IMA Journal of Applied Mathematics Advance Access published November 3, 2009

IMA Journal of Applied Mathematics (2009) 1−16
doi:10.1093/imamat/hxp031

A surface energy approach to the mass reduction problem for elastic bodies

P. D'AMBROSIO† AND D. DE TOMMASI‡
Dipartimento di Ingegneria Civile e Ambientale, Politecnico di Bari, via Re David 200, 70125 Bari, Italy

AND

L. GRANIERI§ AND F. MADDALENA¶
Dipartimento di Matematica, Politecnico di Bari, via Re David 200, 70125 Bari, Italy

[Received on 24 July 2008; revised on 29 May 2009; accepted on 30 September 2009]

We consider the problem of mass reduction for elastic bodies by appearance of cavities. In this work, this problem is related to the minimization of a surface energy, depending on the stress tensor in the original equilibrium configuration. Special cases of mechanical interest are also analysed.

Keywords: Wulff shapes; shape optimization; variational problems in elasticity.

1. Introduction

The problem of mass minimization for elastic bodies is much studied and represents a continuous source of interest with respect to optimization theory as well as to structural design. Over the recent decades, several approaches have been proposed to both theoretical and applied studies on the subject, such as homogenization techniques, topological derivatives and variational methods involving varying domains (see Allaire, 2007; Bendsoe, 1995; Bucur, 2007; Bucur & Buttazzo, 2002; Henrot & Pierre, 2005). In this paper, we deal with the problem of mass reduction of a given elastic body Ω in a variational framework and, in order to take into account the material failure constraint, as usual, we fix the maximum amount of volume to subtract from the initial body.

Usually, the material failure corresponds to a pointwise inequality for the stress field. However, this leads to subtle mathematical issues since it involves the dependence of the solution of the equilibrium problem on the domain. Here, on the basis of *a priori* estimates on the stress field, we also give a justification of the classical approach which fixes the volume we can remove in safety (Section 3). Therefore, the problem under examination becomes the determination of the best geometry and location of the cavities inside the body.

In this paper, we propose a new variational approach based on the minimization of a Wulff-like energy (Section 4). Indeed, the creation of a cavity inside the elastic body increases the total elastic energy and the exceeding energy can be reduced to an energy distribution on the boundary of the cavity.

†Email: p.dambrosio@poliba.it
‡Corresponding author. Email: d.detommasi@poliba.it
§Email: l.granieri@poliba.it
¶Email: f.maddalena@poliba.it

Then, it seems to be reasonable to consider a surface energy as an optimality criterion. On the basis of these considerations, the variational problem takes the form

$$\min_K \left\{ \int_{\partial K} |\mathbf{Tn}| \mathrm{d}\mathcal{H}^{N-1} \middle| K \subset \Omega, |K| = V \right\},$$

where $\mathbf{T}$ is the stress tensor in the original equilibrium configuration of the body, K is the cavity and $\mathbf{n}$ is the outward unit normal to ∂K. Besides the interest in the mass reduction, the above variational problem poses some questions regarding relative isoperimetric properties or generalized Wulff energies which seem not yet completely known (see Fonseca, 1991, and references therein). Moreover, we note that our approach can be related to the variational analysis of cavitation phenomena in elastomeric materials carried in Sivaloganathan & Spector (2000). Finally, we discuss some special cases of mechanical interest (Section 5).

2. Setting of the problem

Let us consider an elastic body occupying a regular region $\Omega \subset \mathbb{R}^3$, or in general a subset of $\mathbb{R}^N$, in equilibrium under the given traction $\mathbf{f}$ on the boundary $\partial\Omega$. Its elastic state is completely determined by the triplet $(\mathbf{u}, \mathbf{E}, \mathbf{T})$ representing the displacement, the strain tensor and the stress tensor fields, respectively, and satisfying the kinematical condition

$$\mathbf{E} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^\top) \quad \text{in } \Omega, \tag{2.1}$$

the constitutive equation

$$\mathbf{T} = \mathbb{C}[\mathbf{E}] \quad \text{in } \Omega \tag{2.2}$$

and the equilibrium conditions

$$\begin{cases} \operatorname{div} \mathbf{T} = 0 & \text{in } \Omega, \\ \mathbf{Tn} = \mathbf{f} & \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

where $\mathbf{n}$ is the outward unit normal to the boundary $\partial\Omega$. Finally, we recall that the elastic strain energy for the solution $(\mathbf{u}, \mathbf{E}, \mathbf{T})$ is given by

$$\mathcal{E}(\Omega) = \frac{1}{2} \int_\Omega \mathbf{T} \cdot \mathbf{E} \, \mathrm{d}x,$$

so that, according to the well-known principle of minimum complementary energy in linear elasticity (see Necas & Hlavacek, 1981), it results that

$$\mathcal{E}(\Omega) = \min \left\{ \frac{1}{2} \int_\Omega \mathbf{T} \cdot \mathbb{C}^{-1}[\mathbf{T}] \mathrm{d}x \middle| \mathbf{T} \text{ satisfies (2.3)} \right\}. \tag{2.4}$$

The existence of minimizers for the previous variational problem is well established (see, for instance, Necas & Hlavacek, 1981). In the sequel, we will always assume that the triplet $(\mathbf{u}, \mathbf{E}, \mathbf{T})$ and the set $\Omega \setminus K$ enjoy the regularity requirements needed for the subsequent developments.

Here, we are interested in analysing the consequence of making a 'hole' inside the body and, in particular, if this can be done in an optimal way, i.e. optimizing a quantity which is particularly significant for the elastic state.

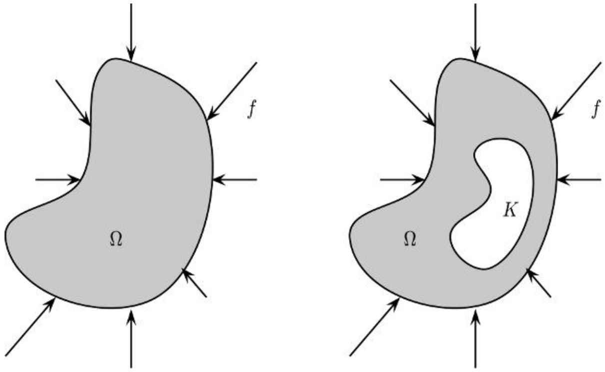

FIG. 1.

