# Higher dimensional problems with volume constraints – Existence and Γ-convergence


Marc Oliver Rieger
Department of Mathematics
ETH Zürich
marc.rieger@math.ethz.ch



January 24, 2006



**Abstract**

We study variational problems with volume constraints (also called level set constraints) of the form

$$\text{Minimize } E(u) := \int_\Omega f(u, \nabla u)\, dx,$$
$$|\{u = 0\}| = \alpha, \quad |\{u = 1\}| = \beta,$$

on $\Omega \subset \mathbb{R}^n$, where $u \in H^1(\Omega)$ and $\alpha + \beta < |\Omega|$. The volume constraints force a phase transition between the areas on which $u = 0$ and $u = 1$.

We give some sharp existence results for the decoupled homogenous and isotropic case $f(u, \nabla u) = \psi(|\nabla u|) + \theta(u)$ under the assumption of $p$-polynomial growth and convexity of $\psi$. We observe an interesting interaction between $p$ and the regularity of the lower order term which is necessary to obtain existence and find a connection to the theory of dead cores. Moreover we obtain some existence results for the vector-valued analogue with constraints on $|u|$.

In the second part of this article we derive the $\Gamma$-limit of the functional $E$ for a general class of functions $f$ in the case of vanishing transition layers, i.e. when $\alpha + \beta \to |\Omega|$. As limit functional we obtain a nonlocal free boundary problem.

# 1 Introduction

We consider variational problems with level set constraints of the type

$$\begin{aligned} \text{Minimize } E(u) := \int_\Omega f(u(x), \nabla u(x))\, dx, \\ |\{x \in \Omega,\ u(x) = a\}| = \alpha, \\ |\{x \in \Omega,\ u(x) = b\}| = \beta, \end{aligned} \tag{1.1}$$

where $u \in H^1(\Omega)$ and $\alpha + \beta < |\Omega|$. The difficulty of this problem is the special structure of its constraints: A sequence of functions satisfying these constraints can have a limit which fails to satisfy the constraints.
Such minimization problems but with only one volume constraint have been studied by various authors, see e.g. [3]. In the last years problems with two or more constraints have caught attention [4, 15, 14, 11, 10, 13], partially motivated by physical problems related to immissible fluids [8] and mixtures of micromagnetic materials [2].
These problems have a very different nature than problems with only one volume constraint: In the case of one volume constraint, only additional boundary conditions or the design of the energy can induce transitions of the solution between different values. Two or more volume constraints, on the other hand, force transitions of the solution by their very nature. Ambrosio, Marcellini, Fonseca and Tartar [4] studied this class of problems for the first time and proved an existence result for the problem of two (or more) level set constraints with an energy density $f = f(|\nabla u|)$. Moreover they derived the $\Gamma$-limit for a vanishing transition layer in the special case $f = |\nabla u|^2$. It turned out that unlike usual variational problems, lower order terms pose hard difficulties for the analysis and can lead, even in very easy examples, to nonexistence [11, 10]. However, under certain regularity assumptions on the energy density the existence results were extended to energy functionals depending on $\nabla u$ and $u$ [11]. For the special case of one space dimension a somewhat complete analysis of existence, uniqueness, local minimizers and the $\Gamma$-limit has been given in [10]. It turned out that there is a strong link between existence and the regularity of the lower order term. One of the goals of this paper is to investigate this link in higher dimensional problems. We prove an existence result for a special class of energies under minimal regularity assumptions. The proof is based on the use of a Maximum Principle for solutions of elliptic equations recently established by Pucci and Serrin [12]. We also consider extensions to vector-valued problems of the form

$$\begin{aligned} \text{Minimize } E(u) := \int_\Omega \psi(|\nabla u|) + \theta(|u|)\, dx, \\ |\{x \in \Omega,\ |u(x)| = a\}| = \alpha, \\ |\{x \in \Omega,\ |u(x)| = b\}| = \beta. \end{aligned} \tag{1.2}$$

Similar problems arise in the analysis of mixtures of micromagnetic materials, compare [1].

In the second part of this paper we study the $\Gamma$-limit of general energy densities as the two phases $\alpha$ and $\beta$ tend to saturate the whole domain. It turns out that the limit problem is nonlocal, hence a standard extension of the $\Gamma$-limit in the one-dimensional problem (see [10]) by a simple slicing argument is not possible. Instead our proof has to rely on methods from geometric measure theory.

# 2 Sharp existence results

In this section we present some new existence results partially extending [10] to the higher dimensional case. As in [10] we consider for simplicity only decoupled functionals of the form $f(u,\nabla u) = \psi(|\nabla u|) + \theta(u)$ where $\psi$ is strictly convex and takes its minimum at zero. We define

$$H(t) := \int_0^{\psi'(t)} (\psi')^{-1}(w)\, dw,$$

where $(\psi')^{-1}$ denotes the inverse of $\psi'$ which is well-defined since $\psi'$ is strictly increasing. $H$ is by definition strictly increasing, hence its inverse $H^{-1}$ is well-defined. We prove the following result:

**Theorem 2.1 (Existence)** *Let $\delta > 0$. Let $\Omega$ be a bounded open set in $\mathbb{R}^n$, $\alpha, \beta > 0$ with $\alpha + \beta < |\Omega|$ and $\theta \in C^{0,1}((0,1), \mathbb{R}_{\geq 0})$. Assume the existence of Lipschitz continuous functions $\theta_1$ and $\theta_2$ with $\theta_1' \geq \theta'$ on $[0,\delta)$ and $\theta_2' \leq \theta'$ on $(1-\delta, 1]$. Moreover let $\theta_1$ be strictly convex on $(0,\delta)$ and $\theta_2$ be strictly convex on $(1-\delta, 1)$ and let $\theta_1$ and $\theta_2$ satisfy the integrability conditions*

$$\int_0^{\delta} \frac{du}{H^{-1}(\theta_1)} = +\infty, \tag{2.1}$$

$$\int_{1-\delta}^{1} \frac{du}{H^{-1}(\theta_2)} = +\infty. \tag{2.2}$$

*Let $\psi$ be Lipschitz continuous with $\psi(0) = \psi'(0) = 0$ and $C_1 t^p \leq \psi(t) \leq C_2 t^p$ for some constants $C_1, C_2 > 0$.*
*Then the volume-constrained minimization problem*

$$\begin{aligned} &\int_\Omega \psi(|\nabla u|) + \theta(u(x))\, dx, \\ &|\{x \in \Omega,\ u(x) = 0\}| = \alpha, \\ &|\{x \in \Omega,\ u(x) = 1\}| = \beta, \end{aligned} \tag{2.3}$$

*admits a solution $u \in W^{1,p}(\Omega, [0,1])$.*

An immediate consequence of this result is the following existence theorem which gives easier sufficient conditions on $\theta$ for the special case of quadratic growth:

**Theorem 2.2 (Existence)** *Let $\Omega$ be a bounded open set in $\mathbb{R}^n$, $\alpha, \beta > 0$ with $\alpha + \beta < |\Omega|$ and $\theta \in C^{0,1}((0,1), \mathbb{R}_{\geq 0})$, locally $C^{1,1}$ at $0$ and $1$ with $\theta'(0) \leq 0$ and $\theta'(1) \geq 0$. Let $\psi$ be Lipschitz continuous with $\psi(0) = \psi'(0)$ and quadratic growth.*
*Then the volume-constrained minimization problem*

$$\begin{aligned} &\int_\Omega \psi(|\nabla u|) + \theta(u(x))\, dx, \\ &|\{x \in \Omega,\ u(x) = 0\}| = \alpha, \\ &|\{x \in \Omega,\ u(x) = 1\}| = \beta, \end{aligned} \tag{2.4}$$

*admits a solution $u \in W^{1,2}(\Omega, [0,1])$.*

Theorem 2.2 is sharp in the following sense: If $\theta \notin C^{1,1}$ locally, but instead in $C^{1,\alpha}$ for some $\alpha < 1$, there are cases of non-existence.
Before we prove these results, we would like to mention the connections to earlier results for the one-dimensional case. A sharp characterization of functions that allow for existence of a volume-constrained problem of the form (2.3) was given in [10]. Theorem 2.1 comes close to this, however its conditions are slightly stronger:
The integrability condition for $\theta$ is the same as in the one-dimensional case (see [10]) but we have to assume the existence of the functions $\theta_1$ and $\theta_2$ since we need the local convexity condition in order to apply a maximum principle (see below). However, this condition is not very strong, as can be seen in Theorem 2.2: Without the sign condition on $\theta'$ it seems possible that a local minimum of $\theta$ in 0 or 1 leads to non-existence if the domain $\Omega$ is chosen appropriately. This was not possible in the one-dimensional case, where only global minima at 0 or 1 were a potential problem, see [10].
The second major difference is that we consider now only functions with values in $[0,1]$. In the one-dimensional case this was not necessary, we only had to assume that $\theta$ (defined on $\mathbb{R}$ rather than on $[0,1]$) has a minimum in $[0,1]$. In higher dimensional situations it is not at all clear that this condition would be sufficient. However, it is possible to give slightly stronger sufficient conditions, see the following Remark:

**Remark 2.3** *If $\theta \in C(\mathbb{R}, \mathbb{R}_{\geq 0})$ satisfies $\theta(z) \geq \theta(0)$ for $z < 0$ and $\theta(z) \leq \theta(1)$ for $z > 1$ then any solution $u \in W^{1,p}((0,1), \mathbb{R})$ of the minimization problem*

$$\begin{aligned} &\int_\Omega \psi(|\nabla u(x)|) + \theta(u(x))\, dx, \\ &|\{x \in \Omega,\ u(x) = 0\}| = \alpha, \\ &|\{x \in \Omega,\ u(x) = 1\}| = \beta, \end{aligned}$$

*satisfies $u \in [0,1]$.*

Proof: Assuming the contrary, the function $v(x) := \min(\max(u(x),0),1)$ would have lower energy than $u$. □

Proof of Theorem 2.1: Our proof relies on a maximum principle for the Euler-Lagrange Equation associated to 2.3, which corresponds to a recent result by Pucci and Serrin [12].

First we extend $\theta$ to a function $\tilde{\theta}$ by

$$\tilde{\theta}(z) := \begin{cases} z^2 - z\theta'(0) + \theta(0), & z < 0, \\ \theta(z), & 0 \leq z \leq 1, \\ (z-1)^2 + (z-1)\theta'(1) + \theta(1), & z > 1. \end{cases}$$

By standard variational methods the relaxed problem

$$\begin{aligned} &\int_\Omega \psi(|\nabla u(x)|) + \tilde{\theta}(u(x))\, dx, \\ &|\{x \in \Omega,\ u(x) = 0\}| \geq \alpha, \\ &|\{x \in \Omega,\ u(x) = 1\}| \geq \beta, \end{aligned}$$

admits a solution $u \in W^{1,p}(\Omega, \mathbb{R})$. By a general regularity result of Mosconi and Tilli [11] the function $u$ is continuous, and by Remark 2.3, which can be applied also to the relaxed case, $u$ takes only values in $[0,1]$.

Now assume that $u$ does *not* solve problem (2.3). Then either $|\{x \in \Omega,\ u(x) = 0\}| > \alpha$ or $|\{x \in \Omega,\ u(x) = 1\}| > \beta$. We consider the first case, i.e. $|\{u = 0\}| := |\{x \in \Omega,\ u(x) = 0\}| = \alpha + \varepsilon$ with $\varepsilon > 0$. Now choose $\eta > 0$ such that an $n$-dimensional ball with radius $\eta$ has volume less than $\varepsilon$, i.e. $|\mathscr{B}(0,\eta)| < \varepsilon$. Take $x \in \Omega$ such that

$$\begin{aligned} \mathscr{B}(x,\eta) \cap \{u = 0\} &\neq \emptyset, \\ \mathscr{B}(x,\eta) \cap \{u \in (0,\delta)\} &\neq \emptyset, \\ \mathscr{B}(x,\eta) \cap \{u \geq \delta\} &= \emptyset. \end{aligned} \tag{2.5}$$

(This is possible for $\eta$ small enough since $u$ is continuous and hence $\{u \in (0,\delta)\}$ is open.)

Now consider variations $u + t\varphi$ with $\varphi \in C_0^\infty(\mathscr{B}(x,\eta))$. Since $u$ is a minimizer of the relaxed problem it satisfies

$$\frac{d}{dt} \int_\Omega \psi(\nabla u + t\nabla\varphi) + \theta(u + t\varphi)\, dx|_{t=0} = 0.$$

This leads to the Euler-Lagrange equality

$$\operatorname{div} A(|\nabla u|)\nabla u - \theta'(u) = 0, \tag{2.6}$$

where $A(|\nabla u|) := \frac{\psi'(|\nabla u|)}{|\nabla u|}$.

By the integrability conditions (2.1)–(2.2) and the local convexity of $\theta_1$ and

$\theta_2$ we deduce that $\theta_1'(0) \le 0$ and $\theta_2'(1) \ge 0$. We consider the first of these inequalities and distinguish the two cases where $\theta_1'(0) < 0$ and $\theta_1'(0) = 0$:
CASE 1: $\theta_1'(0) = 0$
We can apply the regularity theory for degenerate elliptic equations of $p$-Laplacian type (see e.g. [7, 8.9]) to (2.6) to deduce that the solution $u$ has $C^1$-regularity. Moreover $\theta' \le \theta_1'$ on $[0, \delta)$, hence we can apply the maximum principle in [12, Theorem 1] with $\theta_1$ on the domain $\mathscr{B}(x,\eta)$. This gives $u = 0$ on all of $\mathscr{B}(x,\eta)$, contradicting (2.5).
CASE 2: $\theta_1'(0) < 0$
Choose $\eta > 0$ such that $|\mathscr{B}(0,\eta)| < \varepsilon$. Take $x \in \Omega$ such that

$$\begin{aligned} |\mathscr{B}(x,\eta) \cap \{u=0\}| &> 0, \\ \mathscr{B}(x,\eta) \cap \{u=1\} &= \emptyset. \end{aligned} \tag{2.7}$$

