open subset of Σ that contains no corner points neither triple points. Then Γ is locally a C^1 regular curve.*

Sketch of proof for Theorem 14. The proof of Theorem 14 rely on the following strategy : if the diameter of transported set is small in a ball $B(x,r)$, then we know by Theorem 10 that $\Sigma \cap B(x,r)$ is $C^{1,1}$. Moreover the $C^{1,1}$ norm depends only on the length of the longest transportation ray ending in $\Sigma \cap B(x,r)$. On the other hand, if the diameter of transported sets is uniformly bounded from below in a ball $B(x,r)$, i.e. the transported rays are uniformly large, then $\Sigma \cap B(x,r)$ has a "uniform external ball" condition and this is enough to control the oscillation of semi-tangents. The idea is then to glue together the two arguments to control the oscillation of tangents everywhere. In particular there is a little difficulty coming from the non-continuity of $x \mapsto diam(\pi_\Sigma^{-1}(x))$. This application is upper-semicontinuous but not lower. In other words it could happen that some points x_n with very small transported set accumulates to a point x_0 that has a very large one. Therefore one considers in $\Sigma \cap B(x,r)$ some intervals, maximal for the property (3.1). By upper semicontinuity of the diameter, this is a relatively open set in Σ and we know by Theorem 10 that it is $C^{1,1}$. On the other hand by construction, at each boundary point of those maximal intervals, one has a point y such that $diam(\pi_\Sigma^{-1}(y)) \geq C$, and the $C^{1,1}$ regularity holds up to this boundary point, with uniform norm. We then conclude by controling the oscillation of tangents in any situation, by one argument or another depending on the localisation of $y \in \Sigma \cap B(x,r)$, whether it lies in one of the maximal intervals or has a large transported set. We refer the reader to [9, Theorem 31] for more details. □

4 $C^{1,1}$ is optimal

Recently, Tilli [15] proved that any $C^{1,1}$ simple curve is a minimizer for the average distance problem. Consequently it is not possible to prove any further regularity result on Σ, and $C^{1,1}$ is optimal.

Theorem 16. [15, Theorem 1.1] *Let $\gamma : [0,\ell] \to \mathbb{R}^2$ be a curve of class $C^{1,1}$, parameterized by arclentgh, of length ℓ. Let $R > 0$ be a number such that the curvature bound*

$$|\gamma''(t)| \leq \frac{1}{R} \textit{ for a.e. } t \in [0,\ell] \tag{4.1}$$

holds, and suppose that $\ell \leq \pi R$. Then, for every $\lambda \in (0,R)$, the set $\Sigma_\gamma := \gamma([0,\ell])$ is a minimizer for the Problem (1.1), where Ω is defined as

$$\{x \in \mathbb{R}^2; d(x,\Sigma_\gamma) \leq \lambda\}$$

and μ is the Lebesgue measure.

Remark 17. The bound (4.1) and $\ell \leq \pi R$ implies that γ is an injective curve.

Proof of Theorem 16. The proof is simple but very tricky. It relies on the following nice geometrical equality, which is true for any curve γ that fulfills the assumptions of the Lemma,

$$|\{x \in \mathbb{R}^2; 0 < d(x, \Sigma_\gamma) < s\}| = 2\ell s + \pi s^2. \tag{4.2}$$

The second ingredient of Tilli's proof is the following inequality which is true for any compact and connected set Σ such that $\mathcal{H}^1(\Sigma) \leq \ell$,

$$|\{x \in \mathbb{R}^2; 0 < d(x, \Sigma) < s\}| \leq 2\ell s + \pi s^2. \tag{4.3}$$

A reference for inequality (4.3) is for instance [11, Lemma 4.2.]. Now denoting $D := diam(\Omega)$ one can compute for any competitor Σ of Σ_γ,

$$\begin{aligned} \int_\Omega d(x,\Sigma)\,dx &= \int_0^D |\{x \in \mathbb{R}^2; d(x,\Sigma) > s\}| ds \\ &= D|\Omega| - \int_0^D \{x \in \Omega; 0 < d(x,\Sigma) < s\} ds \\ &\geq D|\Omega| - \int_0^D \min(|\Omega|, 2\ell s + \pi s^2) ds. \end{aligned} \tag{4.4}$$

For (4.4) we used (4.3) together with the obvious inequality

$$|\{x \in \Omega; 0 < d(x,\Sigma) < s\}| \leq \min(|\Omega|, |\{x \in \mathbb{R}^2; 0 < d(x,\Sigma) < s\}|)$$

which is true for any $s \geq 0$. Finally we conclude with (4.2), which says that inequality (4.4) is an equality for any s, if and only if $\Sigma = \Sigma_\gamma$ and therefore Σ_γ is a minimizer. $\square$

Remark 18. An analogue of Equality (4.2) remains true if we replace Σ_γ by any convex set, and this will be exploited in Section 7.

