On the asymptotic behavior of the gradient flow of a polyconvex functional

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Abstract

In this paper, we study the asymptotic behavior of the solutions of the system of non-linear partial differential equations studied in [ESG05] for the evolution of a family of diffeomorphisms. We prove existence and regularity of the asymptotic state of solutions and we find an explicit rate of convergence of the time dependent solution to the corresponding final state. We study also a system not considered in [ESG05], linked to a linear Fokker-Planck equation. For this system we show existence of solutions, of the asymptotic state, the regularity and the rate of convergence of the solution to a final state. In both cases, the final states are obtained from the composition of the limit in time of the flow map with the initial data. This structure of the limiting stationary states allows a way of constructing maps with given jacobians as in [ASMV03].

1 Introduction

The present paper deals with the study of the asymptotic behavior of the solution of the following non-linear evolution problem

$$\begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} = \operatorname{div}\left(\Phi'(\det D\boldsymbol{u})(\operatorname{cof} D\boldsymbol{u})^T\right) & \text{in } \mathscr{U} \times (0,+\infty) \\ \boldsymbol{u}(0,\cdot) = \bar{\boldsymbol{u}} & \text{in } \mathscr{U}, \end{cases} \tag{1.1}$$

where $\mathscr{U}$ is a bounded, connected open subset of $\mathbb{R}^d$, $d \geq 1$, and $\Phi : (0,+\infty) \to \mathbb{R}$ is a smooth strictly convex function. The initial datum $\bar{\boldsymbol{u}} : \mathscr{U} \to \mathscr{V}$ belongs to $\operatorname{Diff}(\overline{\mathscr{U}};\overline{\mathscr{V}})$: the class of C^1 diffeomorphisms from $\overline{\mathscr{U}}$ to $\overline{\mathscr{V}}$ such that $\bar{\boldsymbol{u}}(\partial\mathscr{U}) = \partial\mathscr{V}$ and $\det(D\bar{\boldsymbol{u}}) > 0$ (since $\overline{\mathscr{V}}$ is diffeomorph to $\overline{\mathscr{U}}$, then $\mathscr{V}$ is a bounded connected open subset of $\mathbb{R}^d$). The measure of the domain can be normalized by a simple change of variables, so we will reduce to the case $|\mathscr{U}| = 1$ in what follows.

The problem (1.1), studied in [ESG05] and [ALS06], is the *gradient flow*, with respect to the $L^2(\mathscr{U};\mathbb{R}^d)$ metric, of the polyconvex functional

$$I(\boldsymbol{u}) := \int_{\mathscr{U}} \Phi(\det D\boldsymbol{u})\,dx \tag{1.2}$$

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defined on Diff $(\overline{\mathscr{U}};\overline{\mathscr{V}})$. Indeed, at least formally, the problem (1.1), has a natural variational formulation as

$$\frac{d}{dt}\int_{\mathscr{U}} \boldsymbol{u}\cdot\boldsymbol{\xi}\,dx = -\delta I(\boldsymbol{u};\boldsymbol{\xi}) \quad \text{for every } \boldsymbol{\xi}\in C^1(\overline{\mathscr{U}};\mathbb{R}^d) \text{ with } (\operatorname{cof} D\boldsymbol{u})\boldsymbol{n}_{\mathscr{U}} \perp \boldsymbol{\xi} \text{ on } \partial\mathscr{U}, \tag{1.3}$$

where $\delta I(\boldsymbol{u};\boldsymbol{\xi})$ denotes the Euler Lagrange first variation of I along the vector field $\boldsymbol{\xi}$. Indeed, recalling that $D\Phi(\det A)=\Phi'(\det A)(\operatorname{cof} A)^T$ for every positive $d\times d$ matrix A, we compute

$$\begin{aligned}\delta I(\boldsymbol{u};\boldsymbol{\xi}) := \frac{d}{ds}I(\boldsymbol{u}+s\boldsymbol{\xi})\Big|_{s=0} &= \int_{\mathscr{U}} \Phi'(\det D\boldsymbol{u})(\operatorname{cof} D\boldsymbol{u})^T\cdot D\boldsymbol{\xi}\,dx\\ &= -\int_{\mathscr{U}} \operatorname{div}\Big(\Phi'(\det D\boldsymbol{u})(\operatorname{cof} D\boldsymbol{u})^T\Big)\cdot\boldsymbol{\xi}\,dx,\end{aligned} \tag{1.4}$$

and the boundary condition in (1.3) imposed on the test vector field $\boldsymbol{\xi}$ allows to make the integration by parts in (1.4) without boundary term.

In a weak sense, this variational formulation encompasses the natural boundary condition needed in (1.1). More precisely, if the solution of the system of partial differential equations (1.1) is smooth enough and defined as the $L^2(\mathscr{U};\mathbb{R}^d)$-gradient flow of the polyconvex functional (1.2), then this functional has to decrease along solutions of (1.1). Doing the formal computation of its time evolution, we have

$$\frac{d}{dt}I(\boldsymbol{u}(t)) = -\int_{\mathscr{U}}\Big|\operatorname{div}\Big(\Phi'(\det D\boldsymbol{u})(\operatorname{cof} D\boldsymbol{u})^T\Big)\Big|^2\,dx - \int_{\partial\mathscr{U}}\Phi'(\det D\boldsymbol{u})\ \boldsymbol{n}_{\mathscr{U}}^T(\operatorname{cof} D\boldsymbol{u})^T\frac{\partial\boldsymbol{u}}{\partial t}\,d\sigma$$

from which the natural boundary condition to add in (1.1) is

$$(\operatorname{cof} D\boldsymbol{u})\,\boldsymbol{n}_{\mathscr{U}} \perp \frac{\partial\boldsymbol{u}}{\partial t} \qquad \text{on } \partial\mathscr{U}$$

and the solution has to satisfy $\boldsymbol{u}(t,\cdot)\in \operatorname{Diff}(\overline{\mathscr{U}};\overline{\mathscr{V}})$ for every $t\geq 0$.

The existence of a unique solution of problem (1.1) satisfying $\boldsymbol{u}(t,\cdot)\in\operatorname{Diff}(\overline{\mathscr{U}};\overline{\mathscr{V}})$ for any $t>0$ was proved by Evans, Gangbo and Savin [ESG05] by means of a change of variables which transforms the problem (1.1) into a non linear boundary value problem involving a nonlinear diffusion equation on the set $\mathscr{V}$.

In order to make more precise this relation, let us remind the reader the standard notation of image measure through a map. Given two measures μ and ν in the sets $\mathscr{U}$ and $\mathscr{V}$ respectively, we say that a Borel map $T:\mathscr{U}\longrightarrow\mathscr{V}$ *transports* μ onto ν, or that ν is the image measure of μ through the map T, denoted by $\nu=T_{\#}\mu$, if for any Borel measurable set $B\subset\mathscr{V}$, $\nu(B)=\mu(T^{-1}(B))$, or equivalently

$$\int_{\mathscr{U}}\zeta\circ T(x)\,d\mu(x) = \int_{\mathscr{V}}\zeta(y)\,d\nu(y)\,,\quad \forall\zeta\in\mathcal{C}^0_b(\mathscr{V})\,.$$

If the map $T\in\operatorname{Diff}(\overline{\mathscr{U}};\overline{\mathscr{V}})$ and the measures μ and ν are absolutely continuous with respect to Lebesgue measure with densities $\widetilde{\rho}$ and ρ respectively, then $\nu=T_{\#}\mu$ is equivalent by the change of variables theorem to $\rho(T(x))\,\det(DT(x)) = \widetilde{\rho}(x)$. Let us denote by $\mathscr{L}^d_{|\mathscr{U}}$ the absolutely continuous measure associated to the characteristic function of the set $\mathscr{U}$.