Let $K \subset\subset \Omega$ be a compact set representing the hole (see Fig. 1). By subtracting from the body the material inside K, we get a new material body whose equilibrium configuration $\Omega\backslash K$ is characterized by the elastic state $(\mathbf{u}_K, \mathbf{E}_K, \mathbf{T}_K)$. The new stress field $\mathbf{T}_K$ satisfies the equilibrium conditions

$$\begin{cases} \operatorname{div} \mathbf{T}_K = \mathbf{0} & \text{in } \Omega\backslash K, \\ \mathbf{T}_K \mathbf{n} = \mathbf{f} & \text{on } \partial\Omega, \\ \mathbf{T}_K \mathbf{n} = \mathbf{0} & \text{on } \partial K. \end{cases} \tag{2.5}$$

The main problem is 'how to choose K' in the best way, for instance, maximizing the volume of the hole $|K|$ and at the same time optimizing a suitable functional. Certainly, we cannot subtract too much material from the body without exceeding its strength. Therefore, we need to impose a 'failure' constraint that, usually, takes the form of an inequality such as $F(\mathbf{T}_K) \leqslant C$, where the choice of the mapping F relies on constitutive assumptions. We shall assume as failure constraint

$$\|\mathbf{T}_K\|_\infty \leqslant C. \tag{2.6}$$

In order to study a well-posed optimization problem, the set of the equilibrium stress fields corresponding to different choices of $K \subset\subset \Omega$ and satisfying the constraint (2.6) must be closed (with respect to the topology adopted) and this leads to the question: 'how $\mathbf{T}_K$ depends on K?' To give a satisfactory, though not exhaustive, answer to this question is a difficult matter since it requires the study of (2.5) for varying domains. In particular, some continuity property of $\mathbf{T}_K$ with respect to K is needed. For these questions, we refer to the books Bucur & Buttazzo (2002), Henrot & Pierre (2005) and the paper Bucur *et al.* (2001) for applications to elasticity. The approach pursued in these works involves the Hausdorff convergence of closed sets with some additional restriction (i.e. convexity, cone conditions and so on) in order to have convergence of displacements sequences in a Sobolev space.

The optimization problem in this framework seems to lead to extra difficulties due to the lack of control on K to invade all of Ω or to be disconnected or not.

3. Making holes in safety

To estimate the maximum material volume that we can subtract from the body Ω respecting the failure constraint, we could consider a property which is preserved in all the equilibrium configurations. A possible choice (maybe the simplest one) is to consider the mean equilibrium stress. Indeed, we have the following result.

THEOREM 3.1 (Mean stress theorem) Let $K \subset\subset \Omega$ be any given compact subset. Let $\mathbf{T}$ and $\mathbf{T}_K$ be the equilibrium stress fields relative to Ω and $\Omega \backslash K$, respectively. Then,

$$\int_\Omega \mathbf{T}\,\mathrm{d}x = \int_{\Omega\backslash K} \mathbf{T}_K\,\mathrm{d}x.$$

Proof. Let $\mathbf{u}\colon \Omega \to \mathbb{R}^3$ be any affine displacement field and let $\mathbf{E}$ be the corresponding strain tensor. Taking into account conditions (2.3) and (2.5), by using integration by parts, we get

$$\begin{aligned}\mathbf{E}\cdot\int_\Omega \mathbf{T}\,\mathrm{d}x = \int_\Omega \mathbf{T}\cdot\mathbf{E}\,\mathrm{d}x &= \int_{\partial\Omega} \mathbf{T}\mathbf{n}\cdot\mathbf{u}\,\mathrm{d}\mathcal{H}^{N-1}\\ &= \int_{\partial\Omega} \mathbf{T}_K\mathbf{n}\cdot\mathbf{u}\,\mathrm{d}\mathcal{H}^{N-1} - \int_{\partial K} \mathbf{T}_K\mathbf{n}\cdot\mathbf{u}\,\mathrm{d}\mathcal{H}^{N-1}\\ &= \int_{\Omega\backslash K} \mathbf{T}_K\cdot\mathbf{E}\,\mathrm{d}x = \mathbf{E}\cdot\int_{\Omega\backslash K} \mathbf{T}_K\,\mathrm{d}x.\end{aligned}$$

By the arbitrariness of $\mathbf{E}$, we get the thesis. □

We can consider a norm of the mean stress

$$S_K := \frac{1}{|\Omega\backslash K|}\left\|\int_{\Omega\backslash K} \mathbf{T}_K\,\mathrm{d}x\right\| \tag{3.1}$$

as a representative quantity of the stress state of the body. If we choose for the condition (2.6) the following form:

$$S_K \leqslant C, \tag{3.2}$$

in view of (3.2), the best amount of material to subtract from the body is given by

$$|\Omega\backslash K| = \frac{1}{C}\left\|\int_{\Omega\backslash K} \mathbf{T}_K\,\mathrm{d}x\right\|,$$

which yields that

$$V := |K| = |\Omega| - \frac{1}{C}\left\|\int_{\Omega\backslash K} \mathbf{T}_K\,\mathrm{d}x\right\|.$$

3.1 *Bounds on the stress*

Summarizing, we have fixed the maximum volume allowed for the set K by limiting the mean stress. However, limiting the mean stress is not a proper failure criterion since, as it is well known, failure criteria involve pointwise bounds on the stress components, so we need to establish some estimate in order to relate the volume of K with local bounds on the stress. To this aim, we recall the following remarkable result due to Signorini and we refer to the comprehensive treatise Villaggio (1977) for the related proof as well as for several examples and discussions.

Let us introduce some notations. We regard the stress tensor $\mathbf{T}$ as the vector $\mathbf{s} \in \mathbb{R}^6$ whose components are given by $s_i = T_{ii}$ for $i = 1, 2, 3$ and $s_4 = T_{23}$, $s_5 = T_{31}$ and $s_6 = T_{12}$. Moreover, we denote by $\mathbf{q} \in \mathbb{R}^{m+1}$ a vector field whose components constitute an orthogonal system of $m + 1$ functions defined on $\Omega \subset \mathbb{R}^3$, i.e. such that

$$\int_\Omega q_i q_j \,\mathrm{d}x = 0, \quad i \neq j.$$

Let $\mathbf{B} \in \mathbb{R}^{6,6}$ be a given positive-semi-definite symmetric constant matrix and finally, let

$$\mathbf{D} = \frac{1}{|\Omega|}\int_\Omega \mathbf{q} \otimes \mathbf{q} \,\mathrm{d}x, \quad \overline{\mathbf{q} \otimes \mathbf{s}} = \frac{1}{|\Omega|}\int_\Omega \mathbf{q} \otimes \mathbf{s} \,\mathrm{d}x.$$