On the set $\mathscr{B}(x,\eta) \cap \{u=0\}$ we have $\theta'(u) = \theta'(0) \le \theta_1'(0) < 0$. But since on the same set $\operatorname{div} A(|\nabla u|)\nabla u = 0$, we get a contradiction to the Euler-Lagrange equality (2.6).
Hence we have proved in both cases that $|\{u=0\}| = \alpha$. Using the function $\theta_2$ we can prove in the same way that $|\{u=1\}| = \beta$. Thus we have proved existence for the original problem (1.1). □
Theorem 2.2 is now an easy consequence:
PROOF OF THEOREM 2.2: Let $L := \operatorname{Lip}_{(0,\delta)\cup(1-\delta,1)} \theta'$. Choose $\theta_1(z) := \theta(z) - \theta(0) + L\,z^2$, $\theta_2(z) := \theta(z) - \theta(1) + L\,(1-z)^2$ and $\delta > 0$ sufficiently small, then these functions satisfy the conditions of Theorem 2.1: First, both functions are strictly convex, since their derivatives are strictly monotone. Moreover they satisfy the integrability conditions (2.1) resp. (2.2). We prove this for $\theta_1$, the proof for $\theta_2$ is symmetric:
Due to the quadratic growth of $\psi$ and the condition $\psi(0) = \psi'(0) = 0$ we have $\psi'(t) \ge C_1 t$ for a certain constant $C_1 > 0$. This implies a bound on $(\psi')^{-1}$, namely $(\psi')^{-1}(w) \ge w/C_1$. Applying this to the definition of $H$ gives

$$H(t) \;=\; \int_0^{\psi'(t)} (\psi')^{-1}(w)\,dw \;\ge\; \int_0^{\psi'(t)} \frac{w}{C_1}\,dw \;\ge\; \frac{1}{2}C_1 t^2.$$

Using this estimate for the inverse function of $H$ we deduce

$$H^{-1}(t) \le \frac{2}{C_1}\sqrt{t}.$$

Hence we have

$$\int_0^\delta \frac{du}{H^{-1}(\theta_1(u))} \ge \int_0^\delta \frac{C_1}{2\sqrt{\theta_1(u)}} du,$$

and it is therefore sufficient to prove that the latter is infinite. To prove this we have first to distinguish three cases:

Case 1: $\theta(u) \leq \theta(0)$ on $(0,\delta)$
Here we have

$$\sqrt{\theta_1(u)} = \sqrt{\theta(u) - \theta(0) + Lu^2} \leq C_3|u|,$$

where $C_3 := \sqrt{L}$.
Case 2: $\theta(u) > \theta(0)$ on $(0,\delta)$
Here we use the estimate

$$\sqrt{\theta_1(u)} = \sqrt{\theta(u) - \theta(0) + Lu^2} \leq \sqrt{\theta(u) - \theta(0)} + \sqrt{L}|u|,$$

and that because of the regularity of $\theta$ and the assumption that $\theta'(0) \leq 0$

$$\sqrt{\theta(u) - \theta(0)} \leq C_2|u|$$

with some constant $C_2 > 0$.
Combining both we get again $\sqrt{\theta_1(u)} \leq C_3|u|$, this time with $C_3 := C_2 + \sqrt{L} > 0$.
Case 3: remaining situations
This case can be excluded if we only choose $\delta > 0$ sufficiently small, since locally $\theta \in C^{1,1}$.
Using the estimates proved above we obtain (2.1), since

$$\int_0^\delta \frac{C_1}{2\sqrt{\theta_1(u)}} du \ \geq \ \int_0^\delta \frac{C_1}{2C_3|u|} du \ = \ +\infty.$$

Thus Theorem 2.1 can be applied, and a solution $u$ exists. □
It is remarkable that the necessary regularity for $\theta$ depends on the growth properties of $\psi$. In other words: The growth of the leading order term prescribes the necessary regularity for the lower order term! This is not only a technical problem of the proof, very much to the contrary: Theorem 2.1 is sharp, i.e. there are counterexamples to existence if one of the integrability conditions (2.1)–(2.2) is violated – even if $\theta \in C^\infty$, although in the case of quadratic growth $\theta \in C^{1,1}$ is sufficient as we have seen in Theorem 2.2. The following corollary provides such an example. It can be proved copying the methods used in [10]. (The function $H^{-1}$ of Theorem 2.1 becomes in this case simply $\sqrt[4]{\cdot}$.)

**Corollary 2.4** *The one-dimensional volume constrained minimization problem*

$$\int_0^1 |u'|^4 + 256\,|u|^2\,dx,$$
$$|\{x \in (0,1),\ u(x) = 0\}| = \alpha,$$
$$|\{x \in (0,1),\ u(x) = 1\}| = \beta,$$

*with* $\alpha = \beta = 1/10$ *does not admit a solution.*

The results obtained so far can partially be extended to vector-valued problems of the form

$$\begin{aligned} \text{Minimize } E(u) := \int_\Omega \psi(|\nabla u|) + \theta(u)\, dx, \\ |\{x \in \Omega,\ |u(x)| = a\}| = \alpha, \\ |\{x \in \Omega,\ |u(x)| = b\}| = \beta, \end{aligned} \tag{2.8}$$

where now $u\colon \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$.
In fact we have the following theorem:

**Theorem 2.5 (Vector-valued case)** *Let $\Omega$ be a bounded open set in $\mathbb{R}^n$, $p > 1$, $\alpha, \beta > 0$ with $\alpha + \beta < |\Omega|$ and let $\psi$ be Lipschitz continuous with $\psi(0) = \psi'(0) = 0$ and $C_1 t^p \leq \psi(t) \leq C_2 t^p$ for some constants $C_1, C_2 > 0$. Let $\theta \in C^{0,1}(\mathbb{R}^m, \mathbb{R}_{\geq 0})$ satisfy one of the following conditions:*

*(i) The function $\theta$ is* isotropic*, i.e. there exists $\tilde{\theta}$ such that $\theta(P) = \tilde{\theta}(|P|)$ for all $P \in \mathbb{R}^m$ with $a \leq |P| \leq b$.*

*(ii) There exists $\nu \in \mathbb{R}^m$ with $|\nu| = 1$ such that $\theta(P) \geq \theta(|P| \cdot \nu) =: \tilde{\theta}(|P|)$ for all $P \in \mathbb{R}^m$ with $a \leq |P| \leq b$.*

*Moreover let $\tilde{\theta}$ satisfy the analogous conditions of either Theorem 2.1 or Theorem 2.2.*
*Then there exists a solution $u \in W^{1,p}(\Omega, \mathbb{R}^m)$ which solves the vector-valued minimization problem (2.8).*

Proof: First we see that condition (i) is only a special case of condition (ii). Hence we assume condition (ii) is satisfied. The existence of a solution to the relaxed problem follows as in the scalar case. We denote this solution by $v$. Now we define $w := |v| \cdot \nu$. From the isotropy of $\psi$ and condition (ii) it is easy to see that the energy of $w$ cannot be larger then the energy of $v$. This trick is due to Dacorogna and Fonseca (personal communication) and reduces the problem to the scalar case. An application of Theorem 2.1 or 2.2, respectively, concludes the proof. □

We would like to mention that the general vector-valued situation with the constraint as given in (2.8) is much harder. One reason for this is that the solution does not have to be constant on the constraint volumes. Another reason is that continuity for the solutions to the relaxed problem has so far only been obtained for the scalar case using methods that are difficult to apply to the vectorial situation.