With this notation, we can state that the authors of [ESG05] observed the following fact: given a solution $\boldsymbol{u} \in \mathrm{Diff}\,(\overline{\mathscr{U}}; \overline{\mathscr{V}})$ of problem (1.1), defining

$$\boldsymbol{w}(t,\cdot) := (\boldsymbol{u}(t,\cdot))^{-1},$$

then the scalar function ρ defined by

$$\rho(t,y) := \det D\boldsymbol{w}(t,y) = \frac{1}{\det D\boldsymbol{u}(t,\boldsymbol{w}(t,y))}, \quad y \in \mathscr{V}, \tag{1.5}$$

which is the Lebesgue density of the probability (because $|\mathscr{U}| = 1$) measure $\nu_t := (\boldsymbol{u}(t,\cdot))_{\#}\mathscr{L}^d_{|\mathscr{U}}$, solves the nonlinear boundary value problem of diffusion type

$$\begin{cases} \dfrac{\partial}{\partial t}\rho = \Delta P(\rho) & \text{in } (0,+\infty)\times\mathscr{V}, \\ D\rho(t,\cdot)\cdot \boldsymbol{n}_{\mathscr{V}}(\cdot) = 0 & \text{on } (0,+\infty)\times\partial\mathscr{V}, \\ \rho(0,\cdot) = \bar{\rho} := \dfrac{1}{\det D\bar{\boldsymbol{u}}}\circ \bar{\boldsymbol{u}}^{-1} & \text{in } \mathscr{V}, \end{cases} \tag{1.6}$$

where the function P is linked to Φ by the relation $P(s) = -\Phi'(1/s)$. Since $\frac{d}{ds}(-\Phi'(1/s)) = \Phi''(1/s)/s^2$, by the convexity of Φ we have that the map $s \mapsto P(s)$ is monotone increasing, and thus the problem (1.6) is parabolic. We also notice that defining the function $\psi : [0,+\infty) \to \mathbb{R}$, as

$$\psi(s) := s\Phi\Big(\frac{1}{s}\Big), \quad s > 0, \quad \psi(0) = \lim_{s\to 0} s\Phi\Big(\frac{1}{s}\Big) = \lim_{r\to+\infty}\frac{\Phi(r)}{r},$$

we have

$$P(s) = s\psi'(s) - \psi(s).$$

Finally, let us define the functional on measures in the target space $\mathscr{V}$ as

$$\Psi(\nu) := \int_{\mathscr{V}} \psi(\rho(x))\,dx\,, \quad \text{if } \nu = \rho\mathscr{L}^d \tag{1.7}$$

and $+\infty$ otherwise. We will denote by $\Psi(\rho)$ the value of the functional Ψ at the measure $\nu = \rho\mathscr{L}^d$.

A remarkable idea of the approach in [ESG05] is the following: the solution $\boldsymbol{u}$ of problem (1.1) can be built by solving the problem (1.6) as first step, then considering (1.6) as a continuity equation with velocity field given by

$$\boldsymbol{F}(t,y) := -\frac{\nabla P(\rho(t,y))}{\rho(t,y)} = -\nabla\psi'(\rho(t,y)), \tag{1.8}$$

and constructing its associated flow as the second step. Actually, the flow is given by the maps $\boldsymbol{Y} : [0,+\infty)\times\mathscr{V} \to \mathscr{V}$ that are the maximal solutions of the Cauchy problems

$$\begin{cases} \boldsymbol{Y}'(t,y) = \boldsymbol{F}(t,\boldsymbol{Y}(t,y)), \\ \boldsymbol{Y}(0,y) = y, \qquad y \in \mathscr{V} \end{cases} \tag{1.9}$$

and, finally, setting

$$\boldsymbol{u}(t,x) := \boldsymbol{Y}(t,\bar{\boldsymbol{u}}(x)), \tag{1.10}$$

which turns out a solution of problem (1.1). In fact, the key idea is that the equation (1.1) is the $L^2(\mathscr{U},\mathbb{R}^d)$-gradient flow of the functional $I(\boldsymbol{u})$ and the equation (1.6) is the W_2-gradient flow of the functional $\Psi(\nu)$, where W_2 is the Wasserstein euclidean distance between probability measures, see [Vil03], [AGS05]. Moreover, these two gradient flows are in some sense equivalent through the change of variables (1.5), see [ALS06] for precise statements. The main result of [ESG05], can be stated as follows.

Theorem 1.1 (Evans-Gangbo-Savin)**.** *Let us assume that $\mathscr{V}$ is a bounded open set of class $C^{2,\alpha}$ with $0<\alpha<1$; if $\bar{\boldsymbol{u}} \in C^{1,\alpha}(\overline{\mathscr{U}};\overline{\mathscr{V}}) \cap \mathrm{Diff}\,(\overline{\mathscr{U}};\overline{\mathscr{V}})$ and $0<\rho_{\min} \le \det D\bar{\boldsymbol{u}}^{-1} \le \rho_{\max}$, then there exists a unique solution of the distributional formulation* (1.3) *of the problem* (1.1) *such that $\boldsymbol{u}(t,\cdot) \in \mathrm{Diff}\,(\overline{\mathscr{U}};\overline{\mathscr{V}})$ with $\partial_t \boldsymbol{u} \in L^2((0,T)\times\mathscr{U};\mathbb{R}^d)$ for any $T>0$. Moreover $\boldsymbol{u}$ has the representation formula* (1.10)*.*

The assumptions in Theorem 1.1 imply that the flow (1.9) is well defined (see also the discussion on the proof of our Theorem 2.5). Since the solutions of problem (1.1) are defined for $t \in (0,+\infty)$, a natural issue consists in the study of the asymptotic behavior of the solutions for $t \to +\infty$. Moreover, since the solutions to (1.6) are known to converge exponentially fast to their equilibrium solution, a constant value over the domain $\mathscr{V}$, then we may expect that the lagrangian formulation of this problem given by the diffeomorphism $\boldsymbol{u}(t)$ converges also to some final state. If so, this is related to show that the flow map for each single point $\boldsymbol{Y}(t,y)$ has a limiting value as $t\to\infty$ and that the limiting map is smooth enough to give us a limiting diffeomorphism $\boldsymbol{u}_\infty$. The only possibility for the solutions of the Cauchy problems (1.9) to have a limiting value is that the right-hand side is integrable in time, i.e., we need to show that $\boldsymbol{F} \in L^1(0,+\infty;C^1(\overline{\mathscr{V}}))$ for which the convergence rate to stationary states of the solutions of (1.6) will be crucial. Later, we need more properties on the flow map in order to deduce that the limiting map is indeed an element in $\mathrm{Diff}\,(\overline{\mathscr{U}};\overline{\mathscr{V}})$.