THEOREM 3.2 (Signorini's inequality) The state of stress in a material body having configuration Ω satisfies the inequality

$$\int_\Omega \mathbf{Bs} \cdot \mathbf{s} \,\mathrm{d}x \geqslant |\Omega| \mathrm{tr}\{\mathbf{B}[(\overline{\mathbf{q} \otimes \mathbf{s}})^\top \mathbf{D}^{-1} (\overline{\mathbf{q} \otimes \mathbf{s}})]\}. \tag{3.3}$$

It is worth noting that if, for instance, $\mathbf{B}$ is a diagonal matrix with the only non-vanishing elements given by B_{ii} $(i = 1, \ldots, 6)$, then, after setting $M_j = D_{jj}$ $(j = 0, 1, \ldots, m)$, the inequality (3.3) gives the following form:

$$|s_i|_{\max} \geqslant \left(\sum_{j=0}^{m} \frac{(\overline{q_j s_i})^2}{M_j^2} \right)^{\frac{1}{2}}, \tag{3.4}$$

which yields a lower bound for the maximum modulus of the ith stress component. Observe that the inequality (3.4) could be directly derived in this simplified framework. We refer to Villaggio (1977, Chapter IV, Section 18) for other significant forms of the matrix $\mathbf{B}$ leading to lower bound on the maximum normal or tangential stress as well as the second invariant of the stress deviator as occurring in the von Mises criterion. Here, we limit ourselves to illustrate by a simple example how the combined use of the Signorini inequality and a failure criterion allows us to fix the maximum volume of the removable set K.

Let us consider the plane elastic body Ω in Fig. 2, where a and h are the dimensions of the rectangle Ω, loaded by two opposite normal pressure having constant modulus $f = |\mathbf{f}|$. We want to bound the maximum modulus of T_{22} and to this aim we take only one function $q_0 = 1$. Then,

$$M_0^2 = \frac{1}{|\Omega \backslash K|}\int_{\Omega \backslash K} 1 \,\mathrm{d}x = 1,$$

and moreover, to compute $\overline{q_0 s_2}$, we take $\psi = x_2$ and apply the identity (Villaggio, 1977, (16.6))

$$\int_{\partial(\Omega \backslash K)} \psi \mathbf{f} \,\mathrm{d}s = \int_{\Omega \backslash K} \mathbf{T} \nabla \psi \,\mathrm{d}x$$

holding for every C^1-mapping ψ. Therefore, we get

$$\int_{\partial(\Omega \backslash K)} \psi \mathbf{f} \cdot \mathbf{e}_2 \,\mathrm{d}s = fah = \int_{\Omega \backslash K} s_2 \,\mathrm{d}x = |\Omega \backslash K| \overline{1 s_2}$$

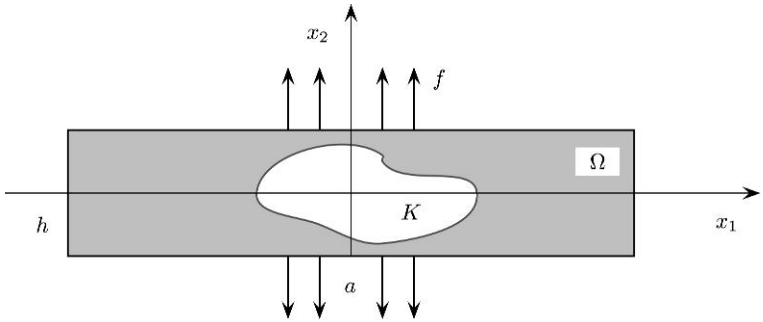

FIG. 2.

and so by (3.4), we obtain the estimate

$$|s_2|_{\max} \geqslant \frac{fah}{3ah - |K|}.$$

Finally, we can conclude by noting that, according with (2.6), if we assume as failure criterion the inequality $|s_2|_{\max} \leqslant s^*$, by combining this condition with the previous one, we obtain the desired estimate on the maximum volume allowed for K. More precisely, we have

$$\frac{fah}{3ah - |K|} \leqslant |s_2|_{\max} \leqslant s^*$$

and so

$$|K| \leqslant \frac{ah(3s^* - f)}{s^*}.$$

4. The main variational problem

We remark that assuming the volume of the hole as the unique descriptor of the problem is not enough. Indeed, the hole K could be placed anywhere inside Ω and this, in general, seems not an optimal choice due to the inhomogeneity of the stress state. In fact, the geometry of K should be related to the stress distribution $\mathbf{T}_K$ and, as we have just observed, the dependence on K makes this problem difficult. To avoid this difficulty, a reasonable choice could be to place the hole where the stress tensor $\mathbf{T}$ of the reference state Ω is lower in order to get a nearly uniform engagement of the material. In the spirit of shape optimization (Bendsoe, 1995; Bucur & Buttazzo, 2002), this leads to minimize the elastic body strain energy. To this aim, let us begin by estimating the change of energy in the elastic body to the creation of an inner hole K. Let us recall that the elastic strain energy at the equilibrium is given by

$$\mathcal{E}(\Omega \backslash K) = \frac{1}{2} \int_{\Omega \backslash K} \mathbf{T}_K \cdot \mathbf{E}_K \, \mathrm{d}x.$$

Moreover, denoting by $\mathcal{E}(\Omega)$ the elastic energy relative to the initial configuration Ω and setting $\Omega_K = \Omega \backslash K$, we have the following simple well-known result.