# 3 The Γ-limit of vanishing transition layers

## 3.1 The isotropic and homogenous case

To study the Γ-convergence for the case where $|\Omega| - \alpha - \beta \to 0$ we need the following lemma, which can be found in [10].

**Lemma 3.1** *Let $f : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function satisfying the following conditions:*

*(i) for every $u$, $f(u,\cdot)$ is convex and increasing;*

*(ii) there exists $c > 0$, and $p > 1$ such that*

$$\frac{1}{c}|\xi|^p - c \le f(u,\xi) \le c(|\xi|^p + 1),$$

*for every $u$, $\xi \in \mathbb{R}$.*

*Then the function $P$ defined for every $t > 0$ by*

$$P(t) := \min\left\{ \int_0^t f(v,v')\,dx : \, v \in W^{1,1}(0,t),\, v(0) = 0,\, v(t) = 1 \right\}, \tag{3.1}$$

*is convex.*
*Moreover the function $\varphi(t) := tP(1/t)$ is increasing and convex.*

Let $\Omega \subset \mathbb{R}^N$ be an bounded open set. For fixed $\alpha$, $\beta \in (0, |\Omega|)$, we define the following functional

$$F_{\alpha,\beta} := \begin{cases} \dfrac{1}{\gamma}\displaystyle\int_\Omega \big(f(u,\gamma|\nabla u|)\big)\,dx & \text{if } u \in \mathcal{A}_{\alpha,\beta}, \\ +\infty & \text{elsewhere in } L^1(\Omega), \end{cases}$$

where $\gamma := |\Omega| - (\alpha + \beta)$ and

$$\mathcal{A}_{\alpha,\beta} := \{u \in W^{1,p}(\Omega) : \, |\{u = 0\}| \ge \alpha \text{ and } |\{u = 1\}| \ge \beta\}.$$

This constraint is the relaxed version of the the original constraint in (1.2). Therefore the $\Gamma$-limit of this functional will coincide with the $\Gamma$-limit of the original problem.

**Theorem 3.2** *Let $f : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function satisfying the conditions of Lemma 3.1 and*

$$f(0,0) = f(1,0) = 0.$$

*Let $\bar{\alpha} \in (0, |\Omega|)$. Then*

$$\Gamma(L^1)\text{-} \lim_{\substack{\alpha \to \bar{\alpha} \\ \beta \to |\Omega| - \bar{\alpha}}} F_{\alpha,\beta} = G_{\bar{\alpha}},$$

*with $G_{\bar{\alpha}}$ given by*

$$G_{\bar{\alpha}} := \begin{cases} \varphi(Per\{u = 0\}) & \text{if } u \in BV(\Omega, \{0,1\}) \text{ and } |\{u = 0\}| = \bar{\alpha}, \\ +\infty & \text{elsewhere in } L^1(\Omega), \end{cases}$$

*where $\varphi$ is defined as in Lemma 3.1.*

**Estimate of the Γ-limit from above**
Let $\alpha_n \to \bar{\alpha}$, $\beta_n \to |\Omega| - \bar{\alpha}$. Denote $\gamma_n := |\Omega| - \alpha_n - \beta_n$. Let us first assume that $\Gamma := \partial^*(\{u = 0\})$ is smooth. We denote $d(x) := \mathrm{dist}(x, \Gamma)$ and define $\varepsilon(\gamma_n)$ such that

$$\int_0^{\varepsilon(\gamma_n)} \mathcal{H}^{N-1}(\{x \in \Omega,\, d(x) = t\})\, dt = \gamma_n.$$

Let $v_n$ be the minimizer of

$$P\left(\frac{\varepsilon(\gamma_n)}{\gamma_n}\right) := \min\Big\{ \int_0^{\varepsilon(\gamma_n)/\gamma_n} f(u, |u'|)\, dx,\quad u(0) = 0,\, u(\varepsilon(\gamma_n)/\gamma_n) = 1 \Big\}.$$

Define $u_n(x) := v_n\left(\frac{d(x)}{\gamma_n}\right)$.
By using this definition and the Coarea Formula we get

$$\begin{aligned} F_{\alpha_n,\beta_n} &:= \frac{1}{\gamma_n} \int_\Omega f(u_n, \gamma_n \nabla u_n) \\ &= \frac{1}{\gamma_n} \int_0^{\varepsilon(\gamma_n)} \bar{f}(v_n(t/\gamma_n), |v_n'(t/\gamma_n)|)\, \mathcal{H}^{N-1}(\{x \in \Omega,\, d(x) = t\})\, dt. \end{aligned}$$

Now we use the fact that

$$\lim_{t \to 0} \mathcal{H}^{N-1}(\{x \in \Omega,\, d(x) = t\}) = \mathcal{H}^{N-1}(\Gamma), \tag{3.2}$$

see [9], Lemma 4.
By this and a transformation of the variable we get for every $\delta > 0$ and for $n$ large enough:

$$\begin{aligned} F_{\alpha_n,\beta_n} &\le \frac{1}{\gamma_n}(1+\delta)\, \mathcal{H}^{N-1}(\Gamma) \int_0^{\varepsilon(\gamma_n)} f(v_n(t/\gamma_n), |v_n'(t/\gamma_n)|) dt \\ &= (1+\delta)\, \mathcal{H}^{N-1}(\Gamma) \int_0^{\varepsilon(\gamma_n)/\gamma_n} f(v_n(s), |v_n'(s)|) ds \end{aligned}$$

By definition 3.1 and since $\gamma_n/\varepsilon(\gamma_n) \to \mathcal{H}^{N-1}(\Gamma)$ (which follows from (3.2)), we get

$$\begin{aligned} \limsup_{n\to\infty} F_{\alpha_n,\beta_n} &\le \lim_{n\to\infty} (1+\delta)\, P(\varepsilon(\gamma_n)/\gamma_n)\, \mathcal{H}^{N-1}(\Gamma) \\ &= (1+\delta)\varphi(\mathcal{H}^{N-1}(\Gamma)). \end{aligned} \tag{3.3}$$

Since $\delta > 0$ is arbitrarily small the Γ-limsup inequality is proved for Γ smooth. The general case follows from a standard density argument based upon the following lemma from [4] (Lemma 4.3):

**Lemma 3.3** *Let $E \subset \Omega$ be a set with finite perimeter such that $0 < \mathcal{L}^N(E) < \mathcal{L}^N(\Omega)$. There exists a sequence of bounded open sets $D_n \subset \mathbb{R}^N$ with smooth boundary such that $\mathcal{L}^N(E) = \mathcal{L}^N(D_n \cap \Omega)$, $\chi_{D_n}$ converges to $\chi_E$ in $L^2(\Omega)$, and*

$$\lim_{n\to\infty} \mathcal{H}^{N-1}(\partial D_n \cap \Omega) = \mathcal{H}^{N-1}(\partial^*(E \cap \Omega)).$$

**Estimate of the $\Gamma$-limit from below**

We shall need the following measure theoretical result whose proof is based upon a standard recovering argument. We will sketch the argument for the reader's convenience and illustrate it in Fig. 1.