This strategy was already used in the case of the heat equation and the Stokes flow in [ASMV03] to construct maps with given jacobians. In fact, our results here allow us also to construct maps with given jacobians by solving these partial differential equations to find the limiting flow maps. We refer to [ASMV03] and the references therein for the motivations and applications of these maps. We will remind the main elements of the strategy in [ASMV03] in the next section. Let us finally mention that the numerical solution of the system (1.1) can be an effective way of computing the solutions of the nonlinear diffusion equations (1.6) and their asymptotic behavior as demonstrated in [CM].

Our main result in the next section states that, under the assumptions of Theorem 1.1, the solution of the problem (1.1) converges, as $t\to+\infty$, to a stationary state $\boldsymbol{u}_\infty$, depending on the initial datum, with exponential rate. Moreover $\boldsymbol{u}_\infty$ enjoy an Hölder regularity property as the initial datum and it satisfies

$$\det(D\boldsymbol{u}_\infty(x)) = c > 0, \qquad \forall x \in \mathscr{U}. \tag{1.11}$$

This condition is indeed a necessary condition for a smooth stationary point of I, see [ESG05].

Furthermore, in Section 3 we will generalize these ideas to linear Fokker-Planck equations. Here, we use a Log-Sobolev type inequality that will be crucial to obtain the exponential convergence towards the equilibrium of their solutions, and thus, to show the integrability in time of the velocity field to start the strategy discussed above.

2 Nonlinear diffusions

Let us start by recalling the main arguments in the existence of maps with fixed Jacobian by [ASMV03]. More precisely, we can find the following result about the existence of the limit diffeomorphism of a flow of a given velocity vector field.

Theorem 2.1 (Large Time Limit of Flow Maps, [ASMV03, Theorem 3]). *Let $\mathscr{V} \subset \mathbb{R}^d$ be a bounded connected domain of class C^2. Assume that $\boldsymbol{F} : (0, +\infty) \to C^1(\overline{\mathscr{V}}; \mathbb{R}^d)$ is a continuous vector-field satisfying*

$$\boldsymbol{F} \in L^1(0, +\infty; C^1(\overline{\mathscr{V}}))$$

and the boundary condition

$$\boldsymbol{F}(t, y) \cdot \boldsymbol{n}_{\mathscr{V}}(y) = 0 \qquad \forall y \in \partial\mathscr{V}.$$

Then the flow

$$\begin{cases} \boldsymbol{Y}'(t, y) = \boldsymbol{F}(t, \boldsymbol{Y}(t, y)) & t \in (0, +\infty), \\ \boldsymbol{Y}(0, y) = y, \qquad y \in \mathscr{V} \end{cases}$$

is well defined. The map

$$\boldsymbol{Y}_\infty(y) := \lim_{t\to+\infty} \boldsymbol{Y}(t, y), \qquad y \in \overline{\mathscr{V}}$$

is well defined and it is a diffeomorphism of $\overline{\mathscr{V}}$ on $\overline{\mathscr{V}}$ of class $C^1(\overline{\mathscr{V}}; \overline{\mathscr{V}})$. Moreover, we have

$$\det(D\boldsymbol{Y}_\infty(y)) = \exp\Big(\int_0^{+\infty} \operatorname{div} \boldsymbol{F}(t, \boldsymbol{Y}(t, y))\, dt\Big). \tag{2.1}$$

Furthermore, if for some $0 < \beta < 1$ there exists a constant $C > 0$ such that

$$\int_0^{+\infty} |\nabla\boldsymbol{F}(s, f(s)) - \nabla\boldsymbol{F}(s, g(s))|\, ds \le C\, \|f - g\|_\infty^\beta, \qquad \forall f, g \in C^0(0, +\infty; \overline{\mathscr{V}}), \tag{2.2}$$

then $\boldsymbol{Y}_\infty$ and its inverse belong to $C^{1,\beta}(\overline{\mathscr{V}}; \overline{\mathscr{V}})$.

We will also need some results concerning regularity for solutions of quasi-linear parabolic equations. We collect in the next Theorem some estimates for variable coefficients linear parabolic evolution problems that are useful in the proof of the main results of the present paper. For the proof in the very general context of the parabolic systems of order $2m$ and more general coefficients, see [Bel79, Section 4, estimates (4.10), (4.12)].

Theorem 2.2 (Regularity Estimates, [Bel79]). *Let $\mathscr{V} \subset \mathbb{R}^d$ be a bounded connected domain of class $C^{2+\alpha}$ with $\alpha \in (0,1)$ and $\beta \in (0,1)$. Denoting by $Q_T = \mathscr{V} \times (0,T)$, if $a : Q_T \to \mathbb{R}$ and $b : Q_T \to \mathbb{R}^d$ are of class $C^{\beta,\beta/2}(Q_T)$, then the classical solution of the problem*

$$\begin{cases} \dfrac{\partial v}{\partial t} = a\Delta v + b \cdot \nabla v & \text{in } Q_T, \\ \nabla v(t,\cdot) \cdot \boldsymbol{n}_{\mathscr{V}}(\cdot) = 0 & \text{on } (0,T) \times \partial \mathscr{V}, \\ v(0,\cdot) = \bar{v} & \text{in } \mathscr{V}. \end{cases}$$

with $\bar{v} \in C^{\alpha}(\overline{\mathscr{V}})$, satisfies the following inequality

$$t^{1-\frac{\alpha-\beta}{2}} \left\| v(t,\cdot) \right\|_{C^{2+\beta}(\overline{\mathscr{V}})} \leq C \left\| \bar{v} \right\|_{C^{\alpha}(\overline{\mathscr{V}})}, \qquad t \in (0,T), \tag{2.3}$$

where C depends only on T, $\mathscr{V}$, $\|a\|_{C^{1+\beta,(1+\beta)/2}(Q_T)}$, $\|b\|_{C^{1+\beta,(1+\beta)/2}(Q_T)}$.

If $a \in C^{1+\beta,(1+\beta)/2}(Q_T)$ and v is a solution of problem

$$\begin{cases} \dfrac{\partial v}{\partial t} = \operatorname{div}(a\nabla v) & \text{in } Q_T, \\ \nabla v(t,\cdot) \cdot \boldsymbol{n}_{\mathscr{V}}(\cdot) = 0 & \text{on } (0,T) \times \partial \mathscr{V}, \\ u(0,\cdot) = \bar{v} & \text{in } \mathscr{V}. \end{cases}$$

then for every $t_0 > 0$ there exists C depending only on the $\|a\|_{C^{1+\beta,(1+\beta)/2}(Q_T)}$ and the domain $\mathscr{V}$ and t_0 such that

$$\left\| v(t,\cdot) \right\|_{C^{2+\beta}(\overline{\mathscr{V}})} \leq C \left\| \bar{v} \right\|_{C^{0}(\overline{\mathscr{V}})}, \qquad \forall t \in [t_0, T). \tag{2.4}$$

We will also need a technical lemma to deal with the regularity of the flow maps.