5 The Euler-Lagrange equation

5.1 Reduction to the penalized problem

As usual in variational calculus on a restricted class, it may happen for a small variation $\Phi_\varepsilon(\Sigma)$ of Σ, that the length constraint $\mathcal{H}^1(\Phi_\varepsilon(\Sigma)) \leq \ell$ is violated. Hence, to compute the Euler Lagrange equation associated to Problem (1.1), a possible strategy is to consider first the penalized functional

$$\mathcal{F}(\Sigma) + \lambda \mathcal{H}^1(\Sigma)$$

for some constant λ, for which any competitor Σ is admissible without length constraint. Then, one has to prove that both problems actually has same first order equations, provided the constant λ is chosen judiciously.

In [3], the first order equation is computed for the penalized functional by Buttazzo, Mainini and Stepanov. We refer for instance to [1] page 355 for the definition and classical properties of the tangential divergence $\mathrm{div}^\Sigma \Phi$.

Proposition 19. [3] *For every compact and connected set $\Sigma \subset \Omega$ and for every $\Phi \in C_0^\infty(\mathbb{R}^2, \mathbb{R}^2)$ one has*

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{F}((\mathrm{Id} + \varepsilon\Phi)(\Sigma))\Big|_{\varepsilon=0} = \int_{\mathbb{R}^2} \langle \Phi(\pi_\Sigma(x)), \frac{\pi_\Sigma(x) - x}{|\pi_\Sigma(x) - x|} \rangle \,\mathrm{d}\mu(x). \tag{5.1}$$

As a consequence, for a given $\lambda > 0$, if Σ is a minimizer for the functional

$$\mathcal{G}(\Sigma') := \int_\Omega d(x, \Sigma')\,\mathrm{d}\mu(x) + \lambda \mathcal{H}^1(\Sigma') \tag{5.2}$$

over all compact and connected sets $\Sigma' \subset \Omega$, then for all $\Phi \in C_0^\infty(\mathbb{R}^2, \mathbb{R}^2)$ one has

$$\int_{\mathbb{R}^2} \langle \Phi(\pi_\Sigma(x)), \frac{\pi_\Sigma(x) - x}{|\pi_\Sigma(x) - x|} \rangle \,\mathrm{d}\mu(x) + \lambda \int_\Omega \mathrm{div}^\Sigma \Phi \,\mathrm{d}\mathcal{H}^1 = 0. \tag{5.3}$$

Now in order to apply Proposition 19 to minimizers of Problem (1.1), one has to find a λ_0 such that the first order equations for the two minimizing problems are the same. This is like what we would obtain by the classical Lagrange multipliers theorem, but here since the Fréchet differentiability of the average distance functional is not known, the penalization process is computed by hand. The idea is very intuitive and perhaps more instructive as well since it gives the explicit value of λ_0 in terms of measure ψ at any endpoint x_0.

To be more precise, let x_0 be an endpoint of Σ that we will assume, up to a translation, being the origin. Following [13], we recall the definition of ν given in Section 3.2, which is the image measure of $\mu \llcorner k^{-1}(\{x_0\})$ by the application $x \mapsto \frac{x}{\|x\|}$. Then we define the vector

$$\bar{v} := \int_{S^1} v \,\mathrm{d}\nu(v).$$

By [13] Theorem 3.2. we know that Σ admits a tangent line at x_0 which direction is given by the vector $-\bar{v}$. Now we define the constant

$$\lambda_0 := \int_{S^1} v.\frac{\bar{v}}{\|\bar{v}\|}\,\mathrm{d}\nu(v) = \|\bar{v}\| \tag{5.4}$$

that actually does not depend on the choice of endpoint. Then we have the following.