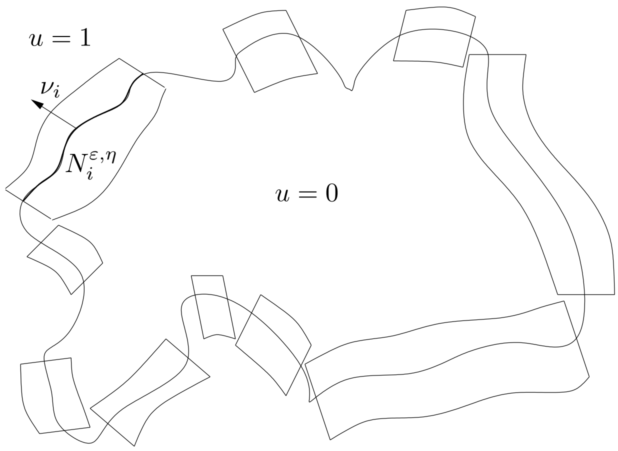


Figure 1: Illustration of the sets $N_i^{\varepsilon,\eta}$, partially convering $\partial^*\{u=0\}$.

**Lemma 3.4** *Let $u \in BV(\Omega, \{0,1\})$ and set $\Gamma := \partial^*\{u = 0\}$. Then for every $\varepsilon, \eta > 0$ we can find a decomposition of $\Gamma$ of the form*

$$\Gamma = \bigcup_{i=1}^{k_{\varepsilon,\eta}} N_i^{\varepsilon,\eta} \cup M_{\varepsilon,\eta},$$

*with the following properties:*

*(i)* $\mathcal{H}^{N-1}(M_{\varepsilon,\eta}) < \varepsilon$;

*(ii)* $N_i^{\varepsilon,\eta} \cap N_j^{\varepsilon,\eta} = \emptyset$ *if* $i \neq j$;

*(iii) for every $i \in \{1, \ldots, k_{\varepsilon,\eta}\}$ the set $N_i^{\varepsilon,\eta}$ is a compact subset of a $C^1$-manifold; more precisely, there exists $\nu_i \in S^{n-1}$ such that $N_i^{\varepsilon,\eta}$ is contained in the graph of a $C^1$-function $g_i$ defined on the plane $\Pi_{\nu_i}$ orthogonal to $\nu_i$;*

*(iv) for every $x \in N_i^{\varepsilon,\eta}$ we have $|\nu(x) - \nu_i| < \eta$.*

**Proof:** We recall first that by De Giorgi's Structure Theorem (see e.g.[6]) the reduced boundary $\partial^*\{u = 0\}$ is $(n-1)$-rectifiable and so, in particular, we can find a decomposition of the form

$$\Gamma = \bigcup_{i=1}^{k_\varepsilon} N_i^\varepsilon \cup M_\varepsilon,$$

where $\mathcal{H}^{N-1}(M_\varepsilon) < \frac{\varepsilon}{2}$ and the $N_i^\varepsilon$ satisfy the following properties:

(i) $N_i^\varepsilon \cap N_j^\varepsilon = \emptyset$ if $i \neq j$;

(ii) for every $i \in \{1, \dots, k_\varepsilon\}$ the set $N_i^\varepsilon$ is a compact subset of the graph of a $C^1$-function.

Using the compactness of $N_i^\varepsilon$, we can find a positive $\delta > 0$ (independent of $i$) such that for every $x, y \in N_i^\varepsilon$ with $|x - y| \leq \delta$ we have $|\nu(x) - \nu(y)| < \min\{\eta, \sqrt{2}\}$. For $x \in \Gamma \setminus M_\varepsilon$ and for $0 \leq s \leq \delta$ we can consider the set $A(x, s) := B(x, s) \cap N_i^\varepsilon$, where $i$ is the (unique) index such that $x \in N_i^\varepsilon$. The family $\{A(x, s) : x \in \Gamma,\, 0 \leq s \leq \delta\}$ forms a fine recovering of $\Gamma \setminus M_\varepsilon$. Therefore we can apply Besicovitch Theorem to extract a finite subfamily $\{A(x_i, s_i)\}_{i=1,\dots,k_{\varepsilon,\eta}}$ of pairwise disjoint sets such that

$$\mathcal{H}^{N-1}\left((\Gamma \setminus M_\varepsilon) \setminus \bigcup_{i=1}^{k_{\varepsilon,\eta}} A(x_i, s_i)\right) < \frac{\varepsilon}{2}.$$

Setting $N_i^{\varepsilon,\eta} := A(x_i, s_i)$ and $\nu_i = \nu(x_i)$, we see that the family $\{N_i^{\varepsilon,\eta}\}_{i=1}^{k_{\varepsilon,\eta}}$ meets all the requirements. □

We are now in a position to prove the Γ-liminf inequality. Suppose that $u_n \to u$ in $L^1(\Omega)$ and a.e. where $u \in BV(\Omega, \{0, 1\})$. We may assume without loss of generality that $F_{\alpha_n,\beta_n}(u_n)$ admits a finite limit. By means of a truncation and smoothing argument we can also assume that $u_n$ is continuous and $0 \leq u_n \leq 1$ for every $n \in \mathbb{N}$. We fix $\varepsilon > 0$ and we find $\eta = \eta(\varepsilon) > 0$ such that

$$\nu_1, \nu_2 \in S^{n-1},\ |\nu_1 - \nu_2| < \eta \Rightarrow \langle \nu_1, \nu_2 \rangle > 1 - \varepsilon. \tag{3.4}$$

We can now find a decomposition of Γ of the form

$$\Gamma = \bigcup_{i=1}^{k_{\varepsilon,\eta}} N_i^{\varepsilon,\eta} \cup M_{\varepsilon,\eta},$$

with the properties stated in the previous lemma.
CLAIM: There exist $\Gamma' \subset \Gamma \setminus M_{\varepsilon,\eta}$ and a subsequence $u_n$ (not relabelled) such that

(i) $\mathcal{H}^{N-1}\left((\Gamma \setminus M_{\varepsilon,\eta}) \setminus \Gamma'\right) < \varepsilon$;

(ii) for every $n$ large enough there exist two positive functions $s_n$ and $t_n$ such that for $x \in N_i^{\varepsilon,\eta} \cap \Gamma'$ we have $u_n(x + t_n(x)\nu_i) = 0$ and either $u_n(x + (t_n(x) + \gamma_n s_n(x))\nu_i) = 1$ or $u_n(x + (t_n(x) - \gamma_n s_n(x))\nu_i) = 1$;

(iii) $\displaystyle\int_{\Gamma'} s_n \, d\mathcal{H}^{N-1} \leq \frac{1}{1-\varepsilon}$ for every $n \geq \bar{n}$;

(iv) $\gamma_n s_n \to 0$ uniformly in $\Gamma'$.