PROPOSITION 4.1 Let $K \subset\subset \Omega$ be any compact subset. Then,

$$\mathcal{E}(\Omega_K) - \mathcal{E}(\Omega) \geqslant 0.$$

Proof. Let $(\mathbf{u}, \mathbf{E}, \mathbf{T})$ and $(\mathbf{u}_K, \mathbf{E}_K, \mathbf{T}_K)$ be the solutions of the elastic problem in the configurations Ω and $\Omega \backslash K$, respectively. To apply the principle of minimum complementary energy (2.4), it is sufficient to consider tensor fields $\mathbf{T} \in L^2(\Omega, \mathbb{R}^9)$ satisfying $\operatorname{div} \mathbf{T} = 0$ in Ω and $\mathbf{Tn} = \mathbf{f}$ on $\partial\Omega$. To this aim, we extend $\mathbf{T}_K$ to zero outside $\Omega \backslash K$. If the boundary of K is at least Lipschitz regular, the new tensor field, still denoted by $\mathbf{T}_K$, belongs to $L^2(\Omega, \mathbb{R}^9) \cap \mathrm{BV}(\Omega, \mathbb{R}^9)$, where $\mathrm{BV}(\Omega, \mathbb{R}^9)$ denotes the space of tensor fields of bounded variation (see, for instance, Attouch *et al.*, 2006; Evans & Gariepy, 1992). In such a case (it suffices that K is a set of finite perimeter), by applying the divergence theorem (see Attouch *et al.*, 2006; Evans & Gariepy, 1992, for fields of bounded variation), because of the boundary condition $\mathbf{T}_K \mathbf{n} = 0$ on ∂K, it follows that $\operatorname{div} \mathbf{T}_\mathrm{K} = 0$ in Ω and $\mathbf{T}_K \mathbf{n} = \mathbf{f}$ on $\partial\Omega$. Therefore, $\mathbf{T}_K$ is a statically admissible stress field for Ω.

By virtue of (2.4), we get that

$$\mathcal{E}(\Omega) \leqslant \frac{1}{2} \int_\Omega \mathbf{T}_K \cdot \mathbb{C}^{-1}[\mathbf{T}_K] \mathrm{d}x = \frac{1}{2} \int_{\Omega \backslash K} \mathbf{T}_K \cdot \mathbb{C}^{-1}[\mathbf{T}_K] \mathrm{d}x = \mathcal{E}(\Omega_K).$$

□

Moreover, by virtue of the following Eshelby-like result, we find that the energy change $\mathcal{E}(\Omega_K) - \mathcal{E}(\Omega)$ is concentrated on the boundary of the hole K (see Lewinski & Sokolowski, 2003, for a detailed study of energy change due to small cavities).

THEOREM 4.2 Let $K \subset\subset \Omega$ be any compact subset and let $(\mathbf{u}, \mathbf{E}, \mathbf{T})$ and $(\mathbf{u}_K, \mathbf{E}_K, \mathbf{T}_K)$ be the solutions of the elastic problem in the configurations Ω and $\Omega \backslash K$, respectively. Then,

$$\mathcal{E}(\Omega_K) - \mathcal{E}(\Omega) = \int_{\partial K} \mathbf{Tn} \cdot \mathbf{u}_K \, \mathrm{d}\mathcal{H}^{N-1}. \tag{4.1}$$

Proof. Let us recall (see, for instance, Temam, 1983) that for every symmetric tensor $\tilde{\mathbf{T}} \in L^2(\Omega, \mathbb{R}^9)$ such that $\operatorname{div} \tilde{\mathbf{T}} \in L^2(\Omega, \mathbb{R}^3)$ and for every displacement $\hat{\mathbf{u}} \in H^1(\Omega, \mathbb{R}^3)$, we have the following identity:

$$\frac{1}{2} \int_\Omega \tilde{\mathbf{T}} \cdot (\nabla \hat{\mathbf{u}} + \nabla \hat{\mathbf{u}}^\top) \mathrm{d}x + \int_\Omega \hat{\mathbf{u}} \cdot \operatorname{div} \tilde{\mathbf{T}} \, \mathrm{d}x = \int_{\partial\Omega} \tilde{\mathbf{T}}\mathbf{n} \cdot \hat{\mathbf{u}} \, \mathrm{d}\mathcal{H}^{N-1}. \tag{4.2}$$

As in Proposition 4.1, we extend $\mathbf{T}_K$ to zero outside $\Omega \backslash K$. Since $\mathbf{T}_K \in \mathrm{BV}(\Omega, \mathbb{R}^9)$, formula (4.2) still holds. Taking into account the conditions (2.3) and (2.5), by formula (4.2), we get that

$$\begin{aligned} &\int_\Omega (\mathbf{T}_K - \mathbf{T}) \cdot \mathbf{E} \, \mathrm{d}x \\ &\quad = \int_{\partial\Omega} \mathbf{T}_K \mathbf{n} \cdot \mathbf{u} \, \mathrm{d}\mathcal{H}^{N-1} - \int_{\partial\Omega} \mathbf{Tn} \cdot \mathbf{u} \, \mathrm{d}\mathcal{H}^{N-1} = 0. \end{aligned}$$

Hence,

$$\int_\Omega \mathbf{T}_K \cdot \mathbf{E} \, \mathrm{d}x = \int_\Omega \mathbf{T} \cdot \mathbf{E} \, \mathrm{d}x. \tag{4.3}$$

Since K is assumed to have Lipschitz regular boundary, we consider a $H^1(\Omega, \mathbb{R}^3)$-extension of $\mathbf{u}_K$ (see, for instance, Attouch *et al.*, 2006; Evans & Gariepy, 1992) on the whole Ω. Note that such extension of the displacement field is independent of the previous extension made on the field $\mathbf{T}_K$. By applying again (4.2) to the fields $\mathbf{T}_K$ and the new field, still denoted by $\mathbf{E}_K$, which is defined on the whole of Ω and is obtained as the symmetric part of the gradient of the above extension of $\mathbf{u}_K$, we also get that

$$\int_\Omega \mathbf{T}_K \cdot \mathbf{E}_K \, dx = \int_\Omega \mathbf{T} \cdot \mathbf{E}_K \, dx. \tag{4.4}$$

Therefore, we can evaluate

$$\mathcal{E}(\Omega_K) - \mathcal{E}(\Omega) = \frac{1}{2}\left(\int_{\Omega\setminus K} \mathbf{T}_K \cdot \mathbf{E}_K \, dx - \int_\Omega \mathbf{T} \cdot \mathbf{E} \, dx\right).$$

Since $\mathbf{T}_K = 0$ on K, by (4.3) and (4.4), we have

$$\mathcal{E}(\Omega_K) - \mathcal{E}(\Omega) = \frac{1}{2}\left(\int_\Omega \mathbf{T} \cdot \mathbf{E}_K \, dx - \int_\Omega \mathbf{T}_K \cdot \mathbf{E} \, dx\right). \tag{4.5}$$

