Lecture Notes on Mean Curvature Flow

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Foreword

Let $\varphi_0 : M \to \mathbb{R}^{n+1}$ be a smooth immersion of an n–dimensional smooth manifold in the Euclidean space. The evolution of $M_0 = \varphi_0(M)$ by mean curvature is a smooth one–parameter family of immersions $\varphi : M \times [0, T) \to \mathbb{R}^{n+1}$ satisfying

$$\begin{cases} \frac{\partial}{\partial t}\varphi(p,t) = \mathrm{H}(p,t)\nu(p,t) \\ \varphi(p,0) = \varphi_0(p) \end{cases}$$

where $\mathrm{H}(p,t)$ and $\nu(p,t)$ are respectively the mean curvature and the unit normal of the hypersurface $M_t = \varphi_t(M)$ at the point $p \in M$, where $\varphi_t = \varphi(\,\cdot\,, t)$.

It can be checked that $\mathrm{H}(p,t)\nu(p,t) = \Delta_{g(t)}\varphi(p,t)$, where $\Delta_{g(t)}$ is the Laplace–Beltrami operator on M associated to the metric $g(t)$, induced by the immersion φ_t. Then, the mean curvature flow may be regarded as a sort of geometric heat equation, in particular it can be shown that it is a parabolic problem and has a unique solution for small time. In addition, the solutions satisfy comparison principles and derivatives estimates similar to the case of parabolic partial differential equations in the Euclidean space.
On the other hand, the mean curvature flow is not really equivalent to a heat equation, since the Laplace–Beltrami operator evolves with the hypersurface itself. In particular, in contrast to the classical heat equation, this flow is described by a nonlinear (quasilinear) evolution system of partial differential equations and the solutions exist in general only in a finite time interval.

Mean curvature flow occurs in the description of the evolution of the interfaces in several multiphase physical models (see e.g. [**94, 111**]), one can indeed date the "genesis" of the subject to the paper of Mullins [**94**]. The main reason for this is the property that it is the gradient–like flow of the $Area$ functional and therefore it arises naturally in problems where a surface energy is relevant. From a physical point of view, it would be interesting also to consider the "hyperbolic" motion by mean curvature, that is, the evolution problem $\partial_t^2\varphi = \mathrm{H}\nu$, but very few results are present in literature at the moment. Algorithms based on the mean curvature flow has been also developed extensively in the field of automatic treatment of digital data, in particular of images. This because of the "regularizing effect" due to its parabolic nature.
Another interesting feature of this flow is its connection with certain reaction–diffusion equations, for instance

$$\frac{\partial u}{\partial t} = \Delta u - \frac{1}{\varepsilon}W'(u)\,,$$

where $W(u) = (u^2-1)^2$ (double–well potential). One can study the singular limits of the solutions of this parabolic equation when ε tends to zero. Under suitable hypotheses, it can be shown that the solutions u_ε with common initial data converge as $\varepsilon \to 0$ to functions which assume only the values ± 1 in regions separated by boundaries evolving by mean curvature (see [**6, 111**]).

Further motivation for the study of the mean curvature flow comes from geometric applications, in analogy with the Ricci flow of metrics on abstract Riemannian manifolds. One can use this flow as a tool to obtain classification results for hypersurfaces satisfying certain curvature conditions, to derive isoperimetric inequalities or to produce minimal surfaces. Like in Hamilton's program for the Ricci flow, a fundamental step in order to apply these techniques is the definition of a flow with surgeries or of a generalized (weak) notion of flow allowing to "pass" through the singularities in a controlled way. There has been much work in this direction by means of techniques based on varifolds, level sets, viscosity solutions (see [**2, 7, 21, 42, 78**]), till the recent results of Huisken and Sinestrari **[75]** about a surgery procedure well suited for topological conclusions.

There are striking analogies between the Ricci flow and the mean curvature flow. Indeed, many results hold in a similar way for both flows and several ideas have been successfully exported from one context to the other. However, at the moment it is not known a formal way of transforming one of them into the other.

In these notes, we will present exclusively the "classical" parametric setting, without discussing the contributions, sometimes quite relevant, coming from other approaches, in particular, the geometric measure theory setting (see [**21, 78**]) and the level sets formulation (see [**23, 42, 99, 120, 122, 123**]).
All the manifolds, quantities and other objects we will consider are smooth, unless otherwise stated. The main tool for the analysis will be a priori estimates (pointwise and integral), very often based on a smart use of the maximum principle in the same spirit of the work of Hamilton for the Ricci flow.