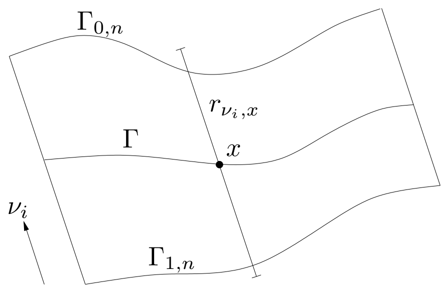


Figure 2: A set $N_i^{\varepsilon,\eta}$ with $\nu_i$ and $r_{\nu_i,x}$ .

Let us set (compare Fig. 2)

$$\Gamma_{0,n} := \{x \in \Omega : u_n(x) = 0\} \qquad \text{and} \qquad \Gamma_{1,n} := \{x \in \Omega : u_n(x) = 1\}.$$

Fix $\tau = \tau(\varepsilon, \eta) > 0$ so small that the sets

$$N_i^{\varepsilon,\eta,\tau} := \{x + t\nu_i : x \in N_i^{\varepsilon,\eta},\, t \in (-2\tau, 2\tau)\} \quad i = 1, \ldots, k_{\varepsilon,\eta}, \tag{3.5}$$

are pairwise disjoint. We denote by $r_{\nu_i,x}$ the straight segment parallel to $\nu_i$ with center in $x$ and length equal to $2\tau$. Let $G_{i,n} \subset N_i^{\varepsilon,\eta}$ be the set on which $\Gamma_{0,n} \cap r_{\nu_i,x}$ and $\Gamma_{1,n} \cap r_{\nu_i,x}$ are both non-empty. We define on $G_{i,n}$

$$s_n(x) \quad := \quad \frac{1}{\gamma_n} \operatorname{dist}(\Gamma_{0,n} \cap r_{\nu_i,x}, \Gamma_{1,n} \cap r_{\nu_i,x}).$$

From this definition it is clear that we can define a function $t_n$ such that

$$u_n(x + t_n(x)\nu_i) = 0 \quad \text{and either}$$

$$u_n(x + (t_n(x) + s_n(x))\nu_i) = 1 \quad \text{or} \quad u_n(x + (t_n(x) - s_n(x))\nu_i) = 1.$$

For simplicity we will in the following discuss only the first case. (Due to the symmetry of $f$ the latter case can be handled in the same way.)

The closedness of $\Gamma_{0,n}$ and $\Gamma_{1,n}$ and the smoothness of $N_i^{\varepsilon,\eta}$ imply that $G_{i,n}$ is closed and hence measurable, and also the measurability of $s_n$ over $G_{i,n}\setminus M_{\varepsilon,\eta}$. Property (ii) is satisfied by construction almost everywhere in $G_{i,n}\setminus M_{\varepsilon,\eta}$. Using the fact that $u_n \to u$, we obtain

$$\lim_{n\to\infty} |\Gamma \setminus \bigcup_i G_{i,n}| = 0.$$

Denoting by $\pi_{\nu_i}$ the orthogonal projection on $\Pi_{\nu_i}$

$$\begin{aligned}
\gamma_n &\geq |\{x \in \Omega : \, 0 < u_n(x) < 1\}| \\
&\geq \sum_{i=1}^{k_{\varepsilon,\eta}} \int_{\pi_{\nu_i}(N_i^{\varepsilon,\eta}\setminus Q^{\varepsilon,\eta})} \gamma_n s_n \, d\mathcal{H}^{N-1} \\
&= \sum_{i=1}^{k_{\varepsilon,\eta}} \int_{N_i^{\varepsilon,\eta}\setminus Q^{\varepsilon,\eta}} \gamma_n s_n \langle \nu(x), \nu_i\rangle \, d\mathcal{H}^{N-1} \\
&\geq (1-\varepsilon) \int_{\Gamma\setminus(M_{\varepsilon,\eta}\cup Q^{\varepsilon,\eta})} \gamma_n s_n \, d\mathcal{H}^{N-1}, \qquad (3.6)
\end{aligned}$$

where the last inequality is a consequence of (3.4). This proves (iii).
Using (3.6) and Egoroff's Theorem we can find a subsequence $u_n$ and $\Gamma' \subset \Gamma \setminus M_{\varepsilon,\eta}$ with all the required properties.
Now set

$$U_n := \bigcup_{i=1}^{k_{\varepsilon,\eta}} \{x + t\nu_i : \, x \in N_i^{\varepsilon,\eta} \cap \Gamma', \, t \in (t_n(x), t_n(x) + \gamma_n s_n(x))\},$$

and choose $n \in \mathbb{N}$ so large that $U_n \subset \bigcup_{i=1}^{k_{\varepsilon,\eta}} N_i^{\varepsilon,\eta,\tau}$ (see (3.5)) with $\tau$ chosen like before. Then, using Fubini's Theorem and the monotonicity of $f$ we can estimate

$$\frac{1}{\gamma_n}\int_\Omega f(u_n, \gamma_n \nabla u_n)\, dx \;\geq\; \frac{1}{\gamma_n}\int_{U_n} f(u_n, \gamma_n \nabla u_n)\, dx$$

$$\geq \sum_{i=1}^{k_{\varepsilon,\eta}} \int_{\pi_{\nu_i}(N_i^{\varepsilon,\eta}\cap\Gamma')} \frac{1}{\gamma_n}\Big(\int_{t_n(g_i(y))}^{t_n(g_i(y))+\gamma_n s_n(g_i(y))} f(u_n(g_i(y)+t\nu_i), \qquad (3.7)$$

$$\gamma_n \partial_{\nu_i} u_n(g_i(y)+t\nu_i))\, dt\Big)\, d\mathcal{H}^{N-1}(y)$$

$$= \sum_{i=1}^{k_{\varepsilon,\eta}} \int_{\pi_{\nu_i}(N_i^{\varepsilon,\eta}\cap\Gamma')} \int_0^{s_n(g_i(y))} f(v_n^y(t), (v_n^y)'(t))\, dt\, d\mathcal{H}^{N-1}(y) =: I, \quad (3.8)$$

where we set $v_n^y(t) := u_n(g_i(y) + t_n(g_i(y)) + \gamma_n t\nu_i)$ ($g_i$ is the function appearing in (iii) of Lemma 3.4). Recalling the definition of $P(t)$ and (3.4) we can continue our estimate as follows

$$\begin{aligned} I &\geq \sum_{i=1}^{k_{\varepsilon,\eta}} \int_{\pi_{\nu_i}(N_i^{\varepsilon,\eta}\cap\Gamma')} P(s_n(g_i(y)))\, d\mathcal{H}^{N-1}(y) \\ &= \sum_{i=1}^{k_{\varepsilon,\eta}} \int_{N_i^{\varepsilon,\eta}\cap\Gamma'} P(s_n(z))\langle \nu(z), \nu_i\rangle\, d\mathcal{H}^{N-1}(z) \\ &\geq (1-\varepsilon)\int_{\Gamma'} P(s_n(z))\, d\mathcal{H}^{N-1}(z); \end{aligned}$$

using the convexity and monotonicity of $P$ (see Lemma 3.1) and property (iii) of the previous claim we get

$$\begin{aligned} I &\geq (1-\varepsilon)\mathcal{H}^{N-1}(\Gamma')P\left(\frac{1}{\mathcal{H}^{N-1}(\Gamma')}\int_{\Gamma'} s_n(z)\, d\mathcal{H}^{N-1}(z)\right) \\ &\geq (1-\varepsilon)\mathcal{H}^{N-1}(\Gamma')P\left(\frac{1}{(1-\varepsilon)\mathcal{H}^{N-1}(\Gamma')}\right). \end{aligned} \tag{3.9}$$