Proposition 20. *Let Σ be a minimizer for the problem* (1.1) *and x_0 be one of its endpoint. Then Equation* (5.3) *holds with $\lambda = \lambda_0$ defined in* (5.4) *and for every $\Phi \in C_0^\infty(\mathbb{R}^2, \mathbb{R}^2)$ compactly supported in the complement of $\{x_0\}$.*

The idea to prove Proposition 20, that was first used by Santambrogio and Tilli in the proof of [13, Theorem 3.5.], is to quantify how much one can win or loose in the functional adding a piece of segment of size r starting at the endpoint x_0 or erasing a piece of curve of size r from the same endpoint. On can prove that the two operations have a cost in $\lambda_0 r + o(r)$ and this is enough to conclude. A precise proof of Proposition 20 can be found in [9].

5.2 A stationary set containing a corner point

Recall that by "corner point" we mean a point in Σ at which the blow up limit is a union of two radius with a strict angle (different from 180 degrees). Although it is not difficult to find some examples of domains Ω where any minimizer Σ necessarily contains a triple point, it remains an open question as to whether a minimizer could actually contain corner points. On the other hand in [3], the first order Euler-Lagrange equation is computed (see Proposition 19 above) and the existence of a stationary set Σ that contains a corner point is exhibited, which is sketched in the picture below.

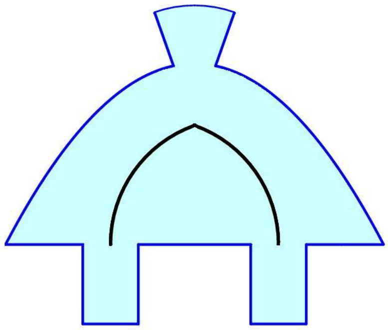


Figure 1: The example of Buttazzo, Mainini and Stepanov [3].

Here the blue part represents the domain Ω and the black curve is Σ, composed by two arcs of circles that meet at the top making a corner point. Each atom for ψ (i.e. the two endpoints and corner point at the top) corresponds to a dirac mass in the curvature of Σ which is compensated by the measure of points projected onto each atom.

5.3 Higher order calculus

The second order derivative of the average distance functional is much more involved, but can also be computed in a quite general framework. This is part of a work in progress with Mainini [10]. In 2002, Buttazzo, Oudet and Stepanov [4] already used a second order argument to exclude very small segments (the idea was to study minimizers with very small length $\ell \to 0$). After that, the second order condition has never been used anymore in this subject, and it may bring some interesting new results.

For sake of simplicity we just write the formula in the case when Σ is a line, and the diffeomorphism ϕ is normal to that line.

Proposition 21. [10] *Let B be a ball such that $\Sigma \cap B$ is a segment. Let $x \in \Omega \setminus \Sigma$ be a fixed point and let $\Phi_\varepsilon := \mathrm{Id} + \varepsilon\Phi$ be a one parameter diffeomorphism with Φ a vector field compactly supported in B and normal to $\Sigma \cap B$. We denote $\Sigma_\varepsilon := \Phi_\varepsilon(\Sigma)$. If $\pi_\Sigma(x) \in \Sigma \cap B$, then*

$$\mathrm{dist}(x, \Sigma_\varepsilon) = \mathrm{dist}(x, \Sigma) - |\Phi(\pi_\Sigma(x))|\varepsilon - \frac{1}{2}\,\mathrm{dist}(x, \Sigma)\,|\partial_\tau \mathbf{\Phi}(\pi_\Sigma(x)|^2\,\varepsilon^2 + o(\varepsilon^2).$$

Notice that the first order term is coherent with Proposition 19. The proof of Proposition 21, even in this simple formulation, is already interesting to study. Indeed, to compute the first order variation of the distance functional, it is enough to approximate $\pi_{\Sigma_\varepsilon}(x) \simeq x + \varepsilon\Phi(x)$. Now if one tries to prove Proposition 21, he will see that this approximation is too weak and one has to develop a Taylor expansion of $\pi_{\Sigma_\varepsilon}(x)$ in terms of ε to be able the reach the second order term in the distance functional. Consequently, if the first order derivative of the distance need no regularity assumption on Σ, the second order derivative will need Σ to be at least $C^{1,1}$. We refer the reader to [10] for more details.

Proposition 21 is useful enough to exclude polygones as convex minimizers of the average distance functional with volume constraint (see Section 7 below).

6 A taste of Multifractal analysis

In this section we try to go further in the structure of Σ in terms of regularity. Using (3.2), one can see that a decay like $\psi(x,r) \leq Cr^\alpha$ implies some $C^{1,\alpha}$ regularity, as says the following proposition.