Up to now, the study of singularities and the classification of their asymptotic shape is almost complete for some classes of evolving hypersurfaces. For others it seems difficult and very far. In Chapter 5 we will try to draw an up–to–date scenario of the "state of the art".

This book grew up from a collection of notes for students, I tried to keep such spirit. This actually means that some discussions will be a little informal and that some points could be too detailed or even pedantic for an expert reader. With the exception of the proofs of some fundamental and deep results (listed in Appendix F), all the material is almost self–contained.

In Chapter 1 we fix the notations and we introduce the basic facts from differential geometry used throughout all the book. Moreover, we define the mean curvature flow, we discuss several examples and we show that any initial, smooth, compact, immersed hypersurface evolves by mean curvature at least for some positive interval of time.

In Chapter 2 we present the maximum principle and its first geometric applications to the mean curvature flow, we compute the evolution equations for the relevant geometric quantities and we discuss their consequences. In particular, the fact that at a singular time the curvature of the evolving hypersurface cannot stay bounded.

Chapter 3 is devoted to the analysis of *type I singularities* of the flow, that is, when the blow up rate of the curvature at the singular time is subject to a suitable natural control. The fundamental Huisken's *monotonicity formula* will play a major role in showing that the singularities are asymptotically modeled on "homothetic shrinkers", that is, hypersurfaces that flow by mean curvature simply by homothety. The classification of such class of hypersurfaces in the special case of positive mean curvature is described with its implications.

Chapter 4 is instead concerned with *type II singularities*, that is, without the above control on the blow up rate of the curvature. Here the goal is to show that, again when the mean curvature of the evolving hypersurface is positive, the singularities are modeled on translating hypersurfaces along their mean curvature flow.

In Chapter 5 we resume many of the conclusions, moreover, we briefly discuss the recent work by Huisken and Sinestrari on the *mean curvature flow with surgeries* and we collect some references to open problems and research directions.

We remark that in all the book special attention is given to the case of evolving curves in the plane. Very often it requires a separate treatment and enjoy better properties than the general higher dimensional case.

The appendices contain Polden's proof of short time existence of a solution for quasilinear parabolic PDE's on manifolds, the precise statements of some results mentioned in the book and a discussion of the Abresch–Langer classification theorem of homothetically shrinking closed curves in the plane.

Further Literature

We definitely suggest to the reader the wonderful survey of White [**121**] for a general overview of the field.

An excellent introduction to the mean curvature flow is provided by the monograph by Ecker [**35**], where many basic results and examples are collected. The second part of the book gives a fairly elementary approach to the difficult field of the regularity theory for weak solutions and, in the author's opinion, it is the natural "next step" for the interested reader. Other nice general references are [**34, 69, 80, 126**].

Two papers which contain a survey of results on the formation of singularities for mean curvature flow (and also discuss several other geometric flows) are the ones by Huisken [**68**] and by Huisken and Polden [**72**]. It is also surely recommendable to read Sections 2 and 3 of Hamilton's fundamental paper [**61**]. Such paper deals with the Ricci flow, but many of the ideas there exposed apply to the mean curvature flow as well.

Two works of central importance on weak solutions are the pioneering monograph by Brakke [**21**] and the memoir by Ilmanen [**78**]; they are of more difficult reading for a beginner.

Another introductory exposition of the mean curvature flow, including topics not treated in the present notes such as the connection with reaction–diffusion equations, is the one by Ambrosio [**6**]. The monograph by Giga [**50**] is also very pleasant to read and it gives a detailed account of the level sets approach to geometric evolutions.

ACKNOWLEDGMENTS. *I wrote these lecture notes collecting material from some introductory courses and seminars on mean curvature flow I gave in Pisa, at the SISSA in Trieste, at the Centre de Recerca Matemàtica in Barcelona and at the Institut Fourier in Grenoble. I am grateful to Gianni Dal Maso, Andrea Malchiodi, Joan Porti, Gérard Besson and Zindine Djadli for these invitations.*

Many people helped me, in particular I wish to thank Gerhard Huisken, Carlo Sinestrari and Brian White for clarifying me several points of the original papers, Charlie Baker, Giovanni Bellettini, Francesco Bonsante, Giovanni Catino, Zindine Djadli, Alessandro Ghigi, Valeria Marcucci, Lorenzo Mazzieri, Reto Müller and Matteo Novaga, for a very valuable criticism and many suggestions.

Special thanks go to Manolo Eminenti, Annibale Magni and Luca Martinazzi for contributing to some sections of the book.

This book was partially written during my visits to the Institut Fourier in Grenoble, the Centre de Recerca Matemàtica in Barcelona and the Institut Henri Poincaré in Paris. In all these places I found a warm hospitality and a perfect environment to work.

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