On the other hand, in view of the symmetry of $\mathbb{C}$, we have

$$\begin{aligned}\int_\Omega \mathbf{T}_K \cdot \mathbf{E} \, dx &= \int_{\Omega\setminus K} \mathbf{T}_K \cdot \mathbf{E} \, dx = \int_{\Omega\setminus K} \mathbb{C}[\mathbf{E}_K] \cdot \mathbf{E} \, dx \\ &= \int_{\Omega\setminus K} \mathbb{C}[\mathbf{E}] \cdot \mathbf{E}_K \, dx = \int_{\Omega\setminus K} \mathbf{T} \cdot \mathbf{E}_K \, dx.\end{aligned}$$

Finally, by the last two equations, we obtain that

$$\mathcal{E}(\Omega_K) - \mathcal{E}(\Omega) = \int_K \mathbf{T} \cdot \mathbf{E}_K \, dx,$$

which, in view of (4.2) applied to the region K, yields that

$$\mathcal{E}(\Omega_K) - \mathcal{E}(\Omega) = \int_{\partial K} \mathbf{T}\mathbf{n} \cdot \mathbf{u}_K \, d\mathcal{H}^{N-1}. \qquad \square$$

The previous considerations and the energy estimates led us to think that, roughly speaking, the creation of a hole inside an elastic body can be related to a kind of surface energy. Indeed, a direct approach could consider the term $\mathbf{u}_K$ in (4.1) as a state variable of an optimal control problem (see Bucur & Buttazzo, 2002). However, this approach seems to be difficult because of the dependence of $\mathbf{u}_K$ on the variable domain $\Omega\setminus K$. Therefore, we observe that according to (4.1), the elastic energy change occurring in the body after the creation of the hole K can be reduced to an energy-like term localized on ∂K. Indeed, by using Poincarè inequality (see, for instance, Evans, 1997) and since the elastic state $\mathbf{u}_K$ is determined up to a constant, we have

$$\begin{aligned}\mathcal{E}(\Omega_K) - \mathcal{E}(\Omega) &= \int_{\partial K} \mathbf{T}\mathbf{n} \cdot \mathbf{u}_K \, d\mathcal{H}^{N-1} \leqslant \|\mathbf{u}_K\|_\infty \int_{\partial K} |\mathbf{T}\mathbf{n}| d\mathcal{H}^{N-1} \\ &\leqslant C\|\nabla \mathbf{u}_K\|_\infty \int_{\partial K} |\mathbf{T}\mathbf{n}| d\mathcal{H}^{N-1}.\end{aligned} \tag{4.6}$$

Now, by virtue of the failure constraint (2.6) and the constitutive equation (2.2), assuming the elasticity tensor $\mathbb{C}$ is invertible, we have

$$\|\nabla \mathbf{u}_K\|_\infty \leqslant M,$$

for some positive constant M. Therefore, since the Poincarè constant is in fact bounded by the volume (actually by the diameter) of $\Omega \backslash K$, according to (4.6), we can estimate the lower bound of the energy increment due to the creation of a hole by minimizing the energy whose density is the norm of the stress vector acting on the boundary of the hole. Then, we are led to study the following minimization problem:

$$\min_K \left\{ \int_{\partial K} |\mathbf{Tn}| \mathrm{d}\mathcal{H}^{N-1} \middle| K \subset \Omega, |K| = V \right\}. \tag{4.7}$$

Let us note that (4.7) is a Wulff problem (Ambrosio, 1997; Brothers & Morgan, 1994; Fonseca, 1991) for the energy density $\Gamma(x, \mathbf{n}(x)) = |\mathbf{Tn}|$, where $|\cdot|$ denotes the Euclidean norm. The Wulff problem is known as describing the equilibrium shape of a perfect crystal of one material in contact with a single surrounding medium for which the dependence of the surface energy on the normal $\mathbf{n}$ relates the surface tension with the bulk crystalline lattice. If the property

$$\Gamma(x, \mathbf{n}(x)) > \alpha > 0 \tag{4.8}$$

holds, the existence of minimizers for the problem (4.7) can be proved by applying the direct methods of the calculus of variations. Indeed, we have compactness among the sets of finite perimeter and by the lower semi-continuity of the surface energy (see Fonseca, 1991), we get minimizers. If (4.8) fails, the existence problem is more involved and in many cases it is an open problem. Different minimization strategies are studied, for instance, in Buttazzo & Guasoni (1997) where K varies among the convex sets or in Granieri & Maddalena (2008) where the set variable is regarded as a Radon measure. In Section 5, we will investigate some geometrical properties of the minimizers in relation to the properties of the stress field.

5. Wulff shapes in stress fields

Now, we shall consider some special cases of stress fields, namely, the homogeneous ones, and we will study the main geometric properties of the optimal holes K given, as in (4.7), by the minimizers of the functional

$$\mathcal{F}(K) = \int_{\partial K} |\mathbf{Tn}| \mathrm{d}\mathcal{H}^{N-1}, \tag{5.1}$$

under the constraint $|K| = V$.

5.1 *Necessary condition of optimality*

For surface integrals like those appearing in (5.1), it is possible to compute the derivatives with respect to the variations of the domain K and then derive the stationarity properties of the minimum configurations. Here, we closely follow the discussion made in Chapter 5 of Henrot & Pierre (2005). To characterize the optimum is quite standard to consider a variation of the domain through a family of maps

$$\phi_\varepsilon(x) := x + \varepsilon \mathbf{v}(x),$$

where $\mathbf{v}$ is a given vector field in $\mathbb{R}^N$ such that for every ε,

$$|\phi_\varepsilon(K)| = |K|.$$

In particular, the above condition yields that

$$\int_{\partial K} \mathbf{v} \cdot \mathbf{n} \, \mathrm{d}\mathcal{H}^{N-1} = \int_K \operatorname{div} \mathbf{v} \, \mathrm{d}x = 0. \tag{5.2}$$

Indeed, let $I(\varepsilon) = |\phi_\varepsilon(K)|$ and observing that $\nabla\phi_\varepsilon = \mathbf{I} + \varepsilon\nabla\mathbf{v}$, we get that

$$\det(\nabla\phi_\varepsilon) = 1 + \varepsilon \operatorname{div}(\mathbf{v}) + \varepsilon^2 l_2(\nabla\mathbf{v}) + \varepsilon^3 l_3(\nabla\mathbf{v}),$$

where l_i denotes the ith dimensional orthogonal invariant, i.e. $l_2(A) = (\mathrm{trace}(A))^2 - \mathrm{trace}(A^2)$ and $l_3(A) = \det(A)$ for every matrix A. Therefore, we evaluate

$$0 = I'(0) = \int_K \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \det(\nabla\phi_\varepsilon)_{|\varepsilon=0} \, \mathrm{d}x = \int_K \operatorname{div} \mathbf{v} \, \mathrm{d}x.$$