Lemma 2.3 (Estimates on the Velocity Field). *Let $\mathscr{V}$ be a bounded connected domain of class $C^{2,\alpha}$ and $v \in C^{2,\beta}(\overline{\mathscr{V}})$ and $m := \min v$, $M := \max v$ with $\alpha \in (0,1)$, $\beta \in (0,\alpha]$. Let P be a C^3 function from $(0,+\infty)$ to $\mathbb{R}$. If $m > 0$, then for every constant $a \in \mathbb{R}$ there exist a constant C, depending only on M, m, P, a and $\mathscr{V}$, such that*

$$\left\| \frac{\nabla P(v)}{v} \right\|_{C^{1,\beta}(\overline{\mathscr{V}})} \leq C \left\| v - a \right\|_{C^{2,\beta}(\overline{\mathscr{V}})}. \tag{2.5}$$

Proof. The case of $P(\rho) = \rho$ is proved in [ASMV03, Lemma 1]. In our general case, we observe that

$$\left\| \frac{\nabla P(v)}{v} \right\|_{C^{1,\beta}(\overline{\mathscr{V}})} \leq \left\| P'(v) \right\|_{C^{0}(\overline{\mathscr{V}})} \left\| \frac{\nabla v}{v} \right\|_{C^{1,\beta}(\overline{\mathscr{V}})} + \left\| P'(v) \right\|_{C^{1,\beta}(\overline{\mathscr{V}})} \left\| \frac{\nabla v}{v} \right\|_{C^{0}(\overline{\mathscr{V}})}.$$

From the smoothness of P and the bounds from above and below on the density, it is easy to see that there exists $C > 0$ depending only on P, m, M, and a such that

$$\left\| P'(v) \right\|_{C^{1,\beta}(\overline{\mathscr{V}})} \leq C \left(1 + \left\| v - a \right\|_{C^{1,\beta}(\overline{\mathscr{V}})} \right)$$

and

$$\left\| P'(v) \right\|_{C^0(\overline{\mathscr{V}})} \leq \sup_{r\in[m,M]} |P'(r)| := \tilde{M}.$$

Since by [ASMV03, Lemma 1] we have

$$\left\| \frac{\nabla v}{v} \right\|_{C^{1,\beta}(\overline{\mathscr{V}})} \leq C \left\| v - a \right\|_{C^{2,\beta}(\overline{\mathscr{V}})},$$

then

$$\left\| \frac{\nabla P(v)}{v} \right\|_{C^{1,\beta}(\overline{\mathscr{V}})} \leq C \tilde{M} \left\| v - a \right\|_{C^{2,\beta}(\overline{\mathscr{V}})} + C \left(1 + \left\| v - a \right\|_{C^{1,\beta}(\overline{\mathscr{V}})} \right) \frac{1}{m} \left\| v - a \right\|_{C^{1}(\overline{\mathscr{V}})}.$$

Using the interpolation inequality

$$\left\| f \right\|_{C^{1,\gamma}(\overline{\mathscr{V}})} \leq C \left\| f \right\|_{C^{2,\beta}(\overline{\mathscr{V}})}^{\frac{1+\gamma}{2+\beta}} \left\| f \right\|_{C^{0}(\overline{\mathscr{V}})}^{1-\frac{1+\gamma}{2+\beta}}$$

for $\gamma = 0$ and for $\gamma = \beta$ and observing that $\left\| v - a \right\|_{C^0(\overline{\mathscr{V}})} \leq M + a$, we obtain (2.5). □

Finally, let us clarify the deep relation between the gradient flows associated to the diffeomorphism equation (1.1) and the nonlinear diffusion equation (1.6). In fact, equation (1.1) is the $L^2(\mathscr{U}, \mathbb{R}^d)$-gradient flow of the functional $I(\boldsymbol{u})$ while equation (1.6) is the W_2-gradient flow of the functional $\Psi(\nu)$ defined in (1.7), see [ALS06]. Before that, let us remind the reader the definition of the euclidean Wasserstein distance W_2.

Let $\mathscr{P}(\mathscr{V})$ denote the set of probability measures in $\mathbb{R}^d$ supported in $\mathscr{V}$. Define the functional W_2 in $\mathscr{P}(\mathscr{V}) \times \mathscr{P}(\mathscr{V})$ by

$$W_2^2(\mu, \nu) = \inf_{\Pi \in \Gamma(\mu,\nu)} \iint_{\mathscr{V}\times\mathscr{V}} |x-y|^2 \, d\Pi(x,y) \ ,$$

where Π runs over the set $\Gamma(\mu, \nu)$ of all *couplings* of the probability measures μ and ν; that is, the set of probability measures in $\mathscr{V} \times \mathscr{V}$ with first marginal μ and second ν. For absolutely continuous probability measures $f \mathscr{L}^d_{|\mathscr{V}}$ and $g \mathscr{L}^d_{|\mathscr{V}}$ we will simply write $W_2(f, g)$ in place of $W_2(f \mathscr{L}^d_{|\mathscr{V}}, g \mathscr{L}^d_{|\mathscr{V}})$. The functional W_2 is a metric on $\mathscr{P}(\mathscr{V})$; it is called the *euclidean-Wasserstein metric*, where the euclidean refers to the exponent 2 on the distance $|x - y|$. We refer to [Vil03, AGS05] for much more information about this distance. Finally, we refer to [AGS05] for the precise meaning of the statement: the equation (1.6) is the W_2-gradient flow of the functional $\Psi(\nu)$ since it is not the objective of this paper. In fact, all we need is the following result:

Lemma 2.4 (Relation between gradient flows, [ALS06])**.** *Let us assume that $\mathscr{V}$ is a bounded open set of class $C^{2,\alpha}$ with $\alpha \in (0,1)$. If $\bar{\boldsymbol{u}} \in C^{1,\alpha}(\overline{\mathscr{U}}; \overline{\mathscr{V}}) \cap \mathrm{Diff}\,(\overline{\mathscr{U}}; \overline{\mathscr{V}})$ and $0 < \rho_{\min} \leq \bar{\rho} = \det D\bar{\boldsymbol{u}}^{-1} \leq \rho_{\max}$ then given the solutions $\boldsymbol{u}(t,\cdot) \in \mathrm{Diff}\,(\overline{\mathscr{U}}; \overline{\mathscr{V}})$ to* (1.1) *with initial datum $\bar{\boldsymbol{u}}$ and $\rho(t,\cdot)$ to* (1.6) *with initial datum $\bar{\rho}$, we have*

$$\Psi(\rho(t,\cdot)) - \Psi(\rho(T,\cdot)) = \int_t^T |\boldsymbol{F}(s,y)|^2 \rho(s,y)\, dy\, ds = \int_t^T \left\| \frac{d}{ds} \boldsymbol{u}(s,\cdot) \right\|_{L^2(\mathscr{U})}^2 ds, \tag{2.6}$$

for all $t, T \in [0, +\infty)$, $t < T$.

In the previous identity (2.6), the first equality is precisely the energy identity for gradient flow solutions of the nonlinear diffusion equation (1.6). The second equality follows from the change of variables (1.5) together with (1.9) and the representation formula (1.10).

These previous ingredients allow us to show the first main result of this paper for nonlinear diffusions.