Since $\varepsilon$ is arbitrarily small and the measure of $\Gamma'$ is arbitrarily close to the measure of $\Gamma$, by combinig (3.8) and (3.9) we complete the proof of the $\Gamma$-liminf inequality. □

## 3.2 Anisotropic energies

In this section we extend the results from the previous section to a class of anisotropic functionals where the energy density $g$ is given by

$$g(u,\xi) := f(u,\psi(\xi)), \tag{3.10}$$

where $\psi$ is a norm given by

$$\psi(\xi) := \sqrt{\langle L\xi, \xi\rangle}, \tag{3.11}$$

with $L\colon \mathbb{R}^n \to \mathbb{R}^n$ a symmetric positive definite linear operator.
For fixed $\alpha$, $\beta \in (0,|\Omega|)$, we define the functional

$$F_{\alpha,\beta} := \begin{cases} \dfrac{1}{\gamma}\displaystyle\int_\Omega \big(g(u,\gamma\nabla u)\big)\, dx & \text{if } u \in \mathcal{A}_{\alpha,\beta}, \\ +\infty & \text{elsewhere in } L^1(\Omega), \end{cases}$$

where $\gamma$ and $\mathcal{A}_{\alpha,\beta}$ are defined as above.

**Theorem 3.5** *Let $f$ satisfy the same conditions as in the previous section, and let $\psi$ be as in (3.11). Let $\bar\alpha \in (0,|\Omega|)$. Then*

$$\Gamma(L^1)\text{-}\lim_{\substack{\alpha\to\bar\alpha \\ \beta\to|\Omega|-\bar\alpha}} F_{\alpha,\beta} = G_{\bar\alpha},$$

*with* $G_{\bar{\alpha}}$ *given by*

$$G_{\bar{\alpha}} := \begin{cases} \varphi\Big((\det L^{-1/2}) \int_{\partial^*\{u=0\}} \psi(\nu(x))\, d\mathcal{H}^{N-1}(x)\Big) & \textit{if } u \in BV(\Omega, \{0,1\}) \\ & \textit{and } |\{u=0\}| = \bar{\alpha}, \\ +\infty & \textit{elsewhere in } L^1(\Omega), \end{cases}$$

*where* $\varphi\colon (0,\infty) \to \mathbb{R}$ *is a monotone function defined by (3.17), below.*

Proof: The main idea of the proof is a change of variables. The following result is well known and can be seen as a consequence of the so-called Generalized Area Formula (see Theorem 2.91 in [5]). Nevertheless for the reader's convenience we give here a simple direct proof based on the Divergence Theorem.

**Lemma 3.6** *Let* $L : \mathbb{R}^N \to \mathbb{R}^N$ *be a symmetric positive definite linear mapping. Let* $\Gamma$ *be an* $(N-1)$*-rectifiable set. Then for every* $\mathcal{H}^{N-1}$*-measurable set* $A \subset \Gamma$ *we have*

$$\mathcal{H}^{N-1}(L(A)) = \det L \int_A |L^{-1}\nu(y)|\mathcal{H}^{N-1}(y).$$

Proof. Using the definition of a rectifiable set we can assume without loss of generality that $\Gamma$ is a $C^1$-manifold.
We consider the pull-back measure $L^\sharp\mathcal{H}^{N-1}$ defined on $\mathbb{R}^N$ as

$$L^\sharp\mathcal{H}^{N-1} : B \mapsto \mathcal{H}^{N-1}(L(B)).$$

It is easy to see that its restriction to $\Gamma$, denoted by $L^\sharp\mathcal{H}^{N-1}\lfloor\Gamma$, is absolutely continuous with respect to $\mathcal{H}^{N-1}\lfloor\Gamma$. We claim that for all $x_0 \in \Gamma$

$$\frac{d(L^\sharp\mathcal{H}^{N-1}\lfloor\Gamma)}{d(\mathcal{H}^{N-1}\lfloor\Gamma)}(x_0) = \det L|L^{-1}\nu(x_0)|. \tag{3.12}$$

Let $r > 0$ be so small that $B(x_0, r) \setminus \Gamma$ has two connected components $B_+$ and $B_-$. Define $D := B(x_0, r) \cap \Gamma$. Denote

$$\Phi := \big\{\eta \in C_0^1(L(B), \mathbb{R}^N),\ \|\eta\|_\infty \le 1\big\}.$$

Using the Divergence Theorem we see that

$$\mathcal{H}^{N-1}(L(D)) = \sup_{\eta\in\Phi} \int_{L(B_+)} \operatorname{div}\, \eta\, dx. \tag{3.13}$$

Given $\eta$ as above, for every $y \in B$ we set

$$\hat{\eta}(y) := \eta(Ly).$$

Note that for every $x \in L(B)$ we have

$$\operatorname{div} \eta(x) = \operatorname{div} \left(L^{-1}\hat{\eta}\right)(L^{-1}x). \tag{3.14}$$

Therefore, using (3.13) and (3.14), we can compute

$$\begin{aligned}
\mathcal{H}^{N-1}(L(D)) &= \sup_{\eta\in\Phi} \int_{L(B)} \operatorname{div} \left(L^{-1}\hat{\eta}\right)(L^{-1}x)\, dx \\
&= \sup_{\eta\in\Phi} \det L \int_B \operatorname{div} \left(L^{-1}\hat{\eta}\right)(y)\, dy \\
&= \sup_{\eta\in\Phi} \det L \int_D \langle L^{-1}\hat{\eta}(y), \nu(y)\rangle\, d\mathcal{H}^{N-1} \\
&= \sup_{\eta\in\Phi} \det L \int_D \langle \hat{\eta}(y), L^{-1}\nu(y)\rangle\, d\mathcal{H}^{N-1} \\
&= \sup_{\substack{\eta\in C^1(B,\mathbb{R}^N)\\ \|\eta\|_\infty\le 1}} \det L \int_D \langle \eta(y), L^{-1}\nu(y)\rangle\, d\mathcal{H}^{N-1} \\
&= \det L \int_D |L^{-1}\nu(y)| d\mathcal{H}^{N-1}(y),
\end{aligned}$$

where the last equality follows by taking $\eta_n := L^{-1}\nu/|L^{-1}\nu|$ as maximizing sequence on $D_n \subset\subset D$, with $D_n$ increasing to $D$. This concludes the proof of (3.12) and therefore of the lemma. □