If K is a minimum, taking $K_\varepsilon = \phi_\varepsilon(K)$, we have that

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} \mathcal{F}(K_\varepsilon)_{|\varepsilon=0} = 0, \tag{5.3}$$

provided that the vector field $\mathbf{v}$ satisfies the condition $\mathbf{v} = 0$ on the boundary of Ω. If K is regular, say with $\mathcal{C}^1$-boundary, (5.3) takes the following form (see Proposition 5.4.18 of Henrot & Pierre, 2005):

$$\int_{\partial K} [H|T\mathbf{n}| + \nabla|T\mathbf{n}| \cdot \mathbf{n}](\mathbf{v} \cdot \mathbf{n}) \mathrm{d}\mathcal{H}^{N-1} = 0, \tag{5.4}$$

where H is the mean curvature and $\mathbf{n}$ is the unit outward normal vector of ∂K. By the arbitrariness of the vector field $\mathbf{v}$, as usual in calculus of variations, the condition (5.2) implies the following optimality necessary condition:

$$H|T\mathbf{n}| + \nabla|T\mathbf{n}| \cdot \mathbf{n} = \text{constant} \quad \text{on } \partial K. \tag{5.5}$$

5.2 *Hydrostatic stress*

Let $\Omega \subset \mathbb{R}^2$ be a rectangle region (Fig. 3) representing the equilibrium configuration of a material body whose equilibrium stress field is given by $\mathbf{T} = \lambda\mathbf{I}$.

In this case, the variational problem (4.7) becomes the isoperimetric problem

$$\min_K \{\mathcal{P}(K) | K \subset \Omega, |K| = V\}, \tag{5.6}$$

where $\mathcal{P}(K)$ is the perimeter of K and $\mathcal{P}(K) = \mathcal{H}^{N-1}(\partial K)$ if the boundary of K is sufficiently regular. If K is regular, say with $\mathcal{C}^1$-boundary, by (5.5), we deduce that $\partial K \cap \Omega$ has constant mean curvature. The same happens whenever T is an isometry. By De Giorgi regularity results (see, for instance, Evans & Gariepy, 1992), this is also true for sets of finite perimeter. It is possible to characterize the optimal solutions as follows.

Let $\mathcal{C}_V \subset \mathbb{R}^2$ be any ball such that $|\mathcal{C}_V| = V$. Of course, by the isoperimetric property of the ball, if $\mathcal{C}_V$ can be contained into the rectangle Ω, then $\mathcal{C}_V$ is the best hole. Otherwise, we have to solve a relative isoperimetric problem. In such a case, for simplicity, consider Ω as an infinite strip $\mathcal{R}$ of height h and consider $\mathcal{C}$ the biggest ball inscribed in $\mathcal{R}$. Let $E \subset \mathcal{R}$ be a regular set such that $|E| = V > |\mathcal{C}|$. By continuity, we can find two vertical lines (see Fig. 4) which enclose two regions E_l and E_r of E both of measure $\frac{|\mathcal{C}|}{2}$. Let D be the distance between such lines and let E_c be the remainder of E, i.e. the region of E enclosed by the two vertical lines.

Now, let $B = \frac{|E_c|}{h}$ and take F as in Fig. 5. We claim that F is the best hole inside the strip $\mathcal{R}$.

Indeed, by the isoperimetric property of the ball, we obtain that

$$\mathcal{P}(E) \geqslant \mathcal{P}(\mathcal{C}) + 2D.$$

On the other hand, we have

$$Bh = |E_r| \leqslant Dh.$$

Therefore, we get

$$\mathcal{P}(E) \geqslant \mathcal{P}(\mathcal{C}) + 2B = \mathcal{P}(F).$$

In the general case of reference domain Ω, explicit computations are more difficult to obtain. Indeed, also in the case of a convex set Ω, it is not known if the solution of the relative isoperimetric problem is convex or not. This is true in the plane by considering the convex hull, but this approach fails in higher dimensions. For an account on these problems, characterizations and regularity of the minimizers, we refer to Peri (2001), Cianchi (1989), Rosales (2003), Stredulinski & Ziemer (1997) and the references therein. Nevertheless, we have always an optimal hole, at least in the class of sets of finite perimeter.

Observe that in the case $\mathbf{T} = \lambda\mathbf{I}$, the optimal hole can touch $\partial\Omega$ since the stress $\mathbf{T}$ is uniformly distributed. Hence, in such a case, our starting elastic problem does not admit any solution since we would have $K \subset\subset \Omega$. Therefore, these examples show the necessity to consider local conditions on the

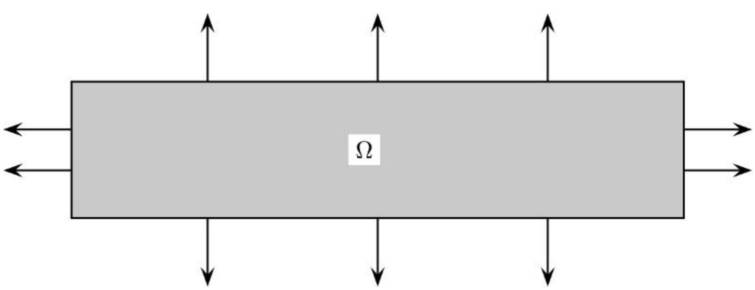

FIG. 3.

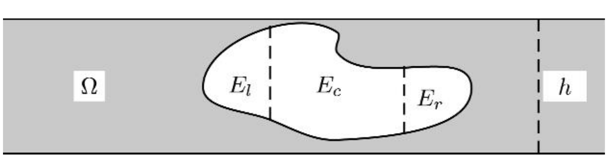

FIG. 4.

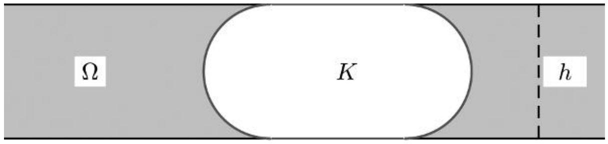

FIG. 5.