Proposition 22. [9, Propositions 17 and 18] *Let $\alpha > 0$. Then $\Sigma \cap B(x_0, r_0)$ is a $C^{1,\alpha}$ regular curve if and only if $B(x_0, r_0)$ contains no triple points nor endpoints and there exists C such that*

$$\sup_{r \leq r_1} \frac{\psi(x,r)}{r^\alpha} \leq C \text{ for all } x \in \Sigma \cap B(x_0, r_0), \text{ and for some } r_1 \leq r_0.$$

This was used in [9] to prove for instance the following fact : if ψ is absolutely continuous with respect to $\mathcal{H}^1$ in a ball B and with density in L^p, then $\Sigma \cap B$ is $C^{1,\alpha}$ regular with $\alpha = \frac{p-1}{p}$.

Let us now assume for simplicity that μ is a bounded measure, so that ψ is also a bounded measure. We introduce the following definitions.

Definition 23. *Let Σ be a minimizer for the Problem* (1.1) *and $\alpha \in [0,1]$.*
i) We will say that a point $x \in \Sigma$ is a $C^{1,\alpha}$-point and we will denote $x \in \overline{R}_\alpha(\Sigma)$ if

$$\limsup_{r \to 0} \frac{\psi(B(x,r))}{r^\alpha} < +\infty.$$

ii) We will say that a point $x \in \Sigma$ is a non-$C^{1,\alpha}$-point and we will denote $x \in \underline{R}_\alpha(\Sigma)$ if

$$\liminf_{r \to 0} \frac{\psi(B(x,r))}{r^\alpha} = +\infty.$$

Notice that Σ may not be necessary $C^{1,\alpha}$ regular in a neighborhood of a $C^{1,\alpha}$-point. On the other hand if $\alpha > 0$ an $G \subset \Sigma$ is a relatively open set containing only $C^{1,\alpha}$-points, uniformly, then G is $C^{1,\alpha}$ regular.

For instance any corner point or endpoint is a $C^{1,0}$-point and at the same time a non-$C^{1,\alpha}$-point for any $\alpha > 0$. In addition by Theorem 10, we know that any triple point is a $C^{1,1}$-point. Notice that any $C^{1,\alpha}$-point is in particular a $C^{1,\alpha'}$-point for all $\alpha' \leq \alpha$, and any non-$C^{1,\alpha}$-point is a non-$C^{1,\alpha'}$-point for all $\alpha' \geq \alpha$. Furthermore it is not possible for a point x to be a $C^{1,\alpha}$-point and a non-$C^{1,\alpha}$-point for the same α.

We would like to estimate how "big" could $\overline{R}_\alpha$ and $\underline{R}_\alpha$ be. For this purpose we will use the notion of Hausdorff dimension. Let us first recall the classical definition of the fractional Hausdorff measure $\mathcal{H}^d$ (see e.g. [2]). For any set A and $\delta > 0$ we define

$$\mathcal{H}^d_\delta(A) := c_d \inf \left\{ \sum_{i\in I} (diam(A_i))^d; \quad diam(A_i) < \delta,\ A \subset \bigcup_{i\in I} A_i \right\},$$

with c_d a certain dimensional constant that we do not need to make explicit. In particular $\delta \mapsto \mathcal{H}^d_\delta(A)$ is nondecreasing and this allows us to define

$$\mathcal{H}^d(A) := \lim_{\delta\to 0} \mathcal{H}^d_\delta(A) = \sup_{\delta>0} \mathcal{H}^d_\delta(A).$$

Then for any set A we define the Hausdorff dimension by

$$\mathcal{H} - dim(A) := \inf\{d \geq 0; \mathcal{H}^d(A) = 0\} = \sup\{d \geq 0; \mathcal{H}^d(A) = +\infty\}.$$

Since ψ is a finite measure, it is clear that the set R_0 of $C^{1,0}$-points is countable. Consequently, it is easy to see that

$$\mathcal{H} - dim(\overline{R}_0) = 0,$$

and in particular the set of corner points has zero Hausdorff dimension.

In the same spirit, by some classical technics about Hausdorff measures, we can easily estimate the dimension of the sets $\overline{R}_\alpha$ and $\underline{R}_\alpha$.