Theorem 2.5 (Asymptotic behavior)**.** *Let us assume that $\mathscr{V}$ is a bounded open set of class $C^{2,\alpha}$ with $\alpha \in (0,1)$. If $\bar{\boldsymbol{u}} \in C^{1,\alpha}(\overline{\mathscr{U}};\overline{\mathscr{V}}) \cap \mathrm{Diff}\,(\overline{\mathscr{U}};\overline{\mathscr{V}})$ and $0 < \rho_{\min} \leq \det D\bar{\boldsymbol{u}}^{-1} \leq \rho_{\max}$ then there exist $\boldsymbol{u}_\infty \in C^{1,\beta}(\overline{\mathscr{U}};\overline{\mathscr{V}}) \cap \mathrm{Diff}\,(\overline{\mathscr{U}};\overline{\mathscr{V}})$ for every $\beta < \alpha$, satisfying* (1.11)*, and there exist constants $C \geq 0$ and $\sigma > 0$, depending on the initial datum, such that*

$$\|\boldsymbol{u}(t,\cdot) - \boldsymbol{u}_\infty\|_{L^2(\mathscr{U};\mathbb{R}^d)} \leq C\, e^{-\sigma t} \qquad \forall t > 0, \tag{2.7}$$

where $\boldsymbol{u}$ is the solution of the problem (1.1) *given by Theorem* 1.1*.*

Remark 2.6 (Strategy and Previous Literature)**.** The representation formula (1.10) of the solution of problem (1.1) suggests that it is natural to study the asymptotic limit of the problem (1.1) by showing the existence and regularity of the map

$$\boldsymbol{Y}_\infty(y) := \lim_{t\to+\infty} \boldsymbol{Y}(t,y).$$

By means of the map $\boldsymbol{Y}_\infty$, the stationary states of the problem (1.1) can be represented by the formula

$$\boldsymbol{u}_\infty(x) = \boldsymbol{Y}_\infty(\bar{\boldsymbol{u}}(x)).$$

Theorem 2.1 was originally applied to the case of the heat equation in [ASMV03, Theorem 1]. Thus the result of [ASMV03] immediately proves our Theorem 2.5 in the particular case of $\Phi(s) = -\log s$, for which the diffusion equation in (1.6) is indeed the heat equation. We prove that Theorem 2.1 applies also to our more general non linear diffusion.

Proof. We show the existence and regularity of the map $\boldsymbol{Y}_\infty$ by applying Theorem 2.1 to the vector field $\boldsymbol{F}$ defined in (1.8), where ρ is the solution of problem (1.6). The strategy can be summarized in the following two main steps:

Step 1.- We have to check that

$$\boldsymbol{F} \in L^1(0,+\infty;C^1(\overline{\mathscr{V}})) \tag{2.8}$$

to show that $\boldsymbol{Y}_\infty$ exists and it is a diffeomorphism.

Step 2.- We have to prove (2.2), i.e., that there exists a constant $C > 0$ such that for all $f, g \in C^0(0,+\infty;\overline{\mathscr{V}})$

$$\int_0^{+\infty} |\nabla \boldsymbol{F}(s,f(s)) - \nabla \boldsymbol{F}(s,g(s))|\, ds \leq C\,\|f-g\|_\infty^\beta$$

to show that $\boldsymbol{Y}_\infty$ and its inverse belong to $C^{1,\beta}(\overline{\mathscr{V}})$.

Since $\bar\rho = \det D\bar{\boldsymbol{u}}^{-1} \in C^{0,\alpha}(\overline{\mathscr{V}})$ satisfies $0 < \rho_{\min} \le \bar\rho \le \rho_{\max}$, the problem (1.6) has a classical solution ρ. By the maximum principle we have that

$$0 < \rho_{\min} \le \rho(t,x) \le \rho_{\max} \qquad \text{in } Q = (0,+\infty)\times\mathscr{V}. \tag{2.9}$$

Then $\rho \in C^{\alpha,\alpha/2}(\overline{Q}_T)$, see [PV93, Theorem 1.3 and Remarks 1.3, 1.4], and there exists a constant γ such that

$$\|\rho\|_{C^{\alpha,\alpha/2}(\overline{Q}_T)} \le \gamma\,\|\rho\|_{L^\infty(Q_T)} \le \gamma\,\|\bar\rho\|_{L^\infty(\mathscr{V})}\,, \tag{2.10}$$

with γ independent on T thanks to (2.9), indeed γ depends on T only through the norm $\|\rho\|_{L^\infty(Q_T)}$, see [DiB93, Cap. III Theorem 1.3], and the estimate (2.10) holds on Q.

Setting $\bar v := P(\bar\rho)$ and $a(t,x) := P'(\rho(t,x))$ we consider the problem

$$\begin{cases} \dfrac{\partial v}{\partial t} = a\Delta v & \text{in } (0,+\infty)\times\mathscr{V},\\ Dv(t,\cdot)\cdot \boldsymbol{n}_{\mathscr{V}}(\cdot) = 0 & \text{on } (0,+\infty)\times\partial\mathscr{V},\\ v(0,\cdot) = \bar v & \text{in } \mathscr{V}. \end{cases} \tag{2.11}$$

Since P' is Lipschitz continuous on $[\rho_{\min},\rho_{\max}]$, from (2.10) we have that $a \in C^{\alpha,\alpha/2}(\overline{Q}_T)$ and there exist two constants c, C such that $0 < c \le a(x,t) \le C$. By the maximum principle we obtain

$$0 < v_{\min} := P(\rho_{\min}) \le v(t,x) \le v_{\max} := P(\rho_{\max}) \qquad \text{in } Q = (0,+\infty)\times\mathscr{V},$$

and the parabolic regularity theory shows that v is a classical solution of (2.11). By the uniqueness of the solutions of problems (1.6) and (2.11) we have that $v(t,x) = P(\rho(t,x))$. Since $a \in C^{\beta,\beta/2}(\overline{Q}_T)$ for every $\beta\in[0,\alpha]$, we can thus apply Theorem 2.2 and by (2.3) we have the following intermediate Schauder-type estimate

$$t^{1-\frac{\alpha-\beta}{2}}\,\|v(t,\cdot)\|_{C^{2+\beta}(\overline{\mathscr{V}})} \le C\,\|\bar v\|_{C^{\alpha}(\overline{\mathscr{V}})}\,, \qquad t\in(0,T), \tag{2.12}$$

with C depending only on T and $\|a\|_{C^{\beta,\beta/2}(\overline{Q}_T)}$ which depends, by (2.10), only on $\bar\rho$.

In order to analyze the behavior near $+\infty$ we fix an integer $k\ge 1$ and we define $a_k(t,x) := P'(\rho(t+k,x))$ and we consider the problem

$$\begin{cases} \dfrac{\partial \rho_k}{\partial t} = \nabla\cdot(a_k\nabla\rho_k) & \text{in } (0,T)\times\mathscr{V},\\ D\rho_k(t,\cdot)\cdot \boldsymbol{n}_{\mathscr{V}}(\cdot) = 0 & \text{on } (0,+\infty)\times\partial\mathscr{V},\\ \rho_k(0,\cdot) = \rho(k,\cdot) - \rho_\infty & \text{in } \mathscr{V}, \end{cases}$$

where ρ_∞ denotes the stationary solution of problem (1.6), which is

$$\rho_\infty := \frac{1}{|\mathscr{V}|}\int_{\mathscr{V}} \bar\rho(x)\,dx = \frac{1}{|\mathscr{V}|}.$$

Due to uniqueness of solution for the problem (1.6) and the fact that ρ_∞ is constant, we have that $\rho_k(t,x) = \rho(t+k,x) - \rho_\infty$. Fixing $T>2$, by the regularity of P and ρ we have, for $\beta\le\alpha$,