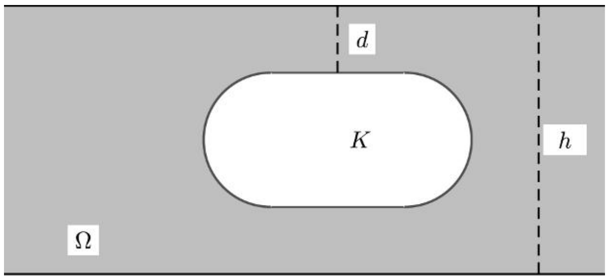

FIG. 6.

stress in order to get solutions of applicative interest. In particular, we expect the stress to be higher and higher whenever the hole approaches the boundary $\partial\Omega$. In the case of $\mathbf{T} = \lambda\mathbf{I}$ discussed above, if for safety requirement the hole has to place at a minimum distance d from the boundary of $\mathcal{R}$, the best hole looks like the one in Fig. 6.

In the general case of constant stress, by diagonalization we can consider $\mathbf{T}$ as a diagonal matrix whose elements are the eigenvalues of $\mathbf{T}$. For $N = 2$, let λ_1 and λ_2 be such eigenvalues. If $\lambda_1 = \lambda_2$, the variational problem reduces to the isoperimetric problem just discussed.

5.3 *Uniaxial stress*

If one eigenvalue vanishes, say $\lambda_2 = 0$, then $|\mathbf{Tn}| = |\lambda_1 n_1|$. Therefore, problem (4.7) can be written as

$$\min_K \left\{ \int_{\partial K} |\mathbf{F}\cdot\mathbf{n}| \mathrm{d}\mathcal{H}^{N-1} \middle| K \subset \Omega, |K| = V \right\}, \tag{5.7}$$

where $\mathbf{F}$ is the constant vector field given by $\mathbf{F} = (\lambda_1, 0)$. These kind of problems are studied in Granieri & Maddalena (2008) in the setting of Radon measures. In the paper Granieri & Maddalena (2008), it is shown that if, e.g. $\overline{\Omega} = [0, 2]^2$ and $V = 1$, an optimal set is given by the rectangle of width 2 and height $\frac{1}{2}$ as shown in Fig. 7.

The optimal set depends not only on the vector field $\mathbf{F}$ but also on the geometry of the reference domain Ω. For instance, in a reference domain as in Fig. 8, since in every rectangle the optimal set is a rectangle of maximal width, the solution is a set similar to the one sketched in Fig. 8. We conjecture that the optimal set will look like the first one sketched in Fig. 9, while the second could be excluded since the necessary condition (5.5) is not satisfied. Indeed, considering for simplicity the ball centred at the origin and radius equal to 1, for $x_1 > 0$, we compute $\nabla|T\mathbf{n}| = \lambda_1(1 - x_1^2, -x_1 x_2)$. Hence, condition (5.5) yields that

$$\lambda_1 H x_1 + \lambda_1(x_1 - x_1^3 - x_1 x_2^2) = \lambda_1(2x_1 - x_1^3 - x_1 + x_1^3) = \lambda_1 x_1,$$

which is not constant on the right side of ∂K.

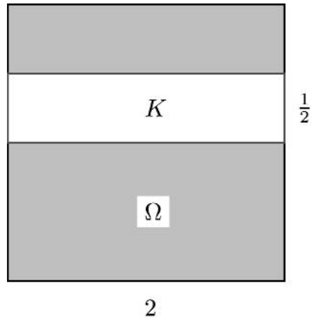

FIG. 7.

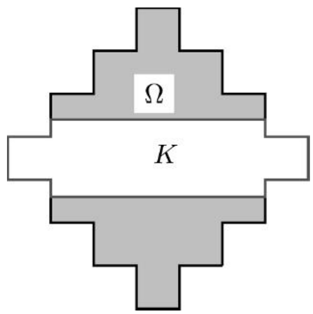

FIG. 8.

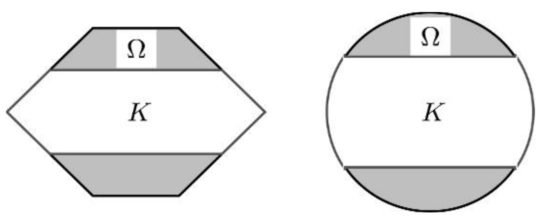

FIG. 9.

5.4 *Uniform stress fields*

By setting $0 < \lambda_1 < \lambda_2$, we need to solve a relative Wulff problem. In fact, if we introduce the norm defined as

$$\|x\| := |\mathbf{T}x| = \sqrt{\lambda_1^2 x_1^2 + \lambda_2^2 x_2^2},$$

the problem (4.7) becomes the following:

$$\min_K \left\{ \int_{\partial K} \|\mathbf{n}\| \mathrm{d}\mathcal{H}^{N-1} \middle| K \subset \Omega, |K| = V \right\}. \tag{5.8}$$

It is known that (see, e.g. Fonseca, 1991; Brothers & Morgan, 1994) in the whole $\mathbb{R}^2$ the solution is given by the Wulff shape

$$W = x_0 + r B_*,$$

where $B_* = \{x \in \mathbb{R}^2 | \|x\|_* \leqslant 1\}$ is the unit ball in the dual norm, while the factor $r > 0$ is chosen to satisfy the volume constraint. Note that the Wulff shape is unique modulo translations. We observe that the norm $\|\cdot\|$ is induced by the scalar product

$$x \cdot y = \lambda_1^2 x_1 y_1 + \lambda_2^2 x_2 y_2.$$

By the definition of dual norm, we have

$$\|x\|_* = \max\{x \cdot y | y \in \mathbb{R}^2, \|y\| = 1\}.$$

The above maximum problem can be explicitly solved (Lagrange multiplier) leading to the following expression:

$$\|x\|_* = \frac{\lambda_2 x_1^2}{\lambda_1(\lambda_2^2 x_1^2 + \lambda_1^2 x_2^2)^{1/2}} + \frac{\lambda_1 x_2^2}{\lambda_2(\lambda_2^2 x_1^2 + \lambda_1^2 x_2^2)^{1/2}}.$$

Therefore, the Wulff shape turns out to be an 'ellipsoid'. Now, since the ellipsoid is symmetric with respect to the vertical axis, if we take Ω as an infinite strip of height h, we can repeat the arguments used in the discussion of the case $\mathbf{T} = \lambda \mathbf{I}$ replacing semicircle with 'semi-ellipsoid'. More general configurations require a careful study of a relative Wulff problem since no general theory seems to be available at the moment to treat similar questions.