We now prove the Γ-liminf inequality. Let $\alpha_n \to \bar{\alpha}$, $\beta_n \to |\Omega| - \bar{\alpha}$. Denote as before $\gamma_n := |\Omega| - \alpha_n - \beta_n$ and $\Gamma := \partial^*(\{u = 0\})$. Suppose that $u_n \to u$ in $L^1(\Omega)$ and a.e. where $u \in BV(\Omega, \{0, 1\})$. We may assume without loss of generality that $F_{\alpha_n,\beta_n}(u_n)$ admits a finite limit.
We now change variables by setting for every $y \in L^{-\frac{1}{2}}\Omega$

$$v_n(y) := u_n\left(L^{\frac{1}{2}}y\right) \quad \text{and} \quad v(y) := u\left(L^{\frac{1}{2}}y\right).$$

Note that $v_n \to v$ in $L^1$ and that for $x \in \Omega$

$$\nabla v_n\left(L^{-\frac{1}{2}}x\right) = L^{\frac{1}{2}}\nabla u_n(x)$$

which yields

$$\left|\nabla v_n\left(L^{-\frac{1}{2}}x\right)\right| = \sqrt{\langle L\nabla u_n(x), \nabla u_n(x)\rangle}.$$

Thus we have

$$\begin{aligned}
\frac{1}{\gamma_n}\int_\Omega g(u_n, \gamma_n\nabla u_n)\, dx &= \frac{1}{\gamma_n}\int_\Omega f\left(v_n\left(L^{-\frac{1}{2}}x\right), \gamma_n\left|\nabla v_n\left(L^{-\frac{1}{2}}x\right)\right|\right) dx \\
&= \frac{\det L^{\frac{1}{2}}}{\gamma_n}\int_{L^{-\frac{1}{2}}\Omega} f(v_n, \gamma_n|\nabla v_n|)\, dy. \qquad (3.15)
\end{aligned}$$

Let us now define

$$h(u,\xi) := f\left(u, (\det L^{\frac{1}{2}})\xi\right).$$

Since $f$ is isotropic, we can write $h$ as function of $u$ and $|\xi|$ and we define

$$\tilde{P}(t) := \inf\left\{\int_0^t h(u,u')\,ds : \, u \in H^1(0,t),\, u(0)=0,\, u(t)=1\right\}.$$

Since the measure of the transition layer of $v_n$ is given by

$$\tilde{\gamma}_n := \frac{\gamma_n}{\det L^{\frac{1}{2}}}$$

and since $f$ is isotropic we can use the results of the previous section to estimate

$$\begin{aligned}
& \liminf_{n\to\infty} \frac{\det L^{\frac{1}{2}}}{\gamma_n} \int_{L^{-\frac{1}{2}}\Omega} f(v_n, \gamma_n|\nabla v_n|)\,dy \\
= \; & \liminf_{n\to\infty} \frac{1}{\tilde{\gamma}_n} \int_{L^{-\frac{1}{2}}\Omega} h(v_n, \tilde{\gamma}_n|\nabla v_n|)\,dy \\
\geq \; & \mathcal{H}^{N-1}\left(L^{-\frac{1}{2}}\Gamma\right) \tilde{P}\left(\frac{1}{\mathcal{H}^{N-1}\left(L^{-\frac{1}{2}}\Gamma\right)}\right) \\
= \; & \det L^{-\frac{1}{2}} \int_\Gamma |L^{\frac{1}{2}}\nu| d\mathcal{H}^{N-1} \tilde{P}\left(\frac{1}{\left(\det L^{-\frac{1}{2}}\right)\int_\Gamma \left|L^{\frac{1}{2}}\nu\right| d\mathcal{H}^{N-1}}\right),
\end{aligned} \tag{3.16}$$

where in the last equality we have used Lemma 3.6.
Defining

$$\varphi(t) := t\tilde{P}\left(\frac{1}{t}\right), \tag{3.17}$$

we deduce finally

$$\liminf_{n\to\infty} \frac{1}{\gamma_n} \int_\Omega g(u_n, \gamma_n \nabla u_n)\,dx \geq \varphi\left((\det L^{-1/2}) \int_\Gamma \psi(\nu(x))\,d\mathcal{H}^{N-1}(x)\right).$$

This concludes the proof of the Γ-liminf inequality. The Γ-limsup inequality can be proved in an analogous way. □

## Acknowledgment

I am grateful to Massimiliano Morini for his crucial contributions to this article. Moreover, I thank Irene Fonseca and James Serrin for interesting and valuable discussions. The initial work on this project has been supported by the Center for Nonlinear Analysis under NSF Grant DMS-9803791. Support by the Scuola Normale Superiore through a postdoctoral scholarship is gratefully acknowledged.

# References

[1] E. Acerbi, I. Fonseca, and G. Mingione, *A model for mixtures of micromagnetic materials allowing existence and regularity*, Progr. Nonlinear Differential Equations Appl., 35 (2002), pp. 1–8. Birkhäuser, Basel.

[2] ——, *Existence and regularity for mixtures of micromagnetic materials*, Preprint 19/2004, Center for Nonlinear Analysis, Carnegie Mellon University, Pittsburgh, USA, 2004.

[3] N. Aguilera, H. W. Alt, and L. A. Caffarelli, *An optimization problem with volume constraint*, SIAM J. Control Optim., 24 (1986), pp. 191–198.

[4] L. Ambrosio, I. Fonseca, P. Marcellini, and L. Tartar, *On a volume-constrained variational problem*, Arch. Ration. Mech. Anal., 149 (1999), pp. 23–47.

[5] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 2000.

[6] L. C. Evans, *Partial Differential Equations*, American Mathematical Society, Providence, R.I., 1998.

[7] E. Giusti, *Metodi diretti nel calcolo delle variazioni*, Unione Matematica Italiana, 1994.

[8] M. E. Gurtin, D. Polignone, and J. Vinals, *Two-phase binary fluids and immissible fluids described by an order parameter*, Math. Models Methods Appl. Sci., 6 (1996), pp. 815–831.

[9] L. Modica, *The gradient theory of phase transitions and the minimal interface criterion*, Arch. Rational Mech. Anal., 98 (1987), pp. 123–142.

[10] M. Morini and M. O. Rieger, *On a volume constrained variational problem with lower order terms*, Applied Mathematics and Optimization, 48 (2003), pp. 21–38.

[11] S. Mosconi and P. Tilli, *Variational problems with several volume constraints on the level sets*, Calc. Var. Partial Differential Equations, 14 (2002), pp. 233–247.

[12] P. Pucci and J. Serrin, *A note on the strong maximum principle for elliptic differential euqations*, Journal de Mathématiques pures et appliquées, 79 (2000), pp. 57–71.

[13] M. O. Rieger, *Abstract variational problems with volume constraints*, ESAIM: Control, Optimisation and Calculus of Variations, 10 (2004), pp. 84–98.

[14] E. Stepanov and P. Tilli, *On the Dirichlet problem with several volume constraints on the level sets*, Proc. Roy. Soc. Edinburgh Sect. A, 132 (2002), pp. 437–461.

[15] P. Tilli, *On a constrained variational problem with an arbitrary number of free boundaries*, Interfaces Free Bound., 2 (2000), pp. 201–212.