Proposition 24. *The following tow inequalities hold.*

$$\mathcal{H} - dim(\underline{R}_\alpha(\Sigma)) \leq \alpha \qquad \text{and} \qquad \mathcal{H} - dim(\overline{R}_\alpha(\Sigma)) \geq \alpha,$$

where $\underline{R}_\alpha(\Sigma)$ is the set of non-$C^{1,\alpha}$-points and $\overline{R}_\alpha(\Sigma)$ is the set of $C^{1,\alpha}$-points.

Proof. The proposition follows from some very classical arguments about Hausdorff measures using the Vitali covering Lemma (see e.g. [2, Theorem 2.4.3]). Indeed, let $\delta, \tau > 0$ be a fixed constant. For any $x \in \underline{R}_\alpha(\Sigma)$ there exists $r(x) < \delta$ such that $\psi(B(x, r(x))) \geq \tau r^\alpha$. Let $\{B(x_i, r(x_i))\}_{i\in I}$ be a disjoint subfamily of $\{B(x, r(x))\}$ such that $\{B(x_i, 5r_i)\}_{i\in I}$ covers $\underline{R}_\alpha(\Sigma)$. This is always possible by the Vitali covering Theorem [2, Theorem 2.2.3]. Then

$$\mathcal{H}^\alpha_\delta(\underline{R}_\alpha(\Sigma)) \leq c_d \sum_{i\in I} (5r(x_i))^\alpha \leq c_d 5^d \frac{1}{\tau} \sum_{i\in I} \psi(B(x_i, r(x_i))) \leq \frac{1}{\tau} c_d 5^d \psi(\Sigma) < +\infty.$$

Letting $\delta \to 0$ we get $\mathcal{H}^\alpha(\underline{R}_\alpha(\Sigma)) \leq c_d 5^d \psi(\Sigma)$, and then letting $\tau \to +\infty$ it comes

$$\mathcal{H}^\alpha(\underline{R}_\alpha(\Sigma)) = 0$$

thus $\mathcal{H} - dim(\underline{R}_\alpha(\Sigma)) \leq \alpha$.

Now we prove the second inequality which is also quite standard (again [2, Theorem 2.4.3]). Let $\alpha' < \alpha$ be given, and fix $\tau > 0$. We define

$$A_h := \{x \in \overline{R}_\alpha(\Sigma); \frac{\psi(B(x,r))}{r^{\alpha'}} < \tau; \forall r \in (0, \frac{1}{h})\}.$$

Since for every $x \in \overline{R}_\alpha(\Sigma)$ and $\alpha' < \alpha$ it holds $\limsup_{r\to 0} \psi(B(x,r))r^{-\alpha'} = 0$, the set $\overline{R}_\alpha(\Sigma)$ is the increasing union of the sets A_h, for $h \geq 1$. Let now U_i be a family of sets such that $diam(U_i) < 1/h$, having at least one point belonging to A_h, whose union contains B_h, and which satisfies

$$c_{\alpha'} \sum_i diam(U_i)^{\alpha'} < \mathcal{H}^{\alpha'}(A_h) + \frac{1}{h} \leq \mathcal{H}^{\alpha'}(\overline{R}_\alpha(\Sigma)) + \frac{1}{h}.$$

The sets $B(x_i, diam(U_i))$ still covers A_h thus,

$$\begin{aligned}
\psi(A_h) &\leq \sum_i \psi(B(x_i, diam(U_i)))^{\alpha'} \\
&\leq \sum_i \tau diam(U_i)^{\alpha'} \\
&\leq \tau c_{\alpha'}^{-1}(\mathcal{H}^{\alpha'}(\overline{R}_\alpha(\Sigma)) + \frac{1}{h})
\end{aligned}$$

hence letting $h \to +\infty$ we obtain that

$$\frac{c_{\alpha'}\psi(\overline{R}_\alpha(\Sigma))}{\tau} \leq \mathcal{H}^{\alpha'}(\overline{R}_\alpha(\Sigma)),$$

and finally letting $\tau \to 0$ we get

$$\mathcal{H}^{\alpha'}(\overline{R}_\alpha(\Sigma)) = +\infty.$$

This holds for any $\alpha' \leq \alpha$ and therefore

$$\mathcal{H} - dim(\overline{R}_\alpha(\Sigma)) \geq \alpha.$$

$\square$

Remark 25. Proposition 24 quantifies the dimension of non-regular points but notice that $\underline{R}_\alpha(\Sigma)$ and $\overline{R}_\alpha(\Sigma)$ could be dense in Σ.