$a_k \in C^{1+\beta,1/2+\beta/2}(\overline{Q}_T)$ and $\|a_k\|_{C^{1+\beta,1/2+\beta/2}(\overline{Q}_T)}$ does not depend on k. Then, we can apply the estimate (2.4) of Theorem 2.2 and we obtain that

$$\|\rho(t+k,\cdot)-\rho_\infty\|_{C^{2+\beta}(\overline{\mathscr{V}})} \le C\,\|\rho(k,\cdot)-\rho_\infty\|_{C^0(\overline{\mathscr{V}})}, \qquad t\in[1,T), \tag{2.13}$$

where C is independent on k. Recalling the fundamental decay estimate for porous medium type equations, see [AR81] and [Váz07, Theorem 16.2 and Remark at page 546], we have

$$\|\rho(t,\cdot)-\rho_\infty\|_{C^0(\overline{\mathscr{V}})} \le Ce^{-\sigma t} \tag{2.14}$$

with C and σ depending only on $\|\bar\rho\|_{L^1(\mathscr{V})}$, from (2.13) we obtain that

$$\|\rho(t+k,\cdot)-\rho_\infty\|_{C^{2+\beta}(\overline{\mathscr{V}})} \le Ce^{-\sigma k}, \qquad t\in[1,T). \tag{2.15}$$

Lemma 2.3 for $a=0$ implies that

$$\|\boldsymbol{F}(t,\cdot)\|_{C^{1+\beta}(\overline{\mathscr{V}})} \le C\,\|v(t,\cdot)\|_{C^{2+\beta}(\overline{\mathscr{V}})}\,. \tag{2.16}$$

From (2.16) and (2.12) we obtain

$$\int_0^2 \|\boldsymbol{F}(t,\cdot)\|_{C^{1+\beta}(\overline{\mathscr{V}})}\,dt < +\infty. \tag{2.17}$$

Now, let us use again Lemma 2.3 for $a=v_\infty=P(\rho_\infty)$ to get

$$\|\boldsymbol{F}(t,\cdot)\|_{C^{1+\beta}(\overline{\mathscr{V}})} \le C\,\|v(t,\cdot)-v_\infty\|_{C^{2+\beta}(\overline{\mathscr{V}})}$$

and by the smoothness of P and (2.9)

$$\|v(t,\cdot)-v_\infty\|_{C^{2+\beta}(\overline{\mathscr{V}})} \le C\,\|\rho(t,\cdot)-\rho_\infty\|_{C^{2+\beta}(\overline{\mathscr{V}})}$$

Then, using (2.15) we get

$$\int_2^{+\infty} \|\boldsymbol{F}(t,\cdot)\|_{C^{1+\beta}(\overline{\mathscr{V}})}\,dt = \sum_{k=1}^{+\infty}\int_k^{k+1} \|\boldsymbol{F}(t,\cdot)\|_{C^{1+\beta}(\overline{\mathscr{V}})}\,dt \le C\sum_{k=1}^{+\infty} e^{-\sigma k} < +\infty. \tag{2.18}$$

In particular (2.8) holds and (2.2), which follows from (2.17) and (2.18), holds for every $\beta < \alpha$. We have thus obtained that $\boldsymbol{Y}_\infty$ exists and it is of class C^β. Then $\boldsymbol{u}_\infty := \boldsymbol{Y}_\infty \circ \bar{\boldsymbol{u}}$ is a diffeomorphism of class C^β.

In order to prove the exponential decay (2.7), we show that

$$\Psi(\rho(t,\cdot)) - \Psi(\rho_\infty) \le Ce^{-2\sigma t}. \tag{2.19}$$

Indeed, since $\int_{\mathscr{V}} \rho(t,x)\,dx = \int_{\mathscr{V}} \rho_\infty\,dx$, using the Taylor formula for ψ we write

$$\begin{aligned}\int_{\mathscr{V}} [\psi(\rho(t,x)) - \psi(\rho_\infty)]\,dx &= \int_{\mathscr{V}} [\psi(\rho(t,x)) - \psi(\rho_\infty) - \psi'(\rho_\infty)(\rho(t,x)-\rho_\infty)]\,dx \\ &= \int_{\mathscr{V}} \frac{1}{2}\psi''(\xi(x))(\rho(t,x)-\rho_\infty)^2\,dx \le \tilde{C}\|\rho(t,\cdot)-\rho_\infty\|^2_{L^2(\mathscr{V})},\end{aligned} \tag{2.20}$$

where $\tilde{C} = \frac{1}{2}\max\{\psi''(r) : r \in [\rho_{\min}, \rho_{\max}]\}$. Since

$$||\rho(t,\cdot) - \rho_\infty||^2_{L^2(\mathscr{V})} \leq |\mathscr{V}| ||\rho(t,\cdot) - \rho_\infty||^2_{L^\infty(\mathscr{V})},$$

we obtain (2.19) from (2.20) and (2.14). By the gradient flow energy identity given in Lemma 2.4 and passing to the limit as $T \to \infty$, we get

$$\Psi(\rho(t,\cdot)) - \Psi(\rho_\infty) = \int_t^{+\infty} |\boldsymbol{F}(s,x)|^2 \rho(s,x)\, dx\, ds.$$

Again, Lemma 2.4 and (2.19) imply

$$\int_t^{+\infty} \left\| \frac{d}{ds}\boldsymbol{u}(s,\cdot) \right\|^2_{L^2(\mathscr{U})} ds \leq C e^{-2\sigma t}.$$

And thus, we conclude

$$\int_{t+n}^{t+n+1} \left\| \frac{d}{ds}\boldsymbol{u}(s,\cdot) \right\|_{L^2(\mathscr{U})} ds \leq \Big(\int_{t+n}^{t+n+1} \left\| \frac{d}{ds}\boldsymbol{u}(s,\cdot) \right\|^2_{L^2(\mathscr{U})} ds \Big)^{1/2} \leq C^{1/2} e^{-(t+n)\sigma},$$

and consequently

$$\int_t^{+\infty} \left\| \frac{d}{ds}\boldsymbol{u}(s,\cdot) \right\|_{L^2(\mathscr{U})} ds = \sum_{n=0}^{+\infty} \int_{t+n}^{t+n+1} \left\| \frac{d}{ds}\boldsymbol{u}(s,\cdot) \right\|_{L^2(\mathscr{U})} ds \leq \frac{C^{1/2}}{1 - e^{-\sigma}} e^{-\sigma t}.$$

Observing that

$$||\boldsymbol{u}(t,\cdot) - \boldsymbol{u}_\infty(\cdot)||_{L^2(\mathscr{U})} \leq \int_t^{+\infty} \left\| \frac{d}{ds}\boldsymbol{u}(s,\cdot) \right\|_{L^2(\mathscr{U})} ds\,,$$

we obtain (2.7).