5.5 *A special case of non-constant stress field*

Let $\lambda_1 = \lambda x_1$ and $\lambda_2 = \lambda x_2$ be the eigenvalues of the stress tensor $\mathbf{T}$. Let us consider the field $\mathbf{F} = (\lambda x_1, \lambda x_2)$. It results that

$$|\mathbf{F} \cdot \mathbf{n}| \leqslant \|\mathbf{T}\mathbf{n}\|. \tag{5.9}$$

If V is the volume to subtract from Ω, we introduce the following problem:

$$\min \left\{ \int_{\partial K} |\mathbf{F} \cdot \mathbf{n}| \mathrm{d}\mathcal{H}^1 \middle| |K| = V \right\}. \tag{5.10}$$

we have the following theorem.

THEOREM 5.1 A solution of problem (5.10) is given by the ball B_r, where $r > 0$ is such that $|B_r| = V$.

Proof. Let $|K| = V$. Since $\operatorname{div} \mathbf{F} = 2$, we have

$$\int_{\partial K} |\mathbf{F} \cdot \mathbf{n}| \mathrm{d}\mathcal{H}^1 \geqslant \left| \int_{\partial K} F \cdot \mathbf{n} \, \mathrm{d}\mathcal{H}^1 \right| = \left| \int_K \operatorname{div} \mathbf{F} \, \mathrm{d}\mathcal{H}^2 \right| = |\lambda| 2V.$$

On the other hand,

$$\int_{\partial B_r} |\mathbf{F} \cdot \mathbf{n}| \mathrm{d}\mathcal{H}^1 = |\lambda| 2\pi r^2 = |\lambda| 2V. \qquad \square$$

We observe that on the ball B_r equality in (5.9) holds. Therefore, for every $|K| = V$, we infer that

$$\int_{\partial B_r} \|\mathbf{Tn}\| \mathrm{d}\mathcal{H}^1 = \int_{\partial B_r} |\mathbf{F}\cdot\mathbf{n}| \mathrm{d}\mathcal{H}^1 \leqslant \int_{\partial K} |\mathbf{F}\cdot\mathbf{n}| \mathrm{d}\mathcal{H}^1 \leqslant \int_{\partial K} \|\mathbf{Tn}\| \mathrm{d}\mathcal{H}^1.$$

Therefore, if $B_r \subset \Omega$, the ball B_r turns out to be the best hole inside Ω. We observe that these arguments also hold in higher dimension. Moreover, the same conclusion holds for stress tensors whose eigenvalues produce a field of the form $\mathbf{F} = \nabla u$, where $u(x) = \log |x|$ when $N = 2$ and $u(x) = \frac{1}{|x|}$ when $N = 3$, so that $\operatorname{div} \mathbf{F} = \Delta u = 0$. For more details, we refer the reader to the paper Granieri & Maddalena (2008).

Acknowledgements

We are grateful to Danilo Percivale for his useful comments and suggestions.

Funding

Research Project Regione Puglia PE110 *Ottimizzazione delle prestazioni meccaniche e termiche di laterizi rettificati*.

REFERENCES

ALLAIRE, G. (2007) *Conception Optimale de Structure*. Berlin: Springer.

AMBROSIO, L. (1997) *Corso Introduttivo alla Teoria Geometrica della Misura*. Pisa, Italy: Scuola Normale Superiore.

ATTOUCH, H., BUTTAZZO, G. & MICHAILLE, G. (2006) *Variational Analysis in Sobolev and BV Spaces*. Philadelphia, PA: SIAM.

BENDSOE, M. P. (1995) *Optimization of Structural Topology, Shape, and Material*. Berlin: Springer.

BROTHERS, J. E. & MORGAN, F. (1994) The isoperimetric theorem for general integrands. *Mich. Math. J.*, **41**, 419–431.

BUCUR, D. (2007) Do optimal shapes exist? *Milan J. Math.*, **75**, 379–398.

BUCUR, D. & BUTTAZZO, G. (2002) *Variational Methods in Some Shape Optimization Problems*. Pisa, Italy: Scuola Normale Superiore.

BUCUR, D., HENROT, A., SOKOLOWSKI, J. & ZOCHOWSKI, A. (2001) Continuity of the elasticity system solutions with respect to the geometrical domain variations. *Adv. Math. Sci. Appl.*, **11**, 57–73.

BUTTAZZO, G. & GUASONI, P. (1997) Shape optimization problems over classes of convex domain. *J. Convex Anal.*, **4**, 343–351.

CIANCHI, A. (1989) On relative isoperimetric inequalities in the plane. *Boll. Un. Mat. Ital.*, **3-B**, 289–325.

EVANS, L. C. (1997) *Partial Differential Equations*. Providence, RI: AMS.

EVANS, L. C. & GARIEPY, R. F. (1992) *Measure Theory and Fine Properties of Functions*. Boca Raton, FL: CRC Press.

FONSECA, I. (1991) The Wulff theorem revisited. *Proc. R. Soc. Lond. A*, **432**, 125–145.

GRANIERI, L. & MADDALENA, F. (2008) Optimal shapes in force fields. *J. Convex Anal.*, **15**, 17–37.

HENROT, A. & PIERRE, H. (2005) *Variation et Optimizasion de Formes*. Mathématiques & Applications, vol. 48. Berlin: Springer.

LEWINSKI, T. & SOKOLOWSKI, J. (2003) Energy change due to the appearance of cavities in elastic solids. *Int. J. Solids Struct.*, **40**, 1765–1803.

NECAS, J. & HLAVACEK, I. (1981) *Mathematical Theory of Elastic and Elastico-Plastic Bodies: An Introduction*. Amsterdam, NY: Elsevier.

PERI, C. (2001) On relative isoperimetric inequalities. *Conferenze del Seminario di Matematica dell'Università di Bari*, vol. 279. Bari, Italy: Bari University.

ROSALES, C. (2003) Isoperimetric regions in rotationally symmetric convex body. *Indiana Univ. Math. J.*, **52**, 1201–1214.

SIVALOGANATHAN, J. & SPECTOR, S. J. (2000) On the optimal location of singularities arising in variational problems of nonlinear elasticity. *J. Elast.*, **58**, 191–224.

STREDULINSKI, E. & ZIEMER, W. P. (1997) Area minimizing sets subject to a volume constraint in a convex set. *J. Geom. Anal.*, **7**, 653–677.

TEMAM, R. (1983) *Problèmes Mathématiques en Plasticité*. Paris, France: Gauthiers-Villars.

VILLAGGIO, P. (1977) *Qualitative Methods in Elasticity*. Leiden: Noordhoff International Publishing.