Remark 26. We could also define the set of exactly-$C^{1,\alpha}$-point by

$$R_\alpha(\Sigma) := \overline{R}_\alpha(\Sigma) \bigcup_{\alpha' > \alpha} \underline{R}_{\alpha'}(\Sigma).$$

The points in $R_\alpha(\Sigma)$ are $C^{1,\alpha}$-points but non-$C^{1,\alpha'}$-points for every $\alpha' > \alpha$. Using both arguments of Proposition 24 to the set $R_\alpha(\Sigma)$ one could prove that

$$\mathcal{H} - dim(R_\alpha(\Sigma)) = \alpha.$$

7 What if we minimize among convex sets ?

In the work in progress with Mainini [10], we focus on a different but related problem, namely minimizing

$$\mathcal{F}(K) + \lambda_1 Vol(K) + \lambda_2 Per(K),$$

with K convex. Here $\mathcal{F}$ still denotes the average distance functional defined in (1.1). One of the first motivation for studying this problem, is that the argument of Tilli [15] provides a proof of the following result.

Theorem 27. [10] *Let $K_0 \subset \mathbb{R}^2$ be a convex set, with $\mathcal{H}^1(\partial K_0) = \ell$ and $|K_0| = V$. Then for any $T > 0$, K_0 solves the minimizing problem*

$$\min_{K \in \mathcal{A}'} \int_{\Omega_T} dist(x, K) d\mu,$$

where $\Omega_T := \{dist(x, K_0) < T\}$, $\mu = \mathscr{L}^2$, and $\mathcal{A}'$ is the class of convex sets K such that $\mathcal{H}^1(\partial K) = \ell$ and $|K| = V$.

Theorem 27 exhibits in particular some minimizers of a certain average distance problem, that contains corner points. One the other hand the conditions on the class $\mathcal{A}'$ seems to be fairly strict. For instance if K_0 is a ball, then $\mathcal{A}'$ contains nothing but this unique element because no other convex sets has same volume and same perimeter than this ball K_0. However if K_0 is not a ball, we believe that the class $\mathcal{A}'$ contains sufficiently many sets to make Theorem 27 substantially interesting.

If we relax the problem to the following less restrictive one,

$$\min \{\mathcal{F}(K) + \lambda_1 Vol(K);\ K \text{ convex}\},$$

then it can be shown using Proposition 21 that a polygone is never a minimizer.

8 List of Open questions

We finish this paper with some open questions that might be interesting to study. Some of them were asked to the author by E. Stepanov.

1) About the set of corner points ? What can we say in general about the set of corner points ? For instance is this set a relatively closed set in Σ ? is it a finite set ? at least under some conditions on the domain Ω and measure μ ?

2) A minimiser that contains a corner point ? It is still an open question as to know whether minimizers could contain corner points or not. It would be interesting to know wether the example of Buttazzo, Mainini and Stepanov (Figure 1) is actually a real minimizer. It would be interesting to see for instance if it could be excluded or not by some second order variational argument.

3) Key Lemma in higher dimension ? As we said before most of the topological description for minimizers in dimension $N > 2$ would follow from an analogue of Lemma 7 in higher dimension. The proof available now (in [7]) only holds for $N = 2$ and a proof in higher dimension would be of great interest.

4) Improve exponent p **?** We don't know if the assumption $p > 4/3$ in Theorem 3 is optimal or not. An eventual improvement would need an improvement of Lemma 5, which is the reason of this assumption.

5) Regularity of convex minimizers ? Any regularity (or irregularity) result concerning the minimizing problem of Section 7 would be really appreciable (see also [10]). With both perimeter and volume constraints we know that any convex is a minimizer so that no regularity result could hold. But with only a volume constraint the regularity of minimizers is still not known.

5) Average distance in a metric measured space ? The definition of the Average distance minimizing Problem (1.1) is well defined in any metric measurable space. It would be interesting to known whether a similar topological description as Theorem 3 holds for the minimizers in this context. Of course this is even not known in the simple metric space $\mathbb{R}^3$ because of the "Key Lemma" that is not known to hold in higher dimension (Question 3). But this problem does not occur any more for the penalized functional $\mathcal{F}(\Sigma)+\lambda\mathcal{H}^1(\Sigma)$, which could be a starting point to study minimizers in the context of general metric spaces.

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