Finally, let us observe that $\frac{\partial \rho}{\partial t} + \boldsymbol{F} \cdot \rho = -\rho \operatorname{div} \boldsymbol{F}$, and thus

$$\frac{d}{dt} \log \rho(t, \boldsymbol{Y}(t,y)) = -(\operatorname{div} \boldsymbol{F})(t, \boldsymbol{Y}(t,y)),$$

for all $y \in \mathscr{V}$. Integrating in $[0,t]$ and taking the limit $t \to \infty$ using (2.1), we finally get

$$\det(D\boldsymbol{Y}_\infty(y)) = \exp\Big(\int_0^{+\infty} \operatorname{div} \boldsymbol{F}(t, \boldsymbol{Y}(t,y))\, dt \Big) = \frac{\rho_\infty}{\bar{\rho}(y)},$$

for all $y \in \mathscr{V}$. Now, (1.11), follows from the representation formula $\boldsymbol{u}_\infty := \boldsymbol{Y}_\infty \circ \bar{\boldsymbol{u}}$ and the definition of $\bar{\rho}$. □

3 The Fokker-Planck case

In this section, we consider the extension of the Evans-Gangbo-Savin approach to study existence and representation formula for solution of the system

$$\begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} = \operatorname{div}\Big(- \dfrac{1}{\det D\boldsymbol{u}} (\operatorname{cof} D\boldsymbol{u})^T \Big) - \nabla V(\boldsymbol{u}) & \text{in } \mathscr{U} \times (0, +\infty) \\ \boldsymbol{u}(0, \cdot) = \bar{\boldsymbol{u}} & \text{in } \mathscr{U}, \end{cases} \tag{3.1}$$

where $V : \overline{\mathscr{V}} \to \mathbb{R}$ is a given confinement potential $V \in C^2(\overline{\mathscr{V}})$. Following the same approach of Section 2 we study the asymptotic behavior of the solution of system (3.1) under the additional assumption that

$$\mathscr{V} \text{ is convex and } D^2 V(x) \geq \lambda I_d \tag{3.2}$$

with $\lambda > 0$ (I_d denotes the $d \times d$ identity matrix).

Recalling that $A(\operatorname{cof} A)^T = (\det A) I_d$ for every $d \times d$ matrix A, the equation in (3.1) can be rewritten as

$$\frac{\partial \boldsymbol{u}}{\partial t} = \operatorname{div}\Big(- (D\boldsymbol{u})^{-1} \Big) - \nabla V(\boldsymbol{u}).$$

The problem (3.1) is the *gradient flow*, with respect to the $L^2(\mathscr{U}; \mathbb{R}^d)$ metric, of the perturbation with a lower order term depending on V of the polyconvex functional (1.2) in the case $\Phi(s) = -\log s$,

$$I(\boldsymbol{u}) := -\int_{\mathscr{U}} \log(\det D\boldsymbol{u})\, dx + \int_{\mathscr{U}} V(\boldsymbol{u})\, dx$$

defined on Diff $(\overline{\mathscr{U}}; \overline{\mathscr{V}})$. Analogously to the previous section, we can observe that when a solution $\boldsymbol{u}$ of problem (3.1) is known, defining

$$\boldsymbol{w}(t, \cdot) := [\boldsymbol{u}(t, \cdot)]^{-1},$$

then the scalar function ρ defined by

$$\rho(t, y) := \det D\boldsymbol{w}(t, y) = \frac{1}{\det D\boldsymbol{u}(t, \boldsymbol{w}(t, y))}, \quad y \in \mathscr{V}, \tag{3.3}$$

which is the Lebesgue density of the measure $\nu_t := (\boldsymbol{u}(t, \cdot))_{\#} \mathscr{L}^d_{|\mathscr{U}}$, solves the linear boundary value problem of Fokker-Planck diffusion type

$$\begin{cases} \dfrac{\partial \rho}{\partial t} = \Delta \rho + \operatorname{div}(\rho \nabla V) & \text{in } (0, +\infty) \times \mathscr{V}, \\ (\nabla \rho(t, \cdot) + \rho(t, \cdot) \nabla V(\cdot)) \cdot \boldsymbol{n}_{\mathscr{V}}(\cdot) = 0 & \text{on } (0, +\infty) \times \partial \mathscr{V}, \\ \rho(0, \cdot) = \bar{\rho} := \dfrac{1}{\det D\bar{\boldsymbol{u}}} \circ \bar{\boldsymbol{u}}^{-1} & \text{in } \mathscr{V}, \end{cases} \tag{3.4}$$

(in the notation of the previous Section, since $\Phi(s) = -\log s$, consequently $P(s) = s$). Let us finally remark that the linear Fokker-Planck equation is the W_2-gradient flow of the functional

$$\Psi_V(\nu) := \int_{\mathscr{V}} [\rho(y) \log \rho(y) + V(y) \rho(y)]\, dy, \quad \text{if } \nu = \rho \mathscr{L}^d$$

and $+\infty$ otherwise.

In this case, the approach similar to the one of Evans, Gangbo and Savin reads as follows: the solution $\boldsymbol{u}$ of problem (3.1) can be built by solving the problem (3.4) as the first step, constructing the flow, as the second step, of the vector field $\boldsymbol{F}_V$ associated to the solution of problem (3.4)

$$\boldsymbol{F}_V(t,y) := -\nabla \log \rho(t,y) - \nabla V(y), \tag{3.5}$$

given, by definition of flow, by the maps $\boldsymbol{Y} : [0,+\infty) \times \mathscr{V} \to \overline{\mathscr{V}}$ that are the maximal solutions of the Cauchy problems

$$\begin{cases} \boldsymbol{Y}'(t,y) = \boldsymbol{F}_V(t,\boldsymbol{Y}(t,y)), \\ \boldsymbol{Y}(0,y) = y, \qquad y \in \mathscr{V} \end{cases}$$

and, finally, setting

$$\boldsymbol{u}(t,x) = \boldsymbol{Y}(t,\bar{\boldsymbol{u}}(x)), \tag{3.6}$$

which turns out a solution of problem (3.1). The existence result can be stated as follows.

Theorem 3.1. *Let us assume that $\mathscr{V}$ is a bounded open set of class $C^{2,\alpha}$ with $0 < \alpha < 1$ with $V \in C^2(\bar{\mathscr{V}})$. If $\bar{\boldsymbol{u}} \in C^{1,\alpha}(\overline{\mathscr{U}};\overline{\mathscr{V}}) \cap \mathrm{Diff}\,(\overline{\mathscr{U}};\overline{\mathscr{V}})$ and $0 < \rho_{\min} \leq \det D\bar{\boldsymbol{u}}^{-1} \leq \rho_{\max}$ then there exists a unique solution of the (distributional formulation of) problem (3.1) such that $\boldsymbol{u}(t,\cdot) \in \mathrm{Diff}\,(\overline{\mathscr{U}};\overline{\mathscr{V}})$ with $\partial_t \boldsymbol{u} \in L^2((0,T)\times\mathscr{U};\mathbb{R}^d)$ for any $T > 0$. Moreover $\boldsymbol{u}$ has the representation formula (3.6) and it satisfies*

$$\Psi_V(\rho(t,\cdot)) - \Psi_V(\rho(T,\cdot)) = \int_t^T |\boldsymbol{F}_V(s,y)|^2 \rho(s,y)\,dy\,ds = \int_t^T \left\| \frac{d}{ds}\boldsymbol{u}(s,\cdot) \right\|^2_{L^2(\mathscr{U})} ds, \tag{3.7}$$

for all $t,T \in [0,+\infty)$, $t < T$.

Proof. We give only a sketch of the proof. First of all, we observe that with the change of variable $\tilde{\rho} = \rho e^V$, the problem (3.4) can be rewritten as follows, where the new density $\tilde{\rho}$ satisfies an Ornstein-Ulenbeck type equation,

$$\begin{cases} \dfrac{\partial \tilde{\rho}}{\partial t} = \Delta\tilde{\rho} + \nabla\tilde{\rho}\cdot\nabla V & \text{in } (0,+\infty)\times\mathscr{V}, \\ \nabla\tilde{\rho}(t,\cdot)\cdot \boldsymbol{n}_{\mathscr{V}}(\cdot) = 0 & \text{on } (0,+\infty)\times\partial\mathscr{V}, \\ \tilde{\rho}(0,\cdot) = \bar{\rho}(\cdot)e^{V(\cdot)} & \text{in } \mathscr{V}, \end{cases} \tag{3.8}$$

and the vector field (3.5) can be rewritten as

$$\boldsymbol{F}_V(t,y) := -\nabla \log \tilde{\rho}(t,y).$$

Since $\tilde{\rho}(0,\cdot)$ is bounded from above and bounded away from 0, and the maximum principle holds for problem (3.8) we have the same bounds for every fixed $t > 0$. Since V is smooth, the estimate (2.3) of Theorem 2.2 yields

$$t^{1-\frac{\alpha-\beta}{2}} \|\tilde{\rho}(t,\cdot)\|_{C^{2+\beta}(\overline{\mathscr{V}})} \leq C\,\|\tilde{\rho}(0,\cdot)\|_{C^{\alpha}(\overline{\mathscr{V}})}, \qquad t \in (0,T).$$

Then

$$\int_0^T \|\boldsymbol{F}_V(t,\cdot)\|_{C^1(\overline{\mathscr{V}})} < +\infty,$$

and the flow $\boldsymbol{Y}$ is well defined for every $t \geq 0$. Since, by the regularity of the solution of problem (3.8) and the maximum principle for this equation, for every $t > 0$ and for every $y \in \overline{\mathscr{V}}$ we can solve the backward problem

$$\begin{cases} y'(s) = \boldsymbol{F}_V(s, y(s)), & s \in (0,t] \\ y(t) = y, & y \in \overline{\mathscr{V}}, \end{cases}$$

the map $\boldsymbol{Y}(t,\cdot)$ is a diffeomorphism of $\overline{\mathscr{V}}$ on itself. Reasoning as in the paper [ESG05], it is straightforward to show that (3.6) solves the weak formulation of the system (3.1). The last identity is direct from the change of variables (3.3) together with (3.6). □

In the same way we can state our asymptotic result for solution of system (3.1).

Theorem 3.2. *Let us assume that $\mathscr{V}$ is a convex bounded open set of class $C^{2,\alpha}$ with $0 < \alpha < 1$ and $V \in C^2(\bar{\mathscr{V}})$ a given confinement potential satisfying* (3.2)*. If $\bar{\boldsymbol{u}} \in C^{1,\alpha}(\overline{\mathscr{U}};\overline{\mathscr{V}}) \cap \mathrm{Diff}\,(\overline{\mathscr{U}};\overline{\mathscr{V}})$ and $0 < \rho_{\min} \leq \det D\bar{\boldsymbol{u}}^{-1} \leq \rho_{\max}$ then there exist $\boldsymbol{u}_\infty \in C^{1,\alpha}(\overline{\mathscr{U}};\overline{\mathscr{V}}) \cap \mathrm{Diff}\,(\overline{\mathscr{U}};\overline{\mathscr{V}})$ and a constant $C \geq 0$, depending on the initial datum, such that*

$$\|\boldsymbol{u}(t,\cdot) - \boldsymbol{u}_\infty\|_{L^2(\mathscr{U};\mathbb{R}^d)} \leq Ce^{-\lambda t} \qquad \forall t > 0,$$

where $\boldsymbol{u}$ is the solution of the problem (3.1) *given by Theorem* 3.1*. The final states $\boldsymbol{u}_\infty$ satisfies*

$$e^{-V(\boldsymbol{u}_\infty(x))}\det(D\boldsymbol{u}_\infty(x)) = c > 0, \qquad \forall x \in \mathscr{U}. \tag{3.9}$$

Proof. First of all we define the stationary state for the problem (3.4) given by

$$\rho_\infty(y) := Z\, e^{-V(y)}, \qquad y \in \mathscr{V},$$

where Z is chosen to normalize ρ_∞ to be a density of unit mass in $\mathscr{V}$. We again are going to apply Theorem 2.1 for which we need to check the integrability of $\boldsymbol{F}_V$ at $+\infty$. We observe that, by Theorem 2.2 applied to the solution of problem (3.8) with initial datum $\tilde{\rho}(k,\cdot) - \tilde{\rho}_\infty$, there exists $C > 0$ such that

$$\|\tilde{\rho}(t+k,\cdot) - \tilde{\rho}_\infty\|_{C^{2,\beta}(\overline{\mathscr{V}})} \leq C\,\|\tilde{\rho}(k,\cdot) - \tilde{\rho}_\infty\|_{C^0(\overline{\mathscr{V}})}\,, \qquad \text{for all } t \geq 1\,,$$

for every $k \in \mathbb{N}$. We show that there exist $C > 0$ and $\sigma > 0$ such that

$$\|\tilde{\rho}(t,\cdot) - \tilde{\rho}_\infty\|_{C^0(\overline{\mathscr{V}})} \leq Ce^{-\sigma t}. \tag{3.10}$$

Since V is a confinement potential satisfying (3.2), then the following logarithmic Sobolev inequality holds

$$\Psi_V(\rho) - \Psi_V(\rho_\infty) \leq \frac{1}{2\lambda}\int_{\mathscr{V}} |\nabla \log \rho + \nabla V|^2 \rho\, dy, \tag{3.11}$$

for all positive densities $\rho \in L^1_+(\mathscr{V})$ with unit mass for which the right-hand side is well defined. This inequality can be seen in [CJMTU, AGS05]. Since the linear Fokker-Planck evolution satisfies

$$\frac{d}{dt}\int_{\mathscr{V}} [\rho(t,y)\log\rho(t,y) + V(y)\rho(t,y)]\, dy = -\int_{\mathscr{V}} |\nabla \log\rho(t,y) + \nabla V(y)|^2 \rho(t,y)\, dy\,,$$

then, by the inequality (3.11), we easily obtain that

$$\Psi_V(\rho(t,\cdot)) - \Psi_V(\rho_\infty) \le e^{-2\lambda t}\,(\Psi_V(\bar\rho) - \Psi_V(\rho_\infty)) \tag{3.12}$$

and the Csizar-Kullback inequality, see [CJMTU] for instance, yields

$$\|\rho(t,\cdot) - \rho_\infty\|_{L^1(\mathscr{V})} \le C\, e^{-\lambda t}. \tag{3.13}$$

Recalling the interpolation inequality [Nir59, Bre83],

$$\|\rho(t,\cdot) - \rho_\infty\|_{C^0(\overline{\mathscr{V}})} \le C\,\|\rho(t,\cdot) - \rho_\infty\|^{\frac{d}{d+1}}_{C^1(\overline{\mathscr{V}})}\,\|\rho(t,\cdot) - \rho_\infty\|^{\frac{1}{d+1}}_{L^1(\overline{\mathscr{V}})}\,,$$

the uniform boundedness of the C^1 norm for $t \ge 1$, the definition of $\tilde\rho$ and (3.13), we obtain (3.10). We can then repeat for $\tilde\rho$ the final part of the proof of Theorem 2.5 to show the integrability in time of the flow map $\boldsymbol{F}_V$. The proof of the convergence of $\boldsymbol{u}(t,\cdot)$ towards $\boldsymbol{u}_\infty$ with the exponential rate of convergence λ follows as in the proof of Theorem 2.5 from (3.12) and (3.7). The formula (3.9) can be obtained by the same argument used at the end of the proof of Theorem 2.5